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Bass' *NK* groups and *cdh*-fibrant Hochschild homology

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Abstract The K-theory of a polynomial ring $R[t]$ contains the K-theory of R as a summand. For R commutative and containing $\mathbb{Q}$, we describe $K_*(R[t])/K_*(R)$ in terms of Hochschild homology and the cohomology of Kähler differentials for the cdh topology.

We use this to address Bass' question, whether $K_n(R) = K_n(R[t])$ implies $K_n(R) = K_n(R[t_1, t_2])$. The answer to this question is affirmative when R is essentially of finite type over the complex numbers, but negative in general.

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In 1972, H. Bass posed the following question (see [4], question $(\mathrm{VI})_n$):

$$\text{Does } K_n(R) = K_n(R[t]) \text{ imply that } K_n(R) = K_n(R[t_1, t_2])?$$

One can rephrase the question in terms of Bass' groups NK_n, introduced in [3]:

$$\text{Does } NK_n(R) = 0 \text{ imply that } N^2K_n(R) = 0?$$

More generally, for any functor F from rings to an abelian category, Bass defines $NF(R)$ as the kernel of the map $F(R[t]) \to F(R)$ induced by evaluation at $t = 0$, and $N^2F = N(NF)$. Bass' question was inspired by Traverso's theorem [26], from which it follows that $N\,\mathrm{Pic}(R) = 0$ implies $N^2\,\mathrm{Pic}(R) = 0$.

In this paper, we give a new interpretation of the groups $NK_n(R)$ in terms of Hochschild homology and the cohomology of Kähler differentials for the *cdh* topology, for commutative $\mathbb{Q}$-algebras. This allows us to give a counterexample to Bass' question in the companion paper [8] (see Theorem 0.2 below).

To state our main structural theorem, recall from [30] that each $NK_n(R)$ has the structure of a module over the ring of big Witt vectors $W(R)$. It is convenient to use the countably infinite-dimensional $\mathbb{Q}$-vector spaces $t\mathbb{Q}[t]$ and $\Omega^1_{\mathbb{Q}[t]}$. If M is any R-module, then $M \otimes t\mathbb{Q}[t]$ and $M \otimes \Omega^1_{\mathbb{Q}[t]}$ are naturally $W(R)$-modules by [12].

Theorem 0.1 *Let R be a commutative ring containing $\mathbb{Q}$. Then there is a $W(R)$-module isomorphism*

$$N^2K_n(R) \cong (NK_n(R) \otimes t\mathbb{Q}[t]) \oplus \big(NK_{n-1}(R) \otimes \Omega^1_{\mathbb{Q}[t]}\big).$$

Thus $K_n(R) = K_n(R[t_1, t_2])$ iff $NK_n(R) = NK_{n-1}(R) = 0$ iff $N^2K_n(R) = 0$.

In addition, the following are equivalent for all $p > 0$:

(a) $K_n(R) = K_n(R[t_1, \dots, t_p])$.
(b) $NK_n(R) = 0$ *and* $K_{n-1}(R) = K_{n-1}(R[t_1, \dots, t_{p-1}])$.
(c) $NK_q(R) = 0$ *for all q such that $n - p < q \le n$.*

The equivalence of (a), (b) and (c) is immediate by induction, using the formula for N^2K_n, and is included for its historical importance; see [27]. Theorem 0.1 also holds for the K-theory of schemes of finite type over a field; see Theorem 4.2 below.

Theorem 0.1 allows us to reformulate Bass' question as follows:

$$\text{Does } NK_n(R) = 0 \text{ imply that } NK_{n-1}(R) = 0?$$

Theorem 0.2 (a) *For any field F algebraic over $\mathbb{Q}$, the 2-dimensional normal algebra*

$$R = F[x, y, z]/(z^2 + y^3 + x^{10} + x^7 y)$$

has $K_0(R) = K_0(R[t])$ but $K_0(R) \neq K_0(R[t_1, t_2])$.

(b) *Suppose R is essentially of finite type over a field of infinite transcendence degree over $\mathbb{Q}$. Then $NK_n(R) = 0$ implies that R is K_n-regular and, in particular, that $K_n(R) = K_n(R[t_1, t_2])$.*

Part (a) is proven in the companion paper [8], using Theorem 0.1, while part (b) is proven below as Corollary 6.7.

The proof of Theorem 0.1 relies on methods developed in [7] and [9], which allow us to compute the groups NK_n and $N^p K_n$ in terms of the Hochschild homology of R, and of the cdh-cohomology of the higher Kähler differentials Ω^p, both relative to $\mathbb{Q}$. The groups $NK_n(R)$ have a natural bigraded structure when $\mathbb{Q} \subset R$, and it is convenient to take advantage of this bigrading in stating our results. The bigrading comes from the eigenspaces $NK_n^{(i)}(R)$ of the Adams operations ψ^k (arising from the λ-filtration) and the eigenspaces of the homothety operations $[r]$ (i.e. base change for $t \mapsto rt$). This bigrading will be explained in Sects. 1 and 5; the general decomposition for Adams weight i has the form:

$$NK_n^{(i)}(R) \cong TK_n^{(i)}(R) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]. \tag{0.3}$$

Here $TK_n^{(i)}$ denotes the *typical piece* of $NK_n^{(i)}(R)$, defined as the simultaneous eigenspace $\{x \in NK_n^{(i)}(R) : [r]x = rx,\ r \in R\}$. (See Example 1.6.) We provide a concrete description of the typical pieces in Theorem 5.1, reproduced here:

Theorem 0.4 *If R is a commutative $\mathbb{Q}$-algebra, then $NK_n^{(i)}(R)$ is determined by its typical pieces $TK_n^{(i)}(R)$ and (0.3). For $i \neq n, n+1$ we have:*

$$TK_n^{(i)}(R) \cong \begin{cases} HH_{n-1}^{(i-1)}(R) & \text{if } i < n, \\ H_{\mathrm{cdh}}^{i-n-1}(R, \Omega^{i-1}) & \text{if } i \geq n+2. \end{cases}$$

For $i = n, n+1$, we have an exact sequence:

$$0 \to TK_{n+1}^{(n+1)}(R) \to \Omega_R^n \to H_{\mathrm{cdh}}^0(R, \Omega^n) \to TK_n^{(n+1)}(R) \to 0.$$

Table 1 The groups $TK_n^{(i)}(R)$ for $n \le 3$, $\dim(R)=2$

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
$TK_3^{(i)}(R)$	0	$HH_2^{(1)}(R)$	$\mathrm{tors}\,\Omega_R^2$	$\Omega^3_{\mathrm{cdh}}(R)/\Omega^3_R$	$H^1_{\mathrm{cdh}}\Omega^4$	0
$TK_2^{(i)}(R)$	0	$\mathrm{tors}\,\Omega_R^1$	$\Omega^2_{\mathrm{cdh}}(R)/\Omega^2_R$	$H^1_{\mathrm{cdh}}\Omega^3$	0	
$TK_1^{(i)}(R)$	$\mathrm{nil}(R)$	$\Omega^1_{\mathrm{cdh}}(R)/\Omega^1_R$	$H^1_{\mathrm{cdh}}\Omega^2$	0		
$TK_0^{(i)}(R)$	R^+/R	$H^1_{\mathrm{cdh}}\Omega^1$	0			
$TK_{-1}^{(i)}(R)$	$H^1_{\mathrm{cdh}}\mathcal{O}$	0				
$TK_{-2}(R)$	0					

The special case $NK_0 = \bigoplus NK_0^{(i)}$ of Theorem 0.4 is that for R essentially of finite type over a field of characteristic zero, with $d=\dim(R)$,

$$NK_0(R) \cong \left((R^+/R_{\mathrm{red}}) \oplus \bigoplus_{p=1}^{d-1} H^p_{\mathrm{cdh}}(R,\Omega^p) \right) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]. \tag{0.5}$$

Here R^+ is the seminormalization of R_{red}; we show in Proposition 2.5 that $R^+ = H^0_{\mathrm{cdh}}(R,\mathcal{O})$. The dimension zero case of Theorem 0.4 is also revealing:

Example 0.6 If $\dim(R)=0$ then we get $NK_n(R) \cong HH_{n-1}(R,I) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]$ for all n, where I is the nilradical of R. It is illuminating to compare this with Goodwillie's Theorem [14], which implies that $NK_n(R) \cong NK_n(R,I) \cong NHC_{n-1}(R,I)$. The identification comes from the standard observation (1.2) that the map $HH_* \to HC_*$ induces $NHC_*(R,I) \cong HH_*(R,I) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]$.

The calculations of Theorem 0.4 for small n are summarized in Table 1 when $\dim(R)=2$. We will need the following cases of 0.4 in [8], to prove Theorem 0.2(a).

Theorem 0.7 *Let R be normal domain of dimension 2 which is essentially of finite type over an algebraic extension of $\mathbb{Q}$. Then*

(a) $NK_0(R) = NK_0^{(2)}(R) \cong H^1_{\mathrm{cdh}}(R,\Omega^1) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]$ *and*
(b) $NK_{-1}(R) = NK_{-1}^{(1)}(R) \cong H^1_{\mathrm{cdh}}(R,\mathcal{O}) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]$.

Here is an overview of this paper: Sect. 1 reviews the bigrading on the Hochschild and cyclic homology of $R[t]$ (and $X \times \mathbb{A}^1$), and Sect. 2 reviews the *cdh*-fibrant analogue. Section 3 describes the sheaf cohomology of the fibers $\mathcal{F}_{HH}(X)$, $\mathcal{F}_{HC}(X)$, etc. of $HH(X) \to \mathbb{H}_{\mathrm{cdh}}(X,HH)$, etc. In Sect. 4 we use these fibers to prove Theorem 0.1, by relating $NK_{n+1}(X)$ to

$H^{-n}\mathcal{F}_{HH}(X)$. We also show that Bass' question is negative for schemes in Lemma 4.5.

In Sect. 5, we give the detailed computations of the typical pieces $TK_n^{(i)}(R)$ needed to establish (0.5) and Table 1; these computations employ the main result of [10]. In Sect. 6, we prove Theorem 0.2(b), that the answer to Bass' question is positive provided we are working over a sufficiently large base field. Finally, Sect. 7 describes how Theorem 0.7 changes if R is of finite type over an arbitrary field of characteristic 0: the map $NK_0(R) \to H^1_{\mathrm{cdh}}(R, \Omega^1_{/F}) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]$ is onto, and an isomorphism if $NK_{-1}(R) = 0$.

Notation

All rings considered in this paper should be assumed to be commutative and noetherian, unless otherwise stated. Throughout this paper, k denotes a field of characteristic 0 and F is a field containing k as a subfield. We write Sch/k for the category of separated schemes essentially of finite type over k. If $\mathcal{F}$ is a presheaf on Sch/k, we write $\mathcal{F}_{\mathrm{cdh}}$ for the associated cdh sheaf, and often simply write $H^*_{\mathrm{cdh}}(X, \mathcal{F})$ in place of the more formal $H^*_{\mathrm{cdh}}(X, \mathcal{F}_{\mathrm{cdh}})$.

