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W. G. Dwyer, J. P. C. Greenlees and S. B. Iyengar

Abstract

We study differential graded algebras (DGAs) whose homology is an exterior algebra over a commutative ring R on a generator of degree n, and also certain types of differential modules over these DGAs. We obtain a complete classification with $R = \mathbb{Z}$ or $R = \mathbb{F}_p$ and $n \geqslant -1$. The examples are unexpectedly interesting.

1. *Introduction*

We mainly study differential graded algebras (DGAs) whose homology is an exterior algebra over either $\mathbb{F}_p$ or $\mathbb{Z}$ on a class of degree -1, as well as (left) modules over these DGAs whose homology is $\mathbb{Z}/p$ in degree 0. In the case of an exterior algebra over $\mathbb{F}_p$, we find many DGAs, each having one such module; in the case of an exterior algebra over $\mathbb{Z}$, one DGA having many such modules. In both cases, the enumeration of possibilities involves complete discrete valuation rings with residue field $\mathbb{F}_p$. There are remarks about other types of DGAs in Section 6.

In more detail, a DGA is a chain complex A of abelian groups together with a multiplication map $A \otimes A \to A$ that is both unital and associative. A morphism $f\colon A \to B$ of DGAs is a map of chain complexes which respects multiplication and unit; f is said to be *an equivalence* if it induces isomorphisms $H_i A \cong H_i B$, $i \in \mathbb{Z}$. The homology $H_* A$ is a graded ring, and A is said to be *of type* β_S, for S a commutative ring, if $H_* A$ is an exterior algebra over S on a class of degree -1. The notation β_S is meant to suggest that A captures a type of Bockstein operation. In topology, the 'Bockstein' is an operation β on mod p homology, of degree -1 and square 0, which arises from a generator of $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/p)$.

A *module* over A is a chain complex X together with an action map $A \otimes X \to X$ with the usual unital and associativity properties. Morphisms and equivalences between modules are defined in the evident way. The module X is of *type* $(M, 0)$ for an abelian group M if there are isomorphisms of abelian groups

$$H_i X \cong \begin{cases} M, & i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

We are particularly interested in DGAs of type β_S for $S = \mathbb{F}_p$ or $S = \mathbb{Z}$, and modules over these DGAs of type $(\mathbb{Z}/p, 0)$.

Theorem 1.1. *There is a natural bijection between*

(i) *equivalence classes of DGAs of type* $\beta_{\mathbb{F}_p}$ *and*

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(ii) *isomorphism classes of complete discrete valuation rings with residue field* $\mathbb{F}_p$.

Up to equivalence, each such DGA has a unique module of type $(\mathbb{Z}/p, 0)$.

REMARK 1.2. A *complete discrete valuation ring* $\mathcal{O}$ is a principal ideal domain with a unique nonzero prime ideal $\mathfrak{m}$, such that $\mathcal{O}$ is complete with respect to the topology determined by powers of $\mathfrak{m}$. If $\mathcal{O}/\mathfrak{m} \cong \mathbb{F}_p$, then $\mathcal{O}$ is isomorphic either to $\mathbb{F}_p[[t]]$ or to a totally ramified extension of finite degree of the ring $\mathbb{Z}_p^{\hat{}}$ of p-adic integers; see [**8**, II.4-5]. The surprise in Theorem 1.1 is the sheer profusion of DGAs. In Remark 3.3, there is an explicit description of how to pass from a ring $\mathcal{O}$ to a DGA.

The tables are turned when it comes to DGAs of type $\beta_{\mathbb{Z}}$.

THEOREM 1.3. *Up to equivalence, there is only a single DGA* A *of type* $\beta_{\mathbb{Z}}$. *There is a natural bijection between*

(i) *equivalence classes of modules over* A *of type* $(\mathbb{Z}/p, 0)$ *and*
(ii) *isomorphism classes of pairs* $(\mathcal{O}, \pi)$, *where* $\mathcal{O}$ *is a complete discrete valuation ring with residue field* $\mathbb{F}_p$, *and* π *is a uniformizer for* $\mathcal{O}$.

REMARK 1.4. A *uniformizer* for $\mathcal{O}$ is a generator of the maximal ideal $\mathfrak{m}$. The surprise in Theorem 1.3 is the profusion of modules, since there are many pairs $(\mathcal{O}, \pi)$ as above. For instance, if $\pi \in \mathbb{Z}_p^{\hat{}}$ is divisible by p but not by p^2, then $(\mathbb{Z}_p^{\hat{}}, \pi)$ is such a pair. Since $\mathbb{Z}_p^{\hat{}}$ has no nontrivial automorphisms, these pairs are distinct for different choices of π.

The object A of Theorem 1.3 can be taken to be the DGA $\mathcal{F}$ that contains copies of $\mathbb{Z}$ in degrees 0 and -1, is trivial elsewhere, and has zero differential. In Remark 5.5, there is an explicit description of how to pass from a pair $(\mathcal{O}, \pi)$ to a module over $\mathcal{F}$.

REMARK 1.5. If A is a *connective* DGA, that is, $H_i A = 0$ for $i < 0$, and X is a module over A such that $H_i X = 0$ for $i \neq 0$, then X is determined up to equivalence by the isomorphism class of $H_0 X$ as an (ordinary) module over $H_0 A$; see, for instance, [**5**, 3.9]. This is in strong contrast to what happens in the nonconnective setting of Theorem 1.3.

