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Control of Responses of Smart Plate Structures Under Non-Stationary Random Excitations

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CONTROL OF RESPONSES OF SMART PLATE STRUCTURES UNDER NON-STATIONARY RANDOM EXCITATIONS

by

Xiaojian Yang

A THESIS

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This thesis is concerned with an investigation of the control of responses of plate structures with piezoelectric layers and under complicated excitations modeled as a non-stationary random process. The plate structures and piezoelectric layers are both discretized by the mixed formulation finite element method (FEM).

The investigation consists of three parts. The first part is a literature survey and theoretical development. The second part is the eigenvalue solution and computation of uncontrolled response statistics of laminated plate structures under nonstationary random excitations. The final part is the introduction and application of the stochastic central difference (SCD) method that was presented by To (1986, 2000) for the computation of response statistics. The responses computed by using the SCD method are compared with those obtained by the Runge-Kutta fourth order (RK4) numerical integration algorithm.

In the first part, publications specifically concerned with smart structures under various deterministic and stochastic excitations were reviewed. The theoretical development required for the present investigation are drawn from previous publications and introduced so as to provide a foundation for the response statistics computation subsequently. In this phase of the investigation, the three-node flat triangular piezoelectric shell finite element of To and Liu (2003), and To and Chen (2007) are
applied. This mixed formulation based shell finite element has three nodes every one of which has seven degrees-of-freedom (dof). The latter include three translation, three rotation, and one electric dof. Without the electric dof the laminated composite shell finite element reduces to that developed by To and Wang (1998). The latter laminated composite shell finite element is able to provide correctly the six rigid-body modes.

The latter six rigid-body modes were confirmed in the eigenvalue solution during the second part of the investigation. A cantilever plate structure with and without the piezoelectric layers acting as sensor and actuator was studied. The differences in natural frequencies between the structure with and without the piezoelectric layers are of particular interest. Having verified the correctness of the eigenvalue solution, response statistics of mean squares of displacements are evaluated and compared with those in the literature.

In the third part of the investigation, two computer programs were developed based on the SCD method of To (1986, 2000), and the RK4 algorithm. Comparisons are made between responses with and without the applied voltage to the smart plate structures. The efficiency of computation of responses applying the SCD method and RK4 numerical integration scheme is examined.
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CHAPTER 1 INTRODUCTION

1.1 Motivation

The sensing and actuating using piezoelectric have been investigated during last thirty years. The major method to analyze it is finite element method. The advantages of using piezoelectric sensors are interface circuitry, low cost, high sensitivity, and high bandwidth. Its application in structure health monitoring [1] has been performed by Park et al. The use of piezoelectric elements in strain sensor is investigated in their study. Sirohi [2] et al. (2000) developed a novel piezoelectric strain sensor for simultaneous damping and tracking control of nanopositioner. Irschik (2002) reviewed the static and dynamic shape control of structures by piezoelectric actuation [3]. Liew [32] et al. (2004) studied the shape and vibration control of the functionally graded material plates with piezoelectric sensors and actuators with classical laminated plate theory. He also examined the effects of the constituent volume fractions and the influence of feedback control gain on the static and dynamic responses. Yong [3] et al. (2008) developed the simultaneous sensing and actuation with a piezoelectric tube scanner.

Bilgen [31] et al. (2011) investigated several piezoelectric materials and substrate configurations for optimal design of light weight, low power aerodynamic applications. Piezoelectric power generation actuator was developed for an underwater thrust platform by Erturk and Delporte [7] (2011).

Self-sensing or control using piezoelectric material has promising application
ability in ultrasmall sensing area. Faegh [6] et al. (2013) developed the self-sensing piezoelectric microcantilever biosensor for detection. It detects tip deflection of piezoelectric microcantilever caused by the absorbed ultrasmall mass.

Stochastic dynamic analysis has been applied to multi-degree of freedom systems mainly by using Monte Carlo simulation or statistical equivalent linearization. To obtain the stationary solution for such systems, the spectral density of the response can be computed from the spectral density of the excitation using frequency transfer functions in the frequency domain. A non-stationary solution in the frequency domain and time domain is hardly feasible for larger FE-systems.

For all these studies listed above, there is no work done using finite element method to analyze large degree of freedom system. Especially, dynamic response of system under non-stationary excitation is not available.

1.2 Objectives

The objectives of the present thesis are three folds. The first objective is to study and adopt the mixed formulation finite element method (FEM) developed previously by To and associates [28, 31]. Explicit expressions of the piezoelectric shell finite element stiffness and mass matrices were obtained in [28] and [31], and they were applied in this thesis. The second objective is concerned with the eigenvalue solutions of discretized plate structures. Natural frequencies and mode-shapes are determined for these plate structures. The third and final objective of the present investigation deals with the studies of uncontrolled and controlled vibration responses of the plate structures under pointed non-stationary random excitations. The control of vibration responses is through the
application of the triangular piezoelectric shell finite elements. In order to study the difference between the responses obtained by the stochastic central difference (SCD) method of To and Liu [35] and those determined by another independent algorithm two digital computer program have been employed. The first one based on the SCD method is written in Fortran language while the second computer program based on the Runge-Kutta fourth-order (RK4) has been implemented in MATLAB.

1.3 Literature Survey

1.3.1 Smart plate and shell finite elements

Tzou and Tseng [13] (1990) presented the shell or plate integrated with distributed piezoelectric sensors and actuators using finite element technique. They derived a new piezoelectric finite element with internal degrees of freedom. Control algorithm is provided which is a constant gain feedback control.

Koconis [9] et al. (1993) presented a model to describe composites plate and piezoelectric shells. The analytical method he developed can be used to calculate the changes in shape for specified applied voltages to the actuators. This method was based on two-dimensional, linear, shallow shell theory. Solutions to the governing equations were obtained via the Ritz method.

Hwang and Park [16] (1993) studied the finite element modeling of piezoelectric sensors and actuators. They provided the finite element formulation for vibration control of a laminated plate with piezoelectric sensors/actuators. Four-Node, 12-degree-of-freedom quadrilateral plate bending elements with one electrical degree of freedom is
used in this model. Control moments are induced by piezoelectric actuator at the ends of the actuator. So the placement, number and size of actuators are important for control system design. They presented the static response with various sensor/actuator geometries. Modal state space analysis is studied to derive the damped frequencies and modal damping ratios.

To and Liu [20] (1994) developed a hybrid strain based three-node flat triangular shell elements. The elements are based on the Hellinger-Reissner hybrid strain formulation. It combines a triangular bending element and a plane stress element. It suppresses shear-locking thin limit so that it can be applied to thin and thick shells.

Tzou and Ye [19] in 1996 used the triangle shell elements to analyze laminated piezoelectric structures. Vibration control and effect of actuator length and distribution are investigated. Coupling and control spillover of lower natural modes are also observed.

Reday[10] et al. in 1997 developed a finite-element model based on the classical laminated plate theory for the active vibration control of a composite plate containing distributed piezoelectric sensors and actuators. The formulation is derived from the variational principle. It also takes into account the piezoelectric mass and stiffness. They used a close-loop negative velocity feedback control algorithm coupling the direct and converse piezoelectric effects to control the dynamic response of an integrated structure. The numerical method they used to solve the equation is the Newmark-β method.

Chee [14] et al. in 1998 reviewed the modeling method of structures with piezoelectric sensors and actuators. They went through the linear constitutive equations of piezoelectric from energy consideration, nonlinear models. They briefly introduced the piezoelectric materials and poling such as distributed Polyvinylidene Fluoride (PVDF)
layers and monolithic piezoceramics, piezoelectric rod 1-3 composites, piezoelectric fiber composites and inter-digitated electrode. Theoretical structural model and finite element model of piezoelectric structure are studied.

Chee [11] et al. (1999) presented a theoretical formulation to model composite smart structures in which the piezoelectric actuators and sensors are treated as constituent parts of the entire structural system. The mathematical model is based on a high order displacement field coupled with a layerwise linear electric potential. They used Hamilton’s variational principle and finite element (FE) formulation. In the FE analysis, he used two-Noded Hermitian layerwise Noded element for an n-layered beam. His work direction is only focused on the static beam structure.

Benjeddou [17] 2000 made a first attempt to survey and discussed the formulations and applications of the finite element modeling of adaptive structural elements. It shows that the element types that used in modeling of piezoelectric structure are shell element, quad element and beam element. But only several of them include the drilling degree of freedom.

Ng [18] et al. in 2002 presented a flat-shell element for the active control of functionally graded material shells. It is based on the first-order shear deformation theory. The frequency response characteristics of the functionally graded material shell containing the piezoelectric sensors/actuators are analyzed in the frequency domain.

Pradlwarter et al. [8] 2002 developed a computational procedure to estimate stochastic dynamic response of large non-linear system. He used the Monte Carlo simulation to linearize the non-linear element and equations. He employed so called Karhunen-Loeve vectors to represent the stochastic response.
Yi and Yao [15] in 2002 presented an eight-Node solid-shell finite element models. They focused on resolve the locking problems of the solid-shell elements in laminated materials and improve accuracy. They employed the natural strain method and hybrid stress method.

Della and Shu [12] in 2006 studied the vibration of beams with embedded piezoelectric sensors and actuators using a micromechanics approach. The natural frequency of the beam is determined from the variational principle in Rayleigh quotient form. Eshelby’s equivalent inclusion method is used for including piezoelectric effect. Euler-Bernoulli beam theory and Rayleigh-Ritz approximation technique are used in this analysis. Their results show that the size and volume fraction of piezoelectric inclusions significantly influence the natural frequency of the beam.

1.3.2 Stochastic optimal control strategies

Bismut [21] 1978 derived various maximum principles from a general Pontragin principle for Ito equations. These principles are used for optimal stochastic control. And applications of duality to optimal stochastic control were given. He defined the general system by Ito equation. Generalized stochastic maximum principles were compared with Kushner’s maximum principle. Duality here means that the system is controllable and observable.

Skelton and Holtz [22] 1987 studied the covariance control theory. They introduced a theory for designing linear feedback controllers so that a root-mean-square values or covariance of the system states and output can be achieved.

Suhardjo et al. [36] in 1990 studied a base acceleration resulting from a seismic
activity. The earthquake excitation is based on a Gaussian white noise process. The equation of motion for the structural system is augmented with the modeled Gaussian white noise. Feedback control law is incorporated into the control loop with the observer designed to estimate the states.

To and Chen [23] in 2007 presented an optimal control of random vibration in plate and shell structures with distributed piezoelectric components. They employed the Skelton’s state covariance assignment method to directly achieve control goals in terms of the root-mean-square values. The models are computed using this control method, which are plate and cylindrical shell structures with distributed piezoelectric components and under stationary random excitations.

Nkundineza and To [24] in 2012 investigated a stochastic optimal control method for linear multi-degree-of-freedom systems under nonstationary random excitations. The feedback matrix was designed based on the achievement of the objectives for individual states in the system. Lyapunov equation is solved. They applied this method to two-degree-of-freedom systems representing buildings under earthquake excitations.

### 1.3.3 Responses of discretized plate structures under non-stationary excitations

To and Orisamolu [28] 1986 presented the results of first and second moments of a general two-degree-freedom system subject to both parametric and non-parametric non-stationary random excitations. Each non-stationary random excitation is modeled as a product of a deterministic envelope modulating function and a Gaussian white noise delta-correlated process. The equation of motion is transformed into a standard Ito differential equation. The moment equations are numerically evaluated by the fifth order
Runge-Kutta method.

To and Liu[27] 1994 developed an technique that incorporates stochastic central difference method and time co-ordinate transformation to determine time-dependent variances and covariance of responses of beam and plate structures discretized by finite element method.

CHAPTER 2 DEVELOPMENTS OF SHELL ELEMENTS WITH PIEZOELECTRIC EFFECTS

In this chapter, the focus is on linear dynamic analysis of shell structures with piezoelectric effect. As the geometry of the shell structures in various industries are frequently complicated, analytical solutions are infeasible and therefore the versatile finite element method (FEM) is employed. Before the derivation of the element matrices, the variational principle of multi-field functional will be introduced first in Section 2.1. In Section 2.2, mixed formulation based shell finite elements with piezoelectric effect are introduced. Section 2.3 is concerned with the formulation of the three-node laminated triangular shell element. Derivation of the consistent mass and stiffness matrices of the three-node flat triangular shell element is presented in Section 2.4.

2.1 Variational Principles and Multi-field Functional

To establish the equation of motion of a linear or nonlinear system, the theoretical principle should be selected to deal with specified problems. There are four major variational principles. They consist of variables—displacement, stress and strain:

- Principle of minimum potential energy (MPE),
- Principle of minimum complimentary energy (MCPE),
- Hellinger-Reissner Principle (HR), and
- Hu-Washizu Principle (HP).

