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Frequency Shifts in Plate Crystal Resonators Induced by Electric, Magnetic, or Mechanical Fields in Surface Films

Nan Liu, Jiashi Yang, Yuantai Hu, Xuedong Chen, and Wei Jiang

***Abstract*—We study frequency shifts in plate crystal resonators with surface films. The films are multiphysical, including the effects of inertia, stiffness, intrinsic stress, piezoelectric coupling, and piezomagnetic coupling. Mindlin's two-dimensional equations for a crystal plate with two elastic surface films are generalized to include the multiphysical effects of the films. They are used to study thickness-shear vibrations of a rotated Y-cut quartz plate with initial fields resulting from the mechanical, electric, and magnetic fields in the surface films. Frequency shifts caused by the initial fields are calculated and examined. Results show that plate crystal resonators with multiphysical surface films may be used for electric/magnetic field sensing.**

I. Introduction

Crystal resonators are key elements of electrical circuits called oscillators providing frequency standards for time-keeping and frequency operations. Oscillators are important components of telecommunication devices, satellites, radars, and other electronic equipment. These devices are often used in harsh environment or on objects in motion. For these applications, high frequency stability of the crystal resonators against environmental effects like a temperature change or acceleration is desired [1]. During the last few decades, the applications of crystal resonators for sensing grew rapidly [2]–[5]. For sensor applications, the resonant frequencies of crystal resonators must be made sensitive to environmental changes for high sensitivity. Thickness-shear (TSh) vibration modes of a rotated Y-cut quartz plate are the most widely used structure and modes of crystal resonators and sensors.

Either as a resonator or a sensor, a crystal plate is often covered with surface films. These films may be metal electrodes or chemically/biologically selective materials to attract certain substances for sensing. They have their own physical properties. For thin films, the frequency effect of the inertia alone is the foundation of quartz crystal microbalances, mass sensors used in chemical/biological sensing. When the films are not thin, their stiffness also needs to be considered. The inertia and stiffness of the films are relatively simple effects that can be modeled by the linear theory of elasticity. They have been studied relatively thoroughly [4], [5], and can be easily determined through calibration once a device is made.

There are other frequency effects of the surface films that are more complicated. These include the film intrinsic stresses [6], [7] resulting from manufacturing processes and the deformation of the films caused by electric or magnetic fields when the films have piezoelectric/piezomagnetic couplings [8], [9]. The intrinsic stresses and multiphysical couplings in the films manifest themselves in a crystal resonator through the stresses and strains they produce, called initial or biasing fields in resonators. When biasing fields are present, the resonator frequencies are slightly different from those without biasing fields. The behavior of crystal resonators with the presence of biasing fields is governed by the theory for small fields superposed on a bias [10] which needs to be derived from a nonlinear theory [11]. Because of the complexity of these theories, the effects of biasing fields in resonators resulting from surface films are rarely studied. Film intrinsic stress and thermal expansion were treated separately in [7] and [12] for resonator applications.

Recently it has been shown through simple, one-dimensional analyses that plate crystal resonators structurally integrated with piezoelectric/piezomagnetic films possessing strong piezoelectric/piezomagnetic couplings may be considered for electric/magnetic field sensing [13], and that it is possible to use surface piezoelectric films to manipulate frequency shifts in crystal resonators caused by other effects, e.g., accelerations [14]. A crystal plate with a layer of a different material is also useful in other frequency designs of resonators, e.g., in temperature [15] or other compensation techniques in which frequency shifts from different origins are used to cancel each other out.

In this paper, we explore further the ideas of electric/magnetic field sensing and frequency manipulation in [13] and [14]. To predict the behavior of real devices of finite sizes, we analyze the more realistic situation of finite, two-dimensional plates for frequency shifts caused by multiphysical effects in surface films. For this purpose, we generalize an earlier version of first-order, two-dimensional equations by Mindlin [16] for coupled extension, TSh, and flexural vibrations of a crystal plate with elastic surface films (see Fig. 1) by including intrinsic stresses and piezo-

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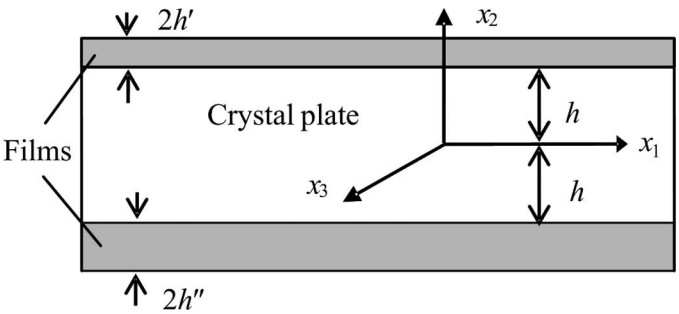

Fig. 1. A crystal plate with asymmetric surface films.

electric/piezomagnetic couplings in the films. The equations are then reduced to the case of coupled extension and elementary flexure without shear deformations, and are used to determine the biasing fields induced by the surface films. With these biasing fields found, frequency shifts caused by electric, magnetic, and intrinsic stresses in the films are calculated and examined.

II. Two-Dimensional Equations for a Crystal Plate With Surface Films

In this section, we generalize the two-dimensional equations for crystal plates with elastic films in [16] to include multiphysical effects in the films. Although the generalization is only in the constitutive relations of the films and is straightforward, it is somewhat tedious, and a concise but complete presentation is given because [16] is not conveniently available.

