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Mordechai Katzman
University of Sheffield, M.Katzman@sheffield.ac.uk

Karl Schwede
The Pennsylvania State University, schwede@math.psu.edu

Anurag K. Singh
University of Utah, singh@math.utah.edu

Wenliang Zhang
University of Nebraska, Lincoln, wzhang15@unl.edu

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MORDECHAI KATZMAN, KARL SCHWEDE, ANURAG K. SINGH and WENLIANG ZHANG

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BY MORDECHAI KATZMAN†

Department of Pure Mathematics, University of Sheffield, Hicks Building,
Sheffield, S3 7RH.
e-mail: M.Katzman@sheffield.ac.uk

KARL SCHWEDE‡

Department of Mathematics, The Pennsylvania State University, University Park,
PA 16802, U.S.A.
e-mail: schwede@math.psu.edu

ANURAG K. SINGH§

Department of Mathematics, University of Utah, 155 S 1400 E, Salt Lake City,
UT 84112, U.S.A.
e-mail: singh@math.utah.edu

AND WENLIANG ZHANG∥

Department of Mathematics, University of Nebraska, Lincoln, NE 68505, U.S.A.
e-mail: wzhang15@unl.edu

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Abstract

Let $R$ be a local ring of prime characteristic. We study the ring of Frobenius operators $\mathcal{F}(E)$, where $E$ is the injective hull of the residue field of $R$. In particular, we examine the finite generation of $\mathcal{F}(E)$ over its degree zero component $\mathcal{F}^0(E)$, and show that $\mathcal{F}(E)$ need not be finitely generated when $R$ is a determinantal ring; nonetheless, we obtain concrete descriptions of $\mathcal{F}(E)$ in good generality that we use, for example, to prove the discreteness of $F$-jumping numbers for arbitrary ideals in determinantal rings.

1. Introduction

Lyubeznik and Smith [LS] initiated the systematic study of rings of Frobenius operators and their applications to tight closure theory. Our focus here is on the Frobenius operators on the injective hull of $R/m$, when $(R, m)$ is a complete local ring of prime characteristic.

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Definition 1.1. Let $R$ be a ring of prime characteristic $p$, with Frobenius endomorphism $F$. Following [LS, section 3], we set $R(F^e)$ to be the ring extension of $R$ obtained by adjoining a noncommutative variable $\chi$ subject to the relations $\chi r = r^{pe} \chi$ for all $r \in R$.

Let $M$ be an $R$-module. Extending the $R$-module structure on $M$ to an $R(F^e)$-module structure is equivalent to specifying an additive map $\varphi : M \to M$ that satisfies

$$\varphi(rm) = r^{pe} \varphi(m), \quad \text{for each } r \in R \text{ and } m \in M.$$ 

Define $\mathcal{F}^e(M)$ to be the set of all such maps $\varphi$ arising from $R(F^e)$-module structures on $M$; this is an Abelian group with a left $R$-module structure, where $r \in R$ acts on $\varphi \in \mathcal{F}^e(M)$ to give the composition $r \circ \varphi$. Given elements $\varphi \in \mathcal{F}^e(M)$ and $\varphi' \in \mathcal{F}^e(M)$, the compositions $\varphi \circ \varphi'$ and $\varphi' \circ \varphi$ are elements of the module $\mathcal{F}^{e+e'}(M)$. Thus,

$$\mathcal{F}(M) = \mathcal{F}^0(M) \oplus \mathcal{F}^1(M) \oplus \mathcal{F}^2(M) \oplus \cdots$$

has a ring structure; this is the ring of Frobenius operators on $M$.

Note that $\mathcal{F}(M)$ is an $\mathbb{N}$-graded ring; it is typically not commutative. The degree 0 component $\mathcal{F}^0(M) = \text{End}_R(M)$ is a subring, with a natural $R$-algebra structure. Lyubeznik and Smith [LS, section 3] ask whether $\mathcal{F}(M)$ is a finitely generated ring extension of $\mathcal{F}^0(M)$. From the point of view of tight closure theory, the main cases of interest are where $(R, m)$ is a complete local ring, and the module $M$ is the local cohomology module $H^0_m(R)$ or the injective hull of the residue field, $E_R(R/m)$, abbreviated $E$ in the following discussion. In the former case, the algebra $\mathcal{F}(M)$ is finitely generated under mild hypotheses, see Example 1.2.2; an investigation of the latter case is our main focus here.

It follows from Example 1.2.2 that for a Gorenstein complete local ring $(R, m)$, the ring $\mathcal{F}(E)$ is a finitely generated extension of $\mathcal{F}^0(E) \cong R$. This need not be true when $R$ is not Gorenstein: Katzman [Ka] constructed the first such examples. In Section 3 we study the finite generation of $\mathcal{F}(E)$, and provide descriptions of $\mathcal{F}(E)$ even when it is not finitely generated: this is in terms of a graded subgroup of the anticanonical cover of $R$, with a Frobenius-twisted multiplication structure, see Theorem 3.3.

Section 4 studies the case of $\mathbb{Q}$-Gorenstein rings. We show that $\mathcal{F}(E)$ is finitely generated (though not necessarily principally generated) if $R$ is $\mathbb{Q}$-Gorenstein with index relatively prime to the characteristic, Proposition 4.1; the dual statement for the Cartier algebra was previously obtained by Schwede in [Sc, remark 4.5]. We also construct a $\mathbb{Q}$-Gorenstein ring for which the ring $\mathcal{F}(E)$ is not finitely generated over $\mathcal{F}^0(E)$; in fact, we conjecture that this is always the case for a $\mathbb{Q}$-Gorenstein ring whose index is a multiple of the characteristic, see Conjecture 4.2.

In Section 5 we show that $\mathcal{F}(E)$ need not be finitely generated for determinantal rings, specifically for the ring $\mathbb{F}[X]/I$, where $X$ is a $2 \times 3$ matrix of variables, and $I$ is the ideal generated by its size 2 minors; this proves a conjecture of Katzman, [Ka, conjecture 3.1]. The relevant calculations also extend a result of Fedder, [Fe, proposition 4.7].

One of the applications of our study of $\mathcal{F}(E)$ is the discreteness of $F$-jumping numbers; in Section 6 we use the description of $\mathcal{F}(E)$, combined with the notion of gauge boundedness, due to Blickle [Bl12], to obtain positive results on the discreteness of $F$-jumping numbers for new classes of rings including determinantal rings, see Theorem 6.4. In the last section, we obtain results on the linear growth of Castelnuovo-Mumford regularity for rings with finite Frobenius representation type; this is also with an eye towards the discreteness of $F$-jumping numbers.
To set the stage, we summarize some previous results on the rings \( \mathcal{F}(M) \).

Example 1.2. Let \( R \) be a ring of prime characteristic.