Applying properly and under certain requirements, these four principles can be applied to different problems.
2.1.1 Principle of minimum potential energy (MPE)

The Euler-Lagrange (EL) equation can be used to express the minimum potential energy principle. The minimum potential energy has a compatible displacement field alone as its variable. Its functional for linear dynamic problems is written in the following form:

$$\pi_p(u) = \Gamma - U + W$$

$$= \int \frac{1}{2} \rho \{u\}^T \{u\} dV - \int \frac{1}{2} \{S\}^T [c] \{S\} dV + \int \{u\}^T \{b\} dV + \int \{u\}^T \{t\} ds \quad (2.1)$$

The symbol used in this Chapter:

- $\Gamma$ is the kinetic energy
- $U$ is the potential energy
- $W$ is the work done by external force
- $\rho$ is density
- $u$ is a displacement vector
- $[c]$ is elastic stiffness matrix
- $b$ is force vector.
- $t$ is vector of surface traction.
- $V$ is the reference volume
- $S, \Sigma$ is the surface where the force is applied.
- $[s]$ is elastic compliance matrix
- $[e]$ is the electric matrix
- $[\varepsilon]$ is the strain matrix
\{\phi\} \text{ is the electric potential vector}

\{Q\} \text{ is the electric surface charge vector}

\mathcal{S}_q \text{ is the surface where the electric charge is applied}

### 2.1.2 Principle of minimum complimentary energy (MCPE)

Similar to the MPE, the principle of minimum complimentary energy has the stress field as a variable. It is in equilibrium and traction boundary condition. Its functional for linear dynamic problems is written below:

\[
\pi_c(T) = \int \frac{1}{2} \rho [\dot{\mathbf{u}}^T \mathbf{u}] dV - \int \frac{1}{2} [\mathbf{T}^T \mathbf{S}] [\mathbf{T}] dV + \int \{\mathbf{u}\}^T \{b\} dV + \int_{\mathcal{S}} \{\mathbf{u}\}^T \{t\} ds
\]

\[(2.2)\]

### 2.1.3 Hellinger-Reissner principle (HR)

Using a Lagrange multiplier, the stresses equilibrium condition is set as a constraint condition. The Hellinger-Reissner principle (MCPE) can be obtained.

### 2.1.4 Hu-Washizu principle (HW)

If the kinematic compatibility condition and displacement boundary condition are constraint using Lagrange multipliers, the Hu-Washizu Principle can be obtained. The displacement boundary condition constraints are stresses and surface tractions respectively.

### 2.1.5 Remarks

With different boundary condition constrained, different principle can be obtained. To deal with different problems, the boundary conditions should be assumed properly for the
specified problems. In general, the principle of minimum potential energy (MPE) and the principle of minimum complimentary energy (MCPE) are used to as prime principles of linear solid mechanics. The Hellinger-Reissner principle (HR) and the Hu-Washizu principle (HW) are applied as multifield variational principles.

2.2 Mixed Formulation Based Shell Finite Elements with Piezoelectric Effects

2.2.1 Principle of minimum potential energy for piezoelectric continua

For piezoelectric continua, the potential energy is replaced by the electric enthalpy such that:

\[ H = \frac{1}{2} \{S\}^T \{c\} \{S\} - \{E\}^T \{\varepsilon\} \{S\} - \frac{1}{2} \{E\}^T \{\varepsilon\} \{E\} \quad (2.3) \]

Substituting equation (2.3) to equation (2.1), one has

\[ \pi_p (u) = \Gamma - H + W \]

\[ = \int_v \frac{1}{2} \rho \{\dot{u}\}^T \{\dot{u}\} dV - \int_v (\frac{1}{2} \{S\}^T \{c\} \{S\} - \{E\}^T \{\varepsilon\} \{S\} - \frac{1}{2} \{E\}^T \{\varepsilon\} \{E\}) dV 
+ \int_u \{u\}^T \{t\} ds + \int_{S_e} \{\phi\}^T \{Q_e\} ds \quad (2.4) \]

2.2.2 Hu-Washizu principle for piezoelectric continua

The Hu-Washizu principle is involved with the potential energy principle. The independent variables include stress, strain, and displacement. Because it permits the freedom of these otherwise related quantities, certain inconsistency is introduced. For
example, boundary conditions will be violated. To relieve, to some degrees, the boundary condition violation, Lagrange multiplier is introduced.

Unlike the displacement-based element, the strain is not related to displacement through the linear operator $[L_u]$. This causes a change in the strain energy:

$$
\int_{V} \left( \{S\} - [L_u] \{u\} \right)^T \{T\} dV
$$

and to enforce the displacement compatibility at the prescribed element boundary, another Lagrange multiplier is employed corresponding to the traction $\{t\}$

$$
\int_{S_e} \left( \{\bar{u}\} - \{u\} \right)^T \{t\} ds
$$

where $\{\bar{u}\}$ is the prescribed displacement vector.

The Hu-Washizu functional can be obtained by adding (2.5) and (2.6) to equation (2.4), thus,

$$
\pi_{elm}(u, S, T, \phi) = \Gamma - H + W
$$

$$
= \left[ \frac{1}{2} \rho \{\bar{u}\}^T \{\bar{u}\} dV - \int_{V} \left( \frac{1}{2} \{S\}^T [\epsilon] \{S\} - \{S\}^T \{T\} \right) \right.
$$

$$
+ \{T\}^T [L_u] \{u\} - \{E\}^T [\epsilon] \{S\} - \frac{1}{2} \{E\}^T [\epsilon] \{E\} \right] dV
$$

$$
+ \int_{S_e} \left( \{\bar{u}\} - \{u\} \right)^T \{t\} ds + \int_{V} \{u\}^T \{b\} dV + \int_{S_e} \{u\}^T \{t\} ds + \int_{S_e} \phi^T \{Q_e\} ds.
$$

To specify equation (2.7), two common approaches can be made, one is the hybrid stress approach in which the strain terms will be eliminated from the equation by being expressed in stress terms, and the other is the hybrid strain approach, in which the stress terms are eliminated by being expressed in strain terms.
2.2.3 Hybrid strain functional

By disregarding the Lagrange multiplier traction of equation (2.6) from (2.7), and using (2.10), the hybrid strain functional can be obtained as

\[
\pi_{HS}(u, S, \phi) = \Gamma - H + W
\]

\[
= \int_V \frac{1}{2} \rho \{\dot{u}\}^T \{\dot{u}\} dV - \int_V \left( \frac{1}{2} \{S\}^T \{c\} \{S\} + \{S\}^T \{L_u\} \{u\} - \{E\}^T \{e\} \{L_u\} \{u\} - \frac{1}{2} \{E\}^T \{e\} \{E\} \right) dV
\]

\[
+ \int_V \{u\}^T \{b\} dV + \int_{s_i} \{u\}^T \{t\} ds + \int_{s_v} \{\phi\}^T \{Q_s\} ds
\]

where the subscript HS means hybrid strain.

Equation (2.8) has two independent fields: the displacement and strain fields as a result of hybrid strain approach. This approach is more natural than the hybrid stress approach because the strains are the ones which can be physically measured. Another reason for applying this approach is that, strains are normally continues throughout the structures while stresses are often discontinuous.

Now, applying the Hamilton principle to the hybrid strain functional (2.8), such that

\[
\delta \int_{s_i}^{s_f} dt \left[ \int_V \frac{1}{2} \rho \{\dot{u}\}^T \{\dot{u}\} dV - \right.
\]

\[
\left. \int_V \left( \frac{1}{2} \{S\}^T \{c\} \{S\} + \{S\}^T \{L_u\} \{u\} - \{E\}^T \{e\} \{L_u\} \{u\} - \frac{1}{2} \{E\}^T \{e\} \{E\} \right) dV + \right.
\]

\[
\left. \int_V \{u\}^T \{b\} dV + \int_{s_i} \{u\}^T \{t\} ds + \int_{s_v} \{\phi\}^T \{Q_s\} ds \right] = 0.
\]

Integrating by parts on the first term on the right hand side of equation (2.9), one has
\[ V_1 = \delta \int_0^{t_2} dt \left[ \frac{1}{\nu} \rho \{ \ddot{u} \}^T \{ \dot{u} \} \right] dV = \int_0^{t_2} dt \left[ \frac{1}{\nu} \rho \{ \ddot{u} \}^T \{ \dot{u} \} \right] dV \]

\[ = \int \frac{1}{\nu} \rho \{ \ddot{u} \}^T \{ \dot{u} \} \, dV - \int_0^{t_2} dt \int \frac{1}{\nu} \rho \{ \ddot{u} \}^T \{ \dot{u} \} \, dV \]

\[ = -\int_0^{t_2} dt \int \frac{1}{\nu} \rho \{ \ddot{u} \}^T \{ \dot{u} \} \, dV \]  

(2.10)

since \{ \ddot{u} \} vanishes at times \( t_1 \) and \( t_2 \)

\[ V_2 = \delta \int_0^{t_2} dt \int \left( -\frac{1}{2} \{ S \}^T \{ e \} \{ S \} + \{ S \}^T \{ e \} [L_u] \{ u \} \right) \]

\[ -\{ E \}^T \{ e \} [L_u] \{ u \} - \frac{1}{2} \{ E \}^T \{ e \} \{ E \} \right) \, dV \]

\[ = \int_0^{t_2} dt \int \left( -\{ S \}^T \{ e \} \{ S \} + \{ S \}^T \{ e \} [L_u] \{ u \} \right) \]

\[ + \{ S \}^T \{ e \} [L_u] \{ \ddot{u} \} - \{ E \}^T \{ e \} [L_u] \{ \ddot{u} \} \]

\[ -\{ E \}^T \{ e \} [L_u] \{ u \} - \{ E \}^T \{ e \} \{ E \} \right) \, dV \]  

(2.11)

The remaining terms can be written as:

\[ V_3 = \delta \int_0^{t_2} dt \int \left[ \{ u \}^T \{ b \} dV + \int \{ u \}^T \{ l \} ds + \int \{ \phi \}^T \{ Q_s \} ds \right] \]

\[ = \int_0^{t_2} dt \left[ \{ \ddot{u} \}^T \{ b \} dV + \int \{ \ddot{u} \}^T \{ l \} ds + \int \{ \ddot{\phi} \}^T \{ Q_s \} ds \right] \]  

(2.12)

Substituting (2.10), (2.11) and (2.12) into (2.9), we have
Note that if the electrical quantities are deleted from equation (2.13), the generalized variational principle of elasticity with hybrid strain method can be obtained.

### 2.2.4 Formulation of hybrid finite elements

The hybrid finite element approach is introduced here. The vector \{u\} is interpolated within the element by the nodal displacements \{q_u\} via the shape function \([N_u]\), therefore,

\[
\{u\} = [N_u] \{q_u\},
\]

(2.14)

Similarly, the electric vector \{\phi\} can be expressed by the nodal electrical potential \{q_\phi\}, and shape function \([N_\phi]\):

\[
\{\phi\} = [N_\phi] \{q_\phi\},
\]

(2.15)

The assumed strain vector \{S\} can be expressed within the element from the generalized strain parameters \{\beta\} and the interpolation matrix \([P_\beta]\):
\{S\} = [P_\beta] \{\beta\} \quad (2.16)

By defining

\[ [B_u] = [L_u] [N_u] , \] \quad (2.17)

and \[ [B_\phi] = [L_\phi] [N_\phi] \] \quad (2.18)

so that the “electric field-electric potential” relation can be expressed as

\[ \{E\} = - [L_\phi] \{\phi\} = [L_\phi] \{N_\phi\} \{q_\phi\} = [B_\phi] \{q_\phi\} \] \quad (2.19)

Substituting the above shape functions and definitions into (2.13):

\[
0 = - \int_V \rho \{N_u\}^T \{N_u\} dV \{\dot{q}_u\} + \int_V \{\delta\beta\}^T [c] [P_\beta] dV \{\beta\} \\
- \{\delta\beta\}^T \int_V [P_\beta] [c] [B_u] dV \{q_u\} - \{\beta\}^T \int_V [P_\beta] [c] [B_u] dV \delta q_u \\
- \{\delta q_\phi\}^T \int_V [B_\phi]^T [e] [B_u] dV \{q_u\} - \{\delta q_\phi\}^T \int_V [B_\phi]^T [e] [B_u] dV \delta q_u \\
+ \{\delta q_\phi\}^T \int_V [B_\phi]^T [e] [B_\phi] dV \{q_\phi\} + \{\delta q_u\}^T \int_V \{N_u\}^T \{b\} dV \\
+ \{\delta q_u\}^T \int_{S_1} \{N_u\}^T \{t\} ds + \{\delta q_\phi\}^T \int_{S_1} \{N_\phi\}^T \{Q_\phi\} ds \quad (2.20)
\]

The following definitions are applied:

\[
[m] = \int_V \rho \{N_u\}^T \{N_u\} dV ,
\]

\[
[H] = \int_V [P_\beta]^T [c] [P_\beta] dV ,
\]

\[
[G_\tau] = \int_V [P_\beta]^T [c] [B_u] dV ,
\]
\[ [k_{\phi u}] = \int_V [B_{\phi}]^T [\varepsilon] [B_u] dV , \]
\[ [k_{\phi \phi}] = \int_V [B_{\phi}]^T [\varepsilon] [B_{\phi}] dV , \]
\[ \{F\} = \int_V [N_u]^T \{b\} dV + \int_{\Sigma} [N_u]^T \{t\} ds , \]
\[ \{Q\} = \int_{\Sigma} [N_{\phi}]^T \{Q_s\} ds \]

where \([m]\) is the consistent element mass matrix, \([H]\) the element stiffness matrix, \([G_e]\) the leverage matrix, \([k_{\phi u}]\) the piezoelectric stiffness matrix, \([k_{\phi \phi}]\) the dielectric stiffness matrix, \([F]\) the forcing vector, and \([Q]\) the electric charge vector.

