A Case Study of the Mathematical Learning of Two Teachers Acquiring Mathematical Knowledge for Teaching

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A CASE STUDY OF THE MATHEMATICAL LEARNING OF TWO TEACHERS
ACQUIRING MATHEMATICAL KNOWLEDGE FOR TEACHING

David R. Hartman, Ph.D.
University of Nebraska, 2010

Adviser: Ruth M. Heaton

This study offers an analysis of the learning of practicing teachers as they acquire a deeper knowledge of mathematics. While some professional developers have shifted part of their focus to helping practicing teachers acquire a deeper knowledge of mathematics (e.g., Stein & Silver, 1996), the results from studies often describe what translates from the professional development experience into classroom practice and measureable gains in student achievement (e.g., Desimone et al., 2002). Studies showing improvements in pedagogy and student learning are important. However, studying what teachers are learning and how they learn is important in developing understanding of the content and process of teachers’ learning.

This case study describes the mathematical learning of two middle level mathematics teachers while participating in a National Science Foundation-funded math professional development institute based on recommendations of the Conference Board of the Mathematical Sciences (CBMS) (2001). The overarching question guiding this research is: How do teachers deepen their understanding of the mathematics content they teach? The analysis is guided, in part, by mathematical habits of mind (e.g., Cuoco et al.,
or ways teachers engage in mathematical practices to strengthen their understanding of mathematical content and communication to others.

Data collected from two teachers to study their learning include: teachers’ written coursework and reflections, observations of teachers’ work and interactions solving problems, and interviews and classroom observations. Qualitative data analysis (Stake, 1995) indicates three findings. First, both teachers embrace collaboration as a tool to learn mathematics. Second, the teachers’ habits of mathematical learning become evident in practices they deploy while learning mathematics. Both teachers utilize making connections, using representations, and testing cases to learn mathematics. Simultaneously, the teachers’ learning looks different from each other: one displays a persistent nature in solving problems while the other consistently looks for ways to link mathematical learning to teaching. Furthermore, Both teachers’ written work indicates a deepening understanding of mathematical ideas (CBMS, 2001) and a growing ability to communicate mathematics to others.
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This dissertation is dedicated to:
  My wife, Katie,
  and my daughters,
  Lauren and Brianna
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Chapter 1: How Do Practicing Teachers Learn Mathematical Knowledge for Teaching?

This study investigates how teachers learn mathematics while part of a professional development program that places priority on the development of mathematical knowledge for teaching middle level mathematics. This opening chapter first outlines the weakened state of teaching and learning of mathematics in K-12 schools and recent calls for changes in professional development. Second, it defines the nature of the mathematical knowledge scholars suggest teachers should possess. Next, it reviews research studies of teachers who have participated in professional development programs aimed to deepen mathematical knowledge. Finally, this chapter identifies the problem statement and research questions of this study.

Teachers’ Mathematical Knowledge: A Call for Change

An examination of literature reveals two important themes about the current state of mathematics instruction in K-12 schools. One is that teachers need to know more mathematics (e.g., Conference Board of the Mathematical Sciences (CBMS), 2001). The other is that teachers need to enhance their mathematics instruction (e.g., Ball, Lubienski, & Mewborn, 2001). Many mathematics teachers, especially at the elementary and middle level, are ill-prepared for the work of teaching mathematics because of their own weak mathematical backgrounds. While some do have undergraduate degrees in mathematics, most elementary and middle level teachers enter the profession without strong mathematical preparation as part of their teacher education (e.g., Ball, 1993; Cuoco, 2003; Ma, 1999). In many respects, teachers with limited mathematical backgrounds tend to have limited flexibility in what they can do with mathematics in classroom practice. In
turn, students are limited in how much and what they can learn (Hill, Rowan, & Ball, 2005). Ball, Lubienski, and Mewborn (2001) have classified students’ mathematical experiences in these types of classrooms as “ordinary” (p. 435).

Research in mathematics education poses questions about teachers’ roles in these ordinary classrooms. The findings indicate an emphasis on procedure-based instruction. For example, Webb, Nemer, Kerting, Ing, and Forrest (2004) found that “Teachers rarely encourage students to verbalize their thinking or problem-solving strategies, or to ask questions…[Teachers] use procedure-bounded discourse” (p. 69). These teachers do not encourage students to communicate mathematically; they deter learners from taking active roles in the learning process.

Other research studies students’ experiences while learning mathematics in these ordinary classrooms. The findings reveal passive learning.

Students come to expect that there is one right method for solving problems, that the method should be supplied by the teacher, and that, as students, they should not be expected to spend their time figuring out the method or taking the responsibility for determining the accuracy or reasonableness of their work. (Stein, Grover, & Henningsen, 1996, p. 457) Students tend to view mathematics as a recipe to follow rather than a discipline to explore. Stein, Baxter, and Leinhardt (1990) suggest student learning is limited in these ordinary settings. Stein et al. also connect instructional decisions with teachers’ mathematical knowledge. “Our findings corroborate the conclusions of other studies that have suggested that limited, poorly organized teacher knowledge often leads to
instruction characterized by few, if any, conceptual connections, less powerful representations, and over routinized student responses” (p. 659). Teachers with limited mathematical knowledge often provide limited mathematical instruction.

Hill, Rowan, and Ball (2005) studied students who learn from non-“ordinary” teachers. They found that students demonstrate gains in mathematical achievement when they learn from teachers who possess a strong mathematics background. Teachers can be more effective in the classroom if they have a solid understanding of mathematics (Ball, Thames, & Phelps, 2008). A common characteristic of these effective mathematics teachers is their flexibility in everyday teaching tasks, which range from error analysis to deciding how to present mathematical ideas to students (Ball, Thames, & Phelps).

Research in mathematics education does offer images of teachers who organize and manage mathematics classrooms in ways that promote student understanding. These images reveal some teachers who are “willing to accept students’ methods and try to understand how and why they work and how they are related to the other methods used in class” (Perkins & Flores, 2002, p. 263). These particular teachers focus classroom experiences on inquiry, problem solving, conceptual understanding, and simply making sense of mathematics in the first place (National Council of Teachers of Mathematics (NCTM), 2000). “In such classrooms, teachers provide students with numerous opportunities to solve complex and interesting problems; to read, represent, discuss, and communicate mathematics; and to formulate and test the validity of personally constructed mathematical ideas” (p. 43). The goal of these teachers is to help students develop a conceptual understanding, or a “complete understanding” (p. 456), as described
by Stein, Grover, and Henningsen (1996). Teachers in these classrooms typically have a
deepened knowledge of mathematics as well as an increased capacity to communicate
mathematics to students.

There have been numerous calls to improve mathematics instruction in K-12
settings from mathematical organizations, educational researchers, and governing bodies.
For example, the National Council of Teachers of Mathematics (NCTM) has long called
for improvements in classroom instruction and teachers’ mathematical knowledge
(NCTM, 2000). The NCTM reports that students have been learning mathematics without
understanding for decades. “In fact, learning without understanding has been a persistent
problem since at least the 1930s” (p. 20). NCTM suggests deepening teachers’ content
knowledge will help improve mathematics instruction. “To be effective, teachers must
know and understand deeply the mathematics they are teaching and be able to draw on
that knowledge with flexibility in their teaching tasks” (NCTM, 2000, p. 16).

The 1995 Third International Math and Science Study (TIMSS) and the 1999
TIMSS video study provide evidence that U.S. teachers are poorly prepared for the work
of teaching mathematics to students (Stigler & Hiebert, 1999). Researchers who studied
TIMSS data found U.S. students lacked opportunities to discuss connections among
mathematical ideas and reason about mathematical concepts (Hiebert et al., 2003). As a
result, researchers have called for professional developers to help teachers develop
deeper, more flexible mathematical knowledge as well as support teachers’ efforts to
improve their mathematics teaching (Silver, 1998).
Individual researchers, including Ma (1999), have added their voices to the call for improvement. Based on a comparative study of mathematical knowledge for teaching focused on Chinese and American elementary teachers, Ma found that American elementary teachers lack a deep understanding of mathematics. “The change [in classroom teaching] that we are expecting can only occur if we work on changing teachers’ knowledge of mathematics” (p. 153). Ma’s research highlights the need for individual teachers, school leaders, and professional developers to attend to teachers’ lack of deep mathematical understanding in order to improve instruction.

Mathematical organizations and educational researchers have not been the only ones concerned with the current state of mathematics instruction and student achievement. Congress and various national advisory groups on the current state of mathematics education in the U.S. have taken positions on the topic as well. For example, the No Child Left Behind Act (NCLB) (2001) has placed pressure on K-12 schools to raise student achievement. One piece of this act calls for schools to have highly qualified teachers in every mathematics classroom. In addition, the RAND Mathematics Study Panel (2003) echoed concerns about weaknesses of teachers’ mathematical knowledge. “Numerous studies show that many teachers in the United States lack adequate knowledge of mathematics for teaching mathematics” (p. xvi). This study panel called for researchers to address questions that focused on “the role that teachers’ knowledge of mathematics, their knowledge of students’ mathematics, and their knowledge of students’ out-of-school practices play in their instructional capabilities” (p. xvi). The panel offered
an invitation for researchers to explore in more depth the nature of practicing teachers’
knowledge of mathematics.

More recently, the first recommendation in Rising Above the Gathering Storm:
Energizing and Employing America for a Brighter Future (National Research Council,
2007), a document intending to address how to improve the country’s global economic
situation, stated that increasing America’s talent pool in math and science depends on
vastly improving K-12 mathematics and science education.

We need to recruit, educate, and retain excellent K–12 teachers who
fundamentally understand biology, chemistry, physics, engineering, and
mathematics. The critical lack of technically trained people in the United
States can be traced directly to poor K–12 mathematics and science
instruction. (p. 114)

Poor instruction frequently stems from teachers who have weak disciplinary knowledge.
This committee’s recommendation suggested K-12 instruction would improve as a result
of helping teachers develop a richer understanding of the subject matter they teach.

**Territory for this Study**

How to deepen teachers’ mathematical knowledge and improve teachers’
pedagogical practices are both topics worthy of inquiry, and both could be studied
concurrently. However, focusing on a single dimension of the complex knowledge
needed for teaching affords the opportunity to unpack complexities and nuances. The
territory for this study is focused on understanding what it means to deepen teachers’
mathematical knowledge for teaching middle level mathematics.
Teachers enter the profession having learned mathematics as part of their own elementary and secondary school experiences as well as in their teacher preparation programs. Yet what teachers have been bringing from their past mathematical learning experiences (e.g., Stein, Grover, & Henningsen, 1996) and their teacher preparation experiences (e.g., Cuoco, 2001) is not sufficient; thus more and more researchers (e.g., Ball & Bass, 2000) are asking the question, what mathematics do teachers need to know? There are a variety of responses to this question. Some responses are vague whereas others are more specific. Researchers who study TIMSS data suggest professional developers should enhance teachers’ knowledge of mathematics content (e.g., Silver, 1998). However, they offer more specific recommendations as to how to improve teachers’ mathematics instruction (e.g., collaborating with peers and attending summer institutes) than on what mathematics content teachers should know.

Ma (1999) responds to the question of what mathematics teachers should know by describing a deeper knowledge and understanding of mathematics. The type of knowledge she suggests represents a much richer understanding of the subject than what can be gained from just completing a series of mathematics courses. Ma describes what she calls a “Profound Understanding of Fundamental Mathematics” (p. 120) considering three dimensions: depth, breadth, and thoroughness. While Ma’s work is exclusively focused on elementary teachers, middle school and high school teachers encounter mathematics that is even more complex than that found in elementary schools. All teachers need deep mathematical understanding, encompassing all three of Ma’s
dimensions, to have knowledge flexible enough to effectively help students learn meaningful mathematics.

CBMS and the NCTM are two groups that do offer specific recommendations on what K-12 teachers should know. The CBMS steering committee outlines specific recommendations as to the mathematics teachers need to know (2001). While focused on pre-service education, the committee acknowledges time and resources prevented them from translating the recommendations to the professional development of practicing mathematics teachers (2001). CBMS recommendations include:

(i) Prospective elementary grade teachers should be required to take at least 9 semester-hours on fundamental ideas of elementary school mathematics.

(ii) Prospective middle grades teachers of mathematics should be required to take at least 21 semester-hours of mathematics, that includes at least 12 semester-hours of fundamental ideas of school mathematics appropriate for middle grades teachers.

(iii) Prospective high school teachers of mathematics should be required to complete the equivalent of an undergraduate major in mathematics that includes a 6-hour capstone course connecting their college mathematics courses with high school mathematics. (p. 8)

Likewise the NCTM (2005) offers similar recommendations for K-12 teachers. They expect high school teachers to complete college coursework equivalent to one who majors in mathematics, middle school teachers to complete college coursework
equivalent to one who minors in mathematics, and elementary teachers to complete at least three college-level mathematics courses (NCTM).

There is existing research connecting teachers’ knowledge of mathematics and students’ achievement in mathematics; yet much of this research does not offer specifics as to how teachers should learn mathematics or what it looks like to learn mathematics.

Although many studies demonstrate that teachers’ mathematical knowledge helps support increased student achievement, the actual nature and extent of that knowledge—whether it is simply basic skills at the grades they teach, or complex and professionally specific mathematical knowledge—is largely unknown. (Ball, Hill, & Bass, 2005, p. 16)

Thus, there is a need for research to explore in greater depth the nature of mathematical knowledge teachers need to learn and how to go about learning it. This study is concerned with practicing teachers’ attempts to acquire a deeper knowledge of mathematics.

**Deepening Mathematical Knowledge for Teaching Beyond Pre-service Experiences**

Professional developers¹ have led efforts to build teacher capacity to improve mathematics education in K-12 schools for many years (Cuban, 1993). Many recent efforts have focused either on pre-service teachers (e.g., Charalambous, Panaoura, & Philippou, 2009; Graeber, 1999; Nicol & Crespo, 2006) or on in-service teachers at the

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¹ This study defines professional developers as collegiate and university faculty, K-12 school personnel, and private companies who engage in the business of providing continuing education to practicing teachers.
elementary level (e.g., Hill, Rowan, & Ball, 2005; Mosenthal, 1995; Prawat & Jennings, 1997).

At the middle level, much emphasis to improve teaching and learning has been through National Science Foundation (NSF) funded curricula projects (i.e., Pathways to Algebra and Geometry; Mathematics in Context; MathScape; Connected Math; Math Thematics). There are reports that proclaim these products are good; yet these are generally reports from the product’s own authors. One example is the Connected Math Project (CMP). While CMP author’s report students who learn using CMP materials outperform those using non-CMP curricula (Connected Math, 2006, paragraph 1), the U.S. Department of Education Institute of Educational Sciences’ What Works Clearinghouse (2010) attributes minimal influence of CMP on students’ mathematics achievement.

Outside researchers have posed questions as to the teachers’ role in implementing these standards-based curricula. Their findings show mathematical knowledge is a factor in how teachers use the curriculum (Remillard, 1999). Romberg (1997) suggests, “If a reform curriculum is to be successfully implemented in middle schools, the weak mathematical background and out-of-date beliefs about mathematics that many of the current teachers have must be challenged and systematically addressed” (p. 371). These findings further highlight the need for professional developers to help practicing teachers deepen mathematical knowledge as they implement curriculum, whether reform or not.

Moving beyond curricula, examining results from studies of teachers participating in professional development reveals two themes. Some research describes what translates
from the professional development experience into classroom instruction (e.g., Carpenter, Fennema, & Franke, 1996; Desimone et al., 2002; Guskey, 2002; Paul & Volk, 2002; Silver & Stein, 1996; Smith, 2008). Other research documents gains in student achievement (e.g., Campbell, 1996; Cohen & Hill, 2000; Saxe, Gearhart, & Nasir, 2001; Wayne et al., 2008). While research revealing improvements in pedagogy as well as improvements in student learning as a result of professional development are important, what mathematics teachers are learning and images of how they are learning the mathematics is lacking.

There are some examples of professional development research that offer images of teachers translating professional development experiences into classroom practice. For example, Silver and Stein (1996) used QUASAR’s (Quantitative Understanding: Amplifying Student Achievement and Reasoning) middle level setting to promote and study the nature of cognitively rich mathematical tasks and their use in middle level math classrooms. “In addition to attending workshops and learning through actual classroom teaching, QUASAR teachers participate[d] in a variety of project-level ‘work’ activities [such as developing curriculum materials and designing assessments]” (Stein & Brown, 1997, p. 157). QUASAR researchers wanted to provide students with an alternative approach to mathematics instruction, much different than the conventional mathematics instruction used by the vast majority of middle grades mathematics teachers (Silver & Stein, 1996). The researchers found QUASAR teachers often reduced the difficulty level of challenging mathematical tasks when they presented those tasks to their students (Stein, Grover, & Henningsen, 1996). Part of that reduction was related to the teachers’
own limited knowledge of mathematics. As teachers learned mathematics while participating in QUASAR workshops, researchers chose to focus on the translation of the professional development experience to classroom practice rather than the teachers’ learning of mathematics.

A second example, Cognitively Guided Instruction (CGI), investigated practicing teachers as they interacted with knowledge about how children think (Fennema, Franke, Carpenter, & Carey, 1993). Carpenter and Fennema helped elementary teachers become more familiar with how children develop understanding of addition and subtraction concepts (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). CGI researchers showed how teacher knowledge of student thinking in mathematics informed instructional decisions. Again, researchers offered images of the transfer to classroom practice, not on the ways participants deepened their knowledge of mathematics.

In addition to images of teachers translating professional development to classroom practice, research also offers images of what students are learning from teachers who learn more mathematics as part of these professional development programs. Using the QUASAR Cognitive Assessment Instrument (QCAI), Silver and Stein (1996) found “clear evidence that students developed an increased capacity for mathematical reasoning, problem solving, and communication” (p. 505). Carpenter et al. (1989) found students who learned from CGI teachers outperformed students who learned from control teachers. In both of these examples, researchers provided images of what the student learning looked like.

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2 This is not intended to be a criticism of the QUASAR researchers.
Research focused on teachers involved in professional development programs aimed to improve mathematical knowledge primarily focuses on the translation of knowledge to classroom practices and student learning and what that translation looks like. The research places little, if any, focus on what it looks like to acquire that deeper knowledge. With the resounding call for teachers to have a deeper knowledge of mathematics for teaching, it is timely for a study to examine closely the learning experiences of teachers in a professional development program.

**Investigating Teachers’ Learning of Mathematical Knowledge for Teaching**

Ma’s (1999) theory that teachers must possess a profound understanding of fundamental mathematics has captured the attention of professional developers for much of the past decade; yet the field of mathematics education lacks information about what the learning of this type of mathematics looks like. In short, there are few, if any, images of how teachers learn the mathematics they need to know. What does the learning look like? It is important for future professional development efforts to have images of what kinds of learning experiences promote the deepening of mathematical knowledge as well as what may hold back mathematical growth and how this learning varies among individual teachers. Understanding how teachers learn mathematics for teaching contributes to the field’s understanding of what it takes to build the capacities of teachers.

Offering images of two in-service middle level teachers learning mathematics is the focus of this study. A need exists to examine the learning of mathematics by practicing teachers who are part of a professional development project that takes seriously the task of strengthening teachers’ mathematical and pedagogical knowledge. It
is important to understand what it looks like for practicing teachers to learn or relearn mathematics themselves. The purpose of this case study is to describe the mathematical learning of two middle level mathematics teachers while they were participating in a professional development project funded by the National Science Foundation that takes CBMS (2001) recommendations seriously.

The grand tour question for this study is: How do teachers deepen their current understanding of the mathematics content they have to teach? Three sub-questions relate to middle level mathematics teachers’ learning of mathematics content for teaching through a professional development project:

- How do teachers extend their mathematics knowledge to the richer, profound understanding needed for teaching?
- What is the relationship between these teachers’ learning of mathematics and their attitudes, beliefs, and dispositions towards mathematics?³
- What factors hinder or enable teachers’ mathematical learning?

By investigating these questions through building a case of two teachers’ experiences of learning mathematics, this study contributes to existing research by offering images of teachers learning the mathematics they need to know, images underrepresented in current literature. The next chapter takes a closer look at the literature surrounding the territory of this inquiry.

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³ Refer to Appendix A for working definitions of attitudes, beliefs, dispositions, and habits.
Chapter 2: A Review of Literature Related to Mathematical Knowledge for Teaching and Professional Development

The goal of this second chapter is to suggest ways this study complements and enriches research in mathematics education. The chapter begins by looking at research on the influence a teacher’s mathematical knowledge has on instruction. It then focuses on different types of mathematical knowledge scholars suggest practicing teachers should possess. Third, it outlines research that highlights shortcomings in professional development experiences related to deepening mathematical knowledge of practicing teachers. Fourth, it examines existing research studies on the development of practicing teachers’ mathematical knowledge. Finally, this chapter identifies gaps in existing research and specifies ways this study helps to fill those gaps in knowledge about how practicing teachers’ learn mathematics.

The Influence of Teachers’ Mathematical Knowledge on Instruction

The relationship between teachers’ mathematical knowledge and their ways of instructing students has been a topic of interest for many researchers (e.g., Ball & Bass, 2000; Ball, Thames, & Phelps, 2008; Cobb, Wood, Yackel, & McNeal, 1992; Herbst, 2004; Stein, Baxter, & Leinhardt, 1990). “That teachers’ own knowledge of the subject affects what they teach and how they teach seems so obvious as to be trivial. However, the empirical support for this obvious fact has been surprisingly elusive” (Ball & Bass, 2000, p. 86). The National Mathematics Advisory Panel’s (2008) final report offers a similar message, “It is clear that teachers’ knowledge of mathematics is positively related to student achievement. However, evidence…remains uneven and has been surprisingly difficult to produce” (p. 37). There are current calls for researchers to more clearly define
the relationship between teachers’ mathematical knowledge, instructional practice, and
student achievement (National Mathematics Advisory Panel).

**Weaknesses in Practicing Teachers’ Mathematical Knowledge**

Educational researchers report instruction delivered by U.S. teachers is limited
(e.g., Ball, Hill, & Bass, 2005; Ma, 1999; Stein, Baxter, & Leinhardt, 1990). Alexander
and Fuller (2004) suggest many teachers offer limited instruction due to a limited
knowledge of mathematics. Nationwide, nearly 70% of middle school math classes,
defined as grades 5-8, are taught by teachers who have no major or even certification in
mathematics (U.S. Department of Education Institute of Educational Sciences’ National
Center for Education Statistics (NCES), 2010).

While one may expect elementary teachers, and even some middle level teachers,
to have limited mathematics knowledge due to the limited number of mathematics
courses completed at the college level, researchers also found weak mathematical
knowledge held by secondary teachers (Ball, Lubienski, & Mewborn, 2001; McCrory,
Zhang, Francis, & Young, 2009). The NCES (2010) reports over 30% of secondary
teachers do not have a major or certification in mathematics. These unfortunate findings
lead to an alarming hypothesis: too many U.S. mathematics teachers do not have the
necessary knowledge to teach in ways that promote students’ deep understanding of
mathematics. In many instances, under-qualified teachers are teaching mathematics to
their students.⁴

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⁴ A second hypothesis could be that U.S. mathematics teachers do possess a strong
understanding of mathematics but just do not know how to teach.
A lack of mathematical knowledge impacts classroom practice. Teachers’ lack of knowledge leads to inadequate responses to student questions (Ball & Bass, 2000). Further, some teachers are unable to respond flexibly to students’ wrong answers (Ball, Hill, & Bass, 2005). Teachers who have weak mathematical backgrounds may incorrectly introduce concepts, or insufficiently teach procedures. “They may also insist on using the prescribed method and not deviate from established practice in a school” (Perkins & Flores, 2002, p. 263). Teachers cannot teach what they do not know. Consequently, they have limited capacity to carry out in meaningful ways the complex tasks of teaching mathematics (Ball, Hill, & Bass, 2005).

**Teachers’ Experiences Learning Mathematics**

Teachers first learn mathematics as part of their own K-12 experiences. Prospective teachers enter college teacher education programs with ideas about what teachers and learners of mathematics do. “Teachers’ thirteen years as learners of K-12 mathematics provided them with images and models – conscious or unconscious—of what it means to teach and learn mathematics” (NCTM, 1991, p. 124). These images and models ultimately influence decisions prospective teachers make as they start their career teaching mathematics in their own classrooms (NCTM, 1991; 2000).

Teachers’ knowledge and beliefs about mathematics are indeed influenced by their past learning experiences. However, pre-service, college experiences do not produce the same influence on teacher knowledge of mathematics as K-12 experiences. Ball, Lubienski, and Mewborn (2001) suggest there is actually little impact from pre-service, college experiences:
Pre-service teacher education typically has a weak effect on teachers’
knowledge and beliefs...by the time they begin professional education,
teachers have already clocked more than 2000 hours in a specialized
“apprenticeship of observation,” which not only has instilled traditional
images of teaching and learning but also has shaped their understanding of
mathematics. Because this understanding of mathematics is the
mathematics they will teach, what they have learned about the subject
matter in elementary and high school turns out to be a significant
component of their preparation for teaching. (Ball, Lubienski, &
Mewborn, 2001, p. 437)

Pre-service teachers’ own K-12 learning experiences likely have as much, if not more,
impact than their teacher education programs.

Prospective teachers who major in elementary education may only take one or
two mathematics courses specifically designed for elementary teachers\(^5\) while those who
major in secondary education often take many mathematics courses. At either level,
however, the learning in these college mathematics courses frequently resembles the
same type of learning experienced during the K-12 years (Borko et al., 1992). As pre-
service secondary teachers moved into their own classrooms, Cuoco (2001) found these
teachers over-utilizing worksheets and demonstrating a watch-and-do style of pedagogy,

\(^5\) Determining the number of mathematics courses required for elementary certification is
a complicated task due to the wide range of certification programs. However, prospective
K-8 teachers completed a greater number of mathematics courses in 2005 as compared to
2000. “While the [CMBS’] course recommendations had not been completely
implemented by fall 2005, the nation was closer to them than in the base-study in fall
2000” (Lutzer, Rodi, Kirkman, & Maxwell, 2007, p.51).
which he traced back to the “kind of mathematics one learns as an undergraduate” (p. 169).

Prospective teachers’ beliefs about teaching and learning, developed from years of experiencing mathematics education from a learner’s perspective, change little during the pre-service experience (Ball, Thames, & Phelps, 2008; Raymond, 1997). Unfortunately many prospective teachers leave pre-service programs and enter the classroom teaching mathematics the way they were taught (Cuoco, 2001).

**Mathematical Knowledge of Practicing Teachers**

The next part of this chapter focuses on the mathematics that scholars suggest practicing teachers should possess. Mathematics teachers must develop a deeper knowledge of mathematics than what they developed as part of their own K-12 mathematics experiences. Yet, is completing college mathematics coursework enough to prepare teachers to actually teach mathematics? The answer is likely no. “Studies that used the mathematics courses that teachers have taken as a proxy for their mathematical knowledge showed mixed results regarding the relationship of teachers’ content knowledge to their students’ achievement at the elementary and middle school level” (National Mathematics Advisory Panel, 2008, p. 36). Teachers need specialized content knowledge.

Scholars offer suggestions as to other domains of mathematical knowledge needed by practicing teachers. This literature review focuses on three complementary perspectives on the mathematical knowledge required of practicing teachers: a profound understanding of fundamental mathematics (Ma, 1999), recommendations offered by
national groups (e.g., CBMS, 2001; National Research Council, 2001; Common Core State Standards Initiative (CCSSI), 2010), and mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008). Each perspective moves well beyond just completing mathematics courses at the college level.

**Profound Understanding of Fundamental Mathematics**

Ball, Lubienski, and Mewborn (2001) found many teachers do not possess a fundamental understanding of mathematics. “An overview [of many studies] reveals pervasive weaknesses in U.S. teachers’ understanding of fundamental mathematical ideas and relationships” (p. 444). Ma (1999) views mathematics as a discipline to be understood, not just learned. She suggests that teachers need to understand the big ideas of mathematics and be able to represent mathematics to students as a coherent and connected activity. The idea of fundamental knowledge is at the forefront of Ma’s research. She describes what she calls a profound understanding of fundamental mathematics (PUFM) in terms of considering three dimensions of teachers’ knowledge of a mathematical topic: depth, breadth, and thoroughness (Ma, 1999).

I define understanding a topic with depth as connecting it with more conceptually powerful ideas of the subject…Understanding a topic with breadth, on the other hand, is to connect it with those of similar or less conceptual power…Depth and breadth, however, depend on thoroughness—the capacity to “pass through” all parts of the field—to weave them together. Indeed, it is this thoroughness which “glues” knowledge of mathematics into a coherent whole. (p. 121)
Ma argues that teachers with all three dimensions have the necessary understanding to help students navigate through school mathematics.

Ma (1999) highlights the conceptual nature of learning mathematics and encourages teachers to help students make conceptual connections across mathematical topics. Ma suggests teachers should help students be able to use as well as recognize strengths and weaknesses of multiple approaches to solve problems. Teachers with PUFM are “ready to exploit an opportunity to review concepts that students have previously studied or to lay the groundwork for a concept to be studied later” (p. 124). Teachers need to know and understand the mathematics they teach, at levels much deeper than is being learned by their students.

Ma suggests the development of PUFM takes time. “Chinese teachers develop PUFM during their teaching careers—stimulated by a concern for what to teach and how to teach it, inspired and supported by their colleagues and teaching materials” (p. 143). She indicates that PUFM does not and cannot merely develop during Chinese teacher preparation. The same holds true of pre-service programs in the in the U.S.. Knowledge consistent with PUFM cannot be learned entirely within what pre-service programs are able to offer future teachers (NCTM, 2000). The best pre-service teacher education programs can prepare future teachers to be learners who will learn and develop throughout their teaching careers.

Ma (1999) identifies three distinct opportunities for Chinese teachers to attain PUFM. The first is by learning mathematics from their colleagues. A second is when teachers acquire PUFM from students. “I had not expected that the teachers would tell me
that they had learned mathematics from their students, but they did” (p. 138). A third time is when teachers simply do mathematics (Ma). U.S. teacher preparation programs as well as in-service professional development programs may benefit from intentionally creating opportunities for teachers to acquire mathematical knowledge for teaching in each of these three contexts.

**Recommendations from National Groups**

Several national groups, including the Conference Board of the Mathematical Sciences (2001), National Research Council (2001), and Common Core State Standards Initiative (2010), have offered a complementary perspective on the mathematical knowledge needed for teaching. The CBMS (2001) perspective moves the discussion toward specific coursework and other learning experiences offered by colleges and teacher preparation programs. This national committee created a document outlining specific recommendations as to the mathematics that prospective teachers need to know.

The mathematical knowledge needed for teaching is quite different from that required by college students pursuing other mathematics-related professions. Prospective teachers need a solid understanding of mathematics so that they can teach it as a coherent, reasoned activity and communicate its elegance and power. (p. xi)

Influenced by Ma (1999), CBMS committee members note that prospective teachers need knowledge beyond what is offered in college courses. While the document addresses pre-service education, the CBMS (2001) recognizes practicing teachers need continued
professional development to enrich what they might have learned in their initial teacher preparation programs.

The CBMS document offers both general and specific details as to the knowledge teachers should possess, a knowledge that exceeds what is needed to deliver K-12 curricula. First, and foremost, prospective teachers “must have classroom experiences in which they become reasoners, conjecturers, and problems solvers” (p. 56). At the middle level, defined as grades 5-8, teachers must have a more sophisticated understanding of mathematics than their elementary counterparts have, yet an understanding that is equally rich to that of their high school counterparts. “Teaching middle grades mathematics requires preparation different from, not simply less than, preparation for teaching high school mathematics, and certainly reflecting more depth than that needed by teachers of earlier grades” (p. 25). The CBMS suggests that far too many middle level teachers have been given the knowledge to merely teach the mathematics of the elementary curriculum and not of more challenging mathematics in the middle grades. The committee recommends that middle level mathematics teachers be given enough education to classify and equip them as mathematics specialists.

Regardless of grade level taught, the CBMS (2001) recommends that prospective teachers be given opportunities to develop mathematical habits of mind. “Along with building mathematical knowledge, mathematics courses for prospective teachers should develop the habits of mind of a mathematical thinker and demonstrate flexible, interactive styles of teaching” (p. 8). Prospective teachers must have experiences as part of their preparation to teach that place a priority on becoming engaged with mathematics
in order to deeply understand it. Pre-service teachers must experience the learning of mathematics in college classrooms that encourage them to develop the habits of mind of a mathematical thinker.

Mathematics is not only about numbers and shapes, but also about patterns of all types. In searching for patterns, mathematical thinkers looks for attributes like linearity, periodicity, continuity, randomness, and symmetry. They take actions like representing, experimenting, modeling, classifying, visualizing, computing, and proving. Teachers need to learn to ask good mathematical questions, as well as find solutions, and to look at problems from multiple points of view. Most of all, prospective teachers need to learn how to learn mathematics…Prospective teachers need to experience such instruction in their college mathematics classes and to learn that there are multiple ways to engage students in mathematics. (p. 8)

The CBMS recommendations do more than prescribe which mathematics should be learned before one enters the classroom. These recommendations stress the need to deeply understand the content one shall teach.

The National Research Council (2001) addresses what mathematics teachers should know in terms of five interwoven, yet independent strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (See Figure 1).
The National Research Council’s term “mathematics proficiency” (p. 5) connects with others’ definitions of what it means to know and understand mathematics. The strands of conceptual understanding and procedural fluency connect to the NCTM’s (2000) content standards and help to extinguish the fiery debate between which is more important, as they are both important. The strands of strategic competence and adaptive reasoning connect to the NCTM’s process standards. The productive disposition strand connects with the CBMS’ habits of mind recommendation.

Most recently, the CCSSI (2010) released the Common Core State Standards for Mathematics. This document offers yet another viewpoint as to the mathematical knowledge teachers should possess. The CCSSI standards “rest on important ‘processes
and proficiencies’ with longstanding importance in mathematics education” (p. 6). The ideas the CCSSI present fall in line with the ideas found within the NCTM’s (2000) five content and five process standards and the National Research Council’s (2001) five intertwined strands of mathematical proficiency.

Mathematical Knowledge for Teaching

Ball, Thames, and Phelps (2008) are developing a theory of mathematical knowledge for teaching that goes much deeper than simply building knowledge through college coursework.

Just knowing a subject well may not be sufficient for teaching…In addition, teachers need to know mathematics in ways useful for, among other things, making mathematical sense of student work and choosing powerful ways of representing the subject so that it is understandable to students. It seems unlikely that just knowing more advanced math will satisfy all of the content demands of teaching…What seems most important are knowing and being able to use the mathematics required inside the work of teaching. (p. 404)

This theory searches for deeper connections between teachers’ knowledge of mathematics and use of that mathematics while teaching. The ideas for this theory are emerging from collaborative efforts of faculty from the education and mathematics departments.

Organizations, such as the NCTM (2000), have long written about teacher knowledge consistent with ideas found within Ball et al.’s emerging theory:
Teachers need several different kinds of mathematical knowledge—knowledge about the whole domain; deep, flexible knowledge about curriculum goals and about the important ideas that are central to their grade level; knowledge about the challenges students are likely to encounter in learning these ideas; knowledge about how the ideas can be represented to teach them effectively; and knowledge about how students’ understanding can be assessed. (p. 17)

While it is true that teachers must have curricular knowledge of mathematics (Ball, Thames, & Phelps, 2008) that has been deepened during the college years (CBMS, 2001), teachers must also possess pedagogical knowledge. “To what extent does teaching—and hence, learning to teach—depend on the development of knowledge of subject matter? On the other hand, to what extent does it rely on the development of pedagogical method?” (Ball & Bass, 2000, p. 85). Ball, Bass, and their colleagues are currently attempting to address these questions.

The interest in knowledge for teaching initially gained prominent attention when Shulman (1987) introduced scholars to the theoretical construct of pedagogical content knowledge in the 1980s. While a knowledge of the subject matter of mathematics refers to one’s depth and breadth of understanding of mathematical concepts and processes, a teacher’s pedagogical content knowledge is directly related to his or her ways of taking subject matter and making it accessible to students. Pedagogical content knowledge is not found in isolation as teachers must know how to use content knowledge in the tasks of teaching.
Ball, Thames, and Phelps (2008) have led an effort, focused on the teaching of elementary mathematics, to develop a refined theory of Shulman’s (1987) initial idea. After much collaboration and study, these researchers have asked specific questions to help fuel the development of ideas: “1. What are the recurrent tasks and problems of teaching mathematics? What do teachers do as they teach mathematics? 2. What mathematical knowledge, skills, and sensibilities are required to manage these tasks?” (Ball, Thames, & Phelps, 2008, p. 394-395). While attempting to answer these questions, Ball, Thames, and Phelps (2008) revisited Shulman’s (1987) original two categories of knowledge for teaching: subject matter and pedagogical content knowledge. They subdivided the categories, identifying domains of content knowledge for teaching. Their theory includes a total of six specific domains (See Figure 2).

![Figure 2. Characterizations of Mathematical Knowledge for Teaching](Mathematical Sciences Research Institute, 2009, p. 12)

Ball, Thames, and Phelps’ (2008) work points to the complexity of teaching mathematics and offers insight into ways to improve the content and pedagogical preparation of both
pre-service and in-service teachers. “We hypothesized that teachers’ opportunities to learn mathematics for teaching could be better tuned if we could identify those types more clearly” (p. 399). This hypothesis offers more insight into the depths and complexities of knowledge and helps to articulate why, and in what ways, mathematical knowledge for teaching must move beyond the typical mathematical content knowledge learned in college mathematics courses.

The recommendations on what teachers need by these researchers and the national committee, in addition to recommendations from the NCTM (2000), National Research Council (2001), and CCSSI (2010), do not necessarily match what teachers actually take away from college as well as professional development learning experiences. The next part of this chapter addresses professional development experiences as they relate to the deepening of mathematical knowledge of practicing teachers.

**Professional Development**

The sights and sounds of professional development have changed as the decades have progressed, yet one common theme has not changed—far too many professional development opportunities are considered a waste of time (e.g., Wilson & Berne, 1999; Borko, 2004). Researchers offer insight as to the limitations of professional development. Not surprisingly, Cohen and Hill (2001) found teachers who attend the same professional development experiences with their colleagues leave with very different levels of knowledge and capacities to make change in the classroom. Loucks-Horsley et al. (2003) report the focus on mathematical content is frequently missing for practicing teachers as they attend professional development activities. Bay (2000) and Burrill (2001) state
professional development does not aid teachers in deepening their understanding of curriculum or how to conduct classroom discourse in ways that help students make meaningful mathematical connections.

Two areas of concern, poor student performance and weak professional development, have prompted the call for changes in learning opportunities for teachers. Stigler and Hiebert (1999) found that American mathematics students do not perform as well as students in other countries. Further, classroom interactions tend to look different in those other countries than in American classrooms (Hiebert & Stigler, 2004). “Results…showed that high-achieving countries teach 8th-grade mathematics in different ways…[American teachers] should focus on ensuring that students have some opportunities to solve challenging problems that require them to construct mathematical relationships—to develop conceptual understanding” (p. 12). Hiebert and Stigler’s findings are consistent with the NCTM’s (2000) process standards, which outline ways for educators to teach mathematics conceptually. Hiebert and Stigler, as well as the NCTM, call for teachers to put more focus on conceptual instruction.

Obstacles to Overcome

Despite calls for change, the shift towards conceptual learning in the classroom is not likely to happen easily. “Teachers, both practicing and pre-service, will need support if we expect them to teach concepts such as these in a manner consistent with the mathematics education community's goals for student learning” (Stein, Baxter, & Leinhardt, 1990, p. 660). Professional development can serve as a vehicle to initiate change in classroom practices. Unfortunately, too many professional development
opportunities tend not to have any sort of sustained impact on teaching (Farmer, Hauk, & Neumann, 2005).

Arbaugh (2003) cites isolationism as one obstacle that prevents many practicing mathematics teachers from developing into better teachers. American schools are steeped in a culture of teacher isolationism (Cohen, 1988; Cuban, 1993). Teachers need to interact with one another (NCTM, 2000). The NCTM (2000) highlights many benefits for breaking down isolation barriers, including the richness of understanding that occurs for teachers from reflection and analysis of learning and teaching situations in collaborative settings. “Collaborating…is a powerful, yet neglected, form of professional development in American schools. The work and time of teachers must be structured to allow and support professional development that will benefit them and their students” (p. 19). To improve the current state of professional development practices, reformers recommend that leaders decrease teachers’ isolationism and increase teachers’ opportunities for working together.

Farmer, Hauk, and Neumann (2005) addressed issues of sustainability of what is learned in professional development as another obstacle for professional developers to overcome:

[The typical] professional development design [consists of] short time span, direct instruction training workshops most often offered as professional development for teachers. The one-day-workshop model, still ubiquitous in professional development, asks teacher-participants to “reform” without providing the necessary time, active engagement, and
intellectual scaffolding of highly contextualized discussions of content and process awareness. (p. 66)

Unfortunately, many teachers view professional development opportunities simply as make it and take it opportunities. Teachers seek specific activities to take back to their classrooms. “It may not be that teachers embark upon a professional development activity in order to change their attitudes or beliefs, although this is a common hoped-for outcome on the part of providers” (Farmer, Gerretson, & Lassak, 2003, p. 332). When professional development is not focused on substantive content, it translates to a much bigger problem when it comes to making changes in a teacher’s practice. Participating in a single workshop does not afford teachers the time to make necessary changes (Farmer, Gerretson, & Lassak). Teachers need to participate in sustained experiences. To improve the current state of professional development practices, researchers point to the need to replace short-term learning experiences with sustained ones.