If H is a functor on Sch/k and R is an algebra essentially of finite type, we occasionally write $H(R)$ for $H(\mathrm{Spec}\, R)$. For example, $H^*_{\mathrm{cdh}}(R, \Omega^i)$ is used for $H^*_{\mathrm{cdh}}(\mathrm{Spec}\, R, \Omega^i)$. Note that, because the cdh site is noetherian (every cover has a finite subcovering) H^*_{cdh} sends inverse limits of schemes over diagrams with affine transition morphisms to direct limits.

If H is a contravariant functor from Sch/k to spectra, (co)chain complexes, or abelian groups that takes filtered inverse limits of schemes over diagrams with affine transition morphisms to colimits (as for example K, HH, $\mathbb{H}_{\mathrm{cdh}}(-, HH)$, and $\mathcal{F}_{HH}$), then for any k-algebra R, we abuse notation and write $H(R)$ for the direct limit of the $H(R_\alpha)$ taken over all subrings R_α of R of finite type over k. (If R is essentially of finite type, the two definitions of $H(R)$ agree up to canonical isomorphism.) In particular, we will use expressions like $\mathbb{H}_{\mathrm{cdh}}(R, HH)$ for general commutative $\mathbb{Q}$-algebras even though we do not define the cdh-topology for arbitrary $\mathbb{Q}$-schemes.

We use cohomological indexing for all chain complexes in this paper; for a complex C, $C[p]^q = C^{p+q}$. For example, the Hochschild, cyclic, periodic, and negative cyclic homology of schemes over a field k can be defined using the Zariski hypercohomology of certain presheaves of complexes; see [34] and [7, 2.7] for precise definitions. We shall write these presheaves as $HH(/k)$, $HC(/k)$, $HP(/k)$ and $HN(/k)$, respectively, omitting k from the notation if it is clear from the context.

It is well known (see [33, 10.9.19]) that there is an Eilenberg-Mac Lane functor $C \mapsto |C|$ from chain complexes of abelian groups to spectra, and from presheaves of chain complexes of abelian groups to presheaves of spectra. This functor sends quasi-isomorphisms of complexes to weak homotopy equivalences of spectra, and satisfies $\pi_n(|C|) = H^{-n}(C)$. For example, applying π_n to the Chern character $K \to |HN|$ yields maps $K_n(R) \to H^{-n}HN(R) = HN_n(R)$. In this spirit, we will use descent terminology for presheaves of complexes.

1 The bigrading on NHH and NHC

Recall that k denotes a field of characteristic 0. In this section, we consider the Hochschild and cyclic homology of polynomial extensions of commutative k-algebras. No great originality is claimed. Throughout, we will use the chain level Hodge decompositions $HH = \prod_{i\geq 0} HH^{(i)}$ and $HC = \prod_{i\geq 0} HC^{(i)}$.

The Künneth formula for Hochschild homology yields

$$NHH_n^{(i)}(R) \cong \big(HH_n^{(i)}(R) \otimes t\mathbb{Q}[t]\big) \oplus \big(HH_{n-1}^{(i-1)}(R) \otimes \Omega^1_{\mathbb{Q}[t]}\big). \tag{1.1}$$

From the exact SBI sequence $0 \to NHC_{n-1} \xrightarrow{B} NHH_n \xrightarrow{I} NHC_n \to 0$ (see [33, 9.9.1]), and induction on n, the map I induces canonical isomorphisms for each i:

$$NHC_n^{(i)}(R) \cong HH_n^{(i)}(R) \otimes t\mathbb{Q}[t]. \tag{1.2}$$

Remark 1.3 Both (1.1) and (1.2) generalize to non-affine quasi-compact schemes X over k. Indeed, NHH and NHC satisfy Zariski descent because HH and HC do and because, for any open cover $\{U_i \to X\}$, the collection $\{U_i \times \mathbb{A}^1 \to X \times \mathbb{A}^1\}$ is also a cover. Thus we have

$$\begin{aligned} NHH^{(i)}(X) &\cong \mathbb{H}_{\mathrm{Zar}}(X, NHH^{(i)}) \\ &\cong \mathbb{H}_{\mathrm{Zar}}(X, HH^{(i)}) \otimes t\mathbb{Q}[t] \oplus \mathbb{H}_{\mathrm{Zar}}(X, HH^{(i-1)})[1] \otimes \Omega^1_{\mathbb{Q}[t]} \\ &\cong HH^{(i)}(X) \otimes t\mathbb{Q}[t] \oplus HH^{(i-1)}(X)[1] \otimes \Omega^1_{\mathbb{Q}[t]}, \end{aligned}$$

and $NHC^{(i)}(X) = \mathbb{H}_{\mathrm{Zar}}(X, NHC^{(i)}) \cong \mathbb{H}_{\mathrm{Zar}}(X, HH^{(i)}) \otimes t\mathbb{Q}[t] \cong HH^{(i)}(X) \otimes t\mathbb{Q}[t]$.

It is easy to iterate the construction $F \mapsto NF$. For example, we see from (1.1) and (1.2) that

$$\begin{aligned} N^2HC_n^{(i)}(R) \cong &\big(HH_n^{(i)}(R) \otimes t\mathbb{Q}[t] \otimes t\mathbb{Q}[t]\big) \\ &\oplus \big(HH_{n-1}^{(i-1)}(R) \otimes t\mathbb{Q}[t] \otimes \Omega^1_{\mathbb{Q}[t]}\big). \end{aligned} \tag{1.4}$$

By induction, we see that $HH_{n-j}^{(i-j)}(R)\otimes(t\mathbb{Q}[t])^{\otimes(p-j)}\otimes(\Omega^1_{\mathbb{Q}[t]})^{\otimes j}$ will occur $\binom{p-1}{j}$ times as a summand of $N^pHC_n^{(i)}(R)$ for all $j\geq 0$. We may write this as the formula:

$$N^pHC_n^{(i)}(R)\cong\bigoplus_{j=0}^{p-1}HH_{n-j}^{(i-j)}(R)\otimes_k\wedge^j k^{p-1}\otimes(t\mathbb{Q}[t])^{\otimes(p-j)}\otimes\left(\Omega^1_{\mathbb{Q}[t]}\right)^{\otimes j}. \tag{1.5}$$

Cartier operations on NHH and NHC

Let $W(R)$ denote the ring of big Witt vectors over R; it is well known that in characteristic 0 we have $W(R)\cong\prod_1^\infty R$. (See [30, p. 468] for example.) Cartier showed in [5] that the endomorphism ring Cart(R) of the additive functor underlying W consists of column-finite sums $\sum V_m[r_{mn}]F_n$, using the *homotheties* $[r]$ (for $r\in R$), and the Verschiebung and Frobenius operators V_m and F_m. Restricting the sum to $m\geq m_0$ yields a descending sequence of ideals of Cart(R), making it complete as a topological ring; $W(R)$ is the complete topological subring of all sums $\sum V_m[r_m]F_m$; see [5].

We will be interested in the intermediate (topological) subring Carf(R) of all row and column-finite sums $\sum V_m[r_{mn}]F_n$. As observed in [12, 2.14], there is an equivalence between the category of R-modules and the category of continuous Carf(R)-modules given by the constructions in the following example. (A left module M is *continuous* if the annihilator ideal of each element is an open left ideal.)

Example 1.6 If M is any R-module, $N=M\otimes t\mathbb{Q}[t]$ is a continuous Carf(R)-module (and hence a $W(R)$-module) via the formulas:

$$[r]t^i=r^it^i,\qquad V_m(t^i)=t^{mi},\qquad F_m(t^i)=\begin{cases}mt^{i/m} & \text{if } m|i,\\ 0 & \text{else.}\end{cases}$$

The ring $W(R)=\prod_1^\infty R$ acts on $M\otimes t\mathbb{Q}[t]$ by $(r_1,\dots,r_n,\dots)*\sum m_it^i=\sum(r_im_i)t^i$. Conversely, every continuous Carf(R)-module N has a "typical piece" M, defined as the simultaneous eigenspace $\{x\in N:[r]x=rx, r\in R\}$, and $N\cong M\otimes t\mathbb{Q}[t]$.

Recall that we can define operators $[r]$ on $NHH_n(R)$ and $NHC_n(R)$, associated to the endomorphisms $t\mapsto rt$ of $R[t]$. There are also operators V_m and F_m, defined via the ring inclusions $R[t^m]\subset R[t]$ and their transfers. These operations commute with the Hodge decomposition. The following result follows immediately from [12, 4.11] using the observation that everything commutes with Adams operations.

Proposition 1.7 *The operators* $[r]$, V_m *and* F_m *make each* $NHC_n^{(i)}(R)$ *into a continuous* $\mathrm{Carf}(R)$*-module, and hence a* $W(R)$*-module. The* R*-module* $HH_n^{(i)}(R)$ *is its typical piece, and the canonical isomorphism* $NHC_n^{(i)}(R) \cong HH_n^{(i)}(R) \otimes t\mathbb{Q}[t]$ *of* (1.2) *is an isomorphism of* $\mathrm{Carf}(R)$*-modules, the module structure on the right being given in Example* 1.6.

A similar structure theorem holds for $NHH_n(R)$ and its Hodge components, using (1.1). However, it uses a non-standard R-module structure on the typical piece $HH_n(R) \oplus HH_{n-1}(R)$; see [12, 3.3] for details.

Remark 1.7.1 The conclusions of Proposition 1.7 still hold for $NHC_n^{(i)}(X)$ and $HH_n^{(i)}(X)$ when X is any scheme, where $W(R)$ and $\mathrm{Carf}(R)$ refer to the ring $R = H^0(X, \mathcal{O})$. That is, $HH_n^{(i)}(X)$ is an R-module and $NHC_n^{(i)}(X)$ is a continuous $\mathrm{Carf}(R)$-module, isomorphic to $HH_n^{(i)}(X) \otimes t\mathbb{Q}[t]$.

This scheme version of Proposition 1.7 is not stated in [12], which was written before the cyclic homology of schemes was developed in [34]. However, the proof in [12] is easily adapted. Since the operators V_m, F_m and $[r]$ are defined on the underlying chain complexes in [12, 4.1], they extend to operations on the Hochschild and cyclic homology of schemes. The identities required to obtain continuous $\mathrm{Carf}(R)$-module structures all come from the Künneth formula for the shuffle product on the chain complexes (see [12, 4.3]), so they also hold for the homology of schemes.

2 *cdh*-fibrant HH and NHC

Now fix a field F containing k; all schemes will lie in the category Sch/F (essentially of finite type over F), in order to use the *cdh* topology on Sch/F of [24]. All rings will be commutative F-algebras; because they are filtered direct limits of finitely generated F-algebras, we can consider their *cdh*-cohomology.

If C is any (pre-)sheaf of cochain complexes on Sch/F, we can form the *cdh*-fibrant replacement $X \mapsto \mathbb{H}_{\mathrm{cdh}}(X, C)$ and write $\mathbb{H}^n_{\mathrm{cdh}}(X, C)$ for the nth cohomology of this complex. (The fibrant replacement is taken with respect to the local injective model structure, as in [7, 3.3].) For example, the *cdh*-fibrant replacement of a *cdh* sheaf C (concentrated in degree zero) is just an injective resolution, and $\mathbb{H}^n_{\mathrm{cdh}}(X, C)$ is the usual cohomology of the *cdh* sheaf associated to C.