Substitute definition (2.21) into (2.20), it becomes
\[ 0 = - \{\delta q_u\} [m] \{\dot{q}_u\} + \{\delta \beta\}^T [H] \{\beta\} - \{\delta \beta\}^T [G_e] \{q_u\} - \{\delta q_u\}^T [G_e]^T \{\beta\} \]
\[ - \{\delta q_u\} [k_{\phi u}] \{\dot{q}_u\} - \{\delta q_\phi\}^T [k_{\phi \phi}] \{q_u\} + \{\delta q_\phi\}^T [k_{\phi \phi}] \{q_\phi\} + \{\delta q_u\} \{F\}^T \]
\[ + \{\delta q_\phi\}^T \{Q\} \]

Since the variations \(\{\delta q_u\}, \{\delta \beta\}^T, \{\delta q_\phi\}\) are arbitrary inside the volume \(V\), one has
\[ \begin{bmatrix} 0 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\beta} \\ \ddot{q}_u \\ \ddot{q}_\phi \end{bmatrix} + \begin{bmatrix} -H & G_e & 0 \\ 0 & -k_{\phi u}^T & 0 \\ G_e^T & 0 & -k_{\phi \phi} \end{bmatrix} \begin{bmatrix} \beta \\ q_u \\ q_\phi \end{bmatrix} = \begin{bmatrix} 0 \\ F \\ Q \end{bmatrix} \]

and eliminate \(\{\beta\}\), it gives
\[ \begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{q}_u \\ \ddot{q}_\phi \end{bmatrix} + \begin{bmatrix} k_{uu} & k_{\phi u}^T \\ k_{\phi u} & -k_{\phi \phi} \end{bmatrix} \begin{bmatrix} q_u \\ q_\phi \end{bmatrix} = \begin{bmatrix} F \\ Q \end{bmatrix} \]

where \(k_{uu} = [G_e]^T [H]^{-1} [G_e] \).
2.2.5 Sensor and actuator equations

Equation (2.24) introduces a general form of the element equation of motion. By applying the different electric boundary conditions, the sensor equation and the actuator equation can be obtained.

2.2.5.1 Sensor equations

For the sensor layer, the direct piezoelectric potential is applied. Since no surface charge or electric field is applied to the sensor layer, \(\{Q\}\) is zero so that the electric DOF can be eliminated from equation (2.24) by using

\[
\{q_\phi\} = \left[k_{\phi\phi}\right]^{-1} \left[k_{\phi u}\right] \{q_u\}
\] (2.25)

Substitute (2.25) into (2.24). It results the equation of the sensor layer

\[
[m] \{\ddot{q}_u\} + [c_{uu}] \{\dot{q}_u\} + \left( [k_{uu}] + [k_{\phi u}]^T \left[k_{\phi\phi}\right]^{-1} \left[k_{\phi u}\right] \right) \{q_u\} = \{F\}
\] (2.26)

2.2.5.2 Actuator equation

There are two ways to activate the actuator layer. One way is to apply voltage directly, and the other way is to apply surface charge on the top or bottom of the piezoelectric layer.

When the voltage is applied, compared with the applied voltage, the self-generated voltage caused by the mechanical deformation is relatively small so that it can be ignored.

\[
[m] \{\ddot{q}_u\} + [c_{uu}] \{\dot{q}_u\} + [k_{uu}] \{q_u\} = \{F\} - \left[k_{\phi u}\right]^T \{q_\phi\}
\] (2.27)

When the actuator is activated by the surface charge, the electric potential can be
expressed as
\[
\{q_\phi\} = [k_{\phi\phi}]^{-1} \left( [k_{\phi u}] \{q_u\} - \{Q\} \right)
\]  \hspace{1cm} (2.28)

Substituting (2.28) into (2.27), resulting the equation for the actuator layer

\[
[m] \{\dot{q}_u\} + [c_{uu}] \{q_u\} + \left( [k_{uu}] + [k_{\phi u}]^T [k_{\phi\phi}]^{-1} [k_{\phi u}] \right) \{q_u\} = \{F\} + [k_{\phi u}]^T [k_{\phi\phi}]^{-1} \{Q\}
\]  \hspace{1cm} (2.29)

2.3 Three-node Triangular Laminated Shell Elements

The hybrid strain formulation of flat shell elements shows much promise among many element formulations. Among the flat shell elements, the most frequently used ones are the triangular and quadrilateral elements. Compared with quadrilateral elements, the triangular elements are relatively easier to deal with complex boundary conditions. Furthermore, the lower order quadrilateral flat shell element may lead to mesh locking.

Allman developed a simple triangular element based on linear displacement. This element includes drilling degree of freedom (DDOF). To incorporate Allman’s element into a shell formulation, To and Liu developed a transverse shear deformable triangular shell element by adding a plate bending element to Allman’s membrane element. In reference, the membrane and bending components are based on the hybrid strain formulation while in two elements the DDOF are based on the displacement formulation. The results from the analysis of the thin and moderate-thick shells in reference are very accurate and converge at a higher rate than those included in the comparison.

The piezoelectric triangular laminated shell element considered here is based on that of To and Liu’s formulation. It is formulated by incorporating one electric DOF into their formulation and then stacking such elements one on top of another. Figure 2.1 shows such a flat triangular shell element in space.
In the following sub-sections, the geometrical descriptions of the triangular shell element, constitutive equations, strain-displacement relationship and element matrices are introduced.
2.3.1 Global and local coordinates

A laminated triangular element in the global and local coordinate systems is shown in Figure 2.1. The global coordinates of the three nodes of the element are expressed in matrix form as \( \{X_i\}^T = \{X_i, Y_i, Z_i\} \) and the local coordinates of the nodes of the element \( \{x_i\}^T = \{x_i, y_i, z_i\} \), where \( \{X_i\} \) is the vector of the global coordinates \( X_i, Y_i \) and \( Z_i \), while \( \{x_i\} \) is the vector of the local coordinates \( x_i, y_i \) and \( z_i \) at Node \( i \).

Node 1 is chosen to be the origin of the local coordinate system and the x-axis coincides with side 1-2. The z-axis is chosen to be normal to the surface 1-2-3 and the y-axis if is then normal to the x-z plane. Both systems are right-handed.

The two coordinate systems are related to each other by the following equation:

\[
\{X\} = \{X_i\} + \left[ T_g \right] \{x\} 
\]

(2.30)
where \( \{X\} \) and \( \{x\} \) are global and local coordinates, respectively, with \( \{X\} \) containing global coordinate of Node 1, and \( [T_g] \) is the transformation matrix. It is given below as

\[
[T_g] = \begin{bmatrix}
\cos(x,X) & \cos(y,X) & \cos(z,X) \\
\cos(x,Y) & \cos(y,Y) & \cos(z,Y) \\
\cos(x,Z) & \cos(y,Z) & \cos(z,Z)
\end{bmatrix}
\]  

(2.31)

where \((x,X)\) denotes the angle between the local positive x-axis and global positive X-axis, and so on.

The relationship between displacement vector \( \{u\} \) in the local coordinate system and the displacement \( \{U\} \) in the global coordinate system is given by the equation

\[
\{u\} = [R] \{U\}
\]  

(2.32)

where

\[
[R] = \begin{bmatrix}
[T_g] & 0 \\
0 & [T_g]^T
\end{bmatrix}
\]  

(2.33)

The relationship between nodal displacement vector \( \{q\} \) in the local coordinate system and nodal displacement \( \{Q\} \) in the global coordinate system is related by

\[
\{q\} = [R_g] \{Q\}
\]  

(2.34)

where

\[
[R_g] = \begin{bmatrix}
[R] & 0 & 0 \\
0 & [R] & 0 \\
0 & 0 & [R]
\end{bmatrix}
\]  

(2.35)

Since the force components must perform the same amount of work in both coordinate systems and therefore

\[
\{F\}^T \{Q\} = \{f\}^T \{q\}.
\]  

(2.36)
Substitute (2.36) into (2.35), one has

\[ \{ F \}^T = [ R_g ] \{ f \}^T. \]  

(2.37)

without loss of generality, the dynamic system is considered undamped. Thus, the local coordinate system equation is

\[ [m] \{ \dot{q} \} + [k] \{ q \} = \{ f \}. \]  

(2.38)

Substituting (2.37) and (2.36) into (2.38)

\[ [R_g]^T [m] [R_g] \{ \dot{q} \} + [R_g]^T [k] [R_g] \{ q \}^T = \{ F \} \]  

(2.39)

so that the e’th element matrices in the global coordinate system become

\[ [M^e] = [R_g]^T [m] [R_g], \]  

(2.40a)

and

\[ [K^e] = [R_g]^T [k] [R_g]. \]  

(2.40b)

The assembled mass and stiffness matrices for the system are

\[ [M] = \sum_{e=1}^{N} [M^e] \]  

and \[ [K] = \sum_{e=1}^{N} [K^e] \]  

(2.41)
where the summation symbol should not be regarded as the conventional one since it is applied here to denote that the element matrices in the global coordinate system are added element by element so that the order of the matrix in the LHS of equation (2.41) is much larger than that for the individual element defined by equations (2.40a) and (2.40b).

2.3.2 Natural coordinates and local coordinates

Natural coordinates can be used to simplify the solution for the triangular shell element. The natural coordinate variables are defined as

\[
\begin{pmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{pmatrix} = \frac{1}{A} \begin{pmatrix}
A_1 \\
A_2 \\
A_3
\end{pmatrix}
\] (2.42)
where A is the total area of the triangular element, while \( A_1, A_2 \) and \( A_3 \) are the areas defined in Figure 2.2. Notice that \( \xi_1 + \xi_2 + \xi_3 = 1 \).

The relationship between the natural coordinates and the local coordinate system is

\[
\begin{pmatrix}
1 \\
x \\
y
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1 \\
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3
\end{pmatrix}
\begin{pmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{pmatrix}
. \tag{2.43}
\]

Equation (2.43) gives

\[
\begin{pmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{pmatrix}
= \frac{1}{2A}
\begin{pmatrix}
x_2 y_3 - x_3 y_2 & y_2 - y_3 & x_3 - x_2 \\
x_3 y_1 - x_1 y_3 & y_3 - y_1 & x_1 - x_3 \\
x_1 y_2 - x_2 y_1 & y_1 - y_2 & x_2 - x_1
\end{pmatrix}
\begin{pmatrix}
1 \\
x \\
y
\end{pmatrix} \tag{2.44}
\]

where A is given as

\[
A = \frac{1}{2}
\begin{vmatrix}
1 & 1 & 1 \\
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3
\end{vmatrix}
. \tag{2.45}
\]

Because the local coordinate system is specifically chosen that \( x_i = y_i = y_2 = 0 \), so that (2.44) can be simplified as

\[
A = \frac{1}{2}
\begin{vmatrix}
1 & 1 & 1 \\
0 & x_2 & x_3 \\
0 & 0 & y_3
\end{vmatrix}
= \frac{1}{2} x_2 y_3 
. \tag{2.46}
\]

Equation (2.43) can also be simplified as

\[
\begin{pmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{pmatrix}
= \frac{1}{2A}
\begin{pmatrix}
x_2 y_3 - y_3 & x_3 - x_2 \\
0 & y_3 & -x_3 \\
0 & 0 & x_2
\end{pmatrix}
\begin{pmatrix}
1 \\
x \\
y
\end{pmatrix}
. \tag{2.47}
\]
The following partial differential of $\xi_i$ with respect to $x$ and $y$, is required for subsequent use

$$
\begin{pmatrix}
\xi_{1,x} \\
\xi_{2,x} \\
\xi_{3,x}
\end{pmatrix} =
\begin{pmatrix}
-\frac{1}{x_2} & 0 & \frac{x_2}{x_2}
\end{pmatrix},
\begin{pmatrix}
\xi_{1,y} \\
\xi_{2,y} \\
\xi_{3,y}
\end{pmatrix} =
\begin{pmatrix}
\frac{x_2}{x_2} & 0 & \frac{x_2}{x_2}
\end{pmatrix}.
$$

(2.48)

One of the advantages of using natural coordinates lies in the ability to evaluate the integral equation, exactly and explicitly. For the triangular element

$$
\frac{1}{A} \int_A \xi^m_1 \xi^n_2 \xi^p_3 dA = \frac{2m!n!p!}{(m+n+p+2)!}
$$

(2.49)

where $m$, $n$ and $p$ can be 0 or positive integers.

### 2.4 Derivation of Element Mass and Stiffness Matrices

The flat triangular shell element shown in Figure 2.1 has three nodes each of which has six DOF. The interpolation functions are constructed in the following.