**Using Professional Development to Strengthen Teachers’ Mathematical Knowledge**

An aspect deemed to be central in future professional development is deepening teachers’ mathematical knowledge for teaching. Teaching mathematics outside of the confines of one’s own experiences learning mathematics, in addition to relying heavily on the textbook, requires teachers to have a deeper knowledge of mathematics (Ball & Bass, 2000). While the NCLB term “highly qualified” is interpreted and defined differently from state to state, most states have included in their definitions passing various levels of mathematics coursework to meet certification requirements to teach mathematics (U.S. Department of Education, 2006).
The National Mathematics Advisory Panel (2008) “recommends that a sharp focus be placed on systematically strengthening…ongoing professional development for teachers of mathematics at every level, with special emphasis on ways to ensure appropriate content knowledge for teaching” (p. 40). The RAND Mathematics Study Panel (2003) states, “The most fundamental effort…is identifying and shaping professional learning opportunities for teachers…to develop the requisite mathematical knowledge, skills, and dispositions to teach each of their students effectively” (p. xvii). Practicing teachers need to participate in professional development opportunities that deepen their mathematical knowledge and understanding.

One possible vehicle to help practicing teachers strengthen their mathematical knowledge may be through the use of professional learning communities. An increased amount of attention has recently been focused on this school-based unit of change (e.g., Arbaugh, 2003; Boerst, 2003; Hollins et al., 2004; Supovitz, 2002). Arbaugh (2003) describes a professional learning community as a “group of educators who come together on a regular basis to support each other as they work collaboratively to both develop professionally and to change their practice” (p. 141). Teachers are not isolated; rather, they are encouraged to interact with one another in small groups over a sustained period of time (Boerst, 2003; Supovitz, 2002). In sum, teachers meeting within professional groups make the time to discuss ideas with one another and reflect on current practice (e.g., Arbaugh, 2003; Boerst, 2003; Hollins et al., 2004).

The current focus of the school-based learning communities is on improving instructional practices through collaborative interactions with others (Supovitz, 2002).
“Sharing our knowledge of teaching with others enhances the quality of (those) ideas because others contribute their own knowledge of teaching” (Boerst, 2003, p. 500).

Boerst uses the phrase, “harness a wealth of professional expertise” (p. 500).

If working together in a school-based community is a viable way for schools to enact change, then it is reasonable to gain insights for how a community of learners could help one another to strengthen mathematical knowledge. Further, the sustained characteristic of teacher communities, as described by Boerst (2003), may represent a key ingredient for successfully implementing reform ideas that researchers, including Ball and Cohen (1999), have been calling for:

Unless ways can be found, through professional development, to help teachers to sustain such work, traditional instruction is likely to persist in frustrating educational reform, and reformers’ visions are likely to continue not to permeate practice broadly or deeply. (p. 6)

Depending on how they are implemented, professional learning communities may help teachers enact reformers’ visions of improved teaching and learning and offer a way to address the issue of sustainability that professional developers face (Farmer et al., 2003). Professional learning communities may also enact the national call to strengthen teachers’ knowledge of mathematics (e.g., National Research Council, 2001; CBMS, 2001; CCSSI, 2010).

While professional learning communities may open the door for improved instruction and deeper mathematical understanding, there is limited research on such communities. The field lacks specific images of what it looks like for teachers to
participate in these professional communities or how collaborating in such sustained settings can lead to a deepening of mathematical knowledge for teaching.

**Long Term Professional Development Programs for Experienced Mathematics Teachers**

Researchers who have studied professional development programs have looked at the effects of a variety of models and programs. A wide range of interests emerged. Some researchers have studied the impact of professional development focused on equity by looking at the effects of race, gender, and ethnicity on students’ learning (e.g., Loucks-Horsely et al., 2003; Nelson, 1997), while others have looked at teacher change by studying professional development focused on various theories of teaching and learning (e.g., Ball & Cohen, 1999; National Staff Development Council, 2001). Still other researchers have studied contextual features of translating professional development into classroom practice by examining factors such as the role of a school’s principal in providing more time and support for teachers’ ongoing development (e.g., DuFour, 2001). While there have been numerous studies on professional development, one question stands out with respect to reviewing research related to the context of this study: Where and how does deepening teachers’ mathematical knowledge intersect studies on professional development?

There are many current examples of professional development programs that address deepening mathematical knowledge of experienced teachers. For example, The Rice University Mathematics Leadership Institute and Oregon Mathematics Leadership Institute Partnership are two examples of Math and
Science Partnerships (MSP), recently funded by the National Science Foundation (NSF), that provide opportunities for practicing teachers to deepen their mathematical knowledge. However, most studies from these and other current professional development programs have not been published to date. However, studies from similar past projects offer insight into what researchers have learned from studying teachers who deepen mathematical knowledge as part of professional development programs. Project IMPACT (Campbell, 1996), QUASAR (Silver & Stein, 1996), and CGI (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989), are such examples.

Investigators with Project IMPACT (Increasing the Mathematical Power of All Children and Teachers) studied how elementary teachers, when provided research on how students learn, help students develop a conceptual knowledge of mathematics (Campbell & White, 1997). Campbell (1996) reported several findings, ranging from increases in student achievement to changes in instruction. QUASAR (Quantitative Understanding: Amplifying Student Achievement and Reasoning) was “a national education reform project aimed at fostering the development and implementation of enhanced mathematics

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7 The Oregon State University and Rice University MSP projects were among some of the first projects funded by the NSF and are likely in the process of making sense of data from each of these respective projects; thus, it is probable that these and other MSP projects will publish their findings in the near future.
instructional programs in middle schools that serve students from disadvantaged backgrounds” (Stein & Brown, 1997, p. 156). QUASAR researchers studied how teachers used materials from the project to make instructional changes in addition to how students’ conceptual understanding changed (Silver & Stein, 1996). Investigators with CGI studied elementary teachers who deepened their knowledge about how children think (Fennema, Franke, Carpenter, & Carey, 1993). Carpenter and Fennema (1989) reported many findings from the study including: CGI teachers allowed more time for problem solving, CGI teachers were more cognizant of their beliefs about children’s understanding when teaching mathematics, CGI students outperformed non-CGI students, and CGI students became more confident in their mathematics abilities.

Project IMPACT, QUASAR, and CGI are three visible examples of sustained efforts to promote long-term change in teaching and learning mathematics. The research focus for all three of these professional development opportunities was on teacher as well as student change. While results from these studies have been important to the field of mathematics education, the investigators did not examine what the practicing teachers’ learning of mathematics looked like. Unlike these past studies, this study places focus on practicing teachers’ learning of mathematics, rather than the translation of that learning to teachers’ classroom practices or their students’ achievement.

**Practicing Teachers’ Attempts to Deepen Mathematical Knowledge**

Lubienski and Bowen (2000) reviewed 3011 articles related to mathematics education in the ERIC (Education Resources Information Center) database, published from 1982 to 1998. Teacher mathematical knowledge was not studied frequently enough
to merit a category in their coding. In a non-exhaustive review of more recent research in mathematics education, this study identified several articles focused on student understanding of mathematics (e.g., Barrett, Clements, Klanderman, Pennisi, & Polaki, 2006) and student use of mathematics (e.g., Joram, Gabriele, Bertheau, Gelman, & Subrahmanyam, 2005). Researchers focused on mathematics learning at all levels, ranging from pre-school (e.g., Sophian, 2002) through college (e.g., Williams, 2001). Some researchers (e.g., Kato, Kamii, Ozaki, & Nagahiro, 2002) even studied the mathematical knowledge of international students. Unfortunately, a focus on teachers’ acquisition of mathematical knowledge for teaching was missing.

There were some studies related to deepening teachers’ mathematical knowledge. Yet, the focus of these studies was still not explicitly or exclusively on practicing teachers’ learning of mathematics. Some researchers focused on examining teacher knowledge in relation to teachers’ personal convictions about teaching. For example, Herbst (2004) examined teacher knowledge alongside beliefs about teaching mathematics. Other researchers focused on teachers’ learning of mathematics in conjunction with aspects that enhance the learning experience. For example, Chamberlin, Farmer, and Novak (2008) examined the potential benefits of using assessments in relation to teachers’ development of mathematical knowledge during professional development. Still other researchers focused on pre-service teachers’ mathematical learning (e.g., Chinnappan, 2003; Philipp et al., 2007). Frykholm (2005) studied six cohorts of pre-service elementary teachers as they learned mathematics related to curriculum activities found within the reform-oriented curriculum, Mathematics in
Context. “Most students in the study…were struck by the holes in their own knowledge, particularly related to conceptual understanding that went beyond the recitation of formulas and algorithms” (p. 28). Frykholm used case study methodology as the basis of his descriptions of what weakened mathematical knowledge looks like. While each of these examples indicates there are some studies that have focused on teachers’ learning of mathematics, the main focus has not been to describe what that learning looked like.

Conclusion

This overview of research points to many areas of interest in the field of mathematics education related to professional development. Several areas do relate to teachers’ mathematical knowledge. Yet, one specific area is currently underrepresented. Studies do not offer images of what it looks like for practicing teachers to actually engage in the learning of mathematics. Just as Putnam, Heaton, Prawat, and Remillard (1992) wrote cases to allow them, as observers, to explore interrelationships among teachers’ knowledge of mathematics, beliefs about mathematics and beliefs about teaching and learning, this study uses a case to explore the nature of practicing teachers’ attempts to learn the mathematics needed for teaching in a professional development program that takes the CBMS (2001) recommendations seriously. This study has the potential to serve as a platform to improve professional development efforts for in-service mathematics educators by offering images of what enables or hinders practicing teachers in the process of learning challenging mathematics.
Chapter 3: An Inquiry Into Teachers’ Learning of Mathematics

The focus of this case study is on how two middle level teachers learn mathematics while participating in a professional development project aimed at helping teachers deepen their understanding of the mathematics needed for teaching. This chapter discusses the details of the methodology.

Using Case Study to Explore Practicing Teachers’ Learning of Mathematics

The overarching question for this inquiry was: How do teachers deepen their current understanding of the mathematics content they have to teach? Since this study set out to closely examine the nature of practicing teachers’ learning of mathematics, a qualitative approach was appropriate to use, as little is known about this topic (Creswell, 2005; Stake, 1995) of practicing teachers’ learning of mathematics. The case study methodology allowed this study to explore and report fine details (Creswell, 2005; Morse & Richards, 2002; Stake, 1995) of in-service teachers’ attempts to learn challenging mathematics.

The Professional Development

The two teachers studied here participated in the Math in the Middle Institute Partnership (M²), funded by the National Science Foundation (NSF). M² is an example of a professional development program that places priority on the development of mathematical and pedagogical knowledge needed for teaching middle level mathematics. The M² Institute was designed for middle level (5-8) mathematics teachers and was comprised of a collection of twelve courses in which in-service teachers studied
mathematics and pedagogy, culminating in a master’s degree. Teachers took the courses spread over three summers and two academic years.

Institute courses were developed with CBMS (2001) recommendations at the forefront. Courses emphasized the NCTM’s process standards (NTCM, 2000), promoted the development of the habits of mind of mathematical thinkers (CBMS, 2001), and provided opportunities for teachers to develop a deeper understanding of mathematics (Lewis, Heaton, McGowan, & Jacobson, 2004). Institute courses were intended to offer “the mathematics content needed by middle grades teachers so they can lead their students to make sense of mathematics, to be able to communicate effectively about mathematics, and be able to use mathematics appropriately” (CBMS, 2001, p. 102).

Institute courses were also aimed to provide participants a means to develop leadership skills. The first three mathematics courses of the M^2 Institute, Mathematics as a Second Language, Functions, Algebra and Geometry for Middle-Level Teachers, and Experimentation, Conjecture and Reasoning, are the courses from which this study examined teachers’ learning of challenging mathematics.\(^8\)

**Habits of Mind**

M^2 instructors promoted the development of mathematical habits of mind across all mathematics courses of the M^2 Institute. These dispositions represented a broad view of what it meant to do mathematics. The project’s co-PIs’ working definition for the set

\(^8\) Chapter four discusses the structure of the professional development.
of skills and dispositions that a mathematical thinker with habits of mind would possess included:

1. Understands which tools are appropriate when solving a problem
2. Is flexible in his or her thinking
3. Uses precise mathematical definitions
4. Understands there exists multiple paths to a solution
5. Is able to make connections between what one knows and the problem
6. Knows what information in the problem is crucial to its being solved
7. Is able to develop strategies to solve a problem
8. Is able to explain solutions to others
9. Knows the effectiveness of algorithms within the context of the problem
10. Is persistent in his or her pursuit of a solution
11. Displays self-efficacy while doing problems
12. Engages in meta-cognition by monitoring and reflecting on the processes of conjecturing, reasoning, proving, and problem solving

The construct of “habits of mind” can be found in work that dates back as early as Dewey (1916/1944). To intellectually grow, one must have the “capacity to acquire habits or develop definite dispositions” (p. 46). Contemporary researchers, Costa and Kallick (2000), identified 16 interdisciplinary habits of mind, including being persistent, taking responsible risks, and communicating with precision, that “lead to productive actions” (p. 8). Mathematicians (e.g., Hardy, 1940/1942; Poyla, 1945) and mathematics

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9 This list of 12 mathematical habits of mind was retrieved from http://scimath.unl.edu/MIM/habits.php on 10/4/2010.
educators (e.g., Cuoco, Goldenberg, & Mark, 1996; Driscoll, 1999) have also added their perspectives to the conversation about habits of mind, specific to mathematics. Cuoco et al. (1996) suggested curricula be centered on developing mathematical habits of mind. “A curriculum organized around habits of mind tries to close the gap between what the users and makers of mathematics do and what they say…A habits of mind curriculum is devoted to giving students a genuine research experience” (p. 376). Most recently, the Common Core State Standards Initiative (CCSSI, 2010) has addressed mathematical habits through their recommendations regarding mathematical practices.

$M^2$ instructors were deliberate about implicitly and explicitly exposing participants to their working definition of habits of mind, as part of classroom discussions and activities as well as required homework assignments. Homework assignments often included specific problems aimed to push participants’ development of mathematical habits of mind. Ideally, the paths to the solutions of these problems were not obvious and required much thought, perseverance, as well as some creativity.

**Participants**

This study featured a single case of two teachers’ experiences learning mathematics during the first three courses of the $M^2$ Institute. The selection of the teachers for this study came from participants who, at the time of data collection, were in their first year of learning mathematics in the $M^2$ Institute. Participants in their first year of participants were chosen, as studying these participants was likely to offer more accurate descriptions of mathematical learning and teaching experiences prior to their start of the $M^2$ Institute than studying participants at other points in relationship to their participation in the institute. A careful attempt was made to do purposeful sampling
(Creswell, 1998; Merriam, 1998). Data collection occurred immediately following teachers’ completion of the third mathematics course.

Three criteria informed the decision as to which teachers to select. First, I narrowed my search to include only those participants within two hours of my location. It was not practical to choose teachers who lived too far from me, as I interviewed them as well as observed them teach multiple times. Second, I used data from surveys collected from all participants during the initial weeks of the M² Institute experience to gather participants’ perspectives on their own mathematical knowledge, learning, and understanding (see three series of reflective prompts in Appendix B). Analysis of this self-reflection data assisted me in identifying teachers who differed along the dimension of “self-assessed” strength in mathematics. Finally, I reviewed the background information of each participant as reported on the application for the M² Institute (e.g., undergraduate degree, teaching endorsement, years of experience, teaching assignments). While I noticed some variation (e.g., endorsements, years of experience), I found most teachers had an elementary education (K-6) background as well as an elementary certification (K-6). Thus, I needed to identify additional factors along which they might differ.

I identified two teachers to study: Becki Zander¹⁰ and Linda Anderson. Mrs. Zander and Mrs. Anderson had several characteristics in common, including the fact they were both female, had numerous years of experience, and graduated from college with an elementary education degree and certification. Several differences marked demographic

¹⁰ All names are pseudonyms.
variation between the two teachers as well. This allowed me to explore how the learning of mathematics might vary across two teachers with contrasting views of themselves as learners of mathematics. Thus, studying these two teachers offered interesting perspectives and contrasts regarding the learning of mathematics. In addition, identifying one city teacher as well as one rural teacher allowed me to explore what is afforded teachers in learning mathematics as part of professional development across varied contexts, including face-to-face and computer-facilitated learning opportunities.

**Data Collection**

Multiple forms of data, including interviews, observations, and artifacts, helped me develop an in-depth understanding of emergent issues (Creswell, 2006) related to the study of these two teachers’ mathematical learning. Based on my research questions, I collected several types of data. Table 3.1 represents the data I collected during the 2007-2008 school year. Each dot represents a form of data I collected for a particular purpose related to my study.

There was a substantial amount of data collected from each teacher as part of their participation in the professional development project prior to the start of my study. Before I selected Mrs. Zander and Mrs. Anderson as research subjects, they, along with all other participants in their cohort, had already completed the first four courses of the M² Institute: three mathematics courses (i.e., MSL, FAGMLT, ECR) and a curriculum inquiry course. A variety of written work was collected from each participant during each of these first four courses, including solutions to individual mathematics assignments as
Table 3.1

*Data Collected*

<table>
<thead>
<tr>
<th>Specific Areas of Study</th>
<th>Data Types</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Previously collected data</td>
</tr>
<tr>
<td>Participants’ background</td>
<td>● ● ● ●</td>
</tr>
<tr>
<td>Participants’ knowledge of mathematics</td>
<td>●</td>
</tr>
<tr>
<td>Participants’ learning of mathematics</td>
<td>●</td>
</tr>
<tr>
<td>Participants’ organization to learn mathematics</td>
<td>●</td>
</tr>
<tr>
<td>Participants’ use of mathematics in the classroom</td>
<td>●</td>
</tr>
</tbody>
</table>

well as end of course problem sets,\textsuperscript{12} reflections about the learning of mathematics, and assignments related to curriculum and instruction. I analyzed a subset of this existing data as part of my study.

Numerous non course-related data were also collected from each participant prior to the start of their participation in M$^2$ as part of the ongoing data collection for the professional development project, separate from this study. For example, each teacher

\textsuperscript{11} Examples of previously collected data are discussed following Table 2.
\textsuperscript{12} End of course problem sets included a selection of problems similar to problems worked during class and for evening homework assignments.
submitted an application including demographic information and written responses to questions prompting teachers to reflect on their practice and mathematics knowledge. Participants submitted a baseline videotape of teaching, and completed an inventory related to teacher beliefs and a pre-test of mathematical knowledge for teaching. I asked each participant to complete three separate reflections on their perceptions of learning mathematics as part of the first two mathematics courses during the summer of 2007 (refer to Appendix B for the reflection prompts). I also recorded observational field notes of participant interactions while working on homework problems following select days of the first two mathematics courses.13 These varied data helped me describe each teacher’s background in chapter four and enabled me to triangulate my claims and assertions in chapters five and six.

Stake (1995) urges case study researchers to have a “connoisseur’s appetite” (p. 56) for data collection. I collected additional data that helped me understand the issues surrounding the research questions for my study. In particular, I collected data that helped me describe each teacher’s milieu, learning of mathematics, and use of mathematical knowledge in teaching. I used interviews, observations, and a problem solving session to supplement data previously collected. I collected this data over a three-month period during the spring of 2008.

To inform each case, I interviewed Mrs. Zander and Mrs. Anderson before I

13 These field notes focused on the interactions of all 35 teachers during varied homework study sessions of the first two mathematics courses. This was well before I had selected the participants for this study. After I had selected Mrs. Anderson and Mrs. Zander, I then recorded a new set of observational field notes of the two teachers working problems together during a problem solving session in May 2008. This later data was used for analysis in Chapter 6.
closely examined their mathematics work from the first three M² mathematics courses. I asked them to talk about their learning experiences prior to as well as during their involvement in Math in the Middle. The specific interview questions can be found in Appendix C. I audiotaped and then immediately transcribed each interview. Following this initial interview, I generated a list of follow-up questions (see Appendix D for follow-up questions) and emailed them to both teachers following the initial interview. I also interviewed each teacher’s principal (see Appendix E for interview questions) to add depth to my description of each teacher’s background and context. I again audiotaped and transcribed each principal interview.

At the conclusion of each participant interview in early February, I asked each teacher to generate a list of mathematical topics they planned to cover in the remaining months of the 2007-2008 school year. Both teachers planned to teach probability, perimeter, and area before the end of the year. I videotaped each participant teaching these topics. I also chose other dates to observe and videotape. Some of the times were selected because they were directly before and after the identified lessons on probability, perimeter, and area. Other times were selected based solely on convenience. Table 3.2 gives the dates of classroom observations.

I personally did most of the videotaping of the lessons and chose to focus the camera on the speaker at all times, including student-to-student as well as teacher-to-student conversations. For each non-videotaped observation, I audiotaped the lesson while I recorded field notes. In my field notes, I recorded teacher moves and
Table 3.2

Data Collection Dates

<table>
<thead>
<tr>
<th></th>
<th>Becki Zander</th>
<th>Linda Anderson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observations followed by</td>
<td>Thursday, 2/28/2008</td>
<td>Wednesday, 2/27/2008</td>
</tr>
<tr>
<td>brief interview</td>
<td>Friday, 2/29/2008</td>
<td>Thursday, 2/28/2008</td>
</tr>
<tr>
<td></td>
<td>Monday, 3/31/2008</td>
<td>Thursday, 4/03/2008</td>
</tr>
<tr>
<td></td>
<td>Wednesday, 4/02/2008</td>
<td>Wednesday, 4/09/2008</td>
</tr>
<tr>
<td></td>
<td>Wednesday, 5/14/2008</td>
<td>Thursday, 5/01/2008</td>
</tr>
<tr>
<td>Videotaped lessons</td>
<td>Monday, 3/03/2008</td>
<td>Thursday, 3/13/2008</td>
</tr>
<tr>
<td></td>
<td>Tuesday, 3/04/2008</td>
<td>Monday, 4/14/2008</td>
</tr>
<tr>
<td></td>
<td>Friday, 4/11/2008</td>
<td>Tuesday, 4/15/2008</td>
</tr>
<tr>
<td></td>
<td>Monday, 4/14/2008</td>
<td>Wednesday, 4/16/2008</td>
</tr>
<tr>
<td></td>
<td>Wednesday, 4/16/2008</td>
<td>Tuesday, 4/22/2008</td>
</tr>
<tr>
<td></td>
<td>Wednesday, 5/07/2008</td>
<td>Tuesday, 5/06/2008</td>
</tr>
<tr>
<td></td>
<td>Thursday, 5/08/2008</td>
<td>Thursday, 5/08/2008</td>
</tr>
</tbody>
</table>

conversations. Following each observation, I asked each teacher questions including, How do you think the lesson went? Did anything surprise you? Do you think the students were understanding the topic of the day? The purpose of the videotaped lessons and observations was simply to become familiar with the ways each teacher communicated and presented mathematics to students as well as to look for connections between Institute learning experiences and middle level teaching experiences.

At the beginning of my study, I had data on each teacher’s learning of mathematics. Yet, these data were in the form of artifacts, submitted coursework from the first three \( M^2 \) mathematics courses. I wanted to personally observe Mrs. Anderson and Mrs. Zander as they engaged in solving mathematics problems. Thus, I organized a problem solving session\(^{14} \) in May 2008. The data I collected in this live session doing mathematics helped me triangulate themes that emerged from analyzing their artifacts of

\(^{14}\) There are more details regarding the problem solving session in chapter 6.
doing mathematics. I collaborated with instructors of the first three M\textsuperscript{2} mathematics courses to identify a set of math problems to ask the participants to solve during this problem solving session (see Appendix F for the problems). These problems were appropriate to use, as they were similar to problems the participants originally solved during the three mathematics courses they had taken six to twelve months earlier.

I held a three-hour problem solving session on 5/22/2008. I first conducted an informal focus group interview. The purpose of this interview was to learn how M\textsuperscript{2} experiences had influenced the participants’ learning and teaching of mathematics. I then observed the participants as they worked on the math problems listed in Appendix F. The purpose of the observation was to observe first hand how each participant interacted with the mathematics and with each other. Before the math work session ended, I allowed the participants to reflect on this experience doing mathematics. I audiotaped and videotaped the informal interview, the teachers’ work on math problems, and their reflections on the experience.

**Data Analysis**

“There is no particular moment when data analysis begins. Analysis is a matter of giving meaning to first impressions as well as to final compilations” (Stake, 1995, p. 71). My analysis began as I looked at data to select the participants for this study and continued until the time I wrote the thick, rich description for this case. I also kept in mind that qualitative research should not be viewed as a recipe (Stake, 1995). “Each researcher needs through experience and reflection to find the forms of analysis that work for him or her” (p. 77). I relied on outside sources to assist me, including reading other
case studies (e.g., Smith, 2008; Wilson, 1990) as well as talking to experienced researchers about how to approach data analysis and conversing with a mathematician to verify my own mathematical thinking and reasoning. The lens through which I analyzed my data focused on understanding teachers’ understanding of mathematics and the process they use to come to understand.

The major components of the data analysis process included: (a) using open and axial coding to analyze non-math related data (e.g., application; teacher survey; interview) as well as to analyze participants’ mathematics work from the first three mathematics courses (Creswell, 2006; Strauss & Corbin, 1990), (b) analysis of mathematical content and written communication of that content based on a possible solution to the math problem posed, and (c) using a frequency count to identify teachers’ mathematical practices and habits for doing mathematics. The following sections describe, in further detail, the data analysis process.

**Open and Axial Coding**

Stake (1995) describes data analysis as the process of breaking apart the data and then putting it back together in a more meaningful way. My interpretation of Mrs. Zander and Mrs. Anderson’s non-mathematical and mathematical work came from looking for common themes across the data. By doing this, I was able discover the uniqueness of each participant, thus supporting my research question related to wanting to understand differences across participants. I could then compare individual participants’ learning of mathematical content, and ways of learning, noting similarities and differences. I used coding to categorize data (Stake, 1995; Merriam, 1998; Creswell, 2006).
I first used open coding (Creswell, 2006). I knew my research questions led to several pre-determined codes that I could use (Stake, 1995), including attitudes, beliefs, and dispositions towards mathematics. However, I chose to define and use codes that emerged from the data itself (Stake, 1995). After I transcribed individual and focus group interviews, and collected artifacts, including course assignments, teacher reflections, and written communication on the online platform for the third mathematics course, I assigned a code that essentially represented what was going on in each line or paragraph of data. My goal with open coding was simply to see what came out of the data without trying to make connections (Stake, 1995). I then turned to axial coding, a process in which I clustered the categories identified through the open coding process (Strauss & Corbin, 1990).

I began with Mrs. Zander’s responses to the initial beliefs inventory and teacher survey,\textsuperscript{15} I first used open coding to assign a brief description. I then grouped those descriptions into common themes. Rather than repeat this process with the Mrs. Anderson’s data, I continued my focus on data associated with Mrs. Zander. Considering my role as a researcher, I recognized the difficulty in separating the complexities of examining two participants at the same time. Therefore I kept my focus on Mrs. Zander. I used open coding with the transcript of my initial interview with her, documents from email correspondence, and all non-coursework artifacts collected by the project. I then

\textsuperscript{15} The beliefs inventory and teacher survey were questionnaires given to participants upon selection to Math in the Middle but before participation in the first course, and annually thereafter.
used the axial codes from the beliefs inventory and teacher survey to create even broader codes of Mrs. Zander’s non-coursework data (see Appendix G for an excerpt).

Next, I shifted my attention to Mrs. Zander’s mathematics work from the first three mathematics courses. Several themes emerged from open coding, including the use of patterns, making connections, and working independently. After the initial coding process, I then attempted to combine codes into broader categories (i.e., axial coding). Finally, I sorted the individual pieces of evidence found under each theme chronologically by course (see Appendix H for an excerpt).

I then repeated the process with Mrs. Anderson’s data. I attempted to analyze this data with a fresh set of eyes; I started from scratch and did not bring in any codes from my analysis of Mrs. Zander’s data. Thus, I wanted to limit the influence of my analysis of Mrs. Zander’s data on my analysis of any other data. However, I did follow the same coding procedures. I also followed these coding procedures when I analyzed data from the problem solving session.

I considered validation an important procedure to follow during my analysis of the data. “All researchers recognize the need not only for being accurate in measuring things but logical in interpreting the meaning of those measurements” (Stake, 1995, p. 108). My intention was to give an accurate description of the case and issues related to my research questions. I implemented two methods to help triangulate my findings. First, I used data source triangulation (Stake, 1995). By design, I was able to look across multiple data sources to triangulate my findings.
A second method of triangulation I used was member checking (Merriam, 1998; Stake, 1995). I asked the participants of this study to check my work. I shared my preliminary analysis of their learning of mathematics. Out of this member check, I learned more about the participants that contributed to background information in chapter 4 and the analysis of how they learned mathematics in chapter 6. I asked each teacher to review my preliminary analysis for accuracy.\textsuperscript{16} I was sensitive to their responses; I edited sections that Mrs. Zander and Mrs. Anderson considered as “alternative language or interpretation” (Stake, 1995, p. 115).

**Mathematical Content and Communication**

To help answer the main research question posed for this study, I took an in-depth look at the mathematics perspective of the solutions submitted by each participant. I first used CBMS (2001) recommendations to identify important topics for participants to understand. I then identified a typical problem from each of the three mathematics courses that represented those important topics. Before taking a closer look at the participants’ solutions, I wrote a possible solution for each of the three problems myself. I then worked in conjunction with a mathematician to analyze strengths and weaknesses of each participant’s solution and their mathematical communication of the solutions.

**Ways of Doing Mathematics**

To help answer the main research question posed for this study, I also took an in-depth look at the participants’ ways of doing mathematics. I compared categories

\textsuperscript{16} Chapter 5 contains an analysis of the teachers’ mathematical work from the perspective of what specific mathematics they learned and how they communicated it. A mathematician reviewed the preliminary analysis for accuracy.
generated from open and axial coding in this study with categories of doing mathematics as presented by others, including the NCTM’s (2000) process standards (e.g., representations, communication, and connections) and Cuoco, Goldenberg, and Mark’s (1996) mathematical habits of mind (e.g., being a pattern sniffer, experimenter, and describer). I applied Rolle’s (2008) method of identifying frequency of practices to identify which ways of doing mathematics were most visible and frequent in Mrs. Zander’s and Mrs. Anderson’s written mathematics work and live interactions during the problem solving session. Identifying practices and their frequencies enabled me to address all three sub-questions for this study.

**Limitations**

No research can be completely objective (Creswell, 1998). My dual role as a M² project staff member and researcher may have blurred my vision to stay completely objective. If I had entered this study with no prior experience with the professional development, I likely would have collected the same data and used the same process of analysis. Did the fact I knew that participants would be exposed to M²’s working definition of habits of mind of a mathematical thinker influence my interpretation of the data? Or did the fact that I had prior exposure to most institute courses as well as learners from earlier cohorts influence what I did or found in this study? Each of these questions may point to strengths of my dual role rather than limitations. I was able to look at the mathematical learning of two teachers from the point of view of an insider as well as the perspective of an outside researcher.
I often took a mental step back and examined my work with a critical lens focused on researcher bias. I understood that it would be impossible to paint value-free descriptions and interpretations; yet this is the nature of qualitative research. “Phenomena need accurate description, but even observation interpretation of those phenomena will be shaped by the mood, the experience, and the intention of the researcher. Some of these wrappings can be shucked, but some cannot” (Stake, 1995, p. 95). I approached this study as objectively as I could.

Following each non-videotaped observation, I met with the teacher to debrief and get the teacher’s perspective of the day’s lesson. I did not want my own pre-conceived notions about how one should teach mathematics to interfere with what really went on. When I could not stay to debrief in person, I sent an email to each teacher, asking her to clarify specific episodes I observed during that lesson. I wanted my lens of observation to be as clear from personal bias as possible and informed as much as possible by the teachers themselves.

I centered my analysis on just two teachers to offer a rich description of the ways that teachers do mathematics. It was important to me to offer vivid images of these two teachers. The goal of this case study was to understand the mathematical learning of these two teachers. My purpose was not to generalize. In fact, generalization demands a different form of inquiry.

The next three chapters collectively represent the results of this case study. Chapter 4 offers background information needed to better understand the images of two teachers’ learning of mathematical content and their ways of doing mathematics. Chapter
5 focuses on each teacher’s mathematical learning from the first three mathematics courses of the institute. Chapter 6 then focuses on the ways these teachers learned the mathematics. In other words, chapter 5 represents the content of these teachers’ learning whereas chapter 6 represents the processes these two teachers used as they learned.
Chapter 4: Background Information Relevant to this Case Study

This chapter discusses three areas of background information to help provide context for chapters five and six. First, the chapter provides information about the courses and structure of the professional development program that provided opportunities for learning for the subjects of this study. Second, it provides background information on the two middle level teachers featured in this study. Finally, it addresses my role as a Math in the Middle staff member.

The Math in the Middle Institute Partnership

An overarching goal of the M² Institute, a collection of twelve mathematics and pedagogy courses,¹⁷ is to offer participants a coherent program of study to deepen their mathematical knowledge for teaching and to develop their leadership skills. Instructors designed the mathematics courses to help teachers learn and deepen mathematical content knowledge, communication skills, and habits of mind. To place Linda Anderson and Becki Zander’s learning in context, it is important to discuss both the mathematical content and structure of the first three mathematics courses. The first two courses were summer courses while the third course was an academic year course, held during the fall semester.

The First Two Mathematics Courses

The first two courses were titled Mathematics as a Second Language (MSL) and Functions, Algebra and Geometry for Middle-Level Teachers (FAGMLT), respectively. In the MSL course, instructors focused on foundational topics from arithmetic, algebra,

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¹⁷ See Appendix I for a full description of the twelve courses.
and geometry. Instructors helped participants strengthen skills in reasoning and proof, problem solving, communication, and making connections among areas of mathematics. The term “habits of mind” was first introduced during MSL. In the second course, FAGMLT, instructors helped participants deepen their conceptual understanding of the algebraic concepts introduced during the first course. Special attention was given to functions, measurement, and geometric modeling in algebra.

MSL was a one-week course offered in June while FAGMLT was a two-week course offered in July. FAGMLT was also paired with Curriculum Inquiry. Participants focused on the mathematics course in the morning and the pedagogy course in the afternoon for two consecutive weeks. Classes met each day from 8-5, with homework assigned each night.

By design, each mathematics course was led by an instructional team consisting of mathematics faculty and graduate students as well as a master K-12 teacher. There were usually five instructors available to help thirty-five participants learn mathematics. Collaboration was a theme consistent throughout the twenty-five month professional development program. From day one, $M^2$ staff encouraged participants to collaborate in mathematics learning teams to learn mathematics alongside their fellow $M^2$ peers as well to assist one another in examining instructional and assessment practices. Course instructors incorporated group activities during class time, often found in the course binder,$^{18}$ and encouraged participants to work homework problems together in the binder.

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$^{18}$ Participants were given a large three-ring binder on the first day of each Math in the Middle course. Binders contained participant information, the course syllabus, expectations, and notes and handouts related to the course material. (Some courses, but
evenings. Most participants, except those lived in Lincoln, roomed in University housing during the summer sessions.\textsuperscript{19} Many who stayed formed nightly study groups to work on the homework problems with each other.\textsuperscript{20} At least one instructor was usually present during the homework sessions.

The nightly homework problems were consistent with the type of problems participants worked on during the day with the exception of several habits of mind (HOM) problems. These HOM problems were more challenging, typically required more time to solve than the other homework problems, and often were not directly related to mathematical topics studied during the day. Instructors asked participants to put forth a good effort toward completing each nightly assignment; however no more than three to four hours. The instructors used stars to mark the most essential problems, or the ones participants should first complete. All solutions to the homework problems were considered drafts. Participants met together in small groups the following morning, led by an instructor, to discuss solutions. Participants then submitted rough drafts to get feedback from the teaching assistants. Some participants were then asked to revise one or more of their solutions.

Instructors asked some participants to present solutions in front of the whole group following the morning homework group meetings. Participants typically displayed their solution on large paper or under a document camera. Instructors encouraged the

\textsuperscript{19} Linda Anderson stayed on campus during the summer sessions while Becki Zander commuted daily.
\textsuperscript{20} Linda Anderson was present at all nightly study sessions while Becki Zander only attended some of the nightly sessions.
presenters to explain the process they used to solve the problems. Sometimes, other participants volunteered alternative methods for solving the problem. Showing multiple ways of solving a problem was another theme consistent throughout the Institute.

At the end of each of the first two mathematics courses, instructors assigned an end of course set of problems (i.e., EOC). These EOC problems represented a cross-section of the mathematics covered during the course. Instructors considered the solutions that participants submitted to be in final draft form. While participants could collaborate on the EOC, each participant submitted his or her own set of solutions. Instructors also asked participants to select a few of the notable nightly homework problems. These were known as the “Favorite Five,” from MSL, and the “Sample Six,” from FAGMLT. Participants were expected to submit initial as well as final drafts for each of these solutions in addition to a brief reflection addressing why the participants liked those particular problems. Instructors also asked participants to write a general, one or two page reflection addressing the course as a whole. Instructors gave participants several weeks following the conclusion of a summer course to complete the end of course assignment.

The Third Mathematics Course

The third course, Experimentation, Conjecture and Reasoning (ECR), was offered during the fall semester. The purpose of this course was to help participants further develop problem solving, reasoning and proof, and communication skills. ECR was a hybrid course, as instructors introduced many of the mathematical topics during an on-
campus, kick-off weekend and then had participants deepen their understanding of those topics as part of the distance learning portion of the course.

Many structures of this academic year course were similar to the structures of the summer courses. First, mathematics faculty and graduate students worked together as an instructional team. Second, instructors encouraged participants to collaborate with one another to learn mathematics. Third, participants’ submissions for individual assignments were considered drafts with opportunities for revision. And fourth, the instructors considered the participants’ solutions to the end of course problem set, representative of the type of problems assigned during the weekly assignments, a final draft.

Due to the nature of an academic year course, there were also differences between the structure of the ECR and the structure of the two summer courses. Two differences are important to discuss with respect to this study. First, participants were not just learning mathematics, as they did during the summer. Participants were also full-time teachers working during the academic year. Instead of learning mathematics during an intensive one or two week summer course setting, participants learned mathematics slowly over several months. Instructors assigned a new set of problems every couple of weeks. Second, distance separated most participants from each other as well as from the instructors. Instructors did not have the luxury of a captive audience following the kick-off weekend, as they introduced participants to the big ideas of the course. The participants also did not have the luxury of having their peers physically present to collaborate with, both during class and the nightly homework sessions. Much of the
instruction for the course was not via classroom instruction. Rather, participants used a
textbook, online resources, and assignments with posed questions to learn mathematics.

To address the obstacle of distance for this distance-learning course, ECR instructors provided an electronic support system for participants to use. Participants could collaborate with one another using an online learning environment (i.e., Blackboard) and conferencing system (i.e., Adobe Connect, previously called Macromedia Breeze). Participants were able to post questions or tentative solutions to electronic discussion boards for instructors, graduate students, and peers within the course to see. Instructors set up weekly meeting times when small groups of participants could log in to Adobe Connect at the same time and have a virtual face-to-face, voice-to-voice homework session. Instructors could answer questions and provide additional instruction in a real-time setting. Naturally, many of the Lincoln teachers did not collaborate using the online learning environment. They could meet face-to-face, in the same room, as they lived a few short miles from one another. This collaboration was very similar to the ways of the summer session. Staff members periodically attended some of these sessions, offering help on the spot.

Participants

A second area of background information relevant to this study is related to the M² participants featured in this study. There were a total of 156 teachers spread across six cohorts of M² Institutes. The teachers came from schools located across the state of Nebraska. There was mix of both rural and city teachers. Some teachers were relatively
new to the profession, with only a couple of years of experience, while others had nearly thirty years of experience.

About 20% of the teachers entered the Institute having taught high school mathematics and had an extensive background of the middle level mathematics teachers should know, while about 50% came in having taught only K-6 mathematics and with very little exposure to middle level mathematics teachers should know. Still others entered knowing the mathematics but not in necessarily deep ways. Thus, some participants entered with a weak mathematical background. Course instructors recognized this fact. However, the instructors had to start somewhere, even though they knew some participants would have gaps in background understanding. As instructors noticed holes in participants’ background knowledge, they helped those participants strengthen that missing understanding.

The next part of this chapter examines the backgrounds of the two participants for this study, Linda Anderson and Becki Zander. Offering an overview of each teacher’s background provides a context for the analysis, findings, and discussion that follow in the remaining chapters.

