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Stable Cohomology Of Local Rings And Castelnuovo-Mumford Regularity Of Graded Modules

Luigi Ferraro
University of Nebraska-Lincoln, luigi.ferraro@huskers.unl.edu

STABLE COHOMOLOGY OF LOCAL RINGS AND CASTELNUOVO-MUMFORD REGULARITY OF GRADED MODULES

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Luigi Ferraro

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Luigi Ferraro, Ph.D.

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Advisers: L. L. Avramov and S. B. Iyengar

This thesis consists of two parts:

1) A bimodule structure on the bounded cohomology of a local ring (Chapter 1),

2) Modules of infinite regularity over graded commutative rings (Chapter 2).

Chapter 1 deals with the structure of stable cohomology and bounded cohomology. Stable cohomology is a $\mathbb{Z}$-graded algebra generalizing Tate cohomology and first defined by Pierre Vogel. It is connected to absolute cohomology and bounded cohomology. We investigate the structure of the bounded cohomology as a graded bimodule. We use the information on the bimodule structure of bounded cohomology to study the stable cohomology algebra as a trivial extension algebra and to study its commutativity.

In Chapter 2 it is proved that if a graded, commutative algebra R over a field k is not Koszul, then the nonzero modules $\mathfrak{m}M$, where M is a finitely generated R-module and $\mathfrak{m}$ is the maximal homogeneous ideal of R, have infinite Castelnuovo-Mumford regularity.

DEDICATION

To my family and my friends.

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Introduction

In this thesis we use homological tools to study commutative rings. Commutative algebra originated with the study of systems of polynomial equations in several variables. Their solutions sets are called algebraic varieties. Varieties are studied by the closely related field of algebraic geometry. The introduction of homological algebra in the 1950s gave a fresh approach to the subject and provided new directions for exploration.

In commutative algebra we study modules over a commutative ring R which are analogues of vector spaces over a field k. Finite dimensional vector spaces are all of the form k^n for some n. It is not true that any finitely generated R-module is of the form R^n. Modules of the form R^n are called free modules. One way to study a module is to approximate it by free modules. Iterations of such a construction leads to objects called free resolutions. To understand resolutions we look for invariants attached to them, usually through some cohomology theory.

Algebraic geometry studies varieties by studying rings of functions defined on a neighbourhood of the points on the variety. These rings are local rings, i.e. rings with an unique maximal ideal. In this thesis we study local rings or graded rings with unique maximal homogeneous ideal.

In Chapter 1 we study the structure of the stable cohomology of a local ring. Stable cohomology was introduced, in an unpublished work, by P. Vogel. In [19] Goichot calls it Tate-Vogel cohomology. It is a generalization of Tate cohomology for modules over finite group rings. In commutative algebra this cohomology was studied by Avramov and Veliche [10] under the name stable cohomology, for it brings out its relation to the stabilization of module categories.

For modules over graded rings, there is an invariant arising from free resolutions, namely the Castelnuovo-Mumford regularity, a numerical invariant that controls how complex a free resolution is. In Chapter 2 we prove that a certain class of modules has infinite regularity over quotients of polynomial rings. This construction is used to prove that if a ring admits a module with infinite regularity then any nonzero module has either infinite regularity or is an extension of modules with infinite regularity.

Next we give more specific descriptions of the main results in each part.

Stable Cohomology

Let $(R, \mathfrak{m}, k)$ be a local Noetherian commutative ring with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. The ring R is *Gorenstein* if the injective dimension of R as ma module over itself is finite. In Chapter 1 we study the structure of the stable cohomology of a Gorenstein ring.

Stable cohomology $\widehat{\mathrm{Ext}}_R$ and absolute cohomology Ext_R are linked to another cohomology theory, bounded cohomology. In [10] Avramov and Veliche give a description of bounded cohomology as a left module. In Chapter 1 we give a description of bounded cohomology as a right module and use this structural information to prove Theorem 1.5.2. Before stating Theorem 1.5.2 we recall that the depth of a connected k-algebra A is the minimum i such that $\mathrm{Ext}^i_A(k, A) \neq 0$ and it is denoted by depth A. The absolute cohomology $\mathrm{Ext}_R(k, k)$ has a structure of graded k-algebra so $\operatorname{depth} \mathrm{Ext}_R(k, k)$ is a well-defined numerical invariant of this algebra. Now we can state the theorem:

Theorem. *If R is a Gorenstein ring of dimension d with* $\operatorname{depth} \operatorname{Ext}_R(k,k) \geq 2$ *then the stable cohomology algebra is a trivial extension algebra*

$$\widehat{\operatorname{Ext}}_R(k,k) \cong \operatorname{Ext}_R(k,k) \ltimes \Sigma^{1-d} \operatorname{Tor}^R(k,k).$$

This theorem is used to give a complete description of the stable cohomology algebra for complete intersection rings, which was previously known only for hypersurfaces. Noteworthy consequence of Theorem 1.5.2 is

Corollary 0.0.1. *The algebra $\widehat{\operatorname{Ext}}_R(k,k)$ is graded-commutative if and only if R is a complete intersection defined by relations that are at least cubic.*

Castelnuovo-Mumford Regularity

In Chapter 2 we turn our attention to quotients of polynomial rings. Let R be the ring $k[x_1,\dots,x_n]/I$ with $\deg x_i = d_i > 0$ and I a homogeneous ideal. We denote by $\mathfrak{m}$ the maximal homogeneous ideal. If M is a graded R-module then $\operatorname{reg}_R M$ denotes the Castelnuovo-Mumford regularity of M. The ring R is said to be a *Koszul ring* if $\operatorname{reg}_R k < \infty$. It is known that over Koszul rings every module has finite regularity and that this characterizes Koszul rings.

If R is not a Koszul ring then there are modules of infinite regularity. In Chapter 3 we identify a class of modules with infinite regularity.

We say that a finitely generated module M is *tightly embeddable* if there exists a finitely generated module L such that $\mathfrak{m}M \subsetneqq \mathfrak{m}L \subseteq M \subseteq L$. Avramov noticed in [3] that these modules have special homological properties. Nonzero modules of the form $\mathfrak{m}L$ for some L are tightly embeddable (by Nakayama). Any nonzero module M is either tightly embeddable or it is an extension of tightly embeddable modules, we just need to notice that k is tightly embeddable by taking $L = R/((x_2, \ldots, x_n) + \mathfrak{m}^2)$.

Theorem. *If R is not a Koszul ring, then any tightly embeddable module has infinite regularity.*

This gives us a simple recipe to construct modules with infinite regularity over rings which are not Koszul: if L is any R-module with $\mathfrak{m}L \neq 0$ then $\mathfrak{m}L$ has infinite regularity. This Theorem also shows that any nonzero module either has infinite regularity or is an extension of modules with infinite regularity.

Chapter 1

A bimodule structure on the bounded cohomology of a local ring

1.1 Introduction

Stable cohomology associates to every pair (M, N) of R-modules, modules

$$\widehat{\operatorname{Ext}}_R^n(M, N), \quad n \in \mathbb{Z}$$

which are zero if M or N have finite projective dimension. There is a canonical transformation $\iota : \operatorname{Ext}_R \to \widehat{\operatorname{Ext}}_R$ of absolute cohomology to stable cohomology.

We focus on local commutative Noetherian rings $(R, \mathfrak{m}, k)$ that are not regular, since in that case $\widehat{\operatorname{Ext}}_R^n(\_, \_) = 0$ for every n. Under this hypothesis Martsinkovsky in [27] proved that $\iota : \operatorname{Ext}_R(k, k) \to \widehat{\operatorname{Ext}}_R(k, k)$ is an injective map. We want to understand the algebra structure of $\widehat{\operatorname{Ext}}_R(k, k)$ and to do that we need to determine the $\operatorname{Ext}_R(k, k)$-bimodule structure of the cokernel of ι. The left module structure of this cokernel was already studied in [10]. We describe the right module structure and use it to determine the structure of $\widehat{\operatorname{Ext}}_R(k, k)$ for Gorenstein rings for which ι is split as a map of $\operatorname{Ext}_R(k, k)$-bimodules.

We then direct our attention to the complete intersection case, in which we give a complete characterization of the algebra structure of $\widehat{\mathrm{Ext}}_R(k,k)$. It depends on the codimension of the ring.

1.2 DG Lie Algebras and Modules

Let R be a commutative ring. Let $\mathfrak{g}$ be a DG Lie algebra over R with differential $\partial^{\mathfrak{g}}$ (see [2, Chapter 10] for the definition of DG Lie algebra). A DG R-module M is a (right) DG Lie $\mathfrak{g}$-module if there exists a map

$$M \otimes_R \mathfrak{g} \to M$$

satisfying the following conditions, for $m \in M$ and $\theta, \xi \in \mathfrak{g}$, where we denote $m \otimes \theta$ by $m \cdot \theta$:

1) $\partial^M(m \cdot \theta) = \partial^M(m) \cdot \theta + (-1)^{|m|} m \cdot \partial^{\mathfrak{g}}(\theta)$, where ∂^M is the differential of M,

2) $m \cdot [\theta, \xi] = (m \cdot \theta) \cdot \xi - (-1)^{|\theta||\xi|}(m \cdot \xi) \cdot \theta$,

3) $m \cdot \theta^{[2]} = (m \cdot \theta) \cdot \theta$, for $\theta \in \mathfrak{g}^{\mathrm{odd}}$,

The definition of DG left $\mathfrak{g}$-module is similar.

If M is a DG left $\mathfrak{g}$-module we can turn it into a DG right $\mathfrak{g}$-module in the following way

$$m \cdot \theta := -(-1)^{|\theta||m|} \theta \cdot m, \quad m \in M, \theta \in \mathfrak{g},$$

A routine computation shows that this is indeed an action.

If M and N are DG right $\mathfrak{g}$-module then $M \otimes_R N$ is a DG right $\mathfrak{g}$-module with action

$$(m \otimes n) \cdot x := m \otimes (n \cdot x) + (-1)^{|x||n|}(m \cdot x) \otimes n, \quad m \in M, n \in N, x \in \mathfrak{g},$$

similarly for tensor product of left modules.

If $\mathfrak{g}$ is a graded Lie k-algebra with k a field, we denote by $U\mathfrak{g}$ its universal enveloping algebra (see [2, Chapter 10] for the definition). Notice that a Lie $\mathfrak{g}$-module is just a $U\mathfrak{g}$-module.