A. Equations for the Crystal Plate

We begin with a summary of the two-dimensional equations for coupled extensional, flexural, and TSh vibrations of a quartz crystal plate alone without surface films. The very small piezoelectric coupling in quartz is neglected, as is usual in frequency analysis of quartz devices. Cartesian tensor notation is employed, with the summation convention for repeated indices. A comma followed by an index denotes a partial derivative with respect to the coordinate associated with the index. i, j, k, and l take the values 1, 2, and 3; a, b, c, and d take the values 1 and 3 but not 2. Consider the crystal plate of thickness $2h$ in Fig. 1. x_1 and x_3 are in the middle plane. x_2 is along the plate normal. The components of displacement, u_i, are approximated by the first two terms of a power series in x_2:

$$u_i = u_i^{(0)}(x_1, x_3, t) + x_2 u_i^{(1)}(x_1, x_3, t), \tag{1}$$

and the components of stress, T_{ij}, by the stress-resultants, $T_{ij}^{(0)}$, and stress couples, $T_{ij}^{(1)}$:

$$T_{ij}^{(0)} \equiv \int_{-h}^{h} T_{ij} \mathrm{d}x_2, \quad T_{ij}^{(1)} \equiv \int_{-h}^{h} x_2 T_{ij} \mathrm{d}x_2. \tag{2}$$

The stress-equations of motion, $T_{ij,i} = \rho \ddot{u}_j$, are approximated by the following equations:

$$T_{ij,i}^{(0)} + F_j^{(0)} = 2h\rho \ddot{u}_j^{(0)}, \quad i, j = 1, 2, 3, \tag{3}$$

$$T_{ab,a}^{(1)} - T_{2b}^{(0)} + F_b^{(1)} = \frac{2}{3} h^3 \ddot{\rho} u_b^{(1)}, \quad a, b = 1, 3, \tag{4}$$

where a dot denotes time derivative, $F_j^{(0)}$ and $F_b^{(1)}$ are surface loads defined by

$$F_j^{(0)} \equiv T_{2j}|_h - T_{2j}|_{-h}, \tag{5a}$$

$$F_b^{(1)} \equiv hT_{2b}|_h + hT_{2b}|_{-h}, \tag{5b}$$

and ρ is the mass density. The stress-strain relations, $T_{ij} = c_{ijkl} S_{kl}$, are approximated by

$$\begin{aligned} T_{ij}^{(0)} &= 2h g_{ijkl}^{*} S_{kl}^{(0)}, \\ T_{ab}^{(1)} &= \frac{2}{3} h^3 \gamma_{abcd} S_{cd}^{(1)}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} g_{ijkl}^{*} &\equiv \kappa_{i+j-2}^{m} \kappa_{k+l-2}^{n} g_{ijkl}, \quad \text{(not summed)} \\ g_{ijkl} &\equiv c_{ijkl} - c_{ij22} c_{22kl} / c_{2222}, \\ \gamma_{abcd} &\equiv c_{abcd} - c_{ab2j} c_{2kcd} (c_{2j2k})^{-1}. \end{aligned} \tag{7}$$

g_{ijkl} are obtained from the stress relaxation $T_{22}^{(0)} = 0$. They are then modified into g_{ijkl}^{*} by the shear correction factors κ_1 and κ_3 [16], [17]. γ_{abcd} is obtained from the stress relaxation $T_{2j}^{(1)} = 0$. In (7), m and n are given by

$$m = \cos^2(ij\pi/2), \quad n = \cos^2(kl\pi/2). \tag{8}$$

The values of κ_a will be provided later. The strains in (6) are given in terms of the displacements by

$$S_{ij}^{(0)} = \frac{1}{2}(u_{i,j}^{(0)} + u_{j,i}^{(0)} + \delta_{2j} u_i^{(1)} + \delta_{2i} u_j^{(1)}), \tag{9a}$$

$$S_{ab}^{(1)} = \frac{1}{2}(u_{a,b}^{(1)} + u_{b,a}^{(1)}), \tag{9b}$$

in which δ_{ij} is the Kronecker symbol.

B. Equations for the Films

Fields associated with the upper and lower films will be designated by primes and double primes, respectively. The film thickness, $2h'$ and $2h''$, are usually much thinner than the crystal plate. Therefore, the films are modeled by the equations of extension. The bending and twisting moments, $T_{ab}'^{(1)}$ and $T_{ab}''^{(1)}$, and the TSh stress-resultants, $T_{a2}'^{(0)}$ and $T_{a2}''^{(0)}$, may be neglected in the films. Thus, the equations for the upper film are

$$\delta_{jb}^{0} T_{ab,a}'^{(0)} + F_j'^{(0)} = 2h'\rho' \ddot{u}_j'^{(0)}, \tag{10}$$

$$F_j'^{(0)} \equiv T_{2j}'|_{h'} - T_{2j}'|_{-h'}, \tag{11}$$

$$T_{ab}'^{(0)} = 2h'(\tau_{ab}' + \gamma_{abcd}' S_{cd}'^{(0)} - e_{kab}' E_k' - h_{kab}' H_k'), \tag{12}$$

$$S'^{(0)}_{ab} = \frac{1}{2}(u'^{(0)}_{a,b} + u'^{(0)}_{b,a}), \tag{13}$$

and for the lower film,

$$\delta^0_{jb} T''^{(0)}_{ab,a} + F''^{(0)}_j = 2h''\rho''\ddot{u}''^{(0)}_j, \tag{14}$$

$$F''^{(0)}_j \equiv T''_{2j}|_{h''} - T''_{2j}|_{-h''}, \tag{15}$$

$$T''^{(0)}_{ab} = 2h''(\tau''_{ab} + \gamma''_{abcd} S''^{(0)}_{cd} - e''_{kab} E''_k - h''_{kab} H''_k), \tag{16}$$

$$S''^{(0)}_{ab} = \frac{1}{2}(u''^{(0)}_{a,b} + u''^{(0)}_{b,a}), \tag{17}$$

where $\delta^0_{jb} = \delta_{cb}$ when $j = c$ and $\delta^0_{jb} = 0$ when $j = 2$. Eqs. (12) and (16) are more general than the corresponding ones in [16] because of the inclusion of the thin-film intrinsic stress τ'_{ab}, the electric field E'_k, and the magnetic field H'_k. e'_{kab} and h'_{kab} are thin-film piezoelectric and piezomagnetic constants. They are obtained from the stress relaxation $T^{(0)}_{2j} = 0$. We assume that the intrinsic stress τ'_{ab}, the strain $S'^{(0)}_{cd}$, the electric field E'_k, and the magnetic field H'_k are all infinitesimal. Only their first-order effects on the resonant frequencies are of interest. τ'_{ab}, E'_k, and H'_k are considered known and they act as loads on the structure. For the lower film, the situation is similar.

