Tame Filling Functions and Closure Properties

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TAME FILLING FUNCTIONS AND CLOSURE PROPERTIES

by

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A DISSERTATION

Presented to the Faculty of
The Graduate College at the University of Nebraska
In Partial Fulfilment of Requirements
For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professors Mark Brittenham and Susan Hermiller

Lincoln, Nebraska
August, 2015
Let $G$ be a group with a finite presentation $\mathcal{P} = \langle A \mid R \rangle$ such that $A$ is inverse-closed. Let $f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ be a nondecreasing function. Loosely, $f$ is an intrinsic tame filling function for $(G, \mathcal{P})$ if for every word $w$ over $A^*$ that represents the identity element in $G$, there exists a van Kampen diagram $\Delta$ for $w$ over $\mathcal{P}$ and a continuous choice of paths from the basepoint $*$ of $\Delta$ to points on the boundary of $\Delta$ such that the paths are steadily moving outward as measured by $f$. The isodiametric function (or intrinsic diameter function) introduced by Gersten and the extrinsic diameter function introduced by Bridson and Riley are useful invariants capturing the topology of the Cayley complex. Tame filling functions are a refinement of the diameter functions, which were introduced by Brittenham and Hermiller and are used to gain insight on how wildly maximum distances can occur in van Kampen diagrams. Brittenham and Hermiller showed that tame filling functions are a quasi-isometry invariant and that if $f$ is an intrinsic (respectively extrinsic) tame filling function for $(G, \mathcal{P})$, then $(G, \mathcal{P})$ has an intrinsic (respectively extrinsic) diameter function equivalent to the function $n \mapsto \lceil f(n) \rceil$. In contrast to diameter functions, it is unknown if every pair $(G, \mathcal{P})$ has a finite-valued tame filling function.

In this thesis I show that two group constructions, namely graph products (a generalization of direct and free products) and certain free products with amalgamation, preserve finite-valued intrinsic tame filling functions.
DEDICATION

To my parents, Nabeeh and Zakiyyah Nu’Man.
ACKNOWLEDGMENTS

In the name of Allah, the Most Gracious, the Most Merciful.

I would like to begin by expressing my sincere gratitude to my advisors Mark Brittenham and Susan Hermiller. Your support and patience throughout these past several years has been incomparable, and I can honestly say that one of the reasons I completed this major milestone in my life is due to having you both as advisors. Thank you for the countless hours spent editing poster presentations, seminar talks, job application materials and certainly not least my dissertation. I am honored to have you both as advisors, and role models, and I look forward to continuing to work with you in the future. I would also like to thank the members of my committee, Matthew Dwyer, John Meakin and Mark Walker and unofficial committee member Jamie Radcliffe for his support and suggesting we bring “reese’s pieces to Anisah thesis.” To Yvonne Lai and Jim Lewis, thank you for being amazing teaching mentors and for pushing me to become a more committed and intentional educator. I am also grateful to all of the UNL Math faculty members and staff for cultivating an environment in which graduates students can succeed.

I am blessed to have an amazing support system, and I would be remiss if I did not mention them. I would like to thank my community back home at the Atlanta Masjid of Al-Islam; special thanks to my middle and high school math teachers, Bro. Haazim and Sis. Jamila, and my homeroom teacher Sis. Qur’an. To my professors at Spelman College, in and outside of the mathematic department, Sylvia Bozeman, Donna Akiba Harper, Monica Stephens Cooley, Colm Mulcahy, Cynthia Neal Spence and Mohammed Tesemma, thank you for encouraging me to pursue my Ph.D. in mathematics. To my fellow Women Math Warriors (WMW) and EDGE family thank you for supporting me in going to England to do research with my advisors; special
thanks to Ulrica Wilson, Shelby Wilson, Ami Radunskaya, and Talithia Williams for your advise while on the job market. To my Lincoln family and friends, I will forever be grateful that you welcomed me into theirs homes and provided me with a home away from home. I want to especially thank my sorority sisters in the Lincoln Alumnae Chapter of Delta Sigma Theta Sorority, Inc. for their support, bringing me closer to my line sister Ashley, and introducing me to one of the greatest sisterhoods in the world.

From the UNL math department, I am thankful to: the GST community especially GST Jrs Ashley Weatherwax Johnson, Melanie DeVries, Nathan Corwin, Maranda Franke, Nick Owad, Anne Kerian, and Muhammad Inam; my officemates (over the years) Lauren Keough, Becky Egg, Annika Denkert, Pei Pei, Sara Reynolds, Jeremy Trageser, and Areeba Ikram; the wonderful and patient office staff Marilyn, Liz, Lori, and Tom; the NCUWM organizing committee for allowing me to serve on the committee for two years; and everyone else! Thanks so much!

Lastly, I must thank my family and friends. To my father, who is no longer with us, I pray that each day I live a life that you would be proud of; to my mom, you a true inspiration and I would be honored if I become half the woman you are; to my brothers and sisters, Rashidah, Khalilah, Haniyyah, Micah, and Sabir, thank you. I’m no long a “professional student.”
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Chapter 1

Introduction

In 1911 Dehn [6] posed three algorithmic problems for finitely presented groups: the word problem, the conjugacy problem, and the isomorphism problem. The word problem for a group asks if there is an algorithm which, given a “word” in the generators of the group, determines whether or not the word represents the trivial element. It is known that the word problem for finitely presented groups is algorithmically unsolvable in general, and proofs can be found in [1] and [13]. One way geometric group theorists approach decision problems is by studying group properties that are invariant under quasi-isometries, and in particular filling functions. Examples of filling functions include the Dehn function (or isoperimetric function), the isodiametric function (also called intrinsic diameter function), and the extrinsic diameter function [2], [3]. Gersten showed that if any of these filling invariants are computable, then the word problem has a solution [7].

Filling functions for a finitely presented group record geometric properties of van Kampen diagrams with respect to a given presentation. Filling functions that are invariant under group presentation are of particular interest because they display underlying algorithmic and algebraic properties of a group, and in [7] Gersten showed that
the isoperimetric and isodiametric (intrinsic diameter) functions are quasi-isometry
invariant up to equivalence of functions. In [3], Bridson and Riley show that the
extrinsic diameter function is also a quasi-isometry invariant up to equivalence of
functions. Therefore, it makes sense to say a finitely presented group satisfies a lin-
ear, quadratic, polynomial, exponential, etc., isoperimetric (respectively isodiametric
and extrinsic diameter) function. Furthermore, in [7] Gersten showed that if \( f \) is an
isoperimetric function for \((G, \mathcal{P})\) then \( g(n) = M f(n) + n \) is an isodiametric function
for a group \( g \) with presentation \( \mathcal{P}, (G, \mathcal{P}) \), where \( M \) is the maximum length of a
relator in \( \mathcal{P} \). Conversely, Cohen in [5] and Gersten in [7] showed that if \( f \) is an
isodiametric function for \((G, \mathcal{P})\), then there exists positive constants \( a, b > 0 \) such
that \( g(n) = a^{b(n)+n} \) is an isoperimetric function for \((G, \mathcal{P})\).

Bridson and Riley [3] showed that the intrinsic diameter (isodiametric) function
is an upper bound for the extrinsic diameter function. Additionally, in [3] they con-
struct the first example of a group \( G \), with presentation \( \mathcal{P} \), for which we have positive
constants \( a, b > 0 \) such that the extrinsic diameter function is bounded above by
\( n \mapsto n^b \) with the intrinsic diameter function at least \( n \mapsto n^{a+b} \).

The intrinsic and extrinsic diameter functions bound maximum distances within
a van Kampen diagram and Cayley complex, respectively, but do not give insight to
the manner in which maximum distances can occur. In [4], Brittenham and Hermiller
define tame filling functions as a refinement of diameter filling functions in an effort
to understand how wildly maximum distances can occur and to gain traction on the
word problem. For a finitely presented group \( G \) with presentation \( \mathcal{P} \), an intuitive
description of an intrinsic tame filling function (respectively, extrinsic tame filling
function) is a nondecreasing function \( f \) such that for every word \( w \) that represents
the identity element in \( G \), there exists a van Kampen diagram \( \Delta \) with base point \( * \)
and a continuous choice of paths from \( * \) to points on the boundary of \( \Delta \) such that
the paths are steadily moving outward as measured by \( f \). A complete definition of a tame filling function is given in Definition 2.5 on page 10. In [4], Brittenham and Hermiller show that almost convex groups, groups that admit a finite complete rewriting system, and more generally stackable groups, all have a well-defined intrinsic (respectively, extrinsic) tame filling function.

Chapter 2 provides background and definitions used throughout the paper, including a minor excursion on diameter functions and the formal definition of tame filling functions. Furthermore, in Chapter 2 we present known results about diameter functions, tame filling functions, and how they relate to each other.

Let \( f : \mathbb{N}^\left[1\right] \rightarrow \mathbb{N}^\left[1\right] \) be a function. We say \( f \) is subnegative if for all \( n, m \) in \( \mathbb{N}^\left[1\right] \) we have \( f(n) + f(m) \leq f(n + m) \).

**Definition 1.1.** Let \( f : \mathbb{N}^\left[1\right] \rightarrow \mathbb{N}^\left[1\right] \) be any function. The subnegative closure of \( f \), denoted by \( \overline{f} \), is the least subnegative function greater than or equal to \( f \).

The subnegative closure of \( f \) can be expressed by

\[
\overline{f}(n) = \max \left\{ \sum_{i=1}^{k} f(n_i) \mid \sum_{i=1}^{k} n_i = n \right\}.
\]

In Chapter 3, we show that the class of groups with finite-valued tame filling functions is closed under taking graph products. Given a finite simple graph \( \Lambda \) and a collection of finitely presented groups \( \{G_i\}_{i=1}^{m} \) associated to the vertices of \( \Lambda \), the graph product of groups \( \{G_i\}_{i=1}^{m} \) associated to \( \Lambda \) is the quotient \( G\Lambda \) of the free product of the groups of \( G_i \) by the normal closure of the set

\( \{[g_i, g_j] = 1 \mid (v_i, v_j) \in E(\Lambda) \text{and} \ g_i \in G_i, g_j \in G_j \} \). We make use of the subnegative closure in proving the following result for graph products.
Theorem 3.5 Let $\Lambda$ be a finite simple graph. Let $\{G_i\}_{i=1}^m$ be a family of finitely generated groups associated to the vertices of $\Lambda$ with presentations $P_i = \langle A_i | R_i \rangle$. Let $h_i : \mathbb{N}[\frac{1}{4}] \rightarrow \mathbb{N}[\frac{1}{4}]$ be an intrinsic tame filling function for $(G_i, P_i)$ for $1 \leq i \leq m$. Then $h : \mathbb{N}[\frac{1}{4}] \rightarrow \mathbb{N}[\frac{1}{4}]$ defined by

$$h(n) = \sum_{i=1}^m h_i(n) + n$$

is equivalent to an intrinsic tame filling function for the graph product $G_\Lambda$ of the groups $\{G_i\}_{i=1}^m$, with respect to the presentation

$$P = \langle \bigcup_{i=1}^m A_i \big| \bigcup_{i=1}^m R_i \cup \{[g_i, g_j] = 1 \mid g_i \in G_i, g_j \in G_j \text{ and } G_i, G_j \text{ are associated to adjacent vertices in } \Lambda \} \rangle.$$ 

The work of Meier in [11] gives a bound on the isodiametric function for a graph product in terms of the isodiametric functions of the respective vertex groups. The proof of Theorem 3.5 above extends Meier’s proof on a bound for the isodiametric function using extra information contained in tame filling invariants.

Given a group $G = \langle A | R \rangle$ with subgroup $H$, a set $S$ of words over $A^*$ is a left transversal for $H$ in $G$ if and only if the map $s \mapsto sH$ from $S$ to $G/H$ is a bijection. A left transversal $S$ is geodesic if all the words in $S$ label geodesics in the associated Cayley graph of $G$. In Chapter 4 we show that the class of tame filling functions is closed under taking certain free products with amalgamation and prove the following result.
Theorem 4.1. Let $f_{\alpha}$ and $f_{\beta}$ be intrinsic tame filling functions for $G_{\alpha} = \langle A_{\alpha}|R_{\alpha} \rangle$ and $G_{\beta} = \langle A_{\beta}|R_{\beta} \rangle$, respectively. Let $H = \langle h_1, \ldots, h_m \rangle$ be a finitely generated group and let $\hat{\alpha} : H \hookrightarrow G_{\alpha}$ and $\hat{\beta} : H \hookrightarrow G_{\beta}$ be injective homomorphisms. Let $G = G_{\alpha} *_{H} G_{\beta}$ with presentation

$$
P = \langle A_{\alpha}, A_{\beta}, h_1, \ldots, h_m | R_{\alpha} \cup R_{\beta} \cup \{ h_i = \hat{\alpha}(h_i), h_i = \hat{\beta}(h_i) \ \forall \ 1 \leq i \leq m \} \rangle.
$$

Suppose that there is a prefix-closed set of geodesic normal forms for $G$ of the form

$$
N = \{ t_1 t_2 \ldots t_n y_h | t_i \text{ alternates between } T_{\alpha} \setminus \epsilon \text{ and } T_{\beta} \setminus \epsilon, \ y_h \in N_H \},
$$

where, for $\gamma \in \{ \alpha, \beta \}$, $T_{\gamma}$ is a set of geodesic left transversals for $G_{\gamma}/\hat{\gamma}(H)$, containing the empty word, with respect to the following presentation for $G_{\gamma}$

$$
P_{\gamma} = \langle A_{\gamma}, h_1, \ldots, h_m | R_{\gamma} \cup \{ h_i = \hat{\gamma}(h_i) \ \forall \ 1 \leq i \leq m \} \rangle
$$

and $N_H$ is a set of prefix-closed geodesic normal forms for $H$. Then $(G, P)$ has an intrinsic tame filling function equivalent to $f(n) = \overline{f_{\alpha}}(n) + \overline{f_{\beta}}(n) + n$. 


Chapter 2

Background

Let $G$ be a finitely presented group with presentation $\mathcal{P} = \langle A | R \rangle$, where $A$ is a finite, inverse closed, generating set for $G$ and $R$ is a finite set of defining relations closed under inverses and cyclic conjugation. We will always assume that our generating set $A$ is closed under inversion. A word $w$ over $A$ is a finite concatenation of letters from the generating set $A$, and the length of $w$, denoted $\ell(w)$, is the number of letters in the word. If two words $u$ and $v$ represent the same group element, then we write $u =_G v$. If they are identically the same word, then we write $u = v$.

The Cayley graph of $G$ with respect to the presentation $\mathcal{P}$, denoted $\Gamma(G, A)$, is the graph whose vertices are elements of $G$ with an oriented edge from vertex $g$ to vertex $h$ labeled by a generator $a \in A$ if and only if $h =_G ga$ in $G$. By attaching a 2-cell, labeled by $r \in R$, at each vertex in $\Gamma(G, A)$ we obtain the Cayley complex $X(G, A)$ for $G$. Let $V(\Gamma(G, A))$ and $E(\Gamma(G, A))$ denote the vertex set and directed edge set of the Cayley graph $\Gamma(G, A)$, respectively. Additionally, let $e(g, a)$ denote the directed edge in $\Gamma(G, A)$ from vertex $g$ labeled by $a$. Given an edge $e$, we denote the initial and terminal vertices of $e$ as $\text{int}(e)$ and $\text{ter}(e)$, respectively. Identify each edge of $\Gamma(G, A)$ with the unit interval $[0, 1]$. Given elements $g$ and $h$ in $G$, the word
metric on $\Gamma(G, A) = X(G, A)^{(1)}$, denoted $d_{\Gamma}$, is the length of the shortest path in $\Gamma(G, A)$ from vertex $g$ to vertex $h$.

