University of Nebraska - Lincoln

DigitalCommons@University of Nebraska - Lincoln

US Army Research

U.S. Department of Defense

2011

An alternative approach to integrating plasticity relations

Richard Becker

US Army Research Laboratory, richard.c.becker@us.army.mil

Follow this and additional works at: <https://digitalcommons.unl.edu/usarmyresearch>

Part of the [Operations Research, Systems Engineering and Industrial Engineering Commons](#)

Becker, Richard, "An alternative approach to integrating plasticity relations" (2011). *US Army Research*. 106.

<https://digitalcommons.unl.edu/usarmyresearch/106>

This Article is brought to you for free and open access by the U.S. Department of Defense at DigitalCommons@University of Nebraska - Lincoln. It has been accepted for inclusion in US Army Research by an authorized administrator of DigitalCommons@University of Nebraska - Lincoln.

Contents lists available at ScienceDirect

International Journal of Plasticity

journal homepage: www.elsevier.com/locate/ijplas

An alternative approach to integrating plasticity relations

Richard Becker*

US Army Research Laboratory, High Rate Mechanics and Failure Branch, Aberdeen Proving Ground, MD 21005, United States

ARTICLE INFO

Article history:

Received 24 August 2010

Received in final revised form 14 January 2011

Available online xxxx

Keywords:

Non-quadratic

Plasticity integration

Flow potential

ABSTRACT

A new plasticity integration algorithm is proposed based upon observations from the closed form integration of a generalized quadratic yield function over a single time step. The key to the approach is specification of the normal to the plastic flow potential as a function of the current state and strain increment. This uniquely defines the direction of the stress tensor for a convex, non-faceted flow potential. The stress magnitude and plastic strain increment are computed to satisfy the yield function. A non-quadratic, isotropic, associative flow model is coded to demonstrate accuracy and time step convergence following a step change in loading path. The model is used in additional simulations of strain localization in an expanding ring and a perforated plate.

Published by Elsevier Ltd.

1. Introduction

Traditional plasticity models are defined in terms of rate equations, and the stress and any history dependent state variables must be integrated through time (Hill, 1950). In the context of a displacement based finite element code, a strain increment or an average strain rate over the time increment is provided to the plasticity algorithm, and updated stress and state variables are returned. The stability and accuracy of the material time integration algorithm may limit the time step. Consequently, the computational efficiency of simulations can be affected by the material time integration method.

Time integration algorithms for general plasticity equations typically involve subtracting a plastic strain increment from the total strain increment to arrive back at the yield surface. Extensive literature reviews of strength models and integration procedures in finite element codes are given by Yu (2002) and Kojić (2002), respectively. More recent plasticity integration algorithms have also been described (e.g. Ulz, 2009; Mosler and Bruhns, 2010; Brannon and Leelavanichkul, 2010). Details can vary considerably, but the underlying concept is common: a return mapping to the yield surface from an initial projection. Textbook descriptions (e.g. Dunne and Petrinic, 2005; Hill, 1950) often provide a 2-D depiction of the yield surface, a stress increment projecting some small distance outward from the surface, and a vector in the direction of the plastic strain increment returning to the surface, Fig. 1a. The plastic strain direction is normal to the flow potential surface. This is the same as the yield surface for associative-flow plasticity models.

The manner in which finite element analyses are run can depart significantly from this textbook illustration. Many finite element codes will take strain increments on the order of 10% or more (Abaqus, 2009). If the material yields at a strain of 0.1%, the vector projecting off the yield surface may be 100 times the radius of the surface. The plastic strain direction must project back to the surface. A slight error in the direction could result in missing the surface entirely, Fig. 1b. This projection at large strain increments is a major difficulty for integrating anisotropic material models (Kojić, 2002).

The problem is simplified considerably for J2-Flow theory and an associated flow rule. This yield surface is a hyper-sphere in deviatoric stress space. A vector passing through the center of the yield surface is colinear with the surface normal. Hence, a plastic strain increment directed toward the center of the yield surface intersects the surface at a stress consistent with the

* Fax: +1 410 278 2460.