Generalizations. We have some remarks in Section 6 about DGAs with other types of exterior algebra homology.

The arguments below can be interpreted in the setting of stable homotopy theory, and they lead to a classification of associative ring spectra of type $\beta_{\mathbb{Z}}$ or $\beta_{\mathbb{F}_p}$ (appropriately interpreted) and of module spectra over these ring spectra of type $(\mathbb{Z}/p, 0)$. No new examples come up; all of these ring spectra and module spectra are obtained in a standard way [**9**] from DGAs.

1.1. *DG-objects, equivalences, and formality*

As suggested above, a map between differential graded (DG) objects of any kind is said to be an *equivalence* if it induces an isomorphism on homology groups. Two objects are *equivalent* if they are related by a zig–zag $\longleftarrow\longrightarrow\longleftarrow \cdots \rightarrow$ of equivalences. A DG-object X is *formal* if it is equivalent to a DG-object Y of the same kind that has zero differentials. Of course, in this case the graded constituent Y_i of Y must be isomorphic to $H_i X$. For instance (cf. Remark 1.5), if A is a connective DGA and X is a module over A such that $H_i X$ vanishes except for a single value of i, then X is formal as an A-module.

If A is a DGA such that $H_iA = 0$ for $i \neq 0$, then A is formal as a DGA. To see this, let $A' \subset A$ be the subcomplex given by

$$A'_i = \begin{cases} A_i, & i > 0, \\ \ker(\partial\colon A_0 \longrightarrow A_{-1}), & i = 0, \\ 0, & i < 0. \end{cases}$$

Then A' is a DGA, and there is a zig–zag of DGA-equivalences

$$A \xleftarrow{\sim} A' \xrightarrow{\sim} H_0A,$$

where the object on the right is a ring, treated as a DGA concentrated in degree 0.

If A is a DGA such that H_*A is a polynomial algebra $\mathbb{Z}[x]$ on a class x of (arbitrary) degree n, then A is also formal: choose a cycle $\chi \in A_n$ representing x and construct an equivalence $(\mathbb{Z}[x], 0) \to (A, \partial)$ by $x \mapsto \chi$. On the other hand, if S is a commutative ring other than $\mathbb{Z}$ and $H_*A \cong S[x]$, then it is not necessarily the case that A is formal, even for $S = \mathbb{F}_p$ (see Section 6 for examples); the above argument applies only if A itself is an algebra over S, or at least equivalent to an algebra over S.

1.2. *Relationship to Moore–Koszul duality*

Suppose that k is some chosen field. Say that a DGA A over k is *admissible* if H_iA is finite-dimensional over k for all i and $H_0A \cong k$. Applying duality over k to the bar constructions in [**6**] produces bijections between equivalence classes of the admissible DGAs indicated below.

$$\{A \mid H_iA = 0,\ i > 0 \ \&\ i = -1\} \underset{B \mapsto \operatorname{End}_B(k)}{\overset{A \mapsto \operatorname{End}_A(k)}{\rightleftarrows}} \{B \mid H_iB = 0,\ i < 0\}.$$

In topology, for instance, this gives the relationship between the cochain algebra A of a simply connected space X and the chain algebra B of the loop space ΩX.

Our technique is to push the boundaries of the above Moore–Koszul duality construction. In proving Theorem 1.1 (Sections 2 and 3), we start with a DGA A of type $\beta_{\mathbb{F}_p}$, more or less construct by hand an action of A on (something equivalent to) $\mathbb{Z}/p$, and show that the DGA $B = \operatorname{End}_A(\mathbb{Z}/p)$ has its homology concentrated in degree 0, and so is essentially an ordinary ring $\mathcal{O}$ (Subsection 1.1). It turns out that $\mathcal{O}$ is a complete discrete valuation ring, and that it determines A up to equivalence via the formula

$$A \xrightarrow{\sim} \operatorname{End}_{\mathcal{O}}(\mathbb{Z}/p).$$

Note that it is *not* necessarily the case that A is a DGA over $\mathbb{F}_p$. In Section 4, the same technique succeeds in classifying DGAs of type $\beta_{\mathbb{Z}}$. The proof in Section 5 of the second part of Theorem 1.3 relies on ideas from [**4**] which fit modules into a type of Moore–Koszul duality setting.

1.3. *Notation and terminology*

We work in the context of [**4**, **5**]. Rings are tacitly identified with DGAs concentrated in degree 0 (Subsection 1.1). An ordinary module M over a ring R is similarly tacitly identified with the chain complex X over R with $X_0 = M$ and $X_i = 0$ for $i \neq 0$. Hom is the derived homomorphism complex and $\otimes$ is the derived tensor product. If X is a module over the DGA (or ring) A, then $\operatorname{End}_A(X)$ denotes the DGA obtained by taking a projective model for X and forming the usual DGA of endomorphisms of this model. See [**10**, 2.7.4], but re-index to conform to our convention that differentials always reduce degree by 1. (Up to equivalence, the DGA $\operatorname{End}_A(X)$ depends only on the equivalence types of A and of X; this can be proved for instance with the bimodule argument of [**7**, 3.7.6].) If R is a ring and M is an ordinary

R-module, then there are isomorphisms

$$H_i \operatorname{End}_R(M) \cong \operatorname{Ext}_R^{-i}(M, M).$$

We write $\operatorname{Ext}^0_R(M, M)$ or $H_0 \operatorname{End}_R(M)$ for the ordinary endomorphism ring of M over R.