#### 2.4.1 Assumed displacement field

The membrane element developed by Allman uses quadratic polynomials natural coordinates to interpolate the displacement inside the element

$$
u = u_1 \xi_1 + u_2 \xi_2 + u_3 \xi_3 \xi_1 \xi_2 + \frac{1}{2} (l_{12} \cos \gamma_{12}) (\psi_{z2} - \psi_{z1}) \xi_3 + \frac{1}{2} (l_{23} \cos \gamma_{23}) (\psi_{z3} - \psi_{z2}) \xi_3 \xi_2 + \frac{1}{2} (l_{31} \cos \gamma_{31}) (\psi_{z1} - \psi_{z3}) \xi_3 \xi_1 \xi_2 ,$$
\[
\nu = v_1 \xi_1 + v_2 \xi_2 + v_3 \xi_3 + \frac{1}{2} (l_{12} \sin \gamma_{12}) (\psi_{z_2} - \psi_{z_1}) \xi_1 \xi_2 \\
+ \frac{1}{2} (l_{23} \sin \gamma_{23}) (\psi_{z_3} - \psi_{z_2}) \xi_2 \xi_3 + \frac{1}{2} (l_{31} \sin \gamma_{31}) (\psi_{z_1} - \psi_{z_3}) \xi_3 \xi_1.
\]

(2.50)

where \( l_{ij} \) represents the length of the side \( i-j \), and \( \gamma_{31} \) represents the angle between the outward normal \( n_{ij} \) of the side \( i-j \) and the positive x-axis which is shown in Figure 2.2.

In the foregoing,

\[
l_{ij} \cos \gamma_{ij} = -y_{ij} \quad l_{ij} \sin \gamma_{ij} = x_{ij}
\]

(2.51)

where \( x_{ij} = x_i - x_j \), \( y_{ij} = y_i - y_j \).

Define

\[
p_1 = -\frac{1}{2} \xi_1 (y_{31} \xi_3 - y_{12} \xi_2), \quad q_1 = \frac{1}{2} \xi_1 (x_{31} \xi_3 - x_{12} \xi_2),
\]

\[
p_2 = -\frac{1}{2} \xi_2 (y_{12} \xi_1 - y_{23} \xi_3), \quad q_2 = \frac{1}{2} \xi_2 (x_{12} \xi_1 - x_{23} \xi_3),
\]

\[
p_3 = -\frac{1}{2} \xi_3 (y_{23} \xi_2 - y_{31} \xi_1), \quad q_3 = \frac{1}{2} \xi_3 (x_{23} \xi_2 - x_{31} \xi_1).
\]

(2.52)

Equation (2.50) can be written as

\[
u = u_1 \xi_1 + u_2 \xi_2 + u_3 \xi_3 + p_1 \psi_{z_1} + p_2 \psi_{z_2} + p_3 \psi_{z_3},
\]

\[
v = v_1 \xi_1 + v_2 \xi_2 + v_3 \xi_3 + q_1 \psi_{z_1} + q_2 \psi_{z_2} + q_3 \psi_{z_3}.
\]

(2.53)
But \( \psi_{zi} \) in Allman’s element is not the true rotation from the vertex which can be expressed as \( \bar{\omega}_i \) where

\[
\bar{\omega}_i = \frac{1}{2} (v_{i,x} + u_{i,y}).
\]  

(2.54)

Allman proposed an improved formulation for \( \psi_{zi} \). However, this formulation which involves cubic shape function instead of quadratic one is much more complicated. To and Liu [20] proposed another solution for this problem. In their formulation, the strain caused by the difference between \( \psi_{zi} \) and \( \bar{\omega}_i \) can be compensated by the “normal rotational shear strain”

\[
\varepsilon_{xy}^d = \psi_z - \frac{1}{2} (v_{x,x} + u_{y,y})
\]  

(2.55)

where the superscript \( d \) denotes DDOF.

Then the element developed by To and Liu was derived by combining the membrane element with the normal rotation shear component and the plate bending element. The shape function for all the six degrees of freedom can be obtained as follows

\[
u = v_1 \xi_1 + v_2 \xi_2 + v_3 \xi_3,
\]

\[
 v = v_1 \xi_1 + v_2 \xi_2 + v_3 \xi_3,
\]

\[
 w = w_1 \xi_1 + w_2 \xi_2 + w_3 \xi_3,
\]

\[
 \psi_x = \psi_{x1} \xi_1 + \psi_{x2} \xi_2 + \psi_{x3} \xi_3,
\]

\[
 \psi_y = \psi_{y1} \xi_1 + \psi_{y2} \xi_2 + \psi_{y3} \xi_3,
\]

\[
 \psi_z = \psi_{z1} \xi_1 + \psi_{z2} \xi_2 + \psi_{z3} \xi_3.
\]  

(2.56)

or in the matrix form

\[
\{ u \} = \left[ N_u \right] \{ q_u \}
\]  

(2.57)
where

\[
[N_u] = \begin{bmatrix} [N_u]_1 & [N_u]_2 & [N_u]_3 \end{bmatrix}
\]

and

\[
[N_u]_i = \begin{bmatrix}
\xi_i & 0 & 0 & 0 & 0 & p_i \\
0 & \xi_i & 0 & 0 & 0 & q_i \\
0 & 0 & \xi_i & 0 & 0 & 0 \\
0 & 0 & 0 & \xi_i & 0 & 0 \\
0 & 0 & 0 & 0 & \xi_i & 0 \\
0 & 0 & 0 & 0 & 0 & \xi_i
\end{bmatrix}
\]

(2.58)

To improve the insufficiency of bending action due to the use of low order interpolation proposed in reference [28], To and Liu [20] propose a strategy to associate the out-of-plane displacement \( w \) with the rotations \( \psi_x \) and \( \psi_y \) that is,

\[
w = w_1 \xi_1 + w_2 \xi_2 + w_3 \xi_3 - p_1 \psi_{x1} - p_2 \psi_{x2} - p_3 \psi_{x3} - q_1 \psi_{y1} - q_2 \psi_{y2} - q_3 \psi_{y3}
\]

(2.59)

The shape function for this case is

\[
[N_u]_i = \begin{bmatrix}
\xi_i & 0 & 0 & 0 & 0 & p_i \\
0 & \xi_i & 0 & 0 & 0 & q_i \\
0 & 0 & \xi_i & 0 & 0 & 0 \\
0 & 0 & 0 & \xi_i & 0 & 0 \\
0 & 0 & 0 & 0 & \xi_i & 0 \\
0 & 0 & 0 & 0 & 0 & \xi_i
\end{bmatrix}
\]

(2.60)

### 2.4.2 Assumed strain field

The selection of the independently assumed strain field is very important for hybrid strain finite elements. To suppress all kinematic deformation modes, the number of strain modes should satisfy the following condition [20]
\[ m \geq n - r \]  

(2.61)

where \( m \) is the number of assumed strain modes , \( n \) is the number of generalized displacements , and \( r \) is the rigid-body modes of the element.

As mentioned in reference [28], the number of assumed strain modes should preferably be a minimum, or the additional modes may lead to a more stiff and expensive element. For the element from To and Liu [20], \( m \) is chosen to be 10, and \( n = 18 \), so that \( r = 8 \), 6 of which are rigid body modes (6 zero eigenvalues) and two of which are so-called spurious or hourglass modes. The assumed strain field can be expressed as:

\[
\{ S \} = [P] \{ \beta \},
\]

where

\[
\{ S \}^T = \{ S_x \ S_y \ S_{xy} \ S_{yz} \ S_{zx} \ \mathbf{S}_{xy} \},
\]

\[
\{ \beta \}^T = \{ \beta_1 \ \beta_2 \ \beta_3 \ \beta_4 \ \beta_5 \ \beta_6 \ \beta_7 \ \beta_8 \ \beta_9 \ \beta_{10} \},
\]

\[
[P] = \begin{bmatrix}
1 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & z & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & y_{13}(1-2\xi_2) & y_{32}(1-2\xi_2) & y_{21}(-1 + 2\xi_1 + 2\xi_2) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x_{13}(1-2\xi_2) & x_{32}(1-2\xi_2) & x_{21}(-1 + 2\xi_1 + 2\xi_2) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(2.62)

where the ten parameters represent each individual strain measures. The parameters \( \beta_1 \) through \( \beta_3 \) are associated with membrane, \( \beta_4 \) through \( \beta_6 \) with bending, \( \beta_7 \) through \( \beta_9 \) with transverse shear, and \( \beta_{10} \) is associated with normal rotation.

To eliminate the spurious modes To and Liu [20] replaced the hybrid strain based normal rotations with the displacement formulation. This is the approach adopted in the present investigation.
2.4.3 Assumed electric potential field

Since the piezoelectric layers are usually very thin, the electric field can be assumed as linearly distributed along the thickness, so that

\[
\phi^k = \frac{1}{2}(1 + \xi)\phi^k_t + \frac{1}{2}(1 - \xi)\phi^k_b
\]  

(2.63)

where \( \phi^k_t \) and \( \phi^k_b \) represent the electrical potentials of the top and bottom layers,

\[
\xi = \frac{z - z_k}{h_k}, \quad \text{with} \quad z_k \quad \text{being the position of the k’th layer’s mid-position in the z direction and} \quad h_k \quad \text{the thickness of the k’th layer}. \quad \text{Thus,} \quad -1 \leq \xi \leq 1.
\]

In practice, either the top or bottom layer is grounded, so that one of the two potentials is not required in the formulation, which leaves the other to be the voltage for sensors or the applied electric field for actuators.

Assuming \( \phi^k_b \) is grounded, (2.63) can be written as:

\[
\phi^k = \frac{1}{2}(1 + \xi) \phi^k_t.
\]  

(2.64)

The electric potential of any point in the triangular element, which can be expressed by linear interpolation function

\[
\phi^k = \xi_1\phi^k_{1k} + \xi_2\phi^k_{2k} + \xi_3\phi^k_{3k}
\]

\[
= \frac{1}{2}(1 + \xi_1)\phi^k_{1k} + \frac{1}{2}(1 + \xi_2)\phi^k_{2k} + \frac{1}{2}(1 + \xi_3)\phi^k_{3k}
\]  

(2.65)

or in the matrix form

\[
\phi^k = [N_\phi]\{q^k_\phi\}
\]  

(2.66)

where
\[
\left\{ q^k \right\} = \left\{ \phi_{t1}, \phi_{t2}, \phi_{t3} \right\}
\]

and

\[
[N_\phi] = \begin{bmatrix}
\frac{1}{2} (1 + \xi_1) \xi_1 & \frac{1}{2} (1 + \xi_2) \xi_2 & \frac{1}{2} (1 + \xi_3) \xi_3 \\
\end{bmatrix}.
\] (2.67)

### 2.4.4 Local strain-displacement relations

According to laminate shell theory, any point through the thickness can be expressed as:

\[
u = u_0 + z \psi_y, \quad v = v_0 - z \psi_x, \quad w = w_0,
\]

\[
\psi_x = \psi_{x0}, \quad \psi_y = \psi_{y0}, \quad \psi_z = \psi_{z0}
\] (2.68)

where the superscript 0 denotes the mid-surface of the layer.

By differentiating (2.68), the linear in-plane strains are

\[
\varepsilon_x = u_x = u_{0,x} + z \psi_{y,x}, \quad \varepsilon_y = v_y = u_{0,x} - z \psi_{x,y},
\]

\[
\varepsilon_{xy} = u_{y,x} + v_{x,y} = u_{0,y,x} + v_{0,x} - z \psi_{x,x} + z \psi_{y,y}.
\]

The transverse strains are

\[
\varepsilon_z = w_z = w_{0,z} = 0,
\]

\[
\varepsilon_{yz} = w_{y,z} + v_z = w_{0,y} - \psi_x,
\]

\[
\varepsilon_{xz} = w_{x,z} + u_z = w_{0,x} - \psi_y.
\] (2.69)

Notice that the transverse normal strain is zero due to the fact that w is assumed to be constant through the thickness.
Assuming the displacement within the shell element is linear, the membrane, bending and transverse shear actions can be analyzed separately.