1. For each \( e \geq 0 \), the left \( R \)-module \( \mathcal{F}^e(R) \) is free of rank one, spanned by \( F^e \); this is [LS, example 3.6]. Hence, \( \mathcal{F}(R) \cong R[F] \).

2. Let \( (R, m) \) be a local ring of dimension \( d \). The Frobenius endomorphism \( F \) of \( R \) induces, by functoriality, an additive map

\[
F: H^d_m(R) \longrightarrow H^d_m(R),
\]

which is the natural Frobenius action on \( H^d_m(R) \). If the ring \( R \) is complete and \( S_2 \), then \( \mathcal{F}^e(H^d_m(R)) \) is a free left \( R \)-module of rank one, spanned by \( F^e \); for a proof of this, see [LS, example 3.7]. It follows that

\[
\mathcal{F}(H^d_m(R)) \cong R[F].
\]

In particular, \( \mathcal{F}(H^d_m(R)) \) is a finitely generated ring extension of \( \mathcal{F}^0(H^d_m(R)) \).

3. Consider the local ring \( R = \mathbb{F}[[x, y, z]]/(xy, yz) \) where \( \mathbb{F} \) is a field, and set \( E \) to be the injective hull of the residue field of \( R \). Katzman [Ka] proved that \( \mathcal{F}(E) \) is not a finitely generated ring extension of \( \mathcal{F}^0(E) \). By [ABZ, theorem 3.5], \( \mathcal{F}(E) \) is either 1-generated or infinitely generated as a ring extension of \( \mathcal{F}^0(E) \) in the Stanley–Reisner case.

Remark 1.3. Let \( R^{(e)} \) denote the \( R \)-bimodule that agrees with \( R \) as a left \( R \)-module, and where the right module structure is given by

\[
x \cdot r = r^{\varphi e} x, \quad \text{for all } r \in R \text{ and } x \in R^{(e)}.\]

For each \( R \)-module \( M \), one then has a natural isomorphism

\[
\mathcal{F}^e(M) \cong \text{Hom}_R \left( R^{(e)} \otimes_R M, M \right)
\]

where \( \varphi \in \mathcal{F}^e(M) \) corresponds to \( x \otimes m \mapsto x \varphi(m) \) and \( \psi \in \text{Hom}_R \left( R^{(e)} \otimes_R M, M \right) \) corresponds to \( m \mapsto \psi(1 \otimes m) \); see [LS, remark 3.2].

Remark 1.4. Let \( R \) be a Noetherian ring of prime characteristic. If \( M \) is a Noetherian \( R \)-module, or if \( R \) is complete local and \( M \) is an Artinian \( R \)-module, then each graded component \( \mathcal{F}^e(M) \) of \( \mathcal{F}(M) \) is a finitely generated left \( R \)-module, and hence also a finitely generated left \( \mathcal{F}^0(M) \)-module; this is [LS, proposition 3.3].

Remark 1.5. Let \( R \) be a complete local ring of prime characteristic \( p \); set \( E \) to be the injective hull of the residue field of \( R \). Let \( A \) be a complete regular local ring with \( R = A/I \). By [Bl1, proposition 3.36], one then has an isomorphism of \( R \)-modules

\[
\mathcal{F}^e(E) \cong \frac{I[\varphi^e]:_A I}{I[\varphi^e]}.
\]

2. Twisted multiplication

Let \( R \) be a complete local ring of prime characteristic; let \( E \) denote the injective hull of the residue field of \( R \). In Theorem 3.3 we prove that \( \mathcal{F}(E) \) is isomorphic to a subgroup of the
Thus, the twisted characteristic assumption is used in verifying that + and * are distributive. Moreover, for elements \( a \in T(\mathcal{R})_e \) and \( b \in T(\mathcal{R})_{e'} \) one has

\[
ab p_{e'} \in \mathcal{R}_{p_{e-1}+p_{e'}(p_{e-1})} = \mathcal{R}_{p_{e+e'}-1} 
\]

and hence

\[
T(\mathcal{R})_e \star T(\mathcal{R})_{e'} \subseteq T(\mathcal{R})_{e+e'}. 
\]

Thus, \( T(\mathcal{R}) \) is an \( \mathbb{N} \)-graded ring; we abbreviate its degree \( e \) component \( T(\mathcal{R})_e \) as \( T_e \). The ring \( T(\mathcal{R}) \) is typically not commutative, and need not be a finitely generated extension ring of \( T_0 \) even when \( R \) is Noetherian:

**Example 2.2.** We examine \( T(\mathcal{R}) \) when \( R \) is a standard graded polynomial ring over a field \( \mathbb{F} \). We show that \( T(\mathcal{R}) \) is a finitely generated ring extension of \( T_0 = \mathbb{F} \) if \( \text{dim } R \leq 2 \), and that \( T(\mathcal{R}) \) is not finitely generated if \( \text{dim } R \geq 3 \).

1. If \( R \) is a polynomial ring of dimension 1, then \( T(\mathcal{R}) \) is commutative and finitely generated over \( \mathbb{F} \): take \( R = \mathbb{F}[x] \), in which case \( T_e = \mathbb{F} \cdot x^{p_{e-1}} \) and

\[
x^{p_{e-1}} \star x^{p_{e'-1}} = x^{p_{e+e'-1}} = x^{p_{e-1}} \star x^{p_{e'-1}}.
\]

Thus, \( T(\mathcal{R}) \) is a polynomial ring in one variable.

2. When \( R \) is a polynomial ring of dimension 2, we verify that \( T(\mathcal{R}) \) is a noncommutative finitely generated ring extension of \( \mathbb{F} \). Let \( R = \mathbb{F}[x, y] \). Then

\[
x^{p-1} \star y^{p-1} = x^{p-1}y^{p-1-p} \quad \text{whereas} \quad y^{p-1} \star x^{p-1} = x^{p-1}y^{p-1},
\]

so \( T(\mathcal{R}) \) is not commutative. For finite generation, it suffices to show that

\[
T_{e+1} = T_1 \star T_e, \quad \text{for each } e \geq 1.
\]

Set \( q = p^e \) and consider the elements

\[
x^iy^{p-1-i} \in T_i, \quad 0 \leq i \leq p-1 \quad \text{and} \quad x^iy^{q-1-j} \in T_e, \quad 0 \leq j \leq q-1.
\]

Then \( T_1 \star T_e \) contains the elements

\[
(x^iy^{p-1-i}) \star (x^iy^{q-1-j}) = x^{i+p+j}y^{p+q-p-j-i-1},
\]

for \( 0 \leq i \leq p-1 \) and \( 0 \leq j \leq q-1 \), and these are readily seen to span \( T_{e+1} \). Hence, the degree \( p-1 \) monomials in \( x \) and \( y \) generate \( T(\mathcal{R}) \) as a ring extension of \( \mathbb{F} \).
(3) For a polynomial ring $\mathcal{R}$ of dimension 3 or higher, the ring $\mathcal{T} (\mathcal{R})$ is noncommutative and not finitely generated over $\mathbb{F}$. The noncommutativity is immediate from (2); we give an argument that $\mathcal{T} (\mathcal{R})$ is not finitely generated for $\mathcal{R} = \mathbb{F} [x, y, z]$, and this carries over to polynomial rings $\mathcal{R}$ of higher dimension.