If X is a chain complex or graded abelian group, then we write $\Sigma^i X$ for its i-fold shift: $(\Sigma^i X)_j = X_{j-i}$, $\partial(\Sigma^i x) = (-1)^i \Sigma^i(\partial x)$. For the sake of clarity, we attempt as much as reasonably possible to make a notational distinction between the field $\mathbb{F}_p$ and the abelian group $\mathbb{Z}/p = \mathbb{Z}/p\mathbb{Z}$.

2. *A module of type* $(\mathbb{Z}/p, 0)$

In this section, A is a DGA of type $\beta_{\mathbb{F}_p}$. We construct a module X over A of type $(\mathbb{Z}/p, 0)$ and show that for any module Y over A of type $(\mathbb{Z}/p, 0)$ there is an equivalence $e_Y : X \to Y$. However, the equivalence e_Y is not unique in any sense, even up to homotopy. See [**5**, 3.3, 3.9] for more general constructions of this type.

The construction is inductive. Suppose that Z is a module over A with

$$H_i Z \cong \begin{cases} \mathbb{Z}/p, & i = 0, -1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Choose a map $\kappa : \Sigma^{-1} A \to Z$ that induces an isomorphism

$$\mathbb{Z}/p \cong H_0 A \cong H_{-1} \Sigma^{-1} A \xrightarrow{H_{-1}\kappa} H_{-1} Z,$$

and let JZ be the mapping cone of κ. There is a natural map $Z \to JZ$ that is an isomorphism on H_0 and trivial on the other homology groups (in particular $H_{-1}Z \to H_{-1}JZ$ is zero), and so the homology exact sequence of the triangle

$$\Sigma^{-1} A \longrightarrow Z \longrightarrow JZ$$

shows that $H_* JZ$ again vanishes except for copies of $\mathbb{Z}/p$ in degrees 0 and -1. Starting with $Z = A$, iterate the process to obtain a sequence

$$A \longrightarrow JA \longrightarrow J^2 A \longrightarrow \cdots$$

of chain complexes and chain maps, and let $X = \operatorname{hocolim}_k J^k A$. (In this case, the homotopy colimit can be taken to be a colimit, that is, the ascending union.) It is immediate that X has type $(\mathbb{Z}/p, 0)$.

Suppose that Y is an arbitrary module of type $(\mathbb{Z}/p, 0)$. There is certainly a map $A \to Y$ that induces an isomorphism on H_0, so to construct an equivalence $X \to Y$ it is enough to show that if Z satisfies (2.1) and $f \colon Z \to Y$ induces an isomorphism on H_0, then f extends to $f' \colon JZ \to Y$. (By induction, this will guarantee that the map $A \to Y$ extends to a map $X = \operatorname{hocolim}_k J^k A \to Y$.) The map f extends to f' if and only if the composite

$$\Sigma^{-1} A \xrightarrow{\kappa} Z \xrightarrow{f} Y$$

is null homotopic. But the group of homotopy classes of A-module maps $\Sigma^{-1} A \to Y$ vanishes, since it is isomorphic to $H_{-1} Y$.

3. *Exterior algebras over* $\mathbb{F}_p$

In this section, we prove Theorem 1.1. The proof depends on two lemmas.

Obtaining a ring from a DGA. Suppose that A is a DGA of type $\beta_{\mathbb{F}_p}$. According to Section 2, up to equivalence there is a unique module X over A of type $(\mathbb{Z}/p, 0)$. Let $\mathcal{E} = \operatorname{End}_A(X)$ be the derived endomorphism algebra of X.

LEMMA 3.1. *Let A, X, and $\mathcal{E}$ be as above.*

(1) *The homology group $H_i\mathcal{E}$ vanishes for $i \neq 0$, and the ring $H_0\mathcal{E}$ is a complete discrete valuation ring with residue field $\mathbb{F}_p$.*
(2) *The natural map $A \to \operatorname{End}_{\mathcal{E}}(X)$ is an equivalence.*

Obtaining a DGA from a ring. Suppose that $\mathcal{O}$ is a complete discrete valuation ring with residue class field $\mathbb{F}_p$. Let X denote $\mathbb{Z}/p$ with the unique possible $\mathcal{O}$-module structure, and let $A = \operatorname{End}_{\mathcal{O}}(X)$ be the derived endomorphism algebra of X.

LEMMA 3.2. *Let $\mathcal{O}$, X, and A be as above.*

(1) *The DGA A is of type $\beta_{\mathbb{F}_p}$.*
(2) *The natural map $\mathcal{O} \to \operatorname{End}_A(X)$ is an equivalence.*

Proof of Theorem 1.1. The existence and uniqueness of the module of type $(\mathbb{Z}/p, 0)$ is from Section 2. For the rest, Subsection 1.1 and Lemmas 3.1 and 3.2 provide inverse constructions matching up appropriate DGAs with appropriate rings. □

For minor efficiency reasons, we first prove Lemma 3.2 and then Lemma 3.1.