1.3 Stable and bounded cohomology

In this section we recall the construction of stable cohomology. Let R be a commutative ring, and let L and M be R-modules. Choose projective resolutions P and Q of L and M, respectively. Recall that a homomorphism $P \to Q$ of degree n is a family $\beta = (\beta_i)_{i\in\mathbb{Z}}$ of R-linear maps $\beta_i \colon P_i \to Q_{i+n}$; that means an element of the R-module

$$\mathrm{Hom}_R(P,Q)_n = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(P_i, Q_{i+n})\,.$$

This module is the n-th component of a complex $\mathrm{Hom}_R(P,Q)$, with differential

$$\partial(\beta) = \partial^Q\beta - (-1)^{|\beta|}\beta\partial^P.$$

The maps $\beta\colon P \to Q$ with $\beta_i = 0$ for $i \gg 0$ are called *bounded maps* and they form a subcomplex with component

$$\overline{\mathrm{Hom}}_R(P,Q)_n = \coprod_{i\in\mathbb{Z}} \mathrm{Hom}_R(P_i, Q_{i+n}) \quad \text{for} \quad n \in \mathbb{Z}\,.$$

We write $\widehat{\mathrm{Hom}}_R(P,Q)$ for the quotient complex. It is showed in [10] that this complex is independent of the choices of P and Q up to R-linear homotopy, and so is the exact sequence of DG $\mathrm{End}_R(Q) - \mathrm{End}_R(P)$-bimodules

$$0 \longrightarrow \overline{\mathrm{Hom}}_R(P,Q) \longrightarrow \mathrm{Hom}_R(P,Q) \longrightarrow \widehat{\mathrm{Hom}}_R(P,Q) \longrightarrow 0\,. \tag{1.3.0.1}$$

The *stable cohomology* of the pair (L, M) is the graded R-module $\widehat{\mathrm{Ext}}_R(L, M)$ with

$$\widehat{\mathrm{Ext}}{}^n_R(L,M) = \mathrm{H}^n(\widehat{\mathrm{Hom}}_R(P,Q)) \quad \text{for each} \quad n \in \mathbb{Z}\,.$$

The *bounded cohomology* of the pair (L, M) is the graded R-module $\overline{\mathrm{Ext}}_R(L, M)$ with

$$\overline{\mathrm{Ext}}{}^n_R(L,M) = \mathrm{H}^n(\overline{\mathrm{Hom}}_R(P,Q)) \quad \text{for each} \quad n \in \mathbb{Z}\,.$$

The sequence (1.3.0.1) defines an exact sequence

$$\begin{gathered}\overline{\mathrm{Ext}}_R(L,M) \xrightarrow{\eta_R} \mathrm{Ext}_R(L,M) \xrightarrow{\iota_R} \widehat{\mathrm{Ext}}_R(L,M) \xrightarrow{\eth_R} \\ \Sigma\overline{\mathrm{Ext}}_R(L,M) \xrightarrow{\Sigma\eta_R} \Sigma\,\mathrm{Ext}_R(L,M)\end{gathered} \tag{1.3.0.2}$$

of graded $\mathrm{Ext}_R(M, M)$-$\mathrm{Ext}_R(L, L)$-bimodules.

We refer to [10] for a treatment on stable cohomology.

1.4 A bimodule structure on the complex of bounded maps

Let $(R, \mathfrak{m}, k)$ be a commutative local Noetherian ring. For the rest of the paper F will denote the acyclic closure of k, i.e. a DG algebra minimial free resolution of k with divided powers, see [2, 6.3] for details. We denote by $\mathrm{Der}^\gamma_R(F)$ the subcomplex of $\mathrm{End}_R(F)$ of the Γ-derivations, i.e. R-linear endomorphisms of F satisfying the Leibniz rule and respecting the divided power structure of F; see [2, 6.2.2] for details. The complex $\mathrm{Der}^\gamma_R(F)$ is a DG Lie R-subalgebra of $\mathrm{End}_R(F)$, where the Lie structure on

$\mathrm{End}_R(F)$ is defined as

$$[\theta, \xi] := \theta\xi - (-1)^{|\theta||\xi|}\xi\theta,$$

$$\zeta^{[2]} := \zeta^2,$$

$$\text{for } \theta, \xi \in \mathrm{End}_R(F), \zeta \in \mathrm{End}_R(F)^{\mathrm{odd}}.$$

Let N be a finitely generated R-module and G a free resolution of N. We want to define a structure of DG $\mathrm{End}_R(G)$-$\mathrm{Der}^{\gamma}_R(F)$-bimodule on $\mathrm{Hom}_R(F, R) \otimes_R (F \otimes_R G)$. For $\alpha \in \mathrm{End}_R(G), \theta \in \mathrm{Der}^{\gamma}_R(F), \varphi \in \mathrm{Hom}_R(F, R), f \in F, g \in G$, we set the left and right products as follow:

$$\alpha \cdot (\varphi \otimes f \otimes g) := (-1)^{|\alpha|(|\varphi|+|f|)}\varphi \otimes f \otimes \alpha(g),$$
$$(\varphi \otimes f \otimes g) \cdot \theta := (-1)^{|\theta|(|f|+|g|)}((\varphi\theta) \otimes f \otimes g - \varphi \otimes \theta(f) \otimes g).$$

The right action is the tensor product action as defined in Section 1, with right action on $F \otimes_R G$ obtained by changing the canonical left action to a right action as explained in section 1: for $\theta \in \mathrm{Der}^{\gamma}_R F, f \in F, g \in G$

$$\theta \cdot (f \otimes g) := \theta(f) \otimes g.$$

Proposition 1.4.1. *Let A be a DG R-algebra, $\mathfrak{g}$ a DG Lie R-algebra. Assume that A is also a DG right $\mathfrak{g}$-module such that if $a, b \in A$ and $\theta \in \mathfrak{g}$ then*

$$(ab)\theta = a(b\theta) + (-1)^{|\theta||b|}(a\theta)b. \tag{1.4.1.1}$$

Let M be a DG right A-module that is also a DG right $\mathfrak{g}$-module. Let N be a DG left A-module that is also a DG right $\mathfrak{g}$-module. Let $a \in A, m \in M, n \in N, \theta \in \mathfrak{g}$ and assume that M and N satisfy the following compatibility conditions:

1) $(an)\theta = (-1)^{|\theta||n|}(a\theta)n + a(n\theta)$

2) $(ma)\theta = (-1)^{|a||\theta|}(m\theta)a + m(a\theta)$

Then the DG right $\mathfrak{g}$-module structure of $M \otimes_R N$ induces a DG right $\mathfrak{g}$-module structure on $M \otimes_A N$.

Proof. Condition (1.4.1.1) is needed to ensure that if $a, b \in A$, $n \in N$ and $\theta \in \mathfrak{g}$ then $((ab)n)\theta = (a(bn))\theta$. In fact

$$\begin{aligned}((ab)n)\theta &= (-1)^{|\theta||n|}((ab)\theta)n + (ab)(n\theta)\\ &= (-1)^{|\theta|(|b|+|n|)}((a\theta)b)n + (-1)^{|\theta||n|}(a(b\theta))n + (ab)(n\theta).\end{aligned}$$

The first equality comes from the compatibility condition 1) and the second equality follows from (1.4.1.1). On the other hand

$$\begin{aligned}(a(bn))\theta &= (-1)^{|\theta|(|b|+|n|)}(a\theta)(bn) + a((bn)\theta)\\ &= (-1)^{|\theta|(|b|+|n|)}(a\theta)(bn) + (-1)^{|\theta||n|}a((b\theta)n) + a(b(n\theta)).\end{aligned}$$

The first equality comes from the compatibility condition 1) and the second equality follows from (1.4.1.1). The two expressions are the same since N is a DG A-module.

Similarly one can prove $(m(ab))\theta = ((ma)b)\theta$ with $m \in M$.

Recall that tensor product over A is defined by the exactness of the sequence

$$M \otimes_R A \otimes_R N \xrightarrow{\eta} M \otimes_R N \to M \otimes_A N \to 0$$

where η is the map

$$\eta : m \otimes a \otimes n \mapsto ma \otimes n - m \otimes an.$$

We need to prove that the image of η is a DG right $\mathfrak{g}$-module. Indeed

$$\begin{aligned}
\eta(m \otimes a \otimes n)\theta &= (ma \otimes n)\theta - (m \otimes an)\theta \\
&= (-1)^{|\theta||n|}(ma)\theta \otimes n + ma \otimes n\theta - (-1)^{|\theta|(|a|+|n|)} m\theta \otimes an - m \otimes (an)\theta \\
&= (-1)^{|\theta||n|+|a||\theta|}(m\theta)a \otimes n + (-1)^{|\theta||n|} m(a\theta) \otimes n + \\
&\quad + ma \otimes n\theta - (-1)^{|\theta||a|+|\theta||n|} m\theta \otimes an + \\
&\quad - (-1)^{|\theta||n|} m \otimes (a\theta)n - m \otimes a(n\theta) \\
&= (-1)^{|\theta||n|+|a||\theta|}(m\theta)a \otimes n - (-1)^{|\theta||a|+|\theta||n|} m\theta \otimes an + \\
&\quad + ma \otimes n\theta - m \otimes a(n\theta) + \\
&\quad + (-1)^{|\theta||n|} m(a\theta) \otimes n - (-1)^{|\theta||n|} m \otimes (a\theta)n.
\end{aligned}$$

This proves the proposition. □

As a corollary we get

Corollary 1.4.2. *The complex* $\mathrm{Hom}_R(F, R) \otimes_F (F \otimes_R G)$ *is an* $\mathrm{End}_R(G)$-$\mathrm{Der}_R^{\gamma}(F)$-*bimodule with structure induced by the bimodule structure of*

$$\mathrm{Hom}_R(F, R) \otimes_R (F \otimes_R G).$$

We want to point out that F is a DG left $\mathrm{Der}^{\gamma}(F)$-module by evaluation, so its right structure is

$$f \cdot \theta := -\theta \cdot f = -\theta(f) \qquad\qquad f \in F, \theta \in \mathrm{Der}^{\gamma}_{R}(F).$$

The $\mathrm{End}_R(G)$-$\mathrm{Der}^{\gamma}_{R}$(F)-bimodule structure of $\mathrm{Hom}_R(F, G)$ and $\overline{\mathrm{Hom}}_R(F, G)$ is given by left and right composition.

Theorem 1.4.3. *The following map is a isomorphism of DG* $\mathrm{End}_R(G)$-$\mathrm{Der}^{\gamma}_{R}(F)$-*bimodules:*

$$\omega : \mathrm{Hom}_R(F, R) \otimes_F (F \otimes_R G) \to \overline{\mathrm{Hom}}_R(F, G) \tag{1.4.3.1}$$

$$\varphi \otimes x \otimes y \mapsto (f \mapsto (-1)^{|f||y|}\varphi(xf)y).$$

Proof. The map ω is bijective since it is the compositions of the following maps

$$\mathrm{Hom}_R(F, R) \otimes_F (F \otimes_R G) \xrightarrow{\cong} \mathrm{Hom}_R(F, R) \otimes_R G \xrightarrow{\cong} \overline{\mathrm{Hom}}_R(F, G).$$

The first map is tensor cancellation and the second map is bijective by [10, 1.3.3].

In the following $\alpha \in \mathrm{End}_R(G); \varphi \in \mathrm{Hom}_R(F, R); f, x \in F; y \in G; \theta \in \mathrm{Der}^{\gamma}_{R}(F)$.