C. Equations for the Crystal Plate With Surface Films

The equations of the crystal plate and the surface films are joined together using the following interface continuity conditions of tractions and displacements:

$$T'_{2j}|_{-h'} = T_{2j}|_h, \quad T''_{2j}|_{h''} = T_{2j}|_{-h}, \tag{18}$$

$$u'^{(0)}_i = u^{(0)}_i + h\delta^0_{ia} u^{(1)}_a, \quad u''^{(0)}_i = u^{(0)}_i - h\delta^0_{ia} u^{(1)}_a. \tag{19}$$

Through (18), the surface loads of the crystal plate, $F^{(0)}_j$ and $F^{(1)}_b$ in (3) and (4), may be expressed in terms of the surface loads $F'^{(0)}_j$ and $F''^{(0)}_j$ of the films. Then the stress equations of motion, (10) and (14), of the films may be incorporated into (3) and (4) by substitution, thus forming five stress-equations of motion of the crystal plate with the films. In these equations, the accelerations may be expressed in terms of $u^{(0)}_i$ and $u^{(1)}_a$ by means of (19). The stress equations of motion may then be converted to five displacement equations of motion for the five displacements, $u^{(0)}_i$ and $u^{(1)}_a$, through the use of the three sets of stress-strain-displacement relations and (19). This is carried out in the following.

From (5a) and (18), and from (5b) and (18), we obtain

$$F^{(0)}_j = T'_{2j}|_{-h'} - T''_{2j}|_{h''}, \tag{20}$$

$$F^{(1)}_b = hT'_{2b}|_{-h'} + hT''_{2b}|_{h''}. \tag{21}$$

From the sum and difference of (11) and (15),

$$T'_{2j}|_{-h'} - T''_{2j}|_{h''} = \mathcal{F}^{(0)}_j - F'^{(0)}_j - F''^{(0)}_j, \tag{22}$$

$$h\,T'_{2b}|_{-h'} + h\,T''_{2b}|_{h''} = \mathcal{F}^{(1)}_b - h(F'^{(0)}_b - F''^{(0)}_b), \tag{23}$$

where

$$\mathcal{F}^{(0)}_j \equiv T'_{2j}|_{h'} - T''_{2j}|_{-h''}, \tag{24}$$

$$\mathcal{F}^{(1)}_b \equiv h\,T'_{2b}|_{h'} + h\,T''_{2b}|_{-h''}, \tag{25}$$

are the loads on the outer surfaces of the films. Hence, from (20)–(23),

$$F^{(0)}_j = \mathcal{F}^{(0)}_j - (F'^{(0)}_j + F''^{(0)}_j), \tag{26}$$

$$F^{(1)}_b = \mathcal{F}^{(1)}_b - h(F'^{(0)}_b - F''^{(0)}_b). \tag{27}$$

From the sum and difference of (10) and (14),

$$\begin{aligned} F'^{(0)}_j + F''^{(0)}_j = &-\delta^0_{jb}(T'^{(0)}_{ab} + T''^{(0)}_{ab})_{,a} + 2h'\rho'\ddot{u}'^{(0)}_j \\ &+ 2h''\rho''\ddot{u}''^{(0)}_j, \end{aligned} \tag{28}$$

$$F'^{(0)}_b - F''^{(0)}_b = -(T'^{(0)}_{ab} - T''^{(0)}_{ab})_{,a} + 2h'\rho'\ddot{u}'^{(0)}_b - 2h''\rho''\ddot{u}''^{(0)}_b. \tag{29}$$

Substituting these expressions into (26) and (27), and using (19), we have

$$\begin{aligned} F^{(0)}_j = &\mathcal{F}^{(0)}_j + \delta^0_{jb}(T'^{(0)}_{ab} + T''^{(0)}_{ab})_{,a} - 2h\rho R_{\mathrm{S}}\ddot{u}^{(0)}_j \\ &- 2h^2\rho R_{\mathrm{D}}\delta^0_{ja}\ddot{u}^{(1)}_a, \end{aligned} \tag{30}$$

$$F^{(1)}_b = \mathcal{F}^{(1)}_b + h(T'^{(0)}_{ab} - T''^{(0)}_{ab})_{,a} - 2h^3\rho R_{\mathrm{S}}\ddot{u}^{(1)}_b - 2h^2\rho R_{\mathrm{D}}\ddot{u}^{(0)}_b, \tag{31}$$

where we have denoted

$$R_{\mathrm{S}} = (\rho'h' + \rho''h'')/\rho h, \quad R_{\mathrm{D}} = (\rho'h' - \rho''h'')/\rho h. \tag{32}$$

R_{S} and R_{D} are the mass ratios of the sum and difference of the films to the crystal plate.

