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THREEFOLD FLOPS VIA MATRIX FACTORIZATION

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Abstract

The explicit McKay correspondence, as formulated by Gonzalez-Sprinberg and Verdier, associates to each exceptional divisor in the minimal resolution of a rational double point, a matrix factorization of the equation of the rational double point. We study deformations of these matrix factorizations, and show that they exist over an appropriate "partially resolved" deformation space for rational double points of types A and D. As a consequence, all simple flops of lengths 1 and 2 can be described in terms of blowups defined from matrix factorizations. We also formulate conjectures which would extend these results to rational double points of type E and simple flops of length greater than 2.

The structure of birational maps between algebraic varieties becomes increasingly complicated as the dimension of the varieties increases. There is no birational geometry to speak of in dimension one: if two complete algebraic curves are birationally isomorphic, then they are biregularly isomorphic. In dimension two we encounter the phenomenon of the blowup of a point, and every birational isomorphism can be factored into a sequence of blowups and blowdowns. In dimension three, however, we first encounter birational maps which are biregular outside of a subvariety of codimension two (called the *center* of the birational map). When the center has a neighborhood with trivial canonical bundle, the birational map is called a *flop*. The focus of this

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paper will be the case of a *three-dimensional simple flop*, in which the center is an irreducible curve (necessarily a smooth rational curve). One of the motivations for studying this case is a theorem of Kawamata [17], which says that all birational maps between Calabi–Yau threefolds can be expressed as the composition of simple flops (in fact, of simple flops between non-singular varieties).

Important examples of simple flops were provided by Laufer [22], and three-dimensional simple flops were studied in general by Reid [25] and by Pinkham [24]. One fundamental property is that the center of the flop can be contracted, leaving a (singular) variety X which is dominated by both of the varieties involved in the original flop. X has a hypersurface singularity, and thus can be locally described as $\{f = 0\}$ for some polynomial f which vanishes at the origin. In fact, as observed by Kollár and Mori (see [20]), the defining polynomial can always be put into the form

$$f = x_1^2 + g(x_2, \ldots, x_m) \tag{0.1}$$

in appropriate coordinates, and the flop is then induced by the automorphism $x_1 \to -x_1$ of the singular variety X.

However, even in dimension three, most hypersurfaces of the form (0.1) do not admit *simple small resolutions*, that is, blowups in which the singular point is replaced by a non-singular rational curve, so the cited result of Kollár and Mori leaves the classification problem open. In 1992, Katz and the second author proved a detailed classification theorem for three-dimensional simple flops [16] in terms of a simple invariant of the singularity: the *length*, which is an integer between 1 and 6. Although this classification theorem in principle gives a complete description of three-dimensional simple flops, the present work began with a realization that the existing classification theorem did not specify sufficient detail about a small neighborhood of the center of a simple flop to answer some fundamental questions arising from string theory.

In the language of string theorists, those questions come from the study of a physical model in which a collection of D-branes is made to wrap a rational curve which is the center of a simple flop. The geometry of a neighborhood of the center dictates the physics of the model, but not enough information about such neighborhoods was available to answer some important physical questions about this model. We were led to the present work by an analogy between this situation and another situation considered by string theorists: the so-called *Landau–Ginzburg models* in string theory. These models also involve a single polynomial (analogous to f) and also have a D-brane theory which is tricky to describe. In the case of the Landau–Ginzburg models, Kontsevich [21] proposed (and a number of physicists investigated, beginning with [15] and summarized in [13]) that the D-branes in the theory should be described by the category of matrix factorizations [11] associated to the polynomial. By analogy, we hoped that matrix factorizations might be useful in studying simple flops, which has proven to be the case.

The idea behind matrix factorizations is quite simple. Although an irreducible polynomial $f(x_1, \ldots, x_m)$ can never be factored (by definition) in the ring $K[x_1, \ldots, x_m]$, there may well be *matrix factorizations*, that is, equations of the form

$$\Phi\Psi = f I_k, \tag{0.2}$$

where Φ and Ψ are $k \times k$ matrices with entries in $K[x_1, \ldots, x_m]$. A familiar example of matrix factorization is provided by the construction of Clifford algebras, the elements of which can be regarded as determining matrix factorizations for a quadratic polynomial $f(x_1, \ldots, x_m) = x_1^2 + \cdots + x_m^2$. In this paper, we consider matrix factorizations for rational double points and their deformations.

The classification theorem for three-dimensional simple flops [16] is based on Reid's observation [25] that the general hyperplane section of the contracted variety X has a rational double point singularity, and that each small resolution induces a partial resolution of that rational double point (dominated by the minimal resolution). Pinkham [24] analyzed the deformation theory of such partial resolutions, which provided the starting point for the classification theorem of [16]. In the present work, we begin with the observation that each partial resolution of a rational double point (dominated by the minimal resolution) has, by the McKay correspondence, an associated maximal Cohen–Macaulay module which can be described as a matrix factorization for the equation of the rational double point. We conjecture that Pinkham's deformation theory for partial resolutions is actually a deformation theory of the corresponding matrix factorization, and we prove this conjecture for rational double points of type A_{n-1} and D_n.

In the course of the proof, we encounter some "universal flops" of lengths 1 and 2; the analogous universal flops of lengths 3, 4, 5, and 6 are still conjectural (and would involve the E_6, E_7 and E_8 singularities). The universal flop of length 1 is very well known: it is the hypersurface singularity

$$xy - z^2 + t^2 = 0, \tag{0.3}$$

which admits two small resolutions by blowing up the ideals $(x, z + t)$ or $(y, z + t)$, respectively. A theorem of Reid [25] guarantees that every flop of length 1 is locally the pullback of the universal one by a map from X to the hypersurface in equation (0.3); our theorem about flops of length 2 is similar.

Recently, Bridgeland [5] has given a construction of flops as describing the passage from an initial space Y to a moduli space Y^+ for certain objects in the derived category of coherent sheaves on Y, and this construction and some related ideas have been applied in the physical situation described above [1, 2]. Our universal flops, and more generally the description of flops in terms of matrix factorizations, should enable one to make Bridgeland's construction and its applications much more explicit. (In particular, our approach seems closely related to van den Bergh's alternate proof [27] of Bridgeland's theorem.) We leave this for future investigation.

The outline of the paper is as follows. In section 1, we review the theory of matrix factorizations, and how the McKay correspondence for rational double points may be interpreted in terms of the existence of matrix factorizations. In section 2, we construct a geometric operation, the *Grassmann blowup*, associated to a matrix factorization, and show that it can be used to recover the partial resolutions of rational double points directly from the matrix factorization. In section 3, we review the basic technique for studying flops by considering the general hyperplane section of the contracted variety. In section 4, we consider the deformation theory for a matrix factorization, treating the case of A_{n-1} in detail and then conjecturing results for the other rational double points. Section 5 is devoted to the construction of the universal flop of length 2 and some of its properties. Finally, in section 6 we prove our conjectures for the case of rational double points of type D_n. In an appendix, we list the matrix factorizations for rational double points of types E_6, E_7, and E_8, essentially drawn from [12].

1. Matrix factorizations and the McKay correspondence

A *maximal Cohen–Macaulay module over a Noetherian local ring R* is an R-module whose depth is equal to the Krull dimension of R. We will primarily be interested in the case in which R is the localization at the origin of $S/(f)$, where $S = K[x_1, \ldots, x_m]$ is a polynomial ring over a field K, and $f \in S$ vanishes at the origin. For such a ring R, a maximal Cohen–Macaulay R-module can be lifted to an S-module M which is supported on the hypersurface $X = \{f = 0\}$; moreover, M will be locally free on the smooth locus of X.

Any such M can be presented as the cokernel of a map between two S-modules of the same rank:

$$S^{\oplus k} \xrightarrow{\Psi} S^{\oplus k} \to M \to 0. \tag{1.1}$$

Since $fM = 0$, $fS^{\oplus k} \subset \Psi(S^{\oplus k})$ which implies that there is a map

$$S^{\oplus k} \xrightarrow{\Phi} S^{\oplus k} \tag{1.2}$$

such that $\Phi \circ \Psi = f\,\mathrm{id}$. The pair (Φ, Ψ) is called a *matrix factorization*, and it turns out that $\Psi \circ \Phi = f\,\mathrm{id}$ as well, so that (Ψ, Φ) is also a matrix factorization.

Theorem (Eisenbud [11]). *There is a one-to-one correspondence between isomorphism classes of maximal Cohen–Macaulay modules over R with no free summands and matrix factorizations (Φ, Ψ) with no summand of the form $(1, f)$ (induced by an equivalence between appropriate categories).*

Matrix factorizations have been used to give an explicit realization of the McKay correspondence. Let G be a finite subgroup of $SU(2)$. The quotient singularity $\mathbb{C}^2/G$ has a number of interesting properties. First, if $\pi : Y \to \mathbb{C}^2/G$ is the minimal resolution of singularities, then $R^i\pi_*\mathcal{O}_Y = 0$ for $i > 0$, i.e., the singularity is *rational.* Moreover, each $\mathbb{C}^2/G$ is a hypersurface singularity of multiplicity 2—hence the name *rational double point,* since these two properties of the singularity characterize this class of singular points [10].