Now we check the left linearity:

$$\begin{aligned}\omega(\alpha \cdot (\varphi \otimes x \otimes y))(f) &= \omega((-1)^{|\alpha|(|\varphi|+|x|)}\varphi \otimes x \otimes \alpha(y))(f)\\ &= (-1)^{|\alpha|(|\varphi|+|x|)+|f||\alpha(y)|}\varphi(xf)\alpha(y),\end{aligned}$$

and

$$
\begin{aligned}
(\alpha\cdot\omega(\varphi\otimes x\otimes y))(f) &= \alpha((-1)^{|f||y|}\varphi(xf)y) \\
&= (-1)^{|f||y|+|\alpha||\varphi(xf)|}\varphi(xf)\alpha(y).
\end{aligned}
$$

An elementary computation shows that the signs coincide.

Now for the right action

$$
\begin{aligned}
\omega((\varphi\otimes x\otimes y)\cdot\theta)(f) &= \omega((-1)^{|\theta|(|x|+|y|)}((\varphi\theta)\otimes x\otimes y-\varphi\otimes\theta(x)\otimes y))(f) \\
&= (-1)^{|\theta|(|x|+|y|)}(\omega((\varphi\theta)\otimes x\otimes y)(f)-\omega(\varphi\otimes\theta(x)\otimes y)) \\
&= (-1)^{|\theta|(|x|+|y|)+|f||y|}(\varphi\theta(xf)y-\varphi(\theta(x)f)y) \\
&= (-1)^{|\theta|(|x|+|y|)+|f||y|}\varphi(\theta(xf)-\theta(x)f)y \\
&= (-1)^{|\theta|(|x|+|y|)+|f||y|+|\theta||x|}\varphi(x\theta(f))y,
\end{aligned}
$$

where the last equality holds because θ is a derivation. On the other hand

$$
\begin{aligned}
(\omega(\varphi\otimes x\otimes y)\cdot\theta)(f) &= \omega(\varphi\otimes x\otimes y)(\theta(f)) \\
&= (-1)^{|\theta(f)||y|}\varphi(x\theta(f))y,
\end{aligned}
$$

an elementary computation shows that the signs coincide. □

Set $\pi(R)=\mathrm{H}(\mathrm{Der}^{\gamma}_R(F))$; it is a graded Lie k-algebra. Recall that by Sjödin [30] $U\pi(R)=\mathrm{Ext}_R(k,k)$, and recall also that, by the discussion at the end of Section 1, a right module over $\pi(R)$ is the same as a right module over $U\pi(R)$. The isomorphism in Theorem 1.4.3 yields, in homology, an isomorphism of $\mathrm{Ext}_R(N,N)$-$\mathrm{Ext}_R(k,k)$-bimodules.

Definition 1.4.1. *Let A be a DG R-algebra, C a right DG A-module and D a left DG A-module. Then the following map is called a Künneth map*

$$\kappa : \mathrm{H}(C) \otimes_{\mathrm{H}(A)} \mathrm{H}(D) \to \mathrm{H}(C \otimes_A D)$$

$$[c] \otimes [d] \mapsto [c \otimes d].$$

Where $c \in C$ and $d \in D$.

Theorem 1.4.4. *Let F be the acyclic closure of k and G a minimal free resolution of N. Then the Künneth map*

$$\kappa : \mathrm{H}(\mathrm{Hom}_R(F, R)) \otimes_{\mathrm{H}(F)} \mathrm{H}(F \otimes_R G) \to \mathrm{H}(\mathrm{Hom}_R(F, R) \otimes_F (F \otimes_R G))$$

is an isomorphism of $\mathrm{Ext}_R(N, N)$-$\mathrm{Ext}_R(k, k)$-*bimodules.*

Proof. A straightforward computation shows the bilinearity of the map. We prove that it is bijective. Let I be a minimal injective resolution of R.

$$
\begin{tikzcd}
\mathrm{H}(\mathrm{Hom}_R(F,R)) \otimes_{\mathrm{H}(F)} \mathrm{H}(F \otimes_R G) \arrow[r, "\kappa"] \arrow[d, "\varphi_1"'] & \mathrm{H}(\mathrm{Hom}_R(F,R) \otimes_F (F \otimes_R G)) \arrow[d, "\varphi_2"] \\
\mathrm{H}(\mathrm{Hom}_R(F,I)) \otimes_{\mathrm{H}(F)} \mathrm{H}(F \otimes_R G) \arrow[r] & \mathrm{H}(\mathrm{Hom}_R(F,I) \otimes_F (F \otimes_R G)) \\
\mathrm{H}(\mathrm{Hom}_R(k,I)) \otimes_k \mathrm{H}(F \otimes_R G) \arrow[r] \arrow[u, "\psi_1"] \arrow[dd] & \mathrm{H}(\mathrm{Hom}_R(k,I) \otimes_F (F \otimes_R G)) \arrow[u, "\psi_2"'] \arrow[d] \\
 & \mathrm{H}(\mathrm{Hom}_R(k,I) \otimes_R G) \arrow[d, equal] \\
\mathrm{H}(\mathrm{Hom}_R(k,I)) \otimes_k \mathrm{H}(k \otimes_R G) \arrow[d, equal] & \mathrm{Hom}_R(k,I) \otimes_R G \arrow[d, equal] \\
\mathrm{Hom}_R(k,I) \otimes_k k \otimes_R G \arrow[r, "id"] & \mathrm{Hom}_R(k,I) \otimes_k k \otimes_R G
\end{tikzcd}
$$

Where φ_1, φ_2 are induced by the map $R \to I$ and ψ_1, ψ_2 are induced by the map $F \to k$. The map φ_1 is an isomorphism because the quasi-isomorphism $R \to I$ induces a quasi-isomorphism $\mathrm{Hom}_R(F,R) \to \mathrm{Hom}_R(F,I)$ because of the choice of F. Similarly for ψ_1. The maps φ_2, ψ_2 are isomorphisms because $F \otimes_R G$ is a semifree F-module since it is a graded-free module bounded below over a non negative DG algebra (see [5] for the definition of semifree DG module. See also [5, Theorem 8.1]). The bottom map is the identity. The complex $\mathrm{Hom}_R(k,I) \otimes_R G$ has zero differentials because G is minimal and $\mathrm{Hom}_R(k,I)$ is a complex of k-vector spaces. The tensor product $\mathrm{H}(\mathrm{Hom}_R(k,I)) \otimes_k \mathrm{H}(F \otimes_R G)$ is isomorphic to $\mathrm{H}(\mathrm{Hom}_R(k,I)) \otimes_k \mathrm{H}(k \otimes_R G)$ since the tensor product is over a field and $\mathrm{H}(F \otimes_R G) \cong \mathrm{H}(k \otimes_R G)$. The last equality on the left follows by the minimality of I and G. The commutativity of the diagram follows by the naturality of the Künneth map. This proves that all the horizontal maps are isomorphisms. $\square$

Corollary 1.4.5. *The composition*

$$\mathrm{H}(\omega) \circ \kappa : \operatorname{Ext}_R(k, R) \otimes_k \operatorname{Tor}^R(k, N) \to \overline{\operatorname{Ext}}_R(k, N)$$

where κ *is the Künneth map and* ω *is the map in* (1.4.3.1)*, is an isomorphism of* $\operatorname{Ext}_R(N, N)$-$\operatorname{Ext}_R(k, k)$*-bimodules.*

In [10] Avramov and Veliche ask: what is the right $\operatorname{Ext}_R(k, k)$-module structure on $\operatorname{Ext}_R(k, R) \otimes_k \operatorname{Tor}^R(k, N)$ that would make it isomorphic to bounded cohomology? The previous corollary answers this question.

1.5 Gorenstein rings

The stable cohomology of a pair of modules is zero if R is regular. From now on we will assume that R is a singular ring (i.e. not regular). If R is Gorenstein, then $\operatorname{Ext}_R(k, R) \cong \Sigma^{-d} k$ with $d = \dim R$ where Σ is the suspension functor, hence by Corollary 1.4.5

$$\overline{\operatorname{Ext}}_R(k, N) \cong \Sigma^{-d} \operatorname{Tor}^R(k, N)$$

as $\operatorname{Ext}_R(N, N)$-$\operatorname{Ext}_R(k, k)$-bimodules. This is because $\Sigma^{-d} k \otimes_k \operatorname{Tor}^R(k, N)$ is isomorphic to $\Sigma^{-d} \operatorname{Tor}^R(k, N)$ as $\operatorname{Ext}_R(N, N)$-$\operatorname{Ext}_R(k, k)$-bimodules.

We will use the following notation

$$\mathcal{S} = \widehat{\operatorname{Ext}}_R(k, k), \quad \mathcal{E} = \operatorname{Ext}_R(k, k), \quad \mathcal{B} = \overline{\operatorname{Ext}}_R(k, k).$$

In [10, 5.1.8] (and [27, Theorem 6]) it is proved that the map η_R for the pair (k,k) in the sequence (1.3.0.2) is zero, yielding an exact sequence of $\mathcal{E}$-bimodules

$$0 \to \mathcal{E} \xrightarrow{\iota} \mathcal{S} \to \Sigma\mathcal{B} \to 0. \tag{1.5.0.1}$$

Definition 1.5.1. *Let k be a field and $\mathcal{A}$ a graded k-algebra with $\mathcal{A}^0 = k$ and $\mathcal{A}^i = 0$ for all $i < 0$. Let $\mathcal{M}$ be a graded left $\mathcal{A}$-module. Set*

$$\Gamma^i\mathcal{M} := \{\mu \in \mathcal{M} \mid \mathcal{A}^{\geq i}\mu = 0\} \quad \text{and} \quad \Gamma\mathcal{M} = \bigcup_{i=0}^{\infty} \Gamma^i\mathcal{M}.$$

The left torsion $\mathcal{E}$-subbimodule of $\mathcal{S}$ is

$$\mathcal{T} := \Gamma\mathcal{S}.$$

Lemma 1.5.1. *If $\mathcal{S} = \iota(\mathcal{E}) \oplus \mathcal{T}'$ for some graded $\mathcal{E}$-subbimodule $\mathcal{T}'$ of $\mathcal{S}$, then $\mathcal{T}' = \mathcal{T}$ and*

$$\mathcal{T}' \cong \Sigma^{1-d}\operatorname{Tor}^R(k,k)$$

as graded $\mathcal{E}$-bimodules.