2.1 Diameter functions

Given a word $w \in A^*$ satisfying $w =_G \epsilon$, a van Kampen diagram $\Delta$ for $w$ over the presentation $\mathcal{P}$ is a connected, simply connected, planar 2-complex with a vertex $*$ on the boundary of $\Delta$, $\partial \Delta$, such that the $\partial \Delta$ is labeled by the word $w$ starting at $*$ and for every 2-cell $\sigma$ of $\Delta$, the label of $\partial \sigma$ is a word in $R$. For any van Kampen diagram $\Delta$ let $d_\Delta$ denote the path metric on the 1-skeleton of $\Delta$. Furthermore, let $\pi_\Delta : \Delta \to X(G, A)$ be the canonical map such that $\pi_\Delta(*) = \epsilon$ and $n$-cells are mapped to $n$-cells preserving both label and direction.

To display the relationship between diameter functions and tame filling functions we provide the definition of the intrinsic diameter function (or isodiametric function, originally defined in [7]) and extrinsic diameter function defined by Bridson and Riley in [3].

Definition 2.1. Given a group $G$ the intrinsic (respectively, extrinsic) diameter function, with respect to the presentation $\mathcal{P}$, is the minimal nondecreasing function $f : \mathbb{N} \to \mathbb{N}$ such that for all $w \in A^*$ with $w =_G \epsilon$, there exists a van Kampen diagram $\Delta$ for $w$ over $\mathcal{P}$ such that for all vertices $v \in \Delta^{(0)}$ we have $d_\Delta(*, v) \leq f(\ell(w))$ (respectively, $d_X (\epsilon, \pi_\Delta(v)) \leq f(\ell(w))$).

Since $\ell(w) \in \mathbb{N}$ and $A$ is a finite set, there are finitely many words $w$ such that $w =_G \epsilon$ and $\ell(w) \leq n$. This gives us that there is a minimal value for $f(n)$, and so the intrinsic and extrinsic diameter functions are well-defined. As stated in the introduction, in [3] Bridson and Riley showed that the extrinsic diameter function is
a lower bound for the intrinsic diameter function and that there exists a group \( G \) for which we have positive constants \( a, b > 0 \) such that the extrinsic diameter is bounded above by \( n \mapsto n^b \) with intrinsic diameter function at least \( n \mapsto n^{a+b} \).

### 2.2 Tame filling functions

To reach a formal definition of a tame filling function for \((G, \mathcal{P})\), given a word \( w = G \epsilon \) and a van Kampen diagram \( \Delta \) for \( w \) with respect to the presentation \( \mathcal{P} \) we need a continuous choice of paths from the base point \( * \) to points on the \( \partial \Delta \). The continuous choice of paths used is called a 1-combing and was originally defined by Mihalik and Tschantz in [12]. We adopt the following definitions from [4].

**Definition 2.2.** [12] Let \( Y \) be a combinatorial 2-complex and \( Z \) a subcomplex of \( Y^{(1)} \). A 1-combing of the pair \((Y, Z)\) at a basepoint \( y_0 \in Y \) is a continuous map \( \Psi : Z \times [0, 1] \to Y \) such that:

\( (c1) \) \( \Psi(p, 0) = y_0 \) and \( \Psi(p, 1) = p \) for all \( p \in Y \),

\( (c2) \) When \( y_0 \in Z \) we have \( \Psi(y_0, t) = y_0 \) for all \( t \in [0, 1] \), and

\( (c3) \) When \( p \in Z^{(0)} \), we have \( \Psi(p, t) \in Y^{(1)} \) for all \( t \in [0, 1] \).

Considering the unit interval \([0, 1]\) as an interval of time, a tame filling function \( f \) bounds the manner in which the continuous paths of a 1-combing are able to travel outward, away from the basepoint. In order to measure the tameness of the paths in a 1-combing, we need to have a way to measure distances to points within \( \Gamma(G, A) \) and 2-cells in \( X(G, A) \). This is done by using the following coarse distance, which was introduced in [4].
Definition 2.3. [4] Let $Y$ be a combinatorial 2-complex with basepoint $y$ and let $p$ be any point in $Y$. The coarse distance, denoted $\tilde{d}_Y(y, p)$, is defined as follows.

- For $p \in Y^{(0)}$, let $\tilde{d}_Y(y, p) = d_Y(y, p)$ where $d_Y$ denotes the path metric on $Y^{(1)}$;
- For $p$ in the interior of the edge $e$ in $Y$ with endpoints $\text{int}(e)$ and $\text{ter}(e)$, let
  \[ \tilde{d}_Y(y, p) = \min\{d_Y(y, \text{int}(e)), d_Y(y, \text{ter}(e))\} + \frac{1}{2}; \]
- For $p$ in the interior of a 2-cell $\sigma$ in $Y$, let
  \[ \tilde{d}_Y(y, p) = \max\{d_Y(y, q) \mid q \in \partial\sigma \setminus Y^{(0)}\} - \frac{1}{4}. \]

One goal for defining tame filling functions is to measure how wildly or tamely maximum distances can occur in a van Kampen diagram and Cayley complex. A sense of tameness needs to be established; this is done with the following tameness condition, introduced in [4].

Definition 2.4. [4] A 1-combing $\Psi$ of the pair $(Y, Z)$ at the basepoint $y_0 \in Y$ is $f$-tame for a nondecreasing function $f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ if for all $p \in Z$ and $0 \leq s < t \leq 1$ we have

\[ \tilde{d}_Y(y_0, \Psi(p, s)) \leq f \left( \tilde{d}_Y(y_0, \Psi(p, t)) \right). \]

The $f$-tame condition requires that if a path reaches a distance greater than $f(n)$ (from the basepoint) at a time $s$, then it cannot return to points within distance $n$ (from the basepoint) at a later time $t$. For the pair $(G, \mathcal{P})$ a combed filling is a set $\mathcal{F} = \{ (\Delta_w, \Psi_w) \mid w \in A^*, w =_G \epsilon \}$, where $\Delta_w$ is a van Kampen diagram for $w$ over $\mathcal{P}$ and $\Psi_w$ is a 1-combing of $(\Delta_w, \partial \Delta_w)$ at the basepoint $\ast$. Using combed fillings and the $f$-tame condition, Brittenham and Hermiller [4] defined an intrinsic (respectively, extrinsic) tame filling function as follows.
Definition 2.5. [4] A nondecreasing function \( f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}] \) is an intrinsic [respectively, extrinsic] tame filling function for the group \( G \) with presentation \( \mathcal{P} = \langle A|R \rangle \) if for every word \( w \in A^* \) with \( w =_G \epsilon \), there exists a combed filling \( \mathcal{F} \) such that the 1-combing \( \Psi_w \) of \( (\Delta_w, \partial D_w) \) at the basepoint \( \ast \) [respectively, \( \pi_{\Delta} \circ \Psi \) of \( (X(G, A), \Gamma(G, A)) \) at the basepoint \( \epsilon \)] is \( f \)-tame.

Tame filling functions are quasi-isometry invariant up to equivalence [4], where we say two functions \( f, g : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}] \) are equivalent, \( f \sim g \), if there exists a constant \( C \) such that for all \( n \in \mathbb{N}[\frac{1}{4}] \) we have \( f(n) \leq Cg(Cn+C)+C \) and \( g(n) \leq Cf(Cn+C)+C \). If \( f \sim g \) then we also have \( \overline{f} \sim \overline{g} \), where \( \overline{f}, \overline{g} \) are the subnegative closures of \( f \) and \( g \), respectively.

Since tame filling functions are a refinement of the intrinsic and extrinsic diameter function, it is important to point out a few observations. Tame filling functions are defined on \( \mathbb{N}[\frac{1}{4}] \) to account for how distances are measured under the coarse distance in Definition 2.2. Furthermore, tame filling functions do not depend on the length of \( w \), as diameter functions do, and therefore it is not known whether or not every group \( G \) with finite presentation \( \mathcal{P} \) admits a finite-valued intrinsic (respectively, extrinsic) tame filling function. It is known from [[4], Corollary 4.5] that if \( (G, \mathcal{P}) \) has a well-defined extrinsic tame filling function then \( (G, \mathcal{P}) \) is tame combable as defined by Mihalik and Tschantz in [12]. Lastly, Brittenham and Hermiller in [[4], Proposition 3.2] show that if \( f \) is an intrinsic (respectively extrinsic) tame filling function for \( (G, \mathcal{P}) \), then the function \( \hat{f}(n) = \lceil f(n) \rceil \) is an upper bound for the intrinsic (respectively extrinsic) diameter function for \( (G, \mathcal{P}) \).
2.3 Edge combings

Tame filling functions require a combed filling in which the provided 1-combings are $f$-tame. The following definitions and results, from [4], allow us to reduce the type of van Kampen diagrams used to ensure a finitely presented group admits an intrinsic (respectively, extrinsic) tame filling function. This enables us to establish the existence of a tame filling function by looking at fewer van Kampen diagrams.

For a group $G$ with finite generating set $A$ and Cayley graph $\Gamma(G,A)$, a set of normal forms for $G$ over $A$ is a choice of one reduced word $y_g$ in $A^*$ for each group element $g$ of $G$ with $y_g =_G g$. Let $N = \{y_g | g \in G\} \subseteq A^*$ be a set of normal forms (including the empty word) for $G = \langle A|R\rangle$ that label simple paths in $X(G,A)$. An $N$-diagram is a van Kampen diagram $\Delta$ for the word $y_g ay_g^{-1}$ for some $g \in G$ and $a \in A$. Let $v_g$ denote the terminal vertex on the boundary of $\Delta$ of the path on the boundary of $\Delta$ starting at the basepoint $*$ labeled by $y_g$. Let $\hat{e}$ denote the edge in $\partial \Delta$ with initial vertex $v_g$ labeled by $a$ and let $u_g$ be the terminal vertex of $\hat{e}$.

**Definition 2.6.** [4] Let $\Delta$ be an $N$-diagram for $G = \langle A|R\rangle$. An edge 1-combing of $\Delta$ is a continuous function $\Theta : \hat{e} \times [0,1] \to \Delta$ such that

1. $\Theta(p,0) = *$ and $\Theta(p,1) = p$ for all $p \in \hat{e}$,
2. if $*$ is in $\hat{e}$ then $\Theta(\ast,t) = *$ for all $t \in [0,1]$, and
3. $\Theta((v_g),\cdot), \Theta((u_g),\cdot) : [0,1] \to \Delta$ follow the paths in $\partial \Delta$ labeled by $y_g$ and $y_ga$ from $*$, respectively.
Definition 2.7. [4] A combed $\mathcal{N}$-filling is a collection $\mathcal{E} = \{(\Delta_e, \Theta_e) | e \in E(X(G, A))\}$ such that for each $e = e(g, a)$ for some $g \in G$ and $a \in A$, $\Delta_e$ is an $\mathcal{N}$-diagram and $\Theta_e$ is an edge combing of $\Delta_e$ that satisfies the following gluing condition:

(g1) For every pair of edges $e$ and $e'$ in $E(X(G, A))$ with a common endpoint $g$ and for all $t \in [0, 1]$ we have $\pi_{\Delta_e} \circ \Theta_e (\hat{g}_e, t) = \pi_{\Delta_{e'}} \circ \Theta_{e'} (\hat{g}_{e'}, t)$, where $\hat{g}_e$ and $\hat{g}_{e'}$ are the vertices of $\Delta_e$ and $\Delta_{e'}$, respectively, that map to $g$ in $X$.

A combed $\mathcal{N}$-filling is geodesic if each word in the set of normal forms $\mathcal{N}$ labels a geodesic path in the associated Cayley graph. The following result from [4] connects combed $\mathcal{N}$-fillings and combed fillings.

Lemma 2.8. [[4], Proposition 4.3] Let $G$ be a finitely presented group with presentation $\mathcal{P} = \langle A | R \rangle$, and let $f : \mathbb{N}[\frac{1}{4}] \rightarrow \mathbb{N}[\frac{1}{4}]$ be a nondecreasing function. The following are equivalent, up to equivalence of functions:

1. $f$ is an intrinsic [respectively, extrinsic] tame filling function for $(G, \mathcal{P})$.

2. $(G, \mathcal{P})$ has a geodesic combed $\mathcal{N}$-filling $\mathcal{E} = \{(\Delta_e, \Theta_e) | e \in E(X(G, A))\}$ such that each edge 1-combing $\Theta_e$ [respectively, $\pi_{\Delta_e} \circ \Theta_e$] is $f$-tame.

Remark 2.9. In the proof of Lemma 2.8, (1) implies (2), uses shortlex normal forms.

We utilize Lemma 2.8 in the proofs of Theorem 3.5 and Theorem 4.1 in the following sections.
Chapter 3

Graph products

In this section I prove that the tame filling property is preserved under the graph product construction.

Let $\Lambda$ be a finite graph with no loops and no multiple edges. Let $V(\Lambda) = \{v_1, \ldots, v_m\}$ and $E(\Lambda)$ denote the vertex set and edge set of $\Lambda$, respectively. Let $e(v_i, v_j)$ denote the edge in $\Lambda$ between vertices $v_i$ and $v_j$. For each vertex $v_i$ we associate a finitely presented group $G_i$. The graph product of groups $\{G_i\}_{i=1}^m$ associated to $\Lambda$ is the quotient, $G_{\Lambda}$, of the free product of the groups $G_i$ by the normal closure of the set $\{[g_i, g_j] = 1 \mid e(v_i, v_j) \in E(\Lambda) \text{ and } g_i \in G_i, g_j \in G_j\}$. Graph products were originally defined by Green in [8] and are a generalization of direct products (in which $\Lambda$ is a complete graph) and free products (in which $\Lambda$ is a graph with no edges).

For each vertex group $G_i$, let $P_i = \langle A_i | R_i \rangle$ be a presentation for $G_i$. We may assume that each set $R_i$ does not contain any relations of the form $a_j = a_k$ or $a_j = 1_{G_i}$, for generators $a_j, a_k \in A_i$. If it does, we can use Tietze transformations to obtain a presentation for $G_i$ with either $a_j$ or $a_k$ removed. Let $A = \bigcup_{i=1}^m A_i$ and $R = \bigcup_{i=1}^m R_i$. 
We can present $G\Lambda$ as

\[
G\Lambda = \langle A \mid R \cup \{[a_i, a_j] = 1 \forall a_i \in A_i, a_j \in A_j \text{ such that } (v_i, v_j) \in E(\Lambda)\} \rangle. \tag{3.1}
\]

For the family of groups $\{G_i\}_{i=1}^m$, let $<_G$ be a total ordering on the finite set of groups. The following definitions and results of Hermiller and Meier are in [9], pg. 9 and 10.

**Definition 3.1.** [9] Given a word $w$ in $A^*$, a subword $w'$ of $w$ is a local word if it is written in letters coming from a single vertex group and there is no longer subword of $w$, containing $w'$, written in letters coming from a single vertex group.