3.1. *Proof of Lemma* 3.2

For part (1), observe that as usual there are isomorphisms

$$H_i \operatorname{End}_{\mathcal{O}}(X) \cong \operatorname{Ext}^{-i}_{\mathcal{O}}(\mathbb{Z}/p, \mathbb{Z}/p).$$

But there is a short projective resolution of $\mathbb{Z}/p$ over $\mathcal{O}$

$$0 \longrightarrow \mathcal{O} \xrightarrow{\ \pi\ } \mathcal{O} \longrightarrow \mathbb{Z}/p \longrightarrow 0\ . \tag{3.1}$$

By inspection, then, $\operatorname{Ext}^i_{\mathcal{O}}(\mathbb{Z}/p, \mathbb{Z}/p)$ vanishes unless $i = 0$ or $i = 1$, and in these two exceptional cases the group is isomorphic to $\mathbb{Z}/p$.

Since $\mathcal{O}/\mathfrak{m} \cong \mathbb{F}_p$ is a field and hence a regular ring, Lemma 3.2(2) is [**5**, 4.20].

REMARK 3.3. If $\mathcal{O}$ is as in Lemma 3.2, an explicit model for $\operatorname{End}_{\mathcal{O}}(\mathbb{Z}/p)$ can be derived from (3.1) as follows:

$$\begin{array}{cc} 1 & \langle L \rangle \\ & \Big\downarrow {\scriptstyle \partial L = \pi D_1 + \pi D_2} \\ 0 & \langle D_1, D_2 \rangle \\ & \Big\downarrow {\scriptstyle \partial D_i = (-1)^i \pi U} \\ -1 & \langle U \rangle \end{array} \qquad \begin{array}{ll} D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & D_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{array}$$

This is a DGA which is nonzero only in degrees $1, 0, -1$; the notation $\langle - \rangle$ denotes the free $\mathcal{O}$-module on the enclosed generators. From a multiplicative point of view, the DGA is as indicated a graded form of the ring of 2×2 matrices over $\mathcal{O}$.

Proof of Lemma 3.1(1). Let Λ denote the graded algebra H_*A and $x \in \Lambda_{-1}$ an additive generator. Write $\mathbb{Z}/p$ for the (ordinary) graded Λ-module $\Lambda/\langle x \rangle \cong H_*X$. For general reasons (see Subsection 3.2) there is a left-half plane Eilenberg–Moore spectral sequence

$$E^2_{-i,j} = \mathrm{Ext}^i_\Lambda(\Sigma^j \mathbb{Z}/p, \mathbb{Z}/p) \Rightarrow H_{j-i}\mathcal{E}.$$

(Note that Ext here is computed in the category of graded modules over a graded ring.) This is a conditionally convergent spectral sequence of bigraded algebras with differentials

$$d_r : E^r_{i,j} \longrightarrow E^r_{i-r,j+r-1},$$

and it abuts to the graded algebra $H_*\mathcal{E}$. There is a free resolution of $\mathbb{Z}/p$ over Λ

$$\cdots \xrightarrow{x} \Sigma^{-2}\Lambda \xrightarrow{x} \Sigma^{-1}\Lambda \xrightarrow{x} \Lambda \longrightarrow \mathbb{Z}/p \longrightarrow 0,$$

which leads by calculation to the conclusion that the bigraded algebra $\mathrm{Ext}^i_\Lambda(\Sigma^j\mathbb{Z}/p, \mathbb{Z}/p)$ is a polynomial algebra on the extension class

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \Sigma\Lambda \longrightarrow \Sigma\mathbb{Z}/p \longrightarrow 0,$$

in $\mathrm{Ext}^1_\Lambda(\Sigma\mathbb{Z}/p, \mathbb{Z}/p)$. It follows that $E^2_{i,j}$ vanishes in the above spectral sequence for $i \neq -j$, the spectral sequence collapses, and $E^\infty = E^2$ is concentrated in total degree 0. Hence, $H_i\mathcal{E}$ vanishes for $i \neq 0$, and $H_0\mathcal{E}$ is a ring $\mathcal{O}$ with a decreasing sequence of ideals

$$\cdots \subset \mathfrak{m}_k \subset \cdots \subset \mathfrak{m}_2 \subset \mathfrak{m}_1 \subset \mathcal{O},$$

such that $\mathfrak{m}_k\mathfrak{m}_\ell \subset \mathfrak{m}_{k+\ell}$, $\mathrm{Gr}(\mathcal{O}) \cong \mathbb{F}_p[t]$, and $\mathcal{O} \cong \lim_k \mathcal{O}/\mathfrak{m}_k$.

Let $\pi \in \mathfrak{m}_1 \subset \mathcal{O}$ be an element that projects to a generator of $\mathfrak{m}_1/\mathfrak{m}_2 \cong \mathbb{Z}/p$, and map $\mathbb{Z}[s]$ to $\mathcal{O}$ by sending s to π. It is easy to argue by induction on k that the composite $\mathbb{Z}[s] \to \mathcal{O} \to \mathcal{O}/\mathfrak{m}_k$ is surjective, so that $\mathcal{O}/\mathfrak{m}_k$ is commutative and $\mathcal{O} \cong \lim_k \mathcal{O}/\mathfrak{m}_k$ is commutative as well. The ring $\mathcal{O}$ is a noetherian domain because it has a complete filtration $\{\mathfrak{m}_k\}$ such that $\mathrm{Gr}(\mathcal{O})$ is a noetherian domain [**2**, Chapter III, Section 2, Corollary 2]. The ideal $\mathfrak{m}_1/\mathfrak{m}_k$ is nilpotent in $\mathcal{O}/\mathfrak{m}_k$, and hence an element $x \in \mathcal{O}/\mathfrak{m}_k$ is a unit if and only the image of x in $\mathcal{O}/\mathfrak{m}_1 \cong \mathbb{F}_p$ is nonzero. It follows directly that $x \in \mathcal{O}$ is a unit if and only if the image of x in $\mathcal{O}/\mathfrak{m}_1$ is nonzero, and so $\mathcal{O}$ is a local ring with maximal ideal $\mathfrak{m} = \mathfrak{m}_1$. Finally, by induction on k there are exact sequences

$$\mathcal{O}/\mathfrak{m}_k \xrightarrow{\pi} \mathcal{O}/\mathfrak{m}_k \longrightarrow \mathcal{O}/\mathfrak{m} \longrightarrow 0.$$