Hochschild and cyclic homology, as well as differential forms, will be taken relative to k. For $C = HH^{(i)}$, it was shown in [9, Theorem 2.4] that

$$\mathbb{H}_{\mathrm{cdh}}(X, HH^{(i)}) \cong \mathbb{H}_{\mathrm{cdh}}(X, \Omega^i)[i]. \tag{2.1}$$

This has the following consequence for $C = NHH^{(i)}$ and $NHC^{(i)}$.

Lemma 2.2 *Let $H^{(i)}$ denote either $HH^{(i)}$ or $HC^{(i)}$, taken relative to a subfield k of F. Then $\mathbb{H}_{\mathrm{cdh}}(X\times\mathbb{A}^1, H^{(i)}) = \mathbb{H}_{\mathrm{cdh}}(X, H^{(i)}) \oplus \mathbb{H}_{\mathrm{cdh}}(X, NH^{(i)})$, and*:

$$\begin{aligned}\mathbb{H}_{\mathrm{cdh}}(X, NHH^{(i)}) &\cong \big(\mathbb{H}_{\mathrm{cdh}}(X,\Omega^i)[i]\otimes t\mathbb{Q}[t]\big)\\ &\qquad \oplus \big(\mathbb{H}_{\mathrm{cdh}}(X,\Omega^{i-1})[i]\otimes \Omega^1_{\mathbb{Q}[t]}\big);\\ \mathbb{H}_{\mathrm{cdh}}(X, NHC^{(i)}) &\cong \mathbb{H}_{\mathrm{cdh}}(X,\Omega^i)[i]\otimes t\mathbb{Q}[t].\end{aligned}$$

Proof The displayed formulas follow from (1.1), (1.2) and (2.1), using the fact that $-\otimes t\mathbb{Q}[t]$ commutes with $\mathbb{H}_{\mathrm{cdh}}$. Thus it suffices to verify the first assertion. By resolution of singularities, we may assume that X is smooth.

Recall from [7, 3.2.2] that the restriction of the *cdh* topology to Sm/k is called the *scdh*-topology. The product of any *scdh* cover of X with $\mathbb{A}^1$ is an *scdh* cover of $X\times\mathbb{A}^1$, and both $HH^{(i)}$ and $HC^{(i)}$ satisfy *scdh*-descent by [9, Theorem 2.4]. Now by Thomason's Cartan-Leray Theorem [25, 1.56] we have

$$\begin{aligned}\mathbb{H}_{\mathrm{cdh}}(X\times\mathbb{A}^1, H^{(i)}) &\cong \mathbb{H}_{\mathrm{cdh}}(X, H^{(i)}(-\times\mathbb{A}^1))\\ &\cong \mathbb{H}_{\mathrm{cdh}}(X, H^{(i)}) \oplus \mathbb{H}_{\mathrm{cdh}}(X, NH^{(i)}).\end{aligned}$$

This gives the first assertion. Alternatively, we may prove the first assertion by induction on $\dim(X)$, using the definition of scdh descent to see that for smooth X we have $H^{(i)}(X) = \mathbb{H}_{\mathrm{cdh}}(X, H^{(i)})$ and

$$\mathbb{H}_{\mathrm{cdh}}(X\times\mathbb{A}^1, H^{(i)}) = H^{(i)}(X\times\mathbb{A}^1) = H^{(i)}(X)\oplus NH^{(i)}(X).$$

In particular, the first assertion holds when $\dim(X)=0$. $\square$

Remark 2.2.1 If R is any commutative F-algebra, the formulas of Lemma 2.2 hold for $X=\mathrm{Spec}(R)$ by naturality. This is because we may write $R=\varinjlim R_\alpha$, where R_α ranges over subrings of finite type over F, and $\mathbb{H}_{\mathrm{cdh}}(X,-) = \varinjlim \mathbb{H}_{\mathrm{cdh}}(\mathrm{Spec}(R_\alpha),-)$.

Corollary 2.3 *If $X=\mathrm{Spec}(R)$ is in Sch/F, the modules $\mathbb{H}^n_{\mathrm{cdh}}(X, HH^{(i)})$ and $\mathbb{H}^n_{\mathrm{cdh}}(X, NHC^{(i)})$ are zero unless $0\le n+i<\dim(X)$ and $i\ge 0$.*

If $n\ge\dim(X)$ and $n>0$ then $\mathbb{H}^n_{\mathrm{cdh}}(X, HH)=0$.

Proof Because $\mathbb{H}^n_{\mathrm{cdh}}(X,\Omega^i)[i] = H^{i+n}_{\mathrm{cdh}}(X,\Omega^i)$, this follows from (2.1), Lemma 2.2 and the fact that $H^n_{\mathrm{cdh}}(X,\Omega^i)=0$ for $n\ge\dim(X)$, $n>0$. This bound is given in [7, 6.1] for $i=0$, and in [9, 2.6] for general i. $\square$

Here is a useful bound on the cohomology groups appearing in Lemma 2.2. Given X, let Q denote the total ring of fractions of X_{red}; it is a finite product of fields Q_j, and we let e denote the maximum of the transcendence degrees $\operatorname{tr.deg}(Q_j/k)$.

Lemma 2.4 *Let X be in* Sch/F. *If* $i > e$ *then* $H^n_{\mathrm{cdh}}(X, \Omega^i) = 0$ *for all* n.

Proof By [21, 12.24], we may assume X reduced. Since we may write X as an inverse limit of a sequence of affine morphisms of schemes of finite type with the same ring of total fractions Q, and cdh-cohomology sends such an inverse limit to a direct limit, we may also assume that X is of finite type over F. This implies that $e = \dim(X) + \operatorname{tr.deg}(F/k)$.

The result is clear if $\dim(X) = 0$, since $H^n_{\mathrm{cdh}}(X, -) = H^n_{\mathrm{Zar}}(X, -)$ in that case. Proceeding by induction on $\dim(X)$, choose a resolution of singularities $X' \to X$ and observe that the singular locus Y and $Y \times_X X'$ have smaller dimension. The hypothesis implies that $\Omega^i = 0$ on X'_{Zar}, so $H^n_{\mathrm{cdh}}(X', \Omega^i) = 0$ by [9, 2.5]. The result now follows by induction from the Mayer-Vietoris sequence of [24, 12.1]. □

If R is a commutative ring, we write R_{red} and R^+ for the associated reduced ring and the seminormalization of R_{red}, respectively. These constructions are natural with respect to localization, so that we may form the seminormalization X^+ of X_{red} for any scheme X. Because $X^+ \to X$ is a universal homeomorphism, we have $H^*_{\mathrm{cdh}}(X, -) \cong H^*_{\mathrm{cdh}}(X^+, -)$ for every X in Sch/k, for any field k of arbitrary characteristic. The case $n = 0$ with coefficients $\mathcal{O}_{\mathrm{cdh}}$ is of special interest; recall our convention that $H^0_{\mathrm{cdh}}(X, \mathcal{O})$ denotes $H^0_{\mathrm{cdh}}(X, \mathcal{O}_{\mathrm{cdh}})$.

Proposition 2.5 *For any algebra R, we have* $H^0_{\mathrm{cdh}}(\operatorname{Spec} R, \mathcal{O}) = R^+$. *Moreover, for every X in* Sch/F *we have* $H^0_{\mathrm{cdh}}(X, \mathcal{O}) = \mathcal{O}(X^+)$.

Proof We may assume R and X are reduced. Writing $R = \varinjlim R_\alpha$ as in Remark 2.2.1, we have $R^+ = \varinjlim R^+_\alpha$ and $H^0_{\mathrm{cdh}}(R, \mathcal{O}) = \varinjlim H^0_{\mathrm{cdh}}(R_\alpha, \mathcal{O})$, so we may assume that R is of finite type. Thus the second assertion implies the first. Since $H^0_{\mathrm{cdh}}(-, \mathcal{O})$ and $\mathcal{O}(-^+)$ are Zariski sheaves, it suffices to consider the case when X is affine.

Let $X = \operatorname{Spec} R$ be in Sch/F, with R reduced. There is an injection $R \to Q$ with Q regular (for example, Q could be the total quotient ring of R). By [7, 6.3], $H^0_{\mathrm{cdh}}(\operatorname{Spec} Q, \mathcal{O}) = Q$, so R injects into $H^0_{\mathrm{cdh}}(\operatorname{Spec} R, \mathcal{O})$. This implies that $\mathcal{O}_{\mathrm{red}}$ is a separated presheaf for the cdh topology on Sch/F. Thus, the ring $H^0_{\mathrm{cdh}}(X, \mathcal{O})$ is the direct limit over all cdh-covers $p : U \to X$ of the Čech H^0. (See [1, 3.2.3].)

Fix an element $b \in H^0_{\mathrm{cdh}}(\operatorname{Spec} R, \mathcal{O})$ and represent it by $b \in \mathcal{O}(U)$ for some cdh cover $U \to X$. Now recall from [21, 12.28] or [24, 5.9] that we may assume, by refining the cdh cover $U \to X$, that it factors as $U \to X' \to X$ where $X' \to X$ is proper birational cdh cover and $U \to X'$ is a Nisnevich cover. If the images of $b \in \mathcal{O}(U)$ agree in $U \times_X U$, i.e. b is a Čech cycle for U/X, then its images agree in $U \times_{X'} U$, i.e. it is a Čech cycle for U/X'. But by faithfully flat descent, b descends to an element of $\mathcal{O}(X')$. Thus we can assume that U is proper and birational over X.

Next, we can assume that the Nisnevich cover $p : U \to X$ is finite, surjective and birational. Indeed, since p is proper and birational we may consider the Stein factorization $U \xrightarrow{q} Y \xrightarrow{r} X$. By [2, 4.3] or [18, III.11.5 & proof], $q_*(\mathcal{O}_U) = \mathcal{O}_Y$ and r is finite surjective and birational. By [24, 5.8], r is also a cdh cover. Because $q_*(O_U) = O_Y$, the canonical map $\mathcal{O}_Y(Y) \to q_*(\mathcal{O}_U)(Y) = \mathcal{O}_U(U)$ is an isomorphism. Hence b descends to an element of $\mathcal{O}(Y)$. By Lemma 2.6, b lies in the seminormalization of R. □

Lemma 2.6 *Let A be a seminormal ring and B a ring between A and its normalization. Then the Čech complex $A \to B \to B \otimes_A B$ is exact.*

Proof We use Traverso's description of the seminormalization (see [26, p. 585]): the seminormalization of a ring A inside a ring B is

$$A^+ = \{b \in B \mid (\forall P \in \operatorname{Spec} A)\ b \in A_P + \operatorname{rad}(B_P)\}.$$

Let $b \in B$ such that $1 \otimes b = b \otimes 1$. We have to show that $b \in A_P + \operatorname{rad}(B_P)$, for all primes P of A. Let $J = \operatorname{rad}(B_P)$; since B_P/J is faithfully flat over the field A_P/P, the image of b in B_P/J lies in A_P/P by flat descent. That is, $b \in A_P + J$, as required. □

Remark 2.7 Even if X is affine seminormal, it can happen that $H^i_{\mathrm{cdh}}(X, \mathcal{O}) \neq 0$ for some $i > 0$. For example, if R denotes the subring $F[x, g, yg]$ of $F[x, y]$ for $g = x^3 - y^2$ then it is easy to show that R is seminormal and that $H^1_{\mathrm{cdh}}(\operatorname{Spec}(R), \mathcal{O}) = F$, because the normalization of R is $F[x, y]$ and the conductor ideal is $gF[x, y]$. For another example, the normal ring of Theorem 0.2 has $H^1_{\mathrm{cdh}}(X, \mathcal{O}) \neq 0$, by Theorems 0.1 and 0.7(b).