**Definition 3.2.** [9] Given sets of normal forms $N_i$ for $G_i$ over $A_i$, a word $w$ in $A^*$ is proper if it satisfies the following local condition and obstruction condition:

(L) Each local word $w_i$ over $A_i^*$ of $w$ is in the normal form set $N_i$.

(O) If $w = \ldots w_i \ldots w_j \ldots$, where $w_i$ and $w_j$ are local words in $A_i^*$ and $A_j^*$, respectively, with $G_j \leq_G G_i$ and $v_i$ is adjacent to $v_j$ in $\Lambda$, then there is a local word $w_k$ such that $w = \ldots w_i \ldots w_k \ldots w_j \ldots$, where $v_k$ and $v_j$ are non-adjacent in $\Lambda$.

**Lemma 3.3.** [[9], Proposition 3.2] Let $G\Lambda$ be the graph product of a finite set of groups $\{G_i\}_{i=1}^m$ each with normal forms $\{N_i\}_{i=1}^m$. Then the set of proper words in $A^*$ is a set of normal forms for $G\Lambda$.

**Lemma 3.4.** [[9], Proposition 3.3] Let $G\Lambda$ be the graph product of a finite set of groups, all having geodesic normal forms. Then the normal forms of $G\Lambda$ given by the proper words are geodesic.
Theorem 3.5. Let $\Lambda$ be a finite simple graph. Let $\{G_i\}_{i=1}^m$ be a family of finitely generated groups associated to the vertices of $\Lambda$ with presentations $\mathcal{P}_i = \langle A_i | R_i \rangle$. Let $h_i : \mathbb{N}^{\frac{1}{4}} \to \mathbb{N}^{\frac{1}{4}}$ be an intrinsic tame filling function for $(G_i, \mathcal{P}_i)$ for $1 \leq i \leq m$. Then $h : \mathbb{N}^{\frac{1}{4}} \to \mathbb{N}^{\frac{1}{4}}$ defined by

$$h(n) = \sum_{i=1}^m \tilde{h}_i(n) + n$$

is equivalent to an intrinsic tame filling function for the graph product $G\Lambda$ of the groups $\{G_i\}_{i=1}^m$, with respect to the presentation

$$\mathcal{P} = \langle \cup_{i=1}^m A_i \cup \cup_{i=1}^m R_i \cup \{[g_i, g_j] = 1 \mid g_i \in G_i, g_j \in G_j \text{ and } G_i, G_j \text{ are associated to adjacent vertices in } \Lambda} \rangle.$$ 

Proof. Let $G\Lambda$ be a graph product of the family of groups $\{G_i\}_{i=1}^m$. Since $h_i$ is an intrinsic tame filling function for $(G_i, \mathcal{P}_i)$ by Lemma 2.8 $(G_i, \mathcal{P}_i)$ has a geodesic combed $\mathcal{N}_i$-filling, $\mathcal{E}_i = \{(\Sigma_e, \Omega_e) | e \in E_X\}$, such that each edge 1-combing $\Omega_e$ is $\tilde{h}_i$-tame, where $\tilde{h}_i$ is equivalent to $h_i$, for $1 \leq i \leq m$. Since $f \sim g$ implies $f \sim g$, by a slight abuse of notation, we will simply write the function $h_i$, instead of $\tilde{h}_i$. Let $\mathcal{N}$ be the set of proper words as defined in Definition 3.2. Since the normal forms $\mathcal{N}_i$ are geodesic for $1 \leq i \leq m$, by Lemma 3.4 the set $\mathcal{N}$ is a set of geodesic normal forms for $G\Lambda$.

Using the normal form set $\mathcal{N}$ we will construct a geodesic combed normal $\mathcal{N}$-filling $\mathcal{E} = \{(\Delta_e, \Psi_e) | e \in E_X\}$ such that each edge 1-combing $\Psi_e$ is $h$-tame. For $g \in G\Lambda$, let $y_g$ be the normal form for $g$ in $\mathcal{N}$. Let $x \in A = \cup_{i=1}^m A_i$. This proof is divided into five parts: (1) factor the normal form $y_g$ (2) compute the normal form for $y_gx$, (3) construct a van Kampen diagram $\Delta$ for $y_gx y_{g^{-1}}$, (4) construct an edge 1-combing $\Psi$, and (5) show that $\Psi$ is $h$-tame.
Fix $g \in GA$ and express $y_g = w_{\alpha_1} \ldots w_{\alpha_k}$ where each $w_{\alpha_i}$ is a local word in $N_{\alpha_i}$ for $1 \leq \alpha_i \leq m$. Without loss of generality, let $x \in A_{\alpha_j}$ for some $1 \leq \alpha_j \leq m$.

**Part 1: Factorization of $y_g$.**

First we will express $y_g$ as $y_g = w_\rho w_\tau w_\sigma$. In $y_g x = w_{\alpha_1} \ldots w_{\alpha_k} x$ shuffle the letter $x$ to the left (past a local word $w_{\alpha_i}$) whenever the corresponding vertex $v_{\alpha_i}$ is adjacent to $v_{\alpha_j}$ in the graph $\Lambda$. Once the letter $x$ has been shuffled as far left as possible in $y_g x$ we then shuffle the letter $x$ to the right of a local word $w_{\alpha_i}$ whenever the corresponding vertices are adjacent and $G_{\alpha_i} < G G_{\alpha_j}$. This process will stabilize when we have $y_g x =_{GA} w_{\alpha_1} \ldots w_{\alpha_s} x w_{\alpha_{s+1}} \ldots w_{\alpha_k}$ for some $1 \leq s \leq k$ in which $G_{\alpha_s} < G G_{\alpha_j} < G G_{\alpha_{s+1}}$.

Let $w_\rho = w_{\alpha_1} \ldots w_{\alpha_s-1}$, $w_\tau = w_{\alpha_s}$ and $w_\sigma = w_{\alpha_{s+1}} \ldots w_{\alpha_k}$. It is possible to have $w_\rho, w_\tau$, or $w_\sigma$ be the empty word $\epsilon$ and so we have the following possible factorizations of $y_g$ as $y_g = w_\rho w_\tau w_\sigma$.

(i.) $w_\rho = \epsilon$, $w_\tau = \epsilon$, and $w_\sigma = \epsilon$

(ii.) $w_\rho = \epsilon$, $w_\tau = \epsilon$, and $w_\sigma = w_{\alpha_1} \ldots w_{\alpha_k} \neq \epsilon$

(iii.) $w_\rho = \epsilon$, $w_\tau = w_{\alpha_1} \neq \epsilon$, and $w_\sigma = \epsilon$

(iv.) $w_\rho = \epsilon$, $w_\tau = w_{\alpha_1} \neq \epsilon$, and $w_\sigma = w_{\alpha_2} \ldots w_{\alpha_k} \neq \epsilon$

(v.) $w_\rho = w_{\alpha_1} \ldots w_{\alpha_{k-1}} \neq \epsilon$, $w_\tau = w_{\alpha_k} \neq \epsilon$, and $w_\sigma = \epsilon$

(vi.) $w_\rho = w_{\alpha_1} \ldots w_{\alpha_{s-1}} \neq \epsilon$, $w_\tau = w_{\alpha_s} \neq \epsilon$, and $w_\sigma = w_{\alpha_{s+1}} \ldots w_{\alpha_k} \neq \epsilon$

This completes our factorization of $y_g$ as $y_g = w_\rho w_\tau w_\sigma$. 
Part 2: Computation of \( y_gx \)

Recall \( y_g \) is in \( \mathcal{N} \), where \( \mathcal{N} \) is the set of proper words in \( A^* \). Having written \( y_g = w_\rho w_\tau w_\sigma \), we now compute the normal form for \( y_gx \), where we recall \( x \in A_\alpha \). We will explicitly compute \( y_gx \) using the factorization of \( y_g \) in case (vi) from Part 1 and note that computing the normal form for \( y_gx \) in the other cases is done by allowing at least one of \( w_\rho, w_\tau, \) or \( w_\sigma \) be the empty word.

Suppose we have \( w_\rho = w_\alpha_1 \ldots w_{\alpha_{s-1}} \neq \epsilon, w_\tau = w_{\alpha_s} \neq \epsilon, \) and \( w_\sigma = w_{\alpha_{s+1}} \ldots w_{\alpha_k} \neq \epsilon \) for some \( 1 \leq s \leq k \). By construction we have \( [w_\tau, x] = G_\Lambda 1 \) and in particular \( [w_{\alpha_i}, x] = G_\Lambda 1 \) for all \( s+1 \leq i \leq k \). Let \( y_{w_\tau x} \) be the normal form for \( w_\tau x \) in \( \mathcal{N} \). Then we have

\[
y_gx = w_\rho w_\tau w_\sigma x = G_\Lambda w_\rho w_\tau x w_\sigma = G_\Lambda w_\rho y_{w_\tau x} w_\sigma = G_\Lambda w_\alpha_1 \ldots w_{\alpha_{s-1}} y_{w_\tau x} w_{\alpha_{s+1}} \ldots w_{\alpha_k}. \tag{3.2}
\]

We claim that \( w_\rho y_{w_\tau x} w_\sigma \) is a proper word in \( \mathcal{N} \) and so \( y_gx = w_\rho y_{w_\tau x} w_\sigma \).

Suppose \( w_\tau \in \mathcal{N}_{\alpha_j} \) and \( y_{w_\tau x} \in \mathcal{N}_{\alpha_j} \) is the empty word. Since \( v_{\alpha_j} \) is adjacent to \( v_{\alpha_{s+1}} \) in \( \Lambda \), we have \( [w_\tau, w_{\alpha_{s+1}}] = G_\Lambda 1 \). From line (3.3) above, if \( \alpha_{s-1} = \alpha_{s+1} \) then \( [w_{\alpha_{s-1}}, w_\tau] = G_\Lambda 1 \), contradicting \( y_g \) being in normal form. Therefore, we have that \( \alpha_{s-1} \neq \alpha_{s+1} \) and in \( w_\rho w_\sigma = w_{\alpha_1} \ldots w_{\alpha_{s-1}} w_{\alpha_{s+1}} \ldots w_{\alpha_k} \) each local word is in normal form with consecutive local words representing elements in distinct vertex groups. Furthermore, in \( w_\rho w_\sigma = w_{\alpha_1} \ldots w_{\alpha_{s-1}} w_{\alpha_{s+1}} \ldots w_{\alpha_k} \), if \( G_{\alpha_{s+1}} < G \ G_{\alpha_{s-1}} \) and \( v_{\alpha_{s+1}} \) and \( v_{\alpha_{s-1}} \) are adjacent in \( \Lambda \), then applying condition (O) in Definition 3.2 to the normal form \( y_g \) we must have that \( v_{\alpha_s} \) and \( v_{\alpha_{s+1}} \) are not adjacent in \( \Lambda \). Since \( w_\tau = w_{\alpha_s} \), this contradicts \( [w_\tau, w_{\alpha_{s+1}}] = G_\Lambda 1 \); therefore, we either have \( G_{\alpha_{s+1}} \not< G \ G_{\alpha_{s-1}} \) or \( v_{\alpha_{s-1}} \) and \( v_{\alpha_{s+1}} \) are not adjacent in \( \Lambda \).

To show \( w_\rho w_\sigma \) satisfies the obstruction condition in Definition 3.2, suppose we
have local words \( w_i \) and \( w_\ell \) of \( w_\rho w_\sigma \) such that \( w_\rho w_\sigma = \ldots w_i \ldots w_\ell \ldots \) with \( \ell \leq i \) and \( v_i \) and \( v_\ell \) adjacent in \( \Lambda \). If \( w_i \ldots w_\ell \) is a subword of \( w_\rho \) or \( w_\sigma \), then there exists a local word \( w_t \) such that we have \( w_i \ldots w_\ell = w_i \ldots w_t \ldots w_\ell \) and the vertices \( v_t \) and \( v_\ell \) are non-adjacent in \( \Lambda \). The local word \( w_t \) is also a local word of \( w_\rho w_\sigma \) and so the obstruction condition in Definition 3.2 holds for \( w_\rho w_\sigma \).

If \( w_i \ldots w_\ell \) is not a subword of \( w_\rho \) or \( w_\sigma \), then \( w_i \) and \( w_\ell \) are local words of \( w_\rho \) and \( w_\sigma \), respectively, such that we have \( y_g = \ldots w_i \ldots w_{\alpha_{s-1}} w_\alpha w_{\alpha_{s+1}} \ldots w_\ell \ldots \). If there exists a local word \( w_t \) such that \( y_g = \ldots w_{\alpha_{s+1}} \ldots w_t \ldots w_\ell \) with \( v_i \) and \( v_\ell \) are non-adjacent in \( \Lambda \), then again \( w_t \) is a local word of \( w_\rho w_\sigma \) and the obstruction condition holds for \( w_\rho w_\sigma \). If there does not exist such a local word, then in \( y_g \) we have \( v_t \) is adjacent to \( v_m \) in \( \Lambda \) for all \( \alpha_{s+1} \leq m \leq \ell - 1 \). Recall by construction we have \([w_\alpha, x] = 1\) for all local words in \( w_\sigma \). Furthermore, since \( x \in A_{\alpha_j} \) and \( w_\tau = w_\alpha \in N_{\alpha_j} \) we have \([w_\alpha, w_\tau] = g_\Lambda \) 1 for all local words of \( w_\sigma \). This gives us \( v_t \) is adjacent to \( v_m \) for all \( \alpha_s \leq m \leq \ell - 1 \). Since \( y_g \) is in normal form there must exist a local word \( w_t \) such that \( w_{\alpha_s} \ldots w_{\alpha_{s-1}} = w_i \ldots w_t \ldots w_{\alpha_{s-1}} \) where \( v_t \) and \( v_\ell \) are non-adjacent in \( \Lambda \). The local word \( w_t \) is also a local word in \( w_\rho w_\sigma \) and again the obstruction condition holds for \( w_\rho w_\sigma \). Since conditions (L) and (O) in Definition 3.2 hold for \( y_g x = w_\rho w_\sigma \), we have the normal form for \( y_g x \) in \( \mathcal{N} \) is \( y_g x = w_\rho y_{w_\tau} w_\sigma = w_\rho w_\sigma \).