E-mail address: richard.c.becker@us.army.mil

Fig. 1. Stress increment and plastic return mapping using a plastic strain direction estimated from the beginning of the time step for: (a) a small strain increment typically described in text books and (b) many analyses where the strain increment can be more than an order of magnitude larger than the yield strain.

plastic flow direction. The normality condition is satisfied automatically. Implicit integration schemes have to solve for the magnitude of the plastic strain increment, not the direction. This radial return method (Krieg and Krieg, 1977; Wilkins, 1964) is robust, efficient and widely used for J2-Flow theory with an associative flow rule. Radial return can be used for plasticity models where the yield surface is not a hyper-sphere. This results in a non-associative flow rule because the yield surface and plastic flow potential are not the same. The approach remains efficient and robust, but it may give results different than those of an associative flow rule.

J2-Flow theory and other plasticity models for which the rate form can be expressed as $\dot{\mathbf{s}} = \mathbf{a}\dot{\mathbf{e}} + \dot{\lambda}\mathbf{s}$ may be integrable analytically or semi-analytically over a time step. Solutions can be obtained for J2-Flow theory (Krieg and Krieg, 1977), kinematic hardening (Krieg and Xu, 1997; Auricchio and Beirão da Veiga, 2003; Arioli et al., 2006), the Drucker–Prager model (Rezaiee-Pajand and Nasirai, 2008; Szabó, 2009), and potentially other plasticity models. These methods have demonstrated accuracy advantages over purely numerical integration algorithms and do not suffer the return mapping direction issues, but the range of models which can be integrated analytically or semi-analytically is limited. Analytical solutions do, however, provide valuable insight into the behavior of plasticity relations. It is this insight that motivates the proposed new plasticity integration scheme presented in Section 3.

The aim of this work is to explore a different approach to plasticity model integration. It is based upon observations from the closed form integration of a generalized quadratic plasticity model over a time step described in Section 2. As an exploration, presentation of the method is the focus. Isotropic elasticity, plastic incompressibility and other assumptions are used to keep the relations simple and more transparent. The implementation is in an explicit finite element code, so formulation of a consistent tangent is not considered. The closed form integration is the starting point for the presentation, followed by: observations leading to the new numerical integration approach; a description of the implementation into a finite element code; simulations verifying the implementation; and results from simple localization calculations.

2. Closed form integration for a generalized quadratic yield function

2.1. Rate formulation

The yield function, ϕ , and flow potential, ψ , for a generalized quadratic plasticity model with an associated flow rule (Karafillis and Boyce, 1993; Maudlin and Schiffrl, 1996) can be written as:

$$\phi = \psi = \sqrt{\frac{3}{2} \boldsymbol{\sigma}' : \mathbf{K} : \boldsymbol{\sigma}' - \bar{\sigma}} = \sigma_{eq} - \bar{\sigma} = 0, \quad (1)$$

$\boldsymbol{\sigma}'$ is the deviatoric stress; $\bar{\sigma}$ is the material flow strength (a material property); and $\mathbf{K}$ is fourth order tensor characterizing the material anisotropy. $\mathbf{K}$ is the identity tensor for an isotropic material. $\phi = \psi$ defines an associative flow rule; $\phi \neq \psi$ results in a non-associative plasticity model.

Following the classical plasticity formulation (Hill, 1950), the plastic part of the rate of deformation tensor is normal to the flow potential

$$\mathbf{d}^p = \dot{\lambda} \frac{\partial \psi}{\partial \boldsymbol{\sigma}} = \dot{\lambda} \mathbf{p} = \dot{\lambda} \frac{3}{2} \frac{\mathbf{K} : \boldsymbol{\sigma}'}{\bar{\sigma}}. \quad (2)$$

The plastic multiplier, $\dot{\lambda}$, can be determined from the equivalence of plastic power between the tensor field and material property representations

$$\boldsymbol{\sigma} : \mathbf{d}^p = \dot{\lambda} \boldsymbol{\sigma} : \mathbf{p} = \dot{\lambda} \bar{\sigma}, \quad (3)$$

