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Dynamic Immunization and Transaction Costs With Different Term Structure Models

Eliseo Navarro* and Juan M. Nave†

Abstract‡

A bond portfolio selection model is developed in a dynamic framework using different term structures, but without transactions costs. We show that the optimal portfolios are consistent with Khang's dynamic immunization theorem, i.e., the optimal path consists of making portfolio duration equal to the investor's horizon planning period. The model is then extended to include transaction costs. The resulting optimal portfolios are no longer consistent with Khang's dynamic immunization theorem. In fact, the strategy for constructing the optimal portfolio consists of initially choosing a portfolio with a duration that is smaller than the horizon planning period.

Key words and phrases: bond, portfolio, planning period, strategy, risk, interest rate, stochastic

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1 Introduction

Suppose an investor in a fixed income market has certain obligations due at some specified future date, called the investor's horizon planning period. A key problem facing such an investor is the problem of immunizing 1 (protecting) his or her portfolio of bonds against interest rate risk.

Bierwag and Khang (1979) prove that the process of immunizing a bond portfolio can be described as a maxi-min strategy in a game against nature where the investor's target is to guarantee a minimum return over his or her planning period or, equivalently, to guarantee a minimum value at the end of his or her horizon planning period. Dantzig (1971) shows that this maxi-min solution can be determined by solving an equivalent linear program that depends on the assumption about the term structure of interest rates.

One of the main results concerning the development of portfolio immunization strategies against interest rate risk is due to Khang (1983) and is described by his dynamic global immunization theorem. Khang's strategy consists of a continuous portfolio rebalancing in order to keep portfolio duration equal to the length of the remaining planning period.

Specifically, consider an investor who has a horizon planning period of length $H$. Suppose the forward interest rates structure shifts up or down by a stochastic shift parameter at any time during the investor's planning period. If the investor follows Khang's strategy, then the investor's wealth at the end of his or her planning period will be no less than the amount anticipated on the basis of the forward interest rates structure observed initially (at time 0). Furthermore, the investor's wealth at time $H$ will be greater than the amount anticipated initially if at least one interest shock takes place during the planning period.

The validity of Khang's strategy rests on two key assumptions: (i) If $g(t), t \geq 0$, denotes the forward interest rates structure, and the forward interest rates structure changes to $g^*(t)$, then

$$g^*(t) = g(t) + \delta$$

where $\delta$ is a stochastic shift parameter; and (ii) there are no transaction costs.

The first assumption avoids the problem of the risk of misestimating the term structure behavior, which Fong and Vasicek (1983) call

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1 Immunization consists of making a portfolio's duration (properly defined) equal to the remaining horizon planning period.
the "immunization risk." Assumption (ii) avoids the high costs that a strategy of continuous portfolio rebalancing may incur.

In this paper we investigate the applicability of Khang's strategy under both a static and a dynamic portfolio selection model. Each model is tested under three different assumptions about the term structure of interest rates behavior: a flat term structure, a diffusion process as in Vasicek (1977), and a diffusion process as in Cox, Ingersoll, and Ross (1985). The dynamic portfolio selection model under the flat term structure model behaves according to classical Fisher and Weil (1971) immunization theorem. It behaves according to Boyle's (1978) stochastic immunization in the two alternative stochastic cases. Finally the model is expanded to include transaction costs. We show, through an example, that if transaction costs are high enough, the optimal strategy may differ from that proposed by Khang.

2 The Term Structure Models

Three different term structure models are used in our analysis:

- The first and simplest model assumes a flat term structure and parallel term structure of interest rates shifts;
- The second model assumes a stochastic term structure with instantaneous spot interest rate following a diffusion process as in Vasicek (1977); and
- The third model assumes a stochastic term structure with instantaneous spot interest rate following a diffusion process as in Cox, Ingersoll, and Ross (1985).

2.1 Flat Term Structure Model

This model makes the following assumptions about process of the term structure of interest rates:

A1 The term structure is flat;

\footnote{Bierwag (1987) calls it "stochastic process risk" and defines the stochastic process as "the way in which the term structure shifted from period to period," adding afterward that "it is conceivable that an investor could assume an incorrect stochastic process and, as a consequence, the perceived durations would be different from the actual ones. The investor . . . losses from misestimation (or miscuesstimation) of the correct process can be substantial" (Bierwag, 1987).}
A2 Term structure of interest rates changes consist of parallel movements of the entire term structure, i.e., short-term and long-term interest rates changes are equal; and

A3 The pure expectations hypothesis holds.

A4 There are \( m \) different levels of interest rates \( r_j \) \((j = 1, 2, \ldots m)\) with \( r_1 < r_2 < \cdots < r_m \).

The implication of assumption (A3) is that under a flat term structure model any interest rate change is considered to be unexpected.

Let

\[
\begin{align*}
r(t) &= \text{The spot rate of interest at time } t; \\
t_0 &= \text{The current time; } \\
r_c &= \text{The current spot rate of interest (at time } t_0); \\
P(r(t), t, s) &= \text{The price at time } t \text{ of a pure discount bond maturing at time } s (t \leq s).
\end{align*}
\]

It follows that

\[
\begin{align*}
P(r(t), t, s) &= e^{-(s-t)r(t)} \quad (1) \\
E[r(s)|r(t_0) = r_c] &= r_c, \quad (2)
\end{align*}
\]

for \( s > t_0 \), i.e., interest rates are expected to remain unchanged. Under the flat term structure model, the relative basis risk \(^4\) of a discount bond is given by

\[
-\frac{1}{P} \frac{\partial P}{\partial r} = s - t.
\]

2.2 The Vasicek (1977) Model

Here we make the following assumptions about the term structure:

A5 Instantaneous spot interest rate \( r(t) \) follows a diffusion process so its behavior is described by the following stochastic differential equation:

\[
dr = \beta(y - r)dt + \rho d\tilde{Z} \quad (3)
\]

where \( \beta, y, \) and \( \rho \) are positive constants, and \( d\tilde{Z} \) is a Wiener process with zero mean and variance \( dt \); and

---

^3For a thorough discussion of the different hypotheses about the term structure of interest rates and their implications, see Cox, Ingersoll, and Ross (1981).

^4Basis risk can be defined as the possibility that an institution's margin will rise or fall as a consequence of market rate movements.
A6 There are no arbitrage opportunities.

Equation (3) yields the following expressions for $s > t$:

$$\begin{align*}
E[r(s) | r(t)] &= \gamma + (r(t) - \gamma)e^{-\beta(s-t)} \\
P(r(t), t, s) &= \exp \left[ -F(s-t)(G - r(t)) \\
&\quad - (s-t)G - \frac{\rho^2}{4\beta} F(s-t)^2 \right]
\end{align*}$$

where

$$\begin{align*}
F(x) &= \frac{1}{\beta} [1 - \exp(-\beta x)] \\
G &= \gamma - \frac{\rho^2}{2\beta^2}.
\end{align*}$$

The relative basis risk of a discount bond is now given by

$$-\frac{1}{P} \frac{\partial P}{\partial r} = F(s-t).$$

2.3 The Cox-Ingersoll-Ross (1979) Model

In addition to assumption (A6), we assume the following:

A7 $r(t)$ satisfies the following stochastic differential equation:

$$dr = \kappa(\mu - r)dt + \sigma \sqrt{r} d\tilde{z}$$

where $\kappa$, $\mu$, and $\sigma$ are positive constants.