Suppose \( w_\tau \in N_{\alpha_j} \) and \( y_{w_\tau} \) in \( N_{\alpha_j} \) is not the empty word. Since \( w_{\alpha_{s-1}} \) and \( w_\tau = w_\alpha \) are consecutive local words the associated vertex groups are distinct and therefore the associated vertex groups for \( y_{w_\tau} \) and \( w_{\alpha_{s-1}} \) are also distinct. Similarly, we have the associated vertex groups for \( y_{w_\tau} \) and \( w_{\alpha_{s+1}} \) are distinct. Since consecutive local words represent group elements in distinct vertex groups and are in the prescribed normal forms, condition (L) in Definition 3.2 holds. Lastly, since \( y_g \) satisfies condition (O) in Definition 3.2 and \( y_{w_\tau} = \alpha_\ell \in N_{\alpha_j} \) we have \( w_\rho y_{w_\tau} w_\sigma \) also satisfies condition (O) and therefore the normal form for \( y_g x \) in \( \mathcal{N} \) is \( y_g x = w_\rho y_{w_\tau} w_\sigma \).
Now, suppose \( w_\tau \notin \mathcal{N}_{\alpha_j} \). Let \( y_x \) be the normal form for \( x \) in \( \mathcal{N}_{\alpha_j} \). Since \( x \in A_{\alpha_j} \), we have \( w_\tau \) and \( y_x \) represent group elements in different vertex groups. By construction of \( w_\tau \) in Part (1) we have \( w_\tau = w_{\alpha_s} \) where \( G_{\alpha_s} < G_{\alpha_j} \). Thus, by Definition 3.2 we have the normal form for \( w_\tau x \) in \( \mathcal{N} \) is \( y_{w_\tau x} = w_\tau y_x \). Furthermore, this gives us
\[
y_g x = w_\rho w_\tau w_\sigma x = G\Lambda w_\rho w_\tau x w_\sigma = G\Lambda w_\rho y_{w_\tau x} w_\sigma = G\Lambda w_\rho w_\tau y_x w_\sigma.
\]
By construction of \( w_\sigma = w_{\alpha_{s+1}} \ldots w_{\alpha_k} \), we have \( w_\sigma \) does not have any local words in \( \mathcal{N}_{\alpha_j} \) as a subword since we shuffled the letter \( x \) past a local word whenever the corresponding vertex was adjacent to \( v_{\alpha_j} \) and \( \Lambda \) is a simple graph. In particular, the local word \( w_{\alpha_{s+1}} \) is not in \( \mathcal{N}_{\alpha_j} \) and in \( w_\rho w_\tau y_x w_\sigma = w_{\alpha_1} \ldots w_{\alpha_{s-1}} w_{\alpha_s} y_x w_{\alpha_{s+1}} \ldots w_{\alpha_k} \) each local word \( w_{\alpha_i} \) is in \( \mathcal{N}_{\alpha_i} \) with consecutive local words representing group elements in distinct vertex groups. To show \( w_\rho w_\tau y_x w_\sigma \) satisfies condition (O) suppose we have \( w_\rho w_\tau y_x w_\sigma = \ldots w_i \ldots w_\ell \ldots \) with \( i \geq \ell \) and \( v_{\alpha_i} \) and \( v_{\alpha_\ell} \) adjacent in \( \Lambda \). That is, based on the ordering \( <_G \) on \( \{G_i\}_{i=1}^m \), the local word \( w_\ell \) should come before the local word \( w_i \). If \( w_i \) and \( w_\ell \) are local words of \( y_g \), since \( y_g \) is in normal form there exists a local word \( w_j \) of \( y_g \) such that \( v_j \) and \( v_\ell \) are non-adjacent in \( \Lambda \). We then have \( w_j \) is also local word of \( w_\rho w_\tau x w_\sigma \) and so condition (O) still holds. If \( w_\ell \) is \( y_x \) then \( w_\rho w_\tau x w_\sigma = \ldots w_i \ldots w_{\alpha_{s-1}} w_{\alpha_s} y_x \ldots \) and \( v_{\alpha_{s-1}} \) and \( v_{\alpha_s} \) are non-adjacent in \( \Lambda \). Since conditions (L) and (O) hold for \( w_\rho w_\tau y_x w_\sigma \), we have the normal form for \( y_g x \) in \( \mathcal{N} \) is \( y_{gx} = w_\rho w_\tau y_x w_\sigma \). We note that in the case where the normal form for \( x \) in \( \mathcal{N}_{\alpha_j} \) is \( x \), we simply have \( y_{gx} = w_\rho w_\tau x w_\sigma \).

What we have done in Part (2) is show that the normal form for \( y_g x \) in \( \mathcal{N} \) is \( y_{gx} = w_\rho y_{w_\tau x} w_\sigma \), where \( y_{w_\tau x} \) is the normal form for \( w_\tau x \).

**Part 3: Construction of a van Kampen diagram** \( \Delta \) for \( y_g x y_{gx}^{-1} \)

Recall in Part (1) we factored \( y_g \) as \( y_g = w_\rho w_\tau w_\sigma \). Having written \( y_{gx} = w_\rho y_{w_\tau x} w_\sigma \) in Part(2), we now construct a van Kampen diagram \( \Delta \) for \( y_g x y_{gx}^{-1} \).
Express $w_\sigma$ as $w_\sigma = a_{i_1} \ldots a_{i_s}$, where each $a_{i_t} \in A$. By construction we have $[w_\sigma, x] = 1$, and in particular $[a_{i_t}, x] = 1$ for all $i_1 \leq i_t \leq i_s$. Let $\Sigma_{i_t}$ be a van Kampen diagram with boundary word $a_{i_t} x a_{i_t}^{-1} x^{-1}$, read counterclockwise starting at the basepoint $v_{i_t}$, for each $i_1 \leq i_t \leq i_s$. Additionally, starting at the basepoint $v_{i_t}$ and reading counterclockwise around the boundary of $\Sigma_{i_t}$, label the vertices on $\partial \Sigma_{i_t}$ as $u_{i_t}, u'_{i_t}$ and $v'_{i_t}$, respectively. Let $\Theta_{i_t}$ be the van Kampen diagram obtained by gluing the van Kampen diagrams $\Sigma_{i_1}$ and $\Sigma_{i_2}$ together along the edge $e(u_{i_t}, x)$ in $\Sigma_{i_1}$ and $e(v_{i_2}, x)$ in $\Sigma_{i_2}$. Inductively define the van Kampen diagram $\Theta_{i_t}$ by gluing the van Kampen diagrams $\Theta_{i_t-1}$ and $\Sigma_{i_t}$ together along the edge $e(u_{i_t-1}, x)$ in $\Theta_{i_t-1}$ and the edge $e(v_{i_t}, x)$ in $\Sigma_{i_t}$, for $i_3 \leq i_t \leq i_s$. Let $\Theta_\sigma = \Theta_{i_s}$. We note that $\Theta_\sigma$ has boundary word $w_\sigma x w_\sigma^{-1} x^{-1}$.

Recall we have $y_g = w_\rho w_\tau w_\sigma$, $x \in A_{\alpha_j}$, and $y_{gx} = w_\rho y_{w_r x} w_\sigma$, where $y_{w_r x}$ is the normal form for $w_r x$. When $w_r \in \mathcal{N}_{\alpha_j}$ and $x \in A_{\alpha_j}$ there exists a pair $(\Delta_r, \Omega_r)$ in $\mathcal{E}_{\alpha_j}$ such that $\Delta_r$ is a van Kampen diagram for the word $w_r x y_{w_r x}^{-1}$ over the presentation $\mathcal{P}_{\alpha_j}$ of $G_{\alpha_j}$, and the edge 1-combing $\Omega_r$ is $h_{\alpha_j}$-tame. Here it is possible to have $y_{w_r x}$ being the empty word, in which case the boundary of $\Delta_r$ is $w_r x^{-1}$. When $w_r \not\in \mathcal{N}_{\alpha_j}$, since $\epsilon \in \mathcal{N}_{\alpha_j}$ and $x \in A_{\alpha_j}$ there exists a pair $(\Delta_x, \Omega_x)$ in $\mathcal{E}_{\alpha_j}$ such that $\Delta_x$ is a van Kampen diagram for the word $x y_{x}^{-1}$ over the presentation $\mathcal{P}_{\alpha_j}$ of $G_{\alpha_j}$, and the edge 1-combing $\Omega_x$ is $h_{\alpha_j}$-tame. We note when the normal form $y_x$ is $x$, the van Kampen diagram $\Delta_x$ is a single edge labeled by $x$ with no 2-cells.

We now construct the van Kampen diagram $\Delta$ for $y_g x y_{gx}^{-1}$. When $w_r \in \mathcal{N}_{\alpha_j}$ let $\mu$ be a simple edge path labeled by $w_\rho$ with boundary vertices $*$ and $u_1$. Adjoin the basepoint of the van Kampen diagram $\Delta_r$ to the vertex $u_1$ and call the resulting diagram $\tilde{\Delta}$. When $w_r \not\in \mathcal{N}_{\alpha_j}$ let $\mu$ be a simple edge path labeled by $w_\rho w_\tau$ with boundary vertices $*$ and $u_2$ and adjoin the basepoint of the van Kampen diagram $\Delta_x$ to the vertex $u_2$; call the resulting diagram $\tilde{\Delta}$. 

Next, in both cases when \( w_\tau \in N_{\alpha_j} \) and when \( w_\tau \notin N_{\alpha_j} \), attach the van Kampen diagram \( \Theta_\sigma \) onto \( \tilde{\Delta} \), by gluing the edge \( e(v_{i_1}, x) \) on the boundary of \( \Theta_\sigma \) to the edge labeled by \( x \) on the boundary of \( \tilde{\Delta} \); call the resulting diagram \( \Delta \). To simplify notation, in \( \Delta \) label the vertices \( v_{i_1}, u_{i_s}, u'_{i_s}, v'_{i_1} \) on the boundary of \( \Delta \) as \( u_2, u_3, u_4, \) and \( u_5 \), respectively. When \( w_\tau \in N_{\alpha_j} \), the van Kampen diagram \( \Delta \) is illustrated in Figures 3.1 (a) and 3.1 (b). When \( w_\tau \notin N_{\alpha_j} \), the van Kampen diagram \( \Delta \) is illustrated in Figures 3.2 (a) and 3.2 (b).

**Part 4: Construction of an edge 1-combing \( \Psi \) for \( \Delta \)**

Let \( (\Delta_\tau, \Omega_\tau), (\Delta_x, \Omega_x), \Theta_\sigma, \) and \( \Delta \) be as defined in Part (3). With a slight abuse of notation, we can think of \( \Delta_\tau, \Delta_x, \) and \( \Theta_\sigma \) as subcomplexes of \( \Delta \) with the basepoints \( u_1, u_2, \) and \( u_2 \) respectively. Let \( e_1 = (u_2, x) \) and \( e_2 = e(u_3, x) \) be the edges in \( \Delta \) with initial vertex \( u_2 \) and \( u_3 \) labeled by \( x \), respectively. Let \( J : e_2 \rightarrow [0, 1] \) and \( K : \Theta_\sigma \rightarrow [0, 1] \times [0, 1] \) be homeomorphisms with \( J(u_3) = 0 \) and \( J(u_4) = 1 \) and \( K(u_2) = (0, 0), K(u_3) = (1, 0), K(u_4) = (1, 1) \) and \( K(u_5) = (0, 1) \). Observe, for \( p \in e_2 \) we have \( K(p) = (1, J(p)) \).
For $p \in \Theta_\sigma$, let $(x_p, z_p) = K^{-1}(p)$. Let $r : \Theta_\sigma \to e_1$ be defined by $r(p) = K^{-1}((0, z_p))$. Then $r$ is a retraction of $\Theta_\sigma$ onto the edge $e_1$. Note that the edge $e_1$ is on the boundary of $\Delta_\tau$ and $\Delta_x$ (as a subcomplex of $\Delta$). For $p \in e_2$, define the map $\eta_p : [0, 1] \to \Theta_\sigma$ by $\eta_p(s) = K^{-1}((s, z_p))$. Then $\eta_p$ is a path in $\Theta_\sigma$ from $r(p)$ to $p$. Lastly, let $\phi_{w_\rho}, \phi_{w_\tau} : [0, 1] \to \Delta$ be the continuous maps that follow the paths labeled by $w_\rho$ and $w_\rho w_\tau$ in $\Delta$ from the basepoint $*$ to $u_1$ and $u_2$, respectively, at a constant speed.

Construction of $\Psi$ when $w_\tau \in \mathcal{N}_{\alpha_j}$ or $w_\tau \not\in \mathcal{N}_{\alpha_j}$ and $y_x \neq x$: Let $\Delta$ be the van Kampen diagram shown in Figure 3.1 (b). We will construct an explicit edge 1-combing $\Psi : e_2 \times [0, 1] \to \Delta$ of $\Delta$. Note that the construction of $\Psi$ for $\Delta$ in Figure 3.1 (a) and Figure 3.2 (b) are the special cases in which $y_{w_\tau x} = \epsilon$ and $w_\tau = \epsilon$, respectively. Before our construction, we define a few auxiliary maps.

Notation 3.6. Let $i, j, k$ and $\ell$ be the lengths of $w_\rho$, $w_\tau$, $w_\sigma$, and $y_{w_\tau x}$, respectively. To aid with computation we introduce the following notation:

- $m_g = \frac{i}{i + j + k}$
- $n_g = \frac{j}{i + j + k}$
- $p_g = \frac{k}{i + j + k}$
- $m_{gx} = \frac{i}{i + \ell + k}$
- $n_{gx} = \frac{\ell}{i + \ell + k}$
- $p_{gx} = \frac{k}{i + \ell + k}$
For \( p \in e_2 \), let
\[
\begin{align*}
ap & = m_g + J(p)(m_{gx} - m_g) \quad \text{and} \\
b_p & = (m_g + n_g) + J(p)(m_{gx} + n_{gx} - m_g - n_g).
\end{align*}
\]

Let \( Y_1 = \{(p,s) | 0 \leq s \leq a_p \} \), \( Y_2 = \{(p,s) | a_p \leq s \leq b_p \} \), and \( Y_3 = \{(p,s) | b_p \leq s \leq 1 \} \); all closed subsets of \( e_2 \times [0,1] \) with \( e_2 \times [0,1] = Y_1 \cup Y_2 \cup Y_3 \).

Define the following continuous maps:
\[
\begin{align*}
q_1 : e_2 \times [0,1] & \to Y_1 \text{ by } q_1((p,s)) = (p,a_p s) \\
q_2 : e_2 \times [0,1] & \to Y_2 \text{ by } q_2((p,s)) = (p,a_p + s(b_p - a_p)) \\
q_3 : e_2 \times [0,1] & \to Y_3 \text{ by } q_3((p,s)) = (p,b_p + s(1 - b_p)).
\end{align*}
\]

Let \( i \in \{1, 2, 3\} \). Define the equivalence relation \( \sim_i \) on \( e_2 \times [0,1] \) where \((p,s) \sim_i (p',t)\) if and only if \( q_i((p,s)) = q_i((p',t)) \). Then the map \( \pi_i : e_2 \times [0,1] \to (e_2 \times [0,1])/\sim_i \) is a quotient map for \( i = 1, 2, 3 \). Since \( q_i : e_2 \times [0,1] \to Y_i \) is a continuous map such that \((p,s) \sim_i (p',t)\) implies \( q_i((p,s)) = q_i((p',t)) \) for all \((p,s)\) and \((p',t)\) in \( e_2 \times [0,1] \), by the universal property of quotient maps there exists a unique continuous map \( f_i : (e_2 \times [0,1])/\sim_i \to Y_i \) such that \( q_i = f_i \pi_i \). Note that \( e_2 \times [0,1] \) is a compact space, \( Y_i \) is a Hausdorff space, and so \( k_i \) is in fact a homeomorphism with \( f_i \left( (p,s) \right) = q_i((p,s)) \) and
\[
f_1^{-1}(p,s) = \begin{cases} 
\left( p, \frac{s}{a_p} \right)^{-1}, & a_p \neq 0 \\
\left( p, 0 \right)^{-1}, & a_p = 0
\end{cases}
\]
\[ f_2^{-1}(p, s) = \begin{cases} \left( \frac{(s - a_p)^2}{p} \right), & a_p \neq b_p \\ \left( \frac{1}{p, a_p} - a_p \right)^2, & a_p = b_p \end{cases} \]

\[ f_3^{-1}(p, s) = \begin{cases} \left( \frac{(s - b_p)^3}{p, 1} \right), & b_p \neq 1 \\ \left( \frac{1}{p, b_p} \right), & b_p = 1 \end{cases} \]

where \((p, s)^i\) denotes the equivalence class of \((p, s)\) under \(\sim_i\).