2.4.4.1 The membrane strains

\[
\{\varepsilon_m\} = \begin{bmatrix}
    u_{0,x} \\
    v_{0,y} \\
    u_{0,y} + v_{0,x}
\end{bmatrix} = \begin{bmatrix}
    \frac{\partial}{\partial x} \\
    \frac{\partial}{\partial y} \\
    \frac{\partial}{\partial y} \frac{\partial}{\partial x}
\end{bmatrix} \begin{bmatrix}
    u_0 \\
    v_0
\end{bmatrix}
\] (2.70)

where \( m \) denotes membrane

2.4.4.2 Bending curvature

The bending curvature are given as

\[
\{\kappa\} = \begin{bmatrix}
    \kappa_x \\
    \kappa_y \\
    \kappa_{xy}
\end{bmatrix} = \begin{bmatrix}
    \psi_{y,x} \\
    -\psi_{x,y} \\
    -\psi_{x,x} + \psi_{y,y}
\end{bmatrix} = \begin{bmatrix}
    0 & \frac{\partial}{\partial x} \\
    -\frac{\partial}{\partial y} & 0 \\
    -\frac{\partial}{\partial x} & \frac{\partial}{\partial y}
\end{bmatrix} \begin{bmatrix}
    \psi_x \\
    \psi_y
\end{bmatrix}.
\] (2.71)

2.4.4.3 Transverse shear strains

\[
\{\chi\} = \begin{bmatrix}
    w_{0,y} - \psi_x \\
    w_{0,x} + \psi_y
\end{bmatrix} = \begin{bmatrix}
    \frac{\partial}{\partial y} & -1 & 0 \\
    \frac{\partial}{\partial x} & 0 & 1
\end{bmatrix} \begin{bmatrix}
    w_0 \\
    \psi_x \\
    \psi_y
\end{bmatrix}
\] (2.72)
In addition to the membrane, bending, and transverse shear strains, the additional in-plane shear strains due to the normal rotation, $\psi_z$, must be added to the element formulation

$$\varepsilon_{xy}^d = u_y + v_x = \frac{1}{2} (u_{0,y} - v_{0,x}) + \psi_z = \left[ \begin{array}{ccc} \frac{1}{2} \frac{\partial}{\partial y} & -1 & 0 \\ -1 & \frac{1}{2} \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 1 \end{array} \right] \begin{bmatrix} u_0 \\ v_0 \\ \psi_z \end{bmatrix}$$ (2.73)

By superimposing the four actions, the local strain-displacement relationship can be expressed as

$$\{\varepsilon\} = [L_u]\{u\}$$ (2.74)

where

$$\{\varepsilon\}^T = \{\varepsilon_x \varepsilon_y \varepsilon_{xy} \varepsilon_{yx} \varepsilon_{xz} \varepsilon_{xz} \varepsilon_{xy}\}$$ (2.75)

The linear differential operator $[L_u]$ is defined by

$$[L_u] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & z \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & -z \frac{\partial}{\partial y} & 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & -z \frac{\partial}{\partial x} & z \frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial y} & -1 & 0 & 0 \\ \frac{1}{2} \frac{\partial}{\partial y} & -1 \frac{\partial}{\partial x} & 0 & 0 & 0 & 1 \end{bmatrix}.$$ (2.76)

By substituting (2.59) or (2.60) and (2.76) into (2.20), matrix $[B_u]$, can be obtained for both the following two cases, and $[B_u]$ can be expressed as:

$$[B_u] = \begin{bmatrix} [B_u]_1 & [B_u]_2 & [B_u]_3 \end{bmatrix}.$$ (2.77)
For case 1,

\[
[B_u]_i = [L_u] [N_u]_i,
\]

\[
= \begin{bmatrix}
\xi_{i,x} & 0 & 0 & 0 & z\xi_{i,x} & p_{i,x} \\
0 & \xi_{i,y} & 0 & -z\xi_{i,y} & 0 & q_{i,y} \\
\xi_{i,y} & \xi_{i,x} & 0 & -z\xi_{i,x} & z\xi_{i,y} & p_{i,y} + q_{i,x} \\
0 & 0 & \xi_{i,y} & -\xi_i & 0 & 0 \\
0 & 0 & \xi_{i,x} & 0 & \xi_i & 0 \\
1/2 \xi_{i,y} & -1/2 \xi_{i,x} & 0 & 0 & 0 & 1/2 p_{i,y} - 1/2 q_{i,x} + \xi_i \\
\end{bmatrix},
\]

\[(2.78)\]

For case 2,

\[
[B_u]_i = [L_u] [N_u]_i
\]

\[
= \begin{bmatrix}
\xi_{i,x} & 0 & 0 & 0 & z\xi_{i,x} & p_{i,x} \\
0 & \xi_{i,y} & 0 & -z\xi_{i,y} & 0 & q_{i,y} \\
\xi_{i,y} & \xi_{i,x} & 0 & -z\xi_{i,x} & z\xi_{i,y} & p_{i,y} + q_{i,x} \\
0 & 0 & \xi_{i,y} & -\xi_i - p_{i,y} & -q_{i,y} & 0 \\
0 & 0 & \xi_{i,x} & -p_{i,x} & \xi_i - q_{i,x} & 0 \\
1/2 \xi_{i,y} & -1/2 \xi_{i,x} & 0 & 0 & 0 & 1/2 p_{i,y} - 1/2 q_{i,x} + \xi_i \\
\end{bmatrix}.
\]

\[(2.79)\]

2.4.4.4 The local electric-field and electric-potential relation

The local electric-field and electric-potential relation of the piezoelectric layer can be expressed as

\[
E_x = \frac{\partial \phi}{\partial x}, \quad E_y = \frac{\partial \phi}{\partial y}, \quad E_z = \frac{\partial \phi}{\partial z}
\]

\[(2.80)\]

or in matrix form
\[ \{E\} = - \{L_\phi\} \phi \]  

(2.81)

where

\[ \{E\} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}, \quad \{L_\phi\} = \begin{pmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \]  

(2.82)

Substituting (2.81) and (2.67) into (2.21), the matrix \([B_\phi]\) can be shown as

\[
[B_\phi] = [L_\phi][N_\phi]
\]

\[
= \begin{pmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \begin{bmatrix} \frac{1}{2} (1 + \xi) \xi_1 & \frac{1}{2} (1 + \xi) \xi_2 & \frac{1}{2} (1 + \xi) \xi_3 \\ \frac{1}{2} (1 + \xi) \xi_{1,x} & \frac{1}{2} (1 + \xi) \xi_{2,x} & \frac{1}{2} (1 + \xi) \xi_{3,x} \\ \frac{1}{2} (1 + \xi) \xi_{1,y} & \frac{1}{2} (1 + \xi) \xi_{2,y} & \frac{1}{2} (1 + \xi) \xi_{3,y} \\ \frac{1}{h_k} \xi_1 & \frac{1}{h_k} \xi_2 & \frac{1}{h_k} \xi_3 \end{bmatrix}
\]

(2.83)

where, \( \xi = (z - z_k) \frac{2}{h_k} \).

### 2.4.5 Displacement based rotational degrees of freedom

To incorporate the DDOF into the element, the strain energy functional due to torsional deformation \( \pi_t \) should be considered. The latter can be expressed as a function of the nodal displacement vector \( q \).
\[ \pi_t(u) = \frac{1}{2} G h_k \int_A \{ u^T \} \{ \bar{r}^T \} \{ \bar{r} \} \{ u \} dA \] (2.84)

where \( G \) is the shear modulus, \( h_k \) is the thickness of the \( k \)'th layer, and \( \bar{r} \) is defined as

\[
\{ \bar{r} \} = \frac{1}{2} \left[ \begin{array}{c}
\frac{x_3 - x_2}{x_2 y_3}, \\
\frac{1}{x_2}, \\
0,0,0,2\xi_1 + p_{1,x} - q_{1,y}, \\
-\frac{x_3}{x_2 y_3}, \\
\frac{1}{x_2}, \\
0,0,0,2\xi_2 + p_{2,x} - q_{2,y}, \\
\frac{1}{s_3}, \\
0,0,0,2\xi_3 + p_{3,x} - q_{3,y} \end{array} \right]_{1 \times 18}
\] (2.85)

where all the symbols have already been defined above.

The mixed variational functional becomes

\[ \pi_t(u, S, \phi) = \pi_{HS}(u, S, \phi) + \pi_t(u) \] (2.86)

where \( \pi_{HS}(q, \beta) \) is the hybrid strain energy functional discussed in Sub-section 2.2.3.

The stiffness matrix therefore is

\[ [k] = [k_{HS}] + [k_r] \] (2.87)

where \( [k_r] \) denotes the element stiffness matrix associated with the DDOF and \( [k_{HS}] \) is the element stiffness matrix including the piezoelectric effect.

The element stiffness matrix associated with the DDOF can be obtained from equation (2.84) as

\[ k_r = \int_A G h_k \{ \bar{r}^T \} \{ \bar{r} \} dA. \] (2.88)
2.4.6 Explicit expressions of the element matrix

The explicit expressions of the element matrices $[m]$, $[H]$, $[G_c]$, $[k_0]$, $[k_1]$ and $[k_r]$ have been obtained by To and Liu [20], and To and Chen [23]. These element matrices are employed in the present investigation.
CHAPTER 3  EIGENVALUE SOLUTIONS OF DISCRETIZED PLATE STRUCTURES

The free vibration analysis of discretized plates using the triangle shell element is demonstrated in this chapter. Natural frequencies of cantilever beam and cantilevered square plate are presented and compared with those available in the literature.

3.1 Equation of Motion for Multi-Degree-of-Freedom Systems

The equation of motion for a MDOF system can be obtained as:

$$[M] \ddot{X} + [C] \dot{X} + [K] X = F$$  \hspace{1cm} (3.1)

where $[M]$, $[C]$, and $[K]$ are respectively the assembled mass, damping, and stiffness matrices of the system.

3.2 Mesh Types

There are 4 types of meshes. They are defined as showed in Figure 3.1. As can be observed in the latter figure D mesh has more Nodes and elements.

Figure 3.1  Mesh types: 2 × 2 A mesh, B mesh, C mesh and D mesh
3.3 Six Rigid Body Modes Test

To study the ability of the triangle element, the natural frequencies of six rigid modes are investigated. The following example is used. The cantilever beam is 100 mm long, 5 mm wide and 1 mm thick. The material of the PVDF layers are: Young’s modulus, \( Y = 2 \times 10^9 \text{N/m}^2 \), the Poisson’s ratio \( \nu = 0.29 \), density \( \rho = 1800 \text{kg/m}^3 \), and \( e_{31} = 0.046 \text{C/m}^2 \).

With the geometry and the material properties given above, the six rigid mode natural frequencies are obtained to set all degree of freedoms to be constrained.

Figure 3.2  Cantilever bimorph beam
The results of natural frequencies from the Fortran are showed in Table 3.1. The first six natural frequencies are numerically zero compared with the 7th natural frequency. The Fortran code uses the triangle element with piezoelectric degree, which is described in Chapter 2. These results can conclude that the triangle element used can give the correct number of rigid-body modes. This means that the unconstrained triangular bimorph beam has six rigid-body modes. This is compared with that based on hybrid formulation. The present result is consistent with that found by To and Liu [30].

Table 3.1 Six rigid body and elastic modes

<table>
<thead>
<tr>
<th>Mode</th>
<th>Natural Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode 1</td>
<td>0.135613 × 10^{-3}</td>
</tr>
<tr>
<td>Mode 2</td>
<td>0.652393 × 10^{-3}</td>
</tr>
<tr>
<td>Mode 3</td>
<td>0.809166 × 10^{-3}</td>
</tr>
<tr>
<td>Mode 4</td>
<td>0.123716 × 10^{-2}</td>
</tr>
<tr>
<td>Mode 5</td>
<td>0.169797 × 10^{-2}</td>
</tr>
<tr>
<td>Mode 6</td>
<td>0.192809 × 10^{-2}</td>
</tr>
<tr>
<td>Mode 7</td>
<td>0.110817 × 10^{3}</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Mode 102</td>
<td>0.294425 × 10^{6}</td>
</tr>
</tbody>
</table>
3.4 Natural Frequencies of Simply Supported Bimorph Beam

This structure is identical to that in Section 3.3 above except that now both end of the bimorph beam are simply-supported (SS).

The natural frequencies are calculated and compared with those presented in [30]. These results are included in Table 3.2 below. As can be observed from the latter table very good agreement has been obtained.
<table>
<thead>
<tr>
<th>Mesh</th>
<th>Neq.</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wei Liu</td>
<td>5×1 D</td>
<td>90</td>
<td>17.373</td>
<td>96.422</td>
</tr>
<tr>
<td>Present</td>
<td>5×1 D</td>
<td>90</td>
<td>17.339</td>
<td>97.649</td>
</tr>
<tr>
<td>Analytical solution [28]</td>
<td></td>
<td>17.036</td>
<td>106.516</td>
<td>298.793</td>
</tr>
</tbody>
</table>

Table 3.2 First 3 natural frequencies of bimorph beam

Without piezoelectric effect

Mode sequence (Hz)
3.5 Natural Frequencies of Discretized Plate

The homogeneous isotropic rectangular plate is with simply supported on all edges. The geometry of the rectangular plate is: \( a = 1.016 \text{m (40 inch)}, b = 1.524 \text{m (60 inch)}, \) with \( 5 \times 1 \text{D mesh}. \) The plate composes of three layers of materials. The substrate is steel with the thickness of 0.00635 m. The bottom and top layers are PZT films of 254 \( \mu \text{m} \) thick. The top layer is the actuator layer and the bottom layer is sensor layer. Figure 3 shows a schematic view of the laminated plate. The material properties of the substrate are: Young’s modulus \( E = 1.6 \times 10^{11} \text{N/m}^2; \) Poisson’s ratio \( \nu = 0.3 \) and density \( \rho = 7800.0 \text{ kg/m}^3. \) The material properties of PZT are: Young’s modulus \( E_p = 6.3 \times 10^{10} \text{N/m}^2, \) Poisson’s ratio \( \nu = 0.3, \) density \( \rho = 7600.0 \text{kg/m}^3, \) and piezoelectric stress-constant \( d_{3i} = 1.79 \times 10^{-10}. \) Applying the symmetry conditions, only a quarter of the plate is applied. The boundary conditions for the quarter plate are: \( \psi_x = 0 \) for the symmetric line parallel to the X axis, \( \psi_y = 0 \) for the symmetric line parallel to the Y axis, \( \psi_x = \psi_y = 0 \) at the center point.