Equation (6) yields the following expressions, for $s > t$:

$$\begin{align*}
E[r(s) | r(t)] &= \mu + (r(t) - \mu)e^{-\kappa(s-t)} \\
P(r(t), t, s) &= A(s-t) \exp[-r(t)B(s-t)]
\end{align*}$$

where:

$$\begin{align*}
A(x) &= \left[ \frac{2\lambda \exp[(\kappa - \lambda)x/2]}{(\lambda + \kappa)[1 - \exp(-\lambda x)] + 2\lambda \exp(-\lambda x)} \right]^{-2\kappa\mu/\sigma^2} \\
B(x) &= \frac{2(1 - \exp(-\lambda x))}{(\lambda + \kappa)[1 - \exp(-\lambda x)] + 2\lambda \exp(-\lambda x)} \\
\lambda &= \sqrt{\kappa^2 + 2\sigma^2}.
\end{align*}$$
The relative basis risk is

$$-\frac{1}{P} \frac{\partial P}{\partial r} = B(s-t).$$

3 The Static Model

Consider an investor who wants to allocate an amount of $I$ dollars in a market where $n$ different default-free non-callable coupon-bearing bonds are available. The investor's objective is to construct a portfolio that guarantees a minimum return over his or her planning period or, equivalently, that guarantees minimum value at the end of the investor's horizon planning period.

In the static model, the market can be characterized by the following set of assumptions:

A8 Financial markets are competitive; Individual investors' decisions don't affect interest rates that are given exogenously;

A9 There is perfect divisibility of financial assets;

A10 There are no arbitrage opportunities;

A11 There are no transaction costs; and

A12 Short sales are not allowed.\(^5\)

3.1 Notation

The notation introduced in this section will be used throughout this paper:

\[ n = \text{Number of default-free non-callable coupon bonds;} \]

\[ H = \text{Horizon planning period, which spans the interval } (t_0, H]; \text{ and} \]

\[ I = \text{Investor's initial wealth at } t_0. \]

We assume that the bonds are ordered according to their maturity so bond 1 is the bond with the shortest time to maturity and bond $n$ is

\(^5\)This constraint is imposed in the model as a sufficient condition in order to guarantee that the net income generated by the portfolio is always nonnegative throughout the planning period, which is one of the hypotheses of Khang's theorem.
the bond with the longest term to maturity. For \( i = 1, \ldots, n \), let

\[
T_i = \text{Time to maturity of bond } i \text{ with } T_i \leq T_{i+1} \\
\text{for } i = 1, \ldots, n - 1;
\]

\[
\rho = \text{Number of bonds maturing in } (t_0, H], \rho = 0, 1, \ldots, n;
\]

\[
n_i = \text{Number of bond } i \text{ coupon payments made after } t_0;
\]

\[
\tau_s^{(i)} = \text{Time of bond } i\text{'s } s\text{-th coupon payment after } t_0,
\]

\[
\text{for } s = 1, 2, \ldots, n_i;
\]

\[
\tau_{n_i}^{(i)} = T_i
\]

\[
p_i = \text{Current (at } t_0) \text{ price of one unit of bond } i;
\]

\[
x_i = \text{Number of units of bond } i \text{ in the optimal portfolio; and}
\]

\[
C_i = \text{Size of each coupon payment from bond } i.
\]

Clearly, in order to obtain a duration close to \( H \), some of the \( T_i \)s must exceed \( H \). Also, bonds \( \rho + 1, \ldots, n \) mature after \( H \).

3.2 The Static Model’s Linear Program

The investor’s strategy consists of purchasing an allocation\(^6\) vector \((x_1, x_2, \ldots, x_n)\) of bonds that satisfy the following budget constraint:

\[
\sum_{i=1}^{n} x_ip_i = I. \tag{9}
\]

If just after selecting a strategy at \( t_0 \), interest rates instantaneously change from \( r_c \) to \( r_j \), then portfolio value at the end of the horizon planning period is \( V_j \) such that:

\[
V_j = \sum_{i=1}^{n} x_i v_{ij} \tag{10}
\]

where \( v_{ij} \) denotes the value at the end of the horizon planning period of an investment of \( p_i \) dollars in bond \( i \), i.e.,

\[
v_{ij} = \frac{\sum_{s=1}^{n_i} C_i e^{-r_j(T_i-t_0)} + (1 + C_i)e^{-r_j(T_i-t_0)}}{e^{-r_jH}} \tag{11}
\]

under the flat term structure model, and

\(^6\) A portfolio allocation vector can be considered as a strategy of the investor.
\[ \nu_{ij} = \frac{\sum_{s=1}^{ni-1} C_i P(r, t_0, \tau^{(i)}_s) + (1 + C_i) P(r, t_0, T_i)}{P(r, t_0, H)}. \]  

under the Vasicek and the Cox-Ingersoll-Ross models.

Note that \( \nu_{ij} \) is based on two assumptions: (i) the interest rates remain \( r_j \) until the end of the horizon planning period (in accordance with the pure expectations theory); and (ii) the coupon and principal payments made before the end of the horizon planning period are reinvested at rate \( r_j \) under the flat term structure model. Under stochastic models, coupon and principal payments are assumed to be reinvested at the forward rates corresponding to a term structure of interest rate derived from a instantaneous spot rate equal to \( r_j \).

Let \( V \) denote the minimum final portfolio value the investor wishes to maximize, i.e., \( V \) is a lower bound for the final portfolio value. Thus, \( V \) is independent of interest rate changes and depends on the selected portfolio. The portfolio selection process can be modeled as the following linear program:

**Static Model**

\[
\text{maximize } V \\
\text{subject to } \sum_{i=1}^{n} x_i \nu_{ij} \geq V, \ j = 1, 2, \ldots, m \\
\sum_{i=1}^{n} x_i p_i = I \\
V \geq 0, \ x_i \geq 0, \ i = 1, 2, \ldots, n.
\]

Cox, Ingersoll, and Ross (1979) point out that if we want stochastic duration to serve as a proxy for the basis risk of coupon bonds with the units of time, it is natural to define it as the maturity of a discount bond with the same risk. Therefore, portfolio duration at \( t_0 \) under Vasicek's term structure model is \( D_V \) given by:

\[
D_V = F^{-1} \left( \frac{\sum_{i=1}^{n} x_i W^{(F)}_i}{\sum_{i=1}^{n} x_i \left[ \sum_{s=1}^{ni} C_i P(r, t_0, \tau^{(i)}_s) + P(r, t_0, T_i) \right]} \right)
\]

where \( r \) is the interest rate at \( t_0 \),

\[
W^{(F)}_i = \sum_{s=1}^{ni} C_i P(r, t_0, \tau^{(i)}_s) F(\tau^{(i)}_s - t_0) P(r, t_0, T_i) + F(T_i - t_0)
\]
and

\[ F^{-1}(x) = \frac{\ln(1 - \beta x)}{\beta}. \]