$\dot{\bar{\epsilon}}$ is the equivalent plastic strain rate for a uniaxial stress test specimen. It is work conjugate to the flow stress, $\bar{\sigma}(\dot{\bar{\epsilon}}, \bar{\epsilon}, T, \dots)$. Eq. (3) leads to an expression for the plastic multiplier

$$\dot{\lambda} = \dot{\bar{\epsilon}} \frac{\bar{\sigma}}{\boldsymbol{\sigma} : \mathbf{p}}, \quad (4)$$

$\dot{\lambda} = \dot{\bar{\epsilon}}$ if the J2-Flow theory yield criterion and associative flow rule are used.

For a hypo-elastic formulation the deviatoric part of the rate of deformation tensor, $\mathbf{d}'$, is decomposed additively into elastic and plastic parts. The Jaumann stress rate is given in terms of the elastic part of the rate of deformation tensor which is expanded by substituting the flow rule from Eq. (2)

$$\dot{\boldsymbol{\sigma}}' = \mathbf{L} : \mathbf{d}'^e = \mathbf{L} : (\mathbf{d}' - \mathbf{d}'^p), \quad (5a)$$

$$= \mathbf{L} : \mathbf{d}' - \dot{\bar{\epsilon}} \frac{\bar{\sigma}}{\boldsymbol{\sigma} : \mathbf{p}} \mathbf{L} : \mathbf{p} = \mathbf{L} : \mathbf{d}' - \frac{3}{2} \frac{\dot{\bar{\epsilon}}}{\boldsymbol{\sigma} : \mathbf{p}} \mathbf{L} : \mathbf{K} : \boldsymbol{\sigma}'. \quad (5b)$$

$\mathbf{L}$ is the fourth order elasticity tensor.

2.2. Closed form integration over a time step

The objective is to integrate Eq. (5b) in closed form for one time step, Δt , given the average rate of deformation tensor over the time step. The anisotropy creates a rate form that is not amenable to the analytic techniques cited above (e.g., Arioli et al., 2006; Szabó, 2009). The equations are instead integrated by summation over infinitesimal parts of the time step. It is assumed that the flow strength and the plastic multiplier are constants over the time step. It is also assumed that material rotations are applied prior to integrating the stress. The latter is common practice when using the Jaumann rate. Details of the integration are given in Appendix A. The resulting stress at the end of the time step is

$$\boldsymbol{\sigma}'_{t+\Delta t} = \exp\left(-\frac{3}{2} \frac{\dot{\bar{\epsilon}} \Delta t}{\boldsymbol{\sigma} : \mathbf{p}} \mathbf{L} : \mathbf{K}\right) : \boldsymbol{\sigma}'_t + \frac{2}{3} \frac{\boldsymbol{\sigma} : \mathbf{p}}{\dot{\bar{\epsilon}} \Delta t} \left[\mathbf{I}' - \exp\left(-\frac{3}{2} \frac{\dot{\bar{\epsilon}} \Delta t}{\boldsymbol{\sigma} : \mathbf{p}} \mathbf{L} : \mathbf{K}\right) \right] : \mathbf{K}^{-1} : \mathbf{d}' \Delta t. \quad (6)$$

Subscripts t and $t + \Delta t$ refer to the stress at the beginning and end of the time step, respectively. $\mathbf{I}'$ is a modified fourth order identity tensor operating only on the deviatoric response, and it is defined by

$$\mathbf{I}' \stackrel{\text{def}}{=} \left(\mathbf{I} - \frac{1}{3} \boldsymbol{\delta} \otimes \boldsymbol{\delta} \right), \quad (7)$$

where $\mathbf{I}$ is the fourth order identity tensor and $\boldsymbol{\delta}$ is the second order identity. Some of the Δt 's could be factored out of Eq. (6) and from subsequent equations, but they are retained as an explicit reminder that the expression are intended to be applied over time increments and are not alternative rate equations. The exponentials are reminiscent of those in analytic solutions for J2-Flow theory (e.g., Arioli et al., 2006) and the Drucker–Prager model (e.g., Szabó, 2009), but the anisotropy renders these fourth order tensors rather than scalars.