The first three natural frequencies were found to be 13.6774 \( \text{Hz}, \) 71.6751 \( \text{Hz} \) and 81.8464 \( \text{Hz}, \) respectively. These results were obtained \( 5 \times 1 \text{ D mesh} \) and the total number of unknowns is 111.
3.6 Natural Frequencies of Pinched Cylinder

In this study, a pinch cylinder model is analyzed using lower order triangular element. One over eighth of the pinch cylinder model is used for analysis. The geometrical properties of the cylinder are: radius $R = 0.76\text{m (30inch)}$, length $L = 1.52\text{m (60inch)}$. Similar to the simply supported bimorph beam, the cylinder is also composed of three layers of materials. The substrate is steel with the thickness of $0.0762\text{m (3 inch)}$. The bottom and top layers are PZT films of $254 \ \mu\text{m}$ thick. The top layer is the actuator layer and the bottom layer is sensor layer. The material properties of the substrate are: Young’s modulus $E = 1.6 \times 10^{11} \text{N/m}^2$; Poisson’s ratio $\nu = 0.3$, and density $\rho = 7800.0 \text{kg/m}^3$.

The material properties of PZT are: Young’s modulus $E_p = 6.3 \times 10^{10} \text{N/m}^2$, Poisson’s
ratio $\nu = 0.3$, and density $\rho = 7600.0 \text{ kg/m}^3$, and piezoelectric stress constant $d_{31} = 1.79 \times 10^{-10}$. Applying the symmetry conditions, this one-eighth cylinder is discretized by $12 \times 12A$ mesh. Figure 3.4 shows the schematic view of the pinched cylinder and mesh type $4 \times 4A$. The boundary conditions for the one-eighth of the cylinder (which is shown as ABCD in Figure 3.4) are: The mode shapes are symmetric on these boundaries. On AB, displacement in X axis is constrained. The rotations along Y and Z axis are constrained; On BC, displacement in Z axis is constrained. The rotations along X and Y axis are constrained; On CD, displacement in Y axis is constrained. The rotation along Z axis is constrained; AD is always fixed. In practical application of a pinched cylinder, such as a tube contact with a tank, leak is undesirable. On AD, 6 DOF are constrained. So $X = \psi_y = \psi_z = 0$ for line AB, $Z = \psi_x = \psi_y = 0$ for line BC, $Y = \psi_x = \psi_z = 0$ for line CD, and $X = Y = Z = \psi_x = \psi_y = \psi_z = 0$ for line AD.

The first three natural frequencies were found to be 4.5580 Hz, 4.7474 Hz and 7.3199 Hz, respectively. These results were obtained $12 \times 12A$ mesh and the total number of unknowns is 864. Refine the mesh to be $15 \times 10A$, the first three natural frequencies were found to be 4.4923 Hz, 4.7440 Hz and 7.1319 Hz with total number of unknowns is 900. Refine the mesh to be $16 \times 10A$, the first three natural frequencies were found to be 4.4719 Hz, 4.7375 Hz and 7.0803 Hz with total number of unknowns is 960. Referring to the analytical fundamental frequency 4.2278 Hz, the present result is fairly close.
Figure 3.4 Pinched cylinder

- R = 0.76 m (30 inch)
- L = 1.52 m (60 inch)
CHAPTER 4  TIME-DEPENDENT STATISTICAL RESPONSES OF PLATE STRUCTURES

The dynamic responses of plate structures under non-stationary excitation are presented in this chapter. The equations of motions are obtained using the mixed formulation finite element method in Chapter 2. These equations are solved in two methods--Stochastic Central Difference method and Runger Kutta method. The accuracy and efficiency of them are discussed in this Chapter.

4.1 Discretized Governing Equation of Motion and Stochastic Central Difference Method for Response Statistics

The discretized dynamic system whose matrix equation of motion is

\[ M\ddot{x} + C\dot{x} + Kx = F(t) = F \]  \hspace{1cm} (4.1)

where the assembled mass, damping, and stiffness matrices of the piezoelectric system or system with piezoelectric layers have been constructed by making use of the element matrices derived in Section 2.2 while \( F(t) \) or simply \( F \) is the externally applied random excitation vector; and \( x \) is the displacement vector. The subscript \( s \) denotes the time step, for example, \( x_s \) is the value of \( x \) at time step \( t_s \) such that the time step size \( \Delta t = t_{s+1} - t_s \) and \( t_0 = 0 \).

The time domain discretized version of Eq. (4.1) is

\[ M\ddot{x}_s + C\dot{x}_s + Kx_s = F_s \]  \hspace{1cm} (4.2)

where the subscript \( s \) denotes the time step, for example, \( x_s \) is the value of \( x \) at time step \( t_s \) such that the time step size \( \Delta t = t_{s+1} - t_s \) and \( t_0 = 0 \).
Applying the central difference method to Eq. (4.1) so that

\[ x_s = (\Delta t)^2 N_1 F_s + N_2 x_s + N_3 x_{s-1} \]  \hspace{1cm} (4.3)

where \( N_1 = [M + \frac{1}{2}(\Delta t)C]^{-1} \),

\[ N_2 = N_1[2M - (\Delta t)^2 K] \],

\[ N_3 = N_1[\frac{1}{2}(\Delta t)C - M] \].

Equation (4.3) expresses the displacement vector at the next time step in terms of the displacement vectors at the current and previous time steps.

Before proceeding further one assumes

\[ F_s = A_s w_s \]  \hspace{1cm} (4.4)

where \( A_s \) is a vector of discretized deterministic amplitude modulating functions, \( w_s \) is the zero mean Gaussian discretized white noise (GDWN) process.

Taking the transpose of Eq. (4.3) and making use of Eq. (4.4) one has

\[ x_{s+1}^T = (\Delta t)^2 N_1 F_s^T + N_2 x_s^T + N_3 x_{s-1}^T = (\Delta t)^2 F_s^T N_1^T + x_s^T N_2^T + x_{s-1}^T N_3^T \]  \hspace{1cm} (4.5)

where the superscript \( T \) denotes the ‘transpose of’.

Multiplying equations (4.3) and (4.5), taking ensemble average and rearranging gives [37]

\[
R_{s+1} = N_2 R_s N_2^T + N_3 R_{s-1} N_3^T + (\Delta t)^4 N_1 B_s N_1^T + N_2 D_s N_3^T + N_3 D_s N_2^T + (\Delta t)^2 N_2 \langle x_s F_s^T \rangle N_1^T \\
+ (\Delta t)^2 N_1 \langle F_s x_{s-1}^T \rangle N_2^T + (\Delta t)^2 N_3 \langle x_{s-1} F_{s-1}^T \rangle N_1^T \\
+ (\Delta t)^2 N_1 \langle F_s x_{s-1}^T \rangle N_3^T 
\]

(4.6)

where the angular brackets denote the ensemble average,

\[ R_s = \langle x_s x_s^T \rangle \]
\[ B_s = \langle F_s F_s^T \rangle \]

\[ D_s = \langle x_s x_{s-1}^T \rangle = N_2 R_{s-1} + N_3 D_{s-1}^T + (\Delta t)^2 N_1 (F_{s-1} x_{s-1}^T) \]  \hspace{1cm} (4.7)

For excitation modeled as Gaussian white noise or shot noise or Wiener process or their modulated forms the terms associated with \( F_s x_s^T \) and \( F_s x_{s-1}^T \) become zero as

\[ \langle F_s x_s^T \rangle = 0, \langle F_s x_{s-1}^T \rangle = 0. \]  \hspace{1cm} (4.8)

Applying equation (4.8) to equation (4.6), it reduces to

\[ R_{s+1} = N_2 R_s N_2^T + N_3 R_{s-1} N_3^T + (\Delta t)^4 N_1 B_s N_1^T + N_2 D_s N_2^T + N_3 D_{s-1} N_2^T \]  \hspace{1cm} (4.9)

Where now

\[ D_s = N_2 R_{s-1} + N_3 D_{s-1}^T \]

Equation (4.9) is the covariance matrix expression for displacement responses of multi-degree-of-freedom systems (MDOF) under external nonstationary random excitations. It can be used to recursively obtain the response in the time domain. It may be appropriate to note that when the excitations are stationary the discretized deterministic modulating functions are of unity. Of course, this is a special case to deterministic modulating or envelope functions defined by Eq. (4.4).

### 4.2 State Space Approach for Response Statistics

In the last section the covariance matrix defined by Eq. (4.9) is obtained by the SCD method. In order to provide an independent method for evaluating the response statistics of a MDOF system the state space approach is applied in this section. This is employed
because in Chapter 5 the state equation for the feedback controlled system is required. This means that the system of second order differential equations defined by Eq. (4.1) has to be transformed or cast into a system of first order differential equations such that the discretized covariance matrix of this system of first order differential equation can be obtained as [34]

\[
R_v = \begin{bmatrix}
<xx^T> & <xx^T > \\
<xx^T > & <xx^T >
\end{bmatrix}
\]

\[
R_{11} = <xx^T >, R_{12} = <xx^T >, R_{21} = <xx^T >, R_{22} = <xx^T >
\]  

(4.10)

in which \(R_{11}\) is given by Eq. (4.9), which represents the covariance matrix of the discretized displacements. The remaining submatrices in Eq. (4.10), \(R_{12} = R_{21}^T\), and \(R_{22} = <xx^T >\) can be obtained from

\[
<xx^T > = \frac{1}{\Delta t} \left(x_s^T x_s^T - x_s^T x_{s-1}^T\right).
\]  

(4.11)

The first term inside the brackets is given above. While the second term inside the brackets can be obtain from equation (4.7).

The remaining covariance of velocity matrix can be shown to be

\[
<xx^T > = \frac{1}{\Delta t} \left(x_{s+1}^T x_{s+1}^T - x_{s+1}^T x_s^T - x_s^T x_{s+1}^T + x_s^T x_s^T\right).
\]  

(4.12)

### 4.3 Damping Matrix

Generally speaking, all structures have structural damping. The damping matrix in Chapter 2 is general and it does not have to be proportional. However, for comparison to results published in the literature it is assumed to be proportional, so that

\[
C = \alpha_M M + \alpha_K K
\]  

(4.13)
where $\alpha_M$ and $\alpha_K$ are constants. The constants $\alpha_M$ and $\alpha_K$ can be determined by the first two vibration modes in which the two corresponding natural frequencies are $\omega_1$ and $\omega_2$, respectively. Thus, $\alpha_M$ and $\alpha_K$ can be derived using the following relations

$$\alpha_M + \alpha_K \omega_1^2 = 2\omega_1 \zeta_1$$

$$\alpha_M + \alpha_K \omega_2^2 = 2\omega_2 \zeta_2$$

(4.14)

where $\zeta_1$ and $\zeta_2$ are the critical damping ratios for the two vibrating modes respectively.

It should be noted that for the type of damping defined by Eq. (4.13) damping ratios for the third and higher modes are not independent. This means, in turn, only the first two damping ratios can be selected independently. The proportional damping has been selected for convenient. However, the damping matrix considered in Chapter 2 and in the SCD method introduced above are general.

### 4.4 Time Co-ordinate Transformation

For stiff or when the number of finite elements is large and therefore the highest natural frequency of the discretized structural system is high the time step size for the SCD method is very small. A time co-ordinate transformation was presented by To and Liu [26] to deal with this difficulty.

Assuming the system governed by Eq. (4.1) has its highest natural frequency $\Omega$. Divide both sides of Eq. (4.1) by the square of $\Omega$ and transforms the resulting equation of motion in the $t$-domain to the $\tau$-domain.

$$M\ddot{x} + \frac{1}{\Omega} C\dot{x} + \frac{1}{\Omega^2} Kx = \frac{1}{\Omega^2} F$$

(4.15)
Where the derivative is with respect to \( \tau = \Omega t \).

The variance or covariance of responses for the system described by Eq. (4.9) can then be evaluated with Eq. (4.8) in which the assembled mass matrix \( M \) is identical to that in Eq. (4.1) divided by the highest natural frequency \( \Omega \), the stiffness matrix \( K \) is equal to the original stiffness matrix \( K \) in Eq. (4.1) divided by \( \Omega^2 \), and the excitation vector \( F(\tau) \) is equal to \( F(\tau) / \Omega^2 \).

From Eq. (4.9), it can be shown that [38]

\[
R_y = \Omega R_y.
\]

The relations of time step size \( \Delta t \) and the natural frequency \( \omega \) have been investigated by To and Liu [26] for the systems under wideband stationary and nonstationary random excitations. The relations are

\[
\Delta t = 0.83 - 0.72 \log_{10} \omega \quad 1.0 < \omega < 5.0
\]

\[
\Delta t = 1.0 - 0.053 \omega - 0.12 \omega^2 \quad \omega \leq 1.0
\]

where \( \omega \) is the highest angular natural frequency, of the system, in radian per second while \( t \) is in second. These relations also apply to dimensionless time and angular natural frequency.