Proof. Our hypothesis and (1.5.0.1) implies that $\mathcal{T}'$ is isomorphic to $\Sigma\mathcal{B}$ as graded $\mathcal{E}$-bimodules. By [10, (7.3.2)] $\mathcal{B} = \Gamma\mathcal{B}$, hence the following containments hold

$$\mathcal{T}' = \Gamma\mathcal{T}' \subseteq \Gamma\mathcal{S} = \mathcal{T}.$$

By [10, (7.3.4)] one has $\iota(\mathcal{E}) \cap \mathcal{T} = (0)$, and since $\mathcal{S} = \iota(\mathcal{E}) \oplus \mathcal{T}'$ we deduce $\mathcal{T} \subseteq \mathcal{T}'$; this gives us

$$\mathcal{T} = \mathcal{T}'. \qquad \square$$

Definition 1.5.2. *Let k be a field and $\mathcal{A}$ a graded k-algebra with $\mathcal{A}^0 = k$ and $\mathcal{A}^i = 0$ for all $i < 0$. Let $\mathcal{M}$ be a graded left $\mathcal{A}$-module. The depth of $\mathcal{M}$ over $\mathcal{A}$ as*

$$\operatorname{depth}_{\mathcal{A}} \mathcal{M} = \inf\{n \in \mathbb{N} \mid \operatorname{Ext}^n_{\mathcal{A}}(k, \mathcal{M}) \neq 0\}.$$

Definition 1.5.3. *Let A be a graded k-algebra with k a field. Let M be a graded A-bimodule. The trivial extension algebra of A by M is an algebra denoted by $A \ltimes M$ with underlying bimodule $A \oplus M$ and product given by*

$$(a, m) \cdot (b, n) := (ab, an + mb).$$

Theorem 1.5.2. *If R is a Gorenstein ring with $\operatorname{depth} \mathcal{E} \geq 2$, then the stable cohomology algebra is a trivial extension algebra,*

$$\mathcal{S} \cong \mathcal{E} \ltimes \Sigma^{1-d} \operatorname{Tor}^R(k, k).$$

Proof. By [10, 7.2(3)] if $\operatorname{depth} \mathcal{E} \geq 2$ then $\mathcal{S} = \iota(\mathcal{E}) \oplus \mathcal{T}$ as $\mathcal{E}$-bimodules, hence by Lemma 1.5.1 $\mathcal{T} \cong \Sigma^{1-d} \operatorname{Tor}^R(k, k)$ as graded $\mathcal{E}$-bimodules. By [10, 9.2(3)] $\mathcal{T} \cdot \mathcal{T} = 0$, hence $\mathcal{S}$ is a trivial extension of $\iota(\mathcal{E})$ and $\mathcal{T}$. $\square$

Remark 1.5.3. We recall that R is a complete intersection if its completion $\widehat{R}$ with respect to the $\mathfrak{m}$-adic topology is the quotient of a regular local ring by an ideal generated by a regular sequence.If R is a complete intersection then by [10, 8.3] $\operatorname{depth} \mathcal{E} = \operatorname{codim} R$, hence any complete intersection with $\operatorname{codim} R \geq 2$ satisfies the hypothesis of Theorem 1.5.2. The structure of $\mathcal{S}$ for hypersurfaces is already known, see [12, (10.2.3)] (see also [10, 8.4]). In this case, stable cohomology $\widehat{\operatorname{Ext}}_R(k, k)$ is a central localization of absolute cohomology $\operatorname{Ext}_R(k, k)$.

1.6 Commutativity

We recall that by [30] (see also [18]) the algebra $\operatorname{Ext}_R(k,k)$ is graded-commutative if and only if R is a complete intersection $\widehat{R} = Q/I$ with $(Q, \mathfrak{n})$ regular ring, I generated by a regular sequence, and $I \subseteq \mathfrak{n}^3$.

In the following F is the acyclic closure of k and G a free resolution of N. We compute $\operatorname{Ext}_R(N,N)$ using the complex $\operatorname{End}_R(G)$. We compute $\operatorname{Tor}^R(k,N)$ using the complex $F \otimes_R G$. The $\operatorname{Ext}_R(N,N)$-$\operatorname{Ext}_R(k,k)$-bimodule structure of $\operatorname{Tor}^R(k,N)$ is defined, in a more general setting, in Section 1.4. Let $[\alpha] \in \operatorname{Ext}_R(N,N)$ and $[\Sigma_i f_i \otimes g_i] \in \operatorname{Tor}^R(k,N)$ then

$$[\alpha] \cdot [\Sigma_i f_i \otimes g_i] = [\Sigma_i (-1)^{|\alpha||f_i|} f_i \otimes \alpha(g_i)].$$

Let $[\theta] \in \pi(R)$, then

$$[\Sigma_i f_i \otimes g_i] \cdot [\theta] = -[\Sigma_i (-1)^{|\theta|(|f_i|+|g_i|)} \theta(f_i) \otimes g_i].$$

Lemma 1.6.1. *Let $(R, \mathfrak{m}, k) \to (R', \mathfrak{m}', k')$ be a local homomorphism such that the R-module R' is flat and $R' \otimes_R k \cong k$. Let N be a finitely generated R-module and let N' be the R'-module $R' \otimes_R N$. There are isomorphisms of algebras*

$$\alpha : R' \otimes_R \operatorname{Ext}_R(k,k) \to \operatorname{Ext}_{R'}(k',k')$$

$$\beta : R' \otimes_R \operatorname{Ext}_R(N,N) \to \operatorname{Ext}_{R'}(N',N').$$

The canonical map $\varphi : R' \otimes_R \operatorname{Tor}^R(k,N) \to \operatorname{Tor}^{R'}(k',N')$ is bijective and β-α-covariant.

Proof. We denote by F' and G' the complexes $R' \otimes_R F$ and $R' \otimes_R G$ respectively. Consider the canonical map of algebras

$$\widehat{\beta} : R' \otimes \operatorname{Hom}_R(G, G) \to \operatorname{Hom}_{R'}(G', G').$$

This map fits in the following commutative diagram

$$\begin{tikzcd}
R' \otimes_R \operatorname{Hom}_R(G, G) \arrow[r] \arrow[d, "\widehat{\beta}"'] & R' \otimes_R \operatorname{Hom}_R(G, N) \arrow[d] \\
\operatorname{Hom}_{R'}(G', G') \arrow[r] & \operatorname{Hom}_{R'}(G', N')
\end{tikzcd}$$

where the horizontal maps are quasi-isomorphisms and the right vertical map is an isomorphism. It follows that the left vertical map is a quasi-isomorphism. Now we set $\beta = \mathrm{H}(\widehat{\beta})$.

Consider the canonical map of DG Lie algebras

$$\widehat{\alpha} : R' \otimes_R \operatorname{Der}^{\gamma}_R(F) \to \operatorname{Der}^{\gamma}_{R'}(F').$$

We denote by $\operatorname{Diff}^{\gamma}_R F$ and $\operatorname{Diff}^{\gamma}_{R'} F'$ the modules of Γ-differentials of F and F' respectively, see [2, Proposition 6.2.3] for the definition and its properties. The map $\widehat{\alpha}$ fits in the following commutative diagram

$$\begin{tikzcd}
R' \otimes_R \operatorname{Der}^{\gamma}_R F \arrow[r] \arrow[d, "\widehat{\alpha}"'] & R' \otimes_R \operatorname{Hom}_F(\operatorname{Diff}^{\gamma}_R F, F) \arrow[d] \\
\operatorname{Der}^{\gamma}_{R'} F' \arrow[r] & \operatorname{Hom}_{F'}(\operatorname{Diff}^{\gamma}_{R'} F', F')
\end{tikzcd}$$

where the horizontal maps are isomorphisms because of [2, Proposition 6.2.3]. The same argument used to prove that $\widehat{\beta}$ is a quasi-isomorphism shows that the right map in the previous diagram is a quasi-isomorphism. By commutativity of the diagram then $\widehat{\alpha}$ is a quasi-isomorphism. We set $\alpha = \mathrm{H}(\widehat{\alpha})$.

Tensor cancelation yields an isomorphism

$$\widehat{\varphi} : R' \otimes_R F \otimes_R G \to F' \otimes_{R'} G'.$$

We set $\varphi = \mathrm{H}(\widehat{\varphi})$. A straightforward computation shows that φ is β-α-covariant. □

Definition 1.6.1. *Let A be a graded k-algebra and M a graded A-bimodule. We say that M is symmetric if for every $m \in M$ and $a \in A$*

$$am = (-1)^{|a||m|}ma.$$

Theorem 1.6.2. *If R is a complete intersection $\widehat{R} = Q/I$ with I generated by a regular sequence, $(Q, \mathfrak{n})$ regular ring, $I \subseteq \mathfrak{n}^3$, then $\mathrm{Tor}^R(k,k)$ is a symmetric $\mathrm{Ext}_R(k,k)$-bimodule.*

Proof. First we want to know how $\pi(R)$ acts on $\mathrm{Tor}^R(k,k)$.

By Lemma 1.6.1 we can assume that R is complete and $R = Q/I$ with $\mathfrak{n} = (a_1, \ldots, a_e)$, $I = (f_1, \ldots, f_c)$, Q regular, $f_1, \ldots, f_c$ a Q-sequence and $I \subseteq \mathfrak{m}^2$. Write

$$f_i = \Sigma_{j \le k} r_{ijk} a_j a_k$$

Let F be an acyclic closure of k over R. If $a \in Q$ we denote by $\bar{a}$ the class of a in R.

Then by [31]

$$F = R\langle x_1, \ldots, x_e, y_1, \ldots, y_c \mid \partial(x_i) = \bar{a}_i, \quad \partial(y_i) = \Sigma_{j\le k}\bar{r}_{ijk}\bar{a}_j x_k\rangle$$

is an acyclic closure of k over R. Since R is a complete intersection, by [30] $\pi(R)$ is generated as a k-vectorspace by elements $\xi_1, \ldots, \xi_e$ of degree 1 and elements $\chi_1, \ldots, \chi_c$ of degree 2, where e is the embedding dimension of R and c its codimension. These generators are classes of derivations of F defined as follows

$$\xi_t(x_i) = \delta_{it} \quad \text{and} \quad \xi_t(y_i) = -\Sigma_{j\le t}\bar{r}_{ijt}x_j$$
$$\chi_t(x_i) = 0 \quad \text{and} \quad \chi_t(y_i) = \delta_{it}$$

The generators (as an algebra) of degree 1 of $\mathrm{Tor}^R(k,k)$, which is $\mathrm{H}(F \otimes_R F)$, are the classes of

$$x_i \otimes 1 - 1 \otimes x_i \qquad i = 1, \ldots, e$$

and the generators of degree 2 are the classes of

$$y_i \otimes 1 + \Sigma_{j\le k}\bar{r}_{ijk}x_k \otimes x_j - \Sigma_{j\le k}\bar{r}_{ijk} \otimes x_k x_j - 1 \otimes y_i$$

for $i = 1, \ldots, c$.