A quick check can be made for small and large strain increments to verify that Eq. (6) behaves appropriately in the limits. In the limit of a small strain increment where only the linear term in the exponential expansion is important, the stress at the end of the step is

$$\boldsymbol{\sigma}'_{t+\Delta t} = \boldsymbol{\sigma}'_t - \frac{3}{2} \frac{\dot{\bar{\epsilon}} \Delta t}{\boldsymbol{\sigma} : \mathbf{p}} \mathbf{L} : \mathbf{K} : \boldsymbol{\sigma}'_t + \mathbf{L} : \mathbf{d}' \Delta t. \quad (8)$$

This is the forward integration result. At large strain increments the exponential vanishes, leaving

$$\frac{3}{2} \frac{\dot{\bar{\epsilon}}}{\boldsymbol{\sigma} : \mathbf{p}} \mathbf{K} : \boldsymbol{\sigma}'_{t+\Delta t} = \mathbf{d}'. \quad (9)$$

Comparing this with the Eq. (2) leads to the condition that the total strain rate and the plastic strain rate are the same in the limit of very large strain increments. This is required for large strain increments to keep the stress bounded in Eq. (5a).

The plastic flow direction (flow potential normal) at the end of the time step follows from Eqs. (2) and (6). To simplify the result, the exponential from Eq. (6) is written as the series expansion. From the expansion it is readily seen that the $\mathbf{K}$ premultiplying the stress from Eq. (2) can be placed within the exponential on the left allowing the existing $\mathbf{K}$ to be factored out on the right. The resulting expression is:

$$\begin{aligned} \mathbf{d}'_{t+\Delta t} &= \frac{3}{2} \frac{\dot{\bar{\epsilon}}}{\boldsymbol{\sigma} : \mathbf{p}} \left\{ \exp\left(-\frac{3}{2} \frac{\dot{\bar{\epsilon}} \Delta t}{\boldsymbol{\sigma} : \mathbf{p}} \mathbf{K} : \mathbf{L}\right) : \mathbf{K} : \boldsymbol{\sigma}'_t + \frac{2}{3} \frac{\boldsymbol{\sigma} : \mathbf{p}}{\dot{\bar{\epsilon}}} \left[\mathbf{I}' - \exp\left(-\frac{3}{2} \frac{\dot{\bar{\epsilon}} \Delta t}{\boldsymbol{\sigma} : \mathbf{p}} \mathbf{K} : \mathbf{L}\right) \right] : \mathbf{d}' \right\} \\ &= \exp\left(-\frac{3}{2} \frac{\dot{\bar{\epsilon}} \Delta t}{\boldsymbol{\sigma} : \mathbf{p}} \mathbf{K} : \mathbf{L}\right) : \mathbf{d}'_t + \left[\mathbf{I}' - \exp\left(-\frac{3}{2} \frac{\dot{\bar{\epsilon}} \Delta t}{\boldsymbol{\sigma} : \mathbf{p}} \mathbf{K} : \mathbf{L}\right) \right] : \mathbf{d}' \end{aligned} \quad (10)$$

An important, but not surprising observation is that the plastic flow direction at the end of the time step is not necessarily the same as the average plastic flow direction over the time step, which is given by

$$\mathbf{d}^p = \mathbf{d}' - \mathbf{L}^{-1} : (\boldsymbol{\sigma}'_{t+\Delta t} - \boldsymbol{\sigma}'_t) / \Delta t. \quad (11)$$

The direction from Eq. (10) and the average direction are the same in the limit of small strain increments, but there is no requirement for the flow potential normal at the end of the time step to be in the same direction as the average over the time step. Equating the two is a common unstated assumption made in many plasticity algorithms because some additional constraint is needed to close the set of equations. A second observation from Eq. (10) is that the evolution of the plastic flow direction over a time step can take a simple form with well defined limits at small and large time steps.