The groups involved are finite, so passing to the limit in k gives an exact sequence

$$\mathcal{O} \xrightarrow{\pi} \mathcal{O} \longrightarrow \mathcal{O}/\mathfrak{m} \longrightarrow 0,$$

expressing the fact that π generates the ideal $\mathfrak{m}$. The element π is not nilpotent (because its image in $\mathrm{Gr}(\mathcal{O}) \cong \mathbb{F}_p[t]$ is t) and so by Serre [**8**, I. Section 2] $\mathcal{O}$ is a discrete valuation ring. □

Proof of Lemma 3.1(2). By Subsection 3.1 and Lemma 3.1(1), $\mathrm{End}_\mathcal{E}(X)$ is a DGA of type $\beta_{\mathbb{F}_p}$. It is thus enough to show that the map $H_{-1}A \to H_{-1}\mathrm{End}_\mathcal{E}(X)$ is an isomorphism, or even that this map is nonzero. We will use the notation of Section 2. Note that the mapping cone of the A-module map $\epsilon : A \to JA$ is again A, and that the mapping cone of any nontrivial map $A \to X$ is again equivalent to X, this last by a homology calculation and the uniqueness result of Section 2. Consider the following diagram, in which the rows are exact triangles and the middle vertical map is provided by Section 2.

$$\begin{array}{ccccccc} A & \longrightarrow & JA & \longrightarrow & A & \overset{a}{\dashrightarrow} & \Sigma A \\ \big\downarrow{=} & & \big\downarrow & & \big\downarrow{`\epsilon'} & & \\ A & \xrightarrow{\epsilon} & X & \xrightarrow{\pi} & X & & \end{array} \tag{3.2}$$

Here a denotes (right) multiplication by a generator of $H_{-1}A$. (In the language of [**5**], the lower row here shows that X is proxy-small over A, and Lemma 3.1(2) now follows fairly directly from [**5**, Proof of 4.10]. For convenience, we continue with a direct argument.) After a suitable identification of the mapping cone of ϵ with X, a homology calculation gives that the right vertical map is homotopic to ϵ. The right lower map is entitled to be labeled π, as in Subsection 3.1, because the homology of its mapping cone (namely $H_*\Sigma A = \Sigma H_* A$) evidently represents a nonzero element of

$$\operatorname{Ext}^1_{H_*A}(\Sigma\mathbb{Z}/p, \mathbb{Z}/p).$$

Note that the element π of $\mathcal{O}$ is determined only up to multiplication by a unit, and this is reflected above in the fact that there are various ways to identify the mapping cone of ϵ with X. Applying $\operatorname{Hom}_A(-, X)$ to (3.2) gives

$$\begin{array}{ccccccc} X & \longleftarrow & \operatorname{Hom}_A(JA, X) & \longleftarrow & X & \overset{a^*}{\dashleftarrow} & \Sigma^{-1}X \\ {\scriptstyle =}\uparrow & & \uparrow & & \uparrow{\scriptstyle \epsilon^*} & & \\ X & \overset{\epsilon^*}{\longleftarrow} & \mathcal{E} & \overset{\pi^*}{\longleftarrow} & \mathcal{E} & & \end{array} \tag{3.3}$$

The map ϵ^* is surjective on homology, since by the argument of Section 2 any A-map $A \to X$ extends over ϵ to a map $X \to X$. Let M be the $\mathcal{O}$-module $H_0 \operatorname{Hom}_A(JA, X)$. Applying H_0 to the solid arrows in diagram (3.3) gives a diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \mathbb{Z}/p & \longleftarrow & M & \longleftarrow & \mathbb{Z}/p & \longleftarrow & 0 \\ & & {\scriptstyle =}\uparrow & & \uparrow & & \uparrow{\scriptstyle \text{onto}} & & \\ 0 & \longleftarrow & \mathbb{Z}/p & \longleftarrow & \mathcal{O} & \overset{\pi}{\longleftarrow} & \mathcal{O} & \longleftarrow & 0 \end{array}$$

The diagram implies that as an $\mathcal{O}$-module, M is $\mathcal{O}/(\pi^2)$. In particular, the extension on the top line is nontrivial over $\mathcal{O}$, which, by backing up the exact triangle as in (3.3), implies that $a^* \colon \Sigma^{-1}X \to X$ represents a nonzero element of $H_{-1} \operatorname{End}_{\mathcal{E}}(X)$. But, by construction, a^* is given by left multiplication with a generator of $H_{-1}A$. $\square$

3.2. *Eilenberg–Moore spectral sequence*

If A is a DGA and X, Y are modules over A, then there is an Eilenberg–Moore spectral sequence

$$E^2_{-i,j} = \operatorname{Ext}^i_{H_*A}(\Sigma^j H_* X, H_* Y) \Rightarrow H_{j-i} \operatorname{Hom}_A(X, Y).$$

In a homotopy context, this can be constructed in precisely the same way as an Adams spectral sequence. In an algebraic context, it is constructed by inductively building an exact sequence of A-modules

$$0 \longleftarrow X \longleftarrow F(0) \longleftarrow F(1) \longleftarrow \cdots \longleftarrow F(i) \longleftarrow \cdots, \tag{3.4}$$

such that

(i) each $F(i)$ is a sum of shifts of copies of A and
(ii) applying H_* to (3.4) produces a free resolution of H_*X over H_*A.