Now we check how the derivations act on these generators

$$\xi_j \cdot (x_i \otimes 1 - 1 \otimes x_i) = -1 \otimes \delta_{ij}$$

$$(x_i \otimes 1 - 1 \otimes x_i) \cdot \xi_j = \delta_{ij} \otimes 1$$

$$\chi_t \cdot (y_i \otimes 1 + \Sigma_{j\le k}\bar{r}_{ijk}x_k \otimes x_j - \Sigma_{j\le k}\bar{r}_{ijk} \otimes x_k x_j - 1 \otimes y_i) = -1 \otimes \delta_{ti}$$

$$(y_i \otimes 1 + \Sigma_{j\leq k}\bar{r}_{ijk}x_k \otimes x_j - \Sigma_{j\leq k}\bar{r}_{ijk} \otimes x_kx_j - 1 \otimes y_i) \cdot \chi_t = -\delta_{ti} \otimes 1$$

$$\xi_t \cdot (y_i \otimes 1 + \Sigma_{j\leq k}\bar{r}_{ijk}x_k \otimes x_j - \Sigma_{j\leq k}\bar{r}_{ijk} \otimes x_kx_j - 1 \otimes y_i) =$$
$$- \Sigma_{j\leq t}\bar{r}_{ijt}x_t \otimes 1 - \Sigma_{j\leq t}\bar{r}_{ijt} \otimes x_j + \Sigma_{t\leq k}\bar{r}_{itk} \otimes x_k + \Sigma_{j\leq t}\bar{r}_{ijt} \otimes x_j$$

$$(y_i \otimes 1 + \Sigma_{j\leq k}\bar{r}_{ijk}x_k \otimes x_j - \Sigma_{j\leq k}\bar{r}_{ijk} \otimes x_kx_j - 1 \otimes y_i) \cdot \xi_t = \Sigma_{j\leq t}\bar{r}_{ijt}x_j \otimes 1 - \Sigma_{j\leq t}\bar{r}_{ijt} \otimes x_t.$$

So

$$\xi_j \cdot (x_i \otimes 1 - 1 \otimes x_i) = -(x_i \otimes 1 - 1 \otimes x_i)\xi_j$$

$$\chi_t \cdot (y_i \otimes 1 + \Sigma_{j\leq k}\bar{r}_{ijk}x_k \otimes x_j - \Sigma_{j\leq k}\bar{r}_{ijk} \otimes x_kx_j - 1 \otimes y_i) =$$
$$(y_i \otimes 1 + \Sigma_{j\leq k}\bar{r}_{ijk}x_k \otimes x_j - \Sigma_{j\leq k}\bar{r}_{ijk} \otimes x_kx_j - 1 \otimes y_i) \cdot \chi_t$$

so only the action of elements of degree 1 on elements of degree 2 might break the symmetry, but the r_{ijk} are in $\mathfrak{m}$ because $I \subseteq \mathfrak{n}^3$. So applying $\varepsilon \otimes F$, where $\varepsilon : F \to k$ is the augmentation, to

$$\xi_t \cdot (y_i \otimes 1 + \Sigma_{j\leq k}\bar{r}_{ijk}x_k \otimes x_j - \Sigma_{j\leq k}\bar{r}_{ijk} \otimes x_kx_j - 1 \otimes y_i) =$$
$$- \Sigma_{j\leq t}\bar{r}_{ijt}x_t \otimes 1 - \Sigma_{j\leq t}\bar{r}_{ijt} \otimes x_j + \Sigma_{t\leq k}\bar{r}_{itk} \otimes x_k + \Sigma_{j\leq t}\bar{r}_{ijt} \otimes x_j$$

$$(y_i \otimes 1 + \Sigma_{j\leq k}\bar{r}_{ijk}x_k \otimes x_j - \Sigma_{j\leq k}\bar{r}_{ijk} \otimes x_kx_j - 1 \otimes y_i) \cdot \xi_t = \Sigma_{j\leq t}\bar{r}_{ijt}x_j \otimes 1 - \Sigma_{j\leq t}\bar{r}_{ijt} \otimes x_t.$$

we get zero, the left and right product of a derivation of degree 1 on a cycle of degree 2 is zero. We just proved that the action of $\pi(R)$ is symmetric, but $\mathrm{Ext}_R(k,k) \cong U\pi(R)$ and by [30] $\mathrm{Ext}_R(k,k)$ is graded-commutative, hence the action of $\mathrm{Ext}_R(k,k)$ is symmetric. $\square$

Example 1.6.3. *We show that* $\operatorname{Tor}^R(k,k)$ *is not in general symmetric. Let* $R = k[[x,y]]/(xy)$*, and denote by* $\bar{x}, \bar{y}$ *the classes of* x *and* y *in* R*. The acyclic closure of* k *over* R *is*

$$F = R\langle T_1, T_2, S \mid \partial T_1 = \bar{x}, \partial T_2 = \bar{y}, \partial S = \bar{x}T_2\rangle.$$

Consider the cycle

$$z = S \otimes 1 + T_2 \otimes T_2 - 1 \otimes T_2T_1 - 1 \otimes S$$

and the derivation defined by

$$\xi(T_1) = 0, \quad \xi(T_2) = 1, \quad \xi(S) = -T_1$$

then

$$\begin{aligned}
\xi \cdot z &= -1 \otimes \xi(T_2T_1) - 1 \otimes \xi(S) \\
&= -1 \otimes (\xi(T_2)T_1 - T_2\xi(T_1)) + 1 \otimes T_1 \\
&= -1 \otimes \xi(T_2)T_1 + 1 \otimes T_1 \\
&= 0
\end{aligned}$$

but

$$\begin{aligned}
z \cdot \xi &= -\xi(S) \otimes 1 - \xi(T_2) \otimes T_1 \\
&= -T_1 \otimes 1 - 1 \otimes T_1 \\
&= -(T_1 \otimes 1 + 1 \otimes T_1)
\end{aligned}$$

Denote by π^1 and $\mathfrak{g}^1$ k-vectorspaces of rank e and $\pi^2, \mathfrak{g}^2$ k-vectorspaces of rank c.

Fix basis

$$\pi^1 = \langle \xi_1, \dots, \xi_e \rangle \quad \pi^2 = \langle \chi_1, \dots, \chi_c \rangle$$

$$\mathfrak{g}^1 = \langle x_1, \dots, x_e \rangle \quad \mathfrak{g}^2 = \langle y_1, \dots, y_c \rangle.$$

Denote by E the graded k-algebra $E = \bigwedge \pi^1 \otimes_k \operatorname{Sym}(\pi^2)$ and by T the k-vectorspace $T = \bigwedge \mathfrak{g}^1 \otimes_k \operatorname{Sym}(\mathfrak{g}^2)$, we give T the following graded E-bimodule structure

$$\xi_j x_i = -\delta_{ij}, \quad x_i \xi_j = \delta_{ij},$$

$$\chi_j y_i = -\delta_{ji}, \quad y_i \chi_j = -\delta_{ji},$$

$$\xi_j y_i = 0, \quad y_i \xi_j = 0.$$

With this notation we prove

Theorem 1.6.4. *The following conditions on a local ring R are equivalent:*
1) $\widehat{R} \cong Q/I$ with $(Q, \mathfrak{n})$ regular, I generated by a regular sequence and $I \subseteq \mathfrak{n}^3$.
2) the k-algebra $\widehat{\operatorname{Ext}}_R(k,k)$ is graded-commutative.
When they hold $\widehat{\operatorname{Ext}}_R(k,k) \cong E \ltimes \Sigma^{1-d} T$ if $\operatorname{codim} R \geq 2$. If R is an hypersurface then $\widehat{\operatorname{Ext}}_R(k,k) \cong \bigwedge \pi^1 \otimes_k k[t, t^{-1}]$ with $\deg t = 2$.

Proof. 2) $\Rightarrow$ 1) If $\widehat{\operatorname{Ext}}_R(k,k)$ is graded-commutative then so is $\operatorname{Ext}_R(k,k)$ since it is a subalgebra (see [27]), and by [30] the ring R has the desired form.
1) $\Rightarrow$ 2) if $\operatorname{codim} R \geq 2$, then by 1.5.2

$$\widehat{\operatorname{Ext}}_R(k,k) \cong \operatorname{Ext}_R(k,k) \ltimes \Sigma^{1-d} \operatorname{Tor}^R(k,k)$$

but by [30] if R has the required form then $\operatorname{Ext}_R(k,k) \cong E$, and by 1.6.2

$$\operatorname{Tor}^R(k,k) \cong T.$$

If the ring is an hypersurface, then by [12] $\widehat{\operatorname{Ext}}_R(k,k)$ has the desired form.
The algebras $E \ltimes T$ and $\bigwedge \pi^1 \otimes_k k[t,t^{-1}]$ are clearly graded-commutative. $\square$

1.7 The dual bimodule structure of $\operatorname{Tor}^R(k,k)$

As before F is an acyclic closure of k and G a free resolution of a module N. The complex $G \otimes_R F$ is a DG $\operatorname{End}_R G$-$\operatorname{Der}^{\gamma}_R F$-bimodule with actions

$$\alpha \cdot (g \otimes f) := \alpha(g) \otimes f \quad \alpha \in \operatorname{End}_R G$$

$$(g \otimes f) \cdot \theta := -(-1)^{|\theta||f|} g \otimes \theta(f) \quad \theta \in \operatorname{Der}^{\gamma}_R F$$

where the right action is obtained by twisting a left action.
The complex $\operatorname{Hom}_F(G \otimes_R F, F)$ is a right DG $\operatorname{End}_R G$-module with product

$$(\psi \cdot \alpha)(g \otimes f) := \psi(\alpha(g) \otimes f) \quad \psi \in \operatorname{Hom}_F(G \otimes_R F, F)\ \alpha \in \operatorname{End}_R G, g \in G, f \in F$$

It has two structure of left DG $\operatorname{Der}^{\gamma}_R F$-module, let $\psi \in \operatorname{Hom}_F(G \otimes_R F, F), \theta \in \operatorname{Der}^{\gamma}_R F, f \in F, g \in G$ and define

$$(\theta *_1 \psi)(g \otimes f) := \theta(\psi(g \otimes f))$$

$$(\theta *_2 \psi)(g \otimes f) := -(-1)^{|\theta|(|\psi|+|g|)} \psi(g \otimes \theta(f))$$

where the second action is obtained by acting on the right on $g \otimes f$.

We are going to combine these two left actions into a third one that we are going to use in the next theorem

$$(\theta \cdot \psi)(g \otimes f) := \theta(\psi(g \otimes f)) - (-1)^{|\theta|(|\psi|+|g|)}\psi(g \otimes \theta(f)). \tag{1.7.0.1}$$

Theorem 1.7.1. *The following map is an isomorphism of DG* $\operatorname{Der}^{\gamma}_R F$*-*$\operatorname{End}_R G$*-bimodules*

$$\chi : \operatorname{Hom}_R(G, F) \to \operatorname{Hom}_F(G \otimes_R F, F)$$

$$\varphi \mapsto (g \otimes f \mapsto \varphi(g)f)$$

where $\operatorname{Hom}_R(G, F)$ *has the canonical bimodule structure and* $\operatorname{Hom}_F(G \otimes_R F, F)$ *has the bimodule structure 1.7.0.1.*

Proof. The map χ is bijective because of the canonical isomorphism $F \to \operatorname{End}_F F$ and adjunction. We just need to check left and right linearity.
We start with right linearity, let $\alpha \in \operatorname{End}_R G, \varphi \in \operatorname{Hom}_R(G, F), f \in F, g \in G$

$$\chi(\varphi\alpha)(g \otimes f) = \varphi\alpha(g)f$$

$$(\chi(\varphi)\alpha)(g \otimes f) = \chi(\varphi)(\alpha \cdot (g \otimes f)) = \chi(\varphi)(\alpha(g) \otimes f) = \varphi\alpha(g)f,$$

this proves $\chi(\varphi\alpha) = \chi(\varphi)\alpha$.
Now let $\theta \in \operatorname{Der}^{\gamma}_R F$, then

$$\chi(\theta\varphi)(g \otimes f) = \theta\varphi(g)f$$

$$
\begin{aligned}
(\theta\chi(\varphi))(g \otimes f) &= \theta(\chi(\varphi)(g \otimes f)) - (-1)^{|\theta|(|\varphi|+|g|)}\chi(\varphi)(g \otimes \theta(f)) \\
&= \theta(\varphi(g)f) - (-1)^{|\theta|(|\varphi|+|g|)}\varphi(g)\theta(f) \\
&= \theta\varphi(g)f + (-1)^{|\theta|(|\varphi|+|g|)}\varphi(g)\theta(f) - (-1)^{|\theta|(|\varphi|+|g|)}\varphi(g)\theta(f) \\
&= \theta\varphi(g)f.
\end{aligned}
$$

where the third equality holds as θ is a derivation. This proves that $\chi(\theta\varphi) = \theta\chi(\varphi)$.