Define the following maps:

\[ g_1 : e_2 \times [0, 1] \to \Delta \text{ by } g_1((p, s)) = \phi_{w_p}(s) \]
\[ g_2 : e_2 \times [0, 1] \to \Delta \text{ by } g_2((p, s)) = \Omega_r(r(p), s) \]
\[ g_3 : e_2 \times [0, 1] \to \Delta \text{ by } g_3((p, s)) = \eta_{r(p)}(s) \]

Observe \(g_1, g_2\) and \(g_3\) are continuous since \(\phi_{w_p}, \Omega_r, \) and \(\eta_p\) are continuous maps. Next we show given \((p, s) \sim_i (p', t)\) we have \(g_i((p, s)) = g_i((p', t))\), for \(i = 1, 2, 3\). We explicitly show this in the case \(i = 2\), and note that the cases in which \(i = 1, 3\) are similar.

Suppose \((p, s) \sim_2 (p', t)\). Then \(q_2((p, s)) = q_2((p', t))\), which implies

\[(p, a_p + s(b_p - a_p)) = (p', a_{p'} + t(b_{p'} - a_{p'})).\]

Therefore \(p = p'\) and \(a_p + s(b_p - a_p) = a_{p'} + t(b_{p'} - a_{p'}).\) Since the map \(J : e_2 \to [0, 1]\) is injective, having \(p = p'\) implies \(a_p = a_{p'}\) and \(b_p = b_{p'}\). Furthermore,

\[ a_p + s(b_p - a_p) = a_{p'} + t(b_{p'} - a_{p'}) = a_p + t(b_p - a_p) \]

implies \(0 = (s - t)(a_p - b_p)\). Therefore, \((p, s) \sim_2 (p', t)\) when \((p, s) = (p', t)\) or when \(p = p'\) and \(a_p = b_p\).

Applying the definition of \(a_p\) and \(b_p\) above to the equation \(a_p = b_p\) gives us

\[ 0 = n_g + J(p)(n_{gx} - n_g) = n_g (1 - J(p)) + J(p)n_{gx}. \]

Since \(n_g, J(p), n_{gx} \geq 0\) and \(J(p) \leq 1\) we must have \(n_g(1 - J(p)) = 0\) and \(J(p)n_{gx} = 0\). We can then conclude
Suppose $(p, s) \sim_2 (p', t)$ when we either have: (1) $p = p'$ and $s = t$, (2) $p = p'$, $n_g = 0$, and $J(p) = 0$, (3) $p = p'$, $n_g = 0$, and $n_{gx} = 0$, or (4) $p = p'$, $J(p) = 1$ and $n_{gx} = 0$.

Next, suppose $(p, s) = (p', t)$; then we immediately have $g_2((p, s)) = g_2((p', t))$. Next, suppose $p = p'$, $n_g = 0$, and $J(p) = 0$. Observe, $n_g = 0$ implies $w_\tau = \epsilon$. Also $J(p) = 0$ implies that $p = u_3$ and so $r(p) = u_2$. Since $\Omega_\tau(u_2, \cdot)$ is the path in $\Delta$ from $u_1$ that follows along $w_\tau = \epsilon$ we have $\Omega_\tau(u_2, \cdot)$ is the constant path at $u_1$. Since $p = p'$ we also have $r(p') = u_2$ and $\Omega_\tau(r(p'), \cdot)$ is also the constant path at $u_1$, hence $g_2((p, s)) = g_2((p', t))$.

Finally, suppose we have $p = p'$, $J(p) = 1$ and $n_{gx} = 0$. Observe, $J(p) = 1$ implies that $p = u_4$ and therefore $r(p) = u_5$. Additionally, $n_{gx} = 0$ implies $y_{w,x} = \epsilon$ and so the vertex $u_1$ equals the vertex $u_5$. Since $\Omega_\tau(u_5, \cdot)$ is the path in $\Delta$ from $u_1$ to $u_5$ that follows along $y_{w,x} = \epsilon$ we have $\Omega_\tau(u_5, \cdot)$ is the constant path at $u_1$. Furthermore, since $p = p'$ we have $r(p') = u_5$ and $\Omega_\tau(r(p'), \cdot)$ is also the constant path at $u_1$, and hence $g_2((p, s)) = g_2((p', t))$.

Thus we have shown in all four cases that for $(p, s) \sim_2 (p', t)$ we have $g_2((p, s)) = g_2((p', t))$. A similar argument shows that for $(p, s) \sim_i (p', t)$ we have $g_i((p, s)) = g_i((p', t))$, for $i = 1, 3$. Applying the universal property of quotient maps to $g_i : e_2 \times [0, 1] \to \Delta$ there exists a unique continuous map $\tilde{g}_i : (e_2 \times [0, 1]) / \sim_i \to \Delta$ such that $g_i = \tilde{g}_i \circ \pi_i$ for $i = 1, 2, 3$. In particular, for $i \in \{1, 2, 3\}$ we have $\tilde{g}_i \left( \overline{(p, s)} \right) = g_i((p, s))$. This gives us a commutative diagram of continuous maps shown in Figure 3.3.
Figure 3.3: Induced quotient maps for $i = 1, 2, 3$.

**Definition 3.7.** Now we define the edge 1-combing $\Psi : e_2 \times [0, 1] \to \Delta$ by

$$
\Psi(p, s) = \begin{cases} 
(g_1 \circ f_1^{-1})(p, s) & (p, s) \in Y_1 \\
(g_2 \circ f_2^{-1})(p, s) & (p, s) \in Y_2 \\
(g_3 \circ f_3^{-1})(p, s) & (p, s) \in Y_3
\end{cases}
$$

Recall that $e_2 \times [0, 1] = Y_1 \cup Y_2 \cup Y_3$. Additionally, since $f_i^{-1}$ and $g_i$ are continuous, we have $(g_i \circ f_i^{-1})$ is continuous for $i = 1, 2, 3$. Next, we need to show that these functions agree where both are defined. For $(p, s) \in Y_1 \cap Y_2$ we have $s = a_p$.

When $a_p = 0$ we have $(p, 0) \sim_1 (p, 1)$ and so \((p, 0)^1 = (p, 1)^1\). This gives us

$$
(g_1 \left(f_1^{-1}(p, a_p)\right)) = g_1 \left((p, 0)^1\right) = g_1 \left((p, 1)^1\right) = g_1((p, 1)) = \phi_w(1) = u_1
$$

$$
(g_2 \left(f_2^{-1}(p, a_p)\right)) \overset{a_p = b_p}{=} g_2 \left((p, a_p)^2\right) = g_2((p, a_p)) = \Omega \tau (r(p), 0) = u_1
$$

$$
(g_2 \left(f_2^{-1}(p, a_p)\right)) \overset{a_p \not= b_p}{=} g_2 \left((p, 0)^2\right) = g_2((p, a_p)) = \Omega \tau (r(p), 0) = u_1
$$
When \( a_p \neq 0 \) and \( a_p = b_p \) we have \((p, 0) \sim_2 (p, a_p)\) thus \(\overline{(p, 0)^2} = \overline{(p, a_p)^2}\). Using this information we have

\[
\begin{align*}
(\bar{g}_1 \left( f_1^{-1}(p, a_p) \right)) &= \bar{g}_1 \left( \overline{(p, 1)^1} \right) = g_1 ((p, 1)) = \phi_{w_p}(1) = u_1 \\
(\bar{g}_2 \left( f_2^{-1}(p, a_p) \right)) &= \bar{g}_2 \left( \overline{(p, a_p)^2} \right) = g_2 (\overline{(p, a_p)^2}) = g_2 ((p, 0)) = \Omega_r(r(p), 0) = u_1 \\
(\bar{g}_3 \left( f_3^{-1}(p, a_p) \right)) &= \bar{g}_3 \left( \overline{(p, a_p)^3} \right) = g_3 ((p, 0)) = \eta_{r(p)}(0) = r(p)
\end{align*}
\]

For \((p, s) \in Y_2 \cap Y_3\) we have \(s = b_p\). When \(a_p = b_p\) we have \((p, 1) \sim_2 (p, a_p)\) and so \(\overline{(p, 1)^2} = \overline{(p, a_p)^2}\). Furthermore, when \(b_p = 1\) we have \((p, 1) \sim_3 (p, 0)\) giving us \(\overline{(p, 1)^3} = \overline{(p, 0)^3}\). Using this information we have

\[
\begin{align*}
(\bar{g}_2 \left( f_2^{-1}(p, b_p) \right)) &= \bar{g}_2 \left( \overline{(p, b_p)^2} \right) = g_2 (\overline{(p, b_p)^2}) = g_2 ((p, 1)) = \Omega_r(r(p), 1) = r(p) \\
(\bar{g}_3 \left( f_3^{-1}(p, b_p) \right)) &= \bar{g}_3 \left( \overline{(p, b_p)^3} \right) = g_3 (\overline{(p, b_p)^3}) = g_3 ((p, 0)) = \eta_{r(p)}(0) = r(p)
\end{align*}
\]

When \(a_p \neq b_p\) we have

\[
\begin{align*}
(\bar{g}_2 \left( f_2^{-1}(p, b_p) \right)) &= \bar{g}_2 \left( \overline{(p, b_p)^2} \right) = g_2 (\overline{(p, b_p)^2}) = g_2 ((p, 1)) = \Omega_r(r(p), 1) = r(p) \\
(\bar{g}_3 \left( f_3^{-1}(p, b_p) \right)) &= \bar{g}_3 \left( \overline{(p, b_p)^3} \right) = g_3 (\overline{(p, b_p)^3}) = g_3 ((p, 0)) = \eta_{r(p)}(0) = r(p)
\end{align*}
\]

Since \(Y_1, Y_2\) and \(Y_3\) are closed subsets with \(c_2 \times [0, 1] = Y_1 \cup Y_2 \cup Y_3\) and we have continuous functions \((\bar{g}_1 \circ f_1^{-1})\), \((\bar{g}_2 \circ f_2^{-1})\), and \((\bar{g}_3 \circ f_3^{-1})\) that agree on points in \(Y_1 \cap Y_2\) and \(Y_2 \cap Y_3\), by the Pasting Lemma we have \(\Psi\) is continuous. We define \(\Psi\) in terms of \(a_p\) and \(b_p\) to ensure the gluing condition in the definition of a combed
\(\mathcal{N}\)-filling is satisfied.

It is worth noting that using \((\Delta_x, \Omega_x)\) and \(\psi_{w_r}\) in the construction above instead of \((\Delta_r, \Omega_r)\) and \(\phi_{w_r}\) will give you an edge 1-combing for \(\Delta\) in Figure 3.2 (b).

**Construction of \(\Psi\) when \(w_r \not\in \mathcal{N}_{a_j}\) and \(y_w = x\):** Let \(\Delta\) be the van Kampen diagram illustrated in Figure 3.2 (a). For \(r \in [0, 1]\) define \(\xi_r : [0, 1] \to [0, 1] \times [0, 1]\) by \(\xi_r(s) = (0, 0)(1 - s) + (0, r)s\) and define \(\omega_r : [0, 1] \to [0, 1] \times [0, 1]\) by \(\omega_r(s) = (0, r)(1 - s) + (1, r)s\). Next, for any \(r \in [0, 1]\) define \(\Xi_r : [0, 1] \to [0, 1] \times [0, 1]\) by

\[
\Xi_r(s) = \begin{cases} 
\xi_r(2s) & s \in [0, \frac{1}{2}] \\
\omega_r(2s - 1) & s \in [\frac{1}{2}, 1]
\end{cases}
\]

For \(p \in e_2\), recall \(K(p) = (1, z_p) \in [0, 1] \times [0, 1]\) for some \(z_p \in [0, 1]\). Abusing notation slightly, let \(\Xi_p\) denote the function \(\Xi_{y_p}\).

Let \(i, j, k, \ell, m_g, n_g,\) and \(p_g\) be as defined in 3.6 on page 22. Since \(y_{w_r} = w_rx\), we have \(\ell = \ell(y_{w_r}) = j + 1\). Let

\[
\begin{align*}
\bullet \ m_{gx} &= \frac{i}{i + j + k + 1} & \bullet \ n_{gx} &= \frac{j}{i + j + k + 1} & \bullet \ p_{gx} &= \frac{k}{i + j + k + 1}
\end{align*}
\]

For \(p \in e_2\), let

\[a_p = m_g + n_g + J(p) (m_{gx} + n_{gx} - m_g - n_g)\,.
\]

Similar to the construction of \(\Psi\) in Definition 3.7, let \(Y_1 = \{(p, s)|0 \leq s \leq a_p\}\) and \(Y_2 = \{(p, s)|a_p \leq s \leq 1\}\) and define the following continuous maps:

\[
q_1 : e_2 \times [0, 1] \to Y_1 \text{ by } q_1((p, s)) = (p, a_p s) \quad \text{ and } \quad q_2 : e_2 \times [0, 1] \to Y_2 \text{ by } q_2((p, s)) = (p, a_p + s(1 - a_p))\.
\]
Then for $i \in \{1, 2\}$, $\pi_i : e_2 \times [0,1] \to (e_2 \times [0,1]) / \sim_i$ is a quotient map where $(p, s) \sim_i (p', t)$ if and only if $q_i((p, s)) = q_i((p', t))$. By the universal property of quotient maps there exists a unique continuous map $f_i : (e_2 \times [0,1]) / \sim_i \to Y_i$ such that $q_i = f_i \pi_i$ for $i = 1, 2$. Again, since $e_2 \times [0,1]$ is a compact space and $Y_i$ is a Hausdorff space, $f_i$ is a homeomorphism with $f_i((p, s)) = q_i((p, s))$ and

$$f_1^{-1}(p, s) = \begin{cases} \left( \frac{s}{a_p}, \frac{1}{a_p} \right), & a_p \neq 0 \\ (p, 0), & a_p = 0 \end{cases}$$

$$f_2^{-1}(p, s) = \begin{cases} \left( \frac{s-a_p}{1-a_p} \right)^2, & a_p \neq 1 \\ \left( \frac{1}{p}, 0 \right)^2, & a_p = 1 \end{cases}$$

Now, define the following maps:

$$g_1 : e_2 \times [0,1] \to \Delta \text{ by } g_1((p, s)) = \phi_{w_r}(s)$$

$$g_2 : e_2 \times [0,1] \to \Delta \text{ by } g_2((p, s)) = (K^{-1} \circ \Xi_p)(s).$$

Then $g_i$ is a continuous map such that $(p, s) \sim_i (p', t)$ implies $g_i((p, s)) = g_i((p', t))$ for $i = 1, 2$. By the universal property of quotient maps there exists a unique continuous map $\bar{g}_i : (e_2 \times [0,1]) / \sim_i \to \Delta$ with $\bar{g}_i\left(\frac{(p, s)}{a_p}\right) = g_i((p, s)).$

**Definition 3.8.** Define the edge 1-combing $\Psi : e_2 \times [0,1] \to \Delta$ by

$$\Psi(p, s) = \begin{cases} (\bar{g}_1 \circ f_1^{-1})(p, s) & (p, s) \in Y_1 \\ (\bar{g}_2 \circ f_2^{-1})(p, s) & (p, s) \in Y_2 \end{cases}$$
To show that $\Psi$ is well-defined, for $(p, s) \in Y_1 \cap Y_2$ we have $s = a_p$. When $a_p = 0$ we have $(p, 0) \sim_1 (p, 1)$ and so $(p, 0)^1 = (p, 1)^1$. Similarly, when $a_p = 1$ we have $(p, 0) \sim_2 (p, 1)$ and we have $(p, 0)^2 = (p, 1)^2$. Using this information we have,

\[
\begin{aligned}
(g_1(f_1^{-1}(p, a_p))) &= g_1((p, 0)^1) = g_1((p, 1)) = \phi_{w_2}(1) = u_2.