Finally, substituting (30) and (31) into (3) and (4), we arrive at the following stress equations of motion of the crystal plate carrying the films:

$$\mathcal{T}^{(0)}_{ij,i} + \mathcal{F}^{(0)}_j = 2h\rho(1 + R_{\mathrm{S}})\ddot{u}^{(0)}_j + 2h^2\rho R_{\mathrm{D}}\delta^0_{jb}\ddot{u}^{(1)}_b, \tag{33}$$

$$\mathcal{T}^{(1)}_{ab,a} - \mathcal{T}^{(0)}_{2b} + \mathcal{F}^{(1)}_b = \frac{2}{3}h^3\rho(1 + 3R_{\mathrm{S}})\ddot{u}^{(1)}_b + 2h^2\rho R_{\mathrm{D}}\ddot{u}^{(0)}_b, \tag{34}$$

where the total resultants including contributions from the plate and the films are

$$\mathcal{T}^{(0)}_{ij} \equiv T^{(0)}_{ij} + \delta^0_{ia}\delta^0_{jb}(T'^{(0)}_{ab} + T''^{(0)}_{ab}), \tag{35}$$

$$\mathcal{T}^{(1)}_{ab} \equiv T^{(1)}_{ab} + h(T'^{(0)}_{ab} - T''^{(0)}_{ab}). \tag{36}$$

Substituting into (35) and (36) the constitutive relations (6), (12), and (16), and then the strain-displacement relations (13) and (17), we find, after using (19) and then (9), the following constitutive relations for the plate including the films:

$$\mathcal{T}_{ij}^{(0)} = 2hg_{ijkl}^{*}S_{kl}^{(0)} + 2h\delta_{ia}^{0}\delta_{jb}^{0}(\gamma_{abcd}^{\mathrm{S}}S_{cd}^{(0)} + h\gamma_{abcd}^{\mathrm{D}}S_{cd}^{(1)}) + 2hN_{ij}, \tag{37}$$

$$\mathcal{T}_{ab}^{(1)} = \frac{2}{3}h^{3}\gamma_{abcd}S_{cd}^{(1)} + 2h^{3}\gamma_{abcd}^{\mathrm{S}}S_{cd}^{(1)} + 2h^{2}\gamma_{abcd}^{\mathrm{D}}S_{cd}^{(0)} + 2hM_{ab}, \tag{38}$$

where we have defined

$$\begin{aligned} N_{ij} &= \delta_{ia}^{0}\delta_{jb}^{0}\frac{h'}{h}(\tau'_{ab} - e'_{kab}E'_{k} - h'_{kab}H'_{k}) \\ &\quad + \delta_{ia}^{0}\delta_{jb}^{0}\frac{h''}{h}(\tau''_{ab} - e''_{kab}E''_{k} - h''_{kab}H''_{k}), \end{aligned} \tag{39}$$

$$\begin{aligned} M_{ab} &= h'(\tau'_{ab} - e'_{kab}E'_{k} - h'_{kab}H'_{k}) \\ &\quad - h''(\tau''_{ab} - e''_{kab}E''_{k} - h''_{kab}H''_{k}), \end{aligned} \tag{40}$$

$$\begin{aligned} \gamma_{abcd}^{\mathrm{S}} &\equiv (h'\gamma'_{abcd} + h''\gamma''_{abcd})/h, \\ \gamma_{abcd}^{\mathrm{D}} &\equiv (h'\gamma'_{abcd} - h''\gamma''_{abcd})/h. \end{aligned} \tag{41}$$

N_{ij} and M_{ab} describe the contributions from the films to the extension and bending of the plate. The displacement equations of motion may be obtained by substituting (9) into (37) and (38) and then substituting (37) and (38) into (33) and (34), resulting in:

$$\begin{aligned} &g_{ijkl}^{*}(u_{k,li}^{(0)} + \delta_{2k}u_{l,i}^{(1)}) + \delta_{jb}^{0}\gamma_{abcd}^{\mathrm{S}}u_{c,da}^{(0)} + h\delta_{jb}^{0}\gamma_{abcd}^{\mathrm{D}}u_{c,da}^{(1)} \\ &\quad + \frac{1}{2}h^{-1}\mathcal{F}_{j}^{(0)} + N_{ij,i} = \rho(1+R_{\mathrm{S}})\ddot{u}_{j}^{(0)} + h\rho R_{\mathrm{D}}\delta_{ja}^{0}\ddot{u}_{a}^{(1)}, \end{aligned} \tag{42}$$

$$\begin{aligned} &\frac{1}{3}h^{2}\gamma_{abcd}u_{c,da}^{(1)} - g_{2bkl}^{*}(u_{k,l}^{(0)} + \delta_{2k}u_{l}^{(1)}) + h^{2}\gamma_{abcd}^{\mathrm{S}}u_{c,da}^{(1)} \\ &\quad + h\gamma_{abcd}^{\mathrm{D}}u_{c,da}^{(0)} + \frac{1}{2}h^{-1}\mathcal{F}_{b}^{(1)} + M_{ab,a} \\ &\quad = \frac{1}{3}h^{2}\rho(1+3R_{\mathrm{S}})\ddot{u}_{b}^{(1)} + h\rho R_{\mathrm{D}}\ddot{u}_{b}^{(0)}. \end{aligned} \tag{43}$$

Finally, the shear correction factors κ_1 and κ_3 are determined by requiring the fundamental TSh frequencies of an infinite plate calculated from the approximate plate equations and the exact three-dimensional equations to be equal. For a plate of monoclinic crystals, which includes rotated Y-cut quarts as a special case and is sufficient for purpose of this paper, the correction factors are [16]

$$\kappa_{a}^{2} = \frac{\pi^{2}}{12}\frac{c_{a}\{1 - C_{a}R_{\mathrm{S}}[1 - C_{a}R_{\mathrm{S}} + \frac{1}{4}\pi^{2}C_{a}^{2}(R_{\mathrm{S}}^{2} - R_{\mathrm{D}}^{2})]\}^{2}}{g_{2a2a}[1 + 3R_{\mathrm{S}} + 3R_{\mathrm{D}}^{2}/(1+R_{\mathrm{S}})]}, \tag{44}$$

where

$$\begin{aligned} c_{1} &= c_{2121}, \\ c_{3} &= \frac{1}{2}\{c_{2222} + c_{2323} - [(c_{2222} - c_{2323})^{2} + 4c_{2223}^{2}]^{1/2}\}, \\ C_{j} &= c_{j}/c_{2j2j}. \end{aligned} \tag{45}$$