Set $q = p^e$ where $e \geq 2$. We claim that the element

$$xy^q/p-1z^q/p-1 \in \mathcal{T}_e$$

does not belong to $\mathcal{T}_{e_1} \ast \mathcal{T}_{e_2}$ for integers $e_i < e$ with $e_1 + e_2 = e$. Indeed, $\mathcal{T}_{e_1} \ast \mathcal{T}_{e_2}$ is spanned by the monomials

$$(x^i y^j z^{q_1-i-j}) \ast (x^k y^l z^{q_2-k-l}) = x^{i+q_1} y^{j+q_1} z^{-q_1-i-j-q_1-k-l-1}$$

where $q_i = p^{e_i}$ and

$$0 \leq i \leq q_1 - 1, \quad 0 \leq j \leq q_1 - 1 - i, \quad 0 \leq k \leq q_2 - 1, \quad 0 \leq l \leq q_2 - 1 - k,$$

so it suffices to verify that the equations

$$i + q_1k = 1 \quad \text{and} \quad j + q_1l = q/p - 1$$

have no solution for integers $i, j, k, l$ in the intervals displayed above. The first of the equations gives $i = 1$, which then implies that $0 \leq j \leq q_1 - 2$. Since $q_1$ divides $q/p$, the second equation gives $j \equiv -1 \mod q_1$. But this has no solution with $0 \leq j \leq q_1 - 2$.

3. The ring structure of $\mathcal{F}(E)$

We describe the ring of Frobenius operators $\mathcal{F}(E)$ in terms of the symbolic Rees algebra $\mathcal{R}$ and the twisted multiplication structure $\mathcal{T} (\mathcal{R})$ of the previous section. First, a notational point: $\omega^{[p^e]}$ below denotes the iterated Frobenius power of an ideal $\omega$, and $\omega^{(n)}$ its symbolic power, which coincides with reflexive power for divisorial ideals $\omega$. We realize that the notation $\omega^{[n]}$ is sometimes used for the reflexive power, hence this note of caution. We start with the following observation:

**Lemma 3.1.** Let $(R, \mathfrak{m})$ be a normal local ring of characteristic $p > 0$. Let $\omega$ be a divisorial ideal of $R$, i.e., an ideal of pure height one. Then for each integer $e \geq 1$, the map

$$H^\dim R_m (\omega^{[p^e]}) \longrightarrow H^\dim R_m (\omega^{(p^e)})$$

induced by the inclusion $\omega^{[p^e]} \subseteq \omega^{(p^e)}$, is an isomorphism.

**Proof.** Set $d = \dim R$. Since $R$ is normal and $\omega$ has pure height one, $\omega R_p$ is principal for each prime ideal $p$ of height one; hence $(\omega^{(p^e)}/\omega^{[p^e]}) R_p = 0$. It follows that

$$\dim (\omega^{(p^e)}/\omega^{[p^e]}) \leq d - 2,$$

which gives the vanishing of the outer terms of the exact sequence

$$H^\dim -1 _m (\omega^{(p^e)}/\omega^{[p^e]}) \longrightarrow H^d _m (\omega^{[p^e]}) \longrightarrow H^d _m (\omega^{(p^e)}) \longrightarrow H^d _m (\omega^{(p^e)}/\omega^{[p^e]}),$$

and thus the desired isomorphism.
Definition 3.2. Let $R$ be a normal ring that is either complete local, or $\mathbb{N}$-graded and finitely generated over $R_0$. Let $\omega$ denote the canonical module of $R$. The symbolic Rees algebra

$$\mathcal{R} = \bigoplus_{n \geq 0} \omega(-n)$$

is the anticanonical cover of $R$; it has a natural $\mathbb{N}$-grading where $\mathcal{R}_n = \omega(-n)$.

Theorem 3.3. Let $(R, m)$ be a normal complete local ring of characteristic $p > 0$. Set $d$ to be the dimension of $R$. Let $\omega$ denote the canonical module of $R$, and identify $E$, the injective hull of the $R/\mathfrak{m}$, with $H^d_m(\omega)$.

(1) Then $\mathcal{F}(E)$, the ring of Frobenius operators on $E$, may be identified with

$$\bigoplus_{e \geq 0} \omega^{(1-\varphi^e)} F^e,$$

where $F^e$ denotes the map $H^d_m(\omega) \to H^d_m(\omega^{(\varphi^e)})$ induced by $\omega \to \omega^{(\varphi^e)}$.

(2) Let $\mathcal{R}$ be the anticanonical cover of $R$. Then one has an isomorphism of graded rings

$$\mathcal{F}(E) \cong \mathcal{T}(\mathcal{R}),$$

where $\mathcal{T}(\mathcal{R})$ is as in Definition 2.1.

Proof. By Remark 1.3, we have

$$\mathcal{F}^{\varphi}(H^d_m(\omega)) \cong \text{Hom}_R \left( R^{(e)} \otimes_R H^d_m(\omega), H^d_m(\omega) \right).$$

Moreover,

$$R^{(e)} \otimes_R H^d_m(\omega) \cong H^d_m(\omega^{(\varphi^e)}) \cong H^d_m(\omega^{(\varphi^e)}),$$

where the first isomorphism of by [ILL⁺, exercise 9.7], and the second by Lemma 3.1. By similar arguments

$$\text{Hom}_R \left( H^d_m(\omega^{(\varphi^e)}), H^d_m(\omega) \right) \cong \text{Hom}_R \left( H^d_m(\omega \otimes_R \omega^{(\varphi^{e-1})}), H^d_m(\omega) \right)$$

$$\cong \text{Hom}_R \left( \omega^{(\varphi^{e-1})} \otimes_R H^d_m(\omega), H^d_m(\omega) \right)$$

$$\cong \text{Hom}_R \left( \omega^{(\varphi^{e-1})}, \text{Hom}_R \left( H^d_m(\omega), H^d_m(\omega) \right) \right),$$

with the last isomorphism using the adjointness of Hom and tensor. Since $R$ is complete, the module above is isomorphic to

$$\text{Hom}_R \left( \omega^{(\varphi^{e-1})}, R \right) \cong \omega^{(1-\varphi^e)}.$$

Suppose $\varphi \in \mathcal{F}^{\varphi}(M)$ and $\varphi' \in \mathcal{F}^{\varphi}(M)$ correspond respectively to $aF^e$ and $a'F^e$, for elements $a \in \omega^{(1-\varphi^e)}$ and $a' \in \omega^{(1-\varphi^e)}$. Then $\varphi \circ \varphi'$ corresponds to $aF^e \circ bF^e = ab^{\varphi^e} F^{e+e}$, which agrees with the ring structure of $\mathcal{T}(\mathcal{R})$ since $a \neq b = ab^{\varphi^e}$.