**Linda Anderson’s Background**

Linda Anderson is a rural educator. The community where she teaches is small, small enough such that everybody seems to know everybody else. The principal describes the school as the “hub of the community.” In an age full of hardships with respect to school funding, the principal states this particular school is well off. The make-up of the school includes few minority students and few students who qualify for the free and
reduced lunch program. Approximately thirty teachers serve three hundred students. “Our teachers show interest in what the kids are doing, not only in the classroom, but in activities, church, 4-H, and all those things” (Principal Interview, 3/25/2008). It appears that teachers make personal connections with students, both during and after school hours.

Mrs. Anderson and the principal both shared that parental involvement and support are two bragging rights for the patrons of the school (Initial Interview, 2/22/2008; Principal Interview, 3/25/2008). Many in the community, including over half of the current staff, attended this school themselves. The school has many traditions, and community members want the traditions to continue (Initial Interview, 2/22/2008). The teaching staff includes a mix of young and old with an average tenure of roughly eighteen years. “The young teachers like to try new things whereas the veteran teachers like to do things the ways they have always done them” (Principal Interview, 3/25/2008). The principal thinks his teachers enjoy teaching at his school. He spoke of Mrs. Anderson:

If I would have had a teacher like her in the fifth grade, I would have been more interested in math and I would have been a better math student...She keeps things lively. You can see learning taking place when you walk into her classroom…it’s fascinating what she does. (Principal Interview)

Mrs. Anderson first taught in a neighboring school, which was even smaller than the one she now teaches in. Concerned about job security, she began to inquire about a teaching position in her own community. “I just approached the superintendent at a ball game and said I have a vested interest in [this community]. If a position would ever come
up, would you please inform me?” (Initial Interview). That following year, the superintendent hired her to work in a small grade school located a few miles from the main school. Ironically, this was the same grade school she attended herself. Eight years later, she accepted a position to teach fifth grade in the main K-12 school building. Again, this was the same building where she had attended high school. At the time of this study, she had held the fifth grade position for six years.

**Mrs. Anderson: Personal Learning Experiences**

Reflecting upon her own learning experiences, Mrs. Anderson remembers little about the content of the math courses she took. She did take the highest math course the high school offered at the time. Mrs. Anderson’s mathematics experience growing up left her nothing to uncover or discover. She merely was given all of the mathematics needed to successfully pass the class. She listened to lecture after lecture, watched her teachers work sample problems, and then worked independently on practice problems, which was usually followed by a test. Despite those passive learning experiences, math emerged as one of her favorite subjects. Much of the credit for that was due to the teacher. He was a first year teacher during Mrs. Anderson’s sophomore year in the late seventies.

What I remember most was it seemed like he realized this was a rural community and lot of us were farm kids. I can remember him relating a lot of the math to actual things we did on the farm. For example, fencing was perimeter…he made it real to us…He wanted us to understand…He also was our volleyball coach so many times we had to stay after school. We
would work on math with him. He would take the time to work with us on math...He made a big difference. (Initial Interview, 2/22/2008)

The experiences Mrs. Anderson had in her own high school setting made an impact on her and set the stage for the teacher she would become many years later.

Because of her choice to study elementary education in college, Mrs. Anderson had few experiences learning mathematics once she left high school. She openly described herself as having limited mathematical knowledge as she only took one basic math class at UNL followed by one math methods course. During her teaching career, Mrs. Anderson participated in just three workshops focused on mathematics, including a Teaching Algebraic Thinking workshop and an AIMS workshop. One could say that the mathematics knowledge Mrs. Anderson called upon in her work with fifth graders was primarily based on the knowledge she acquired as a K-12 student and little more.

**Mrs. Anderson: A Passion to Work with Children**

Mrs. Anderson chose to pursue education, as it was one career that would complement her role outside of the classroom. She speaks of her desire to work with children. She lights up, telling about both past and present students. She is willing to go above and beyond for her fifth graders.

In school on Monday, Jonathan came to me and asked about my geese. He persisted and later asked if he could have some. I said he could but his folks would have to agree because they can really tear up a yard. Before he left my class, he asked me if I would charge him for them. I said no. Tuesday, he came to me first thing in the morning and said his folks told
him he could have the geese. I said OKAY as soon as I can catch them.

Well at 9:00 that night my phone was ringing. It was Jonathan asking if I had caught any geese. I hung up the phone and my husband and I were on the hunt. I took a pair over to his house that night. Now he makes sure to let me know how they are doing every week. (Email Correspondence, 4/13/2008)

Mrs. Anderson’s passion for children starts with her own family. “My first priority is family” (Initial Interview, 2/22/2008). Yet family almost dissuaded her decision to apply to Math in the Middle in the first place. She did not want to sacrifice her role at home due to rigors of the M² Institute.

Mrs. Anderson’s professional life was already quite full. She was part of the school improvement steering committee, crisis team, curriculum committee, textbook adoption committee, and novice teacher mentor program (Initial Interview, 2/22/2008). She was also busy at home. She was busy with her own children. Finally, she was busy taking care of herself. She walked several miles every day, read and completed crossword puzzles. Mrs. Anderson admitted reservation in sending her completed application to M² (Initial Interview); however, she hoped she could do it. She had always challenged herself in the past. She was ready to accept a new challenge.

Mrs. Anderson: A Desire to Make Changes

Prior to starting the Math in the Middle Institute, Mrs. Anderson was struggling with her beliefs about teaching and learning (Belief Inventory, Spring 2007; Teacher
There was a sense of conflict in many areas of her beliefs. Ms. May have come at exactly the right time for her. She had few professional development experiences focused on mathematics in the past, and even fewer would be classified as reform-based. She did not know what she really believed about what was best in the classroom (Beliefs Inventory, Spring 2007). She suggested she wanted to give her students every opportunity to develop and refine their knowledge of basic skills (Teacher Survey, Spring 2007). She was not sure as to how much students should be expected to figure out mathematics on their own. She was unsure about the level of direct instruction she should give students with respect to problem solving. She expressed conflict with having students work together too much. While she loved her basal textbooks, as she had grown up with basals and again experienced them in college, she was beginning to question the role of her textbook. Mrs. Anderson was also unsure of her own mathematics background (Beliefs Inventory, Spring 2007). Thus, Mrs. Anderson likely entered the institute with more questions about teaching and learning than she had answers for.

**Becki Zander’s Background**

Becki Zander grew up in a city of approximately 20,000 people, a large city by Nebraska standards. She recalled being a successful math student in her elementary, middle, and high school days. She completed all of the mathematics courses her high school had to offer, including calculus. However, she always had to work hard to

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21 The Beliefs Inventory and Teacher Survey were two instruments all M2 participants were asked to fill out. Participants filled out these Likert-scale (i.e., ranging from strongly agree to strongly disagree) surveys related to statements of belief before the first day of the institute. The analysis in this section was based on my understanding of the Likert-scale responses for each statement that Mrs. Anderson and Mrs. Zander completed.
understand and admitted that she really began to struggle towards the end. “I was the struggler at the bottom just doggie paddling to stay alive. I wanted to achieve, and even though I wasn’t the brightest one, I was still trying to hang in there” (Initial Interview, 2/22/2008).

Mrs. Zander: Personal Learning Experiences

Mrs. Zander described her learning experiences as rather routine with instructors often teaching her tricks, or fast methods, to use to solve problems. “There was a lot of practice and memorization with very little problem-based learning…there were very few group opportunities and even fewer opportunities to communicate understanding verbally or in writing” (FAGMLT, Week 2 Reflection, 7/27/2007). She remembers doing well on elementary timed tests, learning to “plug and chug” her way through formulas in middle school, and having, as she described, that “annoying experience” of taking geometry from the high school football coach, “who taught just enough to get by” (Initial Interview, 2/22/2008).

Mrs. Zander’s favorite subject was science. She completed the honors level of biology, chemistry and physics in high school.

My [science] teachers were award winning. My science background is pretty strong from high school…I do have a love for science. In education classes [in college], you had to write and talk about a former teacher you had, and the biology teacher is the one I’ve referenced numerous times. I learned about teaching from him in addition to the subject matter. (Initial Interview, 2/22/2008)
Mrs. Zander gives many accolades to her high school science teacher. “He just really had high expectations. I think that was a big part of it. He met you at the door; he made relationships. He helped you gain confidence…he believed in you…he invested in you” (Initial Interview). Even today, Mrs. Zander’s own beliefs as a teacher echo her past experience with him. “Confidence and attitude toward math certainly play a role in success as a teacher and as a student” (Initial Interview). Mrs. Zander, too, puts energy into building relationships.

I believe a teacher’s responsibility in education is immeasurable. Being enthusiastic about teaching…taking advantage of teachable moments and the development of students’ curiosity, encouraging students to set goals…and being positive role models…Creating a sense of community while genuinely caring enough to get to know the students proves to be influential and lasting. (Curriculum Inquiry, Final Curriculum Paper, August 2007)

Mrs. Zander has taken her passion for science and reached out to gifted students. She has taught enrichment science courses during the summer as well as coordinated school science fairs. Even in her first year of teaching, she was not afraid to incorporate lab work, and she still continues to ground her science instruction with labs. Mrs. Zander felt prepared to teach as soon as she entered the profession.

**Mrs. Zander: The Path to Her Current Teaching Assignment**

Mrs. Zander ruled out a career in education when she started college. “My dad had always [told me] to become a teacher because you could go anywhere where your
husband went” (Initial Interview, 2/22/2008). She vividly remembers being upset with her father telling him, “I don’t think so; I’m not going to become a teacher” (Initial Interview). Mrs. Zander headed off to UNL to major in Psychology. However, after an inspirational experience with an education professor, Mrs. Zander became hooked on education. “When I took that class with [her], I was like ‘OK, I’m going to change my major.’ That was a really tough phone call to make [to my dad] and say, ‘You were right’” (Initial Interview). Mrs. Zander had a wonderful experience while at UNL and still has fond memories of her math methods course. “The text used for that course is one of the only college texts I still use and have on my bookshelf today” (MSL, Week 1 Reflection, 6/22/2007).

Mrs. Zander graduated from UNL with an elementary education endorsement. During her first nine years of teaching, Mrs. Zander taught in five different buildings. Mrs. Zander accepted a sixth grade position at her current school, a middle school in a large district, after eight years into her teaching career.

Mrs. Zander has been heavily involved in district activities. From attending many workshops and serving on committees to tutoring and being a district mentor for teachers new to the profession, Mrs. Zander has been very visible in the district. Further, she has opened herself up to leadership positions.

[Leadership] wasn’t always a passion for me...When people come near me, I don’t want them to think that I’m going to come and shove it down their throat. But I am here if they need me, and they can ask me questions. (Initial Interview, 2/22/2008)
Mrs. Zander has served as a teacher leader for both mathematics and science. “I’ve enjoyed the opportunity to collaborate with leaders from other buildings and further discuss issues relative to the curriculum” (Application, Spring 2007). She appreciates the opportunity she has had to personally offer feedback to the district specialists as well as to the teachers in her own building. “This has given me the forum to share mathematical issues with my peers within my building as well” (Application).

**Mrs. Zander: Involvement in Professional Development**

One may classify Mrs. Zander as a lifelong learner. She has never stopped in her quest for improving her capacities as a teacher. She has participated in countless professional development workshops and meetings, including Six-trait writing, instructional practices for high ability learners, and implementation of new math and reading curricula. She has completed professional development courses centered on other topics, such as Love and Logic Discipline and Character Counts. While many of these other experiences have not been directly related to mathematics, she has also sought out opportunities to strengthen herself as a mathematics teacher. “I have taken everything that has been offered…I have been just eating it up” (Initial Interview, 2/22/2008). Most recently, Mrs. Zander completed two formal mathematics courses, one focused on number and operations and the other on algebra.

Much of Mrs. Zander’s fuel to continually pursue professional development in the area of mathematics has been the perceived impact on teaching. Her learning experiences in the before-mentioned formal mathematics courses gave her the confidence to begin to make changes in her teaching.
I started to encourage students to share their math thinking with the class and have a “group problem-solving” mentality…I always wanted to talk to my peers about math as a young student, but it was discouraged. [My recent professional development] instructors solidified my belief that it’s helpful to talk about math with your peers and to learn together in teams rather than as an individual. (MSL, End of Course Reflection, 6/23/2007)

Mrs. Zander had tapped into virtually any and all opportunities to improve herself as a teacher outside the realm of beginning a graduate study in education or mathematics. “I have taken advantage of the summer mathematical classes offered by [my school district], but there are no additional classes to be offered at this time” (Application, Spring 2007). This void in offerings combined with the positive experience she has had with the summer mathematics courses played a key role in her decision to apply to Math in the Middle.

Mrs. Zander’s multiple professional development opportunities have been credited, in part, to teaching in a larger school district with many resources. The district opportunities as well as other formal and informal activities, including a focus on the teaching and learning of mathematics, are directly linked to teaching in the large school district. While Mrs. Zander has sought after and taken advantage of numerous district opportunities, she also works alongside colleagues who teach the same subjects and share a common plan time. Mrs. Zander has been able to hold frequent informal discussions with her colleagues on teaching and learning of mathematics. She collaborates often with her colleagues. Recently Mrs. Zander took part in a district wide initiative aimed on
increasing mathematical knowledge and pedagogy for sixth grade teachers. “[They] led us teachers, by modeling math problems that stretched our thinking, our comfort zone, and how it connected to previous experiences” (Teacher Survey, Spring 2007).

Mrs. Zander does, however, recognize the need to deepen her understanding of mathematics that underscores each objective she teaches.

Students that I teach ask questions about the mathematical background of the curriculum I teach and I want to be able to provide them an accurate, thorough and clear explanation. As I recently taught multiplying decimals by decimals, I knew they would ask questions about how their product was smaller than the original decimals being multiplied. I was able to give them the big picture of why to my best ability, but I want to have all the means and most up-to-date research and hands-on explorations so I can confidently teach this skill. (Application, Spring 2007)

Mrs. Zander admits she has weaknesses in her understanding of mathematics. Thus, one reason she pursued Math in the Middle was to deepen her understanding of mathematics.

**Mrs. Zander: Beliefs about Teaching and Learning Mathematics**

Mrs. Zander does not solely view, or use, the textbook as the only resource in her classroom. “The chapter resource book has adequate practice pages in addition to the textbook. These two materials are where I pull the majority of my materials” (Curriculum Inquiry, Final Curriculum Paper, August 2007). Mrs. Zander utilizes many other resources as part of her lesson delivery, ranging from the internet, other media, activities gleaned from formal and informal collaboration, and just her own creativity. This has
been consistent throughout her career. The textbook she used in her first teaching assignment looks very similar to the one she teaches with today. During those first two years, she viewed the text only as a resource, supplementing with ideas. Some of these ideas are rooted in her own college experiences, including the methods class of which she spoke highly.

In a survey completed just prior to starting M², Mrs. Zander humbly described herself as a confident teacher. She felt strongest with her own mathematical knowledge, at the level she teaches as well as beyond, and her leadership ability. She also felt somewhat confident with her capacity to mentor, coach, or simply offer support to both inexperienced and experienced teachers in terms of mathematics content and pedagogy. “I try to show students the mathematical application to our world and get them to think of themselves as talented in math” (Application, Spring 2007). Having taught for more than a decade, Mrs. Zander indicated she was prepared to deal with any and all aspects of teaching mathematics. In terms of specific content strands of mathematics, Mrs. Zander identified that she gave moderate emphasis to computation and using algorithms as well as communication, connections, and reasoning & proof while teaching sixth grade math. On the other hand, she admitted that she struggled some with implementing problem solving, multiple representations, and manipulatives into her teaching.

In the past, Mrs. Zander viewed herself as an outlier. Mrs. Zander often described how her teaching did not always look like other’s teaching.

I’ve always been someone who needs to talk things out. That’s my personality. So our classroom is always louder than everyone else’s. I
always felt guilty for that. I think people have expectations and that’s not my style. When I have a sub, they want it quiet. It’s tough…but those relationships and building community in a classroom…then you can build from that. (Initial Interview, 2/22/2008). 

She talks about how she enjoys collaboration; yet also experiences tension with that as well. She does not want to feel pressured to use someone else’s method, or even follow someone else’s timeline.

One of the major ideas that emerged from looking at Mrs. Zander’s responses to the Beliefs Inventory and Teacher Survey was that conceptual learning is very important to her. She claimed teachers should use real-world contexts to introduce mathematics to students. Further, she believed a problem solving based curriculum would best help students gain a conceptual understanding. She also believed that there are different approaches to solve mathematics problems and students can find and use those different approaches. While she claimed she wants to instruct students in a way for them to discover those approaches on their own, she was unsure how much teacher-led instruction should be given to students to ensure they correctly learn different approaches. A few other ideas emerged while looking through her responses as well: the textbook is not the only resource available to a teacher, students should have opportunities to work with one another in the mathematics classroom, and a teacher must focus on more than just a student’s answer to assess understanding.
Mrs. Zander: The Decision to Apply to Math in the Middle

An indirect result of Mrs. Zander’s prior experiences in professional development is that she has felt more and more validated as to the decisions she makes in her classroom. “[When I started taking those summer classes…] I could see that it was a green light for talking in math…that it’s OK…It’s not cheating. Collaboration is good. That is where it started. I started immediately after the summers doing that” (Initial Interview, 2/22/2008). Applying for Math in the Middle became the next logical step in her professional life.

Mrs. Zander knew about M² from the start, as she knew participants in a previous cohort.

I want to further my education in the realm of math so I cannot only improve my teaching and students’ achievement, but I can also share the knowledge with my peers as a leader. The Math in the Middle graduate program is exactly what I need to accomplish my goals and do what’s best for middle level students. (Application, Spring 2007)

Mrs. Zander also recognized that participation in M² would offer her a once in a career opportunity.

It is really important that they are paying for this. It makes it more feasible. It’s hard to afford going to school and I don’t know if I would have done it until my children were out of school or I was better able to subsidize my own education. (Initial Interview, 2/22/2008)

Mrs. Zander applied for and was subsequently accepted into Math in the Middle.
My Dual Role

A final piece of background information important to this study is my role with Math in the Middle. I was not just a researcher of Mrs. Anderson and Mrs. Zander’s learning of mathematics. I actually began working for Math in the Middle prior to the start of the first cohort. My role in the professional development program varied from year to year. My formal and informal roles included: master teacher, graduate assistant, research assistant, learning team facilitator, technology advisor, assessment coordinator, and simply participant advocate.

Before the start of this study, I knew about Math in the Middle. I knew how the classes were run. I knew about the nightly homework sessions and the academic year online learning environments. I got to know many of the teachers before I selected participants for this study. I was even a MSL “homework group” leader and a graduate assistant for the Curriculum Inquiry course, which ran parallel to the FAGMLT course. Due to my role, I frequently communicated with most participants in person or via technology before, during and after the data collection phases of this study as part of my work for the project. Therefore, I interacted with Mrs. Zander and Mrs. Anderson.

I attempted to separate my role as researcher from my role as a Math in the Middle staff member. Once I selected teachers as research subjects for this study, I stopped interacting directly with Mrs. Anderson and Mrs. Zander as I performed my daily functions for Math in the Middle. I worked with other participants. Yet, I used my knowledge of the ways of the Institute to help inform the methods of data collection and analysis for this study. For example, I knew about the nature of the data that I did not
personally collect and had a sense of how I might use it. Since I had personally organized all research data for each cohort, I knew what was available and how I could easily access it. I also had an understanding of the purpose or intent of particular kinds of data (i.e., the Beliefs Inventory and the Teacher Survey). Thus, I tried to use my involvement in Math in the Middle as a means to better study two middle-level teachers’ attempts to learn important mathematics.
Chapter 5: The Mathematical Learning of Two Teachers

Mrs. Anderson and Mrs. Zander were provided opportunities to learn and demonstrate understanding of mathematics as part of their professional development coursework in the Math in the Middle Institute. The following analysis focuses on their learning of mathematics during the first three mathematics courses of the program. I review problems and accompanying work each teacher submitted as homework or end-of-course problem sets. In this chapter, I focus on just one problem from each course. I classify the problems as typical, as they represent part of the basic content covered in each course. Further, the concepts showcased by these three problems are central to the middle level mathematics curriculum and represent important mathematical ideas CBMS (2001) and NCTM (2000; 2006) publications call for middle level teachers to understand. The solutions are merely representative, not exhaustive, of mathematical work submitted by each of these two teachers.

The First Math Course: A Learning Experience Involving Temperature Conversion

Instructors for Mathematics as a Second Language (MSL) encouraged participants to make mathematical connections, both within and across the topics of arithmetic, algebra, and geometry. As part of their algebra experience, participants were expected to deepen their understanding of linear relationships, including rates of change, graphs, and inverses. The first problem I examine in depth is taken from MSL and is situated under the umbrella of linear functions in the context of a problem on temperature conversion (see Figure 4).
The mathematical concepts that make up the Temperature Conversion problem are important for middle level teachers to understand. The CBMS (2001) recommends that middle grades teachers develop the capacity to recognize connections between algebraic models and physical situations, understand linear patterns of change and their

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**Temperature Conversion**

The Fahrenheit (F) and Celsius (C) temperature scales are directly proportional in the sense that each degree F corresponds to a number of degrees C, and vise versa.

(a) Given that 0ºC corresponds to 32ºF and 100ºC corresponds to 212ºF, what is the rate at which a Fahrenheit temperature changes with respect to the corresponding Celsius temperature?

(b) Draw the graph that gives the geometric picture for the temperature conversion (Fahrenheit represented vertically, Celsius represented horizontally).

(c) What is the slope of the graph? What is the significance of where the graph cuts the vertical axis? The horizontal axis?

(d) Write a formula that converts Celsius to Fahrenheit temperatures.

(e) Next do parts (a) through (d) with the roles of Fahrenheit and Celsius reversed. Now Celsius temperature is represented vertically and Fahrenheit temperature is represented horizontally.

(f) Describe these relationships in terms of inverse processes.

*Figure 4. A Temperature Conversion Problem Assigned on Day 4 of MSL*
inverses, and relate tabular, symbolic, and graphical representations of linear functions. Likewise, the NCTM (2000; 2006) points to the need for middle grades teachers to help students learn to use linear functions, develop an understanding of slope, and become flexible in use of the varied representations of a linear function. A solution for Temperature Conversion can be found in Appendix J.

Mrs. Anderson’s Solution to the Temperature Conversion Problem

Mrs. Anderson obtained correct equations relating Fahrenheit and Celsius; yet she provided limited written documentation to explain how she arrived at the formulas (see the Celsius to Fahrenheit conversion equation, offset by stars, in Figure 5).

Mrs. Anderson correctly computed the slope and built a table. Assuming she knew a linear equation could be obtained from the slope and the y-intercept, her work is then understandable. Participants were taught in this first mathematics course the connection

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24 See Appendix K for Mrs. Anderson’s complete solution to the temperature conversion problem.
between the $y$-intercept and the $b$ value of a linear equation written in the form $y=mx+b$. It is reasonable to conclude Mrs. Anderson made this connection in her work.

Mrs. Anderson’s work in Figure 5 includes the equation $F^o = xC^o + 32$. While her units were incorrectly written (i.e., $F^o = xC^o + 32$ should have been $^oF = x^oC + 32$), one should keep in mind this was overnight homework. Thus, one ought to be lenient with respect to this kind of error. Figure 5 also reveals that Mrs. Anderson checked her work. She wrote $59-32=27$ and $\frac{27}{15} = \frac{9}{5}$ immediately below her conversion equation. She may have used this specific case to verify her equation. While checking a case is an unnecessary step to provide reasoning in this problem, textbook authors often encourage students to check work at the end of a problem (e.g., Bellman et al., 2009). Thus, Mrs. Anderson’s use of a check may not be out of the ordinary for a practicing teacher who is trying to deepen her own understanding of mathematics and is not completely confident in her own solutions (Knuth, 2002).

Mrs. Anderson’s support behind her answer for finding the rate of change is satisfactory; given this is a draft of a solution. She wrote her answer to part (a) on page one in the small space next to the question (see page 1 of Appendix K) and showed her work on page two (refer back to Figure 5). Mrs. Anderson began by setting up a table of temperature values, with $0^oC$ and $32^oF$ as the first entry. The table continued with entries that increased by $5^oC$ and $9^oF$ respectively. The ten-entry table stopped with $45^oC$ and $113^oF$. Mrs. Anderson could not have completed this table until she recognized the rate of $5^oC$ per $9^oF$. Mrs. Anderson’s work on the right hand side of Figure 5 showed that she
may have determined this rate of change when she found the formula for converting
Celsius to Fahrenheit temperatures.

Mrs. Anderson created an accurate geometric picture on page three of her work
(refer to page 3 of Appendix K). The graph she drew is detailed. She placed each
temperature scale on the appropriate axis. She used her table of temperature values to
help graph the line, as the graph includes highlighted coordinates (i.e., exaggerated
points) that correspond with the values in her table. Mrs. Anderson then connected these
points with a line segment. Between (15, 59) and (20, 68) she added a horizontal and
vertical segment to represent the movement, which is the slope or rise over run. She even
wrote “slope = \( \frac{\text{rise}}{\text{run}} = \frac{9}{5} = 1.8 \)” below her line as well as the equation
\( F = \frac{9}{5}C° + 32 \) above the line. Mrs. Anderson also wrote, “where water freezes” at the y-intercept. This fact is
actually one of the answers for part (c). These additional pieces of information
demonstrate her ability to make connections while solving this problem.

Mrs. Anderson’s answers for part (c) are brief. She answered the first question by
writing “slope = \( \frac{\text{rise}}{\text{run}} = \frac{9}{5} \)” on page one next to the question itself as well as on the graph.
She indicated the significance of the y-intercept on the graph (i.e., “where water
freezes”). However, Mrs. Anderson did not comment on the significance of where the
graph cuts the horizontal axis. As indicated earlier, Mrs. Anderson included the formula
for Celsius to Fahrenheit conversion in her work for part (a) and on her graph. She also
put the formula on page one, next to problem (d).
Part (e) of the Temperature Conversion problem prompted participants to rework the problem with the temperature roles reversed. Thus, there was a considerable amount of work needed for this part alone. Much of Mrs. Anderson’s reasoning for part (e) looks similar to her reasoning for parts (a) through (d) (see Figure 6).

![Figure 6. A Portion of Mrs. Anderson’s Work Obtaining a Formula Converting Celsius to Fahrenheit Temperatures](image)

Mrs. Anderson reversed the temperature values in this table. She wrote the equation  
\[ C^\circ = \frac{x}{9}(F^\circ - 32) \]  
similar to the way she wrote  
\[ F^\circ = \frac{x}{9}(C^\circ + 32) \]  
next to her formula to justify why \( \frac{5}{9} \) was the appropriate rate for this part of the problem.

Just like in her work for part (a), Mrs. Anderson used a specific case to check her work. She substituted 77 for the Fahrenheit temperature and found 25 as the corresponding Celsius temperature value. Mrs. Anderson created an inverse graph for this part on a new sheet of graph paper (see page 4 of Appendix K). She used the table values from Figure 6 to plot coordinates and then graph a line. One area of interest is that Mrs. Anderson noted that the two temperature scales would be equal at -40º on this graph. However, she did not offer any written explanation as to how she came up with this information.
Figure 7 depicts a paragraph Mrs. Anderson wrote as part of her part (e) solution.

Despite the “e),” the sentences clearly reference her first graph (refer to page 3 of Appendix K). An interesting observation in this paragraph is Mrs. Anderson’s use of the word “after.” On a graph, “after” typically means moving to the right along the horizontal axis. Mrs. Anderson, however, suggested that “after that point” the temperatures are negative. She viewed “after” as moving to the left. She probably understood the point she was trying to address; however, the way she communicated her understanding lacked clarity.

Part (f) was the final part of this multi-part problem. Mrs. Anderson stated the correct answer next to the original prompt on page one. She documented her reasoning using a representation (see Figure 8).
Mrs. Anderson’s work included a representation of an input-output machine, a representation common in algebra textbooks (e.g., Bellman et al., 2009). This was consistent with a representation she saw M² instructors use during class time. Mrs. Anderson correctly demonstrated that a 10°C input had to be multiplied by \( \frac{9}{5} \) and then increased by 32 to get a 50°F output. She included arrows moving from left to right. Directly below, she reversed the arrows and demonstrated the inverse process. Mrs. Anderson’s use of a specific case, and moving forwards and backwards through an input-output machine, offers a visual representation of the way she understood this part of the problem.

**Reflections on Mrs. Anderson’s Solution to the Temperature Conversion Problem**

There are strengths and limitations of Mrs. Anderson’s solution for this problem. While her solution and reasoning is correct for the most part, Mrs. Anderson’s communication stops short of all it can be. Her answers appear to be written simply for her instructor to see. Learners often first focus their efforts at communication to an
authority figure (e.g., instructor), trying to provide the information one thinks that
authority figure wants to see—just the facts with no elaboration on their reasoning and
proof. Only later do learners learn to communicate to a peer, someone who may not fully
understand the mathematics involved. Eventually, mathematics teachers, such as Mrs.
Anderson, learn to communicate to their students.

There is a difference between needing to improve one’s understanding of
mathematical concepts and needing to improve one’s ability to communicate
mathematically. This solution indicates Mrs. Anderson understood the concepts; however
she needed to embrace the task of communication. She just did not write very much.
Ironically, the part of the problem where she included the most detail (i.e., Figure 7) was
actually the part that she said reflected somewhat of a discomfort with the math involved.

One of the long-range goals of the Math in the Middle Institute was to help
teachers learn to communicate mathematical understanding for themselves and for others.
One should keep in mind that this Temperature Conversion homework problem was
assigned during the first course. Yes, Mrs. Anderson’s communication fell short of the
long-term goals of the program; however, the essence of her understanding of slope,
graphing lines, finding linear equations, and finding inverses was evident.

Another observation is Mrs. Anderson’s use of an input-output diagram. Mrs.
Anderson used this representation to help her illustrate inverse processes. The input-
output diagram, or machine (Bellman et al., 2009), helps a reader see the process of
temperature being converted. One can visualize placing an input value into a machine
where some transformation(s) occur(s). One can then visualize the transformed value
coming out of the machine. Mrs. Anderson’s use of this representation may signify that she is a visual learner or that she tends to utilize structures she saw as part of class discussions in her own solutions.

**Mrs. Zander’s Solution to the Temperature Conversion Problem**

Analysis of Mrs. Zander’s work on the Temperature Conversion problem revealed she is very shaky with respect to many of the mathematical concepts needed to solve this problem. Most of her answers were correct; yet the reasoning she provided indicated gaps in her understanding. And as was the case in Mrs. Anderson’s work, Mrs. Zander’s communication fell short of what instructors might hope to see. There were times it was necessary to interpret what she was trying to communicate. For example to solve part (a), Mrs. Zander began by writing each corresponding pair of temperature values as fractions (see Figure 9 on the next page). The first fraction, \( \frac{0^\circ C}{32^\circ C} \), has no meaning. One must question why the fraction \( \frac{0}{32} \) did not bother Mrs. Zander, as the value is zero.

Nonetheless, Mrs. Zander was attempting to associate temperature values with the variables, \( x \) and \( y \); however, using a fraction for such an association is a poor representation. She would have communicated this much more clearly by using ordered pairs: \((0^\circ C, 32^\circ F)\) and \((100^\circ C, 212^\circ F)\).

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25 See Appendix L for Mrs. Zander's complete solution to the temperature conversion problem.
In retrospect, Mrs. Zander’s imprecise language of trying to set up a rate with the given temperature values likely came from her limited background knowledge of rates and ratios. Course instructors recognized many participants entered the program with a weak background in mathematics. Instructors gradually identified and then helped participants fill in holes in their background knowledge.

Mrs. Zander did correctly find the change in Fahrenheit and Celsius temperatures and then used those differences to write the correct rate of change for the problem, \( \frac{180}{100} \), which she then reduced to \( \frac{9}{5} \). Mrs. Zander attached the adjective-noun theme (Gross & Gross, 2005), an idea first introduced by the instructors of the MSL course, to her rate by associating the adjective with \( \frac{9}{5} \) and the noun with “F degrees per C degrees.” She
defined slope with a series of equivalent rates: \( \frac{\text{rise}}{\text{run}} \), \( \frac{180\,^\circ\text{F}}{100\,^\circ\text{C}} \), \( \frac{18}{10} \), and \( \frac{9}{5} \) F degrees per C degrees. Her use of the adjective-noun association and the series of equivalent rates indicate Mrs. Zander was able to make several connections as she solved this problem. Moschkovich (1998) found learners’ connections among concepts help to deepen understanding of each concept.

Mrs. Zander provided a geometric picture for the linear relation represented by temperature conversion by drawing a line on a coordinate plane (refer to pages 3, 4, and 5 of Appendix L). She correctly graphed her line by connecting the two coordinates, (0, 32) and (100, 212). This was an efficient manner to produce the geometric picture. (Recall Mrs. Anderson plotted several exaggerated points from her table of values to determine the line.) Unlike earlier in the problem, Mrs. Zander made the connection that corresponding temperatures could be written as coordinates in the form of \((x_1, y_1)\), with Celsius linked to the \(x\)-coordinate and Fahrenheit linked to the \(y\)-coordinate.

Mrs. Zander connected the coordinates with a dashed line, used to indicate a discrete function. Since temperature conversion is continuous, it would have been more appropriate to use a solid line. However, using a dashed line is a social convention and a small number of elementary and middle level teachers receive instruction related to discrete lines. Therefore, it is understandable that Mrs. Zander used a dashed line. Also, most of the examples she had seen up to this point in the program were discrete by nature (e.g., the cost of developing rolls of film). Thus her use of a dashed line matched the instruction she received in class. Temperature Conversion was one of the first continuous functions that participants saw.
Figure 10 shows several connecting pieces of information Mrs. Zander included on her graph, specifically a pair of rise over run arrows, the statement of “the stairsteps of slope—up 9 and over 5,” and the equation $F = \frac{9}{5}C + 32$.

These pieces of information provide additional evidence for the notion that Mrs. Zander was able to make connections as she solved this problem.

Mrs. Zander provided the correct slope for part (c) of her solution. Figure 11 shows how she misinterpreted the “significance” part of the question, addressing how she would move the graph vertically instead of addressing the freezing point of water.
Mrs. Zander suggested that a line that did not start at 32 would pass through the origin (0,0) of the coordinate plane. She attempted to address a transformational significance of the y-intercept, instead of a physical significance of water freezing. Her approach stemmed back to what she learned in class, where instructors first introduced conversion using a much simpler context, feet and inches (MSL Course Notebook, 2007). In that graph, and in many other graphs participants saw during class, the line passed through (0,0). Mrs. Anderson was attempting to connect what she was learning in class to this new situation, albeit incorrectly. She was trying to give the authority what she thought “they” wanted.

Mrs. Zander provided a correct formula for converting Celsius to Fahrenheit temperatures in part (d). However, her work for part (e), reversing the roles of Fahrenheit and Celsius, was not correct (see Figure 12).

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26 The two sentences in Figure 11 read, “It starts at 32 on the vertical axis and on 0 horizontally. If it started at (0,0), the point of origin, it would only be showing slope without consideration of adding 32°F.”
Similar to part (a), Mrs. Zander used a fraction to link corresponding temperatures. Again, one should question why $\frac{32}{0}$ did not bother her, as it is undefined. She would have strengthened her work by using ordered pairs to connect temperature values as described earlier. She did correctly reverse the role of the variables (e.g., 32 was now associated with $x$); however, she incorrectly used the slope formula. She should have used $\frac{y_2 - y_1}{x_2 - x_1}$. Figure 12 shows her use of $\frac{x_2 - x_1}{y_1 - y_2}$ to yield an incorrect slope $\frac{-9}{-5}$. Her answer was the same value as $\frac{9}{5}$, the slope from the first part of the problem, instead of $\frac{5}{9}$. Mrs. Zander did not recognize this error.

Mrs. Zander’s graph of the line for part (e) was correct. She placed this graph on the same coordinate plane as she had used earlier in the problem (refer to pages 3, 4, and 5 in Appendix L). Mrs. Zander again connected her two given coordinates with a dashed line. She drew slope arrows and labeled $\frac{-9}{-5}$, to move down 9 and over to the left 5.
However, the slope arrows were merely a representation, as they did not actually move down 9 units or over to the left 5 units. Mrs. Zander became very mechanical here. She did not think about the relationship of her calculated slope \( \left( \frac{-9}{5} \right) \) and the actual slope of her line. This independent view of the concepts involved in this problem indicated Mrs. Zander was not seeing coherence in the mathematics. While her work revealed she made some connections as she solved the problem, there were also connections missing.

Figure 13 provides Mrs. Zander’s written explanation for her take on inverses in this problem.

![Figure 13. Mrs. Zander’s Explanation for Part (f) of the Temperature Conversion Problem]

Analysis reveals Mrs. Zander had an incomplete understanding of inverses. One must be mindful, however, this work was only a draft assigned as part of a nightly assignment during an intense week of mathematics. The paragraph she wrote for part f) related to the
errors she made in part e). The final sentence, however, showed that she was beginning to work through her errors.

Taking a step back, Mrs. Zander understood a portion of the larger concept of inverses. She knew that she needed to reverse the roles of her variables. She knew how to graph the inverse. Her error came from using the wrong slope formula. She likely used her work from Figure 9 to help generate the inverse slope. Figure 14 shows how Mrs. Zander reversed the “horizontal” aspect of her differences, rather than the “vertical” aspect.

\[
\begin{align*}
\left( \frac{x_1}{y_1} \right)_{0^\circ C} & \to \left( \frac{x_2}{y_2} \right)_{100^\circ C} \\
212 - 32 &= 180^\circ F \\
(y_2 - y_1) &\quad 100 - 0 = 100^\circ C \\
(x_2 - x_1) &\quad \text{So slope} = \frac{\text{rise}}{\text{run}}
\end{align*}
\]

**Figure 14. Comparing a Portion of Mrs. Zander’s Work Found in Figures 9 and 12**

The “rise” in the original slope should have become the “run” in the inverse slope.

Mrs. Zander wrote the correct inverse formula, \( C = \frac{5}{9}(F - 32) \), at the bottom of page one of her solution. A peer likely encouraged her to write down this formula since she wrote, “I’m still unsure why it wouldn’t have been \( \frac{5}{9} \) rather than \( \frac{-9}{-5} \)” in her paragraph for part (f). Mrs. Zander tried to justify why the slope was \( \frac{5}{9} \) instead \( \frac{-9}{-5} \). She indicated that the needed to invert the numbers (refer back to Figure 13). This supports
the hypothesis of her failure to reverse roles of “rise” and “run” when finding inverse slope.

**Reflections on Mrs. Zander’s Solution to the Temperature Conversion Problem**

Analysis of Mrs. Zander’s solution indicates she understands portions of the Temperature Conversion problem. However, the larger issues raised by her work on the problem are not so much related to understanding the mathematics needed to do this particular problem; the issues appear to stem back to the background knowledge Mrs. Zander needs to work the problem comfortably. For example, she did not recognize the errors in representing temperatures as fractions, especially when one fraction was an undefined value. This points to her incomplete understanding of rates and ratios, knowledge that is important as one approaches a linear equation problem like this one, and a lack of understanding her instructors identified and later individually addressed with Mrs. Zander and some others with similar limits to their understanding.

Mrs. Zander’s communication was not geared for her peers, let alone her students. Her communication was geared for her instructors, as she provided limited written communication of her reasoning. Yet, the communication Mrs. Zander did provide (i.e., a sentence or two in a couple of places) helped reveal her thinking as she solved parts of this problem and why she did what she did. Her written communication in Figure 13 revealed an incomplete understanding of inverses. Research discussing learners’ incomplete mathematical understanding is not new (e.g., RAND Mathematics Study Panel, 2003; Knuth, Stephens, McNeil, & Alibali, 2006). Some researchers (e.g., Sfard, 2001) have used communication as a means to uncover learners’ mathematical
understanding and misconceptions. Just as interviews of students explaining their work helped De Bock et al. (2002) and Erlwanger’s (1973) studies expose students’ misconceptions, analysis of written work in this study helped expose Mrs. Zander’s lack of understanding of inverses.

Another observation that stands out is the number of connections that were transparent in Mrs. Zander’s work on just one problem. For example, she made many connections within the concept of slope as part of her work for this problem. For example, she specifically identified the “stairsteps of slope” on her graph on page three of Appendix L. She also used small arrows to show the movement from one point to the next. Further, Mrs. Zander gave a series of equivalent descriptions for slope in her work on page one. A second connection was her reference to the Adjective-Noun theme, a key idea from the first course. She was able to connect the slope value of \( \frac{9}{5} \) to the adjective and the label of °F per °C to the noun. Lobato, Ellis, and Munoz (2003) discuss struggles many learners face when trying to make such linear connections. The connections Mrs. Zander made point to a more generalized understanding of the concept of slope.

**The Second Math Course: A Learning Experience Involving Scale Factor**

Instructors for Functions, Algebra, and Geometry for Middle Level Teachers (FAGMLT) designed the second M² course in part to help participants build upon the arithmetic, algebra, and geometry connections that were made during the first course. More focus was placed on geometry during the second course. The second problem I examined in depth is taken from FAGMLT, is situated within the larger topic of
similarity, and requires participants to communicate their understanding of a learning experience involving scale factor of similar geometric figures (see Figure 15).