The original observation of McKay [23] gave a correspondence between the non-trivial irreducible representations of G and the components of $\pi^{-1}(0)$. The correspondence was made very explicit by Gonzalez-Sprinberg and Verdier [12] as follows. The action of the group G on the coordinate ring $\mathbb{C}[s,t]$ of $\mathbb{C}^2$ makes $\mathbb{C}[s,t]$ into a $\mathbb{C}[s,t]^G$-module. As such, it can be decomposed as

$$\mathbb{C}[s,t] = \bigoplus_\rho \mathbb{C}[s,t]_\rho, \tag{1.3}$$

where $\rho : G \to \mathrm{GL}(V_\rho)$ runs over the irreducible (complex) representations of G. Each summand $\mathbb{C}[s,t]_\rho$ takes the form $M_\rho \otimes_{\mathbb{C}} V_\rho$, where M_ρ is a maximal Cohen–Macaulay module for the ring $R = \mathbb{C}[s,t]^G$. Gonzales-Sprinberg and Verdier computed an explicit presentation of each such module, giving a matrix factorization[1] associated to M_ρ. They went on to show that, while M_ρ is not itself locally free, the pullback $\pi^*(M_\rho)/(\mathrm{tors})$ is locally free, and for each component E_i of $\pi^{-1}(0)$,

$$E_i \cdot c_1(\pi^*(M_\rho)/(\mathrm{tors})) = \delta_{ij_\rho}, \tag{1.4}$$

where E_{j_ρ} is the component of $\pi^{-1}(0)$ associated to ρ by the McKay correspondence.

To take a simple example, if $G = \mathbb{Z}/n\mathbb{Z} \subset SU(2)$, the ring $\mathbb{C}[s,t]^G$ is generated by $x = s^n$, $y = t^n$ and $z = st$ subject to the relation $xy - z^n = 0$.

[1] The fact that the computation gives a matrix factorization was not explicitly pointed out in the original paper, but became an important ingredient in subsequent work of Knörrer [19] and Buchweitz–Greuel–Schreyer [7].

(This shows that the quotient is a hypersurface singularity.) We label the representations of G so that ρ_k acts on $V_k \cong \mathbb{C}$ as $u \mapsto e^{2\pi i k/n}u$. It is not hard to see that $M_{\rho_k} := \mathbb{C}[s,t]_{\rho_k}$ is generated over $\mathbb{C}[s,t]^G$ by s^k and t^{n-k}, with relations

$$\begin{aligned} ys^k &= z^k t^{n-k}, \\ z^{n-k}s^k &= xt^{n-k}. \end{aligned} \tag{1.5}$$

This leads to a presentation of M_{ρ_k} as the cokernel of the matrix

$$\begin{bmatrix} y & -z^k \\ -z^{n-k} & x \end{bmatrix}. \tag{1.6}$$

We can easily complete this to a matrix factorization:

$$\begin{bmatrix} x & z^k \\ z^{n-k} & y \end{bmatrix} \begin{bmatrix} y & -z^k \\ -z^{n-k} & x \end{bmatrix} = (xy - z^n)I_2. \tag{1.7}$$

In general, the Gonzalez-Sprinberg–Verdier matrix factorizations have the following property: if the locally free module $\pi^*(M_\rho)/(\mathrm{tors})$ has rank ℓ_ρ, then the matrix factorization is given by a pair of $2\ell_\rho \times 2\ell_\rho$ matrices. Moreover, the integers ℓ_ρ are determined by the McKay correspondence and the relation

$$\pi^{-1}(\mathfrak{m}) = \sum \ell_\rho E_{j_\rho}, \tag{1.8}$$

where $\mathfrak{m}$ is the maximal ideal of $0 \in \mathbb{C}^2/G$.

The Gonzalez-Sprinberg–Verdier matrix factorizations of type A_{n-1} were given above, and those of type D_n can be obtained from the matrix factorizations of section 6 by specialization of parameters (see equations (6.14) and (6.15) below). For completeness, we have listed the Gonzalez-Sprinberg–Verdier matrix factorizations for E_6, E_7, and E_8 in an appendix to this paper.

2. Grassmann blowups

We now describe a blowup associated to a matrix factorization (or a more general resolution of sheaves). Given a matrix factorization (Φ, Ψ), the cokernel of Ψ is supported on the hypersurface $f = 0$; since we are assuming our hypersurface $f = 0$ is reduced and irreducible, the $k \times k$ matrix Ψ will have some generic rank $k - r$ along the hypersurface, and the cokernel of Ψ will have rank r.

More generally, we can consider maps Ψ whose cokernel has proper support $X \subset \mathbb{C}^n$, such that the cokernel has rank r at each generic point of the support.

We do a *Grassmann blowup*[2] *of* Ψ on which there is a locally free sheaf that agrees with $\operatorname{coker}\Psi$ at generic points, as follows. In the product $\mathbb{C}^n \times \operatorname{Gr}(k-r,k)$ we take the closure of the set[3]

$$\{(x,v) \mid x \in X_{\text{smooth}}, v = \ker \Psi_x\}. \tag{2.1}$$

There are natural coordinate charts for this blowup, given by Plücker coordinates for the Grassmannian, and a locally free sheaf defined by pulling back the universal quotient bundle from $\operatorname{Gr}(k-r,k)$.

The following lemma is immediate.

Lemma 2.1. *If $\nu : \widetilde{X} \to X$ is any birational map such that $\nu^*(\operatorname{coker}\Psi)/(\text{tors})$ is locally free, then ν factors through the normalization of the Grassmann blowup of Ψ.*

From this, and the computations in [12], we deduce:

Theorem 2.2. *Let $G \subset SU(2)$ be a finite group, let ρ be a non-trivial irreducible representation of G, let M_ρ be the associated maximal Cohen–Macaulay $\mathbb{C}[s,t]^G$-module, and let $\pi_\rho : X_\rho \to \mathbb{C}^2/G$ be the (normalization[4] of the) Grassmann blowup associated to a matrix factorization of M_ρ. Then X_ρ has only rational double point singularities, $\pi_\rho^{-1}(0)$ is an irreducible curve C_ρ, and the pullback of C_ρ to the minimal resolution coincides with E_{j_ρ}.*

In other words, the Grassmann blowup of the Gonzalez-Sprinberg–Verdier matrix factorization directly produces the associated curve for the McKay correspondence.

Proof. Let $\pi : Y \to \mathbb{C}^2/G$ be the minimal resolution of singularities. As remarked above, Gonzalez-Sprinberg and Verdier showed that $\pi^*(M_\rho)/(\text{tors})$ is locally free. Moreover, by rather intensive computation they showed that there is a curve Γ_ρ meeting E_{j_ρ} in a single point (and disjoint from other components of $\pi^{-1}(0)$) with $c_1(\pi^*(M_\rho))$ supported on Γ_ρ. Thus, for a sufficiently small open neighborhood U of $\overline{\pi^{-1}(0) - E_{j_\rho}}$, the locally free sheaf $\pi^*(M_\rho)/(\text{tors})$ restricts to a trivial bundle of rank ℓ_ρ on U.

It follows that if $\nu_\rho : \widetilde{X}_\rho \to \mathbb{C}^2/G$ is the partial resolution which blows up precisely the curve E_{i_ρ}, then $\pi^*(M_\rho)/(\text{tors})$ pushes forward to a locally free sheaf on $\widetilde{X}_\rho$, which must coincide with $\nu_\rho^*(M_\rho)/(\text{tors})$. Thus, by the lemma, the partial resolution $\nu_\rho : \widetilde{X}_\rho \to \mathbb{C}^2/G$ factors through a map $\widetilde{X}_\rho \to X_\rho$ to the normalization of the Grassmann blowup. Note that $X_\rho \to \mathbb{C}^2/G$ is non-trivial since M_ρ is not itself locally free; since in addition $\nu_\rho : \widetilde{X}_\rho \to \mathbb{C}^2/G$

[2] Note that when the module in question is the pushforward of the tangent bundle from the smooth locus of X, this construction coincides with the more familiar *Nash blowup*.

[3] We could have equally well formulated this definition in terms of $\operatorname{Gr}(r,k)$ and $\operatorname{coker}\Psi_x$, but the equivalent formulation we give is more convenient for computation.

[4] In fact, the Grassmann blowup is normal in this case.

has irreducible exceptional set, it follows that the map $\widetilde{X}_\rho \to X_\rho$ must be an isomorphism, with E_{j_ρ} mapping to the exceptional set C_ρ. □

The proof we have given is unfortunately rather computationally intensive, insofar as it relies on the detailed explicit computations of [12]. We strongly suspect that a more direct proof should be possible, using the ideas of Ito–Nakamura [14], but we have not carried this out.

In the remainder of this paper, we will study deformations of the Gonzalez-Sprinberg–Verdier matrix factorizations, and their Grassmann blowups.

3. Simultaneous resolution and flops

A *simple flop* is a birational map $Y \dashrightarrow Y^+$ between Gorenstein threefolds which induces an isomorphism $(Y - C) \cong (Y^+ - C^+)$, where C and C^+ are smooth rational curves on Y and Y^+, respectively, and

$$K_Y \cdot C = K_{Y^+} \cdot C^+ = 0. \tag{3.1}$$

The curves C and C^+ can be contracted to points (in Y and Y^+, respectively), yielding the same normal variety X.

