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A SOLUTION TO A PROBLEM OF J. R. RINGROSE¹

BY DAVID R. LARSON

We announce a solution to a multiplicity problem for nests posed by J. R. Ringrose approximately twenty years ago. This also answers a question posed by R. V. Kadison and I. M. Singer, and independently by I. Gohberg and M. Krein concerning the invariant subspace lattice of a compact operator. The key to the proof is a result concerning compact perturbations of nest algebras which was recently obtained by Niels Andersen in his doctoral dissertation. The complete proof of the general result as well as of a number of related results will appear elsewhere. A proof for the special case which answers Ringrose's original question is included herein.

Let \( H \) be infinite dimensional separable Hilbert space. A nest \( N \) is a family of closed subspaces of \( H \) linearly ordered by inclusion. \( N \) is complete if it contains \( \{0\} \) and \( H \) and contains the intersection and the join (closed linear span) of each subfamily. The corresponding nest algebra \( \text{alg} \, N \) is the algebra of all operators in \( L(H) \) which leave every member of \( N \) invariant. The core \( C_N \) is the von Neumann algebra generated by the projections on the members of \( N \), and the diagonal \( D_N \) is the von Neumann algebra \( (\text{alg} \, N) \cap (\text{alg} \, N)^* \). \( N \) is continuous if no member of \( N \) has an immediate predecessor or immediate successor. Equivalently, \( N \) is continuous if the core \( C_N \) is a nonatomic von Neumann algebra. \( N \) has multiplicity one (is multiplicity free) if \( D_N \) is abelian, or equivalently, if \( C_N \) is a m.a.s.a.

J. R. Ringrose posed the following question: Let \( N \) be a multiplicity free nest and \( T: H \to H \) a bounded invertible operator. Is the image nest \( TN = \{TN: N \in N\} \) necessarily multiplicity free? Note that \( T(\text{alg} \, N)T^{-1} = \text{alg}(TN) \), so it is natural to say that \( TN \) is the similarity transform of \( N \). Is multiplicity preserved under similarity? We show that the answer is no. It should be noted that a negative answer was conjectured in recent years by several mathematicians including J. Ringrose and W. Arveson.

The following key result is due to N. Andersen [1]. Let \( LC \) denote the compact operators in \( L(H) \).

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THEOREM (ANDERSEN). If $H$ is separable, and if $N, M$ are arbitrary continuous nests in $H$, then there exists a unitary operator $U$ such that $\operatorname{alg} N + L C = U(\operatorname{alg} M + L C)U^* = \operatorname{alg}(UM) + L C$.

Next, we answer Ringrose's question.

THEOREM 1. Let $N$ be a continuous nest of multiplicity 1. Then there exists a positive invertible operator $T \in L(H)$ such that $TN = \{TN : N \in N\}$ fails to have multiplicity 1.

PROOF. By Andersen's theorem there exists a continuous nest $M$ not of multiplicity 1 such that $\operatorname{alg} M + L C = \operatorname{alg} N + L C$. Since for any algebra $A$ we have $A + L C/L C \cong A/A \cap L C$, the algebras $\operatorname{alg} M/\operatorname{alg} M \cap L C$ and $\operatorname{alg} N/\operatorname{alg} N \cap L C$ are algebraically isomorphic. The diagonal $\mathcal{D}_M = \operatorname{alg} M \cap (\operatorname{alg} M)^*$ is a nonabelian von Neumann algebra so contains a nonzero partial isometry $v$ with orthogonal initial and final spaces. Let $\tilde{S} = v + v^* - vv^* - v^*v + I$ and $\tilde{P} = vv^*$. Then $\tilde{S}^2 = I$ and $\tilde{P}S\tilde{P} = 0$. Since $M$ is continuous $\mathcal{D}_M$ contains no compacts, so $\tilde{P}$ has infinite rank. Thus via the algebraic isomorphism between quotients it follows that $\operatorname{alg} N/\operatorname{alg} N \cap L C$ contains elements $\tilde{P}, \tilde{S}$ with $\tilde{P}^2 = \tilde{P} \neq 0, \tilde{S}^2 = I, \tilde{P}\tilde{S}\tilde{P} = 0$.