On the other hand, the portfolio duration under the Cox-Ingersoll-Ross term structure model is \( D_{CIR} \):

\[
D_{CIR} = B^{-1} \left( \frac{\sum_{i=1}^{n} x_i W_i^{(B)}}{\sum_{i=1}^{n} x_i \left[ \sum_{s=1}^{n_i} C_i P(r, t_0, \tau_s^{(i)}) + P(r, t_0, T_i) \right]} \right) \tag{15}
\]

where:

\[
W_i^{(B)} = \sum_{s=1}^{n_i} C_i P(r, t_0, \tau_s^{(i)}) B(\tau_s^{(i)} - t_0) P(r, t_0, T_i) + B(T_i - t_0)
\]

and

\[
B^{-1}(x) = \frac{1}{\lambda} \ln \left[ \frac{2 - (\kappa - \lambda)x}{2 - (\kappa + \lambda)x} \right].
\]

3.3 An Example

To illustrate our ideas, we apply them to a simple example. Assume an investor has \( I = \$1,000,000 \) and a horizon planning period of 18 months. There is a fixed income market with four default-free non-callable 10 percent coupon bonds, as described in Table 1.

| Table 1 |
| 10% Coupon Bonds |
| **With Coupons Paid Semi-Annually** |
| Maturity (In Years) | Macaulay Duration |
| Bond 1 | 0.5 | 0.5000 |
| Bond 2 | 1.0 | 0.9762 |
| Bond 3 | 1.5 | 1.4297 |
| Bond 4 | 2.0 | 1.8616 |

In the case of the series flat term structure, we assume a nominal current interest rate level of 10 percent (compounded semiannually).
Interest rates may move up and down by 100 basis points to 9 percent or to 11 percent. In other words, nominal interest (compounded semiannually) may take only one of three values: $r_1 = 9$ percent; $r_2 = r_c = 10$ percent; $r_3 = 11$ percent.

The optimal solution of the linear program of equation (13) is shown in Table 2. This result is consistent with Fisher and Weil immunization theorem (which states that the optimal solution consists of a portfolio with a duration equal to the horizon planning period).

In the case of the series Vasicek (1977) and Cox-Ingersoll-Ross (1985) term structure model, we use the model parameters estimated in Nowman (1997) for both U.S. (from the Treasury bill market) and U.K. (sterling one month interbank rate). (See Table 3.)

Nature strategies consist of the different values that the current instantaneous spot rate can take which we assume can vary 100 basis points (up or down) from its current level (5.61 percent for the U.S. and 5.99 percent for the U.K.). The optimal solutions are shown in Table 2.

It is important to see that, under stochastic term structure models, portfolio immunization consists of making portfolio duration (properly defined) equal to the remaining horizon planning period.

3.4 Immunization Risk

So far we have assumed a specific term structure behavior where the whole term structure is supposed to depend on a unique factor (short-term interest rate). The nature of the dynamics of interest rates, however, is more complex. Immunization strategies may fail if the

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7We have assumed a one percent change in instantaneous spot interest rate and then recalculated the whole term structure of interest rates according to equations (5) and (8) to determine the new bond prices. Theoretically, the interest rate change considered in these models should be the largest change that is possible within a trading day (or any other suitably short time interval). We are aware that the probability of one percent change in interest rates within a day is, according to model parameters, negligible. Such a drastic change in interest rates is assumed, however, in order to have a similar level of interest rate risk in all three cases analyzed. Despite this, the interest rate risk assumed under the stochastic model is still lower due to the mean reversion effect. According to Boyle (1978), under flat term structure models, a small interest rate change may have a dramatic impact on the price of long-term bonds. The impact of instantaneous spot rate changes on long-term bonds under the stochastic models considered in this paper, however, is diminished by the expected mean reversion of short interest rates.

8There is some international evidence that at least 95 percent of term structure movements can be explained by three factors: parallel shifts, slope changes, and curvature changes. Depending on the country analyzed and the period covered by different studies, parallel shifts can explain between 72 and 97 percent of the variance in interest rate changes. For further detail see Steeley (1990), Strickland (1993), D'Ecclesia and Zenios.
Table 2
Optimal Strategies in a Static Framework

Panel A: Nonstochastic Model

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>2639.536</td>
<td>0</td>
<td>0</td>
<td>7360.465</td>
<td>1.5006</td>
</tr>
</tbody>
</table>

Panel B: Vasicek Model

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S.</td>
<td>0</td>
<td>0</td>
<td>7886.288</td>
<td>1516.520</td>
<td>1.5010</td>
</tr>
<tr>
<td>U.K.</td>
<td>1044.207</td>
<td>0</td>
<td>4745.310</td>
<td>3676.038</td>
<td>1.4999</td>
</tr>
</tbody>
</table>

Panel C: Cox-Ingersoll-Ross Model

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S.</td>
<td>2858.766</td>
<td>0</td>
<td>0</td>
<td>6843.708</td>
<td>1.5012</td>
</tr>
<tr>
<td>U.K.</td>
<td>0</td>
<td>3919.526</td>
<td>0</td>
<td>5545.140</td>
<td>1.5010</td>
</tr>
</tbody>
</table>

Notes: Portfolio durations are calculated as follows: Macaulay duration is used for Panel A; Equation (14) is used for Panel B; and Equation (15) is used for Panel C.

term structure of interest rates behaves differs significantly. This is known as immunization risk.9

To minimize the immunization risk from an unexpected behavior of the term structure, several proposals have been suggested. Most of them consist of selecting among the set of immunized portfolios those that generate payment streams as close as possible to the end of the horizon planning period. A trivial example would be a portfolio consisting entirely of zero coupon bonds maturing at the end of the horizon planning period.

There are several alternative measures of immunization risk. The usually accepted dispersion measure, however, is that proposed by Fong and Vasicek known as $M^2$. By minimizing this quadratic dispersion measure, the effect on final portfolio value of a non-expected (1994), Navarro and Nave (1995), and Sherris (1995) for the U.K., U.S., Italy, Spain, and Australia, respectively.


