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Analysis of a Monolithic, Two-Dimensional Array of Quartz Crystal Microbalances Loaded by Mass Layers With Nonuniform Thickness

Nan Liu, Jiashi Yang, and Ji Wang

Abstract—We study free thickness-shear vibrations of a monolithic, two-dimensional, and periodic array of quartz crystal microbalances loaded by mass layers with gradually varying thickness. A theoretical analysis is performed using Mindlin's two-dimensional plate equation. It is shown that the problem is mathematically governed by Mathieu's equation with a spatially varying coefficient. A periodic solution for resonant frequencies and modes is obtained and used to examine the effects of the mass layers. Results show that the vibration may be trapped or untrapped under the mass layers. The trapped modes decay differently in the two in-plane directions of the plate. The mode shapes and the decay rate of the trapped modes are sensitive to the mass layer thickness.

I. INTRODUCTION

Quartz crystal microbalances (QCMs) are quartz crystal resonators (QCRs), operating with plate thickness-shear (Tsh) modes, used for monitoring thin-film deposition and mass sensing based on the inertial effect of the mass layers on the frequencies of the QCRs. The effects of a mass layer on a QCR are multifold, including the mass density, elastic constants, and thickness. In the simplest description (the well-known Sauerbrey equation), the inertia of a mass layer lowers the frequencies of a QCR [1], [2]. This has been used to measure the product of the mass layer density and thickness. Researchers also developed more sophisticated models based on pure Tsh modes, showing both the inertial and the stiffness effects [3], [4]. More references can be found in review articles such as [5] and [6].

Pure thickness-mode models for QCMs are only valid for uniform mass layers on unbounded plates, without in-plane variations or edge effects. The behavior of real devices is more complicated and can deviate from the pure thickness-mode models. For example, in a typical QCM, the mass layer covers only the central portion of the crystal plate. In this case, the Tsh vibration is nonuniform; it is mainly under the mass layer and decays rapidly away from the edge of the mass layer. This phenomenon is called energy trapping [7]. It is crucial to device mounting, which can be done at the edge of a plate where there is little vibration. Thus, mounting will not affect the vibration of the plate. The in-plane variations of the Tsh modes also cause deviation from the Sauerbrey equation.

II. GOVERNING EQUATIONS

Consider an AT-cut quartz plate, as shown in Fig. 1. It functions as a periodic array of rectangular QCMs. Only
nine of them are shown. The plate has a uniform thickness $2h$ and a mass density $\rho$. There is an identical thin mass layer with a slowly varying thickness on the top of each QCM. The density of the mass layer is $\rho'$. Its varying thickness is $2h'(x_1, x_3)$. The specific form of this function will be given later. The mass layer is assumed to be very thin. Only its inertia will be considered. Its stiffness will be neglected [1], [2]. Quartz is a material with very weak piezoelectric coupling. For a frequency analysis, we will neglect the small piezoelectric couplings as usual. In general, Tsh vibration may be coupled to flexural motion. This coupling depends on the plate dimensions and general, Tsh vibration may be coupled to flexural motion will be neglected [1], [2]. Quartz is a material with very thin. only its inertia will be considered. Its stiffness will be given later. The mass layer is assumed to be very thin. Its inertia will be considered. Its stiffness will be neglected [1], [2]. Quartz is a material with very weak piezoelectric coupling. For a frequency analysis, we will neglect the small piezoelectric couplings as usual. In general, Tsh vibration may be coupled to flexural motion. This coupling depends on the plate dimensions and

$$\omega_0^2 = \omega_\infty^2(1 - 2R_0).$$

(13)

$\omega_0$ is the frequency of the fundamental Tsh mode of the crystal plate with a mass layer of a uniform thickness $2h_0$. We are mainly interested in the so-called energy-trapped modes with the frequency $\omega$ within
\[ \rho \omega_0^2 A_0 + \frac{1}{4} \rho \omega^2_\infty R_0 A_1 + \left( \frac{1}{2} \rho \omega^2_\infty R_0 A_1 + \left[ \rho \omega^2_0 - \gamma_1 \left( \frac{\pi}{a_1} \right)^2 \right] A_1 + \frac{1}{4} \rho \omega^2_\infty R_0 A_2 \right) \cos \frac{\pi x_1}{a_1} \\
+ \sum_{n=2}^\infty \left( \frac{1}{4} \rho \omega^2_\infty R_0 A_{n-1} + \left[ \rho \omega^2_0 - \gamma_1 \left( \frac{n \pi}{a_1} \right)^2 \right] A_n + \frac{1}{4} \rho \omega^2_\infty R_0 A_{n+1} \right) \cos \frac{n \pi x_1}{a_1} = 0. \]  