Remark 3.4. Let $R$ be a normal complete local ring of prime characteristic $p$; let $A$ be a complete regular local ring with $R = A/I$. Using Remark 1.5 and Theorem 3.3, it is now a straightforward verification that $\mathcal{F}(E)$ is isomorphic, as a graded ring, to

$$\bigoplus_{e \geq 0} \frac{I^{[\varphi^e]}}{I^{[\varphi^e]}} :_A I$$

where the multiplication on this latter ring is the twisted multiplication $\ast$. An example of the isomorphism is worked out in Proposition 5.1.
4. \( \mathbb{Q} \)-Gorenstein rings

We analyze the finite generation of \( \mathcal{F}(E) \) when \( R \) is \( \mathbb{Q} \)-Gorenstein. The following result follows from the corresponding statement for Cartier algebras, [Sc, remark 4.5], but we include it here for the sake of completeness:

**Proposition 4.1.** Let \((R, m)\) be a normal \( \mathbb{Q} \)-Gorenstein local ring of prime characteristic. Let \( \omega \) denote the canonical module of \( R \). If the order of \( \omega \) is relatively prime to the characteristic of \( R \), then \( \mathcal{F}(E) \) is a finitely generated ring extension of \( \mathcal{F}^0(E) \).

**Proof.** Since \( \mathcal{F}^0(E) \) is isomorphic to the \( m \)-adic completion of \( R \), the proposition reduces to the case where the ring \( R \) is assumed to be complete.

Let \( m \) be the order of \( \omega \), and \( p \) the characteristic of \( R \). Then \( p \mod m \) is an element of the group \( (\mathbb{Z}/m\mathbb{Z})^\times \), and hence there exists an integer \( e_0 \) with \( pe_0 \equiv 1 \mod m \). We claim that \( \mathcal{F}(E) \) is generated over \( \mathcal{F}^0(E) \) by \( e_0 \). We use the identification \( \mathcal{F}(E) = T(R) \) from Theorem 3.3. Since \( \omega(n) \) is a cyclic module, one has

\[
\omega^{(n+km)} = \omega^{(n)} \omega^{(km)}, \quad \text{for all integers } k, n.
\]

Thus, for each \( e > e_0 \), one has

\[
\mathcal{T}_e = \mathcal{T}_{e_0} \ast \mathcal{T}_{e_0} = \omega^{(1-p^{-e_0})} \ast \omega^{(1-p^n)}
\]

\[
= \omega^{(1-p^{-e_0})} \cdot (\omega^{(1-p^n)})^{[p^{-e_0}]}
\]

\[
= \omega^{(1-p^{-e_0})} \cdot \omega^{(p^{-e_0}(1-p^n))}
\]

\[
= \omega^{(1-p^{-e_0}+p^{-e_0}-p)}
\]

\[
= \omega^{(1-p)}
\]

\[
= \mathcal{T}_e,
\]

which proves the claim.

We conjecture that Proposition 4.1 has a converse in the following sense:

**Conjecture 4.2.** Let \((R, m)\) be a normal \( \mathbb{Q} \)-Gorenstein ring of prime characteristic, such that the order of the canonical module in the divisor class group is a multiple of the characteristic of \( R \). Then \( \mathcal{F}(E) \) is not a finitely generated ring extension of \( \mathcal{F}^0(E) \).

**Veronese subrings.** Let \( \mathbb{F} \) be a field of characteristic \( p > 0 \), and \( A = \mathbb{F}[x_1, \ldots, x_d] \) a polynomial ring. Given a positive integer \( n \), we denote the \( n \)-th Veronese subring of \( A \) by

\[
A_{(n)} = \bigoplus_{k \geq 0} A_{nk};
\]

this differs from the standard notation, e.g., [GW], since we reserve superscripts \((\ )^{(n)}\) for symbolic powers. The cyclic module \( x_1 \cdots x_d A \) is the graded canonical module for the polynomial ring \( A \). By [GW, corollary 3.1.3], the Veronese submodule

\[
(x_1 \cdots x_d A)_{(n)} = \bigoplus_{k \geq 0} [x_1 \cdots x_d A]_{nk}
\]
is the graded canonical module for the subring $A_{(n)}$. Let $m$ denote the homogeneous maximal ideal of $A_{(n)}$. The injective hull of $A_{(n)}/m$ in the category of graded $A_{(n)}$-modules is

$$H^d_m\left((x_1 \cdots x_d A)_{(n)}\right) = \left[H^d_m\left(x_1 \cdots x_d A\right)\right]_{(n)} = \left[\sum_i x_1 \cdots x_d A_{x_1 \cdots x_d}\right]_{(n)},$$

see [GW, theorem 3.1.1]. By [GW, theorem 1.2.5], this is also the injective hull in the category of all $A_{(n)}$-modules.

Let $R$ be the $m$-adic completion of $A_{(n)}$. As it is $m$-torsion, the module displayed above is also an $R$-module; it is the injective hull of $R/mR$ in the category of $R$-modules.

**Proposition 4.3.** Let $F$ be a field of characteristic $p > 0$, and let $A = \mathbb{F}[x_1, \ldots, x_d]$ be a polynomial ring of dimension $d$. Let $n$ be a positive integer, and $R$ be the completion of the $n$-th Veronese subring of $A$ at its homogeneous maximal ideal. Set $E = M/N$ where

$$M = R^e_{x_1 \cdots x_d}$$

and $N$ is the $R$-submodule spanned by elements $x_1^{i_1} \cdots x_d^{i_d} \in M$ with $i_k \geq 1$ for some $k$; the module $E$ is the injective hull of the residue field of $R$.

Then $F^e(E)$ is the left $R$-module generated by the elements

$$1_{x_1^{a_1} \cdots x_d^{a_d}} F^e,$$

where $F$ is the $p$th power map, $a_k \leq p^e - 1$ for each $k$, and $\sum a_k \equiv 0 \mod n$.

**Remark 4.4.** We use $F$ for the Frobenius endomorphism of the ring $M$. The condition $\sum a_k \equiv 0 \mod n$, or equivalently $x_1^{a_1} \cdots x_d^{a_d} \in M$, implies that

$$1_{x_1^{a_1} \cdots x_d^{a_d}} F^e \in \mathcal{F}^e(M).$$

When $a_k \leq p^e - 1$ for each $k$, the map displayed above stabilizes $N$ and thus induces an element of $\mathcal{F}^e(M/N)$; we reuse $F$ for the $p$th power map on $M/N$.