**Scale Factor**

ABCD and PQRS are similar polygons whose perimeters are 40 inches and 30 30 inches, respectively. The area enclosed by ABCD is 8 square inches.

(a) What area is enclosed by PQRS?

(b) Is it possible for the straight-line distance from point A to point C to be 20 inches?

*Figure 15. A Scale Factor Problem* 27 Assigned on 7/26/2007 as Part of FAGMLT

The concepts found in this Scale Factor problem are important for teachers to know and understand. The CBMS (2001) recommends middle level teachers understand proportional reasoning, similarity, and various techniques for measuring area. The NCTM (2000; 2006) calls for middle grades teachers to help students understand relationships between scale factor and corresponding lengths, areas, and volumes of similar figures. The CBMS (2001) calls for middle level teachers to become “familiar with the role of axioms, theorems, and proofs in the geometry curriculum of the secondary school” (p. 113). This particular problem also exposes teachers’ depth of understanding of the triangle inequality theorem, a fundamental Euclidean concept included in a typical high school geometry textbook (e.g., Larson et al., 2011). A solution for the Scale Factor problem can be found in Appendix M.

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27 Part (b) is not related to scale factor.
Mrs. Anderson’s Solution to the Scale Factor Problem

Figure 16 shows Mrs. Anderson’s written attempt to find the area of polygon PQRS.

<table>
<thead>
<tr>
<th>ABCD</th>
<th>40</th>
<th>[4.50^2]</th>
</tr>
</thead>
<tbody>
<tr>
<td>PQRS</td>
<td>30</td>
<td>[4.50^2]</td>
</tr>
</tbody>
</table>

ABC and PQRS are similar.

a. The area \( \frac{40}{30} = \frac{4}{3} \) scale is 4:3

\[
S^2 = \frac{4^2}{3} \quad 4.5^2 = \frac{16}{3} \quad 8 + \frac{16}{3} \quad \frac{4}{3} \times \frac{4}{3} = \frac{9}{2} \quad 4.5
\]

We are looking for the area of the smaller polygon PQRS. We found the scale factor by proportion of the perimeters \( \frac{40}{30} = \frac{4}{3} \). The scale is 4:3 or \( \frac{4}{3} \). Since we are looking for area, we squared the scale; \( \left(\frac{4}{3}\right)^2 = 1.67 \).

Then we took the proportion of \( 8 + \frac{16}{3} = \frac{8}{3} = 4.5 \).

The area of PQRS is \( 4.5^2 \text{m}^2 \).

Figure 16. Mrs. Anderson’s Work for Part (a) of the Scale Factor Problem

Her work is understandable. She listed the equation \( A^2 = s^2 \cdot A_1 \) at the top of her page, likely taken from notes she was given during class. She eventually used this equation to solve for the missing area, although she did not explicitly say that she did use it. Next, Mrs. Anderson organized the given information in a table-like format. Because of the similarity between the polygons, Mrs. Anderson could then use the given perimeters to determine the scale factor of the larger polygon compared to the smaller polygon since the scale factor of the larger to the smaller is 4:3. She then found the

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28 See Appendix N for Mrs. Anderson’s complete solution to the scale factor problem.
square of the scale factor. Finally, she took the area of the larger polygon divided by the square of the scale factor to give the correct answer of 4.5. Thus, she used her original equation $A\Delta_2 = s^2 \cdot A\Delta_1$ to solve this problem (i.e., she divided 8 by $\frac{16}{9}$, a necessary step in solving $A\Delta_2 = s^2 \cdot A\Delta_1$ for $A\Delta_1$).

Mrs. Anderson incorrectly wrote 4.52 as her final answer at the end of her solution. She did, however, have the correct answer of 4.50 in two places, within the paragraph as well as at the top of her paper. Mrs. Anderson’s answer of 4.52 indicates a rounding error when she used her calculator (i.e., $\frac{16}{9} = 1.77$ and $\frac{8}{1.77} = 4.52$). Despite her two conflicting answers, the reasoning Mrs. Anderson used to solve part (a) was correct.

Mrs. Anderson’s solution for the second part of this problem was also reasonable, but again her ability to communicate was not at a level that a peer could pick up her work and completely understand what she did. The general flow of her argument made sense; however, she left out some details that would have strengthened her argument. The first thing Mrs. Anderson did was to list the triangle inequality theorem at the top of her solution (see Figure 17). This is similar to the way that she listed the equation $A\Delta_2 = s^2 \cdot A\Delta_1$ in part (a).
She likely copied the triangle inequality reference directly from the notes given to her during class. Mrs. Anderson argued, “AC cannot be 20 inches because the perimeter of the triangle is less than 40 in.” Mrs. Anderson did not identify which triangle, $\triangle ABC$ or $\triangle ADC$, the term “triangle: referred to. She meant $\triangle ABC$ based on the rest of her paragraph. Her choice to assign $\triangle ABC$ a perimeter “less” than 40 inches changed to a perimeter “equal” to 40 inches in the second sentence. She could have used the phrase “would leave less than 20 inches” rather than “would only leave 20 inches.”

While it is plausible that Mrs. Anderson attempted to prove this problem by contradiction, where one assumes the conclusion in question to be true, it is as likely she proved her answer by merely ruling out the other possibility. She stated how allowing AC to be 20 inches would then rule out the case of the sum of AB and BC to also be 20 inches. She knew that could not happen because of the triangle inequality theorem. The overall logic of her proof was present; however, her ability to communicate this proof could have been stronger. This may not be unexpected, as researchers have shown many
learners struggle with writing proofs (e.g., Recio & Godino, 2001; Silver & Carpenter, 1989; Sowder & Harel, 2003).

One final observation is the note Mrs. Anderson included at the bottom of her solution (see Figure 18).

![Figure 18. Mrs. Anderson’s Citation in the Scale Factor Problem](image)

Mrs. Anderson referenced two inequalities using the words “Ann” and “Class.” “Ann” is likely one of her classmates and “class” is a reminder that Ann helped her work on this problem during class. This would not be unexpected due to the collaborative structure used by instructors throughout the Math in the Middle Institute.

**Reflections on Mrs. Anderson’s Solution to the Scale Factor Problem**

Mrs. Anderson’s written communication is stronger for this problem than it was for the Temperature Conversion problem. She did not just list the mathematical symbols, numbers, and formulas that she used to solve the problem. Mrs. Anderson also included written explanations documenting more of “what” she did and “why” she did it. She provided a better attempt at offering a solution that a peer could understand. However, she did not include all necessary information as part of her written reasoning and proof. For example, she did not indicate which triangle had a perimeter less than 40 inches in part (b). Communicating precision and clarity of mathematics is a learned behavior. Math in the Middle instructors were helping participants develop this skill in each mathematics.
course. While this study was, in no way, designed to show cause and effect; it is reasonable to note that in this solution from the second course, Mrs. Anderson offered more precision and clarity in what she was writing, as compared to her solution for the Temperature Conversion problem from the first course.

Mrs. Anderson’s work revealed that she understood the mathematical concepts required to solve the Scale Factor problem. She found the scale factor by using the perimeters of the polygons. She incorporated the square of the scale factor as part of the work for part (a). She also used the equation $A\Delta_2 = s^2 \cdot A\Delta_1$ to find the unknown area whereas the solution in the back used proportions to find the unknown area. For the second part, both solutions incorporated the triangle inequality theorem.

Analysis of Mrs. Anderson’s work on the Scale Factor problem reveals two final observations. First, there are three specific places where Mrs. Anderson listed an important piece of information (see Figure 19).

![Figure 19. Mrs. Anderson’s Three References in the Scale Factor Problem](image)

She listed the equation $A\Delta_2 = s^2 \cdot A\Delta_1$ in the middle of the top margin. While she did not reference it in her writing, she incorporated it into her work for part (a). Next, she listed a
textbook-like definition of the triangle inequality theorem at the beginning of the second part of her solution. She did reference this theorem in the final sentence of her argument. Finally, she listed a pair of inequalities, based on the triangle inequality theorem, at the bottom part of her paper. Again, she did not explicitly incorporate these into her solution.

Analysis of the larger body of Mrs. Anderson’s work revealed she often cited the resources that aided her as she learned mathematics in the professional development program (e.g., FAGMLT, End of Course Reflection, 7/28/2007; ECR, End of Course Problem #1, 12/22/2007). To strengthen her communication, Mrs. Anderson could be more explicit in connecting such information to her solutions.

The other observation is Mrs. Anderson used the word “we” as she wrote out her solution (refer back to Figure 16). Mrs. Anderson does not indicate with whom she worked on this problem. It is likely she collaborated with fellow Math in the Middle peers on this problem; however, other work indicates she collaborated with family members or even colleagues with whom she teaches. For example, Mrs. Anderson wrote at the bottom of a solution during the third course, “There is no way I could come up with this next part. This was thanks to [a peer’s] help and then further clarification by [a colleague of mine]” (ECR, End of Course Problem #1, 12/22/2007). While this points to her use of collaboration, her dependence on others is yet another example that highlights a need to strengthen her own communication skills.
Mrs. Zander’s Solution to the Scale Factor Problem\textsuperscript{29}

Mrs. Zander also submitted reasonable solutions for both parts of this problem. For part (a), she took time to carefully draw out the two diagrams and restate all of the given information. Figure 20 shows Mrs. Zander’s work when finding the area of polygon PQRS using the given perimeters of each.

![Figure 20. Mrs. Zander’s Work for Finding the Area of PQRS](image)

At the bottom of this page of work, Mrs. Zander accurately found the area of polygon PQRS by taking the area of polygon ABCD multiplied by the square of the scale factor. Her work revealed an answer of 4.5 in\(^2\); yet she put 6 in\(^2\) in her box. Her work above this answer in the box was not accurate. She wrote, “We take ‘s,’ scale factor, and multiply it by the area, 8 in\(^2\), of the polygon.

\[ S^2 = 8 \times \left( \frac{3}{4} \right)^2 = 8 \times \frac{9}{16} = 4.5 \]

\[ \text{Area of PQRS} = 6 \text{ in}^2 \]

\textsuperscript{29} See Appendix O for Mrs. Zander's complete solution to the scale factor problem.
by the area, 8 in\(^2\), of the polygon.” In other words, she found the product of \(\frac{3}{4}\) and 8, and then put 6 in\(^2\) in her answer box. Clearly, Mrs. Zander did not take into consideration the square of the scale factor in this part of her work. This indicates that she did not fully understand the problem in her first attempt. She came back, possibly during class the next morning, got help from a peer or an instructor, and then revised her work.

On the second page of her work, Mrs. Zander indicated that she later realized the scale factor must be squared when finding area (see Figure 21).

![Figure 21. Mrs. Zander’s Reference to Her Notes from Class](image)

She referenced a page out of her notes. Her use of the word “also” indicates she found this information in her notes after a peer or instructor had helped her revise her solution on the first page. Mrs. Zander included three examples using the fraction \(\frac{1}{2}\) in the right hand side of Figure 21. This suggests her preference for concrete examples. Mrs. Zander also made connections between the type of geometry measurement and the exponent on the label for that particular measurement. This is consistent with connections that she included in her solution to the Temperature Conversion problem.
Mrs. Zander began her solution for the second part of this problem by again restating the problem, sketching a diagram for polygon ABCD, and writing the perimeter equation for polygon ABCD. She then immediately stated that sides $\overline{AD}$ and $\overline{DC}$ could not be less than 20 (see Figure 22).

**Figure 22. Mrs. Zander’s Argument for Part (b) of the Scale Factor Problem**

Mrs. Zander did not use the words, “sum of sides,” but that is what she meant. She did not offer any reasoning behind this first statement. Instead, she merely stated, “we know.” Mathematically, this language suggests $\overline{AD} + \overline{BC} < 20$ was a given inequality in the original problem statement, which it was not. Interestingly, Mrs. Zander did not even incorporate $\overline{AD}$ or $\overline{DC}$ into the rest of her argument. This is an inconsistency in her work.

Stepping back, it is clear that Mrs. Zander struggled writing the proof for this problem. The writing of proof is a challenge many learners face (Recio & Godino, 2001; Sowder & Harel, 2003). Mrs. Zander’s proof is on the right track; yet there are assumptions within her argument that are not sufficiently identified and clarified. In the
main portion of her argument, she focused solely on \( \Delta ABC \) and incorporated the triangle inequality theorem, which she noted at the side of her paper. Similar to Mrs. Anderson, Mrs. Zander did not explicitly use the words in her written argument. Mrs. Zander also used an argument similar to Mrs. Anderson’s argument; they both ruled out other possibilities. While her proof does not reflect a polished solution and has room for improvement, it is also evident that Mrs. Zander did understand some of the mathematics that was required to write the proof.

**Reflections on Mrs. Zander’s Solution to the Scale Factor Problem**

Mrs. Zander’s solution to this Scale Factor problem is stronger than her solution to the Temperature Conversion, as her work for the problem assigned in the first course indicated that she had an incomplete understanding of finding an inverse of a given linear equation, among other concepts. Analysis of Mrs. Zander’s solution for the Scale Factor problem revealed she had a reasonable understanding of the concepts involved, even though she needed help to revise her first attempt at solving part (a).

Mrs. Zander also included more written explanation as part of this solution as compared to the solution for the first problem. Her writing gave insight into why she put an incorrect answer in the box for the first part of this problem (refer back to Figure 20). Her sentence before the box revealed why she put an answer of 6 in\(^2\) in the box. Yet, her work below the box was correct and supported her correct answer of 4.5 in\(^2\). Without any additional explanation, one would not know whether or not Mrs. Zander understood the concept in this problem. This again points to the importance of having students communicate their thinking (e.g., Sfard 2001). Similar to Mrs. Anderson’s
communication for the second problem, Mrs. Zander’s communication revealed a solution that a peer could pick up and understand rather than just an instructor.

Similar to her work on the first problem, Mrs. Zander documented several connections she made as she solved the problem. She recognized that if polygon PQR had a perimeter that was \( \frac{3}{4} \) of polygon ABCD, then polygon ABCD had a perimeter that was \( \frac{4}{3} \) of polygon PQR. She also connected the exponent used on the scale factor with the exponent used on units for perimeter, area, and volume. These connections are strengths of her solution as they represent types of connections the CBMS (2000) recommends for middle grades teachers to be able to make.

The Third Math Course: A Learning Experience Involving Probability

Instructors for Experimentation, Conjecture, Reasoning (ECR) provided numerous opportunities for middle level mathematics teachers to increase capacity for mathematical problem solving, reasoning and proof, and communication. Several topics, including two- and three-dimensional geometry, discrete mathematics, and probability thereby set the stage for participants to deepen their ability to understand core mathematics. The third problem I examined in depth is taken from ECR and is situated within the larger topic of probability: You roll a pair of dice 24 times. What is the probability of seeing at least one 11?\(^{30}\) (Problem assigned on 11/26/2007 as part of ECR).

\[^{30}\text{Burger & Starbird, 1999, p. 569 (Problem #37).}\]
The concept in this problem is again considered important for middle level teachers to understand. The CBMS (2001) calls for middle level teachers to be able to calculate and understand probabilities of independent events. The NCTM (2000; 2006) calls for middle grades teachers to help students develop the capacity to use varied representations, including fractions, when giving theoretical probabilities. A solution for the Rolling Dice problem can be found in Appendix P.

**Mrs. Anderson’s Solution to the Rolling Dice Problem**

Mrs. Anderson’s solution is completely understandable for this problem. Her work resembles narrative explanation as opposed to a series of distinct mathematical moves. She began by citing a reading in her textbook and stated how the pages she read assisted her thinking (see Figure 23).

![Table showing sample space of rolling a pair of 6-sided dice](image)

*Figure 23. Mrs. Anderson’s Work for the Probability of Sum of 11 on a Pair of Dice*

This was the first of three references Mrs. Anderson made while writing this solution. She used a table to represent the sample space of rolling a pair of 6-sided dice. She

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31 See Appendix Q for Mrs. Anderson’s complete solution to the rolling dice problem.
clearly identified the two cases with a sum of 11. On the right hand side, Mrs. Anderson noted the probability was 2 out of 36. The first part of this solution was nicely done.

Figure 24 shows that Mrs. Anderson referenced her textbook again as she started to find the probability of not rolling a sum of 11 on a single roll of the dice.

The book also states that it is more helpful sometimes to look at what is not there. In this case, it is more helpful to look at rolling the dice in terms of being successful and not successful. Since there are 24 times to roll the dice, if you roll anywhere from 1 to 24 11s, you are successful. Since there are 36 possible outcomes in rolling 2 dice and 2 times out of the 36 an 11 could be rolled, that is $\frac{2}{36}$ ways _not_ to roll an 11.

**Figure 24. Mrs. Anderson’s Work for the Probability of Sums Not Equal to 11**

This was likely Mrs. Anderson’s way of giving credit to the approach she took to solve the problem. She correctly determined how many ways one cannot roll a sum of 11.

Figure 25 illustrates how she found the probability of not rolling an 11 on any of the 24 rolls by taking that probability $\frac{34}{36}$ to the 24th power.

**Figure 25. Mrs. Anderson’s Final Answer for the Rolling Dice Problem**
Mrs. Anderson must have used her calculator to find a value of .253649. She communicated this was the probability of not rolling an 11 on any of the rolls. She found her final answer by subtracting that decimal from one and multiplying by 100 to convert to percent. She again rounded her final answer. She added a final comment about how an online discussion helped her solve this problem (refer to the final sentence in Figure 25). This was her third reference to how an outside source helped her solve this problem.

**Reflections on Mrs. Anderson’s Solution to the Rolling Dice Problem**

Mrs. Anderson’s answer to the Rolling Dice problem was correct; her supporting documentation was mathematically sound. Her solution nearly matched each aspect of the solution in Appendix P. Mrs. Anderson’s written explanation is a strong part of this solution as she communicated her thinking from start to finish in a clear and reasonable manner. She communicated at a level such that a peer could pick up her work and understand the process. This was in contrast to her written solution for the Temperature Conversion problem.

Mrs. Anderson also continued to cite her use of references to solve problems. Figure 26 shows three different sources of help she referred to in her work on the Rolling Dice problem.

*Figure 26. Three References to Other Sources within Mrs. Anderson’s Work*
She began by referring to her textbook. She again referenced her textbook when she described the approach she chose to take. Finally, she concluded with a statement about how an online discussion aided her understanding. Participants were encouraged to offer justification for work they submitted. In comparing her three references from the Scale Factor problem (refer back to Figure 19 when she listed an equation, the triangle inequality theorem, and referenced Ann and Class) with three references above in the Rolling Dice problem (refer back to Figure 26), Mrs. Anderson likely interpreted justification as showing work and citing who or what (i.e., textbook) helped her solve the problem.

**Mrs. Zander’s Solution to the Rolling Dice Problem**

Mrs. Zander also submitted an accurate solution for the probability problem. She began by stating the total number of outcomes from rolling a pair of six-sided dice as well as the two outcomes that yield a sum of eleven. Figure 27 shows the probability she found as well as a reference, a page from the textbook used in the course.

![Figure 27. Mrs. Zander’s Probability of Rolling a Sum of 11 with Textbook Reference](image)

At this point, Mrs. Zander had reasonably documented the correct probability of rolling a sum of eleven on any roll of a pair of six-sided dice. She wrote this probability in fraction

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32 See Appendix R for Mrs. Zander's complete solution to the rolling dice problem.
form, which she then reduced to simplest form. Following this, Mrs. Zander then listed all 36 outcomes from rolling two six-sided dice, circling the two outcomes with a sum of eleven (see Figure 28).

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*Figure 28. Mrs. Zander’s Second Method of Finding the Probability of a Sum of 11*

Mrs. Zander did not need to give two approaches for finding the probability of $\frac{2}{36}$. The second approach was rather unsophisticated for the context of this problem. Mrs. Zander did not communicate in her writing why she chose to present the two approaches. Was she unsure of her first answer? Or did she simply want to demonstrate her ability to solve problems in more than one way, a theme many participants seemed to take hold of as part of their Math in the Middle experience?

In contrast with Mrs. Anderson’s work, Mrs. Zander then found the total number of ways of rolling a pair of dice 24 times followed by the total number of ways of not
rolling an eleven on any of the 24 rolls (see Figure 29). Thus, she separated the two parts of a probability ratio.

![Equation](image)

*Figure 29. Mrs. Zander’s Explanation Regarding Total Possibilities and Not Getting a Sum of 11 When Rolling a Pair of Dice 24 Times*

Mrs. Zander wrote down the entire decimal she saw on her calculator. Her decision to separate the two parts of the probability as well to list the unreasonably large numbers actually raise questions about what she does or does not understand about the work to find the probability. She also writes in Figure 29 that “each of the 24 rolls have 34 chances of not rolling an 11.” This is not an accurate statement. Chance is the same as probability. She could have written “each of the 24 rolls have a 34 out of 36 chance of not rolling an 11.” Again, her sentence about chance raises questions about the depth of her understanding.
Mrs. Zander finally combined her two different options (i.e., $34^{24}$ and $36^{24}$) and successfully found the probability of not rolling an eleven on any of 24 rolls by writing the ratio of her findings found in Figure 29. And then similar to Mrs. Anderson’s work, Mrs. Zander found her final answer by subtracting the ratio from one. Figure 30 shows a brief summary Mrs. Zander included to support why she subtracted from one.

This sentence was a good conclusion for a correct answer that was somewhat supported by written explanation.

**Reflections on Mrs. Zander’s Solution to the Rolling Dice Problem**

Mrs. Zander’s solution to the Rolling Dice problem was correct and her reasoning was nearly identical to the reasoning in Appendix P. One would be curious to know why Mrs. Zander chose to list every single outcome (refer back to Figure 28). Although this was a rather unsophisticated method, her decision to show two methods was likely tied to what she was learning as part of Math in the Middle. Figure 31 shows the first “Key Idea” she wrote down on the first day of the institute:
On that first day of the Institute, she learned, or at least was reminded by instructors, that there is more than one way to solve many math problems. Mrs. Zander’s decision to show the probability using two strategies was likely influenced by her experiences learning from the Institute’s instructors.

Mrs. Zander’s written communication was also adequate for this solution, although not to the level of Mrs. Anderson’s written communication. Mrs. Zander’s solution flowed and she documented her thinking from start to finish; yet, her solution to the Scale Factor problem included more written communication. The last sentence was one of the strongest aspects of Mrs. Zander’s communication, as she attempted to explain why she subtracted her decimal from one (refer back to Figure 30). The part of her sentence where she explained what the “1” represented as well as the part where she connected the decimal with the percent (i.e., 0.7464 and 74.64%) further showed that she had a strong understanding of some of the core mathematics involved in this problem.

A Look Across Mrs. Anderson and Mrs. Zander’s Mathematical Solutions

Analysis of Mrs. Anderson and Mrs. Zander’s work prompts two questions. First, did the teachers possess the problem-specific knowledge required to solve each of the particular problems? Analysis of the teachers’ work suggests the answer is yes. The mathematical work Mrs. Anderson and Mrs. Zander submitted revealed a varied depth of knowledge needed to solve these particular problems. For example, Mrs. Anderson had a strong understanding of the concept of inverses for linear functions in the first problem. Her function machine diagram on page one of her temperature conversion solution revealed a sense of how she visualized inverses (refer back to Figure 8). Mrs. Zander had
a strong understanding of the concept of slope in the first problem. Her work revealed an ability to use the formula \( \frac{y_2 - y_1}{x_2 - x_1} \) to find the slope between a pair of corresponding temperature values. She was able to connect the meaning of this slope value with other representations, including \( \frac{\text{Rise}}{\text{Run}} \) and able to connect slope with the adjective-noun theme.

Mrs. Anderson’s work on the second problem pointed to a strong understanding of using one-dimensional measures from similar figures to find an unknown area. She recognized the need to square the scale factor. Mrs. Zander demonstrated she was able to find the correct scale factor between two similar figures. Finally, both teachers’ work for the third problem revealed an understanding of how to calculate a probability of independent events.

Analysis of the teachers’ work also revealed there were times the teachers were missing the background knowledge (De Bock et al., 2002) and/or the mathematical sensibilities (CCSSI, 2010) required to solve a particular problem. For example, Mrs. Zander’s work on the first problem reveals several gaps in her mathematical background, which impede her learning of mathematics in that problem. She did not recognize \( \frac{32}{0} \) was an undefined value; thus, not a good representation to link temperature values.

Likewise, Mrs. Anderson’s incorrect rounding on the second problem led to an answer of 4.52, even though she had correctly documented an answer of 4.50 on the line above. Mrs. Anderson did not recognize the discrepancy.

Analysis of Mrs. Anderson and Mrs. Zander’s work in this chapter also prompts the question, “What is the learner’s understanding with respect to the need to
communicate one’s understanding?” This case study was not necessarily designed to
answer that type of question, as I would have needed to look at the solutions
longitudinally. Nonetheless, it does appear that each teacher was growing in her ability to
communicate mathematically. On one hand, Mrs. Anderson’s communication in her first
solution was directed primarily to her instructor. On the other hand, her third solution was
nearly adequate enough for a peer to pick up the written solution and learn the
mathematics needed to solve the problem. Mrs. Zander’s written communication also
showed improvements, although not to the same levels that Mrs. Anderson’s work
showed.

Conclusion

Mrs. Anderson and Mrs. Zander’s written mathematical work indicated a
deepening understanding of mathematical ideas (CBMS, 2001) and a growing ability to
communicate mathematics to others. While there were still gaps in their understanding,
one must be reminded these were not intended to be polished solutions. The solutions
were two teachers’ attempts to demonstrate understanding, completed in a short periods
of time following intense periods of instruction. Yet the evidence indicates that both Mrs.
Zander and Mrs. Anderson learned many core concepts the CBMS (2001) recommends
for middle level teachers to know. Further, as the two teachers started solving problems
in the institute, their solutions included mainly facts and they did not elaborate on their
reasoning and proof. By the third course, their solutions were more complete; including
expanded written explanation offering both reasoning and adequate proof.
Chapter 6: Patterns in Ways Two Teachers Learn Mathematics

The analysis in chapter 5 focused on Mrs. Anderson and Mrs. Zander’s learning of mathematical knowledge for teaching and their ability to communicate their understandings to others. This chapter shifts the focus to the ways these teachers learn mathematics. I am particularly interested in looking for patterns and exceptions in the ways these teachers engaged in mathematical practices and developed individual understanding of particular mathematical content.

The Problem Solving Session

As described in chapter 3, I designed an experience where I could directly observe Mrs. Anderson and Mrs. Zander in the act of doing mathematics. Although I had access to and examined their written work and reflections submitted from the first three courses of the M^2 institute, these data did not capture the dynamic nature of teachers actually doing mathematics. Thus, I invited both teachers to come to the UNL campus on the evening of 5/22/2008, for a problem solving session. This session occurred at a time when the teachers were completing a pedagogy course. At this point, the teachers had learned no additional math content following the third mathematics course. The three of us met for three hours.

I began the problem solving session with a group interview asking questions to prompt participants’ reflections on their learning experiences from the first three mathematics courses as well as their efforts at translating those learning experiences to their teaching practices. Following this one-hour interview, I asked the two participants to

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33 At this stage, the participants were completing their second pedagogy course and fifth overall course of the Institute.
solve math problems. I recorded field notes as I observed the participants work. Although I had been a homework group leader for these two teachers during the MSL course as well as a graduate assistant for the Curriculum Inquiry course they took, I did not want the teachers to view my role as an instructor during this problem session. I stated to both teachers that I viewed my role as merely an observer. I wanted the teachers to work on the mathematics without any influence of an instructor. Having only two teachers present aided my effort to direct my attention on what they were doing as they solved problems. I videotaped the session to help fill in my notes.

I set aside ninety minutes for teachers to solve problems that looked similar to problems from the first three mathematics courses: three from MSL, two from FAGMLT, and three from ECR. It took longer than I anticipated for the teachers to solve problems; during this session they were able to solve a total of four problems, two from FAGMLT, and one each from MSL and ECR (see Appendix F). This portion of the problem solving session took approximately one hour and forty-five minutes. For the remaining fifteen minutes, I asked teachers to reflect on the problem solving session.

I began analysis for this chapter by examining the teachers’ written work from the first three mathematics courses. As part of this analysis, I identified several themes related to the ways the teachers learned mathematics. From the problem solving session data, I then created representations of live interactions of the two teachers doing mathematics similar to what I saw evidence of in their written work. I used my observations during the problem solving session to triangulate the themes that grew out of my analysis of teachers’ written work. I chose to write two vignettes from the session.
based on my analysis of their mathematics work and written reflections from the first three courses. In particular, I selected the teachers’ work on the second and third problems as the focus of two vignettes from the problem solving session, as their work in those problems best illustrate the patterns in their ways of learning mathematics that emerged from my analysis across all data types.

**Vignette 1: Another Instance of Temperature Conversion**

The Newton-Lewis Temperature problem (see Appendix F) is similar to the Fahrenheit-Celsius temperature discussed in chapter 5 because both problems required participants to find rate of change and then write a formula converting one temperature scale to the other. The Fahrenheit-Celsius problem was first assigned during the first mathematics course (MSL), nearly a full year before this problem solving session. The Newton-Lewis problem was the second problem teachers worked during the problem solving session.

Both teachers read through the problem and begin working independently. After several minutes of silence, neither one is writing. I remind them of the overall goal for this problem. “If I give you a temperature in Newton, you should be able to give me the corresponding temperature in Lewis.” Mrs. Zander immediately speaks.

1  BZ:  If it means anything, it was 5/9 and 9/5 when we were doing conversion on the Celsius and Fahrenheit scales. Remember that?

2  LA:  No, I don’t remember.

3  BZ:  So it could have fractions in it.
The teachers work more on the problem. They converse with one another. I hear discussion about temperature differences of 40 and 33 degrees. Mrs. Zander moves the conversation back to Celsius-Fahrenheit.

4  BZ: I’m just stuck on the fact that the 5/9 and the 9/5 has nothing to do with the difference in that problem. [pause] I know that it’s not just a matter of the difference of 33 or 40 in this problem—so I need to get over this road block—I mean just because it’s a difference of 33 doesn’t mean if you have 15 degrees it will just be 15 plus 33.

Both teachers spend a few moments quietly writing. Mrs. Anderson breaks the silence by restating the math problem.

5  LA: Find a formula to convert from degrees Newton to degrees Lewis.

She looks at what she has been writing and I prompt her to share what she has on her paper so far.

6  LA: Well, I just drew it out.

Figure 32 illustrates Mrs. Anderson’s work.
Mrs. Anderson points to each part of her representation explaining how she found differences of 33 and 40 degrees. Mrs. Zander makes no comment. She is still looking at her own paper. I ask Mrs. Zander to share what she has put down (see Figure 33).

$$\frac{33}{20} = 1.65$$

Figure 33. Mrs. Zander’s Representation of Temperatures

BZ: I was just setting up a proportion and then looking at the difference of freezing from Newton to Lewis and boiling from Newton to Lewis.

Mrs. Anderson looks at Mrs. Zander’s paper as she listens. Mrs. Anderson writes her own proportion after hearing Mrs. Zander talk about a proportion.
8    LA:  So would it be boiling to freezing—that would be 33 to 0 and then boiling to freezing would be 20 to -20.

Mrs. Zander and Mrs. Anderson discuss how their proportions are both similar and different, noting how one wrote that Newton to Newton is proportional to Lewis to Lewis while the other noted that Newton to Lewis is proportional to Newton to Lewis. The room then becomes quiet again as each teacher focuses on her own work. Soon, Mrs. Zander breaks the silence.

9    BZ:  So now I’m thinking what if I play around with numbers. Let’s say I have five degrees Newton. What would that be in Lewis?

For the next few minutes the teachers discuss this question. They are unable to find a corresponding temperature in the Lewis scale. Mrs. Anderson points to something on her own paper (see Figure 34).

10   LA:  Would it help to compare it with zero? I mean zero is in the middle of the Lewis scale. Does that get us anywhere?
Mrs. Zander makes no comment; instead she begins writing two fractions at the top of her paper (see Figure 35). She moves the conversation back to the Celsius-Fahrenheit problem.

\[
\frac{-9}{5} + \frac{5}{9}
\]

*Figure 35. Mrs. Zander’s Second Attempt to Connect Back to Fahrenheit-Celsius*

11 BZ: I’m so stuck on this. One of them is minus nine-fifths and the other is plus five-ninths. I’m thinking we need to do a plus and a minus.

Mrs. Anderson looks at what Mrs. Zander has written (Figure 35). Mrs. Anderson takes her pencil and taps on Mrs. Zander’s fractions.

12 LA: So that nine-fifths and five-ninths; how did they get that?
BZ: It was something like Fahrenheit degrees minus 9/5 and Celsius degrees plus 5/9.

LA: No, how did they come up with the 5/9?

BZ: I don’t know. I don’t want to get us totally stuck on that.

LA: But however they got that is how you would do this.

BZ: So like [pause] whatever freezing minus 33/20 or something like that?

I interrupt and ask what they know about Fahrenheit and Celsius. They immediately recall the freezing and boiling points of each scale.

LA: So how did they get 5/9?

The teachers talk with one another as they try to manipulate the freezing and boiling temperatures for Fahrenheit and Celsius scales to get $\frac{5}{9}$. Mrs. Zander thinks she has figured it out.

BZ: That’s how they got the 9/5. [pause] It’s the difference of 32 and 212 and then of 0 and 100 and then you simplify it.

LA: Wait, wait, wait. I don’t see it.

Mrs. Zander explains in greater detail how she arrived at $\frac{9}{5}$. Following the explanation, Mrs. Anderson states how she wants to put the same information in her own paper. She asks to look at Mrs. Zander’s paper.
LA: I have to look at yours. I have to do it like yours.

Mrs. Anderson turns her own sheet of paper over and begins to copy down Mrs. Zander’s work (see Figure 36.1).

<table>
<thead>
<tr>
<th></th>
<th>Fº</th>
<th>32</th>
<th>212</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cº</td>
<td>0</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

*Figure 36.1. Copied Information from Mrs. Zander’s Paper*

After she copies the work, Mrs. Anderson moves to the right hand side of her paper. She begins to make a second chart (see Figure 36.2).

<table>
<thead>
<tr>
<th></th>
<th>Fº</th>
<th>32</th>
<th>212</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cº</td>
<td>0</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

*Figure 36.2. Mrs. Anderson’s Duplication of the First Representation*

Mrs. Anderson discusses how she wants to do the same thing for this new problem. Her first task is deciding which temperature should go highest on top. Mrs. Zander focuses on her own paper; she tries to find the rate in the new problem. Moments later, Mrs. Zander speaks up.

BZ: So I think it’s that 33/40.

Mrs. Anderson does not respond to Mrs. Zander. Instead, she asks a question.

LA: For this one it is Newton and Lewis. Should we put Newton on top since they said it first?

BZ: Yes, but which to go first? Do you do Newton
minus 40 over 33 or 33 over 40?

25 LA: But wait a minute. Let’s do boiling [pause] no [pause]this [pointing back to the F°-C° table] was boiling and that was freezing.

Mrs. Anderson re-reads the original problem and then completes the second table (see Figure 36.3).

<table>
<thead>
<tr>
<th>F°</th>
<th>32</th>
<th>212</th>
<th>N</th>
<th>0</th>
<th>33</th>
</tr>
</thead>
<tbody>
<tr>
<td>C°</td>
<td>0</td>
<td>100</td>
<td>L</td>
<td>-20</td>
<td>20</td>
</tr>
</tbody>
</table>

*Figure 36.3. Mrs. Anderson’s Identical Representations*

Mrs. Anderson shifts her focus from trying to set up a similar table to trying to find a conversion formula. She asks Mrs. Zander numerous questions. Mrs. Zander repeats several of the things she shared while Mrs. Anderson was setting up her second chart including a repeat of Line 24. They discuss a few options to lead them to a formula that will convert Newton to Lewis. After several minutes, they make little progress. Mrs. Anderson looks at me.

26 LA: If we have it set up the same way, shouldn’t that be the answer?

I do not respond. She looks back at her paper. She puts her pencil tip on her chart (i.e., Figure 36.3).

27 LA: OK. We went Fahrenheit to Celsius and Newton to Lewis. To get the formula, it was just the difference, subtracting, right? So from 0 to 33
would be 33 (referencing Newton-Lewis) and then you added to get from 0 to 100 (referencing Fahrenheit-Celsius)

BZ: Oh, oh. Now I got it. (Nodding) Yeah, you’re right. But the problem is, do you do N minus 33/40 equals L or N plus 33/40 equals L? (pause) Wait you multiply N times 33/40?

Mrs. Anderson and Mrs. Zander check a specific case to test a formula (i.e., $L = N \cdot \frac{33}{40}$). They are unsuccessful. I interrupt them and suggest they first check a specific case within the Fahrenheit-Celsius context to be sure they can demonstrate a correct formula there. The teachers work together without any success. At one point, they insert 212°F in their formula and get 117°C. They laugh because they are seventeen off the correct temperature of 100°C. Mrs. Zander questions whether or not they should be using 9/5 within the Fahrenheit-Celsius context.

BZ: I don’t want to get us stuck on something I’m not very sure about.

LA: But you have a very good reason for it to be 9/5.

BZ: I do think, now, that it is 9/5 and 5/9. That’s in the sixth grade curriculum. I should know.

Both teachers ask for help from me, indicating they feel stuck. I direct them to stop working on the problem in the interest of time. I have only a short period of time for the problem solving session. I want to be able to observe them solve more problems. We
quickly go through the details of finding the answer. I tell them their rates of \( \frac{33}{40} \) and \( \frac{40}{33} \) are correct. I then hear several “Ah-ha” comments when I tell that they could have used the point-slope form of a linear equation or at least found the y-intercept and then used the slope-intercept form of a linear equation. We then move on to the next question.

**Vignette 2: An Intersection Between Geometry and Algebra**

The mathematics needed to solve the Length of Rope problem (see Appendix F) is a representation of the type of mathematics participants learned while participating in the second course. The FAGMLT course included topics such as linear functions, shapes and measurement, and similarity and congruence. This particular problem required participants to find an expression representing the distance around a unique shape. This is the third problem the teachers worked on during the problem solving session.

32 LA: This is going to be a formula. Is that what we are looking for?

Mrs. Zander does not respond as she focuses on her representation (see Figure 37). Mrs. Anderson also creates a representation (see Figure 38.1).
Mrs. Anderson initiates sharing what she sees.

33 LA:  This to me just looks like a third. This looks like a third. And this looks like a third.

Mrs. Zander asks for clarification.

34 BZ:  OK, say that again. I’m not sure what that means.
35 LA:  Like this circle right here. (She points) Isn’t this one-third of the circle (see Figure 38.2)?

![Figure 38.2. Mrs. Anderson’s Observation of One-Third](image)

36 BZ:  Oh, the outer arc. It is one-third of the total circumference.
37 LA:  Is it? I mean it looks like it, but is it?
38 BZ:  That’s what I was trying to figure out. At first, I thought it might be one-half. That would have been slick. But it’s less than one-half. I think you’re right. It’s closer to one-third.
39 LA:  OK, Is there some kind of arc thing that would
prove it?

40 BZ: Well, we know there are three of those arcs represented and then...

Mrs. Zander begins to answer the question, but then she stops. It appears that she is stuck. She shifts the conversation from the curved to the straight sections of the diagram.

41 BZ: I was looking at the distance of this portion here. I was trying to figure out what this distance is.

(See Figure 39.)

![Figure 39. Mrs. Zander’s Observation of a Straight Section](image)

Both teachers try to determine the length of a straight section. They work independently, writing information and marking up their diagrams. Mrs. Zander eventually finds the length of the three segments. Figure 40 represents her findings.
42 BZ: From here to here is two r's.

Mrs. Anderson asks for clarification.

43 BZ: (Tracing a segment connecting the centers of two of the circles,) so from this location to this location is 2r. (She moves her pencil and traces a corresponding segment on the outer part of the figure.) And like a 1/3 of the thing; so there is like three two-r's and three-one-third's, which makes a total of one whole. But wait, we are not using the same—this is radius and this is one-third of...(she pauses)

44 LA: (Picking right up...) One-third of the circumference...so we would have 6r plus one circumference right? Because a third and third and a third is a C? (See Figure 41)
Mrs. Zander and Mrs. Anderson approach having the correct answer. They need only to write circumference in terms of $r$ (i.e., $2\pi r$).

45  BZ: Can we plug in $r=1$ or $r=2$? What if we say that the radius is 2? What does that give us?

I interrupted the discussion and ask the teachers to reread the first sentence in the problem. I know the original problem did not have the “$r$” italicized. I begin to think that Mrs. Zander did not see the italicized “$r$” in the original paragraph. They say in unison, “Three cylinder of the same radius are tied together.” They miss the italicized “$r$” as I suspect. I ask them to slowly reread the question. They reread the problem and say in unison, “$r$.” I tell them I failed to emphasize the “$r$” when I retyped the problem. They ask me how this piece of information changes what they have been doing. I only say that the
final answer must be in terms of “r.” Mrs. Zander notes that nothing changes and immediately goes back to what she was working on before I interrupted.

46 BZ: I see what I’m doing as the same thing. I just want to plug in numbers to try things.

While Mrs. Zander is talking about wanting to plug in values, Mrs. Anderson rereads the question.