Recall that a *two-dimensional ordinary double point* is a singular hypersurface with defining polynomial

$$xy - z^2. \tag{3.2}$$

Such a singularity has a versal deformation with defining polynomial

$$xy - z^2 + s = 0, \tag{3.3}$$

where s is the coordinate in the versal deformation space. In 1958, Atiyah [3] noticed that if he made the base change $s = t^2$ in this versal deformation, then the resulting family of surfaces had a *simultaneous resolution of singularities.* That is, factoring the equation

$$xy - z^2 + t^2 = 0 \tag{3.4}$$

as

$$xy = (z + t)(z - t) \tag{3.5}$$

and blowing up the (non-Cartier) divisor described by $x = z + t = 0$, one obtains a family of non-singular surfaces which, for each value of t, resolves the corresponding singular surface. In fact, there are two ways of doing this, for one might have chosen to blow up the divisor $y = z + t = 0$ instead. This produces two threefolds Y and Y^+ related by a simple flop.

Reid [25] studied a generalization of Atiyah's flop with equation of the form

$$xy = z^2 - t^{2n} = (z + t^n)(z - t^n), \tag{3.6}$$

where again the blowups are given by $x = z + t^n = 0$ or $y = z + t^n = 0$. Other, more complicated examples of simple flops were given by Laufer [22], and generalized by Pinkham and the second author [24] and by Reid [25].

A generalization of Atiyah's observation about simultaneous resolutions was made by Brieskorn [6] and Tyurina [26], who showed that for any rational double point, the universal family over the versal deformation space admits a simultaneous resolution after base change. More precisely, each rational double point has an associated Dynkin diagram Γ whose Weyl group $\mathcal{W} = \mathcal{W}(\Gamma)$ acts on the complexification $\mathfrak{h}_{\mathbb{C}}$ of the Cartan subalgebra $\mathfrak{h}$ of the associated Lie algebra $\mathfrak{g} = \mathfrak{g}(\Gamma)$. A model for the versal deformation space is given by

$$\mathrm{Def} = \mathfrak{h}_{\mathbb{C}}/\mathcal{W}, \tag{3.7}$$

and there is a universal family $\mathcal{X} \to \mathrm{Def}$ of deformations of the rational double point over that space.

The deformations of the resolution are given by a representable functor, which can be modeled by

$$\mathrm{Res} = \mathfrak{h}_{\mathbb{C}}, \tag{3.8}$$

and the base change which relates Res to Def coincides with the quotient map $\mathfrak{h}_{\mathbb{C}} \to \mathfrak{h}_{\mathbb{C}}/\mathcal{W}$. In fact, there is a universal simultaneous resolution $\widehat{\mathcal{X}}$ of the family $\mathcal{X} \times_{\mathrm{Def}} \mathrm{Res}$. The construction of this resolution requires some trickiness with the algebra which will in fact be generalized and somewhat explained later in this paper.

Pinkham [24] adapted the work of Brieskorn and Tyurina to cases in which one does not wish to fully resolve the rational double point, but only to partially resolve it. If we let $\Gamma_0 \subset \Gamma$ be the subdiagram for the part of the singularity that is not being resolved, then we can define a functor of deformations of the partial resolution, which has a model

$$\mathrm{PRes}(\Gamma_0) = \mathfrak{h}_{\mathbb{C}}/\mathcal{W}(\Gamma_0), \tag{3.9}$$

and there is a simultaneous partial resolution $\widehat{\mathcal{X}}(\Gamma_0)$ of the family $\mathcal{X} \times_{\mathrm{Def}} \mathrm{PRes}(\Gamma_0)$.

By a lemma of Reid [25], given a simple flop from Y to Y^+ and the associated small contraction $Y \to X$, the general hyperplane section of X through the singular point P has a rational double point at P, and the proper transform of that surface on Y gives a partial resolution of the rational double point (dominated by the minimal resolution). Pinkham [24] used this to give a construction for all Gorenstein threefold singularities with small resolutions (with irreducible exceptional set): they can be described as pullbacks of the universal family via a map from the disk to $\mathrm{PRes}(\Gamma_0)$ (for some $\Gamma_0 \subset \Gamma$ which is the complement of a single vertex).

Note that in the examples of Atiyah and Reid, one starts with the versal deformation of A_1 given by equation (3.3) whose deformation space Def has coordinate s; pulling this back via the map $s = t^{2n}$ yields equation (3.6). The fact that the power of t is even means that the map $s = t^{2n}$ factors through the degree two cover Res $\to$ Def, as expected from the general theory.

Kollár [8] introduced an invariant of simple flops called the *length*: it is defined to be the generic rank of the sheaf on C defined as the cokernel of $f^*(\mathfrak{m}_P) \to \mathcal{O}_Y$, where $\mathfrak{m}_P$ is the maximal ideal of the singular point P. It is easy to see that the length can be computed from the hyperplane section, and it coincides with the coefficient of the corresponding vertex in the Dynkin diagram in the linear combination of vertices which yields the longest positive root in the root system. In the Atiyah and Reid cases, the length is 1.

As mentioned in the introduction, Katz and the second author [16] classified simple flops by showing that the generic hyperplane section for a simple flop of length ℓ is the smallest rational double point which uses ℓ as a coefficient in the maximal root. The proof was computationally intensive, and Kawamata [18] later gave a short and direct proof of this result.

4. Deformations

The versal deformation of the A_{n-1} singularity has defining polynomial

$$xy - f_n(z) = 0, \tag{4.1}$$

where f is a general monic polynomial of degree n whose coefficient of z^{n-1} vanishes. The coefficients of f are the coordinates on Def; the roots of f give coordinates on Res (subject to the constraint that the sum of the roots is zero). Note that the action of the Weyl group $\mathfrak{W}_{A_{n-1}}$ coincides with the standard action of symmetric group $\mathfrak{S}_n$ on the n roots of f; the invariants of this action—the elementary symmetric functions—are the coefficients of f.

The partial resolution corresponding to the k^{th} vertex in the Dynkin diagram corresponds in the invariant theory to the subgroup $\mathcal{W}(\Gamma_0) \subset \mathcal{W}(\Gamma)$, which in this case is

$$\mathfrak{S}_k \times \mathfrak{S}_{n-k} \subset \mathfrak{S}_n. \tag{4.2}$$

The relationship between the invariants of these two groups is neatly summarized by writing

$$f_n(z) = g_k(z)h_{n-k}(z), \tag{4.3}$$

where g and h are monic polynomials of degrees k and $n-k$ whose coefficients of z^{k-1} and z^{n-k-1}, respectively, sum to zero. More precisely, the coefficients of f give generators for the invariant theory of $\mathcal{W}(\Gamma)$ while the coefficients of g and h give generators for the invariant theory of $\mathcal{W}(\Gamma_0)$, with the relationship between them specified by equation (4.3). Thus, the coefficients of g and h give coordinates on the partial resolution space $\mathrm{PRes} = \mathrm{Res}\,/\mathcal{W}(\Gamma_0)$.

It is then easy to see that the matrix factorization (1.7) associated to the k^{th} vertex of A_{n-1} deforms to a matrix factorization defined over PRes: just use

$$\begin{bmatrix} x & g_k(z) \\ h_{n-k}(z) & y \end{bmatrix} \begin{bmatrix} y & -g_k(z) \\ -h_{n-k}(z) & x \end{bmatrix} = \big(xy - f_n(z)\big) I_2. \tag{4.4}$$

This matrix factorization encodes the special form of the equation which was needed in order to find non-Cartier divisors to blow up. In fact, we can reinterpret the traditional description of these blowups as being the Grassmann blowups associated to $\operatorname{coker}(\Psi)$ and $\operatorname{coker}(\Phi)$, which are normal. (This reinterpretation of the traditional description of flops of length 1 is what we will generalize in this paper.)

To see this, we explicitly carry out the Grassmann blowup associated to the matrix

$$\Psi = \begin{bmatrix} y & -g_k(z) \\ -h_{n-k}(z) & x \end{bmatrix}. \tag{4.5}$$

There are two coordinate charts, in which we use as a basis for the kernel of Ψ the vectors $[\alpha\ 1]^T$ and $[1\ \beta]^T$, respectively. In the first coordinate chart, the equation (in matrix form) defining the variety is

$$\Psi \begin{bmatrix} \alpha \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{4.6}$$

or in other words,

$$\begin{aligned} y\alpha - g_k(z) &= 0, \\ -h_{n-k}(z)\alpha + x &= 0. \end{aligned} \tag{4.7}$$

Using the second equation, x can be eliminated, leaving the single equation

$$y\alpha - g_k(z) = 0, \tag{4.8}$$

which defines the versal deformation of an A_{k-1} singularity.

In the second chart, we have the matrix equation

$$\Psi \begin{bmatrix} 1 \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{4.9}$$

or in other words,

$$\begin{aligned} y - g_k(z)\beta &= 0, \\ -h_{n-k}(z) + x\beta &= 0. \end{aligned} \tag{4.10}$$

This time, the first equation allows us to eliminate y, leaving the single equation

$$x\beta - h_{n-k}(z) = 0, \tag{4.11}$$

which defines the versal deformation of an A_{n-k-1} singularity.

Thus, the Grassmann blowup has produced an exceptional $\mathbb{P}^1$ at two points of which the pulled-back family has two residual singularities along the central fiber (A_{k-1} and A_{n-k}), and due to the versal deformations of those singularities appearing on the universal family, every such deformation can be obtained in this way.