**Proof of Proposition 4.3** In view of the above remark, it remains to establish that the given elements are indeed generators for $\mathcal{F}^e(E)$. The canonical module of $R$ is

$$\omega_R = (x_1 \cdots x_d A)_{(n)}R$$

and, indeed, $H^d_m(\omega_R) = E$. Thus, Theorem 3.3 implies that

$$\mathcal{F}^e(E) = \omega_R^{(1-q)} F^e,$$

where $q = p^e$. But $\omega_R^{(1-q)}$ is the completion of the $A_{(n)}$-module

$$\left[\frac{1}{x_1^{q-1} \cdots x_d^{q-1}} A\right]_{(n)} = \left[\frac{1}{x_1^{a_1} \cdots x_d^{a_d}} A_{x_1 \cdots x_d} \mid a_k \leq q - 1 \text{ for each } k, \sum a_k \equiv 0 \mod n\right] A_{(n)},$$

which completes the proof.

**Example 4.5.** Consider $d = 2$ and $n = 3$ in Proposition 4.3, i.e.,

$$R = \mathbb{F}[x^3, x^2y, xy^2, y^3].$$
Then $\omega = (x^2 y, x y^2) R$ has order 3 in the divisor class group of $R$; indeed,

$$\omega^{(2)} = (x^4 y^2, x^3 y^3, x^2 y^4) R \quad \text{and} \quad \omega^{(3)} = (x^3 y^3) R.$$ 

(1) If $p \equiv 1 \mod 3$, then $\omega^{(1-q)} = (xy)^{1-q} R$ is cyclic for each $q = p^e$ and

$$\mathcal{F}^e(E) = \frac{1}{(xy)^{q-1}} F^e.$$ 

Since

$$\frac{1}{(xy)^{p-1}} F \circ \frac{1}{(xy)^{q-1}} F^e = \frac{1}{(xy)^{pq-1}} F^{e+1},$$

it follows that

$$\mathcal{F}(E) = R \left\{ \frac{1}{(xy)^{p-1}} F \right\}.$$ 

(2) If $p \equiv 2 \mod 3$ and $q = p^e$, then $\omega^{(1-q)} = (xy)^{1-q} R$ for $e$ even and

$$\omega^{(1-q)} = \left( \frac{1}{x^{q^2} y^{q-1}}, \frac{1}{x^{q-2} y^{q-2}}, \frac{1}{x^{q-1} y^{q-3}} \right) R$$

for $e$ odd. The proof of Proposition 4-1 shows that $\mathcal{F}(E)$ is generated by its elements of degree $\leq 2$ and hence

$$\mathcal{F}(E) = R \left\{ \frac{1}{x^{p-1} y^{p-1}} F, \frac{1}{x^{p-2} y^{p-2}} F, \frac{1}{x^{p-1} y^{p-3}} F, \frac{1}{x^{p-1} y^{p-1}} F^2 \right\}.$$ 

In the case $p = 2$, the above reads

$$\mathcal{F}(E) = R \left\{ \frac{x}{y} F, \frac{y}{x} F, \frac{1}{x^3 F^2} \right\}.$$ 

(3) When $p = 3$, one has

$$\omega^{(1-q)} = \frac{1}{x^{q^2} y^{q}} (x^2 y, x y^2) R = \left( \frac{1}{x^{q-2} y^{q-1}}, \frac{1}{x^{q-1} y^{q-2}} \right) R$$

for each $q = p^e$. In this case,

$$\mathcal{F}(E) = R \left\{ \frac{1}{x^2 y} F, \frac{1}{x^2 y} F, \frac{1}{x^7 y^8} F^2, \frac{1}{x^8 y^7} F^2, \frac{1}{x^{25} y^{26}} F^3, \frac{1}{x^{26} y^{25}} F^3, \ldots \right\},$$

and $\mathcal{F}(E)$ is not a finitely generated extension ring of $\mathcal{F}^0(E) = R$; indeed,

$$\omega^{(1-q)} \ast \omega^{(1-q')} = \frac{1}{x^{q^2} y^{q}} (x^2 y, x y^2) R \ast \frac{1}{x^{q^2} y^{q}} (x^2 y, x y^2) R$$

for $q = p^e$ and $q' = p^{e'}$, where $e$ and $e'$ are positive integers.
5. A determinantal ring

Let $R$ be the determinantal ring $\mathbb{F}[X]/I$, where $X$ is a $2 \times 3$ matrix of variables over a field of characteristic $p > 0$, and $I$ is the ideal generated by the size 2 minors of $X$. Set $m$ to be the homogeneous maximal ideal of $R$. We show that the algebra of Frobenius operators $\mathcal{F}(E)$ is not finitely generated over $\mathcal{F}^0(E) = \hat{R}$; this proves [Ka, conjecture 3.1]. We also extend Fedder’s calculation of the ideals $I^{[p]} : I$ to the ideals $I^{[q]} : I$ for all $q = p^e$.

The ring $R$ is isomorphic to the affine semigroup ring

$$\mathbb{F}\left[ \frac{sx, sy, sz}{tx, ty, tz} \right] \subseteq \mathbb{F}[s, t, x, y, z].$$

Using this identification, $R$ is the Segre product $A \# B$ of the polynomial rings $A = \mathbb{F}[s, t]$ and $B = \mathbb{F}[x, y, z]$. By [GW, theorem 4.3.1], the canonical module of $R$ is the Segre product of the graded canonical modules $stA$ and $xyzB$ of the respective polynomial rings, i.e.,

$$\omega_R = stA \# xyzB = (s^2txyz, st^2xyz)\hat{R}.$$

Let $e$ be a nonnegative integer, and $q = p^e$. Then

$$\omega_R^{(1-q)} = \frac{1}{(st)^{q-1}}A \# \frac{1}{(xyz)^{q-1}}B$$

is the $R$ module spanned by the elements

$$\frac{1}{(st)^{q-1}x^k y^l z^m}$$

with $k + l + m = 2q - 2$ and $k, l, m \leq q - 1$.