10There are alternative dispersion measures, such as $M$-absolute, derived from different assumptions about term structure movements. Chalmers and Nawalka (1996) test the suitability of the $M$-absolute measure as a first order condition to protect an investment against interest rate risk instead of using it as a second order condition to minimize immunization risk. Other authors have criticized the $M^2$ measure, suggesting the convenience of including an asset with maturity at the end of the horizon planning
Table 3
Parameter Values of the Vasicek and Cox-Ingersoll-Ross Models (With \(r(t)\) Determined in April 1997)

<table>
<thead>
<tr>
<th>Panel A: Vasicek Model</th>
<th>(\beta)</th>
<th>(\gamma)</th>
<th>(\rho^2)</th>
<th>(r(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S.</td>
<td>0.0506</td>
<td>0.0691700</td>
<td>0.0001</td>
<td>0.0561</td>
</tr>
<tr>
<td>U.K.</td>
<td>0.0311</td>
<td>0.1028939</td>
<td>0.0001</td>
<td>0.0599</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Cox-Ingersoll-Ross Model</th>
<th>(\beta)</th>
<th>(\gamma)</th>
<th>(\rho^2)</th>
<th>(r(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S.</td>
<td>0.0373</td>
<td>0.0697051</td>
<td>0.0008</td>
<td>0.0561</td>
</tr>
<tr>
<td>U.K.</td>
<td>0.0279</td>
<td>0.1039427</td>
<td>0.0007</td>
<td>0.0599</td>
</tr>
</tbody>
</table>

Notes: The parameters \(\beta\), \(\gamma\), and \(\rho^2\) were estimated by Newman (1997) using a discrete time model that reduces some of the temporal aggregation bias. The data used are U.S. Treasury bill one month yields from June 1964 to December 1989 and the one month sterling interbank rate from March 1975 to March 1995.

The term structure of interest rates movement is minimized. Fong and Vasicek analyze the effect of a shift consisting of a linear movement of the instantaneous forward rate around the end of the horizon planning period; in this case it is not possible to build an immunized portfolio, but there is a lower bound for the portfolio’s final value that depends on \(M_2^2\).

For \(i = 1, \ldots, n\), the Fong-Vasicek dispersion measure for bond \(i\), \(M_i^2\), is defined as follows:

\[
M_i^2 = \frac{\sum_{s=1}^{n_i} (\tau_s^{(i)} - H)^2 C_i P(r, t_0, \tau_s^{(i)}) + (T_i - H)^2 P(r, t_0, T_i)}{\sum_{s=1}^{n_i} C_i P(r(t), t, \tau_s^{(i)}) + P(r, t_0, T_i)}.
\] (16)

\(M_i^2\) is introduced in the model by penalizing the objective function which becomes:

\[
V - Q \sum_{i=1}^{n} M_i^2 x_i
\] (17)

period is the best strategy against immunization risk. (See Bierwag et al., 1993). In practical terms, the alternative measures lead to similar results.
where $Q > 0$ is a parametric constant that depends on the investor's immunization risk aversion. When $M^2_t$ is used in the model, the optimal portfolio path consists of immunized portfolios of minimum dispersion, independent of the term structure assumption.\(^{11}\) (See Table 4.)

### Table 4

**Optimal Strategies of Minimum Dispersion in a Static Framework**

<table>
<thead>
<tr>
<th>Panel A: Nonstochastic Model</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>8382.429</td>
<td>1617.571</td>
<td>1.4996</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Vasicek Model</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S.</td>
<td>0</td>
<td>0</td>
<td>7914.367</td>
<td>1488.932</td>
<td>1.4995</td>
</tr>
<tr>
<td>U.K.</td>
<td>0</td>
<td>0</td>
<td>7966.707</td>
<td>1494.853</td>
<td>1.4999</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel C: Cox-Ingersoll-Ross Model</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S.</td>
<td>0</td>
<td>0</td>
<td>7916.721</td>
<td>1485.324</td>
<td>1.4999</td>
</tr>
<tr>
<td>U.K.</td>
<td>0</td>
<td>0</td>
<td>7956.605</td>
<td>1506.982</td>
<td>1.5005</td>
</tr>
</tbody>
</table>

*Notes: Dispersion measure is calculated according to Fong and Vasicek $M^2$ equation (16). Portfolio durations are calculated as follows: Macaulay duration is used for Panel A; Equation (14) is used for Panel B; and Equation (15) is used for Panel C.*

### 4 The Dynamic Model

The static portfolio selection model described in Section 3 provides a portfolio that is immunized against interest rate risk, but only at the beginning of the horizon planning period. The dynamic behavior of portfolio duration makes it impossible to keep that portfolio immunized during the entire planning period. Moreover, the immunization solution provided by the static model is valid only for the current interest rate, so the portfolio must be adjusted continuously as the rate

\(^{11}\) Any other decreasing function of $M^2$ could be added to the objective function to penalize portfolio dispersion, as we are only trying to obtain the immunized portfolio of minimum dispersion.
of interest changes.\textsuperscript{12} Our task now is to derive an optimal dynamic portfolio strategy that rebalances the portfolio in order to keep it free of interest rate risk.

\subsection*{4.1 The Rebalancing Points}

Recall the notation from Section 3.1, i.e., bond \( i \) matures at \( T_i \) and pays coupons at times \( \tau_{s}^{(i)} \) for \( s = 1, 2, \ldots, n_i \). Consider all of the \( n \) bonds at \( t_0 \) and arrange the times of their coupon payments in ascending order so that \( t_s \) denotes the time of \( s \)-th coupon payment in \( (t_0, H] \) so that

\[ t_1 = \min\{\tau_{1}^{(1)}, \tau_{1}^{(2)}, \ldots, \tau_{1}^{(n)}\}. \]

Let \( t_k \) denote the time of the last coupon payment in \( (t_0, H] \), i.e.,

\[ k = \text{The integer such that } t_k \leq H \text{ and } t_{k+1} > H. \quad (18) \]

If one of the \( n \) bonds makes a coupon payment at \( H \), then \( t_k = H \); otherwise, \( t_k < H \). Without loss of generality, we assume that at least one bond pays a coupon at \( H \) so that

\[ t_k = H. \quad (19) \]

Equation (19) implies that \( (t_0, H] \) is partitioned into \( k \) intervals. The \( s \)-th interval is \( (t_{s-1}, t_s] \), for \( s = 1, 2, \ldots, k \).

For example, we have three coupon bonds bought at \( t_0 \). Bond 1 (initially a 10 year bond) matures in 7.1 years and makes its remaining coupon payments at times \( (0.1, 0.6, 1.1, \ldots, 7.1) \); Bond 2 (initially a 20 year bond) matures in 15.75 years and makes its remaining coupon payments at times \( (0.25, 0.75, 1.25, \ldots, 15.75) \); and finally Bond 3 (initially a 30 year bond) matures in 17 years and makes its remaining coupon payments at times \( (0.5, 1.0, 1.5, \ldots, 17) \). Ordering all of the coupon payment times gives \( t_0 = 0, t_1 = 0.1, t_2 = 0.25, t_3 = 0.5, t_4 = 0.6, \) etc. If the investor's planning period is \( H = 1.5 \) years, then \( k = 6 \) and \( t_6 = H = 1.5 \). If \( H = 1.7 \) years, however, then \( k = 7 \) because \( t_6 < H \) and \( t_7 = 1.75 > H \).

Let us now express the time of maturity of bond \( i, T_i \), in terms of the \( t_s \)'s for those \( T_i \leq H \). For \( i = 1, 2, \ldots, n \) define

\textsuperscript{12}Only portfolios consisting entirely of zero coupon bonds with maturity at the end of the horizon planning period can be kept immunized over the horizon planning period without any additional rearrangements.
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\[ s_i = \begin{cases} 
    s & \text{if } \exists s \in \{1, 2, \ldots, k\} \text{ such that } t_s = T_i \leq H; \\
    k & \text{if } T_i > H = t_k 
\end{cases} \]  

(20)

Finally, we assume that portfolio rebalancing is only allowed at the beginning of each interval \( t_s, s = 0, 1, \ldots, k - 1 \).