3 The fibers $\mathcal{F}_{HH}$ and $\mathcal{F}_{HC}$

If C is a presheaf of complexes on Sch/F, we write $\mathcal{F}_C$ for the shifted mapping cone of $C \to \mathbb{H}_{\mathrm{cdh}}(-, C)$, so that we have a distinguished triangle:

$$\mathbb{H}_{\mathrm{cdh}}(X, C)[-1] \to \mathcal{F}_C(X) \to C(X) \to \mathbb{H}_{\mathrm{cdh}}(X, C). \tag{3.1}$$

Example 3.1.1 When C is concentrated in degree 0 we have $H^n\mathcal{F}_C = 0$ for all $n < 0$. For $C = \mathcal{O}$ and $X = \operatorname{Spec}(R)$, we see from Proposition 2.5 that

$H^0\mathcal{F}_{\mathcal{O}}(X) = \mathrm{nil}(R)$, $H^1\mathcal{F}_{\mathcal{O}}(X) = R^+/R$, and $H^n\mathcal{F}_{\mathcal{O}}(X) = H^{n-1}_{\mathrm{cdh}}(X, \mathcal{O})$ for $n \geq 2$. Note that, if $X = \operatorname{Spec} R \in \mathrm{Sch}/F$, then $H^n\mathcal{F}_{\mathcal{O}}(X) = 0$ for $n > \dim(X)$ by [7, 6.1].

We now consider the Hochschild and cyclic homology complexes, taken relative to a subfield k of F. For legibility, we write $\mathcal{F}^{(i)}_{HH}$ for $\mathcal{F}_{HH^{(i)}}$, etc. By the usual homological yoga, $\mathcal{F}_{HH}$ is the direct sum of the $\mathcal{F}^{(i)}_{HH}$, $i \geq 0$, and similarly for $\mathcal{F}_{HC}$.

Example 3.1.2 If X is smooth over F then $\mathcal{F}_{HH}(X) \simeq 0$ by [9, 2.4].

Lemma 2.2 and Remarks 2.2.1 and 1.3 imply the following analogue for $N\mathcal{F}$.

Lemma 3.2 *If X is in Sch/F, or if $X = \mathrm{Spec}(R)$ for an F-algebra R, we have quasi-isomorphisms*:

$$\begin{aligned} N\mathcal{F}^{(i)}_{HH}(X) &\cong \left(\mathcal{F}^{(i)}_{HH}(X) \otimes t\mathbb{Q}[t]\right) \oplus \left(\mathcal{F}^{(i-1)}_{HH}(X)[1] \otimes \Omega^1_{\mathbb{Q}[t]}\right); \\ N\mathcal{F}^{(i)}_{HC}(X) &\cong \mathcal{F}^{(i)}_{HH}(X) \otimes t\mathbb{Q}[t]. \end{aligned}$$

Mimicking the argument that establishes (1.4) and (1.5) yields:

Corollary 3.3 *If X is in Sch/F, or if $X = \mathrm{Spec}(R)$ for an F-algebra R,*

$$N^2\mathcal{F}^{(i)}_{HC}(X) \cong \left(\mathcal{F}^{(i)}_{HH}(X) \otimes t\mathbb{Q}[t] \otimes t\mathbb{Q}[t]\right) \oplus \left(\mathcal{F}^{(i-1)}_{HH}(X)[1] \otimes t\mathbb{Q}[t] \otimes \Omega^1_{\mathbb{Q}[t]}\right)$$

and

$$N^p\mathcal{F}^{(i)}_{HC}(X) \cong \bigoplus_{j=0}^{p-1} \mathcal{F}^{(i-j)}_{HH}(X)[j] \otimes_k \wedge^j k^{p-1} \otimes t\mathbb{Q}[t]^{\otimes(p-j)} \otimes \left(\Omega^1_{\mathbb{Q}[t]}\right)^{\otimes j}.$$

The cohomology of the typical pieces $\mathcal{F}^{(i)}_{HH}(R)$ is given as follows.

Lemma 3.4 *If R is an F-algebra and $i \geq 0$, then there is an exact sequence*:

$$0 \to H^{-i}\mathcal{F}^{(i)}_{HH}(R) \to \Omega^i_R \to H^0_{\mathrm{cdh}}(R, \Omega^i) \to H^{1-i}\mathcal{F}^{(i)}_{HH}(R) \to 0.$$

For $n \neq i, i-1$ we have:

$$H^{-n}\mathcal{F}^{(i)}_{HH}(R) \cong \begin{cases} HH^{(i)}_n(R) & \text{if } i < n, \\ H^{i-n-1}_{\mathrm{cdh}}(R, \Omega^i) & \text{if } i \geq n+2. \end{cases}$$

Proof As in Remark 2.2.1, we may assume R is of finite type. Since $HH_i^{(i)}(R) = \Omega_R^i$ for all $i \geq 0$, and $HH_n^{(i)}(R) = 0$ when $i > n$ (see [33, 9.4.15] or [19, 4.5.10]), it suffices to use (2.1) and to observe that $\mathbb{H}_{\mathrm{cdh}}^{-n}(R, HH^{(i)}) = H_{\mathrm{cdh}}^{i-n}(R, \Omega^i)$ vanishes when $n > i$. □

Example 3.5 Let $X = \mathrm{Spec}(R)$ be in Sch/F. Since $HH^{(0)} = \mathcal{O}$, $\mathcal{F}_{HH}^{(0)}(R)$ is described in Example 3.1.1. Applying Corollary 2.3 and Lemma 3.4 for $i > 0$, and using [9, 2.6] to bound the terms, we see that if $d = \dim(R)$ then $H^n\mathcal{F}_{HH}(X) = 0$ for $n > d$. If $d = 1$, then the only nonzero positive cohomology of $\mathcal{F}_{HH}$ is $H^1\mathcal{F}_{HH}(R) = R^+/R$; if $d > 1$, we have:

$$\begin{aligned}
H^1\mathcal{F}_{HH}(R) &\cong (R^+/R) \oplus H_{\mathrm{cdh}}^1(X, \Omega^1) \oplus \cdots \oplus H_{\mathrm{cdh}}^{d-1}(X, \Omega^{d-1}),\\
H^2\mathcal{F}_{HH}(R) &\cong H_{\mathrm{cdh}}^1(X, \mathcal{O}) \oplus H_{\mathrm{cdh}}^2(X, \Omega^1) \oplus \cdots \oplus H_{\mathrm{cdh}}^{d-1}(X, \Omega^{d-2}),\\
&\ \ \vdots\\
H^d\mathcal{F}_{HH}(R) &\cong H_{\mathrm{cdh}}^{d-1}(X, \mathcal{O}).
\end{aligned}$$

Example 3.6 When R is essentially of finite type over F and $\mathrm{tr.deg}(F/k) < \infty$, $H^m\mathcal{F}_{HH}(R)$ is Hochschild homology for large negative m. To see this, observe that $e = \mathrm{tr.deg}(R/k)$, the maximum transcendence degree of the residue fields of R at its minimal primes, is finite. Using Lemmas 2.4 and 3.4, we get $H^{-n}\mathcal{F}_{HH}^{(i)}(R) = 0$ and $H^{-n}\mathcal{F}_{HH}^{(n)}(R) = \Omega_R^n$ for $i > n > e$, and hence

$$H^{-n}\mathcal{F}_{HH}(R) \cong HH_n(R) \quad \text{for all } n > e.$$

If $R = k \oplus R_1 \oplus R_2 \oplus \cdots$ is graded, and $\widetilde{HC}_*(R) = HC_*(R)/HC_*(k)$, it is well known that the map $\widetilde{HC}_*(R) \xrightarrow{S} \widetilde{HC}_{*-2}(R)$ is zero. (See [33, 9.9.1] for example.) In Lemma 3.8 below, we prove a similar property for $\mathcal{F}_{HH}$ and $\mathcal{F}_{HC}$, which we derive from Lemma 3.2 using the following trick.

Standard Trick 3.7 If R is a non-negatively graded algebra, there is an algebra map $\nu : R \to R[t]$ sending $r \in R_n$ to rt^n. The composition of ν with evaluation at $t = 0$ factors as $R \to R_0 \to R$, and so if H is a functor on algebras taking values in abelian groups, then the composition $H(R) \xrightarrow{\nu} H(R[t]) \xrightarrow{t=0} H(R)$ is zero on the kernel $\widetilde{H}(R)$ of $H(R) \to H(R_0)$. Similarly, the composition of ν with evaluation at $t = 1$ is the identity. That is, ν maps $\widetilde{H}(R)$ isomorphically onto a summand of $NH(R)$, and $\widetilde{H}(R)$ is in the image of $(t = 1) : NH(R) \to H(R)$.

Lemma 3.8 *If $R = k \oplus R_1 \oplus \cdots$ is a graded algebra, then for each m the map $\pi_m\mathcal{F}_{HC}(R) \xrightarrow{S} \pi_{m-2}\mathcal{F}_{HC}(R)$ is zero and there is a split short exact* *sequence*:

$$0 \to \pi_{m-1}\mathcal{F}_{HC}(R) \xrightarrow{B} \pi_m\mathcal{F}_{HH}(R) \xrightarrow{I} \pi_m\mathcal{F}_{HC}(R) \to 0.$$

Similarly, there are split short exact sequences:

$$0 \to \tilde{\mathbb{H}}^{m+1}_{\mathrm{cdh}}(R, HC) \xrightarrow{B} \tilde{\mathbb{H}}^{m}_{\mathrm{cdh}}(R, HH) \xrightarrow{I} \tilde{\mathbb{H}}^{m}_{\mathrm{cdh}}(R, HC) \to 0$$

and

$$0 \to \tilde{\mathbb{H}}^{m-1}_{\mathrm{cdh}}(R, \Omega^{<i}) \xrightarrow{B} \tilde{H}^{m-i}_{\mathrm{cdh}}(R, \Omega^{i}) \xrightarrow{I} \tilde{\mathbb{H}}^{m}_{\mathrm{cdh}}(R, \Omega^{\le i}) \to 0.$$

Proof It suffices to show that I is onto and split. By [9, 2.4], $\mathcal{F}_{HH}(k) = \mathcal{F}_{HC}(k) = 0$, so $\tilde{\mathcal{F}}_{HH} = \mathcal{F}_{HH}$ and $\tilde{\mathcal{F}}_{HC} = \mathcal{F}_{HC}$. By the Standard Trick 3.7, it suffices to show that the maps $N\pi_m\mathcal{F}_{HH}(R) \to N\pi_m\mathcal{F}_{HC}(R)$ and $N\mathbb{H}^m_{\mathrm{cdh}}(R, HH) \to N\mathbb{H}^m_{\mathrm{cdh}}(R, HC)$ are split surjections. But this is evident from the decompositions of $N\mathcal{F}^{(i)}_{HC}(R)$ and $\mathbb{H}_{\mathrm{cdh}}(R, NHC^{(i)})$ in Lemmas 3.2 and 2.2.