View $E$ as $M/N$ where $M = R_{s^2txyz}$, and $N$ is the $R$-submodule spanned by the elements $s^i t^j x^k y^l z^m$ in $M$ that have at least one positive exponent. Then $\mathcal{F}^e(E)$ is the left $\hat{R}$-module generated by

$$\frac{1}{(st)^{q-1}x^k y^l z^m}F^e,$$

where $F$ is the $p$th power map, $k + l + m = 2q - 2$, and $k, l, m \leq q - 1$. Using this description, it is an elementary—though somewhat tedious—verification that $\mathcal{F}(E)$ is not finitely generated over $\mathcal{F}^0(E)$; alternatively, note that the symbolic powers of the height one prime ideals $(sx, sy, sz)\hat{R}$ and $(sx, tx)\hat{R}$ agree with the ordinary powers by [BV, corollary 7.10]. Thus, the anticanonical cover of $\hat{R}$ is the ring $\mathcal{R}$ with

$$\mathcal{R}_n = \frac{1}{(s^2txyz)^n}(sx, sy, sz)^n\hat{R}$$

and so

$$\mathcal{T}_e = \frac{1}{(s^2txyz)^{q-1}}(sx, sy, sz)^{q-1}\hat{R}.$$
Thus,
\[ T_{e_1} \ast T_{e_2} = \frac{1}{(s^2txyz)^{q_i-1}}(sx, sy, sz)^{q_i-1} \ast \frac{1}{(s^2txyz)^{q_j-1}}(sx, sy, sz)^{q_j-1} \]
\[ = \frac{1}{(s^2txyz)^{q_i+q_j-1}}(sx, sy, sz)^{q_i-1} \cdot ((sx, sy, sz)^{q_j-1})^{[q_1]} \]
\[ = \frac{1}{(s^2txyz)^{q_i+q_j-1}}(sx, sy, sz)^{q_i-1} \cdot ((sx)^{q_i}, (sy)^{q_i}, (sz)^{q_i})^{q_j-1} \]

where \( q_i = p^{e_i} \). We claim that
\[ T_e = \sum_{e_1=1}^{e-1} T_{e_1} \ast T_{e-e_1}. \]

For this, it suffices to show that
\[ \frac{1}{(s^2txyz)^{q_i-1}}sx(sy)^{q_i/p-1}(sz)^{q_j/p-1} \]
does not belong to \( T_{e_1} \ast T_{e_2} \) for integers \( e_1 < e \) with \( e_1 + e_2 = e \). By the description of \( T_{e_1} \ast T_{e_2} \) above, this is tantamount to proving that
\[ sx(sy)^{q_i/p-1}(sz)^{q_j/p-1} \notin (sx, sy, sz)^{q_i-1} \cdot ((sx)^{q_i}, (sy)^{q_i}, (sz)^{q_i})^{q_j-1}, \]
but this is essentially Example 2.2.3.

**Fedder’s computation.** Let \( A \) be the power series ring \( \mathbb{F}[[u, v, w, x, y, z]] \) for \( \mathbb{F} \) a field of characteristic \( p > 0 \), and let \( I \) be the ideal generated by the size 2 minors of the matrix
\[
\begin{pmatrix}
  u & v & w \\
  x & y & z
\end{pmatrix},
\]
In [Fe, proposition 4.7], Fedder shows that
\[ I^{[p]} : I = I^{2p-2} + I^{[p]}. \]

We extend this next by calculating the ideals \( I^{[q]} : I \) for each prime power \( q = p^e \).

**Proposition 5.1.** Let \( A \) be the power series ring \( \mathbb{F}[[u, v, w, x, y, z]] \) where \( K \) a field of characteristic \( p > 0 \). Let \( I \) be the ideal of \( A \) generated by \( \Delta_1 = vz - wy, \Delta_2 = wx - uz, \) and \( \Delta_3 = uy - vx. \)

1. For \( q = p^e \) and nonnegative integers \( s, t \) with \( s + t \leq q - 1 \), one has
\[ y^s z^t (\Delta_2 \Delta_3)^{q-1} \in I^{[q]} + x^{s+t} A. \]

2. For \( q, s, t \) as above, let \( f_{s, t} \) be an element of \( A \) with
\[ y^s z^t (\Delta_2 \Delta_3)^{q-1} = x^{s+t} f_{s, t} \mod I^{[q]}. \]

Then \( f_{s, t} \) is well-defined modulo \( I^{[q]} \). Moreover, \( f_{s, t} \in I^{[q]} : A, I, \) and
\[ I^{[q]} : A I = I^{[q]} + (f_{s, t} | s + t \leq q - 1) A. \]

For \( q = p \), the above recovers Fedder’s computation that \( I^{[p]} : I = I^{2p-2} + I^{[p]} \), though for \( q > p \), the ideal \( I^{[p]} : I \) is strictly bigger than \( I^{2p-2} + I^{[p]} \).
Proof. (1) Note that the element
\[ y^s z'(\Delta_2 \Delta_3)^{q-1} = y^s z'(w x - u z)^{q-1} (u y - v x)^{q-1} \]
belongs to the ideals
\[ (x, u)^{2q-2} \subseteq (x^{q-1}, u^q) \subseteq (x^{s+t}, u^q) \]
and also to
\[ y^s z'(x, z)^{q-1} (x, y)^{q-1} \subseteq y^s z'(x^t, z^{q-t})(x^s, y^{q-t}) \subseteq (x^{s+t}, z^q, y^q). \]
Hence,
\[ y^s z'(\Delta_2 \Delta_3)^{q-1} \in (x^{s+t}, u^q)A \cap (x^{s+t}, z^q, y^q)A \]
\[ = (x^{s+t}, u^q z^q, u^q y^q)A \]
\[ \subseteq (x^{s+t}, \Delta_1^q, \Delta_2^q, \Delta_3^q)A. \]

(2) The ideals \( I \) and \( I^{[q]} \) have the same associated primes, [ILL⁺, corollary 21-11]. As \( I \) is prime, it is the only prime associated to \( I^{[q]} \). Hence \( x^{s+t} \) is a nonzerodivisor modulo \( I^{[q]} \), and it follows that \( f_{s,t} \mod I^{[q]} \) is well-defined.

We next claim that
\[ I^{2q-1} \subseteq I^{[q]}. \]
By the earlier observation on associated primes, it suffices to verify this in the local ring \( R_I \). But \( R_I \) is a regular local ring of dimension 2, so \( I R_I \) is generated by two elements, and the claim follows from the pigeonhole principle. The claim implies that
\[ x^{s+t} f_{s,t} I \in I^{[q]}, \]
and using, again, that \( x^{s+t} \) is a nonzerodivisor modulo \( I^{[q]} \), we see that \( f_{s,t} I \subseteq I^{[q]} \), in other words, that \( f_{s,t} I^{[q]} :_A I \) as desired.

By Theorem 3.3 and Remark 3.4, one has the \( R \)-module isomorphisms
\[ \omega^{(1-q)}_R \cong \mathcal{F}^o(E) \cong \frac{I^{[q]} :_A I}{I^{[q]}}. \]
Choosing \( \omega^{(-)}_R = (x, y, z)R \), we claim that the map
\[ h : (x, y, z)^{q-1} R \rightarrow \frac{I^{[q]} :_A I}{I^{[q]}}, \]
\[ x^{q-1-s-t} y^s z^t \mapsto f_{s,t} \]
is an isomorphism. Since the modules in question are reflexive \( R \)-modules of rank one, it suffices to verify that the map is an isomorphism in codimension 1. Upon inverting \( x \), the above map induces
\[ Rx \rightarrow \frac{I^{[q]} A_x :_{A_x} I A_x}{I^{[q]} A_x}, \]
\[ x^{q-1} \mapsto (\Delta_2 \Delta_3)^{q-1} \]
which is readily seen to be an isomorphism since \( I A_x = (\Delta_2, \Delta_3) A_x \).
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6. Cartier algebras and gauge boundedness

For a ring $R$ of prime characteristic $p > 0$, one can interpret $\mathcal{F}^e(E)$ in a dual way as a collection of $p^{-e}$-linear operators on $R$. This point of view was studied by Blickle [Bl12] and Schwede [Sc].