4.2 The Constraints

Next we need to construct the constraints for the linear program. To this end, the following notation will be used:

- \( x(s, i) \) = Number of units of bond \( i \) in the portfolio immediately after the rebalancing at time \( t_s \);
- \( b(s, i) \) = Number of units of bond \( i \) bought at \( t_s \);
- \( z(s, i) \) = Number of units of bond \( i \) sold at \( t_s \);
- \( y(s, i) \) = Number of units of bond \( i \) maturing at \( t_s \);
- \( r_s \) = \( E[r(t_s) | r(t_0)] \) and
- \( p(s, i) \) = Price of one unit of bond \( i \) at \( t_s \) assuming that \( r(u) = E[r(u) | r(t_0)] \) for \( u \geq t_0 \).

The definition of \( p(s, i) \) assumes the actual interest rates are equal to the expected interest rates throughout the planning period.

The following set of constraints must be satisfied:

(i) \( x(0, i) = b(0, i) \) for \( i = 1, \ldots, n \);
(ii) \( x(s, i) = x(s - 1, i) + b(s, i) - z(s, i) - y(s, i) \)
    for \( s = 1, \ldots, k \) and \( i = 1, \ldots, n \);
(iii) \( x(s, i) = b(s, i) \) for \( s = s_i, s_i + 1, \ldots, k \) and \( i = 1, \ldots, n \);
(iv) \( z(s, i) = 0 \) for \( s = s_i, \ldots, k \) and \( i = 1, \ldots, \rho \);
(v) \( y(s, i) = 0 \) for \( s \neq s_i, \ldots, k \) and \( i = 1, \ldots, \rho \);
    or for \( s = 1, \ldots, k \) and \( i = \rho + 1, \ldots, n \).

Constraint (iii) represents the number of units of bond \( s \) maturing at \( T_i \) or being sold at \( t_k \). (Note that all bonds must be sold at the end of the horizon planning period.) Constraint (iv) indicates that bond \( i \) cannot be held or traded after it matures. Constraint (v) states that bond \( i \) matures only at a single point in time.

Constraint (i) indicates the number of bonds bought at the beginning of the horizon planning period. Constraint (ii) indicates the purchases
and sales at each subsequent \( t_s \). Constraint (iii) indicates that bond \( i \) cannot be held or traded after it matures. Constrain (iv) indicates that those bonds with maturity before or at \( t_k \) cannot be sold after their maturity. Note that constraints (ii), (iii), and (iv) imply that \( x(s_i - 1, i) = y(s_i, i) \) for \( i = 1, \ldots, p \), because those bonds with maturity at or before \( t_k \) mature at \( s_i \). Constraint (v) indicates that bonds 1 to \( p \) can only mature at \( s_i \). Meanwhile, bonds \( p + 1 \) to \( n \) do not mature at any point during the planning period. Note that for bonds \( p + 1 \) to \( n \) constraints (ii) and (v) imply that \( x(s_i - 1, i) = z(s_i, i) \), i.e., all bonds outstanding at \( t_k \) have to be sold at the end of the horizon planning period).

The initial budget constraint is now:

\[
\sum_{i=1}^{n} x(0, i)p(0, i) = I \tag{21}
\]

where \( I \) is the amount of money available at the beginning of the horizon planning period. The budget constraint must be satisfied not only at \( t_0 \) but during the whole planning period, so we must add the following set of budget constraints for \( s = 1, \ldots, k - 1 \)

\[
(vi) \quad \sum_{i=1}^{n} [b(s, i)p(s, i) - z(s, i)p(s, i) - y(s, i)p(s, i) - C_i x(s - 1, i)] = 0
\]

\[
(vii) \quad \sum_{i=1}^{n} [z(k, i)p(k, i) + y(k, i)p(k, i) + C_i x(k - 1, i)] = V_k
\]

where \( V_k \) is the portfolio value at \( t_k \), i.e., at the end of the horizon planning period. Note that \( p(s, i) \) is the amount of face value of bond \( i \) maturing at \( ts_i \); The constraint (vi) shows that the amount of money invested in new purchases at each \( t_s \), \( \sum b(s, i)p(s, i) \), must come from coupon payments, \( \sum C_i x(s - 1, i) \), sales, \( \sum z(s, i)p(s, i) \), and principal repayment \( \sum y(s, i)p(s, i) \). Constraint (vii) shows the expected value of the portfolio at \( H \).

4.3 The Optimal Portfolio

As in the static model, the investor's aim at each \( t_s \) is to maximize the guaranteed portfolio value at the end of the horizon planning period assuming unexpected interest rate changes only occur immediately after portfolio rebalancing, i.e., just after each \( t_s \).

If we let \( V_S \) be the minimum final portfolio value to guarantee at \( t_s \), then the following set of constraints must be satisfied:

\[
\sum_{i=1}^{n} x(s, i)\nu_{ij}(s) \geq V_S, \quad s = 0, \ldots, k - 1, \quad j = 1, \ldots, m,
\]
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where \( \nu_{ij}(s) \) denotes the final value (at \( H \)) of an investment of \( p(s, i) \) dollars in bond \( i \) at \( t_s \). In addition, the definition of \( \nu_{ij}(s) \) assumes that the instantaneous spot interest rate changes at \( t_s + \) from \( r_s \) to \( r_j \) and no additional unexpected interest rate change occurs until the end of the horizon planning period. Thus

\[
\nu_{ij}(s) = \frac{\sum_{u:T_u > t_s} C_i P(r_j, t_s, \tau_{ui}) + P(r_j, t_s, T_i)}{P(r_j, t_s, H)}.
\]  

(22)

As the investor's aim is to maximize the minimum portfolio final values at each \( t_s \) and to simultaneously minimize immunization risk at each \( t_s \), the objective function can restated as:

\[
\sum_{s=0}^{k} V_s - Q \sum_{s=0}^{k-1} \sum_{i=1}^{n} M^2(s, i) x(s, i)
\]

(23)

where \( M^2(s, i) \) is the Fong-Vasicek dispersion measure for bond \( i \) at \( t_s \), defined as follows:

\[
M^2(s, i) = \frac{\sum_{u:T_u > t_s} (\tau_{ui} - H)^2 C_i P(r_s, t_s, \tau_{ui}) + (T_i - H)^2 P(r_j, t_s, T_i)}{\sum_{u:T_u > t_s} C_i P(r_s, t_s, \tau_{ui}) + P(r_j, t_s, T_i)}
\]

(24)

for \( s = 0, \ldots, k - 1; \ i = s + 1, \ldots, n \). The complete dynamic model is:

**The Dynamic Model:**

\[
\max \sum_{s=0}^{k} V_s - Q \sum_{s=0}^{k-1} \sum_{i=1}^{n} M^2(s, i) x(s, i)
\]

subject to

\[
\sum_{i=1}^{n} x(0, i) p(0, i) = I
\]