47 LA: Three cylinder of the same radius r. So the radius is r...The radius is r...so it IS 6r.

I remind both of them that they cannot have C in their final answer.

48 BZ: Circumference is Pi r squared.

49 LA: No.

50 BZ: Oh, it’s 2 Pi r.

51 LA: Right. Wait, wait, wait.

52 BZ: So it’s 6r plus 2 pi r. Is that what we’re trying to do?

53 LA: But wait a minute. This is also a diameter, right? (pointing to segment on her diagram that connected center to center). Circumference equals diameter times pi.

54 BZ: Or 2 r pi. Some students like that better.

55 LA: OK. But that is a circumference. (She had d•π on
paper; she then wrote $2r\pi$). So it’s plus $6r$ (pause) so now (pause) wait a minute (pause) this is algebra now, right?

56 BZ: Then it’s 6 plus 2 pi.

57 LA: What?

58 BZ: I factored out an r (see Figure 42). I took out an r from both sides. Then if we want r equals, we’re going to have to... (she pauses in mid-sentence)

\[
\begin{align*}
6r + 2\pi r \\
r(6 + 2\pi)
\end{align*}
\]

*Figure 42. Mrs. Zander’s Factored Expression*

After several seconds of silence, I choose to interrupt. I tell them that they have their final answer already. They each look down at their own paper.

59 BZ: This one? (She points to $6r + 2\pi r$.) But I don’t have an r equals.

I ask her why her answer has to be equal to $r$.

60 BZ: I don’t know. Because you said it has to be in terms of $r$.

Mrs. Anderson thinks she has also found the correct answer on her paper as well.

61 LA: It would be this one then (She points to $2\pi r + 6r$).
I confirm that their answers are correct. I then ask them if they could guarantee to me that they were right with respect to what they were saying about one-third of the circumference. Both say “no” in unison.

62 LA: How would we prove that?

63 BZ: That was just a guess.

64 LA: Is there some sort of an arc thing?

Mrs. Anderson and Mrs. Zander work together to create a valid argument. Mrs. Zander looks at a semi-circle in her diagram. She questions whether or not that could help. Mrs. Anderson wonders if having a specific value for the diameter would help them figure it out.

65 BZ: That’s what I kept thinking. What if we plug in a 2?

66 LA: If we have this diameter (pause) that won’t help us know this is 1/3 of the circumference. How would we prove that?

I interrupt again. I notice the time is moving fast. I still want the teachers to solve at least one more problem before our session ends. I tell them this is a problem they can continue to work on at home. Mrs. Anderson is hesitant to stop.

67 LA: You know what? This one bothers me. I want proof for sure. I want PROOF!

We laugh. I hand out the fourth and final problem of the night.
Reflections on Problem Solving Experiences

Three themes emerged from the analysis of Mrs. Anderson and Mrs. Zander’s submitted mathematics work and reflections coupled with their act of solving problems during the problem solving session. The two vignettes offer an illustration of live interactions consistent with what was found in the written data and thereby help to triangulate the three themes. The first theme represents the collaborative interaction between these two teachers as they solved problems. There were brief periods of time when the teachers worked independently on the problems; however, the majority of their time together was marked by a mathematical conversation between them. Collaboration is an important theme that emerged from analyzing their mathematical work, is common to both teachers, and is common to the institute.

The second theme highlights common mathematical practices, or dispositions, these two teachers frequently accessed as they solved math problems. These practices include making connections, using representations, and testing specific cases. Even though the two teachers used similar mathematical practices, there is variation in how each teacher used them.

The final theme represents mathematical practices for learning that are different between the two teachers. On one hand, Mrs. Anderson’s mathematical work revealed a persistent disposition in her learning of mathematics. On the other hand, Mrs. Zander’s mathematical work revealed a disposition of making connections between her learning of mathematics and her teaching of mathematics.
Theme 1: Collaboration

This section offers a detailed description of the first common theme, collaboration. Lines 33-45 and 48-58 in the second vignette are examples of the collaborative interactions between Mrs. Anderson and Mrs. Zander as they solved problems during the problem solving session. It is important to note this theme is consistent with the structures put into place by the instructors of the Math in the Middle Institute. Participants were encouraged to collaborate with one another. Both Mrs. Anderson and Mrs. Zander embraced collaboration as a tool to do mathematics.

Mrs. Anderson’s Use of Collaboration to Help Her Learn to Do Mathematics

Collaboration immediately emerged as a significant theme in Mrs. Anderson’s mathematics work and reflections from the first three courses of the Math in the Middle Institute. Her written artifacts and reflections document at least thirty-eight instances of collaborating with others to learn mathematics. Some of this evidence included brief comments Mrs. Anderson added in the margins of her work whereas other evidence included more detailed descriptions of collaborating she included in course reflections. Mrs. Anderson often included descriptions of her interactions working alongside peers, who she mentioned by name, within her solutions and reflections. She also frequently wrote how she would have been unable to answer many questions without peer support.

Mrs. Anderson admitted on the last day of the first course that she struggled with many of the mathematical concepts covered that week. “I realize I need a lot of help” (MSL, Reflection 1, 6/22/2007). She used collaboration as a mathematical practice to help her succeed. “Learning from peers is such an asset…working with my peers…is
what I need to strengthen my learning” (MSL, End of Course Reflection, 6/23/2007).

Mrs. Anderson’s collaborative nature was visible in each of the first three mathematics courses. Comments such as, “I needed help here” (FAGMLT, Day 1 homework, 7/16/2007) and “There is no way I could come up with the next part: This was thanks to [a peer]…” (ECR, End of Course Problem 1, 12/22/2007), indicate her reliance on peers. At the close of the problem solving session, Mrs. Anderson even stated, “I wouldn’t be able to do these [problems] on my own, any of these” (Problem Solving Session, 5/22/2008). Mrs. Anderson thought she needed to work with others to help her learn mathematics and be successful with her solutions.

Mrs. Anderson’s reflective writing often included lengthy accounts of her collaborative efforts, sometimes a page or longer. She acknowledged her need to learn in collaborative groups as well as offered insight as to how her peers actually helped her. The following is one example among many from the first course:

Solving [the Mind Over Mathematics problem] was truly a “group effort.” We took it back to the dorm and read and reread it. None of us really got it at first. Then we started throwing out ideas about concepts we learned that day…The reason I chose this one [as one of my favorites] is because I felt like I really contributed to the group! While there was no way I could have ever solved this myself, I felt my personal contributions [to the group] were critical to solving the problem. (MSL, Favorite Five, 6/23/2007)

Mrs. Anderson began this reflection by stating that she did work with others. She then closed it by saying that she would not have been able to solve the problem by herself.
Mrs. Anderson conveyed a similar message while working problems from the second course:

Oh my gosh, this was hard! We worked as a group. [The master teacher] pulled us out of the water and [a graduate student] helped too! First we did some trial and error…Then we tried a systematic approach…Still not getting it yet, we were led to focus on the factors… There is just NO WAY I could have done this without help! (FAGMLT, Day 3 homework problem E, 7/18/2007)

She again acknowledged the group effort and as well as her reliance on others.

As a distance-learning course offered during the fall, the third course (i.e., ECR) presented natural obstacles for collaboration. Gone were the times when teachers could sit side-by-side learning mathematics. Mrs. Anderson communicated her frustrations following the first week of the course.

The first week of ECR has been hard. I find the homework problems difficult in the first place. It would be so much better to actually be sitting at a table and bounce ideas off of each other to solve the problems. (Email Correspondence, 9/15/2007)

However, she was able to use electronic discussion boards as a substitute for meeting face-to-face. There were times Mrs. Anderson used this board as a place to ask for help.

Is that what you are all getting? Am I off on this problem?? I am not sure about my formula? \( F_n = F_{n-2} + F_{n-1} - F_{n-5} \) after month 5????...I am willing to fax anything I have. (ECR Blackboard Posting, 9/16/2007)
A peer almost immediately responded to Mrs. Anderson’s call for help and offered a suggestion. Mrs. Anderson then promptly gave feedback to that peer.

Now I see what I am doing different. I have the bunnies dying after they give birth for the 3rd time. I interpreted the question differently. This will make a big difference...Did anyone else interpret it this way? (ECR Blackboard Posting, 9/17/2007)

The discussion board was a good vehicle for participants to post work; however, there were limitations. At times, Mrs. Anderson had difficulties communicating the approach she took to solve a problem. “It would be so much easier to show in person” (ECR Blackboard Posting, 10/09/2007). Despite the limitation, Mrs. Anderson was able to use the online discussion boards as a place for peers and instructors to offer feedback to her solution drafts, clarification on problems she did not understand, and guidance during those times she just needed help.

The online conferencing system allowed teachers to actually see each other, talk to one another, and post mathematics work during real-time work sessions. The first month proved to be full of obstacles in terms of getting the technology to work correctly. “All group members were on but we experienced audio problems. We were only able to communicate via the ‘chat’ part of Breeze” (ECR Blackboard Posting, 9/30/2007). Once the technical glitches were eliminated, Mrs. Anderson expressed appreciation for online conferences.

October 3rd marks the first time that all of us were able to be seen and heard! We discussed problem #22 the most. We had questions about how
much we have to explain to prove that all 5 of the smaller triangles are congruent and why they are Golden Triangles. I don't know that we are all comfortable with it yet. (ECR Blackboard Posting, 10/07/2007)

Mrs. Anderson eventually utilized online conferencing in much the same way she utilized meetings during the evening homework sessions during the first two courses held in the summer. She was able to ask questions, gather input from peers, and learn mathematics. By the end of this third mathematics course, Mrs. Anderson reflected on many positive aspects of collaborating via the online conferencing system.

I feel we had some very valuable discussions…I personally found this so helpful because I would NOT have gone far enough with this problem had it not been for our meeting…our discussion clarified concepts. [One peer] once again came up with an understandable suggestion about how to know when to add or multiply in probability… "or" signals addition whereas "and" signals multiplication…Group, thank you so much!!! (ECR Blackboard Posting, 12/05/2007)

Mrs. Anderson expressed her appreciation of her group and the benefit the online conferencing system had on her learning. She was able to learn mathematics by working alongside others even though she was physically removed from them due to the long distance nature of the course.

Mrs. Anderson identified several peers that she did enjoy working with during the first five days of the Institute. “There were many peers that I felt comfortable asking for
help (God bless them!!)... All were more than willing to work with me to help me understand the content of what I was asking. I appreciate their patience and persistence.” (MSL, End of Course Reflection, 6/23/2007). Mrs. Anderson pointed out the characteristics of patience and persistent. These characteristics mirrored what she wrote about the M^2 instructional staff as well (MSL, End of Course Reflection, 6/23/2007).

Mrs. Anderson also offered a reflection as to those she preferred to work with (FAGMLT, Week 2 Reflection, 7/27/2007). She described a need to be around positive peers. She did not want to feel inferior as a learner of mathematics because she thought she was slow at understanding math. She also identified “approachable” during the problem solving session as a characteristic she sought in people with whom she worked on mathematics.

Learning in a collaborative setting was new for Mrs. Anderson. She described how she had never learned in such a setting before; yet due to the impact she saw collaboration make in her own learning, she was willing to make changes in her teaching.

My biggest thing was groups. I am so OK now with having kids work together in groups. Before this, it was you were in your own desk and you don’t look at someone else’s paper and you do this independently. And now no, I let [my students] work together, especially for problem solving. I’ve really come out of my shell on that. (Problem Solving Session, 5/22/2008)
The collaborative setting Mrs. Anderson experienced as part of the Math in the Middle Institute Partnership led not only to changes in her own learning of mathematics but also to the changes in the ways she structured learning experiences for her own students.

**Mrs. Anderson’s Collaborative Disposition**

Collaboration helps Mrs. Anderson learn mathematics. Her work reveals how collaboration helps her understand how to begin to solve some problems (e.g., a day 3 homework problem from FAGMLT). Experts in cooperative learning (e.g., Johnson & Johnson, 1999) have addressed how working with peers can help students gain better access into solving a problem. Collaboration helps Mrs. Anderson test theories and then to make subsequent revisions (e.g., the Fibonacci bunny problem from ECR). She uses collaboration as a vehicle to promote a deeper understanding of mathematics by clarifying concepts with peers (e.g., the Chairman and Vice Chairman problem from ECR) in much the same way Goos (2004) found high school students assisting one another in a collaborative environment.

Mrs. Anderson uses collaboration to push herself to go further (e.g., a textbook probability problem from ECR) and help her to see alternative methods for solving problems (e.g., the Box Design Logic problem from FAGMLT). Kramarski, Mevarech and Arami (2002) found middle school students who were encouraged to collaborate with one another “suggested different kinds of representations, compared the strategies, and analyzed the pros and cons of each strategy” (p. 241) with each other. Each of these examples highlights a different aspect of how collaboration plays a role in Mrs. Anderson’s acquisition of mathematical knowledge. Further, the benefits for Mrs.
Anderson as she collaborates with peers resemble benefits for students as they engage in cooperative learning activities in the K-12 classroom (e.g., Johnson & Johnson, 1999; Kagan 1994; Siegel, 2005). This theme also offers commentary on the challenges of learning mathematics while part of distance education courses. Mrs. Anderson became accustomed to working with her peers to learn and better understand the mathematics while participating in two math courses held on campus. Mrs. Anderson’s work often revealed that she was more successful learning mathematics when she could collaborate with her peers in a face-to-face setting. The data suggests she was more motivated and invested to learn mathematics, as she had support right in front of her (Jackson & Bruegmann, 2009). However as she learned mathematics during the third course, a distance education course during the academic year, Mrs. Anderson discussed hardships of learning mathematics when removed from the convenience of sitting face-to-face with her peers (Larson & Bruning, 1996). Yet, she was able to collaborate using technology; this helped her overcome the obstacle of not sitting face-to-face. Mrs. Anderson’s work and reflection support the notion that technology can enable one’s learning of mathematics in a distance education environment (Bernard et al., 2004).

Mrs. Zander’s Use of Collaboration to Help Her Learn to Do Mathematics

Mrs. Zander also used collaboration as a tool to learn to do mathematics during the first three courses of the Math in the Middle Institute. However, this theme did not emerge in the same manner that it did from analysis of Mrs. Anderson’s work. Initial analysis of Mrs. Zander’s data indicated she did not learn much mathematics by
collaborating with others. There was little evidence of Mrs. Zander referencing who she may have worked with as she solved problems. Mrs. Zander was present at some of the nightly study group session during the summer courses; however she was relatively quiet. In comparison to Mrs. Anderson’s frequent reference of collaboration almost as a “means to survive” the institute, Mrs. Zander’s work contained very limited references to the help of others. This data was surprising, as Mrs. Zander had clearly communicated on her application as well as during an interview how much she longed to be part of a learning experience that allowed her to collaborate. An analytic memo, written shortly after the first wave of analysis of Mrs. Zander’s artifacts from the first two courses reveals my early struggles to make sense of conflicting data.

A thing that strikes me, though, is her isolation. I just don’t get it. She talked about wanting to work with others growing up, but she didn’t get to. She talked about organizing her classroom using small groups well before her involvement in Math in the Middle. She wants students to communicate with each other while they learn; however, she is more independent when she herself learns. Her work contains few references working with Math in the Middle peers but many references related to the desire to share things with peers she teaches with. She talks about wanting to start working on Habit of Mind problems independently—yet prefers collaborating. This doesn’t make sense to me. (Hartman Analytic Memo, 1/27/2009)

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34 This analytic memo was written before analysis of the problem solving session.
As described in chapter 4, most participants started working on assigned homework immediately following each day of the summer institute. Many participants chose to work together. Mrs. Zander was among those who stayed after class. However, her actions reflected one with more of an independent disposition, not a collaborative one:

- Left early; more to herself (Day 1 Homework Session, MSL, 6/18/2007)
- Quiet, but will check with her group (Day 4 Homework Session, MSL, 6/21/2007)
- Worked alone (Day 1 Homework Session, FAGMLT, 7/16/2007)
- More talkative tonight (Day 3 Homework Session, FAGMLT, 7/18/2007)
- Not present (Day 9 Homework Session, FAGMLT, 7/26/2007)

Of five data entries, Mrs. Zander stayed after class to work four times, with evidence that she worked with others two of those four times. Thus, the evidence suggests Mrs. Zander collaborated to learn mathematics no more than half of the time during a two week course.

Adding her written mathematics work and reflections from the first two courses as well as data from the first interview further highlighted her independent nature.

I think part of her isolation in Math in the Middle is just her personality…In my interview with her, Mrs. Zander describes herself as homebody, longing for one-on-one time with her son, husband, and garden.

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35 This data came from my general notes observing participants during the summer courses. I recorded these notes before I had selected the subjects for this study. I wanted a record of general participant interactions; data not represented by any artifacts of the Math in the Middle Institute Partnership research agenda.
She likes to meet and have coffee with that “one” friend…She has called herself an outlier in the past…She also likes to solve problems her own way. There is one case where the instructor commented how she solved a completely different problem than everyone else. I suspect she did not collaborate with others on that problem…I even think she may have made some errors on her math assignments because she chose to do much of the math on her own. She was confident in her abilities and didn’t want to feel the pressure of working at someone else’s pace. (Hartman Analytic Memo, 1/27/2009)

The last sentence of that memo highlights the role that peers may have had on her choice to work more independently. Mrs. Zander did not want to feel the pressure of working at someone else’s pace. “I get self-conscious if I don’t know something or take a long time to grasp a concept…sometimes a peer at my table would find an equation and it wouldn’t be crystal clear to me” (FAGMLT, Week 2 Reflection, 7/27/2007).

Analysis of Mrs. Zander’s written mathematical work for the third course revealed that she rarely posted anything on the online discussion boards. Yet it is unlikely the discussion board activity alone would capture the essence of Mrs. Zander’s interactions during this third course, as she lived in close proximity to a number of her peers and had the opportunity to meet face-to-face with them during regularly scheduled homework sessions, as opposed to meeting with peers using the online conferencing
system. One would not expect much involvement from her in a discussion board environment, primarily available for those separated by distance.

However, analysis of her written work suggested she learned to do the mathematics of the ECR course in isolation from her peers. She did not reference anyone as she wrote her solutions. And due to the distance learning nature of the course, participants were rarely asked to write reflections summarizing their process of learning. Thus, there was little reflection offered by Mrs. Zander that would have prompted her to discuss her collaborative efforts to learn mathematics as part of the third course.

To assist and triangulate analysis, Mrs. Zander was asked to reflect on her learning experiences as part of the third course. This request came a full year after she submitted her end of course problem set.

I frequently needed help and reassurance on the homework. I often came up with a reasonable answer and solved the problem on my own thinking that I was right. And then I got to the meeting and found it was completely wrong...I often had questions that I didn’t even know where to begin. I probably did 70 percent on my own and needed insight on the others. I ALWAYS needed to meet with the group. There was no way that I had the information correct on all parts without the assistance of the group. Very humbling, but, as in every classroom, you have a variety of levels and, in my case, I learned a lot! (Email correspondence, 2/16/2009)

The words Mrs. Zander used suggested that she was not so independent after all. As she phrased it, she “always” needed to meet with her group. My original perspective was
skewed. I did not find any evidence in Mrs. Zander’s written data to suggest she collaborated during the third mathematics course. In hindsight, I recognize there is variation in what it means to collaborate; there is not one definition of working together or working with others. Further, just because one does not write about who they likely worked with to solve math problems does not indicate they did not collaborate at all. I likely formulated my definition of collaboration based on what I saw in Mrs. Anderson’s data. That definition influenced what I looked for and consequently found in Mrs. Zander’s data. My initial perception that Mrs. Zander did not want to collaborate, which likely developed in contrast to Mrs. Anderson because collaboration was such an obvious part of Mrs. Anderson’s learning, blinded my ability to clearly see evidence in Mrs. Zander’s work. These types of oversights are not uncommon in case study research (Stake, 1995).

Coupling Mrs. Zander’s reflection with data from the problem solving session, it became clear that Mrs. Zander needed her peers to help her learn the math. Examining the initial data from the first three courses a second time revealed many instances when she did showcase her collaborative nature. This is consistent with Hatch (2002), “Now that you have…refined [your] interpretations…, you should go back to the data in a systematic search for places that related directly to the [new] interpretation” (p. 186). Mrs. Zander met with a group of teachers to work on the end of course assignment for the first course on 6/23/2007, the day after class ended. “I really enjoyed this opportunity to compare our work and get ideas and assistance as needed…I liked being able to
collaborate so much better than working independently” (FAGMLT, Week 2 Reflection, 7/27/2007).

Mrs. Zander reflected several times during the second course about the value of having peers to work with. “This box problem was a great problem to work on with a small group and generate ideas that build on one another” (FAGMLT, Sample Six, 7/28/2007). Another example from the second course revealed a time when Mrs. Zander’s first attempt was wrong but her peers helped her to understand.

Bobo and the Time Bomb was not as it would seem to me at first. I thought the solution was simple…I was so very wrong. After getting together with the [local] group of M² participants, I found out [one of my peers] had mastered this one in a much more complex way than I had first imagined. Here it goes. (FAGMLT, End of Course Problem #11, 7/28/2007).

Mrs. Zander’s collaboration nature carried over to the third course as well. Meeting face-to-face was not the only type of interaction Mrs. Zander had with peers during this academic year course. She also used email as a means for communicate in addition to the face-to-face weekly meetings.

After doing some checking on number 1 and 2 this morning, I think we need to start like this—try shifting all the Fibonacci numbers to the right one and starting with 0. It made more sense and provided an explanation for #2. [One peer] says she got 141 for \(n = 5\) on number 7. I think she was
still working on it at midnight...are you up to 227 now? There's got to be a way. (Email Correspondence, ECR, 12/19/2007)

This particular email revealed Mrs. Zander’s willingness to share with others. She did not just take in information; she also shared. These examples collectively indicate the benefit of communication in collaborative groups. Communication is at the heart of students’ learning in groups (Sfard, 2001).

Watching and listening to Mrs. Zander’s interactions with Mrs. Anderson during the problem solving session illustrate that Mrs. Zander collaborates to learn mathematics. She engaged in discussion as much as Mrs. Anderson did. During an informal conversation shortly before receiving the first problem of the night, Mrs. Zander stated, “I understood very quickly that I wasn’t going to make it through [Math in the Middle] program on my own” (Problem Solving Session, 5/22/2008). As she worked problems that night, nearly a full year since she had stepped foot into M², Mrs. Zander collaborated with Mrs. Anderson on each and every problem. Before she left for the evening, she even told Mrs. Anderson, “You need to know; I don’t like doing these on my own; I needed you” (Problem Solving Session, 5/22/2008).

**Mrs. Zander’s Collaborative Disposition**

It is likely Mrs. Zander entered the professional development program excited about her opportunity to work alongside her peers. However, it is also likely the challenge of the mathematics as well as the wide range of peer personalities served as a short-term obstacle in Mrs. Zander’s quest to learn mathematics with her peers. Once she found peers who matched her needs as a learner and then experienced success solving
problems with those particular peers, Mrs. Zander may have become much more collaborative. Thus, collaboration did facilitate Mrs. Zander’s learning of mathematics.

Naturally, this study’s viewpoint assumes a single definition of collaboration, or working together. Research offers another plausible explanation for what could have been going on with Mrs. Zander. Kramarski, Mevarech, and Arami (2002) found students who are exposed to metacognitive instruction have more positive learning experiences in small groups than students without that exposure. Despite data indicating she had always longed to collaborate with peers, it is likely Mrs. Zander never learned specific skills to help her learn alongside peers. She may have used a variety of experiences from the first couple of mathematics courses to learn to effectively collaborate with her peers.

Collaboration assisted Mrs. Zander’s learning of mathematics in at least two ways. As she was able to work with fellow participants, who she classified as great team players (i.e., patient and encouraging), Mrs. Zander learned more mathematics. Peers offered her help with specific details as well as provided reassurance to her along the way. Collaboration also provided a forum for Mrs. Zander to share her own methods for solving problems. She then could receive feedback and move towards a more complete understanding of the mathematics (e.g., comments posted as part of her email correspondence, 2/16/2009). Leiken and Zaslavsky (1997) described similar benefits of peers giving help to and receiving help from fellow peers to learn mathematics in their study of student interactions in a cooperative learning setting. Further, Mrs. Zander was able to hear other methods for approaching problems (e.g., the Bobo and the Timebomb
problem from FAGMLT) similar to benefits afforded students in Kramarski, Mevarech & Arami’s (2002) study of cooperative learning with metacognitive instruction.

A reluctance to collaborate in certain settings may have hindered Mrs. Zander’s mathematical learning at times. Take, for example, the factor of time. Mrs. Zander felt pressure to work at others’ speeds in solving problems. Some of her peers were ready to move on; yet Mrs. Zander was still trying to comprehend (e.g., FAGMLT End of Course Reflection). Did this pressure prevent her from developing a rich understanding of the mathematics?

Mrs. Zander said that she almost always needed to meet with peers in order to check her level of understanding. “There was no way that I had the information correct on all parts without the assistance of the group” (Email correspondence, 2/16/2009). Mrs. Zander even chose to stick with a certain problem solving approach that another peer was using during the problem solving session because she did not want to get confused (Problem 4, Problem Solving Session, 5/22/2008). Thus, she did not take the time to approach a problem using her own conventions. She settled on someone else’s method, a potential drawback of working alongside peers (Joyce, 1999). Mrs. Zander admitted by the end of the problem solving session that she would never have made it through Institute if it were not for help and support from her peers.

During the initial interview, Mrs. Zander mentioned how important it was for learners to talk aloud mathematically (Hufferd-Ackles, Fuson, & Sherin, 2004). Mrs. Zander said this was something that she always needed to do to learn math for herself; yet this was something she was not encouraged to do during her own K-12 experiences.
Collaboration, therefore, was likely a natural fit for Mrs. Zander, as she could talk through problem solving with her peers.

Mrs. Zander wrote that she struggled at times with giving her own students the “big picture” related to some of the topics she taught in her classroom (Initial Interview, 2/22/2008). These big ideas likely represent some combination of ideas similar to Ma’s (1999) profound understanding of fundamental mathematics, Ball, Thames, & Phelps’ (2008) mathematical knowledge for teaching, and the NCTM’s (2000) content and process standards. Mrs. Zander noted this (i.e., the need to strengthen her ability to offer the “big picture”) was one of the reasons she applied to Math in the Middle in the first place. While solving problems during the first three courses, she again wrote that struggled with seeing the big picture (e.g., dialogue as part of her email correspondence, 2/16/2009). It appears that Mrs. Zander used collaboration as a means to fill in some gaps she recognized in her mathematical understanding. She used her peers’ knowledge and skill of explaining as a tool to improve her own capacity to understand and explain the big picture.

Finally, Mrs. Zander’s responses during my initial interview with her as well as her responses to a beliefs inventory suggested that she likely walked into the institute with the belief that teachers should allow students to work in groups and teachers should collaborate with one another. Mrs. Zander experienced a learning environment where she, in the role as learner, was able to work with her peers in a group. She was not afraid to

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36 Examples of responses related to student collaboration on the belief’s inventory included: Students should work with others on math problems. Students should definitely help each other learn math. And students definitely do not learn best when they work alone.
initiate collaboration; yet she was not likely to force her ideas onto others, an attitude she shared with respect to the ways she acts in the role as a team leader in her school (Initial Interview, 2/22/2008). Some of the dispositions Mrs. Zander exhibited as a learner of mathematics in the institute matched the attitudes, beliefs, and dispositions that marked her as a learner of mathematics outside of the institute. Yet, I am reminded that as a qualitative researcher I must try to “preserve the multiple realities, the different and even contradictory views of what is happening” (Stake, 1995, p. 12, italics in original). While this study suggests several possibilities, there may be other reasons behind both Mrs. Zander and Mrs. Anderson’s purpose for collaborating with peers to learn mathematics as part of the first three mathematics courses of the M² Institute.

**Theme 2: Two Teachers’ Common Learning Practices**

A second theme that emerged examines common practices Mrs. Anderson and Mrs. Zander utilized to learn mathematics while part of the professional development program. Analysis of written work coupled with the problem solving session indicated these two teachers often deploy making connections, using representations, and testing specific cases as practices to learn mathematics. While this study reports these practices as common to these two teachers, there is variation in how each teacher actually uses them to learn mathematics.

**Making Connections**

Line 1 in the problem-solving session vignette indicated an immediate connection Mrs. Zander made between the Newton-Lewis problem and a similar problem she had seen in her past. As she attempted to solve this new problem, Mrs. Zander often tried to
remember specific details about that old problem (e.g., Lines 4, 11, and 19). This example of her attempt to connect new information to old information while solving the Newton-Lewis problem helps illustrate a frequent occurrence in Mrs. Zander’s written work from the first three mathematics courses. There are more than twenty-one instances when she explicitly documents a past connection as she attempts to solve a new problem. This generally looks like a specific phrase written next to a solution (e.g., like the railroad method) or like a detailed description of a connection as part of course reflection.

There are times this documentation included a connection between a problem from Math in the Middle and a concept she had seen outside of Math in the Middle, such as relating a Triangle Puzzle problem assigned in the first course to a Suduko puzzle. Other times she explained how she was able to connect an approach she used to solve an old problem, assigned earlier in the course, with the method she used to solve a new problem. Mrs. Zander recognized the approach she used to solve the making cents coin problem (i.e., starting with the largest coin, a quarter, and then adding smaller coins) would also work to help her solve the string of 8’s problem (i.e., starting with the largest set of 8’s, 888, and then adding smaller sets of 8’s).³⁸

The next problem proposed a number of combinations of 8’s strung together to total 1000. I started by using 888 (as it was similar to the quarter in the above problem) since the number would be closest to

³⁷ The strategy one can use to solve the first part of the making cents problem (i.e., moving from largest to smallest coin) is not the strategy one can use to solve the second part of the problem, where the nickel is removed.
³⁸ Refer to Appendix S for making cents and string of 8’s problems.
1000...I started adding 11 because 11x8=88 just as five pennies would equal one nickel in the previous problem. (MSL, Favorite Five, 6/23/2007)

Mrs. Zander did not have to start from scratch; she could translate the strategy, of going from larger to smaller, from one problem to the next. She found success learning mathematics by using this mathematical practice of making connections.

Analysis of Mrs. Anderson’s work also reveals she used the practice of looking for connections to learn mathematics. There are more than nineteen instances where she documented a past connection as she attempted to solve a problem. Generally, Mrs. Anderson’s connections relate the current problem to a specific problem she had seen earlier in the institute. For example, a part of Mrs. Anderson’s end of course reflection from FAGMLT described her path to solve a pentomino problem: 39

After finding the 12 pentominos, we had to explain why there are only these 12 and no others…I studied the 12 pentominos, looking again for some rule…but couldn’t come up with anything…I reread the question…Then I thought of the Shapes from Four Triangles problem. I approached this one the same way. (FAGMLT, End of Course Reflection, 7/28/2007)

Mrs. Anderson used a systematic approach to show that she had all of the possible shapes using four right isosceles triangles on a homework assignment a few days before she was given the pentomino problem. She was able to translate the strategy from one problem to

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39 Refer to Appendix S for the pentomino problem.
the next. This example is very similar to the way Mrs. Zander described how she used the strategy from the coin problem to help her solve the string of 8’s problem.

Mrs. Anderson often indicated in her solutions that she connected some part of a problem with some aspect in her course binder. Figure 43 illustrates one example of when she uses a course handout to help her solve an end of course problem from FAGMLT.

Mrs. Anderson indicated that she used a page from her binder as an entry point to solve this end of course problem related to linear equations. Mrs. Anderson included very similar comments about how class discussions helped her solve problems. “[The instructor’s] class presentation and examples from his handouts on 6/22/2007 were helpful for this problem” (MSL, End of Course Problem #7a, 6/23/2007). In this particular example she even recalled the specific day of class that helped that helped her with a problem on the end of course assignment. Mrs. Anderson’s frequent documentation of what specific problem or page of notes she referenced coupled with her own reflections about making connections suggests she relies on making connections to learn mathematics.
I don’t know if that other students are like me, but I really have to toil over many of [the end of the course problems]. I go back through the binder and find like problems and I go through the handouts for help. (FAGMLT, End of Course Reflection, 7/28/2007)

Mrs. Anderson was aware that she relied on past examples to help her make sense of new situations; she described this process as applying what she learned. “The ‘learning’ part of this problem occurred for me in the morning…I was able to follow along with that portion of the morning’s work and then apply it later on when it came to this problem” (FAGMLT, Sample Six, 7/28/2007). Mrs. Anderson tries to recall specific procedures or problem solving methods from old situations and use them as she solves new problems.

The preceding paragraphs discuss connections Mrs. Anderson made within a course; she also made some connections between courses. While there are no explicit examples in her work describing how she connected a problem or handout from one course to the next, Mrs. Anderson’s end of course reflection for FAGMLT indicated she did make connections between the first and second course: “I feel like the first course helped me with this second course; will this course help me with the next one?” (FAGMLT, End of Course Reflection, 7/28/2007). Due to the intended structure of the Institute, in that the content of the courses overlapped at times and built upon each other, this observation should not be unexpected. Yet, Mrs. Anderson’s words show she did make some of the intended connections. This further supports the notion Mrs. Anderson uses connections as a mathematical practice or disposition to learn mathematics.
Two of the important findings emerge from analysis of using connections to learn mathematics. First, each of these two participants used connections as an entry point to solving some problems. Mrs. Zander and Mrs. Anderson both used existing examples and notes as a way to get started solving the problem. The other finding comes from Mrs. Anderson’s data. She used connections as way to address the question, “What do you do when you get stuck solving a problem?” Mrs. Anderson returned to her notes and previous examples searching for anything that could help her. Each of these findings closely tie to the habits of mind of a mathematical thinker that Cuoco, Goldenberg, and Mark (1996) describe and the CBMS (2001) promotes.

While there is much evidence to show that using connections helps both Mrs. Anderson and Mrs. Zander understand mathematics, there is also evidence to suggest their use of connections reveals an underdeveloped mathematical practice. For example, Mrs. Anderson liked to identify a connection and then solve a new problem using the exact same structure as an old problem. This may appear to be an efficient means to solve a problem; yet as Mrs. Anderson’s words from 12/17/2007 seem to suggest, using similar problem structures appeared to be a crutch.

Note: I found the next three problems, 6, 7, 8, very difficult. I could never have solved this on my own. If it weren’t for the hint and solution in the back of the text, I don’t think I would have solved them. Once I had #6, I used the same process to figure out #7 and #8. After the solution for #7 in the text, I still had to go back and re-do #6. Between #6 and #7, I could follow the process for #8.

(ECR, Bayes’ Theorem Problem Set, 12/17/2007)
Her work for problem #8 in Figure 44 is nearly an identical copy of what her work looked like for problem #7.\textsuperscript{40} (And her work for problem #7 was nearly an identical copy of her work looked like for problem #6, which was heavily influenced by the sample solution in the back of the textbook.)\textsuperscript{41}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure44}
\caption{Mrs. Anderson’s Connections in the Bayes’ Theorem Problem Set, ECR}
\end{figure}

\textsuperscript{40} This may be exactly what the author intended.
\textsuperscript{41} Refer to Appendix S for the ECR #6, #7, and #8 problems.
Notice three instances where Mrs. Anderson documented connections she made to solve these problems. She referenced the solution from the back of the textbook again next to her work for #7, just like she referenced it in her work for #6. She then began problems #7 and #8 by explaining that she would use the same process that she used in #6. Her work in Figure 44 looks identical for both problems, just with different numbers involved. She admits that she cannot solve these problems without help from the back of the book or without utilizing the same structure for each problem. This example suggests there are times when she has to rely on using similar problem structures to solve a new, or related, problem. While Resnick et al. (1989) addresses limitations of over-generalizing from one domain to another, Stylianides and Stylianides (2007) suggest this ability to connect mathematics, or transfer, is an advantage of using, among other things, similar problem structures. “Learning mathematics with understanding involves making connections among ideas; these connections are considered to facilitate the transfer of prior knowledge to novel situations” (p. 106, italics in original). Another advantage of making connections is that is strengthens one’s ability to make sense of new situation (CCSSI, 2010).

Mrs. Anderson’s work in the first vignette offers an illustration of her tendency to make sense of mathematics using similar mathematics. After she copied Mrs. Zander’s representation with data from an old problem (refer back to Figure 36.1), Mrs. Anderson then duplicated the representation and added corresponding data from the new problem (refer back to Figures 36.2 and 36.3). Once she had matching tables, she thought she had
the answer. “If we have it set up the same way, shouldn’t that be the answer?” Mrs. Anderson asked if this mere duplication of a similar representation could actually be the answer itself. This tendency to make sense of mathematics using related work is very similar to the way she solved problems #7 and #8 of the Bayes’ Theorem set (refer back to Figure 44).

Mrs. Anderson made a valid connection between the new problem and the old problem in the vignette, or using her terms, she applied what she knew from one situation to the next. Yet, she stopped there. She thought just getting the two problems to look the same was the answer itself. Mrs. Anderson made no attempt to write a formula connecting the Newton scale with the Lewis scale. This example is an instance when her memory of the details of other problems was inadequate to work the new problem. This reliance on making connections to a past problem to solve a current problem, and the ways she duplicated the work, exposed a limited understanding of the mathematics involved. Coupled with her reflection, Mrs. Anderson admits that she could not solve this problem on her own. She knew she needed to do something with the past connection; yet she did not know how to make sense out of it (CCSSI, 2010).

Mrs. Zander’s use of connections also exposed a limited mathematical understanding needed to solve some problems. There were instances in her work that indicated connecting an old problem to a new problem only provided a portion of what she needed to solve that new problem. Unfortunately, she was unable to fully use the

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42 Seeing that the work was essentially the same is what instructors intended when they created the problem. Yet, they also wanted to know who could use this knowledge. Mrs. Anderson struggled with “using” the knowledge.
connection to a past problem to solve the new problem. In the first vignette, Mrs. Zander knew from the outset there was a connection to the Celsius-Fahrenheit problem (Problem-Solving Session Vignette, Line 1). She also knew the rates of $\frac{5}{9}$ and $\frac{9}{5}$ were important, but that is all she recalled. As the two teachers attempted to solve the problem, Mrs. Zander returned to her incomplete connection on multiple occasions (e.g., Problem-Solving Session Vignette, Lines 11 and 29). She even admitted that this incomplete connection made her feel stuck and she did not want to slow down Mrs. Anderson’s attempt to solve the problem. Yet, Mrs. Zander could not let go of this incomplete connection.

Episodes similar to this were present in Mrs. Zander’s written work as well. Six months after learning the procedure for changing repeating decimals to fractions as part of the first course, Mrs. Zander could not remember all of the specific details while solving a problem on an arithmetic inventory (see Figure 45).

![Figure 45](image)

*Figure 45. Mrs. Zander Incomplete Connection on an Arithmetic Inventory, 1/18/2008*

Mrs. Zander knew her answer was wrong,\(^{43}\) however she could only remember part of the correct procedure. She knew she needed to move the decimal and then cross something

\(^{43}\) A fraction greater than one does not make sense as the original decimal had a value less than one.
out. She even remembered seeing this during the first summer of the institute. Yet, she had to stop working on the problem because she did not remember the rest of the connection. Mrs. Zander was unable to transfer the procedure from her long-term memory to her working memory, a problem noted in research on human cognition (Sweller, van Merrienboer, & Paas, 1998). She knew she had made an error; yet learners of mathematics can benefit from making errors. “Students are able to make use of mathematical errors as springboards for inquiry in a variety of ways…[including] a better understanding of mathematics as a discipline” (Borasi, 1994, p. 199).

Mrs. Zander did not remember the algorithm for converting a repeating decimal to a fraction (i.e., If \( x = 0.\overline{53} \), then \( 100x = 53.\overline{53} \), \( 99x = 53 \), and \( x = \frac{53}{99} \)). Instead of guessing, Mrs. Anderson provided a solution that was on the right track and offered a reflection admitting the incomplete connection. While Cuoco, Goldenberg, and Mark (1996) did not specifically mention this type of “thinking” in their discussion of habits of mind, recognizing errors is consistent with the types of habits they do address. It is positive that Mrs. Zander has some understanding that she should connect the current work with previous work. It is a concern that she did not learn enough from previous work to solve these problems.

Analysis of Mrs. Zander’s work also reveals situations when she is unsure if she should connect a past idea to a new problem. She questioned herself as to whether or not
she should use Gauss’ formula of addition to solve a chess game problem from the first
course.44 (See Figure 46; note her own question mark).

![Figure 46. Mrs. Zander’s Own Questioning about Gauss’ Formula, MSL](image)

While there is a strong connection as to how to find Gauss’ formula, Mrs. Zander cannot
make the switch from an additive situation (i.e., 2+4+6+8+10…) to multiplicative one
(i.e., 1+2+4+8+16…). However, it is a positive, not a negative, that Mrs. Zander is
remembering Gauss’ formula. This evidence indicates Mrs. Zander is learning that new
mathematics ideas are built on old ones (i.e., making connections). At the same time,
however, two limitations show up. First, previous material has not been learned well
enough. And second, problem solving skills are still being developed; thus, Mrs. Zander’s
learning is a work in progress.