Let $A$ and $B$ be elements of $\operatorname{alg} N$ whose images in the quotient are $\tilde{P}$ and $\tilde{S}$ respectively. Then $A^2 - A, B^2 - I$ and $ABA$ are contained in $\operatorname{alg} N \cap L C$, and this is contained in the Jacobson radical $\mathcal{R}_N$ of $\operatorname{alg} N$ since $N$ is continuous. So $B$ is invertible in $\operatorname{alg} N$. Also, a well-known result [cf. 9, Theorem 2.3.9] states that an element of a Banach algebra which is idempotent modulo the radical is equal modulo the radical to an idempotent. So there exists an idempotent $P \in \operatorname{alg} N$ with $A - P \in \mathcal{R}_N$. We have $P \neq 0$ since otherwise $A$ would be in $\mathcal{R}_N$ and hence $\tilde{P}$ above would be a quasinilpotent idempotent, hence 0.

We have $PB \in \mathcal{R}_N$. Set $B_1 = B - PB$. Then $B_1$ is also invertible in $\operatorname{alg} N$, and $PB_1P = 0$. Now set $S = B_1P + PB_1^{-1}(I - P) - B_1PB_1^{-1}(I - P) + I - P$.

We have $PSP = 0$, and it can be verified that $S^2 = I$. (Let $\alpha$ denote the sum of the first two terms, $\beta$ the sum of the remaining terms, and compute $\beta^2 = \beta, \alpha \beta = \beta \alpha = 0, \alpha^2 = I - \beta$.)

Let $R = I - 2P$. Then $R^2 = I, S^2 = I, RS \neq SR$. Let $G$ be the group generated by $R, S$. We have $SRS = I - 2SPS$, and $PSP = 0$, so $PSRS = P = SRSP$. Hence $R$ commutes with $SRS$ since $P$ does. It easily follows that $G = \{I, S, R, RS, SR, SRS, RSR, SRSR\}$.

So $G$ is a finite noncommutative group contained in $\operatorname{alg} N$. Set $T = (\Sigma_{g \in G} g^*g)^{1/2}$. Then $TGT^{-1} = \{TgT^{-1} : g \in G\}$ is a noncommutative group of
unitaries contained in the diagonal of \( \text{alg}(TN) \), and thus \( TN \) fails to have multiplicity 1.

Theorem 1 serves to answer an open question concerning invariant subspace lattices of compact operators due to Kadison and Singer [5] and to Gohberg and Krein [4]. An operator is said to be hyperintransitive if its lattice of invariant subspaces contains a multiplicity one nest.

**Corollary 2.** There exists a nonhyperintransitive compact operator.

**Proof.** Let \( V \) be the Volterra operator. Then \( \text{Lat} \ V \) is a continuous multiplicity one nest. Let \( N = \text{Lat} \ V \) and let \( T \) be an invertible operator such that \( TN \) does not have multiplicity one. Since \( \text{Lat}(TVT^{-1}) = TN \) and since \( TN \) is a maximal nest the similarity \( TVT^{-1} \) is not hyperintransitive.

**Remark.** It was known for a number of years that a negative resolution to the Ringrose problem would yield Corollary 2. I believe that this connection was first observed by J. Erdos, and it was first shown to me by W. Arveson.

We strengthen Theorem 1 as follows.

**Theorem 3.** Let \( N \) be a continuous nest of multiplicity one. Then given \( \epsilon > 0 \) there exists a positive invertible operator \( T \in L(H) \) with \( T - I \) compact and \( ||T - I|| < \epsilon \) such that \( TN = \{TN: N \in N\} \) fails to have multiplicity one.

A nest has purely atomic core if its core is generated by its minimal projections. The following shows that similarity transforms can fail to act "absolutely continuously" on nests.

**Theorem 4.** If \( N \) is a complete uncountable nest with purely atomic core there exists a positive invertible operator \( T \) such that \( TN \) does not have purely atomic core.

A nest \( N \) is said to have the factorization property if every invertible positive operator \( T \) factors \( T = A^*A \) for \( A \in (\text{alg} \ N) \cap (\text{alg} \ N)^{-1} \). Arveson [2] proved that nests of the "simplest type" have the factorization property. We generalize this to countable complete nests, and then show that these are the only ones with this property.

**Theorem 5.** A complete nest has the factorization property if and only if it is countable.

In contrast, if we drop the requirement that \( A^{-1} \) also be in \( \text{alg} \ N \) we obtain.

**Theorem 6.** Let \( N \) be an arbitrary nest. Then every invertible positive operator \( T \) factors \( T = A^*A \) for \( A \in \text{alg} \ N, A \) invertible in \( L(H) \).

The following answers a question of J. Erdos [3].
THEOREM 7. Let \( N \) be a continuous nest. Then the commutator ideal of \( \text{alg} \ N \) is not proper.

REFERENCES


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