### 4.5 Application and Computation Results

A square bimorph plate with geometrical dimensions \( 1.0 \times 1.0 \times 0.005 \) m\(^2\) and material properties \( E = 2 \times 10^{11} \) N/m\(^2\), \( \rho = 7830 \) kg/m\(^3\), piezoelectric stress-constant \( d_{31} = 2.2 \times 10^{-11} \) C/N, and Poisson’s ratio \( \nu = 0.3 \). In the finite element idealization three
cases were considered for the cantilever plate. The first case is to represent the entire plate by 8 high precision triangular plate bending elements (HPTE) of To and Wang [39]. The second case is to use 8 shell elements to idealize the entire plate. The shell element employed was designated as AT+(k_t^3)' in [30]. That is, for the linear analysis, the consistent element stiffness matrix of the triangular shell element, 
\[ k = (k_m)^t + k_b + k_s + (k_t^3)' \]
where the subscripts m, b, and s denote, respectively, the membrane, bending and shear components of the stiffness matrix, and (k_t^3)' is the component associated with drilling degree of freedom (DDOF) which vary linearly over an element. The first three components on the rhs of the latter equation were derived explicitly by applying the hybrid strain formulation whereas the fourth component, (k_t^3) was obtained by employing the displacement formulation.

Figure 4.1 Cantilever plate represented by 2×2A mesh.
In the high precision bending plate triangular element case there are 27 unknown generalized displacements while, in the 8 shell triangular element (8TE) case there 24 unknown generalized displacements to be determined. The boundary conditions applied were: \( U = V = W = \theta_x = \theta_y = \theta_z = 0.0 \) on the clamped side, which is the line by Nodes 1, 2 and 3, where \( U, V \) and \( W \) are deformations along the global axes \( X, Y \) and \( Z \), respectively. Similarly, \( \theta_x, \theta_y \) and \( \theta_z \) are angular deformations about the global axes \( X, Y \) and \( Z \). In addition, \( U = \theta_z = 0.0 \) were imposed on all the Nodes so that no twisting would be allowed to occur. For the 8 triangular shell element model, the first two and the highest natural frequencies as well as Rayleigh damping coefficients corresponding to 5% damping for the first and second modes are listed in Table 4.1.
Table 4.1 Results of 8 lower order triangular shell element model

<table>
<thead>
<tr>
<th>Model</th>
<th>8 shell elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unknown displacements</td>
<td>24</td>
</tr>
<tr>
<td>$\omega_1$ (rad/s)</td>
<td>28.1</td>
</tr>
<tr>
<td>$\omega_2$ (rad/s)</td>
<td>136.5</td>
</tr>
<tr>
<td>$\Omega$ (rad/s)</td>
<td>$2.492 \times 10^5$</td>
</tr>
<tr>
<td>$\lambda_m$</td>
<td>2.33</td>
</tr>
<tr>
<td>$\lambda_k$</td>
<td>$6.075 \times 10^{-4}$</td>
</tr>
</tbody>
</table>
The damping matrix is obtained by using Eq. (4.13). The nonstationary random excitation was assumed to be a concentrated load and was applied transversally at the middle of the free edge opposite to the clamped one. That is, the point of load application is Node 8, which is the Node shared by element 5, 6, 7. The amplitude modulating function was

$$\varphi(t) = 9.4815(e^{-45t} - e^{-60t}).$$

(4.22)

The spectral intensity of the discrete white noise process was $S_0 = 1.0N^2$. Other form of amplitude modulating functions can be incorporated in the present analysis.

Computed variances of $Z$ direction displacements at Nodes 4, 5 and 8 are presented in Figures 4.2, 4.3, 4.4 and 4.5. The results presented in Figures 4.2, 4.3 and 4.4 are all obtained by the SCD method and therefore the Fortran program has been employed. The comparison studies presented in Figures 4.5, 4.6, 4.7, 4.8 employed both the SCD method and RK4 algorithm. The results using 8TE and HPTE calculated by SCD method have very good agreement. Using 8TE for this problem only, the same result can be achieved as shown in Figure. 4.5. The SCD and RK4 calculation are performed in Fortran and MATLAB. Comparing the computational time, SCD shows superior performance than MATLAB. Applying $2\times2$ D mesh for this case, the highest natural frequency is $1.20185\times10^6$. The time step size is $8.3\times10^{-7}$ such that the total number of steps in 1 second is approximately $1.2\times10^6$. With such a large number of time steps, the time dependent response matrix will be very large in MATLAB. The workstation will run out of memory easily, since it takes a lot of memory to store these data. Fortran does not have this problem, because the program stores one step response matrix at each time step rather than creating a time dependent response matrix in the
beginning of the calculation for all time step. However, the computer program written in Fortran does not have the graphic process after calculation. MATLAB can show the graphic result intermediately after the calculation.

A portable workstation Lenovo Thinkpad W530 is used for the current studies. It has Intel Core i7-3610QM, 3.30 GHz, 16 GB memory and 250 GB solid state drive. With the current machine, 4.2 seconds were required to run the code to solve the problem in Fortran, which employs SCD method. The RK4 is applied in MATLAB. When the RK4 algorithm was applied in MATLAB it took 11.015 seconds. But when the mesh was increased to $5 \times 1D$ which had 180 DOF the computational time by using RK4 in MATLAB was about 5 days comparing with the SCD method in Fortran it took about 8 hours to obtain the numerical results.
Table 4.2  Execution times by using SCD and RK4 algorithm for 8TE case

<table>
<thead>
<tr>
<th>Algorithm Executing Time</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>SCD</td>
<td>4.2 s</td>
<td></td>
</tr>
<tr>
<td>RK4</td>
<td>11.0 s</td>
<td></td>
</tr>
</tbody>
</table>
Fig. 4.2 Uncontrolled system variances of Z direction displacement at Node 8 with HPTE mesh and 8TE mesh

Fig. 4.3 Uncontrolled system variances of Z direction displacement at Node 5 with HPTE mesh and 8TE mesh
Fig. 4.4 Uncontrolled system variances of Z direction displacement at Node 4 with HPTE mesh and 8TE mesh

Fig. 4.5 Uncontrolled system variances of Z direction displacement at Node 5 with 8TE mesh
Fig. 4.6 Uncontrolled system variances of Z direction displacement at Node 8 with 8TE mesh

Fig. 4.7 Uncontrolled system variances of Z direction displacement at Node 4 with 8TE mesh
Fig. 4.8 Uncontrolled system variances of Z direction displacement at Node 9 with 8TE mesh
CHAPTER 5  COMPARISON STUDIES OF CONTROLLED RESPONSE STATISTICS

The control dealt with this part of the investigation is achieved through the application of the multi-layer structures whose FE formulation and element equations of motion have been included in Chapter 2. Specifically, the feedback control is performed with the actuating and sensing layers of the smart triangular element.

Section 1 is concerned with the derivation of the state equations for the discretized plate structure. Lyapunov equation for the system is introduced in Section 2. Here the RK4 scheme for the solution of the Lyapunov equation is outlined. Section 3 is concerned with the computed results for the discretized plate structure by applying the SCD method and RK4 algorithm. Comparison of these two sets of results is made. Remarks for this chapter are included in Section 5.4.

5.1 System Equations

Consider the discretized plate structure with piezoelectric layers subjected to random excitations. The matrix equation of motion can be expressed as in Eq. (3.1). Then the covariance matrix of generalized displacement responses can be computed by applying the SCD method introduced in Chapter 4. In order to provide an independent method for comparison the RK4 algorithm can be employed. To this end, one has to cast the system of second order differential equations in Eq. (3.1) into the first order differential equations. Consequently, Eq. (3.1) can written as

$$\dot{Z} = AZ + P$$  \hspace{1cm} (5.1)
where
\[ Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \quad z_1 = X, \quad z_2 = \dot{X}, \quad P = M^{-1} \begin{pmatrix} 0 \\ \eta \end{pmatrix} w, \quad F = \eta w, \]
\[ A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \]
in which \( w \) or \( w(t) \) is the zero mean Gaussian white noise, and \( \eta \) or \( \eta(t) \) is a \( 2n \times 1 \) vector of deterministic modulating functions of time vector, assuming \( n \) is the number of DOF in Eq. (5.1), and \( I \) is the unit matrix of order \( n \times n \).

For discrete signal, the continuous white noise process \( w \) is replaced by the zero mean Gaussian discrete white noise (GDWN) process \( w_D \) so that
\[ \langle w_D \rangle = 0, \quad \langle w_D(t_s)w_D(t_s + \Delta t) \rangle = 2\pi S_o \delta_K(\Delta t) \] (5.2)
with \( S_o \) meaning the spectral density of the GDWN process and \( \delta_K(\Delta t) \) the Kronecker delta function such that \( \delta_K(0) = 1 \) and \( t_s \) is the time at step \( s \) with \( s = 0, 1, 2, \ldots \)

5.2 Lyapunov Equation of Motion and Runge-Kutta Fourth Order Scheme

To derive the Lyapunov equation of motion for the foregoing system, one rewrites Eq. (5.1) as
\[ \frac{dZ}{dt} = AZ + P. \] (5.3)
Taking the transpose of Eq. (5.3), one obtains
\[ \frac{dZ^T}{dt} = Z^T A^T + P^T. \] (5.4)
Multiplying by the vector \( Z \), it gives
\[ Z \frac{dZ^T}{dt} = ZZ^T A^T + ZP^T. \] (5.5)
Post multiplying Eq. (5.3) by the transpose of \( Z \) one has

\[
\frac{dZ}{dt} Z^T = AZZ^T + PZ^T. \tag{5.6}
\]

Adding Eq. (5.5) to (5.6), one has

\[
Z \frac{dZ^T}{dt} + \frac{dZ}{dt} Z^T = ZZ^T A^T + ZP^T + AZZ^T + PZ^T. \tag{5.7}
\]

Since

\[
Z \frac{dZ^T}{dt} + \frac{dZ}{dt} Z^T = \frac{dZZ^T}{dt}
\]

After taking the ensemble averages on both sides of Eq. (5.7), one can show that

\[
\frac{d\bar{R}}{dt} = A\bar{R} + \bar{R} A^T + \sigma, \tag{5.8}
\]

in which the covariance matrix of state vector \( Z \) is defined by \( \bar{R} = \langle ZZ^T \rangle \), \( \sigma \) is the covariance matrix of the applied nonstationary random excitation vector such that its discretized form becomes

\[
\sigma_s = 2\pi S_o \begin{bmatrix} 0 & 0 \\ 0 & M^{-1} \eta_s \eta_s^T (M^{-1})^T \end{bmatrix}.
\]

Equation (5.8) is known as the Lyapunov equation [38].

In order to solve for Eq. (5.8) numerically one can apply the RK4 scheme. The algorithm to obtain \( \bar{R} \) may be written as [38]

\[
t = n\Delta t, \quad f(t, \bar{R}) = \frac{d\bar{R}}{dt} = A\bar{R} + \bar{R} A^T + \sigma, \tag{5.9a, b}
\]

\[
\varphi_1 = (\Delta t) f(t, \bar{R}), \quad \varphi_2 = (\Delta t) f\left(t + \frac{\Delta t}{2}, \bar{R} + \frac{\varphi_1}{2}\right), \tag{5.9c, d}
\]

\[
\varphi_3 = (\Delta t) f\left(t + \frac{\Delta t}{2}, \bar{R} + \frac{\varphi_2}{2}\right), \quad \varphi_4 = (\Delta t) f\left(t + \Delta t, \bar{R} + \varphi_3\right), \tag{5.9e, f}
\]

\[
\bar{R}(t + \Delta t) = \bar{R}(t) + \frac{1}{6} (\varphi_1 + 2\varphi_2 + 2\varphi_3 + \varphi_4). \tag{5.9g}
\]
5.3 Controlled Response Statistics of Square Plate Structures

A square bimorph plate with geometrical dimensions $1.0 \times 1.0 \times 0.005\,\text{m}$ and material properties $E = 2 \times 10^{11}\,\text{N/m}^2$, $\rho = 7830\,\text{kg/m}^3$, Poisson’s ratio $\nu = 0.3$, and piezoelectric stress-constant $d_{31} = 2.2 \times 10^{-11}\,\text{C/N}$. The remaining material and geometrical properties are identical to those presented in Section 4.4.

5.4 Actuator Equation

For structure with piezoelectric layers, the element equation of motion Eq (2.24) in Chapter 2 is

$$
\begin{bmatrix}
m & 0 \\
0 & 0
\end{bmatrix}
\begin{pmatrix}
\ddot{q}_u \\
\ddot{q}_\phi
\end{pmatrix}
+ \begin{bmatrix}
k_{uu} & k_{\phi u}^T \\
k_{\phi u} & -k_{\phi \phi}
\end{bmatrix}
\begin{pmatrix}
q_u \\
q_\phi
\end{pmatrix}
= \begin{pmatrix}
F \\
Q
\end{pmatrix}
$$

where the matrix can be obtained by Eq (2.21).

$$
[m] = \int_V \rho \{N_u\}^T \{N_u\} \,dV ,
$$

$$
[H] = \int_V \begin{bmatrix} P_\beta \end{bmatrix}^T \begin{bmatrix} c \end{bmatrix} \begin{bmatrix} P_\beta \end{bmatrix} \,dV ,
$$

$$
[G_e] = \int_V \begin{bmatrix} P_\beta \end{bmatrix}^T \begin{bmatrix} c \end{bmatrix} \begin{bmatrix} B_u \end{bmatrix} \,dV ,
$$

$$
[k_{\phi u}] = \int_V \begin{bmatrix} B_\phi \end{bmatrix}^T \begin{bmatrix} e \end{bmatrix} \begin{bmatrix} B_u \end{bmatrix} \,dV ,
$$

$$
[k_{\phi \phi}] = \int_V \begin{bmatrix} B_\phi \end{bmatrix}^T \begin{bmatrix} e \end{bmatrix} \begin{bmatrix} B_\phi \end{bmatrix} \,dV ,
$$

$$
\{F\} = \int_V \{N_u\}^T \{b\} \,dV + \int_{S_b} \{N_u\}^T \{t\} \,ds , \text{ and}
$$
\[ \{Q\} = \int_{s_v} [N_\varphi]^T \{Q_s\} ds \]

where \([m]\) is the consistent element mass matrix, \([H]\) the element stiffness matrix, \([G_e]\) the leverage matrix, \([k_{\varphi\varphi}]\) the piezoelectric stiffness matrix, \([k_{\varphi\psi}]\) the dielectric stiffness matrix, \([F]\) the force vector, and \([Q]\) the electric charge vector.