The third sequence is obtained from the second one by taking the ith component in the Hodge decomposition, described in Lemma 2.2. □

Example 3.9 Splicing the final sequences of Lemma 3.8 together, we see that the de Rham complexes are exact:

$$0 \to k \to R \xrightarrow{d} \tilde{H}^0_{\mathrm{cdh}}(R, \Omega^1) \xrightarrow{d} \tilde{H}^0_{\mathrm{cdh}}(R, \Omega^2) \to \cdots \tag{3.9a}$$

$$0 \to H^n_{\mathrm{cdh}}(R, \mathcal{O}) \xrightarrow{d} H^n_{\mathrm{cdh}}(R, \Omega^1) \xrightarrow{d} H^n_{\mathrm{cdh}}(R, \Omega^2) \to \cdots, \quad n > 0. \tag{3.9b}$$

An analogous exact sequence

$$\cdots \to \pi_{m-1}\mathcal{F}_{HH}(R) \xrightarrow{d} \pi_m\mathcal{F}_{HH}(R) \xrightarrow{d} \pi_{m+1}\mathcal{F}_{HH}(R) \to \cdots$$

is obtained by splicing the other sequences in Lemma 3.8. Using the interpretation of their Hodge components, described in Lemma 3.4, produces two more exact sequences:

$$0 \to \mathrm{nil}(R) \to \mathrm{tors}\,\Omega^1_R \to \mathrm{tors}\,\Omega^2_R \to \mathrm{tors}\,\Omega^3_R \to \cdots \tag{3.9c}$$

$$0 \to (R^+/R) \to \Omega^1_{\mathrm{cdh}}(R)/\Omega^1_R \to \Omega^2_{\mathrm{cdh}}(R)/\Omega^2_R \to \cdots. \tag{3.9d}$$

Here we have written $\Omega^i_{\mathrm{cdh}}(R)$ for $H^0_{\mathrm{cdh}}(R, \Omega^i)$, and $\mathrm{tors}\,\Omega^i_R$ is defined as the kernel of $\Omega^i_R \to \Omega^i_{\mathrm{cdh}}(R)$; the notation reflects the fact that if R is reduced then $\mathrm{tors}\,\Omega^i_R$ is the torsion submodule of Ω^i_R (see Remark 5.3.1 below).

4 Bass' groups $NK_*(X)$

In this section, we relate algebraic K-theory to our Hochschild and cyclic homology calculations relative to the ground field $k = \mathbb{Q}$. Consider the trace map

$$NK_{n+1}(X) \to NHC_n(X) = NHC_n(X/\mathbb{Q})$$

induced by the Chern character. In the affine case, it is defined in [29]; for schemes it is defined using Zariski descent. As explained in [29], it arises from the Chern character from the spectrum $NK(X)$ to the Eilenberg-Mac Lane spectrum $|NHC(X)[1]|$ associated to the cochain complex $NHC(X)[1]$. Note that our indexing conventions are such that $\pi_{n+1}|NHC(X)[1]| = H^{-n}NHC(X) = NHC_n(X)$.

Proposition 4.1 *Suppose that $R = \Gamma(X, \mathcal{O})$ for X in* Sch$/F$, *or that $X =$* Spec(R) *for an F-algebra R. Then for all n, the Chern character induces a natural isomorphism*

$$NK_{n+1}(X) \cong H^{-n}\mathcal{F}_{HH}(X) \otimes t\mathbb{Q}[t].$$

This is an isomorphism of graded R-modules, and even Carf(R)*-modules, identifying the operations $[r]$, V_m and F_m on $NK_*(X)$ with the operations on the right side described in Example 1.6.*

Proof By Remark 2.2.1, we may suppose $X \in$ Sch$/F$. By [9, 1.6], the Chern character $K \to HN$ induces weak equivalences $\mathcal{F}_K(X) \simeq |\mathcal{F}_{HC}(X)[1]|$ and $\mathcal{F}_K(X \times \mathbb{A}^1) \simeq |\mathcal{F}_{HC}(X \times \mathbb{A}^1)[1]|$. Since for any presheaf of spectra E we have a natural objectwise equivalence $E(- \times \mathbb{A}^1) \simeq E \times NE$, we obtain a natural weak equivalence from $NK(X)$ to $|N\mathcal{F}_{HC}(X)[1]|$. Now take homotopy groups and apply Lemma 3.2.

As observed in [12, 4.12], the Chern character also commutes with the ring maps used to define the operators $[r]$, V_m, and with the transfer for $R[t^n] \to R[t]$ defining F_m. That is, it is a homomorphism of Carf(R)-modules. Since the transfer is defined via the ring map $R[t] \to M_n(R[t^n])$, followed by Morita invariance, there is no trouble in passing to schemes. $\square$

We now come to one of our main results, which implies Corollary 0.1.

Theorem 4.2 *For all n, $N^2K_n(X) \cong (NK_n(X) \otimes t\mathbb{Q}[t]) \oplus (NK_{n-1}(X) \otimes \Omega^1_{\mathbb{Q}[t]})$, and*

$$N^{p+1}K_n(X) \cong \bigoplus_{j=0}^{p} NK_{n-j}(X) \otimes \wedge^j \mathbb{Q}^p \otimes (t\mathbb{Q}[t])^{\otimes(p-j)} \otimes \left(\Omega^1_{\mathbb{Q}[t]}\right)^{\otimes j}.$$

This holds for every X in Sch/F, *as well as for* $\mathrm{Spec}(R)$ *where R is an arbitrary commutative F-algebra.*

Proof As in Proposition 4.1 it follows that the Chern character induces a natural weak equivalence $N^2K(X) \simeq |N^2\mathcal{F}_{HC}(X)[1]|$. Now take homotopy groups and apply Corollary 3.3. □

Remark 4.2.1 Jim Davis has pointed out (see [11]) that a computation equivalent to 4.2 can also be derived—for arbitrary rings R—from the Farrell-Jones conjecture for the groups $\mathbb{Z}^r$. This particular case is covered by F. Quinn's proof of hyperelementary assembly for virtually abelian groups; see [22].

As an immediate consequence of 4.2 and [3, XII(7.3)], we deduce:

Corollary 4.3 *Suppose that X is in* Sch/F, *or that* $X = \mathrm{Spec}(R)$ *for an F-algebra R. Then*:

(a) *If* $NK_n(X) = NK_{n-1}(X) = 0$ *then* $N^2K_n(X) = 0$.
(b) *If* $NK_n(X) = 0$ *and* $K_{n-1}(X) = K_{n-1}(X \times \mathbb{A}^p)$ *then* $K_n(X) = K_n(X \times \mathbb{A}^{p+1})$.
(c) $K_n(X) = K_n(X \times \mathbb{A}^p)$ *if and only if* $NK_q(X) = 0$ *for all q such that* $n - p < q \leq n$.

Recall that X is called K_n-*regular* if $K_n(X) = K_n(X \times \mathbb{A}^p)$ for all p.

Corollary 4.4 *Suppose that X is in* Sch/F, *or that* $X = \mathrm{Spec}(R)$ *for an F-algebra R. Then the following conditions are equivalent*:

(a) *X is K_n-regular.*
(b) $NK_n(X) = 0$ *and X is K_{n-1}-regular.*
(c) $NK_q(X) = 0$ *for all* $q \leq n$.

Remark 4.4.1 This gives another proof of Vorst's Theorem [27, 2.1] (in characteristic 0) that K_n-regularity implies K_{n-1}-regularity, and extends it to schemes.

The assumption that the scheme be affine is essential in Bass' question—here is a non-affine example where the answer is negative.

Negative answer to Bass' question for non-affine curves

Let X be a smooth projective elliptic curve over a number field k and let L be a nontrivial degree zero line bundle with $L^{\otimes 3}$ trivial. For example, if X is the Fermat cubic $x^3 + y^3 = z^3$, we may take the line bundle associated to the divisor $P - Q$, where $P = (1:0:1)$ and $Q = (0:1:1)$.

Lemma 4.5 *Write Y for the nonreduced scheme with the same underlying space as X but with structure sheaf $\mathcal{O}_Y = \mathcal{O}_X \oplus L = \mathrm{Sym}(L)/(L^2)$, that is, L is regarded as a square-zero ideal.*

Then $NK_7(Y) = 0$ but $N^2K_7(Y) \cong NK_6(Y) \otimes \Omega^1_{\mathbb{Q}[t]}$ is nonzero.

Proof In this setting, the relative Hochschild homology presheaf $HH_n(Y, L)$ is the kernel of $HH_n(Y) \to HH_n(X)$; sheafifying, $\mathcal{HH}_n(Y, L)$ is the kernel of $\mathcal{HH}_n(Y) \to \mathcal{HH}_n(X)$. Since $\Omega^1_X \cong \mathcal{O}_X$ we see from Lemma 5.3 of [9] that $\mathcal{HH}_n(Y, L)$ is: $L^{\otimes 3} \oplus L^{\otimes 5}$ if $n = 4$; $L^{\otimes 5} \oplus L^{\otimes 5}$ if $n = 5$; and $L^{\otimes 5} \oplus L^{\otimes 7}$ if $n = 6$. By Serre duality, $H^*(X, L^{\otimes i}) = 0$ if $3 \nmid i$ (cf. [9, 5.1]). By Zariski descent, this implies that $HH_5(Y, L) \cong H^1(X, \mathcal{HH}_4) \cong H^1(X, L^{\otimes 3}) \cong k$ and $HH_6(Y, L) = 0$. Since $\mathcal{F}_{HH}(Y) \cong HH(Y, L)$, it follows from 4.1 and 4.2 that $NK_7(Y) = 0$ but $NK_6(Y) \cong t\mathbb{Q}[t]$ and $N^2K_7(Y) \cong NK_6(Y) \otimes \Omega^1_{\mathbb{Q}[t]} \cong t\mathbb{Q}[t] \otimes \Omega^1_{\mathbb{Q}[t]}$. □

We conclude this section by refining Proposition 4.1 and Corollary 4.3 to take account of the Adams/Hodge/λ-decompositions on K-theory and Hochschild homology, and by establishing the triviality of $K^{(i)}_*(X)$ for $i \leq 0$.

Recall that by definition, $K^{(i)}_n(X) = \{x \in K_n(X) \otimes \mathbb{Q} : \psi^k(x) = k^i x\}$. For $n < 0$, the Adams operations cannot be defined integrally. However, it is possible to define the operations ψ^k on $K_n(X) \otimes \mathbb{Q}$ for $n < 0$ using descending induction on n and the formula $\psi^k\{x, t\} = k\{\psi^k(x), t\}$ in $K_{n+1}(X \times (\mathbb{A}^1 - 0))$ for $x \in K_n(X)$ and $\mathcal{O}(\mathbb{A}^1 - 0) = F[t, 1/t]$. This definition was pointed out in [32, 8.4].