See, for instance, [**1**, 9.11]. The totalization tF of the double complex F is then a projective (or cofibrant) model for X; filtering tF by $\{tF(\leqslant n)\}_n$ and applying $\operatorname{Hom}_A(-, Y)$ gives a filtration of $\operatorname{Hom}_A(X, Y)$ that yields the spectral sequence.

4. *Exterior algebras over* $\mathbb{Z}$

In this section, we prove the first claim of Theorem 1.3. To be specific, we show that any DGA A of type $\beta_{\mathbb{Z}}$ is equivalent to the formal DGA $\mathcal{F}$ given by the following chain complex concentrated in degrees $0, -1$.

$$\begin{array}{cc} 0 & \mathbb{Z} \\ & \big\downarrow{\scriptstyle \partial=0} \\ -1 & \mathbb{Z} \end{array}$$

The multiplication on $\mathcal{F}$ is the only possible one consistent with the requirement that $1 \in \mathbb{Z} = \mathcal{F}_0$ act as a unit.

We will only sketch the line of reasoning, since it is similar to that in sections Sections 2 and 3, although the conclusion is very different. Along the lines of Section 2 there exists a module X of type $(\mathbb{Z}, 0)$ over A, and this is unique up to noncanonical equivalence. Let $\mathcal{E} = \mathrm{End}_A(X)$. The argument in the proof of Lemma 3.1(1) shows that $H_i\mathcal{E}$ vanishes for $i \neq 0$, and that $H_0\mathcal{E}$ is a ring R with a decreasing sequence of ideals

$$\cdots \subset \mathfrak{m}_k \subset \cdots \subset \mathfrak{m}_2 \subset \mathfrak{m}_1 \subset R,$$

such that $\mathfrak{m}_k\mathfrak{m}_\ell \subset \mathfrak{m}_{k+\ell}$, $\mathrm{Gr}(R) \cong \mathbb{Z}[t]$, and $R \cong \lim_k R/\mathfrak{m}_k$. Let $\sigma \in \mathfrak{m}_1$ be an element that projects to a generator of $\mathfrak{m}_1/\mathfrak{m}_2 \cong \mathbb{Z}$, and map $\mathbb{Z}[s]$ to R by sending s to σ. It is easy to show by induction on k that this map induces isomorphisms $\mathbb{Z}[s]/(s^{k+1}) \to R/\mathfrak{m}_k$, and so induces an isomorphism $\mathbb{Z}[[s]] \to R$. Using the free resolution

$$0 \longrightarrow \mathbb{Z}[[s]] \xrightarrow{\;s\;} \mathbb{Z}[[s]] \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{4.1}$$

of $\mathbb{Z}$ over $\mathbb{Z}[[s]]$ (s acting by zero on $\mathbb{Z}$), one argues as in the proof of Lemma 3.2(1) that $\mathrm{End}_{\mathbb{Z}[[s]]}(\mathbb{Z})$ is a DGA of type $\beta_{\mathbb{Z}}$. As in the proof of Lemma 3.1(2), the natural map $A \to \mathrm{End}_{\mathcal{E}}(X) \sim \mathrm{End}_{\mathbb{Z}[[s]]}(\mathbb{Z})$ is an equivalence. This reasoning applies in particular to $\mathcal{F}$, giving $\mathcal{F} \sim \mathrm{End}_{\mathbb{Z}[[s]]}(\mathbb{Z})$. Hence, A is equivalent to $\mathcal{F}$.

5. *Noncanonical modules of type* $(\mathbb{Z}/p, 0)$

In this section, we complete the proof of Theorem 1.3. The uniqueness statement for A was proved in Section 4, so we only have to handle the classification of modules of type $(\mathbb{Z}/p, 0)$.

Let $R = \mathbb{Z}[s]$ and let R act on $\mathbb{Z}$ via $\mathbb{Z} \cong R/(s)$. Clearly, $\mathrm{Ext}^i_R(\mathbb{Z}, \mathbb{Z})$ is $\mathbb{Z}$ for $i = 0, 1$ and zero otherwise, so that $\mathcal{E} = \mathrm{End}_R(\mathbb{Z})$ is a DGA of type $\beta_{\mathbb{Z}}$. By uniqueness, we may as well take $A = \mathcal{E}$; the following proposition explains why this is useful.

Say that a chain complex N of R-modules is *s-torsion* if for each $i \in \mathbb{Z}$ and each $x \in H_iN$ there is an integer $k(x) > 0$ such that $s^{k(x)}x = 0$.