Inspired by this case, and the D_n case to be discussed in section 6, we formulate the following conjectures:

Conjecture 1. *For every flop of length ℓ, there are two maximal Cohen–Macaulay modules M and M^+ on X of rank ℓ, such that Y (resp. Y^+) is the Grassmann blowup of M (resp. M^+).*

Conjecture 2. *For a flop of length ℓ, the matrix factorizations corresponding to M and M^+ are of size $2\ell \times 2\ell$, and are obtained from each other by switching the factors $(\Phi, \Psi) \to (\Psi, \Phi)$. Moreover, if coordinates are chosen so that the equation of the hypersurface X takes the form $x_1^2 + f(x_2, x_3, x_4) = 0$, then the matrices Φ and Ψ can be chosen to take the form*

$$\Phi = x_1 I_{2\ell} - \Xi, \quad \Psi = x_1 I_{2\ell} + \Xi, \tag{4.12}$$

where Ξ is a $2\ell \times 2\ell$ matrix whose entries are functions of x_2, x_3, x_4. (This is expected thanks to the result of Kollár and Mori (see [20]*), which implies that the flop will be induced by the automorphism $x_1 \to -x_1$.)*

Conjecture 3. *For a partial resolution of a rational double point corresponding to a single vertex in the Dynkin diagram with coefficient ℓ in the maximal root, the versal deformation $\mathcal{X}$ over* PRes *has two matrix factorizations of size $2\ell \times 2\ell$, such that the two simultaneous partial resolutions can be obtained as the Grassmann blowups of the corresponding Cohen–Macaulay modules. (These should be regarded as deformations of the Gonzalez-Sprinberg–Verdier factorizations.) Moreover, the matrices take the special form $\Phi = xI_{2\ell} - \Xi$, $\Phi = xI_{2\ell} + \Xi$ in appropriate coordinates.*

Main Theorem. *Conjectures* 1 *and* 2 *hold for lengths* 1 *and* 2*; Conjecture* 3 *holds for Dynkin diagrams of type* A_{n-1} *and* D_n*.*

Remarks.

(1) By the classification theorem of [16], in order to prove Conjectures 1 and 2 for lengths 1 and 2, it suffices to prove Conjecture 3 for Dynkin diagrams of type A_1 and D_4.

(2) It would follow from Conjecture 3, combined with the classification theorem of [16], that there exists a universal flop of length ℓ for each $1 \le \ell \le 6$. Each such universal flop would be obtained from the universal matrix factorization for Dynkin diagram of length ℓ and types A_1, D_4, E_6, E_7, E_8 (length 5) and E_8 (length 6), respectively.

(3) Note that we have already given most of the proof for the A_{n-1} case of Conjecture 3 in our discussion above. We need one additional detail to complete the proof: making the change of coordinates $x = u - v$, $y = u + v$ allows us to write the A_{n-1} matrix factorization in the form $\Phi = uI_2 - \Xi$, $\Psi = uI_2 + \Xi$, where

$$\Xi = \begin{bmatrix} v & h_{n-k}(z) \\ g_k(z) & -v \end{bmatrix}, \tag{4.13}$$

verifying Conjecture 3 in this case.

5. The universal flop of length 2

In this section, we investigate a certain flop of length 2 which will turn out to be universal in an appropriate sense. Our description of this flop follows an idea of Reid [25, Lemma (5.16)] although this was not the way we originally found the flop.

We start with a quadratic equation in four variables x, y, z, t over the field $\mathbb{C}(u, v, w)$, chosen so that its discriminant is a perfect square. The one we use can be written in matrix form as

$$\begin{aligned} W(x,y,z,t,u,v,w) &:= x^2 + uy^2 + 2vyz + wz^2 + (uw - v^2)t^2 \\ &= \begin{bmatrix} x & y & z & t \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u & v & 0 \\ 0 & v & w & 0 \\ 0 & 0 & 0 & uw - v^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}, \end{aligned} \tag{5.1}$$

and its discriminant (the determinant of the matrix of coefficients) is

$$(uw - v^2)^2. \tag{5.2}$$

(Reid's construction was similar, but had $v = 0$; as Reid observed, this construction includes both the original Laufer examples [22] and their generalizations by Pinkham and the second author [24].) The general quadratic hypersurface in $\mathbb{C}^4$ has two rulings by $\mathbb{C}^2$; since the discriminant is a perfect square, the individual rulings are already defined over $\mathbb{C}(u, v, w)$. In fact, the corresponding rank two sheaf is a maximal Cohen–Macaulay module over the hypersurface defined by equation (5.1) (which we now regard as a hypersurface in $\mathbb{C}^7$).

By Eisenbud's theorem, this maximal Cohen–Macaulay module can be expressed in terms of a matrix factorization (Φ, Ψ). We have computed one such, which takes the form $\Phi = xI_4 - \Xi$, $\Psi = xI_4 + \Xi$, where

$$\Xi = \begin{bmatrix} -vt & y & z & t \\ -uy - 2vz & vt & -ut & z \\ -wz & wt & -vt & -y \\ -uwt & -wz & uy + 2vz & vt \end{bmatrix}. \tag{5.3}$$

An explicit computation shows that

$$-\Xi^2 = \left(uy^2 + 2vyz + wz^2 + (uw - v^2)t^2\right) I_4, \tag{5.4}$$

and hence

$$\Phi\Psi = \left(x^2 + uy^2 + 2vyz + wz^2 + (uw - v^2)t^2\right) I_4. \tag{5.5}$$

A few comments about the geometry are in order. The quadratic form in x, y, z, t has rank 4 for generic values of the parameters, but when $uw - v^2 = 0$, the rank drops to 2. We get rank 1 at $u = v = w = 0$. This implies that over each point of $uw - v^2 = 0$, the fiber of the Grassmann blowup will be contained in a $\mathbb{P}^2$ (corresponding to the choice of $\mathbb{C}^2$ within $\mathbb{C}^3$) and this remains true at $u = v = w = 0$ as well. Moreover, over $x = y = z = t = 0$ we get a fiber contained in $\mathrm{Gr}(2, 4)$. As we will see in detail below, the latter fibers are actually conics embedded into $\mathrm{Gr}(2, 4)$ and all fibers of the Grassmann blowup have dimensions 0 or 1.

Theorem 5.1. *For every threefold flop of length 2, there is a map from the singular space X to the universal flop of length 2 such that the two blowups Y and Y^+ are the pullbacks of the Grassmann blowups of the universal matrices Φ and Ψ.*

The first step in proving Theorem 5.1 is to explicitly carry out the Grassmann blowup of Ψ. There are six coordinate charts for this blowup, corresponding to Plücker coordinates on the Grassmannian, but we will only display the results in detail for two of these charts. For the first, we introduce four new variables $\alpha_{i,j}$ and eight equations $\lambda_{i,j}$ for the blowup by means of

$$[\lambda_{i,j}] = \Psi \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{bmatrix} = 0. \tag{5.6}$$

Note that by multiplying by Φ from the left, we see that the original equation is contained in the ideal generated by the $\lambda_{i,j}$.

It is not difficult to find some elements of the ideal that are divisible by $z^2 + ut^2 = \partial W/\partial w$:

$$\begin{aligned} z\lambda_{1,1} - ut\lambda_{1,2} - t\lambda_{2,1} - z\lambda_{2,2} &= (z^2 + ut^2)(\alpha_{1,1} - \alpha_{2,2}), \\ -ut\lambda_{1,1} - uz\lambda_{1,2} - z\lambda_{2,1} + ut\lambda_{2,2} &= (z^2 + ut^2)(2v - \alpha_{1,2}u - \alpha_{2,1}), \\ (\alpha_{1,2}uz + \alpha_{2,2}ut)\lambda_{1,2} + (-\alpha_{1,2}ut + \alpha_{2,2}z)\lambda_{2,2} &+ ut\lambda_{3,2} - z\lambda_{4,2} \\ &= (z^2 + ut^2)(\alpha_{2,2}^2 + \alpha_{1,2}^2 u - 2\alpha_{1,2}v + w). \end{aligned} \tag{5.7}$$

Thus, if we localize at $z^2 + ut^2 \neq 0$, we can add the following elements to the ideal:

$$\begin{aligned} \lambda_1 &= \alpha_{1,1} - \alpha_{2,2}, \\ \lambda_2 &= 2v - \alpha_{1,2}u - \alpha_{2,1}, \\ \lambda_3 &= \alpha_{2,2}^2 + \alpha_{1,2}^2 u - 2\alpha_{1,2}v + w. \end{aligned} \tag{5.8}$$

Since the Grassmann blowup is irreducible, the elements of this extended ideal must vanish on it, even when we do not impose $z^2 + ut^2 \neq 0$.

In fact, the extended ideal is generated by $\lambda_{1,2}$, $\lambda_{2,2}$, λ_1, λ_2, and λ_3, as we now verify: we have

$$\begin{aligned} \lambda_{1,1} &= \lambda_{2,2} + z\lambda_1 - t\lambda_2, \\ \lambda_{2,1} &= -u\lambda_{1,2} - ut\lambda_1 - z\lambda_2, \\ \lambda_{3,1} &= (\alpha_{1,2}u - 2v)\lambda_{1,2} + \alpha_{2,2}\lambda_{2,2} + (x - vt)\lambda_1 + y\lambda_2 - z\lambda_3, \\ \lambda_{3,2} &= -\alpha_{2,2}\lambda_{1,2} + \alpha_{1,2}\lambda_{2,2} + t\lambda_3, \\ \lambda_{4,1} &= \alpha_{1,1}u\lambda_{1,2} + \alpha_{2,1}\lambda_{2,2} + (-\alpha_{1,2}uz - \alpha_{2,2}ut + 2vz)\lambda_1, \\ &\qquad + (\alpha_{2,2}z - \alpha_{1,2}ut)\lambda_2 - ut\lambda_3, \\ \lambda_{4,2} &= \alpha_{1,2}u\lambda_{1,2} + \alpha_{2,2}\lambda_{2,2} - z\lambda_3. \end{aligned} \tag{5.9}$$

Moreover, since $\lambda_{1,2} = y + \alpha_{1,2}z + \alpha_{2,2}t$ and $\lambda_{2,2} = x + vt - \alpha_{1,2}ut + \alpha_{2,2}z$, we can use the generators of the extended ideal to eliminate y, x, $\alpha_{1,1}$, $\alpha_{2,1}$, and w, respectively, leaving z, t, u, v, $\alpha_{1,2}$ and $\alpha_{2,2}$ as coordinates in this chart, with no further relations among them.

