3-2011

Quantifying Performance in Fading Channels Using the Sampling Property of a Delta Function

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***Abstract*—We apply the sampling property of a delta function to obtain the probability of error in fading channels. Our approach reduces the integration to a sampling. The sampling point is obtained in terms of fading parameters and the average signal-to-noise ratio (SNR) to provide the closed form solution of the performance.**

***Index Terms*—Fading channel, performance, delta function, sampling.**

I. INTRODUCTION

OUR sampling method is remarkably simple compared to other performance analysis methods for wireless communication in fading channels discussed in [1-8]. The bit error rate (BER) obtained from the proposed method, called delta approximation denoted by '$\doteq$', is graphically or numerically displayed and compared to the exact theoretical result for various fading channels. We extend the result to the integer power of the Q-function to exhibit the average symbol error probability (ASEP) of the differentially encoded quadrature phase-shift keying (DE-QPSK).

II. DERIVATION OF DELTA APPROXIMATION

We can express the probability density function (PDF) of the fading channel as

$$p(\beta) = K \exp\{-b\beta\}\beta^{c-1} f(\beta) \tag{1}$$

where β is the magnitude square of the fading gain. K is a constant, and $f(\beta)$ is an auxiliary function that depends on fading characteristics. For example, $K = 1$, $b = 1$, $c = 1$ and $f(\beta) = 1$ for Rayleigh fading channels. Then the integer power probability of error in fading channels can be obtained as

$$P_b^p(\bar{\gamma}) = \int_0^\infty Q^p(\sqrt{\bar{\gamma}\beta})p(\beta)d\beta \tag{2}$$

$$= K\int_0^\infty Q^p(\sqrt{\bar{\gamma}\beta}) \exp\{-b\beta\}\beta^{c-1} f(\beta)d\beta \tag{3}$$

where $Q(\alpha) = \int_\alpha^\infty (2\pi)^{-1/2} e^{-y^2/2}dy$. $\bar{\gamma}$ is the average received SNR. For binary phase-shift keying (BPSK), $\bar{\gamma} = 2\bar{\gamma}_b$ where $\bar{\gamma}_b = E_b/N_o$. E_b is the average bit energy and N_o is the one-sided noise power spectral density (PSD).

Coherent detection: To find the sampling point, we introduce the Q-function approximation,

$$Q_1(\alpha) \approx \frac{1}{\sqrt{2\pi}\alpha}\exp\{-\alpha^2/2\} - \frac{1}{\sqrt{2\pi}(\alpha+1)}\exp\{-(\alpha+1)^2/2\} \tag{4}$$

where the first term (we call it $Q_o(\alpha)$) is the well-known upper bound of the Q-function. We show $Q_o(\alpha) \geq Q_1(\alpha) \geq Q(\alpha)$ in Appendix A. The integer power of the Q-function approximation can be found using the binomial expansion:

$$Q_1^p(\alpha) \approx \sum_{k=0}^{p}\binom{p}{k}(-1)^k\left(\frac{1}{\sqrt{2\pi}\alpha}\exp\{-\alpha^2/2\}\right)^{p-k} \left(\frac{1}{\sqrt{2\pi}(\alpha+1)}\exp\{-(\alpha+1)^2/2\}\right)^k. \tag{5}$$

With a change of variables ($\bar{\gamma}\beta = x$) and $Q_1^p(\alpha)$ into Eq. (3),

$$P_b^p(\bar{\gamma}) \approx \frac{K}{\bar{\gamma}^c}\int_0^\infty Q_1^p(\sqrt{x})\exp\{-b(x/\bar{\gamma})\}x^{c-1}f(x/\bar{\gamma})dx. \tag{6}$$

Applying Eq. (5) to Eq. (6) and with another change of variables ($x = y^N$),

$$P_b^p(\bar{\gamma}) \approx \left(\frac{1}{\sqrt{2\pi}}\right)^p \frac{K}{\bar{\gamma}^c}\sum_{k=0}^{p}\binom{p}{k}(-1)^k\exp\{-k/2\} \int_0^\infty f(y^N/\bar{\gamma})(\sqrt{y^N}+1)^{-k}(\sqrt{y^N})^{-(p-k)}\exp\{-k\sqrt{y^N}\} \exp\{-(p/2+b/\bar{\gamma})y^N\}y^{N(c-1)}Ny^{N-1}dy. \tag{7}$$