(24)

\[ \omega_0 < \omega < \omega_\infty. \]  

(14)

III. Free Vibration Analysis

As a partial differential equation, (12) is separable. Let

\[ \psi_1(x_1, x_3) = F_1(x_1)F_3(x_3). \]  

(15)

Then, from (12), by the standard procedure of separation of variables, we obtain

\[ \gamma_{11} F_{111} + \rho \omega^2_0 F_1 - \rho \omega^2_\infty 2RF_0 f_1(x_1)F_1 = 0, \]
\[ \gamma_{55} F_{333} + \rho \omega^2_0 F_3 - \rho \omega^2_\infty 2RF_0 f_3(x_3)F_3 = 0, \]

(16)

where the separation constants \( \omega_1 \) and \( \omega_3 \) must satisfy the following equation, which gives the resonant frequencies once \( \omega_1 \) and \( \omega_3 \) are determined:

\[ \omega^2 = \omega_0^2 + \omega_1^2 + \omega_3^2. \]  

(17)

To describe a mass layer with a periodically varying thickness, and thus representing an array, we choose

\[ 2h' = \frac{h_0}{2} \left( \cos \frac{\pi x_1}{a_1} + \cos \frac{\pi x_3}{a_3} \right) + h_0 \]
\[ = 2h_0 \left( 1 - \left( \frac{1}{4} \cos \frac{\pi x_1}{a_1} \right) - \left( \frac{1}{4} \cos \frac{\pi x_3}{a_3} \right) \right), \]  

(18)

where \( a_1 \) and \( a_3 \) are the length and the width of an individual QCM, respectively. In a typical QCM, e.g., the one at the origin with \( |x_1| < a_1 \) and \( |x_3| < a_3 \), the mass layer described by (18) assumes its maximal thickness \( 2h_0 \) at the center (0,0) and it vanishes at the four corners of the rectangle. Therefore, (18) describes a mass layer thick at the center and thin at the edges. In other QCMs of the array, we have the same nonuniform mass layer because of the periodicity of (18). From (18) and (9), we identify

\[ f_1 = \frac{1}{4} - \frac{1}{4} \cos \frac{\pi x_1}{a_1}, \quad f_3 = \frac{1}{4} - \frac{1}{4} \cos \frac{\pi x_3}{a_3}. \]  

(19)

Then, (16) can be written as

\[ \gamma_{11} F_{111} + \left( \rho \omega^2_0 + \frac{1}{2} \rho \omega^2_\infty \cos \frac{\pi x_1}{a_1} \right) F_1 = 0, \]  

(20)

where, for convenience, we have introduced \( \omega_1 \) and \( \omega_3 \), which satisfy the following equation from (17):

\[ \omega^2 = \omega_0^2 + \omega_1^2 + \omega_3^2 + \omega^2_\infty R_0. \]  

(22)

Eqs. (20) and (21) are the well-known Mathieu’s equation [26]. For the periodic array in Fig. 1, we are interested in periodic solutions of (20) and (21) that are even functions of \( x_1 \) and \( x_3 \). Therefore, according to Fourier series, we let

\[ F_1 = A_0 + \sum_{n=1}^\infty A_n \cos \frac{n \pi x_1}{a_1}, \]  

(23a)

\[ F_3 = B_0 + \sum_{n=1}^\infty B_n \cos \frac{n \pi x_3}{a_3}, \]  

(23b)

where \( A_n \) and \( B_n \) are undetermined coefficients. Eqs. (20) and (21) can be solved in the same way. We focus on (20) in the following. Substituting (23a) into (20) and using the relevant trigonometric identity to convert the product terms into sums, we obtain (24), see above. Multiplying both sides of (24) by \( \cos (m \pi x_1 / a_1) \) with \( m = 0, 1, 2, \ldots \), integrating the resulting expression over a period \( -a_1, a_1 \), and using the orthogonality of the trigonometric functions, we arrive at the following recurrence relations, which are linear homogeneous equations for \( A_m \):

\[ \rho \omega^2_0 A_0 + \frac{1}{4} \rho \omega^2_\infty R_0 A_1 = 0, \]
\[ \frac{1}{2} \rho \omega^2_\infty R_0 A_0 + \left[ \rho \omega^2_0 - \gamma_1 \left( \frac{\pi}{a_1} \right)^2 \right] A_1 + \frac{1}{4} \rho \omega^2_\infty R_0 A_2 = 0, \]
\[ \frac{1}{4} \rho \omega^2_\infty R_0 A_{m-1} + \left[ \rho \omega^2_0 - \gamma_1 \left( \frac{m \pi}{a_1} \right)^2 \right] A_m + \frac{1}{4} \rho \omega^2_\infty R_0 A_{m+1} = 0, \quad m = 2, 3, 4, \ldots. \]  