We can describe the map from the Grassmann blowup back to the original hypersurface by setting each generator $\lambda_{1,2}$, $\lambda_{2,2}$, λ_1, λ_2, λ_3 of the ideal to zero in turn, and solving the resulting equation for an appropriate variable:

$$y = -\alpha_{1,2}z - \alpha_{2,2}t, \tag{5.10}$$

$$x + vt = \alpha_{1,2}ut - \alpha_{2,2}z, \tag{5.11}$$

$$\alpha_{1,1} = \alpha_{2,2}, \tag{5.12}$$

$$\alpha_{2,1} = 2v - \alpha_{1,2}u, \tag{5.13}$$

$$w = -\alpha_{2,2}^2 - \alpha_{1,2}^2 u + 2\alpha_{1,2}v. \tag{5.14}$$

Equations (5.10) and (5.11) can be recast in matrix form:

$$\begin{bmatrix} y \\ x+vt \end{bmatrix} = \begin{bmatrix} -z & -t \\ ut & -z \end{bmatrix} \begin{bmatrix} \alpha_{1,2} \\ \alpha_{2,2} \end{bmatrix}, \tag{5.15}$$

and whenever $z^2 + ut^2 \neq 0$, these can be solved for $\alpha_{1,2}$ and $\alpha_{2,2}$ in terms of x, y, z, t, u, v. Equations (5.12) and (5.13) can then be used to solve for $\alpha_{1,1}$ and $\alpha_{2,1}$, and when this has been done, equation (5.14) becomes equivalent to the original hypersurface equation (5.1). Thus, the fiber over a given (x, y, z, t, u, v, w) value is a single point unless $z^2 + ut^2 = 0$.

When $z^2 + ut^2 = 0$, if the coefficient matrix

$$\begin{bmatrix} -z & -t \\ ut & -z \end{bmatrix} \tag{5.16}$$

from equation (5.15) has rank one, then the fiber over a given point takes the form

$$(\alpha_{1,2}, \alpha_{2,2}) \mapsto (\alpha_{1,2} + ct, \alpha_{2,2} - cz). \tag{5.17}$$

Substituting this into equation (5.14) yields the additional relation

$$\alpha_{2,2}z - \alpha_{1,2}ut + tv = 0 \tag{5.18}$$

which must hold if the fiber is non-trivial. When this relation and $z^2 + ut^2 = 0$ both hold, the fiber is indeed one-dimensional.

On the other hand, if the matrix (5.16) has rank 0, then $z = t = 0$ which implies $x = y = 0$. Any such point satifies equations (5.10)–(5.13), leaving only equation (5.14), which describes—for each fixed (u, v, w)—a conic embedded in the Grassmannian mapping to $(0, 0, 0, 0, u, v, w)$, as asserted at the beginning of this section.

For later use, we summarize a portion of the structure of one of the other coordinate charts, in less detail than above. For this second chart, we introduce four new variables $\beta_{i,j}$ and eight new equations $\mu_{i,j}$ for the blowup by means of

$$[\mu_{i,j}] = \Psi \begin{bmatrix} 1 & 0 \\ \beta_{1,1} & \beta_{1,2} \\ 0 & 1 \\ \beta_{2,1} & \beta_{2,2} \end{bmatrix}. \tag{5.19}$$

Note that by multiplying by Φ from the left, we see that the original equation is contained in the ideal generated by the $\mu_{i,j}$.

We find some elements divisible by $y^2 + wt^2 = \partial W/\partial u$:

(5.20)

$$\begin{gathered} y\mu_{1,1} + wt\mu_{1,2} + t\mu_{3,1} - y\mu_{3,2} = (y^2+wt^2)(\beta_{1,1}+\beta_{2,2}), \\ -wt\mu_{1,1} + wy\mu_{1,2} + y\mu_{3,1} + wt\mu_{3,2} = (y^2+wt^2)(-\beta_{2,1}+\beta_{1,2}w), \\ (-2vy+\beta_{1,2}wy+\beta_{2,2}wt)\mu_{1,2} + (\beta_{1,2}wt - \beta_{2,2}y)\mu_{3,2} - wt\mu_{2,2} + y\mu_{4,2} \\ = (y^2+wt^2)(\beta_{2,2}^2 + u - 2\beta_{1,2}v + \beta_{1,2}^2 w). \end{gathered}$$

Localizing at $y^2 + wt^2 \neq 0$, we can add the following elements to the ideal:

$$\begin{aligned} \mu_1 &= \beta_{1,1} + \beta_{2,2}, \\ \mu_2 &= -\beta_{2,1} + \beta_{1,2}w, \\ \mu_3 &= \beta_{2,2}^2 + u - 2\beta_{1,2}v + \beta_{1,2}^2 w. \end{aligned} \tag{5.21}$$

As before, the irreduciblity of the Grassmann blowup ensures that the elements of the extended ideal vanish on it, even when we do not impose $y^2 + wt^2 \neq 0$.

In this chart, the extended ideal is generated by $\mu_{1,2}$, $\mu_{3,2}$, μ_1, μ_2, and μ_3, as we now demonstrate.

$$\begin{aligned} \mu_{1,1} &= \mu_{3,2} + y\mu_1 - t\mu_2, \\ \mu_{2,1} &= (-2v+\beta_{1,2}w)\mu_{1,2} - \beta_{2,2}\mu_{3,2} + (x+vt)\mu_1 - z\mu_2 - y\mu_3, \\ \mu_{2,2} &= \beta_{2,2}\mu_{1,2} + \beta_{1,2}\mu_{3,2} - t\mu_3, \\ \mu_{3,1} &= -w\mu_{1,2} + wt\mu_1 + y\mu_2, \\ \mu_{4,1} &= \beta_{2,2}w\mu_{1,2} + \beta_{1,2}w\mu_{3,2} - wz\mu_1 - (x+vt)\mu_2 - wt\mu_3, \\ \mu_{4,2} &= (2v - \beta_{1,2}w)\mu_{1,2} + \beta_{2,2}\mu_{3,2} + y\mu_3. \end{aligned} \tag{5.22}$$

Note that since

$$\begin{aligned} \mu_{1,2} &= \beta_{1,2}y + z + \beta_{2,2}t, \\ \mu_{3,2} &= \beta_{1,2}wt + x - vt - \beta_{2,2}y, \end{aligned} \tag{5.23}$$

the Grassmann blowup is also non-singular in this coordinate chart: we can eliminate z with $\mu_{1,2}$, x with $\mu_{3,2}$, $\beta_{1,1}$ with μ_1, $\beta_{2,1}$ with μ_2, and u with μ_3. This leaves y, t, v, w, $\beta_{1,2}$, and $\beta_{2,2}$ as coordinates in this chart.

6. The D_n case

The main results of this paper concern deformations of matrix factorizations for rational double points of type D_n. In this section, we will prove Conjecture 3 for the D_n case, from which Conjectures 1 and 2 in the length 2 case immediately follow. We will also prove the universality of the flop discussed in section 5 (i.e., Theorem 5.1).

We follow notation for the versal deformation of D_n and the invariant theory for the corresponding Weyl group which was established in [16] (with one minor exception, noted below). The matrix factorizations we use were first found in [9].

The versal deformation of D_n can be written in the form

$$X^2+Y^2Z-Z^{n-1}+2\gamma Y-\sum_{i=1}^{n-1}\delta_{2i}Z^{n-1-i}=X^2+Y^2Z+2\gamma Y-F(Z), \tag{6.1}$$

where $F(Z)$ is a general monic polynomial of degree $n-1$; the coefficients of $F(Z)$, together with γ, give coordinates on the deformation space Def.

The invariant theory for $\mathfrak{W}_{D_n}$ can be described in terms of the monic polynomial

$$ZF(Z)+\gamma^2=Z^n+\sum_{i=1}^{n-1}\delta_{2i}Z^{n-i}+\gamma^2=\prod_{j=1}^{n}(Z+t_j^2), \tag{6.2}$$

which factors over the resolution space Res (the coordinates on which are t_1, $\dots$, t_n). The Weyl group $\mathfrak{W}_{D_n}$ is an extension of the symmetric group $\mathfrak{S}_n$ (acting by permutations of the t_j's) by a group $(\boldsymbol{\mu}_2)^{n-1}$. The latter is the subgroup of $(\boldsymbol{\mu}_2)^n$ (acting on the t_j's by coordinatewise multiplication) which preserves the product $t_1\dots t_n$. The $\mathfrak{W}_{D_n}$-invariant functions are generated by $\delta_{2i}=\sigma_i(t_1^2,\dots,t_n^2)$, the elementary symmetric functions of t_1^2, $\dots$, t_n^2, together with[5] $\gamma=(-1)^n t_1\cdots t_n$, subject to

$$\delta_{2n}=\gamma^2. \tag{6.3}$$

This is the same set of functions given by the coefficients of $F(Z)$ together with γ, verifying that Def $=$ Res $/\mathfrak{W}_{D_n}$ in this case.