With $a = p/2 + b/\bar{\gamma}$, let us define

$$g(y^N) := \exp\{-ay^N\}Ny^{cN-1}. \tag{8}$$

Then, we find

$$\int_0^\infty g(y^N)dy = \frac{\Gamma(c)}{a^c} \tag{9}$$

with the Gamma function defined as $\Gamma(\alpha) := \int_0^\infty y^{\alpha-1}e^{-y}dy$ for $\alpha > 0$. We also show in Appendix B that

$$\lim_{N\to\infty} g(y^N) = \frac{\Gamma(c)}{a^c}\delta\left(y^N - \frac{c}{a}\right). \tag{10}$$

Now, applying the sampling property of a delta function to Eq. (7),

$$P_b^p(\bar{\gamma}) \doteq \left(\frac{1}{\sqrt{2\pi}}\right)^p \frac{K}{\bar{\gamma}^c}\frac{\Gamma(c)}{a^c}\sum_{k=0}^{p}\binom{p}{k}(-1)^k\exp\{-k/2\} \int_0^\infty f(y^N/\bar{\gamma})(\sqrt{y^N}+1)^{-k}(\sqrt{y^N})^{-(p-k)}\exp\{-k\sqrt{y^N}\} \delta\left(y^N - \frac{c}{a}\right)dy = \left(\frac{1}{\sqrt{2\pi}}\right)^p\frac{K\Gamma(c)}{(\bar{\gamma}a)^c}f\left(\frac{c}{a\bar{\gamma}}\right) \left[\sqrt{\frac{a}{c}} - \left(\sqrt{\frac{c}{a}}+1\right)^{-1}\exp\left\{-\left(\sqrt{\frac{c}{a}}+\frac{1}{2}\right)\right\}\right]^p. \tag{11}$$

Manuscript received October 15, 2010. The associate editor coordinating the review of this letter and approving it for publication was Y. Chen.

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Digital Object Identifier 10.1109/LCOMM.2011.011011.101964

TABLE I
PARAMETERS FOR FADING CHANNELS

fading	K	a	c	$f\left(\frac{c}{a\bar{\gamma}}\right)$
Nakagami-m	$m^m/\Gamma(m)$	$p/2+m/\bar{\gamma}$	m	1
Nakagami-n	$(1+n^2)\exp\{-n^2\}$	$p/2+1/\bar{\gamma}$	1	$\exp\{-n^2/(a\bar{\gamma})\}I_o\left(2n\sqrt{(1+n^2)/(a\bar{\gamma})}\right)$
Nakagami-q	$(1+q^2)/(2q)$	$p/2+(1+q^2)^2/(4q^2\bar{\gamma})$	1	$I_o\left((1-q^4)/\{(4q^2)(a\bar{\gamma})\}\right)$

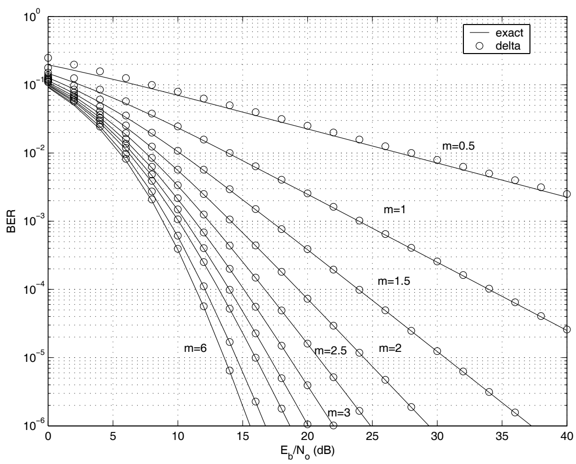

Fig. 1. Nakagami-m fading, BPSK, m=0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 5, 6.

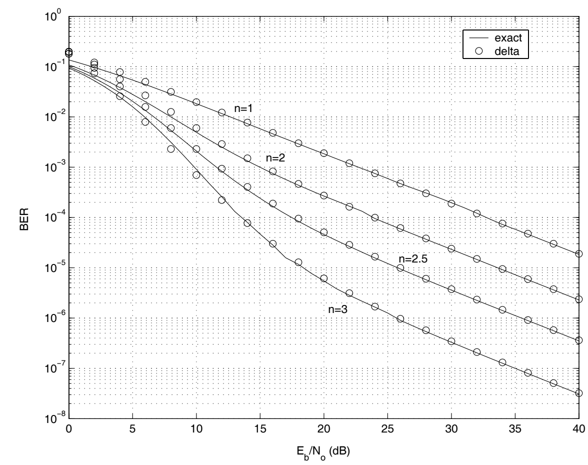

Fig. 2. Nakagami-n fading, BPSK, n=1, 2, 2.5, 3.