3. Proposed time integration algorithm

The strategy for the proposed time integration method is to determine the flow potential normal at the end of the time step and use that in conjunction with the flow potential to calculate a stress direction consistent with the normal. The magnitude of the stress will then be adjusted to satisfy the yield condition and flow strength model. The ensuing development is specialized to plastically incompressible materials. Extension to more general models is possible but beyond the scope of this work. It is important to stress that this method is not a direct integration of the constitutive model; it is a general approximate technique motivated by observations from a closed form solution.

3.1. Yield surface normal

The central component of the model is the flow potential normal at the end of the time step. The expression for the normal, $\mathbf{n}$, at the end of the time step is motivated by Eq. (10) along with relations from Eqs. (2) and (4) and assuming $\mathbf{n} = \mathbf{p} \sqrt{2/3}$.

$$\mathbf{n}_{t+\Delta t} = \alpha \left\{ \exp \left(-\sqrt{\frac{3}{2}} \frac{\bar{\sigma} \dot{\epsilon} \Delta t}{\boldsymbol{\sigma} : \mathbf{p}} \left(\frac{\partial \mathbf{n}}{\partial \boldsymbol{\sigma}'} \right)_t : \mathbf{L} \right) : \mathbf{n}_t + \sqrt{\frac{2}{3}} \frac{\boldsymbol{\sigma} : \mathbf{p}}{\bar{\sigma} \dot{\epsilon} \Delta t} \left[\mathbf{I}' - \exp \left(-\sqrt{\frac{3}{2}} \frac{\bar{\sigma} \dot{\epsilon} \Delta t}{\boldsymbol{\sigma} : \mathbf{p}} \left(\frac{\partial \mathbf{n}}{\partial \boldsymbol{\sigma}'} \right)_t : \mathbf{L} \right) \right] : \mathbf{d}' \Delta t \right\}. \quad (12)$$

The leading factor, α , allows normalization of $\mathbf{n}_{t+\Delta t}$ to a unit tensor at finite time increments where the rate of deformation tensor is not exactly normalized by the plastic strain rate. The normalization by α is only for convenience in the ensuing numerical treatment, the direction is the important feature. An additional extrapolation was made in replacing $\sqrt{3}K/\sqrt{2}\bar{\sigma}$ with the derivative the flow potential normal with respect to stress. The goal in making this generalization is to provide a plausible linearization of the normal in Eq. (12) at small strain increments

$$\mathbf{n}_{t+\delta t} = \mathbf{n}_t + \left(\frac{\partial \mathbf{n}}{\partial \boldsymbol{\sigma}'} \right)_t : \mathbf{L} : \left(\mathbf{d}' - \sqrt{\frac{3}{2}} \frac{\bar{\sigma} \dot{\epsilon}}{\boldsymbol{\sigma} : \mathbf{p}} \mathbf{n}_t \right) \Delta t = \mathbf{n}_t + \left(\frac{\partial \mathbf{n}}{\partial \boldsymbol{\sigma}'} \right)_t : \Delta \boldsymbol{\sigma}'. \quad (13)$$

The middle expression of Eq. (13) is a linearization of Eq. (12), and the $\Delta \boldsymbol{\sigma}'$ in the right hand expression is the stress increment through Eqs. (5a) and (5b). At large strain increments the plastic flow direction must approach the applied strain rate direction in order for the stress to be bounded properly, Eq. (5a). As a result, Eq. (12) should provide an accurate limiting direction for any plastically incompressible flow potential surface.