Mrs. Anderson and Mrs. Zander used the mathematical practice of making
connections between past and present problems as an entry point to solve problems, a
support when they felt stuck while solving, and a tool to help make sense of a new
problem or solution. Thus each used connections to help in the transfer of knowledge
from their long-term to working memories. Making connections offered varied results in

44 Refer to Appendix S for the chess game problem.
each of their paths towards understanding mathematics. There were some instances when making connections likely aided the learning process but other instances when the use of connections revealed their learning was a work in progress. Nonetheless, this way of learning occurred frequently in each of their mathematical work for the first three courses.

**Using Representations**

“I just drew it out” (Problem-Solving Session Vignette, Line 26). When prompted to share her initial work on the Newton-Lewis problem, Mrs. Anderson simply stated that she used a representation (refer back to Figure 32). Mrs. Zander also used a representation as her entry point into this problem (refer back to Figure 33). These examples, taken from the first vignette, help to illustrate a second common mathematical practice both teachers accessed as they learned mathematics during the first three courses: representations. The analysis of their written work revealed that both teachers frequently relied on using representations to help them solve mathematics problems. There were twenty-eight examples in Mrs. Zander’s work indicating use of representations. While nearly one-half of her examples specifically related to using a table, Mrs. Zander’s work also revealed representations in the form of diagrams, flow charts, and visual displays of computational or organizational work. Likewise, there were forty-one examples in Mrs. Anderson’s work, with over one-half relating to tables. Other representations in Mrs. Anderson’s work included diagrams, number lines, and comments relating the mathematics in a problem to an actual physical representation.
Learning to use representations to learn mathematics may have been relatively new to Mrs. Anderson. She participated in a professional development workshop related to algebra the summer before she started Math in the Middle. It appears that one of the goals for this workshop was to help middle grades teachers learn very specific ways to teach algebraic concepts to students (National Training Network, no date). “I was able to see how…visual representation could enhance student learning. I need to incorporate better strategies and practices into my own math program” (Application, Spring 2007). Mrs. Anderson’s words suggest this encounter influenced her practice of using representations as a teacher of mathematics. Data from the first three mathematics courses of Math in the Middle indicate Mrs. Anderson began to incorporate representations as a learner of mathematics (e.g., MSL Final Reflection; FAGMLT End of Course Problem Set).

Mrs. Anderson’s ability to use representations was growing as the first three mathematics courses progressed. There were few references from the first course of Mrs. Anderson’s use of a representation to assist her in solving problems. One of those references was part of the Open and Shut Case problem. A peer showed how using pennies could model opening and shutting of lockers.

Drawing it out on graph paper and erasing our Xs and Os each time a locker was open or closed—that was tedious. So [a peer] came up with using pennies and just putting them on or taking them off the grid to

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46 Refer to Appendix S for the Open and Shut Case problem.
represent an open or closed locker…Such a simple, sensible manipulative
to find the solution efficiently. (MSL, Final Reflection, 6/23/2007)

Nearly a year later Mrs. Anderson was the one who suggested the use of a manipulative
to help solve a problem.47 “Don’t eat the chips, they can be our deer” (Problem Solving
Session, 5/22/2008). It appears Mrs. Anderson began to see the value of using
representations for her own learning as part of the experiences from the first course. “I
benefited from the examples and manipulatives [the instructor] used. All of the visuals
and presentations helped me” (MSL, Final Reflection, 6/23/2007).

Mrs. Anderson may have used representations as an entry point to solve problems.

“Of course, to complete this task I began by drawing 12 clocks and then drew in the
hands at 12:00, 1:15, etc” (FAGMLT, End of Course Problem #11 Bobo and the Time
Bomb, 7/28/2007). Representations also served as a back-up strategy when her first
attempt failed her. “To solve this, after failed attempts of finding every possible
combination of favors for each child, I tried to visualize what was going on” (ECR, End
of Course Problem #7 Birthday Party Blues, 12/22/2007). Mrs. Anderson used a table as
an important resource to solve problems, at least twenty-two times in her mathematics
work. Figure 47 illustrates one example from the second course.

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47 Refer to Appendix S for the Birthday Party Blues and Bobo and the Time Bomb
problems.
I had to make a table and graph to help me visualize the problem. (See Linear Equation graph, 2nd graph attached.)

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>204</td>
</tr>
<tr>
<td>20</td>
<td>304</td>
</tr>
<tr>
<td>30</td>
<td>404</td>
</tr>
<tr>
<td>40</td>
<td>504</td>
</tr>
</tbody>
</table>

The change in y / the change in x. The change in x is +10. The change in y is +100. To write a linear function to represent the information on this table, I came up with \( f(x) = 10(x + 10) + 4 \) or \( 10x + 104 \). The recursive way to express it is \( f(x) = f(x-1) + 100 \).

**Figure 47.** Mrs. Anderson’s Work on End of Course Problem 2b, FAGMLT

Notice how Mrs. Anderson referenced her use of a table and graph to assist her learning.\(^{49}\) In a reflection from this second course, she wrote, “I really liked the clothespin problem because the table helps me see the equation; it increases my level of understanding” (FAGMLT, Sample Six, 7/28/2007). These examples are among many examples of how Mrs. Anderson specifically used tables to solve problems. More generally, tables are useful as they help learners to organize their thinking, identify patterns, and systematically communicate mathematical ideas (NCTM, 2000).

While apparent in her written work in all three courses, Mrs. Anderson did not reflect on her use of representations to solve problems until the end of the second mathematics course.

I guess I really “needed” [the Truncated Solids] problem because it made me feel successful for a change. It felt so good to finally understand something thoroughly and be able to help others with it instead of being the one who needs help all the time… I am guessing it was the visual and manipulatives that made this problem so comprehensible for me. I

\(^{48}\) Mrs. Anderson should have written \( f(x) = f(x + 1) + 100 \).

\(^{49}\) Refer to Appendix S for the FAGMLT EOC #2b problem.
actually held the solid figures in my hands as I did the problem and explained it to my peers. (FAGMLT, Sample Six, 7/28/2007)

But by the end of the third course, she described using representations as an integral part of her problem solving toolbox.

A key element that I learned about mathematics during the process of solving [the Stellated Solids] problem was the importance of being able to ‘visualize’ each step. I was able to make drawings to enhance the explanations…Problems such as this one reinforce the idea that it is still important and beneficial for students of any age to use manipulatives or some other form of visual representation to achieve and solidify concepts.

(ECR, End of Course Reflection, 12/22/2007)

Simply giving students manipulatives, or other representations, will not necessarily improve mathematical understanding (Ball, 1992). “The student’s own internal representation of ideas must somehow connect with the external representation or manipulative” (Moyer, 2001, p. 192). Mrs. Anderson was able to make a connection between her internal understanding and the external visual representations. Also notice Mrs. Anderson’s phrase “of any age.” This is consistent with the NCTM’s (2000) process standard of representation, stretched across all grade levels.

Analysis of Mrs. Zander’s work also indicated her use of representations to help her learn mathematics. Mrs. Zander’s words suggest she was using representations to help her learn mathematics well before she entered the Math in the Middle Institute, “I need to write things out…using tables, graphs, drawings, those kind of things…I’m very
visual” (Initial Interview, 2/22/2008). The course reflections Mrs. Zander wrote as part of
the first three mathematics courses further suggest a disposition of using the visual as a
tool to help her solve problems:

- “I like to use tables or charts if possible as I’m a very visual person” (MSL, Favorite Five, 6/23/2007).
- “I personally learn a lot from hands-on mathematics” (FAGMLT, Sample Six, 07/28/2007).
- “This visual helped me decipher the natural, fake, right and wrong situations” (ECR, Bayes’ Theorem Homework, 12/17/2007).

Again, notice how she identified using representations as tools to learn mathematics in
each course.

There were many examples where Mrs. Zander drew pictures to represent
physical components of the problem. The following illustrates this kind of representation
in Mrs. Zander’s solutions to problems from the second and third course. She drew t-
shirts and clothespins in Figure 48.1 and a hexagonal doghouse and a dog in Figure
48.2.\textsuperscript{50}

\textsuperscript{50} Refer to Appendix S for the Clothespin and Spot’s Dog problems.
These pictures gave Mrs. Zander an entry into solving the problems. She could better handle the abstraction of mathematics. The pictures helped her do mathematics in much the same way Poyla (1969) stressed the importance of becoming an active participant to develop mathematical understanding.

There were other examples when Mrs. Zander used representations to organize her ideas. Often, the representation itself became part of her final solution. One specific example was her use of a table for the String of 8s problem from the first course. As she began working the problem, Mrs. Zander stated the value of the representation so she could organize her work.

I realized that a chart using 888, 88, and 8 would help me make a clear outcome where I could track the numbers I had used. As I charted the numbers, I realized that adding eleven 8s in the 8s column each time I took away one 88 would help me come up with further combinations.

(MSL, Favorite Five, 6/23/2007)
Figure 49 depicts Mrs. Zander’s work on the string of 8s problem. There were at least ten examples in her written work where she used a table to help represent part or all of a problem.

The use of these types of representations aided Mrs. Zander’s understanding of mathematics. They enabled her to organize information, recognize patterns and relationships (NCTM, 2000), and visualize mathematical ideas and relationships (Cuoco, Goldenberg, & Mark, 1996; NCTM, 2000).

**Testing Specific Cases**

A third mathematical practice Mrs. Anderson and Mrs. Zander used to help themselves learn mathematics was testing specific cases. Line 9 in the problem-solving session vignette highlights Mrs. Zander’s desire to “play around with numbers” by testing five degrees Newton (Problem Solving Session, 5/22/2008). Later, both teachers wanted to test a specific case for the radius in the Length of Rope problem (Problem-Solving Session Vignette, Line 65). These examples help illustrate Mrs. Anderson and Mrs.

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51 Refer to Appendix S for the string of 8’s problems.
Zander’s use of special cases to help them solve problems from the first three mathematics courses. Using a specific case to check the validity is common among learners (Knuth, Slaughter, Choppin, & Sutherland, 2002). For this study, there were nearly fifteen examples for each teacher in their written work.

On an end of course problem for FAGMLT, Mrs. Zander used four specific cases to help her justify her answer to a scale factor problem (see Figure 50):

<table>
<thead>
<tr>
<th>large polygon sides</th>
<th>area of large polygon</th>
<th>Divide sides by scale factor of 8</th>
<th>small polygon sides</th>
<th>area of small polygon</th>
</tr>
</thead>
<tbody>
<tr>
<td>80 x 8</td>
<td>640</td>
<td>+ 8</td>
<td>10 x 1</td>
<td>10</td>
</tr>
<tr>
<td>32 x 20</td>
<td>640</td>
<td>+ 8</td>
<td>4 x 2.5</td>
<td>10</td>
</tr>
<tr>
<td>16 x 40</td>
<td>640</td>
<td>+ 8</td>
<td>2 x 5</td>
<td>10</td>
</tr>
<tr>
<td>64 x 10</td>
<td>640</td>
<td>+ 8</td>
<td>8 x 1.25</td>
<td>10</td>
</tr>
</tbody>
</table>

Once I found possible side lengths of the larger polygon, I divided the sides by 8 to get the sides of the smaller polygon. In each case, the sides multiplied together gave me an area of 10 cm² on the smaller polygon.

Figure 50. Mrs. Zander’s Use of Cases on End of Course Problem 8b, FAGMLT

While the original problem prompted participants to consider the area of any polygon, Mrs. Zander narrowed the problem and considered only rectangles. She then found four examples that fit the original criteria for an area of 640 (i.e., 80 x 8; 32 x 20; 16 x 40; 64 x10). While this list was by no means exhaustive, Mrs. Zander had four specific cases she could use to look for a pattern. She eventually determined the area of the smaller polygon to be 10 cm², the correct solution. Using specific cases, or empirical arguments, as reasoning and proof can reveal limitations in student understanding (Healy & Hoyles, 2000); however, the use of specific examples can also reveal a sophisticated understanding, as sometimes it amounts to proof by example.

52 Refer to Appendix S for the FAGMLT EOC #8b problem.
On several occasions, Mrs. Zander tested one or more cases to help her solve an equation (e.g., MSL End of Course Problem #2; ECR Session A #8). She even wrote about why she liked to test cases as part of one of the weekly reflections, “I understand a math idea when I can plug and chug examples to my heart’s content” (FAGMLT, Week 1 Reflection, 7/20/2007). Testing one example, which she sometimes referred to as “playing” or “plugging and chugging,” was evident in both vignettes (Problem-Solving Session Vignette, Lines 9 and 45). Cuoco, Goldenberg, and Mark (1996) identified the characteristic of playing with mathematics as they described their idea of habits of mind. Mrs. Zander may have used specific cases as an entry point to understanding the mathematics. It is also plausible that Mrs. Zander liked to test cases to her “heart’s content” (FAGMLT, Week 1 Reflection, 7/20/2007) as that is how she convinced herself if she had the correct answer or at least was headed in the right direction.

Analysis of written work revealed how both teachers would test a case, then check another, and then finally offer reasoning from a more general point of view. One example is Mrs. Anderson’s reasoning for a problem related to discounting the price of a coat assigned at the end of the first course.

To solve this problem, I first used numbers. I used 80 as the original price…2/5 or (40%) is the actual overall savings from the original prices of the coat…I checked this by using $100 for the original price and the answer was 2/5 again…The reason that 9/20 was wrong was because...

(MSL, End of Course Problem #8, 6/23/2007)

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53 Refer to Appendix S for the ECR Session A #8, MSL EOC #2, and Coat Discount problems.
Mrs. Anderson moved toward a more general argument for the problem. It appears that testing the second case (i.e., $100) prompted her to step back and reason through the problem from a more generalized viewpoint. This may be a sophisticated use of a concrete value. If it is a problem about percents or ratios, the answer one gets for any concrete value is the correct percent. Thus, this can be a great problem solving approach.

Another example is Mrs. Anderson’s solution to a Pythagorean-type problem assigned at the end of the third course (see Figure 51). She again tried specific cases, recognized a pattern, checked more cases, and then finally offered general reasoning.

![Figure 51. Mrs. Anderson’s Reasoning for the End of Course Problem #3, ECR](image)

Refer to Appendix S for the ECR EOC #3 problem.
While Mrs. Anderson did offer a generalized argument, she did make a minor error by referring to the second leg of the triangle as \((n^2 + 2)\) instead of \((n^2 + 2n)\). Nonetheless, this example again points to a great problem solving approach.

Analysis of Mrs. Zander’s solution for this same problem revealed she used cases in a similar manner (see Figure 52).

Figure 52. Mrs. Zander’s Work on the End of Course Problem #3, ECR

Mrs. Zander also tested the first two pairs of consecutive odd numbers like Mrs. Anderson. Mrs. Zander then tested one more case before moving to a general argument. Nathan and Koedinger (2000) found some learners develop an understanding using numeric equations before they understand the same concepts with symbolic equations and verbal expressions.

Offering a generalized argument after providing one or more specific cases did not characterize either of the teachers’ overall bodies of work. Several examples depicted their use of specific cases followed by no explanation of the general rule. This is a common for learners of mathematics (Healy & Hoyles, 2000; Hersh, 1993; Knuth, 2002).
Yet, this again points to the developmental process of learning. These two teachers’ mathematical learning was a work in progress. The fact they did not offer many general arguments in the first three courses of the professional development is not a criticism, but rather a reminder that each teacher had more to learn.

One final example from Mrs. Zander’s work offers evidence that the teachers’ use of specific cases to solve problems was a good strategy for solving the problem, even if they may not have understood the whole process. “My dog was 100 m from home, and my cat was 80 m from home. I called them, and they both ran directly home. If my dog ran twice as fast as my cat, how far from home was my cat when my dog reached home?” Figure 53 displays Mrs. Zander’s reasoning:

| 12.4 | We know that the cat traveled 50 meters in 1 hour of the total 80 meters to get home. There are 30m meters yet to travel. With the cat traveling at 50 miles per hour, you can set up a proportion to find out how long it will take to travel 30 meters (x). The proportion is: 50m/1 hour = 30m/x hours x = 0.6 of an hour or 36 minutes

The dog ran 100 m and the cat ran 80 m. Since the dog runs twice as fast as the cat, let’s say the dog runs 50 mph and ran 100 m in 2 hours. The cat would run 25 mph and I can set up a proportion for the length of time it takes the cat to travel 80 m. It’s 3.2 hours.

If the Dog ran 100 meters per hour then the Cat ran 50 meters per hour.

After the first hour, the cat has 30 meters to go. It takes 1 hour to go 50 meters so it’ll take 0.6 of an hour, or 36 minutes, to go 30 meters more. Therefore, the cat was 30 meters from home and 1 hour and 36 minutes away from home or has 36 minutes left from the time the dog arrived home.

Can you argue/explain how far the cat is from home regardless of the speed?

**Figure 53.** Mrs. Zander’s Work for End of Course Problem #12.4, MSL, 06/23/2007

Mrs. Zander first assigned a specific speed for the dog and the cat instead of any speed. She then based her overall argument on those specific numbers. Notice in the example
how the instructor prompted the teachers to move from a specific to a general argument. Instructors recognized teachers’ learning was not instantaneous. This is important as Yerushalmy (2000) noted it takes time for learners to develop the ability to solve problems in more general ways. The instructor was pushing Mrs. Zander to think about a general argument, even during the first course.

**Theme 3: Two Teachers’ Differing Learning Practices**

This third and final theme offers a detailed description of a disposition of learning mathematics unique to each teacher. Mrs. Anderson’s body of written mathematics work from the first three courses revealed her persistent disposition toward learning mathematics. Oftentimes, she did whatever was necessary to learn. Mrs. Zander’s work revealed a disposition of connecting her learning of mathematics with her teaching of mathematics. She consistently looked for ways to use problems and strategies for learning in her own classroom.

**The Persistent Nature of Mrs. Anderson**

“I mean it looks like it, but is it?...Is there some kind of arc thing?...This one bothers me. I want proof for sure” (Problem-Solving Session Vignette, Lines 37, 64, and 68). Mrs. Anderson did not have a complete understanding of the Length of Rope problem and wanted to keep working on it. She knew there was a reason that would support her conjecture. It bothered her that she could not find it. I sensed she was frustrated when I did not let her continue to work on it. This example illustrates a common theme present throughout Mrs. Anderson’s written work, a persistent disposition to learn mathematics.
This persistent disposition could be found in her mathematics work (thirteen instances) and reflections (seven instances) spread across the first three mathematics courses. Each example pointed to a similar path of learning: Mrs. Anderson was simply willing to try. She was willing to start a problem, even when she did not know where to start. She was willing to seek help from peers and instructors. She was willing to take the time to learn, even if that meant spending more time than what was expected. She was willing to revise her work, time and time again.

The words Mrs. Anderson included in her reflections, discussion board postings, and her actual solutions let others know the amount of time she spent learning mathematics. One example followed the opening weekend of the third course.

I spent the first part of my week on the book homework problems trying to get ready for [the first online meeting]…After we met, I worked till midnight on the homework again. Thursday I worked till about 11:30 on it trying to start getting it polished enough to turn in. I think I am that far with the first 3, but need to polish the last 3 and I still feel kind of questionable about a part on one of my answers. (Email Correspondence, 9/15/2007)

This episode did not represent the first nor last time she spent extended amounts of time learning.

There were times she would just include a sentence or two as part of her solution. “Even after the help of peers, I took this one home and struggled with it. This was my 4th of July entertainment for the evening and that is when it all finally came together for me”
(MSL, End of Course Problem Set). That particular reflection was from the first course; the next was from the third course. “When Mother Nature dumps tons of ice on you on the first day of December…what better to do than work on your ‘Conditional Probability’ homework!! I worked on it for about 6 hours so far today” (ECR Blackboard Posting, 12/01/2007). Mrs. Anderson spent a considerable amount of time learning mathematics in all three courses.

Her reflective writing often offered some insight to why she spent so much time on the problems.

I did spend a lot of time on them. It doesn’t just come to me easily; I really have to think about the problems…I did have to spend more time on the homework each night because I was unable to finish it in the recommended four hours. I was up until at least midnight each day, but I wanted to do all of them. (MSL, End of Course Reflection, 6/23/2007)

She acknowledged the mathematics did not come easy to her. She needed extra time to process and make sense of the mathematics. This evidence suggests Mrs. Anderson’s persistent disposition was partially fueled by the time she was willing to spend learning mathematics.

One specific way Mrs. Anderson processed mathematics became apparent during the problem solving session. She took the time to slowly read the problem aloud and then reread the problem aloud (Problem Solving Session, 5/22/2008). This strategy of rereading was present in her written work as well. “[T]o figure out how many conversations Herbie had was still a puzzle. So I read the problem over and over again”
(MSL, End of Course Problem #13, 6/23/2007). “I really thought I was on to something – but couldn’t come up with anything. I reread the question…” (FAGMLT, End of Course Problem #14, 6/23/2007). This evidence suggests Mrs. Anderson’s persistent disposition was also fueled by strategies she incorporated as she tried to learn mathematics.

One instructor noted Mrs. Anderson’s persistent disposition. “Mrs. Anderson worked hard to learn the material… She took a lot of time to process information and I worked hard not to intervene too early” (FAGMLT Instructor Email Correspondence, 9/27/2007). As Mrs. Anderson worked with her peers, she would tell them to slow down. She recognized her own need to spend time processing the mathematics. Mrs. Anderson used the word “wait” six times during the ninety minutes she worked with her peers during the problem solving session.

Another aspect of Mrs. Anderson’s persistent disposition was her willingness to try new approaches as she solved problems. She did not let unsuccessful first or second attempts prevent her from eventually coming up with a solution. She often included those failed attempts as part of her final solution. She first presented her final solution followed by her other attempts (see Figure 54; refer to Appendix T for her solution).²⁵

Figure 54. Mrs. Anderson’s Note on Her Solution of a Problem from MSL

²⁵ Refer to Appendix S for the Fly and Spider problem.
She was persistent in trying to find the best solution for this particular problem (i.e., the Spider and Fly problem from MSL). While collaboration and using representations likely helped her solve this problem, persistence is what kept her working on the problem from start to finish.

Mrs. Anderson did not display a sense of discouragement as she revised her work for this problem; rather, she spoke of the enjoyment she experienced while solving this problem.

This was one of my favorites because I really thought I had the whole thing right by myself! I was closer than a couple guys at my table and was really confident that I had it! Then—I found out I was WRONG! I had this one down to 19.65 feet and the actual answer is 19.42 feet! And as soon as I heard why, I just thought, “Why didn’t I think of that in the first place!! I know the shortest distance between two points is a straight line!” (MSL, Favorite Five, 6/23/2007)

Mrs. Anderson viewed the revision process as the opportunity to gain understanding. This further indicates a persistent disposition, and one that represented in the National Research Council’s (2001) fifth strand of mathematical proficiency. Mrs. Anderson did not detest the revision process.

Making revisions was a common part of Mrs. Anderson’s work. She first tried a problem on her own. Then, she collaborated with her peers. Finally, she spent time alone again, revising her work and polishing her final answer. This pattern was consistent across her mathematics work from the first three courses, and very much what M2
instructors wanted to have happen. Mrs. Anderson’s persistent disposition likely pushed her to make progress on each problem before seeking help. “I wanted to go through all of the problems independently before I met with any peer groups” (MSL, End of Course Reflection, 6/23/2007). Mrs. Anderson was moving closer to the type of autonomous learner the NCTM (2000) calls for learners to become. Working alone on the problems likely assisted her in identifying weaknesses. She then knew what she wanted to ask about, so that she could fully answer each question. A small hint, extra clue, or confirmation that she was on the right track helped Mrs. Anderson solve a problem. She often documented this persistent pathway as she wrote up her final solutions.

In my opinion this problem was the most difficult problem on this test and without 2 crucial clues, I don’t think that I could have ever solved it! Even after the assistance, I had to study it and contemplate it…I tried a few erroneous pursuits and finally sought the help of peers…So I tried that and still didn’t come up with the right answer…Even after more help from peers, I took this one home and struggled with it. This was my 4th of July entertainment for the evening and that is when it all finally came together for me…But to figure out how many conversations Herbie had was still a puzzle…So I read the problem over and over again, I remembered something…that was the answer…This one was truly a challenge! (MSL, End of Course Reflection, 6/23/2007)

Without a persistent disposition, would Mrs. Anderson have been able to solve this problem? This reflection is representative of other reflections in Mrs. Anderson’s written
work where she documented her process of solving a problem. First, she did not think there was enough information when she read the problem; yet she tried something and made an attempt on her own. Second, she sought help from peers; however, the clues she got from her peers were frequently not enough help. Again, she pressed on. Third, she spent extended time on the problem on her own. Finally, she kept rereading the problem over and over again to see if she missed something. The culmination of these episodes led to her success. Her persistence contributed to her success and seemed to support the development of math practices.

Mrs. Anderson’s persistent disposition also prompted her to consult with Math in the Middle graduates. One time, she solved particular problem for the third course; yet she was not confident in her solution. She then approached a graduate for advice.

[A Math in the Middle Institute graduate] walked me through this again, but I am still so shaky. Is there anyway that this can be presented in class when we come back in January? I would benefit from having this explained again in class, when we’re not under the time constraints. (ECR, End of Course Reflection, 12/22/2007)

Even with the help, Mrs. Anderson was not ready to move on from this problem. She wanted to spend more time with the problem so she could better understand. She wanted the instructor to revisit the problem the next time she saw him. This parallels the way she wanted to know how to prove that the curved sections in the Length of Rope problem were really one-third of the circumference of each circle (Problem-Solving Session Vignette, Line 67). She did not want to leave that problem without knowing.
Considering background data, Mrs. Anderson’s persistent disposition could be found in other areas of her life. Recall the extended time she needed for math while learning mathematics in high school. Her math teacher was also her volleyball coach; he spent time helping her learn mathematics after school and after volleyball practice was over. Recall also that Mrs. Anderson was persistent in going back to school, earning a bachelor’s and master’s degree from college, all while having three children and maintaining a farm (Initial Interview, 2/22/2008). Mrs. Anderson was a persistent learner. Mrs. Anderson kept trying. She kept asking questions. She did not give up. She wanted to learn the mathematics and was willing to access all resources possible to do so.

Mrs. Anderson’s reflective writing from the first three mathematics courses and her engaged presence during the problem solving session is consistent with the following advice from the NCTM (2000):

Students learn more and learn better when they can take control of their learning by defining their goals and monitoring their progress. When challenged with appropriately chosen tasks, students become confident in their ability to tackle difficult problems, eager to figure things out on their own, flexible in exploring mathematical ideas and trying alternative paths, and willing to persevere. (p. 21)

These words reflect what I found in Mrs. Anderson’s written work and reflections. She was challenged by tasks that are appropriate for middle grade teachers to learn (CBMS, 2001). She may not have always portrayed herself as a confident learner; however, she often wrote of her desire to try things on her own
before seeking help (e.g., MSL, End of Course Reflection, 6/23/2007),
explorations of “erroneous pursuits” (i.e., unsuccessful paths) (MSL, End of
Course Reflection), and willingness to spend hours working on problems (e.g.,

Mrs. Anderson’s persistent nature likely helped her cope with the mistakes
she made as she learned. She often reflected on her errors with humor (e.g., her
“4th of July entertainment,” MSL, End of Course Reflection, 6/23/2007). She
possessed the never-give-up attitude the NCTM (2000) describes of effective
learners:

Effective learners recognize the importance of reflecting on their thinking
and learning from their mistakes…view the difficulty of complex
mathematical investigations as a worthwhile challenge rather than an
excuse to give up…When students work hard to solve a difficult problem
or to understand a complex idea, they experience a very special feeling of
accomplishment, which in turn leads to a willingness to continue and
extend their engagement with mathematics. (p. 21)

Mrs. Anderson’s persistent nature likely fueled her drive not to give up no matter how
much time she needed to commit. Ultimately, persistence helped her become a more
successful learner of mathematics.

Mrs. Zander’s Disposition of Thinking about Teaching Math While Learning Math

“I do think, now, that it is $\frac{9}{5}$ and $\frac{5}{9}$. That’s in the sixth grade curriculum. I should
know” (Problem-Solving Session Vignette, Line 31). Mrs. Zander made a connection
between a Fahrenheit-Celsius conversion problem and a concept she teaches to her own students. This example illustrates a major theme found across Mrs. Zander’s mathematics work and reflections from the first three courses. Mrs. Zander frequently thought about her teaching of mathematics as she learned mathematics. Generally, she identified problems and strategies she could use with her own students (twenty instances) or at least share with her colleagues (six instances).

Data analysis revealed a disposition of connecting learning and teaching time and time again. In addition to examples in her mathematics work and reflections, there were explicit comments she wrote in the margins of her notes. Further, Mrs. Zander made comments during interviews and wrote statements in emails that confirmed her desire to connect learning with the teaching of sixth grade students. Mrs. Zander frequently thought about instructional strategies as well as problems she could use with her students and share with her colleagues.

On the first day of the institute, Mrs. Zander’s notes included comments about teaching mathematics (see Figure 55).

![Figure 55. Mrs. Zander’s Comment about Teaching the Concept of Adding Fractions, MSL Day 1, 6/18/2007](image-url)
MSL instructors began by introducing the adjective-noun theme. Mrs. Zander “loved” that theme. She wrote herself a reminder to use this theme when she taught her students the concept of adding fractions. She also referred to this as an “Ah-ha” moment, the first of many from the first three courses (e.g., MSL End of Course Reflection, 6/23/2007; FAGMLT Number Triangle Problem, 7/19/2007).

Later during the first week, Mrs. Zander identified another instructional strategy she wanted to use when she taught sixth graders (see Figure 56).

![Figure 56. Mrs. Zander’s Comment about Teaching Multiplication by Zero and by Negative Numbers, MSL Week 1](image)

Notice how she described this strategy as a “great method for teaching.” Her experiences learning these new methods made an impact on her. She wrote reminders to herself on her notes, sheets of paper that no one else would even see. Analysis of Mrs. Zander’s
work from the second and third course further revealed her tendency to identify instructional methods that she was interested in testing out in her classroom (e.g., FAGMLT, Sample Six Reflection for the Ferris Wheel problem, 7/27/2008).

Many professional development experiences are filled with teachers looking for that single make it and take it activity to immediately use in their own classrooms (e.g., Farmer, Hauk, and Neumann, 2005). That is not a characteristic of high quality professional development (Loucks-Horsley et al., 2003). While Mrs. Zander is making connections to her teaching throughout her professional development experiences, she is part of a sustained professional development program, primarily focused on mathematics content. She is choosing to find things (i.e., teaching strategies) that translate back to her classroom; she is not being given things that directly translate.

Just as she identified several instructional strategies to incorporate into her teaching, Mrs. Zander also identified several problems she wanted to use as she found a natural fit into her curriculum. Mrs. Zander often linked a problem and her curriculum as she wrote reflections for each course, noting the appropriateness of particular tasks to her work with sixth graders:

- The Triangle Game problem will be a great problem for sixth graders to work on and that’s why I chose it as a Day 3 favorite. (MSL, Favorite Five 6/23/2007)

- I think the Number Guess Magic problem will be a fun mathematical trick to share with my sixth graders. They’ll get a kick of using this trick. (FAGMLT, Sample Six, 7/28/2007)
• I was impressed at how this one sequence [i.e., Fibonacci] had an incredible amount of mathematical connections to my curriculum: patterns, fractions, decimals, algebra, problem solving and visual organization. I like that the sequence itself is very simple and my sixth graders could find the pattern with ease. (ECR, End of Course Reflection, 12/22/2007)

She kept her role as a sixth grade teacher in the back of her mind as she moved throughout this long-term professional development program.

She adapted many of the problem solving-type problems, often labeled as habits of mind problems by her instructors, for use in her own teaching. “I’ve retyped some of the ‘Habits of Mind’ problems to have ready for my students and have begun to align them to the objectives in the 6th curriculum” (MSL, End of Course Reflection, 6/23/2007). Figure 57 depicts an example of a problem she adapted from the second course. Notice how she added a section at the top aligning the problem with objectives from her curriculum. She added a piece of clip art. Comparing her adapted version of the problem with the original found in Appendix U, she also changed some of the language as well, in ways that made the problem more accessible to middle level students. In particular, she narrowed the problem to just one single question rather than two. This is quite different from what the professional development literature refers to as make it and take it. Mrs. Zander saw a problem, figured out where to situate it in the scope and sequence of her curriculum, and made it age appropriate. This goes well beyond
(Objectives: place value, division, remainders, fractions, decimals, diagrams)

**Habits of Mind Problem: Ferris Wheel**

Riding the Ferris Wheel

You and your little sister go to the Nebraska State Fair. They have two Ferris wheels this year. One is small and one is large. You get on the large Ferris wheel at the same time your sister gets on the small Ferris wheel. The rides begin as soon as you both get buckled into your seats.

The large Ferris wheel you are on makes 1 revolution (turn) in 60 seconds and the small Ferris wheel makes 1 revolution in 20 seconds.

How many seconds will pass before you and your sister are at the bottom again?

*Figure 57. Example of a Problem Mrs. Zander Adapted for Her Own Classroom*

attending a session directed at handing out a specific problem that one can immediately use without contemplating how best to use and adapt it to meet larger goals (Bay, 2000; Farmer, Gerretson, & Lassak, 2003; Loucks-Horsley et al., 2003).

Mrs. Zander sent this and many more “adapted” problems to her fellow M² peers.

“Here's the Habits of Mind problems I typed and would like to share. Use 'em, modify 'em or feel free to delete 'em. Remember they're geared for 6th grade. Let me know if you've found some cool problems to share” (Email Correspondence, 7/31/2007). She also shared ideas and tasks with her colleagues from school. “I have been listing facts, strategies, proofs, and problems that I can share with my co-workers” (MSL, End of Course Reflection, 6/23/2007). There were times Mrs. Zander simply attached a sticky
note to her written work reminding her of an idea she saw as relevant to her teaching (see Figure 58).

Figure 58. Mrs. Zander’s Reminder to Share with Colleagues

Mrs. Zander’s disposition of connecting learning with teaching was part of her persona well before she entered the professional development program. She hinted that this disposition was one of the reasons she applied for Math in the Middle in the first place. “I want to further my education in the realm of math so I cannot only improve my teaching and students’ achievement, but, I can share the knowledge with my peers as a leader” (Application, Spring 2007). As noted earlier, her role as a teacher included the assignment of being the district math representative for her school (Application, Spring 2007). Thus, Mrs. Zander already had a means by which she could pass along information to her colleagues. Her mathematics work from the first three courses provided numerous examples of how she kept teaching in the back of her mind. She did not just focus on learning mathematics; rather she focused on learning mathematics so that she could better teach mathematics.
Conclusion

Mrs. Anderson and Mrs. Zander’s written mathematical work, reflections, and live interactions indicated both teachers embraced collaboration as a tool to learn mathematics. Further, and in the spirit of Cuoco et al. (1996), Driscoll (1999), and the CBMS (2001), the two teachers’ habits of mathematical learning became evident in mathematical practices they deployed during the first three mathematics courses as well as the problem solving session. Mrs. Anderson and Mrs. Zander both utilized making connections, using representations, and testing specific cases to learn mathematics. Simultaneously, the teachers’ learning looked different from each other as they used the mathematical practices differently. Additionally, Mrs. Anderson displayed a persistent nature in solving problems while Mrs. Zander consistently looked for ways to link her mathematical learning to her teaching.
Chapter 7: Discussion, Implications, and Conclusion

This final chapter offers discussion of Linda Anderson and Becki Zander’s experiences learning mathematics during the first three courses of the Math in the Middle Institute. The first part of this chapter offers a broad discussion of these two teachers as learners of mathematics. The next part of the chapter moves the discussion to the ways these teachers learn mathematics. The third part includes a brief discussion of the intersection of learning mathematics and teaching mathematics through teaching vignettes. Finally, this chapter offers implications for professional development, recommendations for mathematicians working alongside education faculty, and suggestions for future research.

Teachers as Learners of Mathematics: Understanding Mathematical Content

This study revealed that two teachers, both part of a professional development program that takes seriously the recommendations set forth by the CBMS (2001), developed a deeper understanding of mathematical content. The work these teachers submitted for the first three mathematics courses showcased a satisfactory understanding of mathematics needed to solve the given problems. Further, the teachers’ first drafts of solutions, not their polished ones, provided this evidence.\textsuperscript{56}

Strengths of the teachers’ work were consistent with recommendations offered by the CBMS (2001), as both teachers developed, or deepened, an understanding of some of the important mathematical content knowledge middle school teachers should possess.

\textsuperscript{56} If this study would have examined only the polished solutions, it is very likely the evidence would have suggested these two teachers were developing even a deeper understanding of the mathematics involved.
For example, both participants applied proportional reasoning skills and provided explanations for their work (p. 28), developed an understanding of linear functions and rate of change patterns in linear relationships (p. 30), and demonstrated understanding of similarity (p. 32). Each of these strengths also lined up with NCTM (2000) recommendations and indicated a deeper, wider, and more thorough understanding of important mathematical concepts (Ma, 1999).

While some limitations were also present upon examination of the teachers’ draft solutions of written work, one must be reminded these teachers were learning mathematics at the start of the twenty-five month coherent program. It is understandable that neither teacher possessed a complete understanding of the mathematics needed to solve some problems. Areas of weakness included notational errors, incomplete explanations, and a struggle with inverse relationships. These types of errors are not uncommon to learners of mathematics (e.g., Ball, 1993; National Mathematics Advisory Panel, 2008; National Research Council, 2001; Stigler & Hiebert, 1999). Nonetheless, both teachers’ written work revealed understanding of some of the mathematics the CBMS (2001) recommends for teachers to know.

**Teachers as Learners of Mathematics: Communicating Mathematical Understanding**

Both of the participants in this study are capable of communicating mathematics in ways beyond what is expected between a learner and instructor. Portions of their solutions offered evidence supporting the notion these participants can embrace the task of communicating mathematically. Each participant offered reasoning clear enough for peers, and likely suitable for their own middle level students to understand. Further, each
participant expressed herself in a mathematically coherent manner, using precise and appropriate mathematical language.

Communication is a central theme in math reform. One of the five NCTM (2000) process standards is communication. The CBMS (2001) calls for teacher educators and professional developers to emphasize mathematical explanation as part of coursework. The committee suggests that this will likely improve teachers’ own ways of thinking and ways of teaching (CBMS). Further, the CCSSI (2010) calls for learners to “justify their conclusions [and] communicate them to others” (p. 7). Communication is an important part of learning mathematics.

Math in the Middle instructors modeled what mathematical communication looks like and provided multiple opportunities for individual learners and communities of learners to communicate, both verbally and in the context of written solutions. These two participants’ written work revealed growth in mathematical communication. Some solutions from the first course included very little explanation; whereas some solutions from the second and third courses included more complete explanations. As they were able to observe numerous examples of what it looks like to communicate mathematics and then actually communicate mathematics to peers and instructors, these two teachers were able to enhance their own ability to communicate mathematically in ways supported by CBMS (2001), CCSSI (2010), and NCTM (2000).

It is important to note the research questions for this study did not prompt an investigation of teachers’ growth in mathematical communication. Yet, the evidence was overwhelming. Many early solutions were written in a way to communicate an
understanding merely to an instructor, who already understood the mathematics. However, many later solutions were written in a way to communicate to peers as well as an instructor. A peer, with a limited understanding of mathematics, would likely be able to pick up one of these later solutions and understand the mathematics needed to solve the problem. The teachers’ focus in writing changed from explaining what was going on in the solution to a problem to why the problem was solved in a particular way (Knuth, 2002; Ma, 1999; Wood, 1999).

**Teachers as Learners of Mathematics: Mathematical Insensibilities**

Findings revealed both participants have gaps in understanding. This was expected due to the nature of bringing together a diverse group of learners, including many with limited past experiences learning mathematics. Coupled with the fact these teachers were immersed into a challenging program, which met from 9-5 daily, and assigned nightly homework, it is understandable the teachers had gaps in understanding. Analysis of these two teachers’ written work revealed a type of misunderstanding, which can be defined as a lack of mathematical sensibilities. The first Common Core State Standard for mathematical practice (CCSSI, 2010) calls for learners to able to “make sense of problems and persevere in solving them” (p. 6). Mathematical sensibility is a prominent theme throughout the entire CCSSI document. Analyzing these two teachers’ work from the first three mathematics courses of the professional development highlighted several mathematical “insensibilities.” Examples included ways both teachers communicated mathematically (i.e., using the word “after” to mean values to the left) and mathematical representations (i.e., using an undefined fraction, $\frac{32}{0}$, to represent a pairing
of temperature values). There were many instances when these two participants were just not careful (CCSSI, 2010). One must wonder, even at this early point in their professional development program, why did these mistakes not bother them?

One hypothesis, albeit a complex one, considers the participants’ mathematical backgrounds. Neither participant entered this professional development with the mathematical background needed to develop a complete understanding of mathematical concepts involved in what they were learning. The participants had completed very little mathematics coursework during their teacher preparation programs. Prior to this professional development, the participants had been exposed to few of the important concepts the CBMS (2001) calls for teachers to deeply understand. Yet it is not surprising for teachers to enter professional development programs with a limited mathematical background (RAND Mathematics Study Panel, 2003; Ball, Lubienski, & Mewborn, 2001; Romberg, 1997; Silver, 1998).