By [13, 2.3] or [10, 7.2], the Chern character $NK_{n+1}(X) \to NHC_n(X)$ commutes with the Adams operations ψ^k in the sense that it sends $NK^{(i+1)}_{n+1}(X)$ to $NHC^{(i)}_n(X)$ for all $i \leq n$ (and to 0 if $i > n$). Here is the λ-decomposition of the isomorphism in Proposition 4.1:

Proposition 4.6 *Suppose that $X \in \mathrm{Sch}/F$, or that $X = \mathrm{Spec}(R)$ for an F-algebra R. Then for all n and i, the Chern character induces a natural isomorphism*:

$$NK^{(i)}_{n+1}(X) \cong H^{-n}\mathcal{F}^{(i-1)}_{HH}(X) \otimes t\mathbb{Q}[t].$$

In particular, if $i \leq 0$ then $NK^{(i)}_n(X) = 0$ for all n.

Proof By [10], the Chern character $K \to HN$ sends $K^{(i)}(X)$ to $HN^{(i)}(X)$. The proof in [10] shows that the lift $\mathcal{F}_K(X) \to \mathcal{F}_{HN}(X)$, shown to be a weak equivalence in [9, 1.6], may be taken to send $\mathcal{F}^{(i)}_K(X)$ to $\mathcal{F}^{(i)}_{HN}(X)$. Since $HC \to HN$ sends $HC^{(i-1)}$ to $HN^{(i)}$, the weak equivalence $\mathcal{F}_{HC}[1] \simeq \mathcal{F}_{HN}$ identifies $\mathcal{F}^{(i-1)}_{HC}[1]$ and $\mathcal{F}^{(i)}_{HN}$. Finally $\mathcal{F}^{(i-1)}_{HH} = 0$ for $i \leq 0$. □

Corollary 4.7 *Suppose that R is essentially of finite type over F and has dimension d. If $n<0$ then $NK_n^{(i)}(R)=0$ unless $1\le i\le d+n$, in which case*

$$NK_n^{(i)}(R)=H_{\mathrm{cdh}}^{i-n-1}(R,\Omega^{i-1})\otimes t\mathbb{Q}[t].$$

In particular, $NK_n(R)=0$ for all $n\le -d$.

If $d\ge 2$ then:

$$NK_0(R)\cong\big[(R^+/R)\ \oplus H_{\mathrm{cdh}}^1(R,\Omega^1)\oplus\cdots\oplus H_{\mathrm{cdh}}^{d-1}(R,\Omega^{d-1})\big]\otimes t\mathbb{Q}[t],$$

$$NK_{-1}(R)\cong\big[H_{\mathrm{cdh}}^1(R,\mathcal{O})\oplus H_{\mathrm{cdh}}^2(R,\Omega^1)$$

$$\oplus\cdots\oplus H_{\mathrm{cdh}}^{d-1}(R,\Omega^{d-2})\big]\otimes t\mathbb{Q}[t],$$

$$\vdots$$

$$NK_{1-d}(R)\cong H_{\mathrm{cdh}}^{d-1}(R,\mathcal{O})\otimes t\mathbb{Q}[t].$$

If $d=1$ then $NK_0(R)=(R^+/R)\otimes t\mathbb{Q}[t]$ and $NK_n(R)=0$ for $n<0$.

Proof This is straightforward from Proposition 4.6 and Lemma 3.4. □

Remark 4.7.1 The $d=1$ part of Corollary 4.7 holds for any 1-dimensional noetherian ring by [28, 2.8].

Corollary 4.8 *$K_n^{(i)}(X)\cong K_n^{(i)}(X\times\mathbb{A}^p)$ if and only if $NK_{n-j}^{(i-j)}(X)=0$ for all $j=0,\dots,p-1$.*

Theorem 4.9 *For X in Sch/F or $X=\mathrm{Spec}(R)$, and all integers n, we have*:

(1) *For $i<0$, $K_n^{(i)}(X)=0$.*
(2) *For $i=0$, $K_n^{(0)}(X)\cong KH_n^{(0)}(X)\cong H_{\mathrm{cdh}}^{-n}(X,\mathbb{Q})$.*

Here KH denotes the homotopy K-theory of [31]. Theorem 4.9 answers Question 8.2 of [32].

Proof We first show that $K_n^{(i)}(X)\cong KH_n^{(i)}(X)$ when $i\le 0$. Covering X with affine opens and using the Mayer-Vietoris sequences of [31, 5.1], it suffices to consider the case $X=\mathrm{Spec}(R)$.

Since $K(R)_{\mathbb{Q}}$ is the product of the eigen-components, the descent spectral sequence $E^1_{p,q}=N^pK_q(R)_{\mathbb{Q}}\Rightarrow KH_{p+q}(R)_{\mathbb{Q}}$ (see [31, 1.3]) breaks up into one for each eigen-component. If $i\le 0$, the spectral sequence collapses by Proposition 4.6 to yield $K_n^{(i)}(R)\cong KH_n^{(i)}(R)$ for all n.

To determine the groups $KH_n^{(i)}(R)$ when $i\le 0$, we use the *cdh* descent spectral sequence of [17, 1.1]. If $i<0$, then the *cdh* sheaf $K_{\mathrm{cdh}}^{(i)}$ is trivial as

X is locally smooth, so we have $KH_n^{(i)}(R) = 0$ for all n. If $i = 0$ then the *cdh* sheaf $K_{\mathrm{cdh}}^{(0)}$ is the sheaf $\mathbb{Q}_{\mathrm{cdh}}$; see [23, 2.8]. Hence we have $K_n^{(0)}(R) = KH_n^{(0)}(R) = H_{\mathrm{cdh}}^{-n}(X, \mathbb{Q})$. □

5 The typical pieces $TK_n^{(i)}(R)$

In this section, R will be a commutative F-algebra. The default ground field k for Kähler differentials and Hochschild homology will be $\mathbb{Q}$.

As stated in (0.3), the Adams summands $NK_n^{(i)}(R)$ of $NK_n(R)$ decompose as $NK_n^{(i)}(R) = TK_n^{(i)}(R) \otimes t\mathbb{Q}[t]$ for each n and i; the decomposition is obtained from an action of finite Cartier operators precisely as the corresponding one for NHC and NHH, explained in Sect. 1. The typical pieces $TK_n^{(i)}(R)$ are described by the following formulas.

Theorem 5.1 *Let R be a commutative F-algebra. For $i \neq n, n+1$ we have*:

$$TK_n^{(i)}(R) \cong \begin{cases} HH_{n-1}^{(i-1)}(R), & \text{if } i < n, \\ H_{\mathrm{cdh}}^{i-n-1}(R, \Omega^{i-1}) & \text{if } i \geq n+2. \end{cases}$$

For $i = n, n+1$, the typical piece $TK_n^{(i)}(R)$ is given by the exact sequence:

$$0 \to TK_{n+1}^{(n+1)}(R) \to \Omega_R^n \to H_{\mathrm{cdh}}^0(R, \Omega^n) \to TK_n^{(n+1)}(R) \to 0.$$

Proof By Proposition 4.6, $TK_n^{(i)} = H^{1-n}\mathcal{F}_{HH}^{(i-1)}$. The rest is a restatement of Lemma 3.4. □

Remark 5.1.1 If R is essentially of finite type over a field F whose transcendence degree is finite over $\mathbb{Q}$, then the $TK_n^{(i)}(R)$ are finitely generated R-modules. This fails if $\mathrm{tr.deg}(F/\mathbb{Q}) = \infty$ because then $\Omega_{F/\mathbb{Q}}^i$ is infinite dimensional. For instance, Example 0.6 implies that, for $R = F[x]/(x^2)$, we have $TK_2^{(2)}(R) = HH_1(R, x) = F \oplus \Omega_{F/\mathbb{Q}}^1$.

Remark 5.1.2 Observe that Corollaries 4.7 and 4.4 imply that R is K_{-d}-regular. This recovers the affine case of one of the main results in [7].

Here is a special case of the calculations in Theorem 5.1, which proves Theorem 0.7. We will use it to construct the counterexample to Bass' question in the companion paper [8].

Theorem 5.2 *Let F be a field of characteristic* 0 *and R a normal domain of dimension* 2, *essentially of finite type over F. Then*

(a) $H^1\mathcal{F}_{HH}(R/F) \cong H^1_{\mathrm{cdh}}(R, \Omega^1_{/F})$,
(b) $H^2\mathcal{F}_{HH}(R/F) \cong H^1_{\mathrm{cdh}}(R, \mathcal{O})$,
(c) $NK_0(R) \cong H^1_{\mathrm{cdh}}(R, \Omega^1) \otimes t\mathbb{Q}[t]$, *and*
(d) $NK_{-1}(R) \cong H^1_{\mathrm{cdh}}(R, \mathcal{O}) \otimes t\mathbb{Q}[t]$.

Proof Parts (a) and (b) are immediate from Example 3.5 and the fact that R is reduced and seminormal. Parts (c) and (d) follow from (a) and (b) using Proposition 4.1; cf. Corollary 4.7. □

In order to compare the torsion submodules tors Ω^*_R with the typical pieces of $NK_*(R)$, we need the affine case of the following lemma. Following tradition, we write $F(X)$ for the total ring of fractions of X_{red}. That is, $F(X)$ is the product of the function fields of the irreducible components of X_{red}. When $X = \mathrm{Spec}(R)$ is affine, we write Q instead of $F(X)$.

Lemma 5.3 *Let $X \in \mathrm{Sch}/F$; for $F(X)$ as above, the map $\Omega^i_{\mathrm{cdh}}(X) \to \Omega^i_{F(X)}$ is an injection.*

Proof We may assume X reduced, and proceed by induction on $d = \dim(X)$, the case $d = 0$ being trivial. Choose a resolution of singularities $X' \to X$ and let Y be the singular locus of X, with $Y' = Y \times_X X'$. By [24, 12.1], there is a Mayer-Vietoris exact sequence

$$0 \to \Omega^i_{\mathrm{cdh}}(X) \to \Omega^i_{\mathrm{cdh}}(X') \oplus \Omega^i_{\mathrm{cdh}}(Y) \to \Omega^i_{\mathrm{cdh}}(Y') \xrightarrow{\partial} H^1_{\mathrm{cdh}}(X, \Omega^i) \to \cdots .$$

Since $F(Y) \subseteq F(Y')$, $\Omega^i_{F(Y)} \subseteq \Omega^i_{F(Y')}$. Because $\dim(Y') < d$, the inductive hypothesis implies that $\Omega^i_{\mathrm{cdh}}(Y) \to \Omega^i_{\mathrm{cdh}}(Y')$ is an injection. Hence $\Omega^i_{\mathrm{cdh}}(X) \to \Omega^i_{\mathrm{cdh}}(X')$ is an injection. But X' is smooth, so by *scdh* descent for Ω^i (see [9, 2.5]) we have $\Omega^i_{\mathrm{cdh}}(X') \cong \Omega^i(X') \subset \Omega^i_{F(X')} = \Omega^i_{F(X)}$. □

Remark 5.3.1 Lemma 5.3 remains true if, instead of Ω^i, we use $\Omega^i_{/k}$ for $k \subseteq F$. In particular, if $X = \mathrm{Spec}(R)$ is reduced affine, then $\Omega^i_{\mathrm{cdh}}(R/k) = H^0_{\mathrm{cdh}}(R, \Omega^i_{/k})$ injects into $\Omega^i_{Q/k}$. Thus tors$(\Omega^i_{R/k})$, defined as the kernel of $\Omega^i_{R/k} \to \Omega^i_{\mathrm{cdh}}(R/k)$ in (3.9c), is the torsion submodule of $\Omega^i_{R/k}$.