Note that this setup continues to make sense for low values of n: $\mathfrak{W}_{D_3}$ is an extension of $\mathfrak{S}_3$ by $(\boldsymbol{\mu}_2)^2$ which coincides with $\mathfrak{W}_{A_3}=\mathfrak{S}_4$; $\mathfrak{W}_{D_2}$ is an extension of $\mathfrak{S}_2$ by $\boldsymbol{\mu}_2$ and coincides with $\mathfrak{W}_{A_1\cup A_1}=\mathfrak{S}_2\times\mathfrak{S}_2$; and $\mathfrak{W}_{D_1}$ is trivial.

[5]The sign in the definition of γ was not present in [16], but is convenient here. When comparing formulas in this paper to those in [16], one should replace Y by $(-1)^nY$ to compensate for this sign change.

The partial resolution corresponding to the k^{th} vertex in the Dynkin diagram has complementary graph Γ_0 of the form $\Gamma_{A_{k-1}} \cup \Gamma_{D_{n-k}}$, for $1 \le k \le n-2$, or $k = n$. (The partial resolution for the $n{-}1^{\text{st}}$ vertex can be obtained from that for the n^{th} by applying an automorphism to the Dynkin diagram.) The corresponding subgroup in the invariant theory is

$$\mathfrak{S}_k \times \mathfrak{W}_{D_{n-k}} \subset \mathfrak{W}_{D_n}, \tag{6.4}$$

which can be considered for any k between 1 and n. Note, however, that this subgroup in the case of $k = n-1$ corresponds to the complement of *two* vertices in the Dynkin diagram—the two "short legs."

The two polynomials whose coefficients (together with $\eta = (-1)^{n-k} t_{k+1} \cdots t_n$) capture the invariant theory for the subgroup (6.4) are

$$f(U) = \prod_{j=1}^{k} (U - t_j) \text{ and } Zh(Z) + \eta^2 = \prod_{j=k+1}^{n} (Z + t_j^2). \tag{6.5}$$

In particular, the coefficients of f and h, together with η, give coordinates on the space $\mathrm{PRes} = \mathrm{Res}/\mathcal{W}(\Gamma_0)$. To describe the map $\mathrm{PRes} \to \mathrm{Def}$, we relate the coordinates on the two spaces, i.e., we relate the polynomials in equation (6.5) to the original polynomial (6.2), as follows. First write

$$f(U) = Q(-U^2) + UP(-U^2), \tag{6.6}$$

encoding the coefficients of f into the coefficients of P and Q. Note that if $Z = -U^2$, then

$$\prod_{j=1}^{k} (Z + t_j^2) = f(U)f(-U) = Q(Z)^2 + ZP(Z)^2 \tag{6.7}$$

so that

$$\prod_{j=1}^{n} (Z + t_j^2) = (Zh(Z) + \eta^2)(Q(Z)^2 + ZP(Z)^2) \tag{6.8}$$

and that $\gamma = \eta Q(0)$, since $Q(0) = f(0) = (-1)^k t_1 \cdots t_k$. We also write $Q(Z) = ZS(Z) + Q(0)$ when needed: the coefficents of S, together with $Q(0)$, are equivalent to the coefficients of Q.

We can now put the polynomial,

$$F(Z) = \frac{1}{Z}\left(-\gamma^2 + \prod_{j=1}^{n} (Z + t_j^2)\right), \tag{6.9}$$

which appears in the versal deformation of D_n into the form

$$F(Z) = h(Z)\big(Q(Z)^2 + ZP(Z)^2\big) + \eta^2\big(2Q(0)S(Z) + ZS(Z)^2 + P(Z)^2\big); \tag{6.10}$$

this, together with the formula $\gamma = \eta Q(0)$, provides the explicit map from PRes to Def.

For later use, we also observe that the relationship among $P(Z)$, $Q(Z)$, and $f(U)$ is captured by the existence of a polynomial in two variables $G(Z,U)$ satisfying

$$UP(Z) + Q(Z) = (U^2 + Z)G(Z,U) + f(U). \tag{6.11}$$

This invariant theory analysis has provided precisely the functions we need in order to use the universal length 2 matrix factorization. If we substitute

$$\begin{gathered} x = X,\ y = Y - \eta S(Z),\ z = Q(Z),\ t = P(Z), \\ u = Z,\ v = \eta,\ w = -h(Z), \end{gathered} \tag{6.12}$$

in equation (5.1), we find

(6.13)
$$\begin{aligned} &X^2+Z(Y-\eta S(Z))^2+2\eta(Y-\eta S(Z))Q(Z)-h(Z)Q(Z)^2-(Zh(Z)+\eta^2)P(Z)^2 \\ &= X^2+Z(Y^2-\eta^2 S(Z)^2)+2\gamma(Y-\eta S(Z))-h(Z)Q(Z)^2-(Zh(Z)+\eta^2)P(Z)^2 \\ &= X^2 + Y^2 Z + 2\gamma Y - F(Z), \end{aligned}$$

using equation (6.10). Thus, substituting (6.12) into (5.3) gives a matrix factorization of length 2 defined over the space PRes. This will turn out to be the matrix factorization predicted by Conjecture 3 for the D_n case.

Note that over the origin of PRes, our construction specializes to a matrix factorization for the rational double point D_n, of the form $\Phi = XI_4 - \Xi$, $\Psi = XI_4 + \Xi$, where

$$\Xi = \begin{bmatrix} 0 & Y & (-Z)^{\frac{k}{2}} & 0 \\ -YZ & 0 & 0 & (-Z)^{\frac{k}{2}} \\ (-1)^{n-1}(-Z)^{n-\frac{k+2}{2}} & 0 & 0 & -Y \\ 0 & (-1)^{n-1}(-Z)^{n-\frac{k+2}{2}} & YZ & 0 \end{bmatrix} \tag{6.14}$$

if k is even, and

(6.15)
$$\Xi = \begin{bmatrix} 0 & Y & 0 & (-Z)^{\frac{k-1}{2}} \\ -YZ & 0 & (-Z)^{\frac{k+1}{2}} & 0 \\ 0 & (-1)^{n-1}(-Z)^{n-\frac{k+3}{2}} & 0 & -Y \\ (-1)^{n-1}(-Z)^{n-\frac{k+1}{2}} & 0 & YZ & 0 \end{bmatrix}$$

if k is odd. These are the Gonzalez-Sprinberg–Verdier matrix factorizations for D_n.

Note that the "universal flop of length 2" from section 5 is a special case of our construction when $n = 4$ and $k = 2$. In that case, $f(U) = U^2+f_1U+f_0$ has degree 2 and $h(Z) = Z+h(0)$ has degree 1; it follows that $Q(Z) = -Z+Q(0)$

and $P(Z) = P(0)$, where $P(0) = f_1$ and $Q(0) = f_0$. We also have $S(Z) = -1$. Thus,

$$\begin{gathered} x = X,\ y = Y + \eta,\ z = -Z + Q(0),\ t = P(0), \\ u = Z,\ v = \eta,\ w = -Z - h(0), \end{gathered} \tag{6.16}$$

or conversely,

$$\begin{gathered} X = x,\ Y = y - v,\ Z = u,\ \eta = v, \\ h(0) = -u - w,\ P(0) = t,\ Q(0) = z + u, \end{gathered} \tag{6.17}$$

and so it is clear that x, y, z, t, u, v, w is just another set of coordinates for the space spanned by X, Y, Z, η, $h(0)$, $P(0)$, $Q(0)$. Thus, the universal flop of length 2 and the D_4 deformation with $k = 2$ coincide.

The classification theorem of [16], when specialized to flops of length 2, asserts that for every flop $Y \dashrightarrow Y^+$ of length 2, after shrinking Y, there is a map $\pi : Y \to \Delta$ to the unit disk Δ and a map

$$\rho : \Delta \to \mathrm{PRes}(\Gamma_{D_4} - \{\text{central vertex}\}) \tag{6.18}$$

such that the pullback via ρ of the deformation of D_4 coincides with X, and the pullback via ρ of the universal partial resolution coincides with Y. Thus, since we have just shown that our universal flop of length 2 coincides with the deformation of D_4 over $\mathrm{PRes}(\Gamma_{D_4} - \{\text{central vertex}\})$, Theorem 5.1 will follow once we have verified that the Grassmann blowup in the case $k = 2$, $n = 4$ gives the simultaneous partial resolution, i.e., once we have verified Conjecture 3 in this case.