III. Reduction to Extension and Elementary Flexure

Eqs. (33) and (34) or (42) and (43) are coupled equations for the extension $u_a^{(0)}$, flexure $u_2^{(0)}$, and TSh $u_a^{(1)}$. For the applications we are interested in, $u_a^{(0)}$, $u_2^{(0)}$, and $u_a^{(1)}$ are biasing deformations resulting from τ'_{ab}, E'_k, H'_k and similar fields from the lower film. For thin plates, $u_a^{(1)}$ is usually small and can be eliminated, resulting in a simpler theory for coupled extension and elementary flexure without shear. Such a theory will be sufficient for our needs and many other applications. It can be reduced from (33) and (34) as follows. First, we rewrite (33) and (34) as separate equations for $u_a^{(0)}$, $u_2^{(0)}$, and $u_a^{(1)}$:

$$\mathcal{T}_{ab,a}^{(0)} + \mathcal{F}_{b}^{(0)} = 2h\rho(1+R_{\mathrm{S}})\ddot{u}_{b}^{(0)} + 2h^{2}\rho R_{\mathrm{D}}\ddot{u}_{b}^{(1)}, \tag{46}$$

$$\mathcal{T}_{a2,a}^{(0)} + \mathcal{F}_{2}^{(0)} = 2h\rho(1+R_{\mathrm{S}})\ddot{u}_{2}^{(0)}, \tag{47}$$

$$\mathcal{T}_{ab,a}^{(1)} - \mathcal{T}_{2b}^{(0)} + \mathcal{F}_{b}^{(1)} = 0, \tag{48}$$

where, as one of the two approximations needed for the reduction to elementary flexure, we have neglected the rotatory inertia terms on the right-hand side of (34). Eq. (48) now provides the usual shear force-bending moment relation in the elementary theory for flexure. Solving (48) for $\mathcal{T}_{2b}^{(0)}$ and substituting the resulting expression into (47) gives the following equation for elementary flexure:

$$\mathcal{T}_{ab,ab}^{(1)} + \mathcal{F}_{b,b}^{(1)} + \mathcal{F}_{2}^{(0)} = 2h\rho(1+R_{\mathrm{S}})\ddot{u}_{2}^{(0)}. \tag{49}$$

Another approximation needed for the reduction to elementary flexure is that the plate shear strains $S_{a2}^{(0)}$ vanish, namely,

$$S_{a2}^{(0)} = \frac{1}{2}(u_{2,a}^{(0)} + u_{a}^{(1)}) = 0. \tag{50}$$

This implies, through (9), that

$$u_{a}^{(1)} = -u_{2,a}^{(0)}, \tag{51}$$

$$S_{ab}^{(1)} = -u_{2,ab}^{(0)}. \tag{52}$$

With (51) we can write the extensional equation (46) as

$$\mathcal{T}_{ab,a}^{(0)} + \mathcal{F}_{b}^{(0)} = 2h\rho(1+R_{\mathrm{S}})\ddot{u}_{b}^{(0)} - 2h^{2}\rho R_{\mathrm{D}}\ddot{u}_{2,b}^{(0)}. \tag{53}$$

Eq. (53) and (49) are three equations for $u_a^{(0)}$ and $u_2^{(0)}$. The extension $u_a^{(0)}$ and the flexure $u_2^{(0)}$ are coupled on the right-hand side of (53) when the two films are different. Because $S_{a2}^{(0)} = 0$ and $S_{22}^{(0)}$ is not present in (37), from (37) and (39) we have

$$\mathcal{T}_{ab}^{(0)} = 2hg_{abcd}^{*}S_{cd}^{(0)} + 2h(\gamma_{abcd}^{\mathrm{S}}S_{cd}^{(0)} + h\gamma_{abcd}^{\mathrm{D}}S_{cd}^{(1)}) + 2hN_{ab}, \tag{54}$$

$$\begin{aligned} N_{ab} &= \frac{h'}{h}(\tau'_{ab} - e'_{kab}E'_{k} - h'_{kab}H'_{k}) \\ &\quad + \frac{h''}{h}(\tau''_{ab} - e''_{kab}E''_{k} - h''_{kab}H''_{k}). \end{aligned} \tag{55}$$

The equations of motion for $u_a^{(0)}$ and $u_2^{(0)}$ are obtained by substituting (54) and (38) into (53) and (49). $S_{ab}^{(0)}$ and $S_{ab}^{(1)}$ are now given by (9) and (52), respectively, in terms of $u_a^{(0)}$ and $u_2^{(0)}$.