Eq. (12) could be used as the basis for the integration algorithm, but evaluating the exponential of the fourth order tensors in Eq. (12) would be prohibitively expensive in most explicit finite element applications. Further assumptions are made to obtain a scalar argument for the exponential. A more computationally practicable basis for the proposed time integration scheme is

$$\mathbf{n}_{t+\Delta t} = \alpha \left\{ \exp \left(-\frac{3\mu \dot{\epsilon} \Delta t}{\boldsymbol{\sigma} : \mathbf{p}} \right) \mathbf{n}_t + \frac{\boldsymbol{\sigma} : \mathbf{p}}{\bar{\sigma} \dot{\epsilon} \Delta t} \sqrt{\frac{2}{3}} \left[1 - \exp \left(-\frac{3\mu \dot{\epsilon} \Delta t}{\boldsymbol{\sigma} : \mathbf{p}} \right) \right] \mathbf{d}' \Delta t \right\}. \quad (14)$$

Eq. (14) is consistent with the exact expression, Eq. (10), specialized to J2-Flow theory. For a more general constitutive model, Eq. (14) gives the proper flow potential normal direction at a zero time step, $\mathbf{n}_t$, and at large strain increments, $\mathbf{d}' \Delta t$. However, the path along the flow potential connecting these two limits will be in error, particularly at small time steps. This is the trade-off in going from Eqs. (12) to (14). The significance of this path error will be evaluated later through several examples. Eq. (12) would give a better approximation to the path if one could compute it efficiently.

For improved accuracy at small strain increments while using Eq. (14), a traditional backward difference, normal return algorithm is used in place of the new approach. If the strain increment exceeds some value or is beyond the radius of convergence for the traditional normal return algorithm, the new methodology is used. Thus, the traditional forward approach is used where it is accurate and efficient, and the new algorithm provides improved robustness and accuracy at larger strain increments.

3.2. Plastic strain rate estimate

The proposed algorithm using Eqs. (12) or (14) requires an estimate of the plastic strain rate. A reasonable approximation can be obtained through a radial return solution for the time step. In the radial return mapping for an isotropic elastic material, the direction of the plastic flow is assumed to be coaxial with the elastic trial stress, which is defined as;

$$\boldsymbol{\sigma}^T \stackrel{\text{def}}{=} \boldsymbol{\sigma}'_t + 2\mu \mathbf{d}' \Delta t. \quad (15)$$

Consequently, the stress at the end of the time step is in the same direction, Eq. (5a). The strain rate and stress tensor must satisfy the yield function

$$\phi = \mathcal{G}(\boldsymbol{\sigma}) - \bar{\sigma}(\bar{\epsilon}, \dot{\bar{\epsilon}}, T, \dots) = \sigma_{eq} - \bar{\sigma} = 0, \quad (16)$$

as well as the stress increment given by Eq. (5a). If $\phi \leq 0$ the material is elastic; the strain rate is zero and the updated stress is equal to the trial stress. If $\phi > 0$ the material will deform plastically, and the strain rate will be determined.

The yield function is homogeneous order one in stress. Using the assumption that the stress is coaxial with the trial stress, the radial return algorithm reduces to finding the plastic strain rate that satisfies

$$\phi = \mathcal{G}(\boldsymbol{\sigma}^T) - 3\mu \dot{\bar{\epsilon}} \Delta t \left(\frac{\mathcal{G}(\boldsymbol{\sigma}^T)}{\sigma_{vm}^T} \right)^2 - \bar{\sigma}(\bar{\epsilon}, \dot{\bar{\epsilon}}, T, \dots) = 0, \quad (17)$$

σ_{vm}^T is the von Mises stress from the trial stress tensor, Eq. (15). Eq. (17) is solved using a combination Newton–Raphson/Bisection algorithm described in *Numerical Recipes* (Press et al., 1992).

One could stop at this point and determine the stress by

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma}^T \frac{\bar{\sigma}(\bar{\epsilon}, \dot{\bar{\epsilon}}, T, \dots)}{\mathcal{G}(\boldsymbol{\sigma}^T)}. \quad (18)$$

The yield function would be satisfied but the direction of plastic flow does not satisfy the flow potential. For some applications the salient aspects of the constitutive model may be adequately captured by only satisfying the yield function. Otherwise, the Eq. (17) only provides an estimate of the plastic strain rate for use in Eqs. (12) or (14).