(i) \( x(0, i) = b(0, i) \) for \( i = 1, \ldots, n \);

(ii) \( x(s, i) = x(s - 1, i) + b(s, i) - z(s, i) - y(s, i) \) for \( s = 1, \ldots, k \) and \( i = 1, \ldots, n \);

(iii) \( x(s, i) = b(s, i) \) for \( s = s_i, s_i + 1, \ldots, k \) and \( i = 1, \ldots, n \);

(iv) \( z(s, i) = 0 \) for \( s = s_i, \ldots, k \) and \( i = 1, \ldots, p \);

(v) \( y(s, i) = 0 \) for \( s \neq s_i, \ldots, k \) and \( i = 1, \ldots, p \);

or for \( s = 1, \ldots, k \) and \( i = p + 1, \ldots, n \);

(vi) \( \sum_{i=1}^{n} [b(s, i) p(s, i) - z(s, i) p(s, i) - y(s, i) p(s, i) - C_i x(s - 1, i)] = 0 \) for \( s = 1, \ldots, k - 1 \)

(vii) \( \sum_{i=1}^{n} [z(k, i) p(k, i) + y(k, i) p(k, i) + C_i x(k - 1, i)] = V_k \)

(viii) \( x(s, i), b(s, i), z(s, i), y(s, i), V_s \geq 0 \) for \( s = 1, \ldots, k \) and \( i = 1, \ldots, n \).
4.4 The Example Continued

Here $H = 1.5$ years and $t_0 = 0$. From the data in Table 1 we see that $T_1 = 0.5$, $T_2 = 1.0$, $T_3 = 1.5$, and $T_4 = 2.0$. There are only three coupon payments in $(0, 1.5]$ at $t_1 = 0.5$, $t_2 = 1.0$, and $t_3 = 1.5$; it follows that $k = 3$. It is easily seen that, because of the way the bonds are labeled, $s_i = i$ for those bonds with maturity at or before $t_3$.

In the case of the series flat term structure, we again assume a current interest rate level of 10 percent and $r_1 = 9$ percent; $r_2 = r_c = 10$ percent; $r_3 = 11$ percent. The optimal solution paths are reported in Panel A of Table 5. We can see that this result is consistent with Khang's theorem: the optimal portfolio duration consists of making duration equal to the remaining horizon planning period at every $t_s$. The small difference between these two variables is due to the finite number of scenarios of interest rate changes considered.

In the case of the Vasicek (1977) and Cox-Ingersoll-Ross (1985) term structure models, we use the parameters estimated in Nowman (1997) for both U.S. (from the Treasury bill market) and U.K. (sterling one month interbank rate). (See Table 3.) The expected interest rate at the beginning of each interval is given by equation (4) for the Vasicek model and equation (7) for the Cox-Ingersoll-Ross model.

The results are displayed in Panel A of Tables 6 to 9. Again, Khang's theorem is still valid. These results also are consistent with those obtained by Gagnon and Johnson (1994) under a stochastic interest rate in a discrete time framework.\textsuperscript{13}

5 Transaction Costs

The static and dynamic models described in Sections 3 and 4 do not include transaction costs and lead to solutions consistent with Khang's theorem. The next step is to introduce transaction costs into the model and analyze their effects on the optimal solution.

We assume all transaction costs are incurred only at portfolio rearrangement times and are a constant proportion, $\alpha$ ($0 \leq \alpha < 1$), of the volume traded (in dollars) at each $t_s$. We also assume that principal and coupon repayments don't generate transaction costs.

\textsuperscript{13}In particular, Gagnon and Johnson (1994) assume the Black, Derman, and Toy (1990) arbitrage-free evolution model.
Table 5
Optimal Portfolio Path
Under the Flat Term Structure

<table>
<thead>
<tr>
<th>Panel A: No Transactions Costs (α = 0.00%)</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>x(s,1)</td>
<td>x(s,2)</td>
<td>x(s,3)</td>
<td>x(s,4)</td>
<td>Duration</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8382.429</td>
<td>1617.571</td>
<td>1.4996</td>
</tr>
<tr>
<td>0.5</td>
<td>0</td>
<td>9951.796</td>
<td>548.204</td>
<td>0</td>
<td>0.9999</td>
</tr>
<tr>
<td>1</td>
<td>11025.000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.5000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: With Transactions Costs</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>α = 0.15%</td>
<td>s</td>
<td>x(s,1)</td>
<td>x(s,2)</td>
<td>x(s,3)</td>
<td>x(s,4)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>9448.628</td>
<td>536.395</td>
<td>1.4529</td>
</tr>
<tr>
<td>0.5</td>
<td>0</td>
<td>9947.132</td>
<td>536.395</td>
<td>0</td>
<td>0.9994</td>
</tr>
<tr>
<td>1</td>
<td>10470.523</td>
<td>536.395</td>
<td>0</td>
<td>0</td>
<td>0.5232</td>
</tr>
</tbody>
</table>

| α = 0.30% | s | x(s,1) | x(s,2) | x(s,3) | x(s,4) | Duration |
| 0 | 0 | 0 | 9434.535 | 535.554 | 1.4529 |
| 0.5 | 0 | 9931.548 | 535.554 | 0 | 0.9994 |
| 1 | 10453.338 | 535.554 | 0 | 0 | 0.5232 |

| α = 0.45% | s | x(s,1) | x(s,2) | x(s,3) | x(s,4) | Duration |
| 0 | 0 | 0 | 9420.485 | 534.717 | 1.4529 |
| 0.5 | 0 | 9916.016 | 534.717 | 0 | 0.9994 |
| 1 | 10436.212 | 534.717 | 0 | 0 | 0.5232 |

| α = 0.60% | s | x(s,1) | x(s,2) | x(s,3) | x(s,4) | Duration |
| 0 | 0 | 0 | 9940.359 | 0 | 1.4297 |
| 0.5 | 0 | 10434.412 | 0 | 0 | 0.9762 |
| 1 | 10953.022 | 0 | 0 | 0 | 0.5000 |

Notes: (a) The α value represents the level of transaction costs as a percentage of the volume traded; α = 0 means the absence of transaction costs. In this case the optimal strategy is consistent with Khang's theorem, i.e., at each rebalancing point the portfolio has to be restructured in order to keep its duration equal to the remaining horizon planning period; (b) Macaulay duration is used for this table.
Table 6
Optimal Portfolio Path
Under the Vasicek Model Using U.S. Data

Panel A: No Transactions Costs ($\alpha = 0.00\%$)

<table>
<thead>
<tr>
<th>$s$</th>
<th>$x(s,1)$</th>
<th>$x(s,2)$</th>
<th>$x(s,3)$</th>
<th>$x(s,4)$</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>7914.367</td>
<td>1488.932</td>
<td>1.4995</td>
</tr>
<tr>
<td>0.5</td>
<td>0</td>
<td>9363.004</td>
<td>510.227</td>
<td>0</td>
<td>0.9999</td>
</tr>
<tr>
<td>1</td>
<td>10366.868</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.5000</td>
</tr>
</tbody>
</table>