IV. Deformation of a Free Plate

Consider the static deformation of a finite plate with traction-free surfaces all around. The top and bottom surface loads $\mathcal{F}_j^{(0)} = 0$ and $\mathcal{F}_b^{(1)} = 0$. The edge of the plate is geometrically smooth (without corners) and is free from any mechanical resultants. In this case, $\mathcal{T}_{ab}^{(0)} \equiv 0$ and $\mathcal{T}_{ab}^{(1)} \equiv 0$ satisfy the governing equations in (53) and (49), as well as all boundary conditions. We consider the case of uniform intrinsic stresses and electric/magnetic fields in the films. In this case, N_{ab} and M_{ab} are constants. The corresponding constant plate extensional strains $S_{ab}^{(0)}$ and flexural strains $S_{ab}^{(1)}$ are determined by setting (54) and (38) to zero

$$\mathcal{T}_{ab}^{(0)} = 2hg_{abcd}^{*}S_{cd}^{(0)} + 2h(\gamma_{abcd}^{\mathrm{S}}S_{cd}^{(0)} + h\gamma_{abcd}^{\mathrm{D}}S_{cd}^{(1)}) + 2hN_{ab} = 0, \tag{56}$$

$$\mathcal{T}_{ab}^{(1)} = \frac{2}{3}h^3\gamma_{abcd}S_{cd}^{(1)} + 2h^3\gamma_{abcd}^{\mathrm{S}}S_{cd}^{(1)} + 2h^2\gamma_{abcd}^{\mathrm{D}}S_{cd}^{(0)} + 2hM_{ab} = 0. \tag{57}$$

For elementary flexure $S_{2a}^{(0)} = 0$. $S_{22}^{(0)}$ is determined from the stress relaxation condition $T_{22}^{(0)} = 0$. $S_{2j}^{(1)}$ is determined from the stress relaxation conditions $T_{2j}^{(1)} = 0$. Then $S_{ij}^{(0)}$ and $S_{ij}^{(1)}$ are completely known. In the relatively simple case of cylindrical deformations of plates with $u_3 = 0$ and $\partial/\partial x_3 = 0$, from (56) and (57) we obtain

$$S_1^{(0)} = \frac{-\gamma_{11}^{\mathrm{D}}M_1 + h(\gamma_{11}/3 + \gamma_{11}^{\mathrm{S}})N_1}{h(\gamma_{11}^{\mathrm{D}})^2 - h(\gamma_{11}/3 + \gamma_{11}^{\mathrm{S}})(g_{11}^{*} + \gamma_{11}^{\mathrm{S}})}, \tag{58}$$

$$S_1^{(1)} = \frac{(g_{11}^{*} + \gamma_{11}^{\mathrm{S}})M_1 - h\gamma_{11}^{\mathrm{D}}N_1}{h^2(\gamma_{11}^{\mathrm{D}})^2 - h^2(\gamma_{11}/3 + \gamma_{11}^{\mathrm{S}})(g_{11}^{*} + \gamma_{11}^{\mathrm{S}})}. \tag{59}$$

Eqs. (58) and (59) show that γ_{11}^{D} causes coupling between extension and bending. For symmetric films, it vanishes.

V. Frequency Shifts of the Fundamental Thickness-Shear Mode

In this section, we consider TSh vibrations of a rotated Y-cut quartz plate with the presence of biasing fields, $S_{ij}^{(0)}$ and $S_{ij}^{(1)}$, caused by the intrinsic stresses and electric/magnetic fields in the surface films. The TSh vibration is an incremental motion superposed on these biasing fields. For the most widely used fundamental TSh mode, the frequency shift $\Delta\omega$ caused by the biasing fields is given by [18], [19]

$$\frac{\Delta\omega}{\omega_0} = S_1^{(0)} + \frac{1}{2c_{66}}(c_{166}S_1^{(0)} + c_{266}S_2^{(0)} + c_{366}S_3^{(0)} + c_{466}S_4^{(0)}) - \frac{h^2}{\sqrt{3}\pi c_{66}}(c_{165}S_{5,1}^{(1)} + c_{561}S_{1,3}^{(1)} + c_{563}S_{3,3}^{(1)}), \tag{60}$$

where $\Delta\omega$ is normalized by the fundamental TSh frequency, ω_0, of the crystal plate alone when the films are not present:

$$\omega_0 = \frac{\pi}{2h}\sqrt{\frac{c_{66}}{\rho}}. \tag{61}$$

c_{661} and c_{662} are the third-order elastic constants for nonlinear material behavior. They have six indices in the tensor notation and three indices under the compact matrix notation [20] used in (58). The first-order plate strains $S_{ij}^{(1)}$ have contributions only when they are inhomogeneous. In the special case of cylindrical deformations, (60) reduces to

$$\frac{\Delta\omega}{\omega_0} = \left(1 + \frac{c_{166}}{2c_{66}} - \frac{c_{266}c_{21}}{2c_{66}c_{22}}\right)S_1^{(0)}. \tag{62}$$

VI. Examples of Application

In this section, we use the equations derived in the preceding sections to study frequency shifts in a resonator caused by electric/magnetic fields and electrode stresses. Consider a crystal plate of Y-cut quartz, which is a special case of rotated Y-cuts in which the angle of rotation is zero. The linear material constants can be found in [20]. The third-order elastic constants are from [21]. The following relations among the third-order elastic constants exist and are needed [22]:

$$\begin{aligned} c_{661} &= (-2c_{111} - c_{112} + 3c_{222})/4, \\ c_{662} &= (2c_{111} - c_{112} - c_{222})/4. \end{aligned} \tag{63}$$

For the plate thickness, we choose $2h = 1$ mm. The fundamental TSh mode of the plate alone without the surface films is $\omega_0/2\pi = 1.940022 \times 10^6$ Hz.