(25)

For nontrivial solutions of \( A_m \), the determinant of the coefficient matrix of (25) must vanish, which gives the frequency equation that determines \( \omega_1 \). The corresponding nontrivial \( A_m \) obtained from (25) determines \( F_1 \) through (23a). In a similar way, \( \omega_3 \) and \( B_0 \) can be obtained by solving (21). Then, \( \omega \) can be obtained from (22). These will be done numerically on a computer.
IV. Numerical Results

For numerical examples, we fix $2h = 1$ mm and $a_1 = a_3 = 1$ cm, which are typical dimensions for a QCM. We consider the case of $R_0 = 5\%$. In this case, $f_{\infty} = 1654638$ Hz and $f_0 = 1569727$ Hz. Table I shows the results of numerical tests using three, five, or ten terms in the Fourier series. Three resonant frequencies, $f_{11}$, $f_{12}$, and $f_{21}$, are found within $\omega_0 < \omega < \omega_{\infty}$. It can be seen that the frequencies are already indistinguishable when using five and ten terms. Numerical tests also show that when using ten terms in the series, the modes corresponding to the three frequencies also become stable. This is expected because, as will be seen later, the behaviors of the modes we are interested in are relatively simple, with only a few in-plane oscillations, and therefore can be well approximated by a few terms in the Fourier series. Our subsequent calculations will be based on ten terms in the series.

For the three modes found, Fig. 2 shows their vibration distribution in a single QCM, e.g., the one within $|x_1| < a_1$ and $|x_3| < a_3$. The first mode is at the frequency $f_{11}$, the lowest frequency. The vibration is large near the center and decays to almost zero near the edges. Therefore, this is a well-trapped mode. For this mode, neighboring QCMs have little interaction, which is the desired situation for a QCM array. The second mode is at $f_{12}$. It is well trapped along $x_3$ and is nearly zero at $x_1 = \pm a_1$. However, it changes its sign along $x_3$ with two nodal points and has a large amplitude near $x_3 = \pm a_3$. Therefore, for this mode, in the $x_3$ direction, the vibration leaks out of the mass layer and neighboring QCMs begin to interact significantly.

<table>
<thead>
<tr>
<th>Number of terms in the Fourier series</th>
<th>$f_{11}$ (Hz)</th>
<th>$f_{12}$ (Hz)</th>
<th>$f_{21}$ (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1595188</td>
<td>1626922</td>
<td>1631909</td>
</tr>
<tr>
<td>5</td>
<td>1595181</td>
<td>1626864</td>
<td>1631890</td>
</tr>
<tr>
<td>10</td>
<td>1595181</td>
<td>1626864</td>
<td>1631890</td>
</tr>
</tbody>
</table>
Similarly, the third mode at \( f_{21} \) is well trapped along \( x_3 \) but has two nodal points along \( x_1 \) and a large amplitude near \( x_1 = \pm a_1 \).

To see the variations of the three modes found more clearly, in Fig. 3 we plot their \( x_1 \) dependence and \( x_3 \) dependence separately for a 3 × 3 array of QCMS. Clearly, the first mode is trapped in both directions, the second mode is trapped in the \( x_1 \) direction only, and the third mode is trapped in the \( x_3 \) direction only. We note that the modes decay faster in the \( x_3 \) direction than in the \( x_1 \) direction. Therefore, rectangular QCMSs with \( a_3 < a_1 \) are more reasonable than square QCMSs for device miniaturization. Similarly, elliptical QCMSs are more reasonable than circular ones. Fig. 3 also shows that even for the first mode there is still some vibration left between neighboring QCMSs.

Fig. 4 shows the effect of the mass ratio \( R_0 \) on the vibration distribution of the first mode for a 3 × 3 array of QCMSs. A larger \( R_0 \) represents a mass layer that is thicker at the center. The figure shows that a thicker mass layer is associated with faster decay of the vibration amplitude from the center and smaller vibration amplitude between neighboring QCMSs or less interaction among them.

For a more visual presentation, in Fig. 5 we show the vibration distribution of the first mode in a 3 × 3 array of QCMSs.

V. Conclusion

Using Mindlin’s equation for a quartz plate, a periodic array of QCMSs with nonuniform mass layers is governed by Mathieu’s equation with a spatially varying periodic coefficient whose periodic solution can be obtained by Fourier series. Numerical results show that there exist both trapped and untrapped modes. The trapped modes decay differently along \( x_1 \) and \( x_3 \). Therefore, rectangular and elliptical QCMSs are better designs than square and circular ones. The trapped modes decay faster for mass layers that are thicker at the center.

References