Panel B: With Transactions Costs

$\alpha = 0.15\%$

<table>
<thead>
<tr>
<th>$s$</th>
<th>$x(s,1)$</th>
<th>$x(s,2)$</th>
<th>$x(s,3)$</th>
<th>$x(s,4)$</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8893.123</td>
<td>513.351</td>
<td>1.4543</td>
</tr>
<tr>
<td>0.5</td>
<td>0</td>
<td>9344.520</td>
<td>513.351</td>
<td>0</td>
<td>1.0002</td>
</tr>
<tr>
<td>1</td>
<td>10348.526</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.5000</td>
</tr>
</tbody>
</table>

$\alpha = 0.30\%$

<table>
<thead>
<tr>
<th>$s$</th>
<th>$x(s,1)$</th>
<th>$x(s,2)$</th>
<th>$x(s,3)$</th>
<th>$x(s,4)$</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8879.864</td>
<td>512.549</td>
<td>1.4543</td>
</tr>
<tr>
<td>0.5</td>
<td>0</td>
<td>9329.912</td>
<td>512.549</td>
<td>0</td>
<td>1.0002</td>
</tr>
<tr>
<td>1</td>
<td>9810.844</td>
<td>512.549</td>
<td>0</td>
<td>0</td>
<td>0.5238</td>
</tr>
</tbody>
</table>

$\alpha = 0.45\%$

<table>
<thead>
<tr>
<th>$s$</th>
<th>$x(s,1)$</th>
<th>$x(s,2)$</th>
<th>$x(s,3)$</th>
<th>$x(s,4)$</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>9387.450</td>
<td>0</td>
<td>1.4305</td>
</tr>
<tr>
<td>0.5</td>
<td>0</td>
<td>9836.590</td>
<td>0</td>
<td>0</td>
<td>0.9764</td>
</tr>
<tr>
<td>1</td>
<td>10316.518</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.5000</td>
</tr>
</tbody>
</table>

$\alpha = 0.60\%$

<table>
<thead>
<tr>
<th>$s$</th>
<th>$x(s,1)$</th>
<th>$x(s,2)$</th>
<th>$x(s,3)$</th>
<th>$x(s,4)$</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>9373.477</td>
<td>0</td>
<td>1.4305</td>
</tr>
<tr>
<td>0.5</td>
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<td>9821.278</td>
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<td>0.9764</td>
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<td>1</td>
<td>10299.744</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.5000</td>
</tr>
</tbody>
</table>

Notes: (a) The $\alpha$ value represents the level of transaction costs as a percentage of the volume traded; $\alpha = 0$ means the absence of transaction costs. In this case the optimal strategy is consistent with Khang's theorem, i.e., at each rebalancing point the portfolio has to be restructured in order to keep its duration equal to the remaining horizon planning period; (b) Equation (14) is used for duration in this table.
Table 7
Optimal Portfolio Path
Under the Vasicek Model Using U.K. Data

<table>
<thead>
<tr>
<th>Panel A: No Transactions Costs ((\alpha = 0.00%))</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(s)</td>
<td>(x(s,1))</td>
<td>(x(s,2))</td>
<td>(x(s,3))</td>
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<td>(x(s,2))</td>
<td>(x(s,3))</td>
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<td>(s)</td>
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<td>(x(s,2))</td>
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</tbody>
</table>

Notes: (a) The \(\alpha\) value represents the level of transaction costs as a percentage of the volume traded; \(\alpha = 0\) means the absence of transaction costs. In this case the optimal strategy is consistent with Khang's theorem, i.e., at each rebalancing point the portfolio has to be restructured in order to keep its duration equal to the remaining horizon planning period; (b) Equation (14) is used for duration in this table.
Table 8
Optimal Portfolio Path
Under the Cox-Ingersoll-Ross Model Using U.S. Data

<p>| Panel A: No Transactions Costs ($\alpha = 0.00%$) | | | | | |
|---|---|---|---|---|</p>
<table>
<thead>
<tr>
<th>$s$</th>
<th>$x(s,1)$</th>
<th>$x(s,2)$</th>
<th>$x(s,3)$</th>
<th>$x(s,4)$</th>
<th>Duration</th>
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</tbody>
</table>

<p>| Panel B: With Transactions Costs | | | | | |
|---|---|---|---|---|
| $\alpha = 0.15%$ | | | | | |</p>
<table>
<thead>
<tr>
<th>$s$</th>
<th>$x(s,1)$</th>
<th>$x(s,2)$</th>
<th>$x(s,3)$</th>
<th>$x(s,4)$</th>
<th>Duration</th>
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<tr>
<td>0</td>
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<td>0</td>
<td>8910.619</td>
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<tr>
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<td>$\alpha = 0.30%$</td>
<td></td>
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</tr>
<tr>
<td>$s$</td>
<td>$x(s,1)$</td>
<td>$x(s,2)$</td>
<td>$x(s,3)$</td>
<td>$x(s,4)$</td>
<td>Duration</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8897.360</td>
<td>494.133</td>
<td>1.4539</td>
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<td>$s$</td>
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<td>$x(s,2)$</td>
<td>$x(s,3)$</td>
<td>$x(s,4)$</td>
<td>Duration</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
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</tr>
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<td></td>
</tr>
<tr>
<td>$s$</td>
<td>$x(s,1)$</td>
<td>$x(s,2)$</td>
<td>$x(s,3)$</td>
<td>$x(s,4)$</td>
<td>Duration</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
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<td>---</td>
<td>---</td>
<td>---</td>
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<td>10297.982</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.5000</td>
</tr>
</tbody>
</table>

Notes: (a) The $\alpha$ value represents the level of transaction costs as a percentage of the volume traded; $\alpha = 0$ means the absence of transaction costs. In this case the optimal strategy is consistent with Khang's theorem, i.e., at each rebalancing point the portfolio has to be restructured in order to keep its duration equal to the remaining horizon planning period; (b) Equation (15) is used for duration in this table.
<table>
<thead>
<tr>
<th>Panel A: No Transactions Costs (α = 0.00%)</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>x(s, 1)</td>
<td>x(s, 2)</td>
<td>x(s, 3)</td>
<td>x(s, 4)</td>
<td>Duration</td>
</tr>
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<td>0.9999</td>
</tr>
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<td>10433.109</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.5000</td>
</tr>
</tbody>
</table>

| Panel B: With Transactions Costs |
|---|---|---|---|---|---|
| α = 0.15% |  |  |  |  |  |
| s  | x(s, 1) | x(s, 2) | x(s, 3) | x(s, 4) | Duration |
| 0  | 0       | 0       | 8959.023 | 505.935 | 1.4542   |
| 0.5 | 0       | 9414.835| 505.935  | 0        | 0.9998   |
| 1   | 9901.047| 505.935 | 0        | 0        | 0.5234   |

| α = 0.30% |  |  |  |  |  |
| s  | x(s, 1) | x(s, 2) | x(s, 3) | x(s, 4) | Duration |
| 0  | 0       | 0       | 8958.877 | 492.139 | 1.4537   |
| 0.5 | 0       | 9413.339| 492.139  | 0        | 0.9992   |
| 1   | 9898.075| 492.139 | 0        | 0        | 0.5228   |

| α = 0.45% |  |  |  |  |  |
| s  | x(s, 1) | x(s, 2) | x(s, 3) | x(s, 4) | Duration |
| 0  | 0       | 0       | 8945.533 | 491.371 | 1.4537   |
| 0.5 | 0       | 9398.639| 491.371  | 0        | 0.9992   |
| 1   | 9881.896| 491.371 | 0        | 0        | 0.5228   |

| α = 0.60% |  |  |  |  |  |
| s  | x(s, 1) | x(s, 2) | x(s, 3) | x(s, 4) | Duration |
| 0  | 0       | 0       | 9430.492 | 0        | 1.4309   |
| 0.5 | 0       | 9882.616| 0        | 0        | 0.9764   |
| 1   | 10364.794| 0       | 0        | 0        | 0.5000   |