A. Electric Field Sensing

Consider the previously described quartz plate with only one surface film of PZT-5H (Sinocera Inc., Shanghai, China) poled in the thickness direction ($h'' = 0$). Polarized ceramics have much stronger piezoelectric coupling than quartz. When the plate is placed under an external electric field in the thickness direction which we want to measure, the ceramic film tends to expand or contract but the quartz plate does not do so through piezoelectric coupling because its $e_{21} = e_{22} = e_{23} = 0$. Instead, the quartz plate extends and bends under the action of the ceramic film. We solve (56) and (57) on a computer for $S_{ab}^{(0)}$ and $S_{ab}^{(1)}$, and

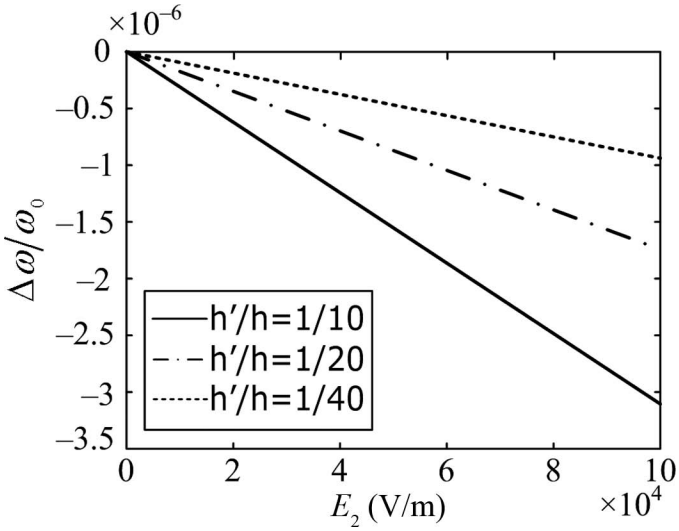

Fig. 2. Frequency shift resulting from electric field.

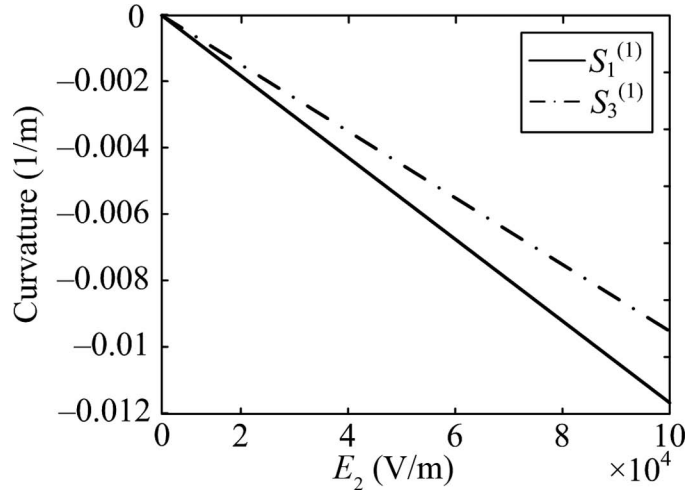

Fig. 3. Curvature resulting from electric field.

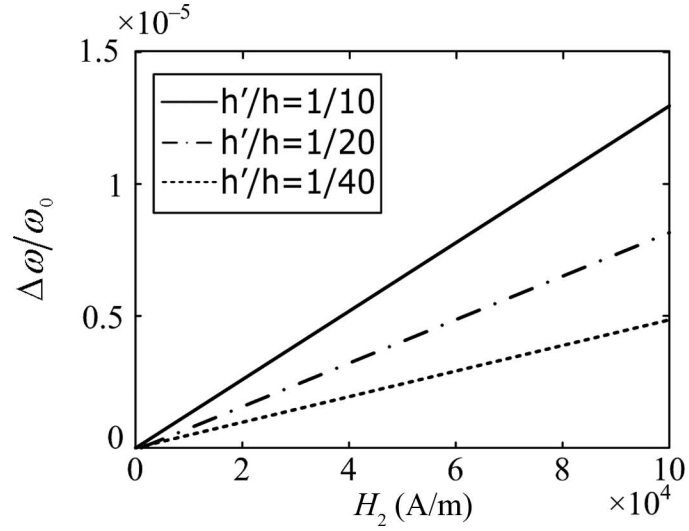

Fig. 4. Frequency shift resulting from magnetic field.

use the stress relaxation conditions to find the other plate strains. Then we calculate the frequency shift using (60) and plot the results in Fig. 2. A linear relationship between the frequency shift and the electric field is predicted, which is ideal for electric field sensing. This linearity is also a consequence of the theory employed, i.e., the biasing fields are assumed to be small and are obtained by the linear theory of elasticity, and only the first-order effect of the biasing fields on the incremental TSh vibration is considered. For large biasing fields, the nonlinear theory of elasticity is needed to determine the biasing fields, and the second- and higher-order effects of the biasing fields must be considered [23]. In that case, a nonlinear relationship between the frequency shift and the biasing fields would be predicted. Such a calculation requires the knowledge of the fourth-order material constants, which at present are not available; therefore, the range of the linear output cannot be determined from the present analysis. For a moderate electric field of 10^5 V/m, or 100 V/mm, the relative frequency shift is of the order of 10^{-6}, which is measurable in crystal resonators whose frequency shifts are typically described in parts per million. Fig. 2 shows that thicker piezoelectric films imply higher sensitivity, as expected. These results agree with reference [13] for cylindrical motions. Because the ceramic film is on one side of the crystal plate only, the electric field also causes bending of the crystal plate. The curvatures of the middle plane of the crystal plate are shown in Fig. 3 for $h'/h = 0.1$. The curvature in the x_3 direction is smaller because c_{33} is larger than c_{11} for Y-cut quartz plates.

B. Magnetic Field Sensing

When a Y-cut quartz plate carries only one piezomagnetic film ($h'' = 0$) of $CoFe_2O_4$ (the material constants can be found in [24]), the analysis is similar. Quartz does not respond to magnetic fields directly. Through the extension or contraction of the piezomagnetic film, the quartz plate extends and bends, resulting in frequency shifts and curvatures. The results are shown in Figs. 4 and 5, respectively, showing that the structure can possibly function as a magnetic field sensor. These results also agree with the results of the cylindrical motions considered in [13].