□

Corollary 1.7.2. *The graded* $\mathrm{Ext}_R(k,k)$-$\mathrm{Ext}_R(M,M)$*-bimodules* $\mathrm{Ext}_R(M,k)$ *and* $\mathrm{Hom}_k(\mathrm{Tor}^R(M,k),k)$ *are isomorphic, where the bimodule actions are induced in homology by the actions on* $\mathrm{Hom}_R(G,F)$ *and* $\mathrm{Hom}_F(G \otimes_R F, F)$*.*

Proof. We only need to prove that

$$\mathrm{H}(\mathrm{Hom}_F(G \otimes_R F, F)) \cong \mathrm{Hom}_k(\mathrm{Tor}^R(M,k),k).$$

Since $G \otimes_R F$ is a semifree DG F-module the functor $\mathrm{Hom}_F(G \otimes_R F, \_)$ preserves quasi-isomorphisms, hence

$$\mathrm{H}(\mathrm{Hom}_F(G \otimes_R F, F)) \cong \mathrm{H}(\mathrm{Hom}_F(G \otimes_R F, k))$$

now it remains to notice that $\mathrm{Hom}_F(G \otimes_R F, k) = \mathrm{Hom}_k(G \otimes_R F, k)$ and

$$\mathrm{H}(\mathrm{Hom}_k(G \otimes_R F, k)) = \mathrm{Hom}_k(\mathrm{H}(G \otimes_R F), k)$$

since k is a field. □

Dualizing the isomorphism in the previous corollary and using 1.5.2 we get

Corollary 1.7.3. *If R is a Gorenstein ring of dimension d with* $\operatorname{depth}\mathcal{E} \geq 2$ *then the stable cohomology algebra is a trivial extension algebra,*

$$\mathcal{S} \cong \mathcal{E} \ltimes \Sigma^{1-d}\mathcal{E}^{\vee}.$$

where $(\_)^{\vee}$ *denotes* $\operatorname{Hom}_k(\_, k)$.

Let $R^!$ be the Koszul dual of R, i.e. the subalgebra of $\operatorname{Ext}_R(k,k)$ generated by $\operatorname{Ext}^1_R(k,k)$. In [21, 3.3] it is proved that $\operatorname{Ext}_R(M,k)$ and $\operatorname{Hom}_k(\operatorname{Tor}^R(M,k),k)$ are isomorphic as right $R^!$-modules where the action on $\operatorname{Ext}_R(M,k)$ is the usual left action twisted with the antipode map. We want to prove that the left $\operatorname{Ext}_R(k,k)$-action of the previous corollary restricts to the left $R^!$-action of [21, 3.3] once we turn the modules from right to left using the antipode map.

Proof. Let e be the embedding dimension of R and $y_1, \ldots, y_e$ an algebra basis of $R^!$ as constructed in [21, 2.9]. Fix $y_i = [\theta_i]$ with $\theta_i \in \operatorname{Der}^{\gamma}_R(F)$, we drop the subscript i.

Let G be a projective resolution of M and F an acyclic closure o k. Consider the following commutative diagram where $\varepsilon : F \to k$ is the augmentation

$$\begin{tikzcd}
\operatorname{Hom}_R(G,F) \arrow[r, "\chi"] \arrow[d, "\operatorname{Hom}_R(G,\varepsilon)"'] & \operatorname{Hom}_F(G \otimes_R F, F) \arrow[d, "\operatorname{Hom}_F(G \otimes_R F,\varepsilon)"] \\
\operatorname{Hom}_R(G,k) \arrow[r] & \operatorname{Hom}_F(G \otimes_R F, k)
\end{tikzcd} \tag{1.7.3.1}$$

The vertical maps are isomorphisms. Take $\alpha \in \operatorname{Hom}_R(F,k)$ of degree $1-n$ and $z \in G$ of degree n. Denote by $\bar{z}$ the image of z in $G \otimes_R k = \operatorname{Tor}^R(M,k)$. Denote by φ the element $\chi(\tilde{\alpha})$ where $\tilde{\alpha}$ is a lifting of α to $\mathrm{Hom}_R(G, F)$. By 1.7.1 we have

$$(\theta\varphi)(g \otimes f) = \theta(\tilde{\alpha}(g))f$$

Let $r_1, \ldots, r_e$ be a minimal generating set of $\mathfrak{m}$, and let $\partial(z) = \Sigma_{j=1}^{e} r_j f_j$ with $f_j \in F_{n-1}$. Now we calculate $[\theta\varphi](\bar{z})$ by lifting $\bar{z}$ to $G \otimes_R F$

$$\begin{aligned}
[\theta\varphi](\bar{z}) &= \varepsilon((\theta\varphi)(z \otimes 1 + \cdots)) \\
&= \varepsilon\theta(\tilde{\alpha}(z)) \\
&= (-1)^{|\alpha|}\varepsilon\tilde{\alpha}(f_i) \\
&= (-1)^{|\alpha|}\alpha(f_i)
\end{aligned}$$

where the first equality follows from the commutativity of the diagram 1.7.3.1, the second from 1.7.1 and degree reasons, the third from [21, 2.12], the fourth from the definition of $\tilde{\alpha}$.

In the proof of [21, 3.3] it is proved that $(\alpha \cdot y_i)(z) = -\alpha(f_i)$, twisting this into a left action using $y_i \cdot \alpha := -(-1)^{|\alpha|}\alpha \cdot y_i$ we get that this left action is the same as the one defined in 1.7.0.1. $\square$

Chapter 2

Modules of infinite regularity over graded commutative rings

2.1 Introduction

Let $k[x_1, \ldots, x_n]$ be a polynomial ring, graded by setting $\deg x_i = d_i \geq 1$, and let $R = k[x_1, \ldots, x_n]/I$ where I is a homogeneous ideal. We denote by $\mathfrak{m}$ the maximal homogeneous ideal of R. The size of a minimal free resolution of a graded R-module M is measured by its graded *Betti numbers* $\beta_{i,j}(M) = \operatorname{rank}_k \operatorname{Ext}_R^{i,j}(M, k)$. Invariants arising from the Betti numbers of M are its *projective dimension* and its *Castelnuovo-Mumford regularity.* In [4] Avramov and Eisenbud prove that the regularity of k is finite if and only if the regularity of every finitely generated module is finite. In [3], Avramov proves that nonzero modules of the form $\mathfrak{m}M$ have infinite projective dimension, provided R is not regular. In this chapter we prove that they also have infinite regularity, provided R is not Koszul.

In section 2 we prove that a nonzero direct summands of a syzygy of k has infinite regularity, provided R is a complete intersection which is not Koszul. We ask whether or not this holds true for any non Koszul ring.

In section 3 we provide a connection between having a nonzero direct summand of a syzygy of k of finite regularity and the vanishing of the graded deviations of R.

2.1 Tightly embeddable modules

In this section we construct a class of modules which, over non Koszul rings, have infinite regularity.

Definition 2.1.1. *Let M be a graded R-module. We say that M is* tightly embeddable *if there exists a finitely generated graded R-module L such that*

$$\mathfrak{m}M \subsetneq \mathfrak{m}L \subseteq M \subseteq L.$$

In that case $M \subseteq L$ is a tight embedding*.*

2.1.1. It follows from Nakayama's Lemma that if L is finitely generated then $\mathfrak{m}^i L \subseteq \mathfrak{m}^{i-1}L$ is a tight embedding for each $i \geq 1$ such that $\mathfrak{m}^i L$ is not zero.

Motivated by results in [3], we explore the relation between tightly embeddable modules and regularity.

We make the convention that if V is a graded k-vector space then $V^j = V_{-j}$:

2.1.2. The Hilbert series of a graded k-vector space V is

$$H_V(s) = \Sigma_j \operatorname{rank}_k V_j s^j.$$

2.1.3. The bigraded Poincaré series of a graded R-module M is

$$P^R_M(s,t) = \Sigma_{i,j} \beta^R_{i,j}(M) s^j t^i \in \mathbb{Z}[s^{\pm 1}][[t]].$$

2.1.4. Let $\Sigma_i a_i t^i$ and $\Sigma_i b_i t^i$ be formal power series in $\mathbb{Z}[[t]]$. If $a_i \leq b_i$ for every i we write

$$\Sigma_i a_i t^i \preceq \Sigma_i b_i t^i$$

or

$$\Sigma_i b_i t^i \succeq \Sigma_i a_i t^i.$$

2.1.5. If A is a graded k-algebra and X a graded A-module, the sth *shift* of X is the graded A-module $(\Sigma^s X)^i = X^{i-s}$. The A-action is defined as $a \cdot x = (-1)^{|a|s} ax$, where $|a|$ denotes the degree of a.

Remark 2.1.6. By a bigraded version of Gulliksen and Levin [20], the algebra $\operatorname{Ext}_R(k,k)$ is a (bi-)graded Hopf k-algebra. This algebra is the universal enveloping algebra of a (bi-)graded Lie algebra $\pi^{*,*}(R)$ where the first degree is homological and the second is the internal degree. This follows from bigraded versions of theorems in [28, (5.18)] (characteristic 0), [1, Theorem 17] (positive odd characteristic) [30, Theorem 2] (characteristic 2).

Notation 2.1.7. We denote by $\mathcal{E}$ the algebra $\operatorname{Ext}_R(k,k)$. For every R-module M we denote by $\mathcal{E}(M)$ the left $\mathcal{E}$-module $\operatorname{Ext}_R(M,k)$.

The following theorem is a graded version of [3, Lemma 6], and has a similar proof.