(g_2(f_2^{-1}(p, a_p)))_{a_p \neq 1} &= g_2((p, 0)^2) = g_2((p, 0)) = K^{-1}(\Xi_p(0)) = K^{-1}(\xi_p(0))
= K^{-1}((0, 0))
= u_2.

\text{When } a_p \neq 0 \text{ we have}
\end{aligned}
\]

\[
\begin{aligned}
(g_1(f_1^{-1}(p, a_p))) &= g_1((p, 1)^1) = g_1((p, 1)) = \phi_{w_2}(1) = u_2

(g_2(f_2^{-1}(p, a_p)))_{a_p \neq 1} &= g_2((p, 0)^2) = g_2((p, 0)) = u_2

(g_2(f_2^{-1}(p, a_p)))_{a_p = 1} &= g_2((p, 1)^2) = g_2((p, 0)^2) = g_2((p, 0)) = u_2
\end{aligned}
\]

Since $Y_1$ and $Y_2$ are closed subsets with $e_2 \times [0, 1] = Y_1 \cup Y_2$ and we have continuous functions $(g_1 \circ f_1^{-1})$ and $(g_2 \circ f_2^{-1})$ that agree on points in $Y_1 \cap Y_2$, by the Pasting Lemma we have $\Psi$ is continuous.

To show that $\Psi$ is a 1-combing of the pair $(\Delta, e_2)$ at the basepoint $*$ let $p$ in $e_2$. Then at time $s = 0$ we have

\[
\Psi(p, 0) = (g_1 \circ f_1^{-1})(p, 0) = g_1((p, 0)^1) = g_1((p, 0)) = \phi_{w_2}(0) = *.
\]
Similarly, at time $s = 1$ we have

$$
\Psi(p, 1) = (\bar{g}_2 \circ f_2^{-1})(p, 1) = g_2 \left( (p, 1)^2 \right) = g_2((p, 1)) = K^{-1}(\Xi_p(1))
= K^{-1}(\omega_p(1))
= K^{-1}(1, y_p)
= p.
$$

Furthermore, we note in the case when $\ast$ is a vertex on the edge $e_2$, then we have $u_1 = u_2 = u_3 = \ast$, $u_4 = u_5$. Therefore, $J(p) = J(u_3) = 0$, $K(\ast) = K(u_2) = (0, 0)$ and for $s \in [0, 1]$ we have,

$$
\Psi(\ast, s) = \bar{g}_2 \left( f_2^{-1} \left( \ast, \frac{s - a_p}{1 - a_p} \right) \right) = g_2 \left( \left( \ast, \frac{s - a_p}{1 - a_p} \right)^2 \right) = (K^{-1} \circ \Xi_\ast) \left( \frac{s - a_p}{1 - a_p} \right)
= K^{-1}((0, 0))
= \ast.
$$

Lastly, for $u_3$ in $e_2$ we have $\Psi(u_3, \cdot) : [0, 1] \to \Delta$ follows the path labeled by $w_\tau$ on the interval $[0, a_p]$ and the path labeled by $w_\sigma$ on the interval $[a_p, 1]$. Similarly, $\Psi(u_5, \cdot)$ follows the path $w_\tau x$ on the interval $[0, a_p]$ and the path labeled by $w_\sigma$ on the interval $[a_p, 1]$. Since $(e_2)^{(0)} = \{u_3, u_5\}$ we have $\Psi(p, t) \subseteq \Delta^{(1)}$ whenever $p \in \{u_3, u_4\}$ and $t \in [0, 1]$. By construction $\Psi$ follows $y_g$ and $y_{gx}$ at a constant speed and so the gluing condition holds. This completes the construction of the edge 1-combing $\Psi$. 
Part 5: $f$-tameness of $(\Delta, \Psi)$

The final step is to show that the function $\Psi$ is $h$-tame, for the function $h : \mathbb{N}[\frac{1}{4}] \rightarrow \mathbb{N}[\frac{1}{4}]$ defined by $h(n) = \sum_{i=1}^{m} \overline{h}_i(n) + n$, where $\overline{h}_i$ is the subnegative closure of $h_i$ for $1 \leq i \leq m$ as defined in Definition 1.1. Note, for $n \in \mathbb{N}[\frac{1}{4}]$ we have $n \leq h(n)$.

Case I: Suppose we have $\Delta$ as illustrated in Figure 3.1 (b) and $\Psi$ as defined on page 26. Let $p$ be a point in $e_2$; we proceed with the following subcases.

* Subcase (a): Suppose $0 \leq s < t \leq a_p$. Then

$$\Psi(p, s) = (\bar{g}_1 \circ f_1^{-1}) \left( p, \frac{s}{a_p} \right) = \phi_{w_{\rho}} \left( \frac{s}{a_p} \right).$$

Since $n \leq h(n)$ for all $n$ and $\phi_{w_{\rho}}$ is the path from the basepoint $*$ to the vertex $u_1$ labeled by $w_{\rho}$ we have,

$$\tilde{d}_\Delta (\star, \Psi(p, s)) \leq \tilde{d}_\Delta (\star, \Psi(p, t)) \leq h \left( \tilde{d}_\Delta (\star, \Psi(p, t)) \right).$$

* Subcase (b): Suppose $a_p \leq s < t \leq b_p$. For $q \in \Delta_\tau$ (as a subcomplex of $\Delta$), every path from the basepoint $*$ to $q$ must go through the vertex $u_1$ and so we have $\tilde{d}_\Delta (\star, q) = \tilde{d}_\Delta (\star, u_1) + \tilde{d}_\Delta (u_1, q)$. Furthermore, since $(\Delta_\tau, \Omega_\tau)$ is in the geodesic combed $N_{\alpha_j}$-filling $E_{\alpha_j}$, we have the edge 1-combing $\Omega_\tau$ is $h_{\alpha_j}$-tame.
Combining these two facts together with the fact that $n \leq h(n)$ gives us,

\[
\tilde{d}_\Delta (\ast, \Psi(p, s)) = \tilde{d}_\Delta (\ast, (\tilde{g}_2 \circ f_2^{-1}) \left ( r(p), \frac{s - a_p}{b_p - a_p} \right )) \\
= \tilde{d}_\Delta (\ast, \Omega_r \left ( r(p), \frac{s - a_p}{b_p - a_p} \right )) \\
= \tilde{d}_\Delta (\ast, u_1) + \tilde{d}_\Delta \left ( \ast', \Omega_\tau \left ( r(p), \frac{t - a_p}{b_p - a_p} \right ) \right ) \\
\leq \tilde{d}_\Delta (\ast, u_1) + h_{\alpha_j} \left ( \tilde{d}_\Delta \left ( \ast', \Omega_\tau \left ( r(p), \frac{t - a_p}{b_p - a_p} \right ) \right ) \right ) \\
\leq h_{\alpha_j} \left ( \tilde{d}_\Delta (\ast, u_1) \right ) + h_{\alpha_j} \left ( \tilde{d}_\Delta \left ( \ast', \Omega_\tau \left ( r(p), \frac{t - a_p}{b_p - a_p} \right ) \right ) \right ) \\
\leq \tilde{h}_{\alpha_j} \left ( \tilde{d}_\Delta (\ast, u_1) \right ) + \tilde{h}_{\alpha_j} \left ( \tilde{d}_\Delta \left ( \ast', \Omega_\tau \left ( r(p), \frac{t - a_p}{b_p - a_p} \right ) \right ) \right ) \\
\leq \tilde{h}_{\alpha_j} \left ( \tilde{d}_\Delta (\ast, u_1) \right ) + \tilde{h}_{\alpha_j} \left ( \tilde{d}_\Delta \left ( \ast', \Omega_\tau \left ( r(p), \frac{t - a_p}{b_p - a_p} \right ) \right ) \right ) \\
= \tilde{h}_{\alpha_j} \left ( \tilde{d}_\Delta \left ( \ast, \Omega_\tau \left ( r(p), \frac{t - a_p}{b_p - a_p} \right ) \right ) \right ) \\
= \tilde{h}_{\alpha_j} \left ( \tilde{d}_\Delta (\ast, \Psi(p, t)) \right ) \\
= \tilde{h} \left ( \tilde{d}_\Delta (\ast, \Psi(p, t)) \right ).
\]

**Subase (c):** Suppose $b_p \leq s < t \leq 1$. Recall that $\Theta_\sigma$, thought of as a subcomplex of $\Delta$, is the van Kampen diagram for the word $w_\sigma x w_\sigma^{-1} x^{-1}$ with basepoint $u_2$, and there are no 2-cells strictly in the interior of $\Theta_\sigma$. Additionally, recall the map $\eta_p$ is the straight line path in $\Theta_\sigma$ from $r(p)$ to $p$, and so for $s < t$ we have $\tilde{d}_\Delta (u_2, \eta_p(s)) \leq \tilde{d}_\Delta (u_2, \eta_p(t))$. 
Therefore, for \( b_p \leq s < t \leq 1 \) we have

\[
\tilde{d}_\Delta (\ast, \Psi(p,s)) = (g_3 \circ f_3^{-1}) \left( p, \frac{s - b_p}{1 - b_p} \right) = \eta_{r(p)} \left( \frac{s - b_p}{1 - b_p} \right) \leq \eta_{r(p)} \left( \frac{t - b_p}{1 - b_p} \right) \\
= \tilde{d}_\Delta (\ast, \Psi(p,t)) \\
\leq h \left( \tilde{d}_\Delta (\ast, \Psi(p,t)) \right).
\]

- **Subcase (d):** Suppose \( a_p \leq s < b_p < t \leq 1 \). Observe for any point \( q \in \Theta_\sigma \) that any path in \( \Delta \) from \(*\) to \( q \) must cross the edge \( e_1 \) in \( \Delta \). Furthermore, from case (c), for \( b_p \leq t \leq 1 \) we know \( \tilde{d}_\Delta (\ast, \Psi(p,b_p)) \leq \tilde{d}_\Delta (\ast, \Psi(p,t)) \). Putting these two facts together with the fact that \( h \) is nondecreasing we have

\[
\tilde{d}_\Delta (\ast, \Psi(p,s)) \leq h \left( \tilde{d}_\Delta (\ast, \Psi(p,b_p)) \right) = h \left( \tilde{d}_\Delta (\ast, \Psi(p,t)) \right).
\]

- **Subcase (e):** Now, suppose \( 0 \leq s \leq a_p \leq t \leq b_p \). Recall from case (a) we know on the interval \([0, a_p]\) the distance \( \tilde{d}_\Delta (\Psi(p, \cdot)) \) is a nondecreasing function. Furthermore, for \( r \in [a_p, 1] \) any path in \( \Delta \) from \(*\) to \( \Psi(p,r) \) must go through the vertex \( u_1 \) and therefore we have \( \tilde{d}_\Delta (\ast, \Psi(p,r)) \geq \tilde{d}_\Delta (\ast, u_1) \). We are then able to conclude, using case (b), that

\[
\tilde{d}_\Delta (\ast, \Psi(p,s)) \leq \tilde{d}_\Delta (\ast, \Psi(p,a_p)) = \tilde{d}_\Delta (\ast, u_1) \leq \tilde{d}_\Delta (\ast, \Psi(p,t)) \leq h \left( \tilde{d}_\Delta (\ast, \Psi(p,t)) \right).
\]

In the above subcases we have shown that the edge 1-combing \( \Psi \) as defined in Definition 3.7 is \( h \)-tame.

**Case II:** Suppose we have \( \Delta \) as illustrated in Figure 3.2 (a) and \( \Psi \) as defined on page 29. Recall, for any \( r \in [0, 1] \) the function \( \Xi_r : [0, 1] \to [0, 1] \times [0, 1] \) is defined
by $\Xi_r(s) = \xi_r(2s)$ for $s \in [0, \frac{1}{2}]$ and $\Xi_r(s) = \omega_r(2s - 1)$ for $s \in [\frac{1}{2}, 1]$. Let $p \in e_2$; we proceed with the following subcases.

**Subcase (a):** Suppose $0 \leq s < t \leq a_p$. Then

$$\Psi(p, s) = \left( g_1 \circ f_1^{-1} \right) \left( p, \frac{s}{a_p} \right) = \phi_{w_r} \left( \frac{s}{a_p} \right).$$

Since $\phi_{w_r}$ is the path from the basepoint $*$ to the vertex $u_2$ labeled by $w_r w_r$ we have

$$\bar{d}_\Delta (\ast, \Psi(p, s)) \leq \bar{d}_\Delta (\ast, \Psi(p, t)) \leq h \left( \bar{d}_\Delta (\ast, \Psi(p, t)) \right).$$

**Subcase (b):** Suppose $a_p \leq s < t \leq 1$. Then $\Psi(p, s) = (g_2 \circ f_2^{-1}) \left( p, \frac{s-a_p}{1-a_p} \right) = (K^{-1} \circ \Xi_p) \left( \frac{s-a_p}{1-a_p} \right)$. Note, for any point $p \in \Theta_\sigma$ we have $\bar{d}_\Delta (\ast, p) = \bar{d}_\Delta (\ast, u_2) + \bar{d}_\Delta (u_2, p)$, since any path in $\Delta$ from $*$ to $p$ must go through the vertex $u_2$. Since the map $(K^{-1} \circ \Xi_p)$ follows the straight line path in $\Theta_\sigma$ from the basepoint $u_3$ to $r(p)$ on $e_1$ and from $r(p)$ to $p$ in $e_2$. Since $\Theta_\sigma$ has no interior 2-cells, for $s < t$ we have

$$\bar{d}_\Delta \left( u_3, K^{-1} \circ \Xi_p \left( \frac{s-a_p}{1-a_p} \right) \right) \leq \bar{d}_\Delta \left( u_3, K^{-1} \circ \Xi_p \left( \frac{t-a_p}{1-a_p} \right) \right),$$
Therefore we can conclude
\[
\tilde{d}_\Delta (\ast, \Psi(p, s)) = \tilde{d}_\Delta \left( \ast, \left( K^{-1} \circ \Xi_p \right) \left( \frac{s-a_p}{1-a_p} \right) \right) \\
= \tilde{d}_\Delta (\ast, u_2) + \tilde{d}_\Delta \left( u_2, \left( K^{-1} \circ \Xi_p \right) \left( \frac{s-a_p}{1-a_p} \right) \right) \\
\leq \tilde{d}_\Delta (\ast, u_2) + \tilde{d}_\Delta \left( u_2, \left( K^{-1} \circ \Xi_p \right) \left( \frac{t-a_p}{1-a_p} \right) \right) \\
\leq \tilde{d}_\Delta \left( \ast, \left( K^{-1} \circ \Xi_p \right) \left( \frac{t-a_p}{1-a_p} \right) \right) \\
= \tilde{d}_\Delta (\ast, \Psi(p, t)) \\
\leq h \left( \tilde{d}_\Delta (\ast, \Psi(p, t)) \right).
\]

**Subcase (c):** For the last case, suppose \(0 \leq s < a_p < t \leq 1\). For \(q \in \Theta_\sigma\) we have any path in \(\Delta\) from the basepoint \(\ast\) to \(q\) must go through vertex \(u_2\) and so \(\tilde{d}_\Delta (\ast, q) \geq \tilde{d}_\Delta (\ast, u_2)\). Therefore we have
\[
\tilde{d}_\Delta (\ast, \Psi(p, s)) \leq \tilde{d}_\Delta (\ast, u_2) \leq \tilde{d}_\Delta (\ast, \Psi(p, t)) \leq h \left( \tilde{d}_\Delta (\ast, \Psi(p, t)) \right).
\]

In the above two cases we have shown that the edge 1-combing \(\Psi\) is \(h\)-tame. That is, we have know shown that the pair \((\Delta, \Psi)\), where \(\Delta\) was constructed in Part (3) and \(\Psi\) was constructed in Part (4), is \(h\)-tame.