C. Electrode Stress

Plate quartz resonators are usually exited by electrodes on the top and bottom of the plates. The electrodes often carry intrinsic stresses caused by their manufacturing processes. These electrode stresses are of the order of 200 MPa [25]. Their effects on resonator frequency stability are an important issue in resonator design. As an example, consider a Y-cut quartz plate with identical electrodes of gold with intrinsic stresses of 200 MPa in both the x_1 and x_3 directions. The frequency shift calculated from (60) is shown in Fig. 6. It is of the order of 10^{-5}, quite significant.

Another important aspect of the electrodes is their thickness. In resonator manufacturing, one electrode is

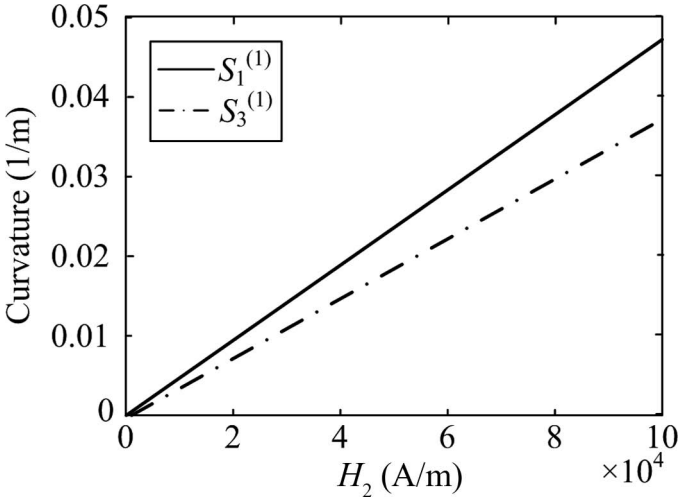

Fig. 5. Curvature resulting from magnetic field.

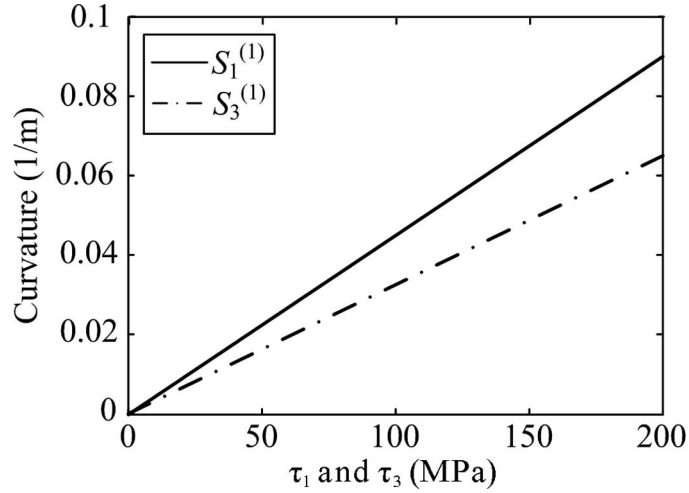

Fig. 7. Curvature resulting from electrode stress of unequal thickness.

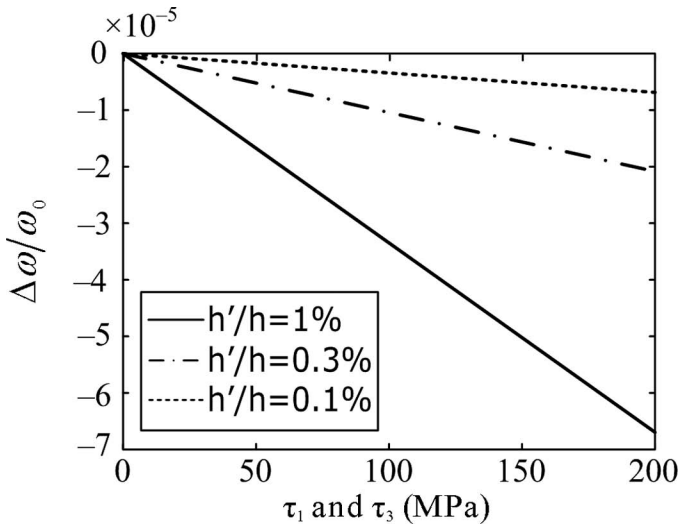

Fig. 6. Frequency shift resulting from identical electrode stress.

pre-deposited with a pre-determined thickness. The electrode on the other side of the plate has a thickness controlled by the desired frequency of the electroded plate. This normally results in a crystal plate with electrodes of unequal thickness. It is known that electrodes with different thickness cause undesirable mode couplings and affect resonator performance [16], [26]. As a numerical example, consider a Y-cut quartz plate with electrodes of different thickness. $h'/h = 0.003$ and $h''/h = 0.001$. The corresponding curvatures are calculated from (56) and (57) and are shown in Fig. 7. If sufficient information about the plate strains and curvatures is measured experimentally, the electrode intrinsic stresses and/or their thicknesses can be calculated from measured data using (56) and (57).

VII. Conclusion

Two-dimensional equations for coupled extension, flexure, and TSh motions of anisotropic crystal plates with multiphysical surface films are derived, reduced to couple extension and elementary flexure without shear, and used in the analysis of frequency shifts in crystal resonators. It is shown that a crystal plate with a piezoelectric/piezomagnetic film may be used as a sensor for electric/magnetic fields. If the film/plate thickness ratio is 1/10 and the electric field is of the order of 100 V/mm, the relative frequency shift is of the order of a few parts per million. If the magnetic field is of the order of 100 A/mm, the relative frequency shift is of the order of 10^{-5}. These agree with some previous analyses in simpler situations, and are detectable frequency shifts in crystal resonators. The equations derived are also useful in the analysis of the effects of electrode stress in crystal resonators and other frequency analysis of plate crystal resonators and sensors in general.

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