Theorem 2.1.8. *If $M \subseteq L$ is a tight embedding, then, with $V = \frac{\mathfrak{m}L}{\mathfrak{m}M}$, there is a coefficient wise inequality*

$$H_V(s) P_k^R(s,t) \preceq P_M^R(s,t) \prod_{i=1}^{n} (1 + s^{d_i} t).$$

Proof. We set the following notation

$$\overline{M} = M/\mathfrak{m}M, \quad \overline{L} = L/\mathfrak{m}L,$$

$$N = L/M, \quad W = L/\mathfrak{m}M.$$

Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \overline{M} \arrow[r] & W \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

This induces a commutative diagram of homomorphisms of bigraded left $\mathcal{E}$-modules.

$$\begin{tikzcd}
\mathcal{E}(L) \arrow[r] & \mathcal{E}(M) \arrow[r, "\eth"] & \Sigma\mathcal{E}(N) \\
\mathcal{E}(W) \arrow[r] \arrow[u] & \mathcal{E}(\overline{M}) \arrow[r, "\delta"] \arrow[u] & \Sigma\mathcal{E}(N). \arrow[u, equal]
\end{tikzcd} \tag{2.1.8.1}$$

As $\mathfrak{m}$ annihilates $\overline{M}$ and N, there are natural isomorphisms $\mathcal{E}(\overline{M}) \cong \mathcal{E} \otimes_k \overline{M}^\vee$ and $\mathcal{E}(N) \cong \mathcal{E} \otimes_k N^\vee$. Let $\mathcal{K}$ denote the universal enveloping algebra of $\pi^{\geq 2,*}(R)$.

Consider the following commutative diagram

$$\begin{tikzcd}
\mathcal{E}(\overline{M}) \arrow[r, "\delta"] & \Sigma\mathcal{E}(N) \\
\overline{M}^\vee \arrow[r, "\delta^0"] \arrow[u] & \mathcal{E}^1 \otimes_k N^\vee \arrow[u]
\end{tikzcd}$$

where the vertical maps are natural injective maps. Fix k-basis

$$\mu_1, \ldots, \mu_m \text{ of } \overline{M}^\vee, \quad \nu_1, \ldots, \nu_n \text{ of } N^\vee, \quad \xi_1, \ldots, \xi_e \text{ of } \mathcal{E}^1.$$

The map δ is $\mathcal{E}$-linear, and so also $\mathcal{K}$-linear. Hence if $\varepsilon \in \mathcal{E}$

$$\delta(\varepsilon \otimes \mu_h) = \varepsilon\delta^0(\mu_h) = \Sigma_{i,j} a_{i,j,h} \varepsilon \xi_j \otimes \nu_i$$

where

$$\delta^0(\mu_h) = \Sigma_{i,j} a_{i,j,h} \xi_j \otimes \nu_i \quad \text{with } a_{i,j,h} \in k.$$

By the Poincaré-Birkhoff-Witt theorem $\mathcal{E}$ is a free $\mathcal{K}$-module with basis

$$\{\xi_{j_1} \cdots \xi_{j_k}\}_{j_1 < \cdots < j_k}. \tag{2.1.8.2}$$

hence $\mathcal{E}(N)$ is a free $\mathcal{K}$-module with basis

$$\{\xi_{j_1} \cdots \xi_{j_k} \otimes \nu_i\}_{j_1 < \cdots < j_k, i}. \tag{2.1.8.3}$$

If $\kappa \in \mathcal{K}$ then

$$\delta(\kappa \otimes \mu_h) = \Sigma_{i,j} a_{i,j,h} \kappa \xi_j \otimes \nu_i.$$

This means that as a $\mathcal{K}$-module $\operatorname{Im} \delta_{|\mathcal{K} \otimes_k \overline{M}^\vee}$ is generated by the elements $\delta^0(\mu_h)$.

We can change the basis of $\overline{M}^\vee$ such that the coordinate vectors of

$$\delta^0(\mu_1), \ldots, \delta^0(\mu_{m'}),$$

with respect to the basis (2.1.8.3), are linearly independent over k and

$$\delta^0(\mu_{m'+1}), \ldots, \delta^0(\mu_m)$$

are all zero. Since the elements $\xi_j \otimes \nu_i$ are part of a $\mathcal{K}$-basis of $\mathcal{E}(N)$ we deduce that $\delta^0(\mu_1), \ldots, \delta^0(\mu_{m'})$ are linearly independent over $\mathcal{K}$.

This shows

$$\operatorname{Im} \delta_{|\mathcal{K} \otimes_k \overline{M}^\vee} = \Sigma \mathcal{K} \otimes_k \operatorname{Im} \delta^0.$$

This means that $\mathcal{E}(N)$ contains a copy of $\mathcal{K} \otimes_k \operatorname{Im} \delta^0$, and by commutativity of the diagram (2.1.8.1) this gives

$$H_{\mathcal{E}(M)}(s,t) \succeq H_{\mathcal{K}\otimes_k \operatorname{Im}\delta^0}(s,t) = H_{\mathcal{K}}(s,t) H_{\operatorname{Im}\delta^0}(s).$$

By 2.1.8.2 there is an equality of formal power series

$$H_{\mathcal{K}}(s,t) = \frac{H_{\mathcal{E}}(s,t)}{H_{\bigwedge \mathcal{E}^1}(s,t)}.$$

Since $\mathcal{E}^1 \cong (\mathfrak{m}/\mathfrak{m}^2)^\vee$ and $H_{X^\vee}(t) = H_X(t^{-1})$ for every R-module X, we deduce

$$H_{\bigwedge \mathcal{E}^1}(s,t) = \prod_i (1 + s^{-d_i} t).$$

Now consider the following chain of equalities of Hilbert series

$$\begin{aligned} H_{\operatorname{Im}\delta^0}(s) &= H_{\overline{M}^\vee}(s) - H_{W^\vee}(s) + H_{N^\vee}(s) \\ &= H_{\overline{M}}(s^{-1}) - H_{W^\vee}(s) + H_N(s^{-1}) \\ &= H_W(s^{-1}) - H_{W^\vee}(s) \\ &= H_V(s^{-1}) + H_{\overline{L}}(s^{-1}) - H_{W^\vee}(s) \\ &= H_V(s^{-1}). \end{aligned}$$

The first equality follows from the exact sequence in cohomology induced by $0 \to M \to L \to N \to 0$, i.e.

$$0 \to N^\vee \to (W)^\vee \to M^\vee \to \operatorname{Im}\delta^0 \to 0,$$

the second equality follows from the already mentioned fact: $H_{X^\vee}(t) = H_X(t^{-1})$ for every R-module X. The third equality follows from

$$0 \to \overline{M} \to W \to N \to 0,$$

the fourth equality follows from

$$0 \to V \to W \to \overline{L} \to 0,$$

and the last equality follows from $H_{W^\vee}(s) = H_{\overline{L}}(s^{-1})$.

Putting everything together we get

$$\begin{aligned} H_{\mathcal{E}(M)}(s,t) &\succeq H_{\mathcal{K}\otimes_k \operatorname{Im}\delta^0}(s,t) \\ &= H_{\mathcal{K}}(s,t) H_{\operatorname{Im}\delta^0}(s,t) \\ &= \frac{H_{\mathcal{E}}(s,t)}{\prod_{i=1}^n (1+s^{-d_i}t)} H_V(s^{-1}). \end{aligned}$$

Now notice that

$$H_{\mathcal{E}(M)}(s,t) = P_M^R(s^{-1},t) \quad \text{and} \quad H_{\mathcal{E}}(s,t) = P_k^R(s^{-1},t).$$

Hence we get

$$P_M^R(s^{-1},t) \succeq \frac{P_k^R(s^{-1},t)}{\prod_{i=1}^n (1+s^{-d_i}t)} H_V(s^{-1}),$$

and replacing s^{-1} with s we get the desired inequality. □

Let M be a finitely generated graded R-module. The *Castelnuovo-Mumford regularity* of M is

$$\operatorname{reg}_R M = \sup\{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

Definition 2.1.2. *A ring R is* Koszul *if the algebra* $\operatorname{Ext}_R(k,k)$ *is generated by*

$\mathrm{Ext}_R^{1,*}(k,k)$.

By [9] if $\operatorname{reg}_R k < \infty$ then $\operatorname{reg}_R k = \Sigma_i(d_i - 1)$, and this happens if and only if R is Koszul.

Corollary 2.1.9. *If M is a tightly embeddable module and $\operatorname{reg}_R M < \infty$ then R is Koszul.*

Proof. Set $r := \operatorname{reg}_R M$. If $\operatorname{reg}_R M < \infty$, then $\beta_{i,j}^R(M) = 0$ for $j - i > r$. If

$$\Sigma_{i,j} a_{i,j} s^j t^i = P_M^R(s,t) \prod_i (1 + s^{d_i} t)$$

then $a_{i,j} = 0$ for $j - i > r'$. By 2.1.8 if

$$\Sigma_{i,j} c_{i,j} s^j t^i = H_V(s) P_k^R(s,t)$$

then $c_{i,j} = 0$ if $j - i > r'$ for some r'. Since $\mathfrak{m}L/\mathfrak{m}M \neq 0$ we deduce $\beta_{i,j}^R(k) = 0$ if $j - i > r''$ for some r'' which is equivalent to $\operatorname{reg}_R k < \infty$. □

2.2 Direct summands of syzygies of the residue field

In this section we prove that nonzero direct summands of syzygies of k have infinite regularity if R is a complete intersection that is not generated by quadrics. Special homological properties of this class of modules were already noticed in [3, 27, 29].

Theorem 2.2.1. *If R is a complete intersection, $M = \Omega^m k$, $\beta\colon M \to N$ is an R-module homomorphism of finitely generated graded R-modules such that for some n the map $\beta^n\colon \mathrm{Ext}_R^n(N,k) \to \mathrm{Ext}_R^n(M,k)$ is not zero, then for some $b \in \mathbb{Z}$ there is a* *coefficient wise inequality*

$$s^b P_k^R(s,t) \preceq P_{\Omega^n N}^R(s,t) \prod_{i=1}^{n}(1+s^{d_i}t).$$

Proof. We set the following notation

$$N' = \Omega^n N, \quad \overline{N}' = N'/\mathfrak{m}N', \quad M' = \Omega^{m+n}k.$$

Let π be the canonical projection $N' \to N'/\mathfrak{m}N'$, and β' be a morphism $M' \to N'$ obtained by extending β to a morphism of free resolutions.

Let ξ denote the composed map

$$\xi : \mathcal{E} \otimes_k (\overline{N}')^\vee \cong \mathcal{E}(\overline{N}') \xrightarrow{\pi^*} \mathcal{E}(N') \xrightarrow{\beta'^*} \mathcal{E}(M') \xrightarrow{\alpha^*} \Sigma^{m+n}\mathcal{E}^{\geq m+n,*},$$

where α^* is an iterated connecting homomorphism, hence an isomorphism. By construction $\xi^0 \neq 0$.