3.3. Stress direction calculation

The most significant computational effort in determining the stress tensor is in finding its direction from the flow potential normal.

$$\frac{d\psi}{d\boldsymbol{\sigma}} \left\| \frac{d\psi}{d\boldsymbol{\sigma}} \right\|^{-1} = \mathbf{n}_{t+\Delta t}. \quad (19)$$

In forming the matrix to solve Eq. (19), the shear terms are multiplied by $\sqrt{2}$, where appropriate, to permit standard matrix operations while taking advantage of the symmetry of the stress tensor. The solution of Eq. (19) is a well defined, purely mathematical problem, particularly for smooth, convex flow potentials. The mechanistic approximations and the dependency on time step and material properties have been separated. Alternative methods can be used to solve Eq. (19) to improve efficiency without affecting the algorithm.

Eq. (19) is solved here using Newton–Raphson iteration for the stress tensor. The primary difficulty is that the matrix created from Eq. (19) has a null space of at least order two. The consistent components of the null space are related to the pressure and to the magnitude of the stress tensor. The pressure does not impact the flow potential for materials which are plastically incompressible. The magnitude of the tensor is removed from Eq. (19) by the explicit normalization. The stress tensor itself is normalized after each iteration to maintain a consistent stress magnitude. Other contributions to the null space may come from combinations of stress increments which do not affect the normal direction.

The null space complicates the solution of Eq. (19). A singular value decomposition (SVD) module from the LAPACK library is used in the Newton–Raphson iteration. The LAPACK DGESVD routine returns the input matrix A decomposed into orthogonal matrices U and V and diagonal matrix w

$$[A] \times \{s\} = [U](w)[V]^T \times \{s\} = \{e\}, \quad (20)$$

$\{s\}$ is the correction to the deviatoric stress and $\{e\}$ is the error for the Newton–Raphson equation. The null space is associated with zeros in the diagonal w matrix. Since U and V are orthogonal matrices and w is diagonal, inversion of Eq. (20) is straight forward. When inverting the w matrix, the reciprocals of the zero values are replaced by zeros. This eliminates the contribution of the null space from the solution. The SVD is computationally intensive.

An alternative approach to deal with the null space is to add penalty terms to the matrix. Fourth order tensors are constructed from the outer product of the second order identity with itself and from the outer product of the deviatoric stress with itself. These are multiplied by a penalty parameter and added to matrix. The penalty parameter is 10^6 times the norm of the error, $\{e\}$, with a minimum penalty of 1.0.

3.4. Plastic strain rate determination

The plastic strain rate is determined by first creating a scalar equation from the yield function, the flow rule, and a plastic work equivalence statement. The normalized stress direction determined in Section 3.3 is denoted $\tilde{\sigma}$ and the deviatoric stress at the end of the time step is given by

$$\sigma' = \beta \tilde{\sigma}, \quad (21)$$

where β is an unknown scale factor. The equivalent stress is defined through Eq. (16) as

$$\sigma_{eq} = \mathcal{G}(\sigma) = \mathcal{G}(\sigma') = \mathcal{G}(\beta \tilde{\sigma}) = \beta \mathcal{G}(\tilde{\sigma}) = \beta \tilde{\sigma}_{eq} \quad (22)$$

and similar relations can be constructed for the von Mises stress

$$\sigma_{vm} = \sqrt{\frac{3}{2} \sigma' : \sigma'} = \beta \sqrt{\frac{3}{2} \tilde{\sigma} : \tilde{\sigma}} = \beta \tilde{\sigma}_{vm}. \quad (23)$$

These relations will be used to determine the scale factor β .