Returning to the case of general $n \geq 4$, we next show that the maximal Cohen–Macaulay module associated to our matrix factorization is reducible for a few special values of k (for each fixed n). To see this, we introduce the following invertible change of basis matrices:

$$\begin{gathered} B_0 := \begin{bmatrix} X - \eta & Y & Q(0) & 1 \\ -1 & 0 & 0 & 0 \\ -Q(0) & 1 & 0 & 0 \\ Y & 0 & 1 & 0 \end{bmatrix}, B_1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ -X - \eta & Y & Q(0) & 1 \\ -Q(0) & 1 & 0 & 0 \\ Y & 0 & 1 & 0 \end{bmatrix}, \\ B_2 := \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2}Z & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, B_3 := \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}. \end{gathered} \tag{6.19}$$

When $k = 1$, we have $P(Z) = 1$, $S(Z) = 0$ and $Q(Z) = Q(0)$, and we find (6.20)

$$B_0 \Phi B_1^{-1} = \begin{bmatrix} \varphi_1 & 0 & 0 \\ 0 & \varphi_2 & 0 \\ 0 & 0 & XI_2 - \xi_1 \end{bmatrix}, \text{ and } B_1 \Psi B_0^{-1} = \begin{bmatrix} \psi_1 & 0 & 0 \\ 0 & \psi_2 & 0 \\ 0 & 0 & XI_2 + \xi_1 \end{bmatrix},$$

where $\varphi_2 = \psi_1 = 1$, $\varphi_1 = \psi_2 = X^2 + Y^2 Z + 2\gamma Y - F(Z)$, and

$$\xi_1 = \begin{bmatrix} \eta - Q(0)Y & -Z - Q(0)^2 \\ Y^2 - h(Z) & -\eta + Q(0)Y \end{bmatrix}. \tag{6.21}$$

Thus, the rank 2 maximal Cohen–Macaulay module (Φ, Ψ) is a direct sum of the rank 1 module $(XI_2 - \xi_1, XI_2 + \xi_1)$ and the trivial modules (φ_1, ψ_1) and (φ_2, ψ_2). This is to be expected, since the corresponding vertex in the Dynkin diagram has coefficient 1 rather than 2.

When $k = n$, we have $h(Z) = 0$, $\eta = 1$ and we find

$$B_2 \Xi B_2^{-1} = \begin{bmatrix} \xi_2 & 0 \\ 0 & \xi_2 \end{bmatrix}, \tag{6.22}$$

where

$$\xi_2 = \begin{bmatrix} -P(Z) & Y - S(Z) \\ -YZ + ZS(Z) - 2Q(Z) & P(Z) \end{bmatrix}. \tag{6.23}$$

This time, the rank 2 maximal Cohen–Macaulay module (Φ, Ψ) is a direct sum of two copies of the rank 1 module $(XI_2 - \xi_2, XI_2 + \xi_2)$. Again, we expected a rank 1 module due to the coefficient of this vertex in the Dynkin diagram being 1.

Finally, for $k = n - 1$, we have $h(Z) = 1$ and we find

$$B_3 \Xi B_3^{-1} = \begin{bmatrix} \xi_3 & 0 \\ 0 & \xi_4 \end{bmatrix}, \tag{6.24}$$

where

$$\xi_3 = \begin{bmatrix} \eta P(Z) + Q(Z) & -Y + \eta S(Z) + P(Z) \\ Z(Y - \eta S(Z) + P(Z)) + 2\eta Q(Z) & \eta P(Z) - Q(Z) \end{bmatrix} \tag{6.25}$$

and

$$\xi_4 = \begin{bmatrix} -\eta P(Z) - Q(Z) & -Y + \eta S(Z) - P(Z) \\ Z(Y - \eta S(Z) - P(Z)) + 2\eta Q(Z) & \eta P(Z) + Q(Z) \end{bmatrix}. \tag{6.26}$$

This time there are two inequivalent summands of smaller rank, and indeed, this subgroup of the Weyl group is *not* obtained by deleting a single vertex from the Dynkin diagram. The two summands correspond to the two short legs of the Dynkin diagram, and blowing up the given maximal Cohen–Macaulay module will necessarily blow up both of the corresponding curves.

We are now ready to carry out the Grassmann blowup, in the case of k between 2 and $n-2$, inclusive. In the first chart, after using λ_1 and λ_2 to eliminate $\alpha_{1,1}$ and $\alpha_{2,1}$, respectively, the Grassmann blowup is defined by the ideal $(\lambda_{1,2}, \lambda_{2,2}, \lambda_3)$, where

$$\lambda_{1,2} = Y - \eta S(Z) + \alpha_{1,2} Q(Z) + \alpha_{2,2} P(Z) \tag{6.27}$$

allows us to eliminate Y, and

$$\lambda_{2,2} = X + \eta P(Z) - \alpha_{1,2} Z P(Z) + \alpha_{2,2} Q(Z) \tag{6.28}$$

allows us to eliminate X. The remaining generator is

$$\lambda_3 = \alpha_{2,2}^2 + \alpha_{1,2}^2 Z - 2\eta\alpha_{1,2} - h(Z), \tag{6.29}$$

which defines the versal deformation of a D_{n-k} singularity,[6] since $h(Z)$ is a monic polynomial of degree $n-k-1$.

In the other chart, after using μ_1 and μ_2 to eliminate $\beta_{1,1}$ and $\beta_{2,1}$, respectively, the Grassmann blowup is defined by the ideal $(\mu_{1,2}, \mu_{3,2}, \mu_3)$. We have

$$\mu_{3,2} = X - \eta P(Z) - \beta_{1,2} h(Z) P(Z) - \beta_{2,2}(Y - \eta S(Z)) \tag{6.30}$$

which allows us to eliminate X on this chart, but

$$\mu_{1,2} = Q(Z) + \beta_{1,2}(Y - \eta S(Z)) + \beta_{2,2} P(Z) \tag{6.31}$$

does not immediately allow elimination. The third generator is

$$\mu_3 = \beta_{2,2}^2 + Z - 2\eta\beta_{1,2} - \beta_{1,2}^2 h(Z). \tag{6.32}$$

To understand the geometry of this chart, we follow an algebraic trick introduced by Tyurina [26][7] and form the combination

$$\widetilde{\mu}_{1,2} = \mu_{1,2} - G(Z, \beta_{2,2})\mu_3, \tag{6.33}$$

where G is the polynomial from equation (6.11). We then use the defining property for G to compute:

$$\begin{aligned} \widetilde{\mu}_{1,2} &= (Y - \eta S(Z))\beta_{1,2} + f(\beta_{2,2}) + 2\eta G(Z, \beta_{2,2})\beta_{1,2} + G(Z, \beta_{2,2}) h(Z) \beta_{1,2}^2 \\ &= (Y - \eta S(Z) + G(Z, \beta_{2,2}) h(Z) \beta_{1,2} + 2\eta G(Z, \beta_{2,2}))\beta_{1,2} + f(\beta_{2,2}). \end{aligned} \tag{6.34}$$

Thus, introducing the variable

$$\widetilde{Y} = Y - \eta S(Z) + G(Z, \beta_{2,2}) h(Z) \beta_{1,2} + 2\eta G(Z, \beta_{2,2}), \tag{6.35}$$

[6] Note that we are including here the cases of $D_3 = A_3$ and $D_2 = A_1 \cup A_1$, and that the deformation theory works correctly in these "degenerate" cases.

[7] This is a variant of the original trick which Tyurina used to show the very existence of simultaneous resolutions for deformations of D_n singularities.

we see that $\widetilde{\mu}_{1,2}$ defines a versal deformation of an A_{k-1} singularity:

$$\widetilde{\mu}_{1,2} = \widetilde{Y}\beta_{1,2} + f(\beta_{2,2}) \tag{6.36}$$

since $f(U)$ is monic of degree k. Note that Z can be implicitly eliminated using μ_3 (i.e., equation (6.32)).

Thus, the Grassmann blowup of Ψ yields a space with a rational curve lying over the origin, such that the central fiber has both an A_{k-1} and a D_{n-k} singularity along that curve, and the deformation induces versal deformations of these A_{k-1} and D_{n-k} singularities—in fact, the same versal deformations encoded by our invariant theory analysis. A precisely analogous thing happens if we blow up Φ instead of Ψ. (This is clear, since $x \mapsto -x$ exchanges the two.) This proves Conjecture 3 in the D_n case, and hence also proves Conjectures 1 and 2 for length 2, as well as Theorem 5.1.

Appendix A. Matrix Factorizations for E_6, E_7, and E_8

In this appendix, we present the Gonzalez-Sprinberg–Verdier matrix factorizations for E_6, E_7, and E_8. Gonzalez-Sprinberg and Verdier left some of the entries in the matrices undetermined, and we have made choices for these. We have also endeavored to make our matrices agree with those found in Chapter 9 of [28], after substituting $Y = x$, $Z = y$, permuting the rows and columns, and subjecting the matrices $\varphi_\ell^\bullet$ (defined below) to an overall sign change.

We write the equation for the rational double point in the form

$$f(X, Y, Z) = X^2 + g(Y, Z), \tag{A.1}$$

and consider a maximal Cohen–Macaulay module of length ℓ. Many such modules can be described in terms of a matrix factorization of $\ell \times \ell$ matrices over the ring $\mathbb{C}[Y, Z]$,

$$\varphi_\ell^\bullet \psi_\ell^\bullet = -g(Y, Z) I_\ell, \tag{A.2}$$

where $\bullet$ is a label (possibly empty) that is used to distinguish among different modules of rank ℓ when needed. Out of these matrices, we construct

$$\Xi_\ell^\bullet = \begin{bmatrix} 0 & \varphi_\ell^\bullet \\ -\psi_\ell^\bullet & 0 \end{bmatrix}, \tag{A.3}$$

which determines the matrix factorization

$$(X I_{2\ell} - \Xi_\ell^\bullet)(X I_{2\ell} + \Xi_\ell^\bullet) = (X^2 + g(Y, Z)) I_{2\ell} \tag{A.4}$$

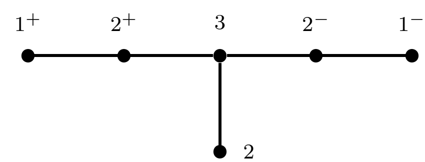
for the rational double point. In a few cases, the smaller matrices $\varphi_\ell^\bullet$ and $\psi_\ell^\bullet$ do not exist, and we give $\Xi_\ell^\bullet$ directly.