Corollary 5.4 *For all $n \geq 1$, $TK_n^{(n)}(R) \cong \ker(\Omega^{n-1}_R \to \Omega^{n-1}_Q)$. In particular if R is reduced, then $TK_n^{(n)}(R)$ is the torsion submodule of Ω^{n-1}_R.*

Proof By Theorem 5.1, $TK_n^{(n)}(R)$ is the kernel of $\Omega^{n-1}_R \to \Omega^{n-1}_{\mathrm{cdh}}(R)$, so Lemma 5.3 applies. □

We introduce some notation to make the statement of the next theorem more readable. The letter e denotes the maximum transcendence degree of the component fields in the total ring of fractions Q of R_{red}. For simplicity, we write $\Omega^i_{\mathrm{cdh}}(X)$ for $H^0_{\mathrm{cdh}}(X,\Omega^i)$, and we have written $\Omega^i_{\mathrm{cdh}}(R)/\Omega^i_R$ for the cokernel of $\Omega^i_R \to \Omega^i_{\mathrm{cdh}}(R)$.

Definition 5.5 For any commutative ring R containing $\mathbb{Q}$, we define:

$$E_n(R) = \Omega^n_{\mathrm{cdh}}(R)/\Omega^n_R \oplus \bigoplus_{p=1}^{\infty} H^p_{\mathrm{cdh}}(R,\Omega^{n+p});$$

$$\widetilde{HH}_n(R) = \ker\big(HH_n(R)\to\Omega^n_Q\big) = \ker\big(\Omega^n_R\to\Omega^n_Q\big)\oplus\bigoplus_{i=1}^{n-1} HH^{(i)}_n(R).$$

Theorem 5.6 *Let R be a commutative ring containing $\mathbb{Q}$. Then for all n*:

$$NK_n(R) \cong \big[\widetilde{HH}_{n-1}(R)\oplus E_n(R)\big]\otimes t\mathbb{Q}[t].$$

If furthermore R is essentially of finite type over a field, and $n \ge e+2$, then $NK_n(R) \cong HH_{n-1}(R)\otimes t\mathbb{Q}[t]$.

Proof Assembling the descriptions of the $TK^{(i)}_n(R)$ in Theorem 5.1 yields the first assertion. The second part is immediate from this and Example 3.6. □

Remark 5.6.1 The Chern character $NK_n(R)\to NHC_{n-1}(R)\cong HH_{n-1}(R)\otimes t\mathbb{Q}[t]$ is an isomorphism for $n\ge e+2$. If $n\le e+1$, neither it nor the map $H^{1-n}\mathcal{F}_{HH}(R)\to HH_{n-1}(R)$ of Proposition 4.1 need be a surjection.

The typical pieces of $NK^{(2)}_1(R)$ and $NK^{(2)}_2(R)$ of Theorem 5.1 and Corollary 5.4 may be described as follows.

Proposition 5.7 *For all reduced F-algebras R, the typical pieces $TK^{(2)}_1(R)=\Omega^1_{\mathrm{cdh}}(R)/\Omega^1_R$ and $TK^{(2)}_2(R)=\mathrm{tors}(\Omega^1_R)$ fit into an exact sequence*:

$$0\to\mathrm{tors}(\Omega^1_R)\to\mathrm{tors}(\Omega^1_{R/F})\to\Omega^1_F\otimes(R^+/R)\to\frac{\Omega^1_{\mathrm{cdh}}(R)}{\Omega^1_R}$$
$$\to\frac{\Omega^1_{\mathrm{cdh}}(R/F)}{\Omega^1_{R/F}}\to 0.$$

Proof We may assume $\operatorname{Spec} R\in \mathrm{Sch}/F$. Recall from [9, 4.2] that there is a bounded second quadrant homological spectral sequence for all p ($0\le i<p$,

$j \geq 0$):

$$ {}_pE^1_{-i,i+j} = \Omega^i_{F/k} \otimes_F HH^{(p-i)}_{p-i+j}(R/F) \quad \Rightarrow \quad HH^{(p)}_{p+j}(R/k). $$

When $p = 1$, this spectral sequence degenerates to yield exactness of the bottom row in the following commutative diagram; the top row is the First Fundamental Exact Sequence for Ω^1 [33, 9.2.6].

$$\begin{tikzcd} & \Omega^1_F \otimes R \arrow[r] \arrow[d] & \Omega^1_R \arrow[r] \arrow[d] & \Omega^1_{R/F} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Omega^1_F \otimes R^+ \arrow[r] & \Omega^1_{\mathrm{cdh}}(R) \arrow[r] & \Omega^1_{\mathrm{cdh}}(R/F) \arrow[r] & 0. \end{tikzcd}$$

The upper left horizontal map is an injection because the left vertical map is an injection. Now apply the snake lemma, using Remark 5.3.1. □

6 Bass' question for algebras over large fields

We will now show that the answer to Bass' question is positive for algebras R essentially of finite type over a field F of infinite transcendence degree over $\mathbb{Q}$.

Recall from Proposition 4.1 that $NK_{n+1}(R) \cong H^{-n}\mathcal{F}_{HH}(R/\mathbb{Q}) \otimes t\mathbb{Q}[t]$. In light of this identification, the version of Bass' question stated before Theorem 0.2 becomes the case $k = \mathbb{Q}$ of the following question:

$$\text{Does } H^m\mathcal{F}_{HH}(R/k) = 0 \text{ imply that } H^{m+1}\mathcal{F}_{HH}(R/k) = 0? \tag{6.1}$$

In Theorem 6.6, we show that the answer to question (6.1) is positive provided R is of finite type over a field F that has infinite transcendence degree over k. The proof is essentially a formal consequence of the Künneth formula in Lemma 6.3.

Lemma 6.2 *Let R be a commutative F-algebra, and suppose k is a subfield of F. Then $H^{-*}\mathcal{F}_{HH}(R/k)$ and $\mathbb{H}^{-*}_{\mathrm{cdh}}(R, HH(/k))$ are graded modules over the graded ring $\Omega^\bullet_{F/k}$.*

Proof As in Remark 2.2.1, we may suppose that R is of finite type over F. Consider the functor on F-algebras that associates to an F-algebra A the Hochschild complex $HH(A/k)$. The shuffle product makes this into a functor to dg-$HH(F/k)$-modules. Since the cdh-site has a set of points (corresponding to valuations by [15, 2.1]), we can use a Godement resolution to find a model for the cdh-hypercohomology $\mathbb{H}_{\mathrm{cdh}}(-, HH(/k))$ which is also a functor to dg-$HH(F/k)$-modules. It follows that there is a model for $\mathcal{F}_{HH}(R/k)$ that is a dg-$HH(F/k)$-module, functorially in R. This implies the assertion, since $\Omega^\bullet_{F/k} = H^{-\bullet}HH(F/k)$. □

Lemma 6.3 (Künneth formula) *Suppose that* $\mathbb{Q} \subseteq k \subseteq F_0 \subseteq F$ *are fields. Let* R_0 *be an* F_0*-algebra, and set* $R = F \otimes_{F_0} R_0$.

(i) *Let* $T = \{t_i\}$ *be transcendence basis of* F/F_0; *writing* $F[dT]$ *for the exterior algebra on the set* $\{dt_i\}$, *we have* $\Omega^\bullet_{F/F_0} = F[dT]$ *and*:

$$\Omega^\bullet_{F/k} \cong F[dT] \otimes_{F_0} \Omega^\bullet_{F_0/k}$$

In particular, the graded algebra homomorphism $\Omega^\bullet_{F_0/k} \to \Omega^\bullet_{F/k}$ *is flat.*

(ii) $HH_*(R/k) \cong \Omega^\bullet_{F/k} \otimes_{\Omega^\bullet_{F_0/k}} HH_*(R_0/k) \cong F[dT] \otimes_{F_0} HH_*(R_0/k)$.

Proof It is classical that $F[dT] = \Omega^\bullet_{F/F_0}$. The tensor product decomposition of part (i) follows from the fact that the fundamental sequence

$$0 \to F \otimes_{F_0} \Omega^1_{F_0/k} \to \Omega^1_F \to \Omega^1_{F/F_0} \to 0$$

is split exact. This proves (i). To prove (ii), choose a free chain dg-F_0-algebra Λ and a surjective quasi-isomorphism of dg-algebras $\Lambda \overset{\sim}{\twoheadrightarrow} R_0$. Then $\Lambda' = F \otimes_{F_0} \Lambda \to F \otimes_{F_0} R_0 = R$ is a free chain model of R as a k-algebra. Write $\Omega^\bullet_{\Lambda/k}$ for differential forms; consider $\Omega^\bullet_{\Lambda/k}$ as a chain dg-algebra with the differential δ induced by that of Λ. Note Λ and Λ' are homologically regular in the sense of [6], so that Theorem 2.6 of [6] applies. Combining this with part (i), we obtain

$$\begin{aligned} HH_*(R/k) &= HH_*(\Lambda'/k) = H_*(\Omega^\bullet_{\Lambda'/k}) \\ &= H_*(\Omega^\bullet_{F/k} \otimes_{\Omega^\bullet_{F_0/k}} \Omega^\bullet_{\Lambda/k}) = \Omega^\bullet_{F/k} \otimes_{\Omega^\bullet_{F_0/k}} H_*(\Omega^\bullet_{\Lambda/k}) \\ &= \Omega^\bullet_{F/k} \otimes_{\Omega^\bullet_{F_0/k}} HH_*(R_0/k). \end{aligned}$$

□

Here is an easy consequence of Lemmas 6.2 and 6.3.

Proposition 6.4 *Suppose* $\mathbb{Q} \subseteq k \subseteq F_0 \subseteq F$ *are field extensions, that* R_0 *is an* F_0*-algebra and* $R = F \otimes_{F_0} R_0$. *Then there is an isomorphism of graded* $\Omega^\bullet_{F/k}$*-modules*

$$F[dT] \otimes_{F_0} H^{-*}\mathcal{F}_{HH}(R_0/k) \cong H^{-*}(\mathcal{F}_{HH}(R/k)).$$

We also need the following lemma to prove the main result of this section.