Proposition 5.1 [**4**, 2.1]. *The assignment $N \mapsto \mathrm{Hom}_R(\mathbb{Z}, N)$ restricts to a bijection between equivalence classes of s-torsion chain complexes over R and equivalence classes of right $\mathcal{E}$-modules.*

Remark 5.2. Here $\mathcal{E}$ acts on $\mathrm{Hom}_R(\mathbb{Z}, N)$ through its action on $\mathbb{Z}$. The inverse to this bijection assigns to a right $\mathcal{E}$-module X the derived tensor product $X \otimes_{\mathcal{E}} \mathbb{Z}$.

It is clear from Section 4 that $\mathcal{E} \sim \mathcal{F}$ is equivalent as a DGA to its opposite algebra, so we can pass over the distinction between right and left $\mathcal{E}$-modules. The question of studying

$\mathcal{E}$-modules of type $(\mathbb{Z}/p, 0)$ thus becomes one of classifying s-torsion chain complexes N over R such that

$$H_i \operatorname{Hom}_R(\mathbb{Z}, N) \cong \begin{cases} \mathbb{Z}/p, & i = 0, \\ 0, & i \neq 0. \end{cases} \tag{5.1}$$

Suppose that N is such a chain complex. Applying $\operatorname{Hom}_R(-, N)$ to the exact sequence

$$0 \longrightarrow R \xrightarrow{\ s\ } R \longrightarrow \mathbb{Z} \longrightarrow 0,$$

and taking homology gives a long exact sequence

$$\cdots \longrightarrow H_i \operatorname{Hom}_R(\mathbb{Z}, N) \longrightarrow H_i N \xrightarrow{\ s\ } H_i N \longrightarrow \cdots,$$

in which the arrow labeled 's' cannot be injective unless $H_i N = 0$. It follows that $H_i N$ vanishes unless $i = 0$, and that $H_0 N$ is an abelian group D on which the operator s acts with the following properties:

(1) $\ker(s)$ is isomorphic to $\mathbb{Z}/p$;
(2) s is surjective; and
(3) $\bigcup_{k \geqslant 0} \ker(s^k) = D$.

Call such a pair (D, s) *distinguished*. In fact, any distinguished pair (D, s) gives a chain complex N over R (D itself concentrated in degree 0) which has property (5.1). Combining this observation with Remark 1.5 and Proposition 5.1 thus provides a bijection between

(i) equivalence classes of modules over $\mathcal{E}$ of type $(\mathbb{Z}/p, 0)$ and
(ii) isomorphism classes of distinguished pairs (D, s).

The proof of Theorem 1.3 is completed by the following two routine lemmas. Recall that $R = \mathbb{Z}[s]$. If $\mathcal{O}$ is a discrete valuation ring with uniformizer π, write $\mathcal{O}/\pi^\infty$ for the quotient $\mathcal{O}[1/\pi]/\mathcal{O}$. (This quotient is the injective hull of $\mathcal{O}/\pi$ as an ordinary module over $\mathcal{O}$.)

LEMMA 5.3. *Suppose that (D, s) is a distinguished pair. Then $\mathcal{O} = \operatorname{Ext}^0_R(D, D)$ is a complete discrete valuation ring with residue field $\mathbb{F}_p$ and uniformizer $\pi = s$. The pair (D, s) is naturally isomorphic to $(\mathcal{O}/\pi^\infty, \pi)$.*

LEMMA 5.4. *Suppose that $\mathcal{O}$ is a complete discrete valuation ring with residue field $\mathbb{F}_p$ and uniformizer π. Then $(D, s) = (\mathcal{O}/\pi^\infty, \pi)$ is a distinguished pair, and the natural map $\mathcal{O} \to \operatorname{Ext}^0_R(D, D)$ is an isomorphism.*

REMARK 5.5. According to Theorem 1.3, any complete discrete valuation ring $\mathcal{O}$ with residue field $\mathbb{F}_p$ and uniformizer π should give rise to a module X of type $(\mathbb{Z}/p, 0)$ over the formal DGA $\mathcal{F}$ of Section 4. Observe that a module over $\mathcal{F}$ is just a chain complex with a self-map f of degree -1 and square 0. Let $(D, s) = (\mathcal{O}/\pi^\infty, \pi)$ be the distinguished pair associated to $\mathcal{O}$ and π. Tracing through the above arguments shows that X can be taken to be the following object

$$\begin{array}{cc} 0 & D \\ & f=\mathrm{id} \curvearrowright \quad \big\downarrow \partial = s \\ -1 & D \end{array}$$

concentrated in degrees 0 and -1.

6. *Other exterior algebras*

Suppose that R is a commutative ring, and say that a DGA A is *of type* $\beta_R(n)$ if H_*A is an exterior algebra over R on a class of degree n. In this section, we briefly consider the problem of classifying such DGAs if $n \neq -1$. If $n = 0$, then A is formal (Subsection 1.1) and hence determined up to equivalence by the ring H_0A, so we may as well also assume $n \neq 0$.

Say that a DGA is of type $P_R(n)$ if H_*A is isomorphic to a polynomial algebra over R on a class of degree n.