By the Poincaré-Birkhoff-Witt Theorem $\mathcal{E} \cong \mathcal{K} \otimes_k \bigwedge \mathcal{E}^1$ as (bi-)graded left $\mathcal{K}$-modules; if R is a complete intersection then $\pi^{i,*}(R) = 0$ if $i \geq 3$, see for example [2, Theorem 7.3.3 and Theorem 10.1.2] and the references given there. Since $\pi^{i,*}(R) = 0$ if $i \geq 3$, $\mathcal{K}$ is a polynomial ring. Hence $\operatorname{Im}\xi$ is a torsion-free $\mathcal{K}$-module (since $\mathcal{E}^{\geq m+n}$ is a submodule of a free module over the polynomial ring $\mathcal{K}$). So any nonzero element in the image of ξ^0 generates a copy of $\mathcal{K}$ in $\operatorname{Im}\xi$, copy whose internal degree might be shifted by some $b \in \mathbb{Z}$. It follows that

$$H_{\mathcal{E}(N')}(s,t) \succeq s^b H_{\mathcal{K}}(s,t) = s^b \frac{H_{\mathcal{E}}(s,t)}{\prod_{i=1}^{n}(1+s^{-d_i}t)}.$$

Now we conclude as in 2.1.8. $\square$

Corollary 2.2.2. *Let R be a complete intersection with $\operatorname{reg}_R k = \infty$. For any $n \geq 0$, each nonzero direct summand of $\Omega^n k$ has infinite regularity.*

We raise the following question

Question 2.2.3. *If $\beta \colon \Omega^m k \to N$ is an R-module homomorphism of finitely generated graded R-modules such that for some n the map*

$$\beta^n \colon \operatorname{Ext}^n_R(N, k) \to \operatorname{Ext}^n_R(\Omega^m k, k)$$

is not zero and N has finite regularity, is then R Koszul?

2.3 On the vanishing of the graded deviations

In this section we relate Question 2.2.3 and the vanishing of the graded deviations of R. Let $R\langle X\rangle$ be an acyclic closure of k (see [2, 6.3] for the definition). Over graded rings we require the differential of the acyclic closure to be a homogeneous map. In order to make the differential homogeneous we have to give the elements of X an internal grading, making X a bigraded set.

Definition 2.3.1. *The (i, j)th graded deviation of R is*

$$\varepsilon_{i,j}(R) := \operatorname{Card}(X_{i,j}),$$

where $X_{i,j}$ is the set of variables in the acyclic closure of homological degree i and internal degree j.

Theorem 2.3.1. *If $M = \Omega^m k$ and $\beta \colon M \to N$ is a homomorphism of finitely gener-*

ated graded R-modules such that for some n the map

$$\beta^n\colon \operatorname{Ext}^n_R(N,k) \to \operatorname{Ext}^n_R(M,k)$$

is not zero, and $\operatorname{reg}_R N < \infty$ *then* $\varepsilon_{i,j}(R) = 0$ *for* $i > m+n$ *and* $i \neq j$.

Proof. We set the following notation

$$N' = \Omega^n N, \quad \overline{N}' = N'/\mathfrak{m}N', \quad M' = \Omega^{m+n}k.$$

Let $\mathcal{V}$ be the universal enveloping algebra of $\pi^{>m+n,*}(R)$. Let ξ denote the composed map

$$\xi : \mathcal{E}(\overline{N}') \xrightarrow{\pi^*} \mathcal{E}(N') \xrightarrow{\beta'^*} \mathcal{E}(M') \xrightarrow{\alpha^*} \Sigma^{m+n}\mathcal{E}^{\geq m+n,*}.$$

Consider the following commutative diagram

$$\begin{tikzcd}
\mathcal{E}(\overline{N}') \arrow[r, "\xi"] & \Sigma^{m+n}\mathcal{E}^{\geq m+n,*} \\
(\overline{N}')^\vee \arrow[u] \arrow[r, "\xi^0"] & \mathcal{E}^{m+n,*} \arrow[u]
\end{tikzcd}$$

By the Poincaré-Birkhoff-Witt theorem a $\mathcal{V}$-basis of $\mathcal{E}^{\geq m+n}$ is given by

$$\begin{aligned}\{\theta_1^{(i_1)}\cdots\theta_n^{(i_n)} \mid |\theta_1| \leq \cdots \leq |\theta_n|, i_j \leq 1 \text{ if } |\theta_j| \text{ odd },\\ \Sigma_j i_j|\theta_j| \geq m+n, \theta_j \in \pi^{<m+n,*}(R)\}.\end{aligned} \tag{2.3.1.1}$$

The $\mathcal{V}$-module $\operatorname{Im}\xi_{|\mathcal{V}\otimes_k(\overline{N}')^\vee}$ is generated by the elements $\xi^0(\nu_h)$ where $\nu_1,\ldots,\nu_m$

is a k-basis of $(\overline{N}')^{\vee}$. We set

$$\xi^0(\nu_h) = \Sigma a_{i_1,\dots,i_n}\theta_1^{(i_1)}\cdots\theta_n^{(i_n)}, \quad a_{i_1,\dots,i_n} \in k.$$

We can change the basis of $(\overline{N}')^{\vee}$ so that the coordinate vectors of $\xi^0(\nu_1),\dots,\xi^0(\nu_{n'})$, with respect to the basis (2.3.1.1), are linearly independent over k and $\xi^0(\nu_{n'+1}),\dots,\xi^0(\nu_n)$ are all zero. Since the elements $\theta_1^{(i_1)}\cdots\theta_n^{(i_n)}$ form a basis of $\mathcal{E}^{\geq m+n,*}$ over $\mathcal{V}$ we deduce that $\xi^0(\nu_1),\dots,\xi^0(\nu_{n'})$ are also linearly independent over $\mathcal{V}$.

This shows

$$\operatorname{Im}\xi_{|\mathcal{V}\otimes_k(\overline{N}')^{\vee}} = \Sigma^{m+n}\mathcal{V}\otimes\operatorname{Im}\xi^0.$$

This means that $\operatorname{Im}\xi$ contains a copy of $\mathcal{V}$ and by construction of ξ so does $\mathcal{E}(N')$.

We now recall that by [2, Theorem 10.2.1] $\dim_k \pi^{i,j}(R) = \varepsilon_{i,j}(R)$.

If there is an even $i > m+n$ such that $\varepsilon_{i,j}(R) \neq 0$ and $i \neq j$ then there is a nonzero element $x \in \pi^{i,j}(R)$. The powers of x belong to $\mathcal{V}$. But a copy of $\mathcal{V}$ is contained in $\mathcal{E}(N')$. Since bideg $x = (i,j)$ then bideg $x^{(l)} = (li, lj)$ and $lj - li = l(j-i)$ goes to ∞ as l goes to ∞, this implies $\operatorname{reg}_R N' = \infty$ which is a contradiction.

If there are infinitely many i's with $i > m+n$, i odd, $i \neq 0$ and $\varepsilon_{i,j}(R) \neq 0$ then there are infinitely many nonzero x_t with $t = 1,2,\dots$ belonging to $\pi^{i_t,j_t}(R)$. The products $x_1x_2\cdots x_s$ belong to $\mathcal{V}$. But a copy of $\mathcal{V}$ is contained in $\mathcal{E}(N')$. The bidegree of this product is

$$\text{bideg } x_1x_2\cdots x_s = (i_1+i_2+\cdots+i_s, j_1+j_2+\cdots+j_s)$$

and since $j_t - i_t \geq 1$ for every t the following inequality holds

$$j_1+j_2+\cdots+j_s - (i_1+i_2+\cdots+i_s) \geq s,$$

and as s goes to ∞, we get $\operatorname{reg}_R N' = \infty$, which is a contradiction.

Now we assume that $\varepsilon_{i,j}(R) \neq 0$ for finitely many i's with $i > m+n$, i odd and $i \neq j$. Let $R\langle X\rangle$ be an acyclic closure of k. Then we can choose an element $y \in X$ of odd degree, off diagonal, of maximal homological degree. We denote by $|.|$ the homological degree of a homogeneous element in the acyclic closure and by deg its internal degree. We claim that $A = R\langle X \backslash \{y\}\rangle$ is a DG subalgebra of the acyclic closure. Indeed if $x_1, \ldots, x_n, y$ are all the odd elements off diagonal then, by [9, Lemma 7], $R\langle X \backslash \{x_1, \ldots, x_n, y\}\rangle$ is a DG subalgebra of the acyclic closure. By maximality of y also A is a DG subalgebra of the acyclic closure. So A is a DG algebra and the acyclic closure of k can be written as $A\langle y\rangle$, which implies (by [9, Lemma 6]) that $|y| = 1$, which is a contradiction since $m \geq 1$. $\square$

Proposition 2.3.2. *Let $d \geq 4$ be an even number. If $\varepsilon_{i,j}(R) = 0$ for $i \geq d$ and $i \neq j$, then $\varepsilon_{i,j}(R) = 0$ for $i \geq d-1$ and $i \neq j$.*

Proof. Let $R\langle X\rangle$ be an acyclic closure of k. Let y be an element of X of homological degree $d-1$, we need to prove that $\deg y = d-1$. We assume that y appears in a boundary of a (bi)homogeneous element x with $|x| \geq d$

$$dx = \Sigma r_i a_i + rwy.$$

There are the following string of (in)equalities

$$\begin{aligned}
\deg w + \deg y &= \deg(wy) \\
&\geq |wy| \\
&= |x| - 1 = \deg x - 1 \\
&= \deg w + \deg y + \deg r - 1 \\
&\geq \deg w + \deg y.
\end{aligned}$$

We deduce

$$\deg w + \deg y = |w| + |y|.$$

If $\deg y > |y|$ then $\deg w < |w|$ which is not possible, hence $\deg y = |y|$.

If y doesn't appear in the boundary of any element then we can write the acyclic closure as $A\langle y\rangle$ with $A = R\langle X\backslash\{y\}\rangle$ and by [9, Lemma 6] $|y| = 1$, hence $d = 2$ which is not possible. □

Set $\varepsilon_i(R) = \Sigma_j \varepsilon_{i,j}(R)$. It is known that if $\varepsilon_i(R) = 0$ for $i \gg 0$ then $\varepsilon_i(R) = 0$ for $i \geq 3$, see [2, Theorem 7.3.3]. Motivated by this and by the previous proposition we raise the following question

Question 2.3.3. *If $\varepsilon_{i,j}(R) = 0$ when $i \gg 0$ and $i \neq j$ is it true that $\varepsilon_{i,j}(R) = 0$ when $i \geq 3$ and $i \neq j$?*

By [2, Theorem 7.3.3] and [2, Theorem 7.3.2]

1) $\varepsilon_i(R) = 0$ for $i \geq 2$ if and only if R is regular,

2) $\varepsilon_i(R) = 0$ for $i \geq 3$ if and only if R is a complete intersection.

By [9, Theorem 2] if $\varepsilon_{i,j}(R) = 0$ when $i \geq 2$ and $i \neq j$ then the ring R is of the form $Q \otimes_k S$ with Q regular and S a standard graded Koszul ring. Motivated by these last results we raise the following question

Question 2.3.4. *If $\varepsilon_{i,j}(R) = 0$ when $i \geq 3$ and $i \neq j$ is the ring R of the form $Q \otimes_k S$ with Q a complete intersection and S a standard graded Koszul ring?*

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