Noncoherent detection: The BER in a noncoherent additive white Gaussian noise (AWGN) channel can be expressed as $P_e(\bar{\gamma}) = 2^{-1}e^{-\bar{\gamma}/2}$ where $\bar{\gamma} = 2\bar{\gamma}_b$ for differential PSK (DPSK) [3, Eq. (14.3-9)]. Then, the integer power of the BER in fading channels can be expressed using Eq. (1) as

$$P_b^p(\bar{\gamma}) = \frac{K}{2^p}\int_0^\infty \exp\{-p\bar{\gamma}\beta/2\}\exp\{-b\beta\}\beta^{c-1}f(\beta)d\beta. \quad (12)$$

With a change of variables ($\bar{\gamma}\beta = x$),

$$P_b^p(\bar{\gamma}) = \frac{K}{2^p}\frac{1}{\bar{\gamma}^c}\int_0^\infty \exp\{-px/2\}\exp\{-bx/\bar{\gamma}\} x^{c-1}f(x/\bar{\gamma})dx. \quad (13)$$

With another change of variables ($x = y^N$),

$$P_b^p(\bar{\gamma}) = \frac{K}{2^p}\frac{1}{\bar{\gamma}^c}\int_0^\infty \exp\{-(p/2+b/\bar{\gamma})y^N\} y^{N(c-1)}f(y^N/\bar{\gamma})Ny^{N-1}dy. \quad (14)$$

Applying the same process in coherent detection, the delta approximation can be obtained as

$$P_b^p(\bar{\gamma}) \doteq \frac{K}{2^p}\frac{\Gamma(c)}{(\bar{\gamma}a)^c}f\left(c/a\bar{\gamma}\right). \quad (15)$$

III. NUMERICAL RESULTS

Nakagami-m fading: The PDF of the fading is [1]

$$p(\beta) = \frac{m^m\beta^{m-1}}{\Gamma(m)}\exp\{-m\beta\}, \qquad \beta \geq 0. \quad (16)$$

Making reference to Eq. (1), we choose $K = m^m/\Gamma(m)$, $c = m$, $a = p/2 + m/\bar{\gamma}$ and $f(\frac{c}{a\bar{\gamma}})$=1. We summarize the parameter selection in different fadings in Table I. The result is compared to the exact performance of BPSK [2, Eq. (5.80)]

$$P_e = \frac{1}{\pi}\int_0^{\pi/2}\left(1+\frac{\bar{\gamma}_b}{m\sin^2\phi}\right)^{-m}d\phi. \quad (17)$$

In Fig. 1 we plot the exact BER and the delta approximation for m= 0.5 to 6. We can see that the delta approximation and the exact performance agree well.

Nakagami-n fading: For Nakagami-n fading channels, the fading PDF is shown as [1]

$$p(\beta) = (1+n^2)\exp\{-n^2\}\exp\{-(1+n^2)\beta\} I_o(2n\sqrt{(1+n^2)\beta}), \qquad \beta \geq 0 \quad (18)$$

and we choose $K = (1+n^2)\exp\{-n^2\}$, $c = 1$, $a = p/2 + 1/\bar{\gamma}$ and $f(\frac{c}{a\bar{\gamma}}) = \exp\{-n^2/(a\bar{\gamma})\}\ I_o(2n\sqrt{(1+n^2)/(a\bar{\gamma})})$. It is important to choose an auxiliary function $f(\cdot)$ as flat as possible with respect to (w.r.t.) the SNR to be robust against the sampling point error that may be introduced by the Q-function approximation. The modified Bessel function $I_o(\cdot)$ is a decreasing function w.r.t. the SNR while $\exp\{-n^2/(a\bar{\gamma})\}$ is an increasing function. As a result, the product of the two is rather flat w.r.t. the SNR and thus less sensitive to the sampling point error. The BER of Nakagami-n with BPSK is shown in Fig. 2 for n=1, 2, 2.5 and 3. The delta approximation agrees well with the exact BER at a moderate and high SNR. The exact performance is numerically obtained from Eq. (2) with $p = 1$ using a double integration with the modified Bessel function.