**Definition 6.1.** Let $R$ be a ring of prime characteristic $p > 0$. For each $e \geq 0$, set $C^R_e$ to be set of additive maps $\varphi : R \to R$ satisfying

$$\varphi(p^r x) = r \varphi(x), \quad \text{for } r, x \in R.$$  

The *total Cartier algebra* is the direct sum

$$C^R = \bigoplus_{e \geq 0} C^R_e.$$  

For $\varphi \in C^R_e$ and $\varphi' \in C^R_e$, the compositions $\varphi \circ \varphi'$ and $\varphi' \circ \varphi$ are elements of $C^R_{e+e'}$. This gives $C^R$ the structure of an $\mathbb{N}$-graded ring; it is typically not a commutative ring. As pointed out in [ABZ, 2.2.1], if $(R, m)$ is an $F$-finite complete local ring, then the ring of Frobenius operators $\mathcal{F}(E)$ is isomorphic to $C^R$.

Each $C^R_e$ has a left and a right $R$-module structure: for $\varphi \in C^R_e$ and $r \in R$, we define $r \cdot \varphi$ to be the map $x \mapsto r \varphi(x)$, and $\varphi \cdot r$ to be the map $x \mapsto \varphi(r x)$.

**Definition 6.2.** Blickle [Bl12] introduced a notion of boundedness for Cartier algebras: Let $R = A/I$ for a polynomial ring $A = \mathbb{F}[x_1, \ldots, x_d]$ over an $F$-finite field $\mathbb{F}$. Set $R_n$ to be the finite dimensional $\mathbb{F}$-vector subspace of $R$ spanned by the images of the monomials

$$x_1^{\lambda_1} \cdots x_d^{\lambda_d}, \quad \text{for } 0 \leq \lambda_j \leq n.$$  

Following [An] and [Bl12], we define a map $\delta : R \to \mathbb{Z}$ by $\delta(r) = n$ if $r \in R_n \smallsetminus R_{n-1}$; the map $\delta$ is a gauge. If $I = 0$, then $\delta(r) \leq \deg(r)$ for each $r \in R$. We recall some properties from [An, proposition 1] and [Bl12, lemma 4.2]:

$$\delta(r + r') \leq \max\{\delta(r), \delta(r')\},$$  

$$\delta(r \cdot r') \leq \delta(r) + \delta(r').$$  

The ring $C^R$ is *gauge bounded* if there exists a constant $K$, and elements $\varphi_{e,i}$ in $C^R_e$ for each $e \geq 1$ generating $C^R_e$ as a left $R$-module, such that

$$\delta(\varphi_{e,i}(x)) \leq \frac{\delta(x)}{p^e} + K, \quad \text{for each } e \text{ and } i.$$  

**Remark 6.3.** We record two key facts that will be used in our proof of Theorem 6.4:

1. If there exists a constant $C$ such that $I^{[p^e]} : A I$ is generated by elements of degree at most $C p^e$ for each $e \geq 1$, then $C^R$ is gauge bounded; this is [KZ, lemma 2.2].

2. If $C^R$ is gauge bounded, then for each ideal $\mathfrak{a}$ of $R$, the $F$-jumping numbers of $\tau(R, \mathfrak{a})$ are a subset of the real numbers with no limit points; in particular, they form a discrete set. This is [Bl12, theorem 4.18].

We now prove the main result of the section:

**Theorem 6.4.** Let $R$ be a normal $\mathbb{N}$-graded that is finitely generated over an $F$-finite field $R_0$. (The ring $R$ need not be standard graded.)

Suppose that the anticanonical cover of $R$ is finitely generated as an $R$-algebra. Then $C^R$ is gauge bounded. Hence, for each ideal $\mathfrak{a}$ of $R$, the set of $F$-jumping numbers of $\tau(R, \mathfrak{a})$ is a subset of the real numbers with no limit points.
Proof. Let \( A \) be a polynomial ring, with a possibly non-standard \( \mathbb{N} \)-grading, such that \( R = A/I \). It suffices to obtain a constant \( C \) such that the ideals \( I^{[p^e]} : A \mathcal{I} \) are generated by elements of degree at most \( C^p e \) for each \( e \geq 1 \).

There exists a ring isomorphism \( \bigoplus_{e \geq 0} \omega^{(1-p^e)} \cong \bigoplus_{e \geq 0} (I^{[p^e]} : A \mathcal{I})/I^{[p^e]} \) by Remark 3.4 that respects the \( e \)th graded components. After replacing \( \omega \) by an isomorphic \( R \)-module with a possible graded shift, we may assume that the isomorphism above induces degree preserving \( R \)-module isomorphisms for each \( e \geq 0 \). While \( \omega \) is no longer canonically graded, we still have the finite generation of the anticanonical cover \( \bigoplus_{n \geq 0} \omega^{(-n)} \). It suffices to check that there exists a constant \( C \) such that \( \omega^{(1-p^e)} \) is generated, as an \( R \)-module, by elements of degree at most \( C^p e \).

Choose finitely many homogeneous \( R \)-algebra generators \( z_1, \ldots, z_k \) for \( \bigoplus_{n \geq 0} \omega^{(-n)} \), say with \( z_i \in \omega^{(-j_i)} \). Set \( C \) to be the maximum of \( \deg z_1, \ldots, \deg z_k \). Then the monomials

\[
z^\lambda = z_1^{\lambda_1} z_2^{\lambda_2} \cdots z_k^{\lambda_k}, \quad \text{with} \quad \sum \lambda_i j_i = p^e - 1
\]
generate the \( R \)-module \( \omega^{(1-p^e)} \), and it is readily seen that

\[
\deg z^\lambda = \sum \lambda_i \deg z_i \leq C \sum \lambda_i \leq C(p^e - 1).
\]

By [KZ, lemma 2-2], it follows that \( C^R \) is gauge bounded; the assertion now follows from [Bl2, theorem 4.18].

Corollary 6.5. Let \( R \) be the determinantal ring \( \mathbb{F}[X]/I \), where \( X \) is a matrix of indeterminates over an \( \mathbb{F} \)-finite field \( \mathbb{F} \) of prime characteristic, and \( I \) is the ideal generated by the minors of \( X \) of an arbitrary but fixed size. Then, for each ideal \( \mathfrak{a} \) of \( R \), the set of \( \mathbb{F} \)-jumping numbers of \( \tau(R, \mathfrak{a}^t) \) is a subset of the real numbers with no limit points.