**Teachers as Learners of Mathematics: Making Mathematical Connections**

“When students can connect mathematical ideas, their understanding is deeper and more lasting” (NCTM, 2000, p. 64). Both participants made connections while learning mathematics in the professional development; yet there were varied purposes for why each teacher made many mathematical connections. I categorize the difference as “making connections between past examples and the current problem with the goal to solve the current problem” as compared to “making connections between the current problem and other mathematical domains with the goal to enhance mathematical understanding.” These two purposes parallel descriptions from the NCTM. One learner
used connections to “understand how mathematical ideas interconnect and build on one another” (p. 65) whereas the other learner used connections to “recognize and apply mathematics in contexts outside of mathematics” (p. 65).

One participant primarily focused on practices that helped her solve problems. She cited pages from the text, formulas offered from peers, and duplicated representations from class work. Each of the connections she made was important because they pointed her in the right direction. In contrast, the other teacher cited practices from her experience in the Institute in a way that said, “Ah-ha…this is neat.” She was able to connect an extraneous part of a solution to a major theme from a class session or connect a problem, or problem solving technique, to a situation from her past. Many of the connections she made for herself aided her in solving the problem but also linked together a myriad of ideas from her previous learning and teaching experiences.

There are instances of the word “connection” in the Common Core Math Standards (CCSSI, 2010) and several more instances in the CBMS (2001) recommendations. Connecting new topics to one’s prior knowledge is not new (e.g., Bruning, Schraw, Norby, & Ronning, 2004). Learners must make connections in order to deepen their knowledge and understanding of important mathematics (Carpenter & Lehrer, 1999; Schoenfeld, 1988; National Mathematics Advisory Panel, 2008). Both teachers made connections; yet one of the big ideas that emerged from examining their mathematical work was that each teacher had a different purpose in making connections.
Summary of Two Teachers’ Learning of Important Mathematical Content

The CBMS (2001) offers recommendations as to the education of mathematics teachers. The NCTM (2000) presents ten standards for school mathematics in addition to six principles for school mathematics, including one each explicitly about teaching and learning. The CCSSI (2010) defines “what students should understand and be able to do in their study of mathematics” (p. 4). The purpose of these reports, in part, is to help shape the current and future state of professional development. This study offered a description of teachers’ learning as part of a professional development program that takes seriously the recommendations offered by these reports.

Upon further examination of the NCTM (2000), CBMS (2001), National Research Council (2001), and CCSSI’s (2010) reports, the analysis of teacher learning in this study provides vivid images of what it looks like to learn mathematics using such things as connections and representations. This study provides images of what it looks like to struggle while learning challenging mathematics, especially when one does not possess the necessary prerequisite skills. However, this case identifies some actions professional development programs can take to help teachers learn challenging mathematics (i.e., a coherent mathematics program, the use of instructional teams, the development of communities of learning, and high but attainable expectations). This study showed that teachers can learn important mathematics through professional development programs that place priority on recommendations such as the CBMS (2001) offers. This study also provided descriptions of what that mathematical learning looks like...
like, including strengths and weaknesses as well as differences between individual learners.

Describing both teachers’ use of connections to learn specific mathematical content only describes the “what” of learning. That description offers insight into what important mathematical content can be learned using mathematical connections. Naturally using connections became a means to “how” the teachers learned important mathematics. Taking another pass through these teachers’ mathematical work revealed ways teachers learned important mathematics. The next part of this chapter moves the focus of the discussion to the ways teachers learn mathematics.

**How Two Teachers’ Learn Mathematics: The Role of Collaboration**

The most obvious themes that emerged from studying two teachers’ learning of important mathematics in the Math in the Middle Institute was collaboration. Both participants not only used collaboration as an avenue to learn mathematics but also relied on collaboration to learn. Each recognized that they would not have been as successful learning the mathematical content without their peers. Yet, they were willing to start tackling problems on their own; moreover, they wanted to go as far as they could independently. They each used collaboration to move them further along the path of learning mathematics, especially when they felt stuck. Further, each participant used collaboration to learn different approaches to solve various problems as well as a platform to hypothesize. The participants could test out their ideas, whether to solve a problem or just verify a certain way of thinking was in line mathematically.
The recommendations offered by the CBMS (2001) address many key areas that will help reshape the education of teachers. Yet by design, the committee tried to avoid giving pedagogical advice. The NCTM (2000) principles and standards, however, called for teachers to create learning environments safe for students to use collaboration as a means to learn mathematics. Yet, little direction was given to educators to achieve this goal. Leaders in the field of cooperative learning (e.g., Johnson & Johnson, 1999) offer recommendations and guidance to practitioners in helping students learn to productively collaborate with one another to learn mathematics. Educators frequently use such guidance to make changes in the classroom. How does a teacher with little experience learning in a collaborative environment make the move to establishing a learning environment where students collaborate?

While it may be true that the answer for a K-12 classroom may be very different from the answer in a professional development program, the analysis of this study may help both settings. This study offered a description of what it means for teachers to learn mathematics in a collaborative setting. One participant of this study expressed the desire to experience collaboration as a learner, often void in her past learning experience. The other participant had rarely considered using collaboration as a tool to learn mathematics before entering the professional development program. During the first three courses, these two participants used collaboration as a tool to develop a deeper, broader, and more thorough understanding of important mathematical concepts (Ma, 1999). The description of teachers as learners offered in this study complement other researchers who discuss the benefits of collaboration for learners of mathematics (e.g., Goos, 2004; Jackson &
Bruegmann, 2009; Kramarski, Mevarech, & Arami, 2002; Leiken & Zaslavsky, 1997). These benefits have indicated, but are not limited to, an increase in engagement, confidence, problem solving strategies, and overall mathematical understanding.

A final thought related to using collaboration to facilitate the learning process is related to the social context of constructivism (e.g., Bruning, Schraw, Norby, & Ronning, 2004). Several educational researchers, including Richards and von Glasersfeld (1980), Kilpatrick (1987), and Cobb, Yackel, and Wood (1992), have addressed constructivism in mathematics education for nearly half of a century. Many aspects of this theory of learning have underlined reform efforts in K-12 mathematics education (NCTM, 1989, 1991, 2000; Simon, 1995). Yet, the central part of this theory has focused on K-12 learners of mathematics (e.g., Cobb, Yackel, & Wood, 1992), not teachers as learners of mathematics. This study revealed how collaboration played a large role in how two teachers learned important mathematics. Both teachers were able to construct mathematical knowledge and construct a deeper understanding of mathematics through collaborative interactions. Some researchers have called for models of teaching based on constructivism (e.g., Simon, 1995). This study describes a model of learning based on constructivism embedded in collaboration.

**How Two Teachers’ Learn Mathematics: Habits**

Another important finding from this study also relates to how teachers learn mathematics. Both teachers accessed their own habits of mathematical learning as they solved problems during the professional development experience. Both participants revealed similar habits of mathematical learning while participating in the professional
development program: seeking and making connections, using representations, and relying on specific cases. Each teacher frequently tapped into each of these resources as they solved problems; frequently enough to consider them habits (Rolle, 2008). Upon reading a problem or looking for an alternative path to solve a problem when they were stuck, each teacher accessed habits that assisted their learning. While accessing and using these practices did not always lead to success, the frequency of their use within the participants’ work and subsequent reflections supported the notion that turning to these practices was habitual.

Dewey (1916/1944) argued that educators must provide experiences meaningful to learners based on learners’ habits of mind. Cuoco, Goldenberg, and Mark (1996) identified eight mathematical habits of mind K-12 students should possess. Driscoll identified several habits of mind specific to learning algebra (1999) and geometry (2007). CBMS (2001) recommended that pre-service experiences and professional development coursework help teachers to develop habits of minds of a mathematical thinker. “Mathematical thinkers…take actions like representing, experimenting, modeling, classifying, visualizing, computing, and proving. Teachers need to…look at problems from multiple points of view. Most of all…teachers need to learn how to learn mathematics” (p. 8). The Common Core Standards (CCSSI, 2010), while not explicitly using the words habits of mind, listed eight standards for mathematical practice bearing some resemblance to the NCTM’s (2000) five process standards and National Research Council’s (2001) five intertwined strands of proficiency as well as researchers’ views on habits of mind.
The case presented in this study describes what it looks like for teachers to use their habits as mathematical learners. The analysis offers insight into strengths and limitations in accessing and using the habits of making connections, using representations, and checking specific cases. On one hand, much of the focus on past research has been on students’ habits of mind. This study, on the other hand, placed focus on the habits practicing teachers accessed and used as students learning, or relearning, important mathematics. The CBMS (2001) called for teachers to “learn to how learn mathematics” (p. 8). This committee also gave recommendations of what should be taught to teachers and, in many ways, how it should be taught (CBMS). The case presented in this study describes what it looks like for teachers to learn how to learn mathematics, in a professional development program that follows the recommendations offered by the CBMS.

This study points to common habits of two different mathematical learners as well as their unique habits. Not all practicing teachers access the same tools, or mathematical practices, to learn mathematics. Some habits will be more visible than others, given the individual learner. In this particular case study, one teacher displayed a strong habit of perseverance while the other displayed a strong habit of thinking about her own teaching as she learned. While this professional development program did not explicitly teach particular habits, Math in the Middle encouraged the development of these habits by regularly providing challenging curricula and opportunities for participants to demonstrate understanding. Each participant’s own ways of learning became more and more clear over time. Some ways of learning were present in the analysis of both
participants (e.g., the previously discussed habits of making connections, using representations, and relying on specific cases). However, different ways of learning across the two participants also emerged (e.g., one teacher’s persistent disposition and the other’s desire to connect her learning of mathematics with her teaching of mathematics). While these two unique habits were very different from each other, neither one was more important than another, as each individual habit of mathematical learning helped an individual participant become more successful learning mathematics.

Summary of How Two Teachers Learned Important Mathematical Content

The CBMS (2001) calls for teacher development courses to help teachers develop ways of learning how to learn mathematics. The CCSSI (2010) describes eight standards for mathematical practice that “rest on important ‘processes and proficiencies’ with longstanding importance in mathematics education” (p. 6). Each report hints on the importance for learners to develop the habits of mind of a mathematical thinker. Again, these reports are shaping the current and future state of professional development. This study offered a description of how teachers learn important mathematics, as they participant in a professional development program that takes seriously the recommendations offered by the CBMS (2001).

Both teachers use collaboration as a means to learn important mathematics. This study describes their collaborative practices. Both teachers access similar and different habits of mathematical learning. These two teachers are examples of what it looks like, in practice, for learners to be developing the habits of mind other researchers have been advocating. This study showed that teachers do use certain mathematical habits to learn
important mathematics and provided detailed descriptions of what those mathematical habits of learning look like, including strengths and weaknesses as well as variance between the individual learners.

**How Two Teachers’ Translate Their Learning of Mathematics to the Classroom: The Intersection of Learning and Teaching**

Do teachers translate their own “ways” of learning to their “ways” of teaching? Evidence from Mrs. Zander’s learning of mathematics indicates she does translate at least some of what and how she learns to what she plans to do in the classroom. As the research questions for this qualitative study were first formulated, the plan was to examine each participant’s “teaching” of mathematics in addition to her “learning” of mathematics. Thus, there was much “teaching” data collected for this study. However, one could not predict just how big the study of teachers’ learning of mathematics would be. In hindsight, examining each teacher’s own “teaching” of mathematics is a study in and of itself. Nonetheless, data on Mrs. Anderson and Mrs. Zander’s teaching of mathematics afford this study an opportunity to offer at least a glimpse into each teacher’s own classroom practices. The purposes of these snapshots are twofold. First, the snapshots reveal evidence that the teachers’ did translate some of their ways learning to their ways of teaching. And second, the snapshots strengthen the call for future studies to examine the intersection of the domains of learning mathematics and teaching mathematics.

**Mrs. Anderson’s Teaching: Episode and Discussion**

Mrs. Anderson teaches fifth grade math. She says she teaches the textbook, *Mathematics: The Path to Math Success!* (Fennell, Ferrini-Mundy, Ginsburg, Greenes,
Murphy, & Tate, 1999), from cover to cover. She especially likes this book because “it does a good job at helping students strengthen their basic facts” (Interview). One of the teaching episodes recorded for this study related to the area of parallelograms and triangles. The two lessons that immediately preceded this lesson focused on perimeter and area of rectangles and squares.

“Today we are going to do the area of a parallelogram.” After she writes the word “parallelogram” on the board, she asks her students to tell her what a parallelogram looks like. Two students raise their hands. Mrs. Anderson calls on Grant. He suggests that a parallelogram “sort of looks like a piece of paper that you’re looking at from an angle.” Mrs. Anderson confirms his answer and immediately moves to the front chalkboard and draws an example (see Figure 59). Her sketch does not include all characteristics of a parallelogram, as the opposite angles are not congruent.

![Figure 59.1. A Parallelogram Mrs. Anderson Drew on the Chalkboard](image)

Mrs. Anderson could have asked Grant to come up to the chalkboard to draw his idea of a parallelogram. Instead, she took the lead. She showed her students what a parallelogram looked like. As she participated in the M² Institute, Mrs. Anderson observed many occasions when instructors handed a marker to the participants and then

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57 The teaching episode occurred at a time that Mrs. Anderson had completed the first three mathematics courses of Math in the Middle and while she was in the process of completing her second pedagogy course.
asked them to share ideas with the rest of the group. Mrs. Anderson’s action did not match what she experienced as a learner of mathematics.

Mrs. Anderson’s sketch of the parallelogram was not particularly good, as the one she sketched included opposite sides that were not congruent. Congruent opposite sides are fundamental to the concept of a parallelogram. Her diagram was actually a trapezoid. Mrs. Anderson learned about important mathematics throughout her M² experience as well as the need to communicate in appropriate and sensible ways. This example highlights a less than desirable translation from the professional development to the classroom.

Mrs. Anderson then turns to her students and asks, “Would you agree that it is a parallelogram?” Some students nod. Mrs. Anderson continues, “The area of a parallelogram is a lot like this [pointing to the area formula written under the rectangle]; but we use a different term. We use base times height.” Mrs. Anderson proceeds to write the area formula in the parallelogram she had previously drawn on the chalkboard (see Figure 59.2).

![Figure 59.2. Mrs. Anderson’s Parallelogram Drawing with Formula](A=bxh)

I’m going to give you the formulas right away. We’re going have one more formula today and that would be the area of a triangle…the area of a triangle is one-half of the base times the height. And we’ll get to that. I’m
going to put a triangle here so that we don’t forget. These are two new
ones that we are going to be focusing on today.

While talking, Mrs. Anderson writes the formula on the board as well as draws a small triangle next to her formula.

Mrs. Anderson made two interesting teacher moves. First, she told her students that a parallelogram was similar to a rectangle; however, the main difference was in the use of base and height instead of length and width. Mrs. Anderson taught her students the area formula for rectangles and squares just one day prior. It appeared that she wanted them to make an early connection between parallelograms and rectangles, without any conceptual connection. Mrs. Anderson merely highlighted the use of different words in the formula; yet, she made no reference as to why these particular words are used.

Second, Mrs. Anderson decided to simply tell her students the formula from the beginning without helping them develop any conceptual understanding. She was very upfront with her students as she told them she would give the formulas right away. She even provided the area formula for a triangle well before she talked about triangles. This is consistent with the way Mrs. Anderson learned mathematics for herself. She liked to list formulas, page numbers, examples from notes, previous homework problems, and even the people that helped her. Many times she listed these things before she began solving a problem. This example highlights a translation of using mathematical practices as a learner of mathematics to teaching practice.

Mrs. Anderson then grabs a red, rectangular wooden block off of her desk (see Figure 60). She states, “Before we start, let’s find the area of this block.” With her
students help, Mrs. Anderson measures the dimensions of the block and determines the area to be 36 cm\(^2\).

*Figure 60. Representation of the Rectangular Block Mrs. Anderson Used*

There is no mention of why it is thirty-six, not even a reference back to the formula \(A = l \times w\). Mrs. Anderson immediately moves on to the first topic of the day, parallelograms.

Mrs. Anderson tells her class, “Just a question here. Would you agree that this block is the same as these two?” Mrs. Anderson grabs two more wooden blocks off of her desk. This time they are yellow, right scalene triangles. She carefully fits them together to make a rectangle, and then holds the pair of triangles up next to the red rectangle for the students to see (refer to Figure 61).

*Figure 61. Representation of the Two Triangular Blocks Mrs. Anderson Used*

Immediately the triangles slip on one another. “I’m going to have to hold them so it doesn’t slip. I’ll hold it right on top.” Mrs. Anderson places the rectangle, made up of the two triangles, directly on top of the original rectangle. “Isn’t it the same size? Is it the exact the same size?” Many students say yes. “So what do you think the area of the two yellow blocks put together would be?” Many students state the correct answer. “Thirty-
six centimeters squared? O.K. I agree with that. Let’s turn this.” Mrs. Anderson separates the two yellow triangles and begins to rearrange them. “Am I adding anything to these?” Students watch closely as she continues to reposition the triangles. “Oh—what do I have here?” Mrs. Anderson repositions the two triangles to form a parallelogram (refer to Figure 62).

![Figure 62. Representation of the Parallelogram Mrs. Anderson Used](image)

Several students say that it is a parallelogram. “There’s my parallelogram Grade 5. Is this a different shape than this? [She again holds up the original red rectangle.] What do you think the area is going to be?” A few students suggest the area will still be thirty-six. “Yeah. We all agree that it’s going to be 36 centimeters squared, right? Why do you say that? It’s a different shape now...Josie?” Josie states that the two triangles combined together make the red block. She thinks it would have the same area since the parts are just moved around. Mrs. Anderson responds to Josie reasoning by saying, “We didn’t gain or lose anything, did we?”

Mrs. Anderson used the manipulative to help her students see a connection between a rectangle and a parallelogram. She, too, relied on using representations and making connections in her own learning of mathematics. This example highlights a translation of learning mathematics to teaching mathematics. Mrs. Anderson’s choice to use the blocks to help students develop an understanding the area of a parallelogram is a much better example of her teaching, as compared to the start of this vignette, which was
more procedural. Figures 61 and 62 indicate a more conceptual view on teaching and learning.

Later in the lesson, Mrs. Anderson moved back to being procedural. She asks her students to look at the first example in the textbook. “They give you the answer Grade 5 because it’s the first one.” Figure 63 shows the first example from this lesson in the book.

![Figure 63. First Example in this Section of the Textbook](image)

What’s the area of that first parallelogram A? What’s the base? Base equals…[one student responds]. What about height?...[three students respond]. So the area is one-two-three [i.e., a visual cue for all students to share the answer at the same time]…fifty-four feet squared [more students shout out the answer.] When I’m recording this, Grade 5, do I square these feet? [pointing to the base] these feet? [pointing to the height] Where do I square them? Ian.

Ian states that Mrs. Anderson should only square the area.

Mrs. Anderson then uses this same problem to ask her students a more challenging problem related to finding area. “O.K., let’s go back to something we did yesterday. Let’s say we don’t know what the height is? We know the base is nine feet. We know the area is fifty-four. How are we going to find the height? Zoe.” Zoe says she...
would take fifty-four divided by nine. “Exactly, we do an inverse operation.” Mrs.
Anderson then goes through the process with her students. “…That gives us our height.
Now you can always check yourself to see if you are right. Does nine times six equal
fifty-four?…What if we don’t know the base?…Did yesterday’s math help us today?” A
few students nod. Mrs. Anderson surveys her group of students, “We don’t know the
base, now what do we do? Oh, I love the hands.” Most hands are up. She calls on Cole.
Cole responds, “You could do six times one, six times two, and keep on going until you
find fifty-four.” Mrs. Anderson agrees but immediately responds, “OK…by trial and
error…what is an easier way to do that?”

“Brian, what’s an easier way?” Brian states that he could divide fifty-four by six.
Mrs. Anderson states, “And again Grade 5, you can check yourself. Does nine times six
equal fifty-four. So do you notice that if you have any two components you can figure out
what the third one is? Can you see that?”

Mrs. Anderson again made several interesting teacher moves. This final
discussion on Mrs. Anderson’s teaching will highlight two moves. First, she used the
example from the textbook as a springboard into a more challenging process, solving for
an unknown side length rather than unknown area. This may indicate she possessed a
more complete understanding of the concept of inverses. A goal of the professional
development was to deepen her understanding of mathematics. While this study does not
claim that participation in the professional development led to this specific teaching
example, it is important to point out that Mrs. Anderson was able to teach important
mathematics (e.g., inverse relationships) to her students.
The second teacher move is noteworthy in how Mrs. Anderson pushed her own students to see past a specific example and look for a more general situation in this teaching episode. Analysis of Mrs. Anderson’s learning of mathematics from the first three courses of the professional development revealed times when she struggled moving from specific examples to the general case. She did not struggle with it at this particular point in her teaching. “If you know any two components, you can figure out what the third one is.” It would be interesting to know if this would have been a teacher move present in her teaching before she entered the Math in the Middle Institute. Did her own experiences with taking specific cases and trying to offer general explanations influence her decisions as a teacher?

**Mrs. Zander’s Teaching: Episode and Discussion**

Mrs. Zander teaches sixth grade math. She follows the district pacing guide and covers a variety of sections from the textbook, *Middle School Math: Course 1* (Bennett, Chard, Jackson, Milgram, Scheer, & Waits, 2004). One of the teaching episodes recorded for this study related to the area of rectangles, parallelograms, and triangles. The previous day’s lesson focused on perimeter. Mrs. Zander brought to closure a few tasks related to the perimeter lesson before she moved on to the area lesson.

Mrs. Zander stands in the front of the room, looks out at her students, and says, “Yesterday we were working on perimeter. So if you could please locate the cardstock paper (i.e., sheet of notes for perimeter)…we just had a few to finish at the bottom.” The students slowly locate and retrieve this sheet of notes (see Appendix V). “So let’s see
how we did…If you did number twelve (see Figure 64) and have an answer, raise your hand.” Mrs. Zander scans the room to see how many hands go up.

![Figure 64. Problem 12 on the Sheet of Notes for Perimeter](image)

Mrs. Zander asks Jackson to share his answer to problem twelve. Jackson gives an answer of twelve. Mrs. Zander responded by asking the rest of the class to nod if they agreed, “so he can see.” Both Mrs. Zander and Jackson look around the room and see several heads nodding. She continues, “O.K. There is lots of agreement. Now can you talk about how you found that? I agree that you are right, but what would you say was your best method of finding it. How did you find the number twelve?” Jackson offers a brief statement as to his reasoning. Mrs. Zander then responds.

O.K. So he was assuming the opposite side was congruent. There is a lot of geometry proof on why that could be…What else?…Because maybe [that opposite side] is just a little off [and it is not congruent]? Can anyone else prove it mathematically?

Carmen raises her hand. After listening to Carmen’s reasoning, Mrs. Zander states, “So it kind of sounds like you did a little guess and check. You thought it was twelve and then proved it afterwards.” Mrs. Zander looks for another student to call on. She calls on Megan who states, “Well I added up twelve, six, thirteen, and six again. I got thirty-seven. Then I did forty-nine minus thirty-seven.” Mrs. Zander nods in agreement with
Megan’s reasoning. “So [Megan] took all of the knowns that were there and then subtracted that from the total.”

Reasoning and proof appeared to be an important component of the brief discussion in this teaching episode. It seems Mrs. Zander was not willing to accept reasoning based on guess and check or just looking at the diagram. She wanted to have mathematical proof. This is consistent with her experiences learning mathematics as part of the professional development. She had to provide justification for the work she submitted. In this episode, it appeared that she carefully listened to her students’ reasoning. She did not want students to merely accept a side length because it looked to be congruent to another side. This indicates Mrs. Zander understood an important concept from geometry middle level teachers should know.

Mrs. Zander then shifts the focus of the lesson to area.

Now the next thing is that we can go ahead and talk about area. When I think of area…area to me is talking about the grass or the sod that will be inside that fence. Or if you are inside your house, and you look at the carpet or hardwood floor or tile or paint walls—that’s area. So I wanted to connect it to something that I already know.

A habit present in Mrs. Zander’s own attempts to learn mathematics was to make connections to physical situations in the real world. In this episode, it appeared that she wanted her students to make those same connections. It would be interesting to know if all of Mrs. Zander’s habits of mathematical learning translate to her ways of teaching.
After quickly reviewing the area formula for rectangles, of which she tells her students, “I bet you know this anyway,” Mrs. Zander moves on. “Now if you go to page 505, you are going to find the area of a parallelogram and the area of a triangle…so [the area of a parallelogram is] base times height.” Mrs. Zander uses her computer mouse to point at the parallelogram.\(^{58}\)

Now the reason we are looking at base times height here—and if you look at this height in this case is not talking about this (i.e., slant height)—but if you look right up here where this mouse is—it’s talking about this right here (i.e., right angle height). Height is talking about this dashed or dotted line. That’s the height and this is the base we are looking at.

Mrs. Zander then offers an explanation that may be geared towards helping her students build a conceptual understanding of area of a parallelogram. She asks her students to look at the diagram she provided for them on the note sheet (see Figure 65).

![Figure 65. Mrs. Zander’s Illustration Connecting a Parallelogram with a Rectangle](image)

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\(^{58}\) Mrs. Zander projected a copy of the worksheet from her computer to the front screen.
Now one thing that I was trying to show is if you take this particular triangle that is cut at the height and move it over here and what do you see down here...[a few students shout out ‘rectangle’]...so it ends up being just like length times width...so it’s really, when I say it’s the same thing as finding the area of a rectangle, that’s not a real great statement, but essentially it’s using the same formula...so you must know the height and you must know the base. And you can move these triangles [referring to her diagram in Figure 65] and cut them off and look at them that way.

Mrs. Zander likely assumed her students knew all about rectangles. She simply provided the area formula without spending any time helping students develop a conceptual understanding of area. It is likely she knows, or expects, that previous teachers provide opportunities for students to develop a conceptual understanding of the area formula for rectangles. In this lesson, she told her students to focus on the first example on the bottom of the page.

Mrs. Zander then used a completely different approach to introduce her students to the area formula for a parallelogram. This approach is similar to what she saw in both MSL and FAGMLT. Mrs. Zander first made an attempt to help her students recognize the actual height of a parallelogram is the ‘right-angle’ height. She then made a deliberate attempt to help students see the connection between the area formula for a parallelogram and the area formula for a rectangle. This episode more closely matches the type of learning experience offered by M2 instructors. She experienced learning in ways that
helped her make connections. In turn, this episode revealed that she offered instruction that likely helped her own students make connections.

Later in the lesson, Mrs. Zander states, “Let’s move on. Here’s where we’re going to talk about some of the figures that we were talking about yesterday.” She directs her students’ attention to problem number seven (see Figure 66).

![Figure 66. Area Worksheet, Problem 7, with Measurements](image)

This is the same figure that we were looking at yesterday when we were talking about perimeter. Now this time I colored it in with gray to show that we are looking at area. Can you talk with your table partners and see if you can figure out what the area of this figure we are looking at? Go ahead and work with neighbors. I’ll give you a couple minutes to work with your neighbors.

The rest of the class period (i.e., nearly twenty minutes) is devoted to this one problem.

Once Mrs. Zander gives students the green light to work on the problem, the classroom environment gets loud. Mrs. Zander slowly moves from group to group listening to students’ conversations. She asks questions as well as offers help. A few groups of students question whether they are supposed to be finding perimeter or area. As Mrs. Zander leaves the second group she helps, she asks a girl from that group to post her
own solution on the front white board. Mrs. Zander also asks a girl from the third group she visits to post work on the board. After several minutes, three different students are posting their solutions on the boards (see Figures 67.1, 67.2, and 67.3).

**Figure 67.1. Dawn’s Work**

**Figure 67.2. Molly’s Work**

**Figure 67.3. Tom’s Work**
Eventually, Mrs. Zander asks for her students’ attention, “As I walked around, I saw lots of different strategies…I’ve got…[Pointing at the board]…Molly and Dawn, they have two different strategies. I also have Tom.” Mrs. Zander asked Dawn if she would share her solution first.

Dawn stands next to her work on the front white board and shares how she found her answer. Her method is actually a combination of perimeter and area. She first focuses on the small rectangle on the right hand side. She multiplies the top length of two and the right width of three. She then adds that result to the product of the left width of three and bottom length of two. She uses all four sides, as one would do to find perimeter of a rectangle. Yet, she multiplies the length and width, as one would do to find area. Dawn then uses this logic to give a value for each of the three remaining small rectangles in the figure. She never addresses the center rectangle. For her final answer, Dawn adds the four small rectangles and places her final answer in the center of the diagram.

Mrs. Zander responds by asking a probing question, “And the center part—where the fifty-six is sitting—did you figure out that space as well?” After Dawn looks at her diagram for several moments, Mrs. Zander asks, “Or did you just do the outside area?” Dawn smiles, “Oh, I forgot to do that.” Mrs. Zander then prompts Dawn to consider revising her work. Mrs. Zander then asks a question of all of the students:

How many of you understood that she split up the rectangles and was looking at area there? And guys, feedback is helpful. Because not everything that is up here is all correct. There can be some things that you might notice that you did differently.
There are many interesting teacher moves in this episode of Mrs. Zander’s instruction. First, it is clear Mrs. Zander promoted collaboration and provided extended time for her students to work on the problem. These two structures were an important part of Mrs. Zander’s own learning experiences while part of Math in the Middle. Again, one cannot assume Mrs. Zander added collaboration or the use of extended time to her teaching as a direct result of her participation in the professional development; yet, it is likely to suggest that her own experiences collaborating with others to learn mathematics helped reinforce the importance of doing so in her own classroom.

Second, splitting up the rectangles and then finding the center area was Mrs. Zander’s method of solving this problem. She actually introduced this problem by telling students that “this strategy” would be a good method. Her first question of Dawn’s work was in regard to the center area. It is possible that Mrs. Zander’s only critique of Dawn’s work was the fact that Dawn did not include the three by four center rectangle as part of her final answer. Mrs. Zander did not address the incorrect strategy of mixing area and perimeter together. Did she actually notice this error? Her last statement indicated that not everything up here was correct. However, that comment was made in a context of encouraging peers to offer peer feedback, not in the context of suggesting there were more critical errors to consider. This indicates Mrs. Zander may not have possessed a complete understanding of the mathematics needed to solve this problem. This incomplete understanding matches what this study found about her learning of mathematics in the first three courses of the professional development. Her learning was
a work in progress, growing from course to course. Was her teaching also a work in progress, growing from year to year?

A final teacher move worthy of discussion is Mrs. Zander’s decision to ask students to post work on the front board and then share in front of their classmates. This teacher move was consistent with her own experience learning mathematics in Math in the Middle. Mrs. Zander even prompted Dawn to revise her work. Mrs. Zander encouraged students to offer their own reasoning. She encouraged them to communicate their own understanding of the mathematics. Mrs. Zander relinquished some of her control as a teacher and allowed students to help teach other students.

**Reflection: The Intersection of Learning and Teaching**

In these two brief episodes of two teachers’ instruction, the participants’ ways of learning could also be found in their ways of teaching. Further, these two teachers used some of the structures in their classroom as Math in the Middle instructors used in the institute. These snapshots of teaching strengthen the call to longitudinally study teachers. Closely examining teachers’ ways of doing mathematics before, during, and after sustained involvement in a professional development program could better inform future professional development. It would be interesting to know to what degree Mrs. Anderson and Mrs. Zander’s involvement in Math in the Middle influenced their teaching from a researcher’s perspective as well as their own.

**Implications for Professional Development**

The results of this study prompt three recommendations for future professional development experiences. First, professional developers should be intentional about the
structures they put into place to support learning. The participants in this study commented how the varied structures aided their learning of mathematics. These structures included the instructional team approach, support provided both during and after class, focus on multiple perspectives, emphasis on improving draft solutions, and encouragement to collaborate to learn mathematics. Both participants frequently reflected on the impact these structures made on their own learning. Further, evidence from examining the participants’ written mathematical work and live interactions solving problems indicated these structures did help them learn many important mathematical concepts. This professional development program designed classroom activities and homework assignments as well as set expectations with the goal of building teachers’ capacities. Yet at the same time, the professional development acknowledged that deepening mathematical knowledge is a work in progress. Future professional development should carefully consider the structures this professional development program put in to place to help teachers learn mathematics.

The second implication for professional development is related to habits of mind. Professional developers must consider seriously the role of teachers’ habits of mathematical learning as they design learning experiences. The construct of mathematical habits of mind has been gaining more and more attention over the past two decades. Now is the time, especially in relation to the release of the Common Core Standards (CCSSI, 2010), to build professional development experiences that enable individuals to use his or her own habits to learn meaningful mathematics. In turn, collaborating with one another will prompt individuals to help peers strengthen
mathematical habits of mind. Professional developers should discuss habits of mind with learners. The habits presented by Cuoco, Goldenberg, and Mark (1996) and the standards for mathematical practice offered by the CCSSI (2010), in addition to the habits identified and described in this study could help individual learners unearth ways of learning that prompt more and deeper learning of important mathematical ideas.

Finally, future professional development programs must attend to helping teachers learn how to collaborate with one another. This study offers two vivid images of collaboration. One participant quickly developed collaborative relationships with many of her peers whereas the other participant took more time in developing those relationships. By the end of the third course, both teachers depended on collaborative relationships to deepen their own mathematical knowledge.

Learning to collaborate with peers is not an easy task. Professional developers should take time to help participants learn what it means to successfully learn alongside peers. From team-builder activities to incorporating various structures, Johnson and Johnson (1999) provide detailed directions on how to build a collaborative environment in the K-12 setting. Professional developers should learn from experts in the field of collaboration (e.g., Johnson & Johnson) and teachers as examples (e.g., Nebesniak & Heaton, 2010) to cultivate an environment that is safe and conducive to learning.

**Implications for Mathematicians Working Alongside Education Faculty**

The sixth recommendation offered by the CBMS (2001) addresses the relationship between mathematics faculty and mathematics education faculty. Both parties are needed to help pre-service teachers learn mathematical content for teaching.
More generally, mathematicians and education faculty should work side-by-side to help practicing teachers learn, and relearn, important mathematics content. This study shows how important it is for mathematicians to think about creating opportunities for learning, not just teach content. “How it is” that people learn is just as important as “what it is” that people learn. Too often, there is distrust between those who primarily focus on mathematics and those who primarily focus on pedagogy (CBMS, 2001). This study describes a professional development experience where mathematicians successfully engage with pedagogy specialists and consider and implement structures to help practicing teachers learn mathematics. The mathematicians place priority on implementing a curriculum filled with important mathematics content and also place priority on their own pedagogy.

**Implications for Future Research**

The results of this study prompt discussion for two future studies. First, this study closely examined what it looks like for two teachers to learn the mathematics that middle level teachers should know. One result revealed descriptions of the ways (i.e., habits) these teachers learned mathematics. Future studies could examine additional learners to offer not only broader, or more general, descriptions of habits of mathematical learning, characterized by more learners, but also descriptions of habits not addressed in this study. Providing examples could benefit learners and professional developers in future professional development sessions.

Secondly, this study focused on the learning of important mathematics. The next step is to learn what translates from this type professional development experience to a
teacher’s classroom. Therefore, the next step is to study teaching. How does the learning, or relearning, of important mathematics content translate to one’s instruction of mathematics? What does it look like for a teacher to use this content knowledge, or deepened understanding of mathematics, in the classroom setting? Smith (2008) studied a similar question (i.e., How do middle grades teachers use their experiences participating in an ambitious professional development program?), yet did not examine teachers’ learning of mathematics in relation to teaching of mathematics. She placed focus, and collected data, around the domain of teaching mathematics, similar to the way this study placed focus, and collected data, around the learning of mathematics. Thus, an area for future study can be found in the intersection of these two domains.

Another question in relation to teaching is: How do ways of learning translate to ways of teaching? How might a teacher’s habits of mind while learning mathematics translate to their instruction of mathematics to students? Rolle (2008) studied a similar question (i.e., What are the pedagogical habits displayed by a middle grades teacher?), yet did not examine a teacher’s own habits of learning. She, too, placed focus, and collected data, around the domain of teaching mathematics. Again, an area of future study can be found in the intersection of two domains: teachers’ habits of mathematical learning and teachers’ pedagogical habits.

**Conclusion**

This study helps answer the question, how do practicing teachers learn mathematics needed for teaching, by offering images of practicing teachers learning important mathematics. These images reveal teachers can and do learn challenging
mathematics as part of a sustained professional development program that takes seriously the development of mathematical knowledge. More importantly, these images capture many of the ways teachers learn mathematics. This study helps describe what it looks like to possess, and then use, mathematical habits of mind. This study helps to describe what it looks like to use collaboration as a tool to learn mathematics. This study also points to the need for future research to tackle the important topic of translating mathematical habits of mind from the context of teachers’ learning to their practices of teaching.
References


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Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics, 46*(1/3), 13-57.


Appendix A: Working Definitions\textsuperscript{59}

Attitude: One’s outlook in terms of learning mathematics or solving specific problems. This can be disguised.

Belief: How good one thinks her mathematical ability is. The assumption about one’s self. This is deep-rooted, harder to disguise or change.

Disposition: A tendency to approach or solve a problem a certain way. This can be learned.

Habit: The frequency of a disposition. This is a regular occurrence; this may become a regular way of working for the learner.

\textsuperscript{59} These working definitions are a result of synthesizing dictionary definitions with the writings of, but not limited to, Cuoco, Goldenberg, and Mark (1996), NCTM (2000), and the National Research Council (2001).
Appendix B: Reflection Prompts Asked During the First Three Weeks of the M² Institute

MSL Week 1 Reflection (6/22/2007)  Name: __________________________

1. List the names of those who have influenced you the most this week.

2. Take a moment to think about new mathematical insights you have gained this week. Now describe what you would consider to be your biggest insight.

3. Describe how learning mathematics this week has been different from your past experiences of learning mathematics. What has it felt like? Think about your experiences in class as well as any experiences with homework groups at night.

FAGMTL Week 1 Reflection (7/20/2007) Name: __________________________

1. List some new mathematical insights you have gained this week. Now describe what you would consider to be your biggest mathematical insight:

2. How do you know when you understand a math idea and when you don't? Explain.

3. Who do you think are the three best “math students” in the FAGMTL math class? Explain why you chose these individuals.

4. Who would you most want to work in a group with during the FAGMTL math class? Explain why you chose these individuals.

5. Who would you most want to work in a group with during the Curriculum Inquiry class? Explain why you chose these individuals.
FAGMTL Week 2 Reflection (7/27/2007)           Name: ____________________________

1. Think about your own K-12 and college experiences learning mathematics. Describe how the past three weeks of M\(^2\) have been different from those past experiences learning mathematics.

2. Compare and contrast “M\(^2\) as a professional development activity focused on math” with “your PAST professional development experiences focused on math” (Your past is not limited to but may include: college practicum classes and methods classes; ESU meetings; building/district meetings)

3. Describe your comfort level in working with others in the process of learning mathematics.

4. At this point of Curriculum Inquiry, what does the word “curriculum” mean to you? What sort of positive and/or negative connotations does the word have for you and why? What sorts of issues, questions, or concerns arise for you when thinking about the word curriculum and mathematics teaching and learning? Use the back of this sheet if needed.
Appendix C: Participant Interview Questions (2/22/2008)

• What do math teachers need to know and be able to do to carry out the work of teaching mathematics effectively?
• How would you describe your “mathematical knowledge?”
• When someone asks, how do you describe what “Math in the Middle” is?
• Why did you choose to go into the field of education?
• What were your experiences like learning K-12 mathematics?
• What were your experiences like learning mathematics in college?
• How prepared “were you” during your first year or two of teaching math?
• Have you had experiences learning mathematics as a teacher (post-college but pre-M²)? Describe those experiences.
• (pre-M²) How have your professional development experiences (as a practicing teacher) helped you become a better math teacher?
• What grade levels / mathematics courses have you taught in your career?
• What prompted you to apply for Math in the Middle?
• What do you think you will gain from the 25+ month experience?
• Talk about learning mathematics through the M² experience so far? What has aided you? What has hindered you?
• Do you feel “more prepared” to teach math now that you have completed 3 mathematics courses as part of M²?
• You wrote that you thought ______ were the best math students last summer? Why did you pick those 3 people?
• How would you describe the building you teach in?
• What role do you have in your school’s mathematics decision-making processes?
• Who would you say are your colleagues?
• Describe the collaboration you have with other teachers at your school.
• Describe the role of your administrator(s) at your school.
• Describe the role of your community at your school.
• Describe the interaction between you and parents at your recent conferences.
• Who or what influences the pedagogical decisions you make in your classroom?
• How would you describe your classroom instruction to someone who has never seen you teach before? (Note: I have not been to your classroom and I have not viewed either of the videos you’ve submitted to M^2. Thus, I have no prior knowledge regarding your current instruction)
• Identify what you perceive as your strengths and weaknesses as a math teacher.
• Some argue that math teachers should have a specialized knowledge of mathematics in order to teach. What do you think about this? Do you think there is a specialized knowledge of mathematics that math teachers should possess?
• Is there anything else yourself or your teaching that you would like to share with me?
Appendix D: Participant Follow-up Interview Questions (3/9/2008)

- What resources or materials influence your teaching practice?
- What role did you have in the textbook selection process?
- How do you use your textbook in planning / delivering lessons?
- Who decides what “you” need to cover in your math class during the year?
- Have you observed other math teachers teaching? If so, what did you learn from that observation?
- Have you collaboratively planned math lessons with other math teachers?
- You are a 5th/6th grade teacher? Do you feel more like a middle level or elementary teacher? Why?
- How do you spend your time when you are not teaching?
- Where do you see yourself 10 years from now?
- If you had to leave this job, but got a chance to train your replacement, what would you want to be sure that person know and/or did?
Appendix E: Principal Interview Questions (3/25/2008; 4/16/2008)

- How would you describe your school to someone who has never been here before?
- How do you describe the school’s place in the larger community?
- What is the role of the parents in the school? Parent teacher conferences?
- If I was at the first day of teacher in-service this year, what would I have heard you share about your goals for this school year?
- Have there been any obstacles or goals that you’ve had to overcome this year?
- How does the textbook selection process work?
- How has NCLB impacted this school?
- How do your teachers currently participate in leadership roles? What roles might some have?
- When did you first hear about M²?
- How familiar are you with the goals of the Math in the Middle project?
- How do you feel about having math specialists at the 6th or even 5th grade level?
- Is there anything else you would want me to know?
Appendix F: Problem Solving Session Math Problems

1. (This problem is similar to MSL’s “Temperature Conversion,” MSL Course Notebook, 2007.) Remember that Newton had devised his own temp scale in which he assigned 0 to be the freezing temperature of water and 33 to be the boiling temperature of water. After our experiment in January, Jeffrey Lewis wanted to get in on the temperature scale act and designed a scale where -20 degrees Lewis was the freezing point of water and +20 degrees Lewis was the boiling point. Find a formula to convert from degrees Newton to degrees Lewis. Find a formula to convert from degrees Lewis to degrees Fahrenheit.