In summary, for \(e \in E(X(G\Lambda))\) with initial vertex \(g\) and label \(x \in A\), let \(y_g\) be the normal form for \(g\) in \(\mathcal{N}\). Use Part (1) to factor \(y_g\) as \(y_g = w_p w_\tau w_\sigma\). Let \((\Delta_e, \Psi_e)\) be the van Kampen diagram and edge 1-combing for \(y_g x y_g^{-1}\) obtained from following Parts (2)-(4). Let \(\mathcal{E} = \{(\Delta_e, \Psi_e) | e \in E(X(G\Lambda))\}\). Then \(\mathcal{E}\) is a geodesic combed \(\mathcal{N}\)-filling for \((G\Lambda, \mathcal{P})\) such that each edge 1-combing \(\Psi_e\) is \(h\)-tame. Applying Lemma 2.8, we have that \(h\) is equivalent to an intrinsic tame filling function for graph product \((G\Lambda, \mathcal{P})\). \(\square\)
Chapter 4

Free Products with Amalgamation

Here we study the behavior of tame filling functions under a special case of amalgamated products. The main result in this chapter is Theorem 4.1. Before we state Theorem 4.1, we recall a few facts about free products with amalgamation.

Let $G = \langle A | R \rangle$ and $H = \langle B | S \rangle$ be finitely presented groups, and let $K$ be a group with injective homomorphisms $\alpha : K \hookrightarrow G$ and $\beta : K \hookrightarrow H$. Then the free product with amalgamation, $G *_{K} H$, can be presented by

$$G *_{K} H = \langle A \cup B | R \cup S \cup \{\alpha(k) = \beta(k) \forall k \in K\} \rangle.$$ 

Given a group $G = \langle A | R \rangle$ with subgroup $H$, a set $S$ of words over $A^*$ is a left transversal for $H$ in $G$ if and only if the map $s \mapsto sH$ from $S$ to $G/H$ is a bijection. A left transversal $S$ is geodesic if all the words in $S$ label geodesics in the associated Cayley graph of $G$. From the normal form theorem for amalgamated free products [10], page 187] we have the following normal forms for elements in $G *_{K} H$.

$$\mathcal{N} = \{t_1 t_2 \ldots t_n k | t_i \text{ alternates between } T_1 \setminus \epsilon \text{ or } T_2 \setminus \epsilon, \ k \in K\},$$

where $T_1$ and $T_2$ are left transversal for $G/K$ and $H/K$, respectively.
Theorem 4.1. Let $f_\alpha$ and $f_\beta$ be intrinsic tame filling functions for $G_\alpha = \langle A_\alpha | R_\alpha \rangle$ and $G_\beta = \langle A_\beta | R_\beta \rangle$, respectively. Let $H = \langle h_1, \ldots, h_m \rangle$ be a finitely generated group and let $\hat{\alpha} : H \hookrightarrow G_\alpha$ and $\hat{\beta} : H \hookrightarrow G_\beta$ be injective homomorphisms. Let $G = G_\alpha \ast_H G_\beta$ with presentation

$$P = \langle A_\alpha, A_\beta, h_1, \ldots, h_m | R_\alpha \cup R_\beta \cup \{h_i = \hat{\alpha}(h_i), h_i = \hat{\beta}(h_i) \ \forall \ 1 \leq i \leq m\} \rangle.$$ 

Suppose that there is a prefix-closed set of geodesic normal forms for $G$ of the form

$$N = \{t_1t_2\ldots t_n y_h | t_i \text{ alternates between } T_\alpha \setminus \epsilon \text{ and } T_\beta \setminus \epsilon, y_h \in N_H\},$$

where, for $\gamma \in \{\alpha, \beta\}$, $T_\gamma$ is a set of geodesic left transversals for $G_\gamma/\hat{\gamma}(H)$, containing the empty word, with respect to the following presentation for $G_\gamma$

$$P_\gamma = \langle A_\gamma, h_1, \ldots, h_m | R_\gamma \cup \{h_i = \hat{\gamma}(h_i) \ \forall \ 1 \leq i \leq m\} \rangle$$

and $N_H$ is a set of prefix-closed geodesic normal forms for $H$. Then $(G, P)$ has an intrinsic tame filling function equivalent to

$$f(n) = f_\alpha(n) + f_\beta(n) + n.$$ 

Before we prove Theorem 4.1, we make an observation. If the set of normal forms $N$ described in the statement of Theorem 4.1 are geodesic normal forms for $G = G_\alpha \ast_H G_\beta$ then

$$N_\alpha = \{ty_h | t \in T_\alpha, y_h \in N_H\} \text{ and } N_\beta = \{ty_h | t \in T_\beta, y_h \in N_H\}, \quad (4.1)$$

are geodesic normal forms for $G_\alpha$ and $G_\beta$, respectively.

Proof of Theorem 4.1. Let $N$ be the set of geodesic normal forms for $(G, P)$ described in Theorem 4.1. Since elements of $N$ are geodesic, from line (4.1), we have a set of
geodesic normal forms, $\mathcal{N}_\alpha$ and $\mathcal{N}_\beta$, for $G_\alpha$ and $G_\beta$, respectively. Let $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$ be the presentations for $G_\alpha$ and $G_\beta$ given in Theorem 4.1. Since $f_\alpha$ is an intrinsic tame filling functions for $G_\alpha$ by Lemma 2.8 ($G_\alpha, \mathcal{P}_\alpha$) has a geodesic combed $\mathcal{N}_\alpha$-filling, $\mathcal{E}_\alpha$, such that each edge 1-combing is $\tilde{f}_\alpha$-tame where $\tilde{f}_\alpha$ is equivalent to $f_\alpha$. Similarly, $(G_\beta, \mathcal{P}_\beta)$ has a geodesic combed $\mathcal{N}_\beta$-filling, $\mathcal{E}_\beta$, such that each edge 1-combing is $\tilde{f}_\beta$-tame where $\tilde{f}_\beta$ is equivalent to $f_\beta$. Again, since $f \sim g$ implies $\tilde{f} \sim \tilde{g}$ with a slight abuse of notation, we will simply denote $\tilde{f}_\alpha$ and $\tilde{f}_\beta$ as $f_\alpha$ and $f_\beta$.

We will construct a geodesic combed $\mathcal{N}$-filling for $(G, \mathcal{P})$ such that each edge 1-combing is $f$-tame. Similar to the proof of Theorem 3.5 we divide this proof into four parts: (1) compute the normal form for $y_gx$, (2) construct a van Kampen diagram $\Delta$ for $y_gxy_{y_g}^{-1}$, (3) construct an edge 1-combing $\Psi$, and (4) show that $\Psi$ is $f$-tame, where $f$ is defined in the statement of Theorem 4.1.

**Part 1: Computation of $y_gx$**

Let $A = A_\alpha \cup A_\beta \cup \{h_1, \ldots, h_m\}$ and let $x \in A$. Let $\bar{A}_\alpha = A_\alpha \cup \{h_1 \ldots h_m\}$ and let $\bar{A}_\beta = A_\beta \cup \{h_1 \ldots h_m\}$. For $g$ in $G$, let $y_g = t_1t_2\ldots t_ny_h$ be the normal form for $g$ in $\mathcal{N}$. We consider the word $y_gx = t_1\ldots t_ny_hx$.

Suppose $t_n \in T_\gamma$, $x \in \bar{A}_\gamma$, and $y_hx \notin \bar{A}_\gamma$ for some $\gamma \in \{\alpha, \beta\}$. Then $t_ny_h \in \mathcal{N}_\gamma$ and there exists $t'y_{h'} \in \mathcal{N}_\gamma$ such that $t_ny_hx =_{G_\gamma} t'y_{h'}$. Therefore, the normal form for $gx$ in $\mathcal{N}$ is $y_{gx} = t_1\ldots t_{n-1}t'y_{h'}$. Next, suppose $t_n \in T_\eta$ and $x \in \bar{A}_\mu$ for $(\eta, \mu) \in \{(\alpha, \beta), (\beta, \alpha)\}$. Then $y_h \in \mathcal{N}_\mu$ and so $y_hx \in G_\mu$. Therefore, there exist $t''y_{h''} \in \mathcal{N}_\mu$ such that $y_hx =_{G_\mu} t''y_{h''}$. This implies that the normal form for $gx$ in $\mathcal{N}$ is $y_{gx} =_{G_\beta} t_1\ldots t_n t''y_{h''}$. Lastly, suppose $t_n \in T_\gamma$ and $x \in \{h_1, \ldots, h_m\}$ for some $\gamma \in \{\alpha, \beta\}$. Observe, $y_h \in \mathcal{N}_\alpha \cap \mathcal{N}_\beta$ and therefore $y_h \in T_\gamma$. We then have $t_ny_h \in \mathcal{N}_\gamma$ and $x$ is in $\bar{A}_\alpha$. The normal form for $y_{gx}$ is then obtained by applying the previous two cases. In summary, we have shown that given $y_g \in \mathcal{N}$ and $x \in A$, we have the following possible normal forms for $y_{gx}$: (1) $y_{gx} = t_1\ldots t_{n-1}t'y_{h'}$ or (2) $y_{gx} = t_1\ldots t_n t''y_{h''}$. 
Part 2: Construction of a van Kampen diagram $\Delta$ for $y_g x y_{g^{-1}}$

Next, we construct a van Kampen diagram for $y_g x y_{g^{-1}}$ for each case above. In Case I, we have $t_n y_h \in N_\gamma$ and $x \in A_\gamma$ for some $\gamma \in \{\alpha, \beta\}$ and so there exists a pair $(\Sigma_\gamma, \Omega_\gamma)$ in $E_\gamma$ such that $\Sigma_\gamma$ is a van Kampen diagram for $t_n y_h x (t'_y h')^{-1}$ and $\Psi_\gamma$ is an $f_\gamma$-tame 1-edge combing of $\Sigma_\gamma$. Let $\Phi$ be a simple edge path labeled by $t_1 \ldots t_{n-1}$ with initial vertex $*$ and terminal vertex $u_2$. Adjoin the basepoint of $\Sigma_\gamma$ to the vertex $u_2$ and let $\Delta$ be the resulting van Kampen diagram, illustrated in Figure 4.1 (I).

In Case II, we have $t_n \in T_\eta$, $y_h \in N_\mu$ and $x \in A_\mu$ for $(\eta, \mu) \in \{(\alpha, \beta), (\beta, \alpha)\}$. Therefore, there exists a pair $(\Sigma_\mu, \Omega_\mu)$ in $E_\eta$ for the word $y_h x (t''_{y h''})^{-1}$. Let $\Phi$ be a simple edge path labeled by $t_1 \ldots t_n$ with boundary $*$ and $u_2$. Let $\Delta$ be the diagram obtained by adjoining the basepoint of $\Sigma_\eta$ to vertex $u_2$ as illustrated in Figure 4.1 (II). It is worth noting that Case II also covers the case when $n = 0$. 
Part 3: Construction of an edge 1-combing $\Psi$ for $\Delta$

Next, we construct an edge 1-combing $\Psi$ for the van Kampen diagram $\Delta$. We will construct $\Psi$ for Case I, in which $\Delta$ as illustrated in Figure 4.1 (I), and note that the construction of $\Psi$ for the other case is similar.

Let $i, j$ and $k$ be the lengths of $t_1 \ldots t_{n-1}, t_n y_h$ and $t'_y h'$, respectively. Let $(\Sigma_{\gamma}, \Omega_{\gamma})$ and $\Delta$ be as defined in Case I of Part 2. With a slight abuse of notation, we can think of $\Sigma_{\gamma}$ as a subcomplex of $\Delta$ with the basepoint $u_2$. Let $e$ be the edge on the boundary of $\Delta$ labeled by $x$ in $y_g x y_g^{-1}$. Furthermore, label the initial and terminal vertices of the edge $e$ as $u_3$ and $u_4$, as illustrated in Figure 4.1. Let $J : e \to [0, 1]$ be a homeomorphism with $J(u_3) = 0$ and $J(u_4) = 1$.

To aid with computation we introduce the following notation:

$$m_g = \frac{i}{i + j}, \quad m_{gx} = \frac{i}{i + k}, \quad n_g = \frac{j}{i + j}, \quad n_{gx} = \frac{k}{i + k}.$$

For $p \in e_2$, let $a_p = m_g + J(p)(m_{gx} - m_g)$. Let $\phi : [0, 1] \to \Delta$ be the continuous constant speed map in $\Delta_1$ from vertices $* \to u_2$ such that $\phi(0) = *$ and $\phi(1) = u_2$. Recall the pair $(\Sigma_{\gamma}, \Omega_{\gamma}) \in \mathcal{E}_{\gamma}$ is associated to the word $w' = (t_n y_h) x (t'_y h')^{-1}$ and $\gamma \in \{\alpha, \beta\}$. Let $Y_1 = \{(p, s) : 0 \leq s \leq a_p\}$ and $Y_2 = \{(p, s) : a_p \leq s \leq 1\}$ and define the following continuous maps:

$$q_1 : e \times [0, 1] \to Y_1$$

by $q_1((p, s)) = (p, a_p s)$

$$q_2 : e \times [0, 1] \to Y_2$$

by $q_2((p, s)) = (p, a_p + s(1 - a_p))$.

Let $i \in 1, 2$. Let $\pi_i : e \times [0, 1] \to (e \times [0, 1])/ \sim_i$ be the quotient map, where $(p, s) \sim_i (p', t)$ if and only if $q_i((p, s)) = q_i((p', t))$. By the universal property of quotient maps there exists a unique continuous map $k_i : (e \times [0, 1])/ \sim_i \to Y_i$ such
that \( q_i = k_i \pi_i \) with \( k_i \left( \overline{(p, s)}^i \right) = q_i((p, s)) \), where \( \overline{(p, s)}^i \) denotes the equivalence class of \((p, s)\) under \( \sim_i \). Since \( e \times [0, 1] \) is a compact space and \( Y_i \) is a Hausdorff space we have \( k_i \) is in fact a homeomorphism and we have

\[
\begin{align*}
    k_1^{-1}(p, s) &= \begin{cases} 
    \left( p, \frac{s}{a_p} \right)^{-1}, & a_p \neq 0 \\
    (p, 0)^{-1}, & a_p = 0
    \end{cases} \\
    k_2^{-1}(p, s) &= \begin{cases} 
    \left( p, \frac{s - a_p}{1 - a_p} \right)^2, & a_p \neq 1 \\
    (p, 1)^2, & a_p = 1
    \end{cases}
\end{align*}
\]

Now, define the following maps:

\[
\begin{align*}
    g_1 : e \times [0, 1] &\to \Delta \text{ by } g_1((p, s)) = \phi(s) \\
    g_2 : e \times [0, 1] &\to \Delta \text{ by } g_2((p, s)) = \Omega_{\gamma}((p, s)).
\end{align*}
\]

Observe \( g_1 \) and \( g_2 \) are continuous since \( \phi \) and \( \Omega_{\gamma} \) are continuous maps. Next, we show given \((p, s) \sim_i (p', t)\) we have \( g_i((p, s)) = g_i((p', t))\), for \( i = 1, 2 \). We explicitly show this in the case \( i = 2 \), and note the case \( i = 1 \) is similar.