Nakagami-q fading: The PDF of Nakagami-q fading channels is presented as [1]

$$p(\beta) = \frac{1+q^2}{2q}\exp\left\{-\frac{(1+q^2)^2\beta}{4q^2}\right\}I_o\left(\frac{(1-q^4)\beta}{4q^2}\right), \beta \geq 0 \quad (19)$$

and we choose $K = (1+q^2)/(2q)$, $c = 1$, $a = p/2 + (1+q^2)^2/(4q^2\bar{\gamma})$, and $f(\frac{c}{a\bar{\gamma}}) = I_o((1-q^4)/\{(4q^2)(a\bar{\gamma})\})$. Since Nakagami-$q$ fading with $q = 1$ corresponds to the Rayleigh fading channel, we compare the delta approximation to the exact performance of BPSK [3, Eq. (14.3-7)]

$$P_e = (1/2)\left(1-\sqrt{\bar{\gamma}_b/(1+\bar{\gamma}_b)}\right). \quad (20)$$

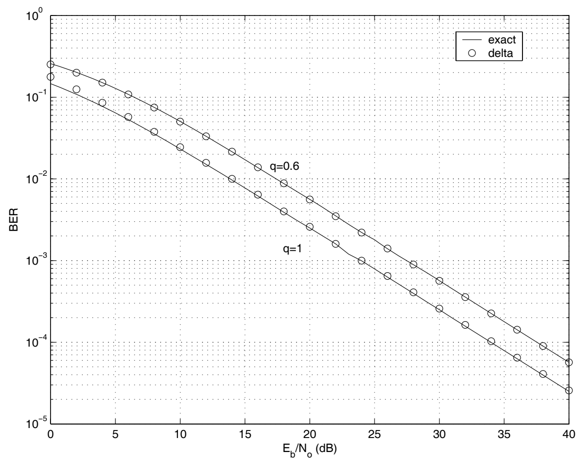

Fig. 3. Nakagami-q fading, q=0.6 (DPSK), q=1 (BPSK).

The delta approximation of DPSK is obtained from Eq. (15) with $p = 1$ and then compared to the exact performance [2, Eqs. (2.12) & (8.201)]

$$P_e = (1/2)\left\{1 + 2\bar{\gamma}_b + (2\bar{\gamma}_b q)^2/(1+q^2)^2\right\}^{-1/2}. \quad (21)$$

The delta approximation agrees well with the exact performance as shown in Fig. 3.

Differentially Encoded QPSK: The ASEP of the DE-QPSK in the AWGN channel is given by [4]

$$P_e = 4Q(\sqrt{\bar{\gamma}}) - 8Q^2(\sqrt{\bar{\gamma}}) + 8Q^3(\sqrt{\bar{\gamma}}) - 4Q^4(\sqrt{\bar{\gamma}}). \quad (22)$$

Therefore, the ASEP of the DE-QPSK in fading channels can be obtained as

$$P(\bar{\gamma}) \doteq 4P_b^1(\bar{\gamma}) - 8P_b^2(\bar{\gamma}) + 8P_b^3(\bar{\gamma}) - 4P_b^4(\bar{\gamma}) \quad (23)$$

with $P_b^p(\bar{\gamma})$ defined in Eq. (11). The ASEP of the DE-QPSK in Nakagami-m fading channels is shown in Tables II and III for m=2.5 and 3.5, respectively. The result is compared to approximations in Chiani [5, Eq. (14)], Börjesson [6, Eq. (9)], Karagiannidis [7, Eq. (12)] and Isukapalli [8, Eq. (11)]. The delta approximation provides a tighter approximation than Chiani and Börjesson for $\bar{\gamma} \geq 10$ dB for both m=2.5 and 3.5. Our method also provides a tighter approximation than Karagiannidis and Isukapalli for $\bar{\gamma} \geq 20$ dB for both m=2.5 and 3.5. Karagiannidis ASEP is obtained with parameter values specified in [7]. Isukapalli ASEP is obtained with the number of terms in Taylor series expansion, N_a=14. The order of its computational complexity is $O(\exp\{N_a \ln(N_a)\})$ that can make Isukapalli method infeasible for a large value of m or SNR.

TABLE II
ASEP OF DE-QPSK (m=2.5)

$\bar{\gamma}$	Exact	Chiani	Börj.	Karag.	Isukap.	Eq. (23)
0	4.814e-1	5.061e-1	5.000e-1	5.259e-1	4.841e-1	5.532e-1
5	2.133e-1	2.436e-1	2.250e-1	2.292e-1	2.130e-1	2.298e-1
10	4.326e-2	5.122e-2	4.570e-2	4.528e-2	4.270e-2	4.430e-2
15	4.410e-3	5.257e-3	4.653e-3	4.546e-3	4.318e-3	4.412e-3
20	3.098e-4	3.694e-4	3.266e-4	3.174e-4	3.022e-4	3.073e-4
25	1.877e-5	2.239e-5	1.978e-5	1.919e-5	1.828e-5	1.857e-5

TABLE III
ASEP OF DE-QPSK (m=3.5)