Equation (2.24) introduces a general form of the element equation of motion. By applying the different electric boundary conditions, the sensor equation and the actuator equation can be obtained.

In this Chapter, only the actuator equation will be discussed.

Apply voltage and charge the piezoelectric surface is the two ways to actuate it.

When the voltage is applied, compared with the applied voltage, the self-generated voltage caused by the mechanical deformation is relatively small so that it can be ignored. \([Q]\) is zero, Eq (2.24) in the current case’s form is

\[
[M] \{\ddot{X}_u\} + [C_{uu}] \{\dot{X}_u\} + [K_{uu}] \{X_u\} = \{F\} - [K_{\varphi\psi}]^T \{X_\varphi\} \tag{5.10}
\]

The SCD method and RK4 scheme are used to solve this equation.

5.5 Application and Computation Results

The cantilever square bimorph plate case in Chapter 4 is studied here. The piezoelectric bimorph plate is made of two piezoelectric layers PVDF laminated together with opposite polarity in the thickness direction. Its geometrical dimensions \(1.0 \times 1.0 \times 0.005\) m and material properties \(E = 2 \times 10^{11}\) N/m², \(\rho = 7830\) kg/m³, piezoelectric stress-constant \(d_{31} = 2.2 \times 10^{-11}\) C/N, permittivity \(\varepsilon_{11} = 1.062 \times 10^{-10}\) F/m, \(\varepsilon_{22} = 1.062 \times 10^{-10}\) F/m, \(\varepsilon_{33} = 1.062 \times 10^{-10}\) F/m and Poisson’s ratio \(\nu = 0.3\). When a positive voltage applied on
the top and bottom surfaces of the plate, the piezoelectric effect will appear. The upper layer will be in compression state and the lower layer will be tension state. It will cause the flexural deflection.

The external force caused by piezoelectric effect is:

\[ F_p = -\left[K_{\phi u}\right]^T \{X_\phi\} \]  \hspace{1cm} (5.11)

It can be calculated from the equations below:

\[ \{X_\phi\} = \left[K_{\phi\phi}\right]^{-1} \left[K_{\phi u}\right] \{X_u\} \]  \hspace{1cm} (5.12)

\[ \left[K_{\phi\phi}\right] \] is a symmetric matrix. Permittivity \(\varepsilon_{11} = 1.062 \times 10^{-10} \text{ F/m}, \varepsilon_{22} = 1.062 \times 10^{-10} \text{ F/m}, \varepsilon_{33} = 1.062 \times 10^{-10} \text{ F/m}\). All its elements are very small. It is an ill condition matrix. So a special algorithm is needed to deal with the inverse of this matrix. However, this is not performed in current study.

Rather than calculate \(F_p\) through Eq (5.11) and Eq (5.12), \(F_p\) is treated as an averagely distributed constant force on each Node. \(F_p\) can be obtained through the equation below [40]:

\[ F_p = \frac{3 \times E \times W \times d_{31} \times T \times V}{2 \times l} \text{ N} \]

where

\(d_{31}\) = piezoelectric coefficient in the "1" direction

L, T, W = length, thickness, and width of piezo-film

V = applied voltage (Volts)

E = Young's modulus of piezo-film (2x10^9 N/m²)
Nodes 1, 2, 3 are fully constrained. Node 1 is shared by elements 1 and 2. So this Node shares \((2/\text{total})\times F_p\). Node 2 shares \((3/\text{total})\times F_p\), Node 3 shares \((1/\text{total})\times F_p\), Node 4 is shared by elements 1, 5 and 6. So this Node shares \((3/\text{total})\times F_p\). Node 5 is shared by elements 1, 2, 3, 6, 7 and 8. So it will share \((6/\text{total})\times F_p\). Node 6 shares \((3/\text{total})\times F_p\). Node 7 shares \((1/\text{total})\times F_p\). Node 8 shares \((3/\text{total})\times F_p\). Node 9 shares \((2/\text{total})\times F_p\). The total is 2+3+1+3+6+3+1+3+2 = 24.

After above calculation, the forcing vector \(F_p\) will replace the term \(-\left[K_{\phi u}\right]^T \{\chi_{\phi}\}\) in Eq (5.10).

The nonstationary random excitation is the same as that presented in Chapter 4.

The calculation of dynamic responses is performed by applying SCD and RK4 in Fortran and MATLAB.

Figures 5.1 to 5.3 are concerned with the comparison between the result of SCD and RK4 for the controlled variance of displacement in the Z-direction at various Nodes and under 100 V. From these plots, it is observed that “identical results” between those evaluated by the SCD method and RK4 algorithm were obtained. Further control of plate with applied voltages of 300V, 900V and 1800V was made. The responses were computed using the SCD method.

Figure 5.5 shows the variance of displacement in Y direction is larger than that in Fig 5.4 which is variance of displacement in Z direction. Since the excitation is applied at Node 8, which is the tip of the plate at Z direction. At an extremely short time, the displacement in Z increases at Node 5. However, the plate is dragged to be elongated, which is Y
direction. If the displacement is in local coordinate system, the displacement in Y direction should be bigger than Z. But in Global coordinate system, difference between the displacements in Y and Z is depended on stiffness difference in these two directions and the angle the plate bends under excitation.

Figures 5.4 to 5.7 present the controlled response by applying larger voltage. The non-stationary random excitation is applied along the Z-direction at Node 5. When a voltage of 100V is applied, the peak variance of displacement drop by about 6.7% compared with the uncontrolled case. When a voltage of 300V is applied, it drops by about 8.3%. With a voltage of 900V, it drops by about 21.3%. With a voltage of 1800V, it drops by around 34.8%.

Figure 5.8 to Figure 5.12 presents the change of variance of velocity at X, Y Z direction at Node 5 and Z direction at Nodes 4, 8, 9. Every figure shows that with the increase of variance of applied voltage, the variance of velocity increases a lot. The reason for it is that when control voltage is applied and increased, the force applied on the plate becomes larger. At an extremely short time, the velocity will increase. In addition, the variance of velocity indicates the rate of how fast the variance of displacement changes. When applied voltage increases, at the same time period, the variance of displacement drops more with larger applied voltage. This proves the results presented by Figure 5.8 to Figure 5.12 are logical and right.
Fig. 5.1 Controlled system variance of Z direction displacement at Node 4 under 100 V using RK4 scheme and SCD method

Fig. 5.2 Controlled system variance of Z direction displacement at Node 5 under 100 V using RK4 scheme and SCD method
Fig. 5.3 Controlled system variance of Z direction displacement at Node 8 under 100 V using RK4 scheme and SCD method

Fig. 5.4 Controlled system variance of Z direction displacement at Node 5 under 100, 300, 900, 1800 V using SCD method
Fig. 5.5 Controlled system variance of Y direction displacement at Node 5 under 100, 300, 900, 1800 V using SCD method

Fig. 5.6 Controlled system variance of Z direction displacement at Node 8 under 100, 300, 900, 1800 V using SCD method
Fig. 5.7 Controlled system variance of Z direction displacement at Node 9 under 100, 300, 900, 1800 V using SCD method

Fig. 5.8 Controlled system variance of Y direction velocity at Node 5 under 100, 300, 900, 1800 V using SCD method
Fig. 5.9 Controlled System Variance of Z direction Velocity at Node 5 under 100, 300, 900, 1800 V using SCD method

Fig. 5.10 Controlled System Variance of Z direction Velocity at Node 4 under 100, 300, 900, 1800 V using SCD method
Fig. 5.11 Controlled System Variance of Z direction Velocity at Node 8 under 100, 300, 900, 1800 V using SCD method

Fig. 5.12 Controlled System Variance of Z direction Velocity at Node 9 under 100, 300, 900, 1800 V using SCD method
CHAPTER 6  CONCLUSIONS AND RECOMMENDATIONS

6.1 Introduction

Current study shows the control of smart plate structure under non-stationary random excitation. The mathematical model of linear systems with piezoelectric material layers is established. It is based on a lower order triangular shell finite element with 7 DOF including piezoelectric effect. The responses of linear system under nonstationary random excitations are presented. These results are obtained using stochastic central difference (SCD) method and Runge-Kutta fourth order (RK4) algorithm. The computed results are compared with results obtained by former studies. The accuracy and efficiency of the SCD method is compared with those of the RK4 scheme. Finally in Chapter 5, the systems investigated have lower degrees of freedom and smart material PVDF. The computational results of variance of responses of controlled plate under applied voltage are presented in Chapter 5. A very good control performance is achieved.

For the first time response statistics such as variances of generalized displacements and velocities of smart plate structures under nonstationary random excitations are obtained. By applying the SCD method, nonstationary random responses of plate structures can be actively controlled via the application of a piezoelectric material layer.

The SCD method has better capability and efficiency than the RK4 scheme using the same engineering workstation.
6.2 Conclusion

Lower order triangular shell finite element with piezoelectric layers can be used to establish the model for structure with smart materials. It has a superior performance than other triangular finite elements. For example, six rigid modes can be correctly obtained. The accuracy of natural frequencies obtained by the lower order triangular shell finite element is also better than those previously reported by other researchers as shown in Chapter 2.

Compared with the high precision plate element, the lower order triangular shell element has a good agreement in regard to the computed responses of plate structures under non-stationary excitation.

The SCD method and RK4 scheme have been applied to solve the equation of motion for the stochastic responses. Computational time is much shorter by using the SCD method than the RK4 algorithm. For example presented, the computational time is 4.2 seconds by the SCD method written in Fortran. It is compared with 11.05 seconds by the RK4 scheme implemented in MATLAB. Thus, the SCD method in Fortran showed a better computational efficiency and capability than the RK4 scheme in MATLAB. MATLAB does not have the capability to handle large DOF systems within a portable engineering workstation. This is because it needs very large memory in a workstation to store all the matrices.

In Chapter 5, an example by applying the actuation equation is presented. The excitation is applied along the Z direction at Node 8, which is at the middle of the free end of the plate. At an extremely short time, the displacement in Z direction increases at Node 5. However, the plate is elongated at the Y direction. If the displacement is in the local
coordinate system, the displacement in Y direction should be bigger than that in the Z direction. But in the global coordinate system, difference between the displacements in Y and Z is dependent of the stiffness difference in these two directions and the angle the plate bends under excitation.

The current example shows very good control performance. The non-stationary random excitation is applied along the Z-direction at Node 5. When a voltage of 100V is applied, the peak variance of displacement drop by about 6.7% compared with the uncontrolled case. When a voltage of 300V is applied, it drops by about 8.3%. With a voltage of 900V, it drops by about 21.3%. With a voltage of 1800V, it drops by around 34.8%.

Figure 5.8 to Figure 5.12 presents the change of variance of velocity in the X, Y, Z directions at Node 5 and Z direction at Nodes 4, 8, 9. Every figure shows that with the increase of variance of applied voltage, the variance of velocity increases significantly. The reason for this is that when control voltage is applied and increased, the force applied on the plate becomes larger. At an extremely short time, the velocity will increase. In addition, the variance of velocity indicates the rate of how fast the variance of displacement changes. When the applied voltage increases, at the same time period, the variance of displacement drops more with larger applied voltage. This proves that the results presented by Figure 5.8 to Figure 5.12 are logical and correct.

The numerical procedures presented in this study enable one to perform the finite element analysis of dynamic response of linear smart systems under nonstationary excitations.

6.3 Recommendations

The above studies show very good control of variance of displacement of a system.
However, when the piezoelectric material layer is very thin computational instability occurs and therefore future work should explore technique(s) to resolve such an issue.

The foregoing studies were focused on linear deformations and thus the investigation of geometrically nonlinear random responses of smart plate and shell structures would be a logical extension.

The stochastic optimal control of linear and geometrically nonlinear random responses of smart plate and shell structures should be investigated.

In this thesis, only numerical solutions are provided. So verification tests should be conducted to compare with the computational results.

The SCD method presented in this study is implemented in Fortran. After computed results from the Fortran program are output in a data file MATLAB is used to produce the figures. An efficiency code in MATLAB should be developed in the future for the direct production of figures from the computed results of the SCD method. In addition, a memory release structure should be developed to avoid the issue of limited memory in MATLAB.

The current study is focused on control of variances of displacements. Future studies should also investigate the control of variances of velocities.

References