Notes: (a) The α value represents the level of transaction costs as a percentage of the volume traded; α = 0 means the absence of transaction costs. In this case the optimal strategy is consistent with Khang’s theorem, i.e., at each rebalancing point the portfolio has to be restructured in order to keep its duration equal to the remaining horizon planning period; (b) Equation (15) is used for duration in this table.
The budget constraints must be modified as follows:

\[
\sum_{i=1}^{n} (1 + \alpha)x(0, i)p(0, i) = I \\
(i) \quad x(0, i) = b(0, i) \text{ for } i = 1, \ldots, n; \\
(ii) \quad x(s, i) = x(s - 1, i) + b(s, i) - z(s, i) - y(s, i) \\
\text{for } s = 1, \ldots, k \text{ and } i = 1, \ldots, n; \\
(iii) \quad x(s, i) = b(s, i) \text{ for } s = s_i, s_i + 1, \ldots, k \text{ and } i = 1, \ldots, n; \\
(iv) \quad z(s, i) = 0 \text{ for } s = s_i, \ldots, k \text{ and } i = 1, \ldots, \rho; \\
(v) \quad y(s, i) = 0 \text{ for } s \neq s_i, \ldots, k \text{ and } i = 1, \ldots, \rho; \\
\text{or for } s = 1, \ldots, k \text{ and } i = \rho + 1, \ldots, n; \\
(vi) \quad \sum_{s=1}^{n} [(1 + \alpha)b(s, i)p(s, i) - (1 - \alpha)z(s, i)p(s, i) \\
- y(s, i)p(s, i) - C_i x(s - 1, i)] = 0 \text{ for } s = 1, \ldots, k - 1 \\
(vii) \quad \sum_{s=1}^{n} [(1 - \alpha)z(k, i) + y(k, i))p(k, i) + C_i x(k - 1, i)] = V_k \\
(viii) \quad x(s, i), b(s, i), z(s, i), y(s, i), V \geq 0 \\
\text{for } s = 1, \ldots, k \text{ and } i = 1, \ldots, n.
\]

Transaction costs have the effect of increasing asset purchase prices by \( \alpha \) while reducing sale prices by \( \alpha \). This new purchase (sale) price can be understood as the bid (ask) price of the bonds plus (minus) fees paid to intermediaries.

5.1 The Example Continued

The dynamic model with transactions costs is applied to the example, and the results are presented in Panel B of Tables 5 to 9 for different \( \alpha \) values (0.15 percent, 0.3 percent, 0.45 percent, and 0.6 percent).

The optimal path depends on the level of the transaction costs, i.e., on the level of \( \alpha \). For \( \alpha = 0 \) we reach Khang's optimal solution: at each rebalancing point portfolio duration must be equal to the remaining horizon planning period. But for values of \( \alpha \) greater than 0.05 percent the optimal path has an initial portfolio that is not immunized because its duration is less than the horizon planning period. The fact that values of \( \alpha \) as low as 0.05 percent lead to a non-immunized portfolio implies that the immunization strategy cannot be optimal, in practical terms.

The difference between the initial portfolio duration and the horizon planning period increases as the level of transaction costs rises. In this simple example four different solutions are obtained; the initial portfolio durations range from 1.5 years (for \( \alpha = 0 \)) to approximately 1.43 years (for \( \alpha = 0.6 \) percent). For sufficiently high transaction costs, the optimal solution is to invest the entire initial budget in a bond with
maturity at the end of the horizon planning period. This is because by investing in bonds with maturity at the end of the horizon planning period, we avoid the transaction costs generated by the reinvestment of those bonds with maturity before the end of the horizon planning period as well as the losses derived from the sales of those bonds still outstanding at the end of the horizon planning period. In the example this is the result we get when $\alpha = 0.6$ percent. To avoid transaction costs the optimal strategy consists of investing in bond 3, i.e., the bond with maturity at $t_3 = H = 1.5$ years.

A possible explanation of why the optimal strategy consists of portfolios with an initial duration less than the horizon planning period\(^{14}\) is that if no cash payment occurs, portfolio duration is equal to the remaining horizon planning period. But if a coupon payment occurs, portfolio duration is increased a finite amount and becomes greater than the horizon planning period.

If portfolio duration is long enough, this problem may be solved by reinvesting coupon payments in bonds with a short duration. If the horizon planning period is short, it will not be possible to keep duration equal to the horizon planning period unless we sell bonds with long durations and invest the proceeds in bonds with shorter duration.

The optimal solution of this model provides an initial portfolio with a duration less than the horizon planning period. As coupon payments are due, its duration increases approaching the horizon planning period without any additional rebalancing, thereby avoiding transaction costs. Also, this fact can be helped by an optimal reinvestment of coupon payments.

These findings are common to all cases analyzed, i.e., they are independent of the term structure of interest rate model assumed. These models provide a first hint to answer the question posed by Maloney and Logue (1989) with respect to the "mismatch duration that is tolerable, given that allowing a modest mismatch will certainly reduce trading costs."

\(^{14}\)At $t = 1$ all optimal portfolios have a duration greater or equal to the remaining horizon planning period. This is caused by the characteristics of the set of bonds considered in this counterexample. At $t = 1$ all the bonds have a duration greater than 0.5. If we had included bonds with a duration at $t = 1$ less than 0.5 (i.e., bonds with quarterly coupon payments) this result could not hold.
6 Summary

This paper develops a dynamic portfolio selection model for interest rate risk management under different term structure of interest rate regimes. This model's results are consistent with Khang's dynamic immunization strategy which consists of a continuous rebalancing to keep portfolio duration equal to the investor's horizon planning period.

The model is then extended in order to analyze the effects of transaction costs on the optimal strategy. Our results suggest that if transaction costs are considered, the strategy of making portfolio duration equal to the horizon planning period is not optimal. Moreover, the optimal path has an initial solution with a portfolio duration less than the horizon planning period. Furthermore, the bigger the level of transaction costs, the bigger the difference between the initial portfolio duration and the horizon planning period. This result holds under different term structure of interest rate models.

References


Navarro and Nave: Dynamic Immunization and Transaction Costs 179