$\bar{\gamma}$	Exact	Chiani	Börj.	Karag.	Isukap.	Eq. (23)
0	4.763e-1	5.045e-1	4.964e-1	5.217e-1	4.793e-1	5.557e-1
5	1.950e-1	2.269e-1	2.063e-1	2.085e-1	1.944e-1	2.128e-1
10	2.889e-2	3.480e-2	3.053e-2	2.977e-2	2.832e-2	2.986e-2
15	1.502e-3	1.802e-3	1.580e-3	1.512e-3	1.452e-3	1.512e-3
20	4.081e-5	4.869e-5	4.285e-5	4.068e-5	3.909e-5	4.069e-5
25	8.398e-7	9.995e-7	8.811e-7	8.342e-7	8.009e-7	8.345e-7

APPENDIX A

$Q_o(\alpha) \geq Q_1(\alpha) \geq Q(\alpha)$*:* Let $e(\alpha) = Q_1(\alpha) - Q(\alpha)$. Then $e(0) = \infty$ and $e(\infty) = 0$. If we assume $de(\alpha)/d\alpha < 0, \forall\alpha \geq 0$,

$$de(\alpha)/d\alpha = (2\pi)^{-1/2}[-\alpha^{-2}\exp\{-\alpha^2/2\} \quad (24)$$

$$+(\alpha+1)^{-2}\exp\{-(\alpha+1)^2/2\} + \exp\{-(\alpha+1)^2/2\}] < 0, \text{ or} \quad (25)$$

$$\{\alpha^2/(\alpha+1)^2\}\exp\{-(\alpha+1/2)\} + \alpha^2\exp\{-(\alpha+1/2)\} < 1. \quad (26)$$

Let $y_1(\alpha)$ and $y_2(\alpha)$ be the first and second term of Eq. (26). Then,

$$y(\alpha) = y_1(\alpha) + y_2(\alpha) < 1. \quad (27)$$

$y_1(\alpha)$ and $y_2(\alpha)$ are unimodal functions for $\alpha \geq 0$ with their peaks at α=1 and α=2, respectively. Thus, $y(\alpha) < y_1(1) + y_2(2) < 1$. Indeed, $de(\alpha)/d\alpha < 0$ for $\alpha \geq 0$. In result, $Q_o(\alpha) \geq Q_1(\alpha) \geq Q(\alpha)$.

APPENDIX B

A limit impulse: To obtain the critical point y_*^N of $g(y^N)$:

$$\frac{dg(y^N)}{dy} = -aNy^{N-1}\exp\{-ay^N\}Ny^{cN-1} + \exp\{-ay^N\}N(cN-1)y^{cN-2} = 0, \quad (28)$$

or $y_*^N = \lim_{N\to\infty} y^N = \frac{c}{a}$. Applying the above result to Eq. (8), we find that

$$\lim_{N\to\infty} g(y_*^N) = \lim_{N\to\infty} \exp\{-c\}N(\frac{c}{a})^c = \infty. \quad (29)$$

From Eq. (28), we can see that $dg(y^N)/dy > 0$ for $0 < y^N < y_*^N$, and $dg(y^N)/dy < 0$ for $y^N > y_*^N$. Together with Eqs. (9) and (29), we obtain Eq. (10).

REFERENCES

[1] Z. Wang and G. B. Giannakis, "A simple and general parameterization quantifying performance in fading channels," *IEEE Trans. Commun.*, vol. 51, no. 8, pp. 1389-1398, Aug. 2003.
[2] M. K. Simon and M. S. Alouini, *Digital Communications over Fading Channels: A Unified Approach to Performance Analysis*, 2nd edition. Wiley, 2000.
[3] J. G. Proakis, *Digital Communications*, 4th edition. McGraw-Hill, 2001.
[4] M. K. Simon, "Integral representation of certain integer powers of the Gaussian Q-function and their applications," *IEEE Commun. Lett*, vol. 6, no. 12, pp. 532-534, Dec. 2002.
[5] M. Chiani, D. Dardari, and M. K. Simon, "New exponential bounds and approximations for the computation of error probability in fading channels," *IEEE Trans. Wireless Commun.*, vol. 2, no. 4, pp. 840-845, July 2003.
[6] P. O. Börjesson and C. E. Sundberg, "Simple approximations of the error function Q(x) for communications applications," *IEEE Trans. Commun.*, vol. COM-27, no. 3, pp. 639-643, Mar. 1979.
[7] G. K. Karagiannidis and A. S. Lioumpas, "An improved approximation for the Gaussian $Q-$function," *IEEE Commun. Lett.*, vol. 11, no. 8, pp. 644-646, Aug. 2007.
[8] Y. Isukapalli and B. D. Rao, "An analytically tractable approximation for Gaussian Q-function," *IEEE Commun. Lett.*, vol. 12, no. 9, pp. 669-671, Sep. 2008.