Proof. Since \( R \) is a determinantal ring, the symbolic powers of the ideal \( \omega^{(-1)} \) agree with the ordinary powers by [BV, corollary 7.10]. Hence the anticanonical cover of \( R \) is finitely generated, and the result follows from Theorem 6.4.

Remark 6.6. It would be natural to remove the hypothesis that \( R \) is graded in Theorem 6.4. However, we do not know how to do this: when \( R \) is not graded, it is unclear if one can choose gauges that are compatible with the ring isomorphism

\[
\bigoplus_{e \geq 0} \omega^{(1-p^e)} \cong \bigoplus_{e \geq 0} (I^{[p^e]} : A \mathcal{I})/I^{[p^e]}.
\]

7. Linear growth of Castelnuovo–Mumford regularity for rings of finite Frobenius representation type

Let \( A \) be a standard graded polynomial ring over a field \( \mathbb{F} \), with homogeneous maximal ideal \( \mathfrak{m} \). We recall the definition of the Castelnuovo-Mumford regularity of a graded module following [Ei, chapter 4]:

Definition 7.1. Let \( M = \bigoplus_{d \in \mathbb{Q}} M_d \) be a graded \( A \)-module. If \( M \) is Artinian, we set

\[
\reg M = \max\{d \mid M_d \neq 0\};
\]

for an arbitrary graded module we define

\[
\reg M = \max_{k \geq 0} \{\reg H^k_{\mathfrak{m}}(M) + k\}.
\]
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Definition 7.2. Let $I$ and $J$ be homogeneous ideals of $A$. We say that the regularity of $A/(I + J^{[p^e]})$ has linear growth with respect to $p^e$, if there is a constant $C$, such that

$$\text{reg} \ A/(I + J^{[p^e]}) \leq Cp^e, \quad \text{for each } e \geq 0.$$  

It follows from [KZ, corollary 2.4] that if $\text{reg} \ A/(I + J^{[p^e]})$ has linear growth for each homogeneous ideal $J$, then $C^{A/J}$ is gauge-bounded.

Remark 7.3. Let $R = A/I$ for a homogeneous ideal $I$. We define a grading on the bimodule $R^{(e)}$ introduced in Remark 1.3: when an element $r$ of $R$ is viewed as an element of $R^{(e)}$, we denote it by $r^{(e)}$. For a homogeneous element $r \in R$, we set

$$\deg' r^{(e)} = \frac{1}{p^e} \deg r.$$  

For each ideal $J$ of $R$, one has an isomorphism

$$R^{(e)} \otimes_R R/J \xrightarrow{\cong} R/J^{[p^e]}$$  

under which $r^{(e)} \otimes \overline{s} \mapsto rs^{p^e}$. To make this isomorphism degree-preserving for a homogeneous ideal $J$, we define a grading on $R/J^{[p^e]}$ as follows:

$$\deg' \overline{r} = \frac{1}{p^e} \deg \overline{r}, \quad \text{for a homogeneous element } r \text{ of } R.$$  

The notion of finite Frobenius representation type was introduced by Smith and Van den Bergh [SV]; we recall the definition in the graded context:

Definition 7.4. Let $R$ be an $\mathbb{N}$-graded Noetherian ring of prime characteristic $p$. Then $R$ has finite graded Frobenius-representation type by finitely generated $\mathbb{Q}$-graded $R$-modules $M_1, \ldots, M_s$, if for every $e \in \mathbb{N}$, the $\mathbb{Q}$-graded $R$-module $R^{(e)}$ is isomorphic to a finite direct sum of the modules $M_j$ with possible graded shifts, i.e., if there exist rational numbers $\alpha_{ij}^{(e)}$, such that there exists a $\mathbb{Q}$-graded isomorphism

$$R^{(e)} \cong \bigoplus_{i,j} M_j(\alpha_{ij}^{(e)}).$$  

Remark 7.5. Suppose $R$ has finite graded Frobenius-representation type. With the notation as above, there exists a constant $C$ such that

$$\alpha_{ij}^{(e)} \leq C, \quad \text{for all } e, i, j;$$  

see the proof of [TT, theorem 2.9].

We now prove the main result of this section; compare with [TT, theorem 4.8].

Theorem 7.6. Let $A$ be a standard graded polynomial ring over an $F$-finite field of characteristic $p > 0$. Let $I$ be a homogeneous ideal of $A$.

Suppose $R = A/I$ has finite graded $F$-representation type. Then $\text{reg} \ A/(I + J^{[p^e]})$ has linear growth for each homogeneous ideal $J$. In particular, $C^k$ is gauge bounded, and for each ideal $a$ of $R$, the set of $F$-jumping numbers of $\tau(R, a')$ is a subset of the real numbers with no limit points.

Proof. We use $J$ for the ideal of $A$, and also for its image in $R$. Let $a'(H^k_m(R/J^{[p^e]}))$ denote the largest degree of a nonzero element of $H^k_m(R/J^{[p^e]})$ under the $\deg'$-grading, i.e.,

$$a'(H^k_m(R/J^{[p^e]})) = \frac{1}{p^e} \text{reg} H^k_m(R/J^{[p^e]}).$$
Since we have degree-preserving isomorphisms \( R^{(e)} \otimes_R R/J \cong R/J^{[p^e]} \), and
\[
R^{(e)} \cong \bigoplus_{i,j} M_i(\alpha_{ij}^{(e)}),
\]
it follows that
\[
H^k_m(R/J^{[p^e]}) \cong H^k_m(R^{(e)} \otimes_R R/J)
\]
\[
\cong \bigoplus_{i,j} H^k_m(M_i(\alpha_{ij}^{(e)} \otimes_R R/J))
\]
\[
\cong \bigoplus_{i,j} H^k_m(M_i/J M_i)(\alpha_{ij}^{(e)}).
\]
The numbers \( \alpha_{ij}^{(e)} \) are bounded by Remark 7.5; thus,
\[
\alpha'(H^k_m(R/J^{[p^e]})) \leq \max_i \{ \alpha'(H^k_m(M_i/J M_i)) + C \}.
\]
Since there are only finitely many modules \( M_i \) and finitely many homological indices \( k \), it follows that \( \alpha'(H^k_m(R/J^{[p^e]})) \leq C' \), where \( C' \) is a constant independent of \( e \) and \( k \). Hence
\[
\operatorname{reg} H^k_m(R/J^{[p^e]}) \leq C' p^e, \quad \text{for all } e, k,
\]
and so
\[
\operatorname{reg} A/(I + J^{[p^e]}) = \max_k \{ \operatorname{reg} H^k_m(R/J^{[p^e]} + k) \} \leq C' p^e + \dim A.
\]
This proves that \( \operatorname{reg} A/J^{[p^e]} \) has linear growth; \([\text{KZ}], \) corollary 2.4 implies that \( C^R \) is gauge bounded, and the discreteness assertion follows from \([\text{Bl2}], \) theorem 4.18.

REFERENCES

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