Lemma 6.5 *Let R be essentially of finite type over $F \supset \mathbb{Q}$, and let $H_n(R)$ denote either $HH_n(R)$ or $H^{-n}\mathcal{F}_{HH}(R)$. Assume that $H_{n_i}(R) = 0$ for some finite set $\{n_1, \dots, n_r\}$ of positive integers. Then there exist an F-algebra of finite type R', and a multiplicatively closed set S such that $R \cong S^{-1}R'$ and $H_{n_i}(R') = 0$ for $1 \le i \le r$.*

Proof Because R is essentially of finite type, it is the localization $R = S^{-1}R''$ of some finite type F-algebra R''. It is well known that $HH_n(S^{-1}R'') \cong S^{-1}HH_n(R'')$ (see [33, 9.1.8]), and $H^{-n}\mathcal{F}_{HH}(S^{-1}R'') \cong S^{-1}H^{-n}\mathcal{F}_{HH}(R'')$ by [9, 2.8–9].

Because R'' is of finite type over F, we may write $R'' = F \otimes_{F_0} R_0$ for some finitely generated field extension F_0 of $\mathbb{Q}$ and some finite type F_0-algebra R_0. Note R_0 is essentially of finite type over $\mathbb{Q}$, whence $H_p(R_0)$ is a finitely generated R_0-module ($p \ge 0$). By Lemma 6.3 and/or Proposition 6.4, $H_p(R'')$ is isomorphic, as an R''-module, to a direct sum of copies of $R'' \otimes_{R_0} H_q(R_0)$ with $q \le p$. In particular, $M = \bigoplus_{i=1}^r H_{n_i}(R'')$ is a finite sum of R''-modules, each of which is a—possibly infinite—direct sum of copies of one finitely generated module.

Given that M has this form, the hypothesis that $S^{-1}M = 0$ implies that there exists a nonzero element $s \in Ann(M) \cap S$. Consider the finite type F-algebra $R' = R''[1/s]$. Then $R \cong S^{-1}R'$ and we have $\bigoplus_i H_{n_i}(R') = M[1/s] = 0$. □

Theorem 6.6 *Suppose $k \subset F$ is an extension with $\operatorname{tr.deg}(F/k) = \infty$, and R is essentially of finite type over F. If $H^n(\mathcal{F}_{HH}(R/k)) = 0$, then $H^m(\mathcal{F}_{HH}(R/k)) = 0$ for all $m \ge n$.*

Proof By Lemma 6.5, we may assume that R is of finite type over F. There is a finitely generated field extension $F_0 \subset F$ of k and a finite type F_0-algebra R_0 such that $R = R_0 \otimes_{F_0} F$. Note that $\operatorname{tr.deg}(F/F_0) = \infty$. By Lemma 6.3 and Proposition 6.4, $\Omega^i_{F/F_0} \otimes_{F_0} H^{n+i}(\mathcal{F}_{HH}(R_0/k))$ is a direct summand of $H^n(\mathcal{F}_{HH}(R/k))$ for each $i \ge 0$. Since $\Omega^i_{F/F_0} \ne 0$ for all i, all the $H^{n+i}(\mathcal{F}_{HH}(R_0/k))$ vanish as well. Similarly, $H^m(\mathcal{F}_{HH}(R/k))$ is a direct sum of copies of the groups $\Omega^j_{F/F_0} \otimes_{F_0} H^{m+j}(\mathcal{F}_{HH}(R_0/k))$ for $j \ge 0$, all of which vanish when $m \ge n$, as we just observed. □

Corollary 6.7 *Let $\mathbb{Q} \subset F$ be a field extension of infinite transcendence degree, and suppose R is essentially of finite type over F. Then $NK_n(R) = 0$ implies that R is K_n-regular.*

Proof Combine Theorem 6.6 with Proposition 4.1 and Corollary 4.4. □

Here is another proof of Corollary 6.7, which is essentially due to Murthy and Pedrini and given in their 1972 paper [20]; they stated the result only for $n \le 1$ because transfer maps for higher K-theory and the $W(R)$-module structure had not yet been discovered. We are grateful to Joseph Gubeladze [16] for pointing this out to the authors.

Lemma 6.8 *If R is an algebra over a field k of characteristic 0, $N^pK_n(R[t]) \to N^pK_n(R \otimes_k k(t))$ is injective.*

Proof The proof in [20, 1.3–1.6] goes through, taking into account that the norm map and localization sequences used there for K_0, K_1 are now known for all K_n. □

Lemma 6.9 *Suppose that k is an algebraically closed field of infinite transcendence degree over $\mathbb{Q}$, and that R is a finitely generated k-algebra. If $NK_n(R)$ is zero, then $K_n(R) \xrightarrow{\simeq} K_n(R[x_1, \dots, x_p])$ for all $p > 0$.*

Proof Muthy and Pedrini prove this in [20, 2.1.]; although their result is only stated for $i \le 1$, their proof works in general. Note that since $NK_n(R)$ has the form $TK_n(R) \otimes t\mathbb{Q}[t]$ by (0.3) (a result which was not known in 1972), $NK_n(R)$ is torsionfree, and has finite rank if and only if it is zero. □

Proof of Corollary 6.7 Let Φ denote the functor N^pK_n. If $k \subset k_1$ is a finite algebraic field extension and R is a k-algebra, then $\Phi(R) \to \Phi(R \otimes_k k_1)$ is an injection because its composition with the transfer $\Phi(R \otimes_k k_1) \to \Phi(R)$ is multiplication by $[k_1 : k]$, and $\Phi(R)$ is a torsionfree group. Since Φ commutes with filtered colimits of rings, $\Phi(R) \to \Phi(R \otimes_k \bar{k})$ is an injection. Thus Lemma 6.9 suffices to prove Corollary 6.7 when R is of finite type. □

7 NK_0 of surfaces

We conclude with a general description for affine surfaces of the canonical map $\Omega^1_F \otimes_F NK_{-1} \to NK_0$. This sheds light on the difference between the cases of small and large base fields, and also explains some results of [35].

If R is a 2-dimensional noetherian ring then $NK_0(R)$ is the direct sum of $NK_0^{(1)}(R) = N\operatorname{Pic}(R)$ and $NK_0^{(2)}(R)$.

Theorem 7.1 *Let R be a 2-dimensional normal domain of finite type over a field F of characteristic 0. There is an exact sequence*:

$$0 \to NK_1^{(2)}(R) \to \left(H^0(R, \Omega^1_{/F})/\Omega^1_{R/F}\right) \otimes t\mathbb{Q}[t]$$
$$\to \Omega^1_F \otimes_F NK_{-1}(R) \to NK_0(R) \to H^1_{\mathrm{cdh}}\left(R, \Omega^1_{/F}\right) \otimes t\mathbb{Q}[t] \to 0.$$

Proof Consider the following short exact sequence of sheaves in $(\mathrm{Sch}/F)_{\mathrm{cdh}}$:

$$0 \to \Omega^1_F \otimes_F \mathcal{O} \to \Omega^1 \to \Omega^1_{/F} \to 0.$$

Applying H_{cdh} yields

$$\begin{aligned} 0 &\to \Omega^1_F \otimes_F R \xrightarrow{\iota} H^0(R, \Omega^1) \to H^0\big(R, \Omega^1_{/F}\big) \\ &\xrightarrow{\partial} \Omega^1_F \otimes_F H^1_{\mathrm{cdh}}(R, \mathcal{O}) \to H^1_{\mathrm{cdh}}(R, \Omega^1) \to H^1_{\mathrm{cdh}}\big(R, \Omega^1_{/F}\big) \to 0. \end{aligned}$$

Note that, because $\Omega^1_R \to \Omega^1_{R/F}$ is onto, the map ∂ kills the image of $\Omega^1_{R/F}$. Similarly, the image of ι is contained in that of Ω^1_R. Thus we obtain

$$\begin{aligned} 0 &\to H^0(R, \Omega^1)/\Omega^1_R \to H^0\big(R, \Omega^1_{/F}\big)/\Omega^1_{R/F} \\ &\to \Omega^1_F \otimes_F H^1_{\mathrm{cdh}}(R, \mathcal{O}) \to H^1_{\mathrm{cdh}}(R, \Omega^1) \to H^1_{\mathrm{cdh}}\big(R, \Omega^1_{/F}\big) \to 0. \end{aligned}$$

Now apply $\otimes t\mathbb{Q}[t]$ and use Theorem 5.1 and parts (c) and (d) of Theorem 5.2. □

Corollary 7.2 *Let R be a 2-dimensional normal domain of finite type over a field F of characteristic 0. If $NK_{-1}(R) = 0$ then $NK_0(R) \cong H^1_{\mathrm{cdh}}(R, \Omega^1_{/F}) \otimes t\mathbb{Q}[t]$.*

Example 7.3 Let R be a 2-dimensional normal domain of finite type over $\mathbb{Q}$, and put $R_F = R \otimes F$. By Propositions 4.1 and 6.4,

$$NK_*(R_F) \cong NK_*(R) \otimes \Omega^*_{F/\mathbb{Q}}. \tag{7.4}$$

Keeping track of the λ-decomposition, as in Theorem 5.1, we see from Theorem 0.7 that

$$\begin{aligned} TK_1^{(2)}(R_F) &\cong TK_1^{(2)}(R) \otimes F \cong H^0(R, \Omega^1) \otimes F / \Omega^1_R \otimes F \\ &\cong H^0\big(R_F, \Omega^1_{/F}\big)/\Omega^1_{R_F/F}. \end{aligned}$$

From Theorem 7.1 we get an exact sequence

$$0 \to \Omega^1_{F/\mathbb{Q}} \otimes_F NK_{-1}(R_F) \to NK_0(R_F) \to H^1_{\mathrm{cdh}}(R_F, \Omega^1_{/F}) \otimes t\mathbb{Q}[t] \to 0. \tag{7.5}$$

Using (7.4) and Theorem 0.7 again, we see that the sequence (7.5) is isomorphic to the sum

$$
\begin{aligned}
&\big(0 \to \Omega^1_{F/\mathbb{Q}} \otimes H^1_{\mathrm{cdh}}(R, \mathcal{O}) \otimes t\mathbb{Q}[t] \\
&\qquad \xrightarrow{\simeq} \Omega^1_{F/\mathbb{Q}} \otimes H^1_{\mathrm{cdh}}(R, \mathcal{O}) \otimes t\mathbb{Q}[t] \to 0 \to 0\big) \\
&\qquad\qquad \oplus \\
&\big(0 \to 0 \to F \otimes H^1_{\mathrm{cdh}}(R, \Omega^1) \otimes t\mathbb{Q}[t] \xrightarrow{\simeq} F \otimes H^1_{\mathrm{cdh}}(R, \Omega^1) \otimes t\mathbb{Q}[t] \to 0\big).
\end{aligned}
$$

For example, for $R_F := F[x, y, z]/(z^2 + y^3 + x^{10} + x^7 y)$ the results of [8] show that:

$$
\begin{aligned}
NK_{-1}(R_F) &= F \otimes t\mathbb{Q}[t], \\
NK_0(R_F) &= \Omega^1_{F/\mathbb{Q}} \otimes t\mathbb{Q}[t] \cong \bigoplus_{p=1}^{\mathrm{tr.\,deg}(F)} F \otimes t\mathbb{Q}[t].
\end{aligned}
$$

In other words, both typical pieces $TK_{-1}(R_F)$ and $TK_0(R_F)$ are F-vectorspaces, but while $\dim_F TK_{-1}(R_F) = 1$ for all F, any cardinal number κ can be realized as $\dim_F TK_0(R_F)$ for an appropriate F.

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