Proposition 6.1. *If $n \notin \{0, -1\}$, then there is a natural bijection between equivalence classes of DGAs of type $\beta_R(n)$ and equivalence classes of DGAs of type $P_R(-n-1)$.*

Proof (*Sketch*). If A is of type $\beta_R(n)$, then the inductive technique of Section 2 produces a left module X_A over A of type $(R, 0)$, that is, a module X_A such that $H_0(X_A)$ is a free (ordinary) module of rank 1 over $R = H_0A$ and $H_i(X_A) = 0$ for $i \neq 0$. If B is of type $P_R(n)$, then it is even easier to produce a right module X_B over B of type $(R, 0)$: just take the mapping cone of $f\colon \Sigma^n B \to B$, where f represents left multiplication by a generator of H_nB. In both cases, the modules are unique up to possibly noncanonical equivalence. Along the lines of Section 3 (cf. Subsection 1.2), calculating with appropriate collapsing Eilenberg–Moore spectral sequences now gives the desired bijection.

$$\{DGAs\, A \text{ of type } \beta_R(n)\} \underset{B\mapsto \mathrm{End}_B(X_B)}{\overset{A\mapsto \mathrm{End}_A(X_A)}{\rightleftarrows}} \{DGAs\, B \text{ of type } P_R(-n-1)\}. \qquad \square$$

Remark 6.2. It is clear from Theorem 1.1 that Proposition 6.1 fails for $n = -1$ and $R = \mathbb{F}_p$, essentially because if A is of type $\beta_{\mathbb{F}_p}(-1)$, then the nonvanishing entries of the Eilenberg–Moore spectral sequence for $H_* \mathrm{End}_A(X_A)$ accumulate in degree 0. This accumulation creates extension possibilities that allow for a profusion of complete discrete valuation rings in the abutment. Similarly, Proposition 6.1 fails for $n = 0$ and $R = \mathbb{F}_p$, because if B is of type $P_{\mathbb{F}_p}(-1)$, then the Eilenberg–Moore spectral sequence for $H_* \mathrm{End}_B(X_B)$ accumulates in degree 0. In this case, though, the accumulation has consequences which are less drastic, because up to isomorphism there are only two possibilities for a ring R with an ideal I such that $I^2 = 0$ and such that the associated graded ring $\{R/I, I\}$ is an exterior algebra on one generator over $\mathbb{F}_p$. The conclusion is that up to equivalence there are only two DGAs of type $P_{\mathbb{F}_p}(-1)$, one corresponding to the true exterior algebra $\mathbb{F}_p[t]/t^2$, and the other to the fake exterior algebra $\mathbb{Z}/p^2$.

6.1. *DGAs of type $\beta_{\mathbb{Z}}(n)$, all n*

Up to equivalence, there is only one of these for each n. The case $n = 0$ is trivial (Subsection 1.1), while $n = -1$ is Theorem 1.3. By Proposition 6.1, for other n these correspond to DGAs of type $P_{\mathbb{Z}}(n)$, but there is only one of these for each n, because they are all formal (Subsection 1.1).

6.2. *DGAs of type $\beta_{\mathbb{F}_p}(n)$, $n \geqslant 0$*

As usual, the case $n = 0$ is trivial: up to equivalence there is only one example. We sketch an argument that if $n > 0$ is odd, then there is only one example, while if $n > 0$ is even, then there are two. In [**3**], Dugger and Shipley describe a Postnikov approach to constructing a connective (Remark 1.5) DGA A; the technique involves starting with the ring H_0A (considered as a DGA with trivial higher homology) and attaching one homology group at a time, working from low dimensions to high. If A is of type $\beta_{\mathbb{F}_p}(n)$, then there is only a single homology group to deal with, namely $\mathbb{Z}/p$ in degree n. By Dugger and Shipley [**3**, Theorem 8] and a theorem of Mandell [**3**, Remark 8.7] the choices involved in attaching H_nA can be identified with the group $HH^{n+2}_{\mathbb{Z}}(\mathbb{F}_p, \mathbb{Z}/p)$; this is Shukla cohomology of $\mathbb{F}_p$ with coefficients in the $\mathbb{F}_p$-bimodule $\mathbb{Z}/p$. In our notation, this group might be written

$$H_{-n-2}\,\mathrm{Hom}_{\mathbb{F}_p\otimes_{\mathbb{Z}}\mathbb{F}_p^{\mathrm{op}}}(\mathbb{Z}/p, \mathbb{Z}/p), \tag{6.1}$$

where the indicated tensor product over $\mathbb{Z}$ is derived. (This is the appropriate variant of Hochschild cohomology when the ring involved, here $\mathbb{F}_p$, is not flat over the ground ring, here $\mathbb{Z}$.) The ring $H_*(\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p^{\mathrm{op}})$ is an exterior algebra over $\mathbb{F}_p$ on a class of degree 1, so the Eilenberg–Moore spectral sequence computes

$$H_*\,\mathrm{Hom}_{\mathbb{F}_p\otimes_{\mathbb{Z}}\mathbb{F}_p^{\mathrm{op}}}(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{F}_p[u],$$

where the degree of u is -2. This immediately shows that the group (6.1) of gluing choices is trivial if n is odd. If n is even, then there are p gluing choices, but $p-1$ of them are identified by the automorphisms of $\mathbb{Z}/p$ as an $\mathbb{F}_p$-bimodule. The conclusion is that if n is even, then there are p gluing choices, but that up to equivalence only two DGAs emerge.

By Proposition 6.1, this also gives a classification of DGAs of type $P_{\mathbb{F}_p}(n)$ for $n \leqslant -2$.

We do not know how to classify DGAs of type $\beta_{\mathbb{F}_p}(n)$ for $n \leqslant -2$.

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