Suppose \((p, s) \sim_2 (p', t)\). Then \( q_2((p, s)) = q_2((p', t))\), which implies
\[
(p, a_p + s(1 - a_p)) = (p', a_{p'} + t(1 - a_{p'})).
\]
Therefore \( p = p' \) and
\[
a_p + s(1 - a_p) = a_{p'} + t(1 - a_{p'}). \]
Since the map \( J : e \to [0, 1] \) is injective, having \( p = p' \) implies that \( a_p = a_{p'} \). Furthermore,
\[
\begin{align*}
    a_p + s(1 - a_p) &= a_{p'} + t(1 - a_{p'}) \\
    &= a_p + t(1 - a_p),
\end{align*}
\]
implies \( 0 = (s - t)(1 - a_p) \). Therefore, \((p, s) \sim_2 (p', t)\) when \((p, s) = (p', t)\) or when \( p = p' \) and \( a_p = 1 \).
Applying the definition of \( a_p \) above to the equation \( a_p = 1 \) gives us \( 0 = n_g + J(p) (n_{gx} - n_g) = n_g (1 - J(p)) + J(p)n_{gx} \). Since \( n_g, J(p), n_{gx}, (1 - J(p)) \geq 0 \), we must have \( n_g (1 - J(p)) = 0 \) and \( J(p)n_{gx} = 0 \). We can then conclude \( (p, s) \sim (p', t) \) when we either have: (1) \( p = p' \) and \( s = t \), (2) \( p = p', \ n_g = 0 \), and \( J(p) = 0 \), (3) \( p = p', \ n_g = 0 \), and \( n_{gx} = 0 \), or (4) \( p = p', \ J(p) = 1 \) and \( n_{gx} = 0 \).

When \( (p, s) = (p', t) \), then we immediately have \( g_2((p, s)) = g_2((p', t)) \). When \( p = p', \ n_g = 0 \), and \( J(p) = 0 \) we have \( n_g = 0 \) which implies \( t_n y_h = \epsilon \). Also \( J(p) = 0 \) implies that \( p = u_3 \), and so we have that \( \Omega_\gamma(u_3, \cdot) \) is the constant path at \( u_3 \). Since \( p = p' \) we also have \( \Omega_\gamma(p', \cdot) \) is the constant path at \( u_2 \), hence \( g_2((p, s)) = g_2((p', t)) \).

Next, suppose \( p = p', \ n_g = 0 \), and \( n_{gx} = 0 \). Since \( n_g = 0 \) and \( n_{gx} = 0 \), then \( t_n y_h = \epsilon \) and \( t' y_{h'} = \epsilon \), respectively. Thus \( t_n y_h x t' y_{h'}^{-1} = \epsilon \) implies that \( x \) is the identity element in \( G \). Since the empty word \( \epsilon \) is the unique normal form for the identity element of \( G \) in \( \mathcal{N} \), we can not have \( x = \epsilon \), so this case cannot occur.

Finally, suppose we have \( p = p', \ J(p) = 1 \) and \( n_{gx} = 0 \). Since \( J(p) = 1 \) we have \( p = u_4 \). Additionally, since \( n_{gx} = 0 \) then \( t' y_{h'} = \epsilon \) and we have the path \( \Omega_\gamma(u_4, \cdot) \), which starts at \( u_3 \) and follows along \( t' y_{h'} = \epsilon \), is the constant path at \( u_3 \). Furthermore, since \( p = p' \) we have \( \Omega_\gamma(p', \cdot) \) is also the constant path at \( u_2 \), and hence \( g_2((p, s)) = g_2((p', t)) \). This completes the proof that for \( (p, s) \sim (p', t) \) we have \( g_2((p, s)) = g_2((p', t)) \).

Since \( g_i : e \times [0, 1] \rightarrow \Delta \) is a continuous map such that \( (p, s) \sim (p', t) \) implies \( g_i((p, s)) = g_i((p', t)) \) for all \( (p, s) \) and \( (p', t) \) in \( e \times [0, 1] \), by the universal property of quotient maps there exists a unique continuous function \( \tilde{g}_i : (e \times [0, 1])/ \sim_i \rightarrow \Delta \) such that \( g_i = \tilde{g}_i \circ \pi_i \) for \( i = 1, 2 \). In particular, we have \( \tilde{g}_i \left( (p, s)^i \right) = g_i((p, s)) \).
We are now able to define the edge 1-combing $\Psi : e \times [0,1] \to \Delta$ by

$$\Psi(p,s) = \begin{cases} 
(g_1 \circ k_1^{-1})(p,s) & (p,s), \in Y_1 \\
(g_2 \circ k_2^{-1})(p,s) & (p,s), \in Y_2
\end{cases}$$

To show $\Psi$ is well-defined, for $(p,s) \in Y_1 \cap Y_2$ we have $s = a_p$. If $a_p = 0$, then observe we have $q_1((p,0)) = (p,0)$ and $q_1((p,1)) = (p,a_p) = (p,0)$ therefore $(p,0) \sim_1 (p,1)$ and so $(p,0) \equiv_1 = (p,1)$. Furthermore, when $a_p = 0$ we have

$$(\bar{g}_1 (k_1^{-1}(p,a_p))) = \bar{g}_1 (\overline{p,0}) = \bar{g}_1 (\overline{p,1}) = g_1((p,1)) = \phi(1) = u_2$$

$$(\bar{g}_2 (k_2^{-1}(p,a_p))) = \bar{g}_2 \left( \overline{p, \frac{a_p - a_p}{1 - a_p}} \right) = \bar{g}_2 \left( \overline{p,0} \right) = g_2 ((p,0)) = \Omega_\gamma ((p,0)) = u_2.$$ 

If $a_p = 1$, we have $(p,0) \sim_2 (p,1)$ and so $(p,0) \equiv_2 = (p,1)$. We additionally have

$$(\bar{g}_1 (k_1^{-1}(p,a_p))) = \bar{g}_1 (\overline{p,1}) = g_1((p,1)) = \phi(1) = u_2$$

$$(\bar{g}_2 (k_2^{-1}(p,a_p))) = \bar{g}_2 (\overline{p,1}) = \bar{g}_2 (\overline{p,0}) = g_2 ((p,0)) = \Omega_\gamma ((p,0)) = u_2.$$
Lastly, if $a_p \neq 0$ and $a_p \neq 1$, we have

$$
(g_1 (k_1^{-1}(p, a_p))) = \bar{g}_1 \left( \frac{(p, 1)}{1} \right) = g_1 ((p, 1)) = \phi(1) = u_2
$$

$$
(g_2 (k_2^{-1}(p, a_p))) = \bar{g}_2 \left( \frac{(p, 0)}{2} \right) = g_2 ((p, 0)) = \Omega, ((p, 0)) = u_2.
$$

Since $Y_1$ and $Y_2$ are closed subsets with $e_2 \times [0, 1] = Y_1 \cup Y_2$ and we have continuous functions $(\bar{g}_1 \circ k_1^{-1})$ and $(\bar{g}_2 \circ k_2^{-1})$ that agree on points in $Y_1 \cap Y_2$, then by the Pasting Lemma the map $\Psi$ is continuous. We have constructed $\Psi$ to be a constant speed path on the endpoints of the edge $e$, and so the gluing condition $(g1)$ of Definition 2.7 is satisfied.

**Part 4: $f$-tameness of $(\Delta, \Psi)$**

The final step is to analyze the $f$-tameness of $\Psi$ for $f : \mathbb{N}^{[\frac{1}{4}]} \to \mathbb{N}^{[\frac{1}{4}]}$ defined by $f(n) = \bar{f}_\alpha(n) + \bar{f}_\beta(n) + n$. Note we have $n \leq f(n)$ for $n \in \mathbb{N}^{[\frac{1}{4}]}$.

Additionally for $n, m \in \mathbb{N}^{[\frac{1}{4}]}$ we have

$$
f(n) + f(m) = \bar{f}_\alpha(n) + \bar{f}_\beta(n) + n + \bar{f}_\alpha(m) + \bar{f}_\beta(m) + m
$$

$$
= \bar{f}_\alpha(n) + \bar{f}_\alpha(m) + \bar{f}_\beta(n) + \bar{f}_\beta(m) + n + m
$$

$$
\leq \bar{f}_\alpha(n + m) + \bar{f}_\beta(n + m) + n + m
$$

$$
= f(n + m),
$$

So $f$ is subnegative. In the following 3 cases we analyze the tameness of $\Psi$.

- **Case (a).** Suppose $p \in e$ and $0 \leq s < t \leq a_p$. Since the map $(\bar{g}_1 \circ k_1^{-1})(p, s) = \phi(s)$ follows the geodesic edge path labeled by $t_1 \ldots t_{n-1}$ on the interval $[0, a_p]$ we have $\bar{d}_{\Delta} (*, \Psi(p, s)) = \bar{d}_{\Delta} (*, \phi(s)) \leq \bar{d}_{\Delta} (*, \phi(t)) = \bar{d}_{\Delta} (*, \Psi(p, t))$ for $s < t$. 

Therefore, since \( n \leq f(n) \)

\[
\tilde{d}_\Delta (\ast, \Psi(p,s)) \leq \tilde{d}_\Delta (\ast, \Psi(p,t)) \leq f \left( \tilde{d}_\Delta (\ast, \Psi(p,t)) \right). \tag{4.2}
\]

- **Case (b).** Suppose \( p \in e \) and \( 0 \leq s < a_p < t \leq 1 \). Since \( 0 \leq s < a_p \) and \( \Psi(p,a_p) = u_2 \) in \( \Delta \), from case (a) we have

\[
\tilde{d}_\Delta (\ast, \Psi(p,s)) \leq f \left( \tilde{d}_\Delta (\ast, \Psi(p,a_p)) \right) = f \left( \tilde{d}_\Delta (\ast, u_2) \right). \tag{4.3}
\]

Furthermore, for \( q \in \Sigma_\gamma \) every path from the basepoint \( \ast \) to \( q \) must go through the vertex \( u_2 \) giving us \( d_\Delta (\ast, q) = d_\Delta (\ast, u_2) + d_\Delta (u_2, q) \). In particular, since \( a_p < t \leq 1 \) we have \( \Psi(p,t) \in \Sigma_\gamma \). This gives us

\[
\tilde{d}_\Delta (\ast, \Psi(p,t)) = \tilde{d}_\Delta (\ast, u_2) + \tilde{d}_\Delta (u_2, \Psi(p,t)). \tag{4.4}
\]

Thus, when \( p \in e \) we have

\[
\tilde{d}_\Delta (\ast, \Psi(p,s)) \leq f \left( \tilde{d}_\Delta (\ast, u_2) \right) \quad \text{from (4.10)}
\]
\[
\leq f \left( \tilde{d}_\Delta (\ast, u_2) \right) + f \left( \tilde{d}_\Delta (u_2, \Psi(p,t)) \right) \quad \text{from (4.8)}
\]
\[
\leq f \left( \tilde{d}_\Delta (\ast, u_2) + \tilde{d}_\Delta (u_2, \Psi(p,t)) \right) \quad \text{from (4.11)}
\]

- **Case (c).** Suppose \( p \in e \) and \( a_p \leq s < t \leq 1 \). Recall \( (\Sigma_\gamma, \Omega_\gamma) \in \mathcal{E}_\gamma \) and therefore the edge 1-combing \( \Omega_\gamma \) is \( f_\gamma \)-tame. Additionally, on the interval \([a_p, 1]\), we have the path \( \Psi(p, \cdot) \) in contained in \( \Sigma_\gamma \). Since every path in \( \Delta \) from \( \ast \) to any point \( q \in \Sigma_\gamma \) must go through the vertex \( u_2 \), for \( s \in [a_p, 1] \) we have
\[ d_\Delta(*, \Omega_\gamma(q, s)) = d_\Delta(*, u_2) + d_\Delta(u_2, \Omega_\gamma(q, s)). \]

Observe the embedding of \( \Sigma_\gamma \) into \( \Delta \) maps the basepoint of \( \Sigma_\gamma, *' \), to the vertex \( u_2 \) and maps \( n \)-cells to \( n \)-cells preserving edge labels and orientation. Therefore, for \( q \in \Sigma_\gamma \) we have

\[ d_\Delta(u_2, q) = d_{\Sigma_\gamma}(*, q). \]

Thus we have

\[ d_\Delta(*, \Psi(p, s)) = d_\Delta(*, (g_2 \circ k_2^{-1})(p, s)) \]

\[ = d_\Delta(*, g_2(p, \frac{s-a_p}{1-a_p})) \]

\[ = d_\Delta(*, \Omega_\gamma(p, \frac{s-a_p}{1-a_p})) \]

\[ = d_\Delta(*, u_2) + d_\Delta(u_2, \Omega_\gamma(p, \frac{s-a_p}{1-a_p})) \]

\[ \leq d_\Delta(*, u_2) + f_\gamma(d_\Delta(u_2, \Omega_\gamma(p, \frac{t-a_p}{1-a_p}))) \]

\[ \leq f_\gamma(d_\Delta(*, u_2)) + f_\gamma(d_\Delta(u_2, \Omega_\gamma(p, \frac{t-a_p}{1-a_p}))) \]

\[ \leq f_\gamma(d_\Delta(*, u_2)) + f_\gamma(d_\Delta(u_2, \Omega_\gamma(p, \frac{t-a_p}{1-a_p}))) \]

\[ \leq f_\gamma(d_\Delta(*, u_2) + d_\Delta(u_2, \Omega_\gamma(p, \frac{t-a_p}{1-a_p}))) \] \quad \text{subnegativity of } f_\gamma

\[ = f_\gamma(d_\Delta(*, \Omega_\gamma(p, \frac{t-a_p}{1-a_p}))) \]

\[ = f_\gamma(d_\Delta(*, g_2(p, \frac{t-a_p}{1-a_p}))) \]

\[ = f_\gamma(d_\Delta(*, \Psi(p, t))) \]

\[ \leq f(d_\Delta(*, \Psi(p, t))). \]

In the above three cases, we showed that the edge 1-combing \( \Psi \) of \( \Delta \) is \( f \)-tame. Therefore, \((G, \mathcal{P})\) has a geodesic combed \( N \)-filling \( \mathcal{E} \) such that each edge 1-combing is \( f \)-tame, and by Lemma 2.8 the function \( f \) is equivalent to an intrinsic tame filling function for \((G, \mathcal{P})\). \( \square \)
Bibliography