2. (This problem is similar to FAGMLT’s “Homework Day #8 D,” FAGMLT Course Notebook, 2007) If possible, sketch each of the following. If it is not possible, then give a reason why it cannot be sketched:
   a) A quadrilateral that has exactly one right angle and no parallel sides.
   b) A quadrilateral that has exactly two right angles and no parallel sides.
   c) A quadrilateral that has exactly three right angles.
   d) A quadrilateral that has exactly one right angle and exactly one pair of parallel sides.

3. (This problem is similar to geometry content from FAGMLT and AMC-type problems from ECR.) Three cylinders of the same radius $r$ are tied together snugly by a rope. The diagram shows a cross-sectional view. What is the length of the rope around the cylinders?
4. (This problem is similar to ECR’s “Baby Bunnies,” Burger & Starbird, 1999, p. 57.)

Deer generally don’t reproduce till they are two years old. They are born in the spring. Mating season is in the fall and an average fold has 2 fawns. The population is approximately 50-50 males to females. So let’s define an adult deer as any deer that is at least 2 years old. Assume you start with $P$ deer in the fall that are all adults, that the male female rate is a constant 50-50 and that each adult female produces 2 fawns each spring. Find the sequence of populations, $P_0 = P, P_1, P_2, P_3, \ldots, P_n$, where $n$ is the number of years. How does this change if each deer lives only 4 years? (Assume the death occurs before mating season.) What if they live only 3 years? (The average life span of a deer in the wild is 3 years.)
Appendix G: Excerpt of Analysis of Becki Zander’s Non-Mathematical Work

Axial codes from Mrs. Zander’s responses to the beliefs inventory and teacher survey:

- Problem-solving / conceptual learning / real-world contexts are very important
- There is a place for procedures and basic skills
- There are different methods to solve problems / students can find those methods
- The answer to a problem is just an answer
- The textbook is not the Bible
- Students should work together
- Students should have access to calculators

Categories that emerged from Mrs. Zander’s non-mathematical work:

<table>
<thead>
<tr>
<th>Background information</th>
<th>Leadership</th>
</tr>
</thead>
<tbody>
<tr>
<td>Past professional development</td>
<td>Current math knowledge</td>
</tr>
<tr>
<td>Teaching mathematics prior to M²</td>
<td>View of M²</td>
</tr>
<tr>
<td>Needs as a math teacher</td>
<td>Current math teaching</td>
</tr>
<tr>
<td>Curriculum</td>
<td>Confidence levels</td>
</tr>
<tr>
<td>Three things I want to change in my teaching</td>
<td>M² as professional development</td>
</tr>
</tbody>
</table>
Appendix H: Excerpt of Analysis of Becki Zander’s Written Mathematics Work

Uses a specific case

MSL Fav. Five “Decimals that do not repeat and have a random pattern are irrational numbers. For example, the decimal 0.909009000900009…does not repeat a pattern, but, rather, it’s a random pattern.”

MSL EOC [BIG!] 2b) “…I found that (2/5)x = 0 and ‘x’ must be 0 in order to solve the problem. Next, I tried to use x=2 and found that 14 does not equal 10. I also tried to subtract 5x from the 7x and got 2x = 0, where x=0. Again, x could only be zero, so there was NO SOLUTION.” [Good work shown here…look at this again]

MSL EOC Metropolis and Gotham City…distance between them…she wrote, “Say, 600 km each way.” [Instructor wrote: What if the distance between the cities is not 600 km?] Later Mrs. Zander wrote, “In addition, you can use the formula \( \frac{2D}{\frac{D}{300} + \frac{D}{600}} \) and plug in any number for D.” [Instructor wrote: Good. Notice:
\[
\frac{2D}{\left( \frac{1}{300} + \frac{1}{600} \right)D}
\]
So the D’s cancel. This means his average speed does not depend on D.]

FAGMT EOC 8b) “Let’s say the two regions are either squares or rectangles. If they are is 640 cm\(^2\), then two sides must be equal this when multiplied. The sides of the smaller polygon are multiplied by the s.f. [scale factor?] of 8” [Then uses diagram…10 cm\(^2\) square and 640 cm\(^2\) square…wrote—side options: 80x8…sets up table…uses several options for 640, including 80x8 32x20 16x40 64x10; divides by scale factor of 8 gets…10x1 4x2.5 2x5 8 x 1.25] “Once I found possible side lengths of the larger polygon, I divided the sides by 8 to get the sides of the smaller polygon. In each case, the sides multiplied together gave me an area of 10 cm\(^2\) on the smaller polygon” [Polygonal regions…simplified to squares…specific case!]

ECR Session A #8) \((F_{n+1})^2 - (F_{n-1})^2 = F_{2n}\) Mrs. Zander uses \((n=4)\, F_{2(4)} = F_8\) which is 21 (the difference) and uses \((n=3)\, F_{2(3)} = F_6\) which is 8 (the difference). She uses arrow to point back up to the list. Instructor wrote: Can you explain why this holds?
Appendix I: Math in the Middle Institute Course Descriptions

MATH 800T: Mathematics as a Second Language (MSL)

This course lays the foundation for developing the "habits of mind of a mathematical thinker," a theme that is further developed in subsequent M² courses. The approach is to understand arithmetic (number) and (introductory) algebra as a means of communicating mathematical ideas (i.e., as a language). The course will stress a deep understanding of the basic operations of arithmetic, as well as the interconnected nature of arithmetic, algebra and geometry. Attention is given to connections with other areas of mathematics and emphasizes the development of an appreciation for the importance of careful reasoning, problem solving and communicating mathematics both orally and in writing.

One distinctive characteristic of the text (and therefore the course) is its use of what is referred to as the "adjective-noun theme". The authors of the text argue that numbers are adjectives that modify nouns (or other adjectives); and that an adjective in isolation (without reference to a noun) leaves an incomplete picture. A second distinctive characteristic of the text is its premise that arithmetic, algebra and geometry are interconnected and that the study of mathematics should reflect their interconnected nature.

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60 Course descriptions were retrieved from http://scimath.unl.edu/MIM/coursematerials.php on 10/27/2009.
MATH 802T: Functions, Algebra and Geometry for Middle-Level Teachers (FAGMLT)

This course builds upon Mathematics as a Second Language. Participants will obtain a deep understanding of the concepts of variable and function, utilize functions in problem solving, the theory of measurement (especially length, area, volume), and develop geometric modeling in algebra. Emphasis is placed on the ways in which these concepts develop across the middle level curriculum.

The course includes a number of classroom connections and classroom discussions (activities which teachers might someday use in their middle school classrooms). Both are designed to deepen the connections between the algebra and geometry being studied to the algebra and geometry in the middle school curricula. Excerpts from Standards-based middle school mathematics curricula are included in the texts to also deepen this connection.

MATH 804T: Experimentation, Conjecture and Reasoning (ECR)

This course focuses on problem solving, reasoning and proof and communicating mathematics. It utilizes the extensive resources of the American Mathematics Competition (AMC) to help middle level mathematics teachers develop problem-solving skills.

TEAC 801: Curriculum Inquiry

This pedagogy course focused on helping students gain a deeper understanding of mathematics curriculum development, including historical and contemporary issues that influence curriculum planning and educational change. Participants consider current
curricular issues in relationship to their own mathematics teaching and learning and how the mathematics learned in other M² courses transfers into the planned and enacted curriculum of one's own teaching practice.

**TEAC 800: Inquiry into Learning and Teaching**

This is a pedagogy course which is focused on inquiry into mathematics teaching and learning. The course has two principle goals: (1) the investigation and articulation of the basic principles of educational research and inquiry; and (2) the analysis, synthesis, and evaluation of key concepts of classroom pedagogy based on research and instructional theories.

**MATH 805T: Discrete Mathematics for Middle-Level Teachers**

This course extends the breadth of knowledge of discrete mathematics in directions beyond, but related to, topics covered in middle-grades curricula. It increases the depth of mathematical experiences through problems rich in opportunities for exploration and communication.

**MATH 806T: Number Theory and Cryptology for Middle-Level Teachers**

This course focuses on basic number theory results which are needed to understand the number theoretic RSA cryptography algorithm (an encryption algorithm which is in use today to secure information sent via the internet). As the number theory results are developed, connections to middle level curricula are emphasized and proofs are carefully selected so that those which are included in the course are particularly
relevant and accessible to middle level teachers. This portion of the course promotes a
deep understanding of the integers and their properties in connection with the operations
of multiplication and division. Elementary ciphers (methods for encoding and decoding)
are included to introduce the nature of cryptology in preparation for understanding the
RSA method. The cryptology related activities are readily adaptable as enrichment
activities for middle level students. The connection of number theory to the RSA
encryption algorithm allows the participants to see and understand a very relevant, real-
world application of mathematics.

TEAC 888: Teacher as Scholarly Practitioner

This course introduces participants to the theory and practice of teacher-led
inquiry. The course prepares teachers to engage in a school-based action research project
that will be conducted during the following spring semester.

STAT 892: Statistics For Middle-Level Teachers

This course offers an introduction to probability and statistics. It follows an
inquiry/discovery design dedicating much of class time to activities, discussion and group
work. The course emphasizes both topics in probability and statistics that are part of the
middle school curriculum and also statistics that are used in education and school-based
research.
Math 807T: Using Mathematics to Understand Our World

This course is designed around a series of projects in which students examine the mathematics underlying several socially-relevant questions which arise in a variety of academic disciplines (i.e. real-world problems). Students learn to extract the mathematics out of the problem in order to construct models to describe them. The models are then analyzed using skills developed in this or previous mathematics courses.

Math 808T: Concepts of Calculus for Middle-Level Teachers

Students in this course will develop conceptual knowledge of the processes of differentiation and integration, along with their applications. The course is designed around a series of explorations (worksheets) through which students are led to "discover" the main ideas of calculus. Instructors' roles are primarily to answer individuals' questions that arise in completing the worksheets, facilitate class discussions as the explorations are completed, and summarize the main ideas developed in the course as the class progresses through the material.

TEAC 889 / Math 896: Integrating the Teaching and Learning of Math: Capstone Course

This is a pedagogy course which is focused on inquiry into mathematics teaching and learning. The course has two principle goals: (1) the investigation and articulation of the basic principles of educational research and inquiry; and (2) the analysis, synthesis, and evaluation of key concepts of classroom pedagogy based on research and instructional theories. Concurrently with this course, teachers will be working on satisfying the master’s exam requirements for their Masters Degree.
Appendix J
Mathematics as a Second Language
A Solution to the Temperature Conversion Problem (Page 1 of 4)

The Fahrenheit (F) and Celsius (C) temperature scales are directly proportional in the sense that each degree F corresponds to a certain number of degrees C, and vise versa.

(a) Given that 0°C corresponds to 32°F and 100°C corresponds to 212°F, what is the rate at which a Fahrenheit temperature changes with respect to the corresponding Celsius temperature?

This is a linear relation. One can compare the change in value between any pair corresponding temperatures, such as (0, 32) and (100, 212), to calculate the rate of change:

\[
\frac{\text{Change in Farhenheit}}{\text{Change in Celsius}} = \frac{212 - 32}{100 - 0} = \frac{180}{100} = \frac{9}{5}.
\]

This tells us that there is a change of 9°F for every 5°C.

(b) Draw the graph that gives the geometric picture for the temperature conversion (Fahrenheit represented vertically, Celsius represented horizontally).

The graph is the line between (0, 32) and (100, 212). See below.

![Graph of Temperature Conversion](image-url)
Mathematics as a Second Language
A Solution to the Temperature Conversion Problem (Page 2 of 4)

(c) What is the slope of the graph? What is the significance of where the graph cuts the vertical axis? The horizontal axis?

The slope for this graph is 9/5, the value found in part a). The slope tells use the rate at which F is increasing with respect to an increase of one unit of the variable C. The point (0, 32) tells us that the temperature 0ºC corresponds to 32ºF. This is the freezing point of water, so it has physical significance. The point \((-\frac{160}{9}, 0)\) tells us that 0ºF corresponds to \(-\frac{160}{9}ºC). This has no real significance.

(d) Write a formula that converts Celsius to Fahrenheit temperatures.

Since the change in F is directly proportional to the change in C, we use the point-slope form for a line to determine the equation.

Given: Slope 9/5 and a point (0,32)

\[ F - F_1 = \frac{9}{5}(C - C_1) \]

\[ F - 32 = \frac{9}{5}(C - 0) \]

\[ F = \frac{9}{5}C + 32 \]

The formula that converts Celsius to Fahrenheit can therefore be written as

\[ F = \frac{9}{5}C + 32. \]
(e) Next do parts (a) through (d) with the roles of Fahrenheit and Celsius reversed.

Now Celsius temperature is represented vertically and Fahrenheit temperature is represented.

(a) \[
\frac{\text{Change in Celsius}}{\text{Change in Fahrenheit}} = \frac{100 - 0}{212 - 32} = \frac{100}{180} = \frac{5}{9}.
\]
There is a change of 5ºC for every 9ºF.

(b) The graph is the line between (32, 0) and (212, 100). See below.

(c) The slope is 5/9. The graph crosses the horizontal axis at (32, 0), telling us that 32ºF corresponds to 0ºC, the freezing point of water. The graph crosses the vertical axis at \( \left(0, -\frac{160}{9}\right) \). Using the point-slope, the formula that converts Fahrenheit to Celsius is:

\[
C = \frac{5}{9} (F - 32)
\]

Given: Slope 5/9 and a point (32,0)

\[
C - C_i = \frac{5}{9} (F - F_i)
\]
(f) Describe these relationships in terms of inverse processes.

The process of converting a temperature from Celsius to Fahrenheit looks like:

\[ C - 0 = \frac{5}{9} (F - 32) \]
\[ C = \frac{5}{9} (F - 32) \]

The reverse of that process is called the inverse process:

The line \( y = x \) is a line of symmetry for graphs of inverse relations:
Appendix K
Mathematics as a Second Language
Mrs. Anderson’s Solution to the Temperature Conversion Problem (Page 1 of 4)

The Fahrenheit (F) and Celsius (C) temperature scales are directly proportional in the sense that each degree F corresponds to a certain number of degrees C, and vice versa.

(a) Given that 0°C corresponds to 32°F and 100°C corresponds to 212°F, what is the rate at which a Fahrenheit temperature changes with respect to the corresponding Celsius temperature? The rate of change from Fahrenheit to Celsius is 9°F to 5°C.

(b) Draw the graph that gives the geometric picture for the temperature conversion (Fahrenheit represented vertically, Celsius represented horizontally). Next page.

(c) What is the slope of the graph? What is the significance of where the graph cuts the vertical axis? The horizontal axis? \(\text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{9}{5}\) slope is 1.8

(d) Write a formula that converts Celsius to Fahrenheit temperatures.

\[ F = \frac{9}{5}C + 32\]

(e) Next do parts (a) through (d) with the roles of Fahrenheit and Celsius reversed. Now Celsius temperature is represented vertically and Fahrenheit temperature is represented horizontally.

(f) Describe these relationships in terms of inverse processes. \(F = \frac{9}{5}C + 32\) and \(C = \frac{5}{9}(F - 32)\) are inverse processes.

(c) Where the line on the graph crosses the horizontal axis, the temperature in Fahrenheit is 0°F. After that point the Fahrenheit temperatures are negative. At this point, both the Celsius and Fahrenheit temperatures are negative. Where the line crosses the vertical axis, the temperature in Celsius is 0°C. After that point any Celsius temperatures are negative. However, the Fahrenheit temperature at that point is 32°F.
### Temperature Conversion

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\[
F = \frac{9}{5}C + 32
\]

\[
C = \frac{5}{9}(F - 32)
\]

<table>
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</table>
Mathematics as a Second Language
Mrs. Anderson’s Solution to the Temperature Conversion Problem (Page 3 of 4)

Temperature Conversion

\[ F = \frac{9}{5} C + 32 \]

Slope = \frac{\text{Rise}}{\text{Run}} = \frac{9}{5} = 1.8
Temperature Conversion

\[ C^\circ = \frac{5}{9}(F^\circ - 32) \]
Appendix L

Mathematics as a Second Language

Mrs. Zander’s Solution to the Temperature Conversion Problem (Page 1 of 5)

Temperature conversion

The Fahrenheit (F) and Celsius (C) temperature scales are directly proportional in the sense that each degree F. corresponds to a certain number of degrees C., and vice versa.

(a) Given that 0°C corresponds to 32°F and 100°C corresponds to 212°F, what is the rate at which a Fahrenheit temperature changes with respect to the corresponding Celsius temperature?

\[ \frac{\text{F}}{\text{C}} = \frac{9}{5} \]

(b) Draw the graph that gives the geometric picture for the temperature conversion (Fahrenheit represented vertically, Celsius represented horizontally).

(c) What is the slope of the graph? What is the significance of where the graph cuts the vertical axis? The horizontal axis?

(d) Write a formula that converts Celsius to Fahrenheit temperatures.

\[ (\frac{F}{9} - 32) \]

(e) Next do parts (a) through (d) with the roles of Fahrenheit and Celsius reversed. Now Celsius temperature is represented vertically and Fahrenheit temperature is represented horizontally.

(f) Describe these relationships in terms of inverse processes.

Since we know this rate, we can make a graph showing the linear problem and decipher slope by subtracting x, from x₂, y, from y₂.

\[ \frac{\text{F}}{\text{C}} = \frac{9}{5} \]

Plotted on the graph:

\[ (x₁, y₁) = (0°, 32°F), (x₂, y₂) = (100°C, 212°F) \]

Slope = \[ \frac{\text{rise}}{\text{run}} \]

Simplify \[ \frac{180°}{100} = \frac{9}{5} \]

Attach New Adj. labels

\[ F = \frac{9}{5}C + 32 \]

\[ C = \frac{5}{9}(F - 32) \]
Mathematics as a Second Language
Mrs. Zander’s Solution to the Temperature Conversion Problem (Page 2 of 5)

Parts a-d 4 graph paper
\[
\begin{align*}
\frac{32}{F} &- \frac{0}{C} \\
\frac{212}{F} - \frac{0}{C} = \frac{32}{F} - \frac{0}{C}
\end{align*}
\]

\[32 - 212 = -180 \text{ (rise)} \quad \frac{-180}{100} = -\frac{9}{5}
\]
\[-100 - 100 = -200 \text{ (run)} \quad \frac{-200}{100} = -2
\]

f) The inverse relationship of the 2 graphed linear problems shows an inverse of the temperature measurement. First of all, I reversed the role of Celsius and Fahrenheit and plotted the points. After subtracting \((x_1 - x_2)\) and \((y_1 - y_2)\) I found the slope to be \(-\frac{9}{5}\)

where it had been \(\frac{9}{5}\) previously also. I'm still unsure why it wouldn't have been \(-\frac{9}{5}\) rather than \(-\frac{9}{5}\), but, I was able to achieve this by inverting the original constant rates as above.
Note: A more detailed view of this graph can be found on the next two pages.
Temperature Conversion

Fahrenheit (Degrees)

Celsius (Degrees)

\( F = \frac{9}{5} C + 32 \)

Slope = \( \frac{9}{5} \)

Steps of slope:
- Up 9 and Over 5

\((100\degree F, 37.7\degree C)\)

\((140\degree F, 60\degree C)\)

\((176\degree F, 80\degree C)\)

\((212\degree F, 100\degree C)\)

\(-40\degree F \) is where the two graphs will intersect
Mathematics as a Second Language
Mrs. Zander’s Solution to the Temperature Conversion Problem (Page 5 of 5)

\[ \text{Slope} = \frac{-9}{-5} \]

\[ \text{[means down 9 over 5]} \]
ABCD and PQRS are similar polygons whose perimeters are 40 inches and 30 inches, respectively. The area enclosed by ABCD is 8 square inches.\(^{61}\)

(a) **What area is enclosed by PQRS?**

Given similar polygons, ABCD and PQRS, the ratio between any side of polygon ABCD and the corresponding side of polygon PQRS is the scale factor that describes the proportional change from ABCD to PQRS. Since scale factor is \(a:b\), one can use the square of the scale factor, or \(a^2:b^2\), to help find the area of PQRS:

\[
\frac{4^2}{3^2} = \frac{16}{9}
\]

Area of ABCD: Area of PQRS = 16:9

8 : Area of PQRS = 16:9

\[
\frac{8}{4.5} = \frac{16}{9}
\]

Thus, the area of PQRS is 4.5 inches\(^2\).

---

\(^{61}\) Note that if the perimeter of ABCD is 40 in, then the area of ABCD would have to be more than 8 in\(^2\) unless the angle at B is extremely small, a possibility that is inconsistent with the representation of the quadrilateral. This inconsistency did not appear to bother either the teachers or the course instructors.
(b) Is it possible for the straight-line distance from point A to point C to be 20 inches?

According to the triangle inequality theorem, the sum of any two sides of a triangle must be greater than the third side. In ΔABC if AC is equal to 20 inches, then AB+BC must be greater than 20 inches. In ΔADC if AC is equal to 20 inches, then CD+DA must be greater than 20 inches. Since segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ make up the four sides of polygon ABCD, $AB+BC+CD+DA$ would be equal to the perimeter of polygon ABCD. If AC is equal to 20 inches, then the perimeter of ABCD would be greater than 40 inches. This contradicts the given statement that polygon ABCD has a perimeter of 40 inches. Therefore the answer is “No, it is not possible for the straight-line distance from point A to point C to be 20 inches.”
Appendix N
Functions, Algebra and Geometry for Middle Level Teachers
Mrs. Anderson’s Solution to the Scale Factor Problem

\[ A \Delta_2 = S^2 \cdot A \Delta_1 \]

Perimeter  Enclosed Area
\[ \begin{array}{ll}
A \quad ABCD & 40 \\
\quad \quad \quad \quad PAPS & 30 \\
\end{array} \]

\[ \left(\begin{array}{c}
8 \text{ in}^2 \\
(4.5 \text{ in})^2 \\
\end{array} \right) \]

ABCD and PAPS are similar

a. The area \( \frac{40}{30} = \frac{4}{3} \) scale is 4:3

\[ S^2 = \frac{4^2}{3^2} \]

\[ = \frac{16}{9} \]

\[ 8 \div \frac{16}{9} \]

\[ \frac{8}{9} \times 16 = \frac{9}{2} = 4.5 \]

We are looking for the area of the smaller polygon PAPS.
We found the scale factor by proportion of the
perimeters \( \frac{40}{30} = \frac{4}{3} \). The scale is 4:3 or \( \frac{4}{3} \).
Since we are looking for area, we squared the scale; \( \frac{4}{3} \)

Then we took the proportion of \( \frac{8}{16} = \frac{9}{2} = 4.5 \)

The area of PAPS is 4.5 in

b. The triangle inequality: if \( A, B, \) and \( C \) are
three noncollinear points, then \( AB + BC > AC \)

AC cannot be 20 inches because the
perimeter of the triangle is less than 40m
If AC were 20m that would only leave 20m to the sum of \( AB \) and \( BC \). That cannot happen because the sum of \( AB + BC \) has to be greater than 20
due to the triangle inequality theorem.

Class:
Ann: \( AD + DC = 720 \)
\( AB + BC = 720 \)
Mrs. Zander’s Solution to the Scale Factor Problem

A. Polygon PQRS

Perimeter = 30 in.
Area = \( a \) in.²

Perimeter = 40 in.
Area = 8 in.²

The Scale Factor for the two polygons is \( \frac{3}{4} \).

Thus, means that PQRS is \( \frac{3}{4} \) of the perimeter of the polygon ABCD or ABCD is \( \frac{4}{3} \) of the perimeter of the polygon PQRS. Each polygon is similar using the scale factor for each side length.

a) The area of ABCD is 8 in.². The area of the polygon PQRS can be calculated knowing the scale factor of \( \frac{3}{4} \) for perimeter and knowing ABCD’s area is 8 in.².

We take \( S \) scale factor and multiply it by the area, 8 in.², of the polygon.

\[
S^2 \cdot 8 = 8 \cdot \left( \frac{3}{4} \right)^2 \quad \text{or} \quad \text{Area of PQRS} = 6 \text{ in.}²
\]
A continued

I also realized after looking at the chart on page 7 of our similarity section in our binder that:

- Scale factor is 1 dimensional \( \frac{1}{3} = 10 \)
- Surface Area is 2 dimensional \( \frac{1}{2} \)
- Volume is 3 dimensional \( \frac{1}{3} \)

These also play a role when labeling inches as squared, cubed or just inches.

b) Perimeter of ABCD is 40 inches. Is it possible to have a straight-line distance from point A to point C being 20 inches?

\[ \overline{AB} + \overline{BC} + \overline{CD} + \overline{DA} = 40 \text{ in. (perimeter)} \]

In this scenario of polygon ABCD, we know that sides \( \overline{AB} \) and \( \overline{BC} \) cannot be less than 20.

In order for \( \overline{BC} + \overline{BA} + \overline{AC} = 40 \), \( \overline{BC} + \overline{BA} \) would have to equal 20.

**Triangle Inequality**: This is not possible since \( \overline{AB} + \overline{BC} \) had to be greater than 20.

The shortest distance between 2 points is a straight line.

\[ \overline{AB} + \overline{BC} \geq 20 \]

Thus, the perimeter cannot be 40.
You roll a pair of dice 24 times. What is the probability of seeing at least one 11? 

There are 36 possible outcomes when rolling a pair of six-sided dice. Two of those outcomes result in a sum of 11, rolling a 5 on one die and a 6 on the other or vice-versa. Probability is recorded as the ratio of favorable outcomes to total outcomes, 2:36 in this case. Each of the twenty-four rolls of the dice is considered to be an independent event, as no particular roll can influence the outcome of another roll. The probability of seeing at least one 11 implies that rolling more than one 11 is also a valid outcome. Finding the sum of 24 independent probabilities is not an efficient manner for solving this problem. A better approach is to find the probability of not rolling a sum of 11 on any single roll. That probability is 34:36 or 17:18. Thus, the probability for 24 rolls is \((17/18)^{24}\), which is approximately 0.2536 or 25.36%. In other words, 25.36% of the time one will not see a sum of 11. Thus, 74.64% of the time (100% - 25.36%) one will see a sum of 11. The probability of seeing at least one 11 during 24 rolls of a pair of six-sided dice is approximately 0.7464.
Appendix Q

Experimentation, Conjecture, Reasoning

Mrs. Anderson’s Solution to the Rolling Dice Problem

You roll a pair of dice 24 times. What is the probability of seeing at least one 11?

I first started by going back to information from the reading pages we were assigned for this lesson. From this I know: You have to roll a 5 and a 6 to get 11.

There are 36 equally ordered outcomes for a pair of dice. Can be shown in table:

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</tbody>
</table>

Out of a pair of dice, there are 36 possible outcomes. 11 is 2 times out of 36 possible outcomes in rolling 2 dice, and 2 times out of the 36 an 11 could be rolled, that is 2/36 ways to roll an 11.

\[ \frac{2}{36} = 0.0555 \]

Since we roll the dice 24 times, 0.0555 is multiplied to the 24th power. Then divide by 24.

\[ \left(0.0555\right)^{24} = 0.253649 \]

This is the probability of not rolling at least one 11. To find the chance of rolling at least one 11, subtract 0.253649... from 1.

\[ 1 - 0.253649... = 0.746357 \times 100 \text{ to find } \% = 74.6\% \]

This can be stated that there is a 74.6% chance of rolling at least one eleven out of 24 rolls. (Class discussion helped me understand this.)
Appendix R
Experimentation, Conjecture, Reasoning
Mrs. Zander’s Solution to the Rolling Dice Problem (Page 1 of 3)

No dice

\[ 6 \times 6 = 36 \text{ options/outcomes} \]

with 2 die

\[ \frac{2}{36} \text{ or } \frac{1}{18} \]

\( 11 \) can be achieved by rolling

1. 5 and 6
2. 6 and 5

There are 24 rolls, how many likely are you to roll an 11?

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### Mrs. Zander’s Solution to the Rolling Dice Problem

#### Experimentation, Conjecture, Reasoning

#37) cont... No dice

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<tr>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>(6</td>
<td>6) Sum of 11</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

= Total of 36 options

Each roll allows you 36 options. For each of the 24 rolls - you have 36 options. Therefore $36^{24}$ or $36 \times 36 \times \ldots \times 36 \times 36$.

$36^{24} = 2.245225771 \times 10^{37}$
Experimentation, Conjecture, Reasoning
Mrs. Zander’s Solution to the Rolling Dice Problem (Page 3 of 3)

#3) Cont...

Possibility of not rolling an 11 = \( \frac{36}{36} \) - 2 options
or 34.

So each of the 24 rolls have
34 chances of NOT rolling an 11.

\[
34^{24} = 5.695603680 \times 10^{36}
\]

\[
\frac{(\text{Not rolling})}{(\text{Total})} \cdot 34^{24} = 0.253649488... \text{ or } 0.2536
\]

1 - 0.2536 = 0.746350512 or 0.7464
25.36% to not roll “11”

1 represents 1 whole or 100% and
when you subtract the change
of not rolling “11” total, you will
have a probability of 0.7464 (74.64%)
representing the chance you
will roll an “11”. 

Nov. 26
Appendix S: Mathematical Problems Referenced in Chapter 6

It All Makes Cents Problem

What is the fewest number of coins that it will take to make 43 cents if you have available pennies, nickels, dimes, and quarters? After you have solved this problem, provide an explanation that proves that your answer is correct. How does the answer (and the justification) change if you only have pennies, dimes and quarters available?

The Eights Problem

How many ways are there to use eights strung together with plus (+) signs to equal 1000? Once you think you have an answer, can you provide a mathematical argument that you are correct?

Pentominos

A domino can be thought of as two squares joined along one side. Similarly, a triomino might be a polygon formed by joining three squares together. (in each case two matching sides must fit together exactly.) Continue in this manner; define what is meant by a tetronimo (i.e., a polygon created by joining four squares together) and a pentomino (a polygon created by joining five squares together). We will say that two pentominos are the same if one can be shifted, rotated and/or flipped to fit exactly onto the other pentomino.

Warm-up: How many pentominos are there? Many of you will know the answer to this. In fact, you may have a set of pentominos in your classroom. Just to level the playing field, we will give you the answer. There are 12 pentominos.
The Challenge of Explaining Why: Once you have convinced yourself that there are exactly 12 pentominos, can you provide an argument that your answer is correct? I.e., Can you give a convincing argument that “all 12 are different” and “there are no more?” Note that this calls for more than statements like, “I tried all flips and rotations and no two are the same,” or “I considered all possibilities and couldn’t create any new pentominos.” Those are “trust me” arguments. What this calls for is a discussion of the attributes of the pentominos that explains why they are different and why there are no more.

ECR page 660 #6, #7, #8

6. Blonde, bleached blonde. You have high standards with respect to truth in advertising, particularly when it comes to hair color. One day at the Laundromat, you meet an attractive blonde stranger named Chris and wonder if you should pursue a relationship. Unfortunately, you have a nagging belief that Chris’s golden locks may have been the result of peroxide—presenting the specter of a dark (haired) future. However, you also know several facts about the incidence of dyed hair and about your ability to detect fraudulent follicles. You know that 90% of blonde people in the world are naturally blonde. You have done a personal survey and learned that you are 80% accurate in your ability to correctly categorize fake hair color as fake and real hair color as real. What is the probability that Chris’s hair is fair and that your bleached beliefs were incorrect? Given these facts, should you pursue your relationship with Chris?

7. Blonde again. Given the scenario in problem 6, now suppose that 70% of blonde people are naturally blonde and that you are able to accurately detect dyed hair
85% of the time. What is the probability that Chris’s hair is fair and that your bleached beliefs were incorrect? Given these facts, should you pursue your relationship with Chris?

8. Bleached again. Given the scenario in problem 6, now suppose that 80% of blonde people are naturally blonde and that you are able to accurately detect dyed hair 50% of the time. What is the probability that Chris’s hair is fair and that your bleached beliefs were incorrect? Given these facts, should you pursue your relationship with Chris?

**The Chess Game Problem**

There once was a humble servant who was also a chess master. He taught his king to play the game of chess. The king became fascinated by the game and offered the servant gold or jewels in payment, but the servant replied that he only wanted rice—one grain for the first square of the chess board, two on the second, four on the third, and so on with each square receiving twice as many as the previous square. The king quickly agreed. How much rice does the king owe the chess master? Suppose it was your job to pick up the rice. What might you use to collect the rice, a grocery sack, a wheelbarrow, or perhaps a Mac truck? Where might you store the rice?

**An Open and Shut Case**

In a certain school there are 100 lockers lining a long hallway. The lockers are numbered 1, 2, 3, ..., 99, 100. All are closed. Suppose that 100 students walk down the hall in single file, one after another. Suppose the first student (who we will call “Student #1” for obvious reasons) opens every locker. The second student (i.e., Student #2) comes
along and closes every 2nd locker beginning with locker #2 (i.e., lockers #2, 4, 6,…, 98, 100). Along comes Student #3 who changes the position of every third locker; if it is open, this student closes it; if it is closed, this student opens it. Student #4 changes the “open or shut” position of every fourth locker, and so forth, until the 100th student changes the position of locker #100. Which lockers are open at the end of this event?

1) Can you extend this? I.e., if in the problem above, we had 1000 (or even 10,000) numbered lockers and people, which lockers would still be open at the end of the event?

2) Explain why these particular numbers are the numbers of the lockers that are open at the end of the event?

Birthday Party Blues

A mother is holding a birthday party with several excited young children. She has n distinctively wrapped party favors to give to n children. She how has a headache, so she quickly hands out each favor package randomly without looking to see if the recipient already has a package. For n=3, 4, and 5, find the probability that each child gets a package.

Bobo and the Time Bomb

After Bobo successfully completed his hemispheric fencing project, he decided to go on a vacation to get some much-deserved rest and relaxation. He flew down to the Bahamas and was lying on the beach catching a few rays when his good friend, Bozo came running up to him. He explained that Bobo was in grave danger because of the fence he had built. It seems that some very powerful drug lords were upset because the
fence was interfering with their drug trafficking. They had vowed to kill Bobo with a bomb.

Bobo ran to his hotel room, locked himself in, and tried to figure out what to do. He noticed that the alarm clock had some funny wires coming out of it and a strange package was concealed below the nightstand. That was it! But he was locked in the room. He frantically looked around for a way out when he saw the note taped to the window.

The note told him that if he could solve a certain problem, he could disable the bomb and save his life. The problem was to determine the EXACT time when the minute hand and hour hand of the clock would be in the same position. Of course this is a fairly common occurrence; it happens 22 times every day starting at 12:00:00:00 (12 o’clock midnight, zero minutes, zero seconds, zero hundredths of a second).

The clock read 2:15, so Bobo knew that he had only about an hour to find the next time the hands would coincide or he would be history. Bobo solved the problem in less than an hour. Your task is to solve the same problem, but you have to find all 22 times and explain your solution. Good luck!

FAGMLT EOC #2b

Complete a linear function \( f(x) = \) ______ representing the information in the table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>204</td>
</tr>
<tr>
<td>20</td>
<td>304</td>
</tr>
<tr>
<td>30</td>
<td>404</td>
</tr>
<tr>
<td>40</td>
<td>504</td>
</tr>
</tbody>
</table>
The Clothespin Problem

(#4 in page 12 in Algebra Connections.) Heez A. Wasure is hanging clothes out on the clothesline to dry. Heez places one clothespin in the middle and two on the sides of each shirt. He links all of the shirts together so that he conserves clothespins. From the picture in the book you can see that he uses seven clothespins for three shirts.


b) Write a recursive rule describing the clothespin pattern.

c) Write an explicit rule representing the clothespin pattern.

d) Suppose that Heez used 77 clothespins for one line of shirts. How many shirts did he hand out to dry?

Spot’s Doghouse

Spot’s doghouse has a regular hexagonal base that measures one yard on each side. He is tethered to a vertex with a two-yard rope. What is the area, in square yards, of the region outside the doghouse that Spot can reach?

FAGMLT EOC #8b

Two polygonal regions are similar with scale factor 8. The area of the larger polygon is 640 cm\(^2\). What is the area of the smaller region? Explain.

ECR Session A #8

By experimenting with numerous examples in search of a pattern, determine a simple formula for \((F_{n+1})^2 - (F_{n-1})^2\) — that is, a formula for the difference of the squares of two Fibonacci numbers.
Find all solutions to the following equations. If there is no solution, explain why.

a. \( 5(x + 4) = 7x - 2 \)

b. \( 7x = 5x \)

c. \( 6(5 + 2x) = 4(2x - 1) + 4(x + 5) \)

Coat Discount

A store marks down the price of a coat by \( \frac{1}{4} \). The coat does not sell, so the following week the store marks the coat down by \( \frac{1}{5} \) of the sale price. Since \( \frac{1}{4} + \frac{1}{5} = \frac{9}{20} \), the overall savings from the original price for the coat is \( \frac{9}{20} \). True or false, and why?

ECR EOC #3

Can you make a triangle with the following leg lengths: one leg is the sum of two consecutive odd integers, the other leg is the product of two consecutive odd integers, and the hypotenuse is two more than the product? Can you make a right triangle with those leg lengths? For two points, provide some experimentation and a conjecture, for 3 points, provide some mathematical reasoning which justifies your conjecture.

The Fly and Spider Problem

A room has walls on the east and west that are 12 feet long. The room is 8 feet high. Along the south end wall (10 feet wide and 8 feet high) there is a spider at a spot 5 feet up and 2 feet in from the southeast corner. Across the room on the north wall, there is a fly that is 4 feet up the wall and 4 feet in from the northwest corner. The spider,
without a web, has decided to walk across to catch the fly. What is the shortest distance
the spider must walk to get to the fly?

While the spider was making the computation, the fly moved along the north wall
to a spot still 4 feet up, but now 3 feet in from the northeast corner. Now what is the
shortest distance the spider must walk to get to the fly?
Appendix T
Mathematics as a Second Language
Mrs. Anderson’s Work on the Spider and Fly Problem (Page 1 of 2)

We had the fly crawl across the ceiling. So to find the shortest distance, we drew a straight line from the spider to the fly. We had to flatten out the ceiling and the north and south wall. So the distance from the spider to the fly is the hypotenuse. Then we used the Pythagorean Theorem to find the distance in feet.

When the fly moved 3 feet in from the northeast corner, the shortest distance was around the east wall. So we flattened out the south, east, and north walls to find the hypotenuse of the new distance to the new fly’s position.

My next page has my wrong (1st attempt) that led me to the correct answer. Ok
Riding the Ferris Wheel: You and your little sister go to a carnival that has both a large and a small Ferris wheel. You get on the large Ferris wheel at the same time your sister gets on the small Ferris wheel. The rides begin as soon as you are both buckle into your seats. Determine the number of seconds that will pass before you and your sister are both at the bottom again.

a. Assume the large wheel makes one revolution in 60 seconds and the small wheel makes one revolution in 20 seconds.

b. Assume the large wheel makes one revolution in 50 seconds and the small wheel makes one revolution in 30 seconds.
Appendix V
Mrs. Zander’s Sheet of Notes for Perimeter

Holt 10.2 – Area

Name _________________

Formulas:
Area of a rectangle = ________________

width

length

Area of a parallelogram = ________________

height

base

(becomes a rectangle)

Area of a triangle = ________________

height

base