For each rational double point, we have included a Dynkin diagram with vertices labeled by ℓ or $\ell^\bullet$ to make the explicit McKay correspondence clear.

The problem of computing matrix factorizations $(\varphi_\ell^\bullet, \psi_\ell^\bullet)$ for $g(Y,Z)$ is known as the computation of the *Auslander–Reiten quiver* [4], and it is in that context that the computations in Chapter 9 of [28] are presented.

Case E_6: $g(Y,Z) = Y^3 + Z^4$.

Dynkin diagram:



Matrices:

$$\Xi_1^\pm = \begin{bmatrix} \pm iZ^2 & -Y^2 \\ Y & \mp iZ^2 \end{bmatrix},$$

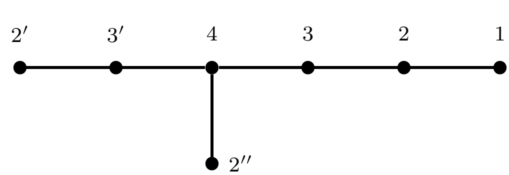
$$\Xi_2^\pm = \begin{bmatrix} \pm iZ^2 & 0 & -Y^2 & 0 \\ 0 & \pm iZ^2 & YZ & -Y^2 \\ Y & 0 & \mp iZ^2 & 0 \\ Z & Y & 0 & \mp iZ^2 \end{bmatrix},$$

$$\varphi_3 = \begin{bmatrix} -Y^2 & -Z^3 & -YZ^2 \\ YZ & -Y^2 & Z^3 \\ Z^2 & -YZ & -Y^2 \end{bmatrix}, \qquad \psi_3 = \begin{bmatrix} Y & 0 & -Z^2 \\ Z & Y & 0 \\ 0 & -Z & Y \end{bmatrix},$$

$$\varphi_2 = \begin{bmatrix} Y^2 & -Z^3 \\ -Z & -Y \end{bmatrix}, \qquad \psi_2 = \begin{bmatrix} -Y & Z^3 \\ Z & Y^2 \end{bmatrix}.$$

Case E_7: $g(Y,Z) = Y^3 + YZ^3$.

Dynkin diagram:



Matrices:

$$\varphi_2' = \begin{bmatrix} Y^2 & -YZ^2 \\ -Z & -Y \end{bmatrix}, \qquad \psi_2' = \begin{bmatrix} -Y & YZ^2 \\ Z & Y^2 \end{bmatrix},$$

$$\varphi_3' = \begin{bmatrix} Y^2 & -YZ^2 & Y^2Z \\ -YZ & -Y^2 & -YZ^2 \\ -Z^2 & -YZ & Y^2 \end{bmatrix}, \qquad \psi_3' = \begin{bmatrix} -Y & 0 & YZ \\ Z & Y & 0 \\ 0 & Z & -Y \end{bmatrix},$$

$$\varphi_4 = \begin{bmatrix} 0 & 0 & Y^2 & -YZ^2 \\ 0 & 0 & -YZ & -Y^2 \\ Y & -Z^2 & 0 & YZ \\ -Z & -Y & -Y & 0 \end{bmatrix}, \qquad \psi_4 = \begin{bmatrix} 0 & -YZ & -Y^2 & YZ^2 \\ Y & 0 & YZ & Y^2 \\ -Y & Z^2 & 0 & 0 \\ Z & Y & 0 & 0 \end{bmatrix},$$

$$\varphi_3 = \begin{bmatrix} -YZ & -Y^2 & -YZ^2 \\ Z^2 & YZ & -Y^2 \\ -Y & Z^2 & -YZ \end{bmatrix}, \qquad \psi_3 = \begin{bmatrix} 0 & -YZ & Y^2 \\ Y & 0 & -YZ \\ Z & Y & 0 \end{bmatrix},$$

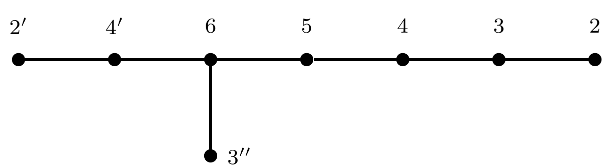
$$\varphi_2 = \begin{bmatrix} -YZ & Y^2 \\ -Y & -Z^2 \end{bmatrix}, \qquad \psi_2 = \begin{bmatrix} Z^2 & Y^2 \\ -Y & YZ \end{bmatrix},$$

$$\varphi_1 = \begin{bmatrix} -Y^2 - Z^3 \end{bmatrix}, \qquad \psi_1 = \begin{bmatrix} Y \end{bmatrix},$$

$$\varphi_2'' = \begin{bmatrix} Y^2 & -YZ^2 \\ -YZ & -Y^2 \end{bmatrix}, \qquad \psi_2'' = \begin{bmatrix} -Y & Z^2 \\ Z & Y \end{bmatrix}.$$

Case E_8: $g(Y,Z) = Y^3 + Z^5$.

Dynkin diagram:



Matrices:

$$\varphi_2' = \begin{bmatrix} -Z^3 & Y^2 \\ -Y & -Z^2 \end{bmatrix}, \qquad \psi_2' = \begin{bmatrix} Z^2 & Y^2 \\ -Y & Z^3 \end{bmatrix},$$

$$\varphi_4' = \begin{bmatrix} 0 & -Z^3 & Y^2 & 0 \\ -Z^2 & 0 & -YZ & -Y^2 \\ Y & Z^2 & 0 & -Z^3 \\ 0 & -Y & -Z^2 & 0 \end{bmatrix}, \qquad \psi_4' = \begin{bmatrix} 0 & Z^3 & -Y^2 & -YZ^2 \\ Z^2 & 0 & 0 & Y^2 \\ -Y & 0 & 0 & Z^3 \\ Z & Y & Z^2 & 0 \end{bmatrix},$$

$$\varphi_6 = \begin{bmatrix} 0 & 0 & 0 & -Y^2 & -YZ^2 & -Z^4 \\ 0 & 0 & 0 & -Z^3 & Y^2 & YZ^2 \\ 0 & 0 & 0 & -YZ & -Z^3 & Y^2 \\ -Y & -Z^2 & 0 & 0 & 0 & -Z^3 \\ 0 & Y & -Z^2 & Z^2 & 0 & 0 \\ -Z & 0 & Y & 0 & Z^2 & 0 \end{bmatrix},$$

$$\psi_6 = \begin{bmatrix} 0 & 0 & Z^3 & Y^2 & YZ^2 & Z^4 \\ -Z^2 & 0 & 0 & Z^3 & -Y^2 & -YZ^2 \\ 0 & Z^2 & 0 & YZ & Z^3 & -Y^2 \\ Y & Z^2 & 0 & 0 & 0 & 0 \\ 0 & -Y & Z^2 & 0 & 0 & 0 \\ Z & 0 & -Y & 0 & 0 & 0 \end{bmatrix},$$

$$\varphi_5 = \begin{bmatrix} Z^3 & Y^2 & 0 & 0 & 0 \\ 0 & -Z^3 & -Y^2 & YZ^2 & Z^4 \\ 0 & -YZ & Z^3 & Y^2 & YZ^2 \\ -Z^2 & 0 & -YZ & Z^3 & -Y^2 \\ Y & -Z^2 & 0 & 0 & 0 \end{bmatrix},$$

$$\psi_5 = \begin{bmatrix} -Z^2 & 0 & 0 & 0 & -Y^2 \\ -Y & 0 & 0 & 0 & Z^3 \\ 0 & Y & -Z^2 & 0 & 0 \\ -Z & 0 & -Y & -Z^2 & 0 \\ 0 & -Z & 0 & Y & Z^2 \end{bmatrix},$$

$$\varphi_4 = \begin{bmatrix} Z^3 & -Y^2 & 0 & 0 \\ 0 & -YZ & Z^3 & Y^2 \\ Y & Z^2 & 0 & 0 \\ -Z & 0 & Y & -Z^2 \end{bmatrix}, \qquad \psi_4 = \begin{bmatrix} -Z^2 & 0 & -Y^2 & 0 \\ Y & 0 & -Z^3 & 0 \\ 0 & -Z^2 & -YZ & -Y^2 \\ Z & -Y & 0 & Z^3 \end{bmatrix},$$

$$\varphi_3 = \begin{bmatrix} -Y^2 & -Z^4 & -YZ^3 \\ -YZ & Y^2 & -Z^4 \\ -Z^2 & YZ & Y^2 \end{bmatrix}, \qquad \psi_3 = \begin{bmatrix} Y & 0 & Z^3 \\ Z & -Y & 0 \\ 0 & Z & -Y \end{bmatrix},$$

$$\varphi_2 = \begin{bmatrix} Y^2 & -Z^4 \\ -Z & -Y \end{bmatrix}, \qquad \psi_2 = \begin{bmatrix} -Y & Z^4 \\ Z & Y^2 \end{bmatrix},$$

$$\varphi_3'' = \begin{bmatrix} -Y^2 & -YZ^2 & -Z^4 \\ -Z^3 & Y^2 & YZ^2 \\ -YZ & -Z^3 & Y^2 \end{bmatrix}, \qquad \psi_3'' = \begin{bmatrix} Y & Z^2 & 0 \\ 0 & -Y & Z^2 \\ Z & 0 & -Y \end{bmatrix}.$$

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