The equivalent plastic strain rate is defined such that the plastic work rate from the tensor expression is equal to that defined by the equivalent scalars, Eqs. (1) and (3),

$$\sigma : \mathbf{d}^p = \sigma_{eq} \dot{\bar{\epsilon}} = \bar{\sigma} \dot{\bar{\epsilon}}. \quad (24)$$

The plastic part of the rate of deformation tensor is obtained by specializing Eq. (11) to an isotropic elastic material and substituting into Eq. (24). Eq. (24) becomes

$$\sigma_{eq} \dot{\bar{\epsilon}} \Delta t = \sigma : (\sigma'_t + 2\mu \mathbf{d}' \Delta t - \sigma') / 2\mu. \quad (25)$$

Making a few substitutions from Eq. (15) and Eqs. (21)–(23), and after some manipulation, Eq. (25) can be written as

$$3\mu \sigma_{eq} \dot{\bar{\epsilon}} \Delta t = \frac{3}{2} \frac{\sigma_{eq}}{\tilde{\sigma}_{eq}} \tilde{\sigma} : \sigma'^T - \frac{\sigma_{eq}^2}{\tilde{\sigma}_{eq}^2} \tilde{\sigma}_{vm}^2. \quad (26)$$

Using the yield function, Eq. (16), this can be cast into a form similar to Eq. (17) to solve for the plastic strain rate

$$\frac{3}{2} \frac{\tilde{\sigma}_{eq}}{\tilde{\sigma}_{vm}^2} \tilde{\sigma} : \sigma'^T - 3\mu \dot{\bar{\epsilon}} \Delta t \frac{\tilde{\sigma}_{eq}^2}{\tilde{\sigma}_{vm}^2} - \bar{\sigma}(\bar{\epsilon}, \dot{\bar{\epsilon}}, T, \dots) = 0. \quad (27)$$

With the plastic strain rate and flow strength known, the deviatoric part of the stress tensor at the end of the time step is determined from Eqs. (21) and (22) as

$$\sigma' = \frac{\bar{\sigma}}{\tilde{\sigma}_{eq}} \tilde{\sigma}. \quad (28)$$

The full stress tensor is recovered by adding the contribution of the hydrostatic stress determined by an equation of state.

3.5. Treatment for reverse loading

If the inner product $\sigma'_t : \mathbf{d}' < 0$, the loading path may project into the yield surface. Development of Eqs. (12) and (14) assumed continuous loading, so these do not apply. Therefore, an alternative solution is needed for reverse loading situations. If the final stress state lies within the yield surface, $\phi < 0$, the increment is elastic and the trial stress becomes the stress at the end of the time step. If the reverse loading increment is sufficiently large, the loading path may cut across the yield surface causing plastic flow in a substantially different direction. The radial return solution of Section 3.2 is adopted for these cases. In such situations the deformation increments are too large to track the path details explicitly over the time step. Consequently, any solution will be approximate, and an efficient and robust method is a practical choice.

4. Examples using isotropic, non-quadratic yield surface

An isotropic, non-quadratic yield surface model with an associated flow rule is used as an example (Hosford, 1972; Karafillis and Boyce, 1993). It is written in terms of principal stresses as:

$$\phi = \psi = \sqrt[q]{\frac{1}{2} [(\sigma_1 - \sigma_2)^q + (\sigma_2 - \sigma_3)^q + (\sigma_3 - \sigma_1)^q]} - \bar{\sigma} = 0, \quad (29)$$

where q is an even integer exponent. This model reduces to J2-Flow theory for a yield surface exponent of 2, which enables comparisons with the well-known radial return method. It also approximates a Tresca yield surface when high exponents are used. Experimental yield surface measurements on aluminum sheet (Barlat et al, 1997) and polycrystal plasticity simulations (Logan and Hosford, 1980) suggest an exponent of approximately eight for FCC materials. Details of the flow potential

derivatives needed for Eq. (19) are given in Appendix B. The yield function and flow rule were implemented in the Arbitrary Lagrange Eulerian code ALE3D (Nichols, 2009) for evaluation and demonstration.

4.1. Yield function representation

The first verification check is to ensure that the yield function is implemented correctly, independent of the details of the integration algorithm. Time varying velocity gradients were imposed independently on the x , y and z -coordinates of all the nodes of a unit cube.

$$\begin{aligned} x\text{-direction} : \quad \frac{d\dot{x}}{dx} &= \cos(t), \\ y\text{-direction} : \quad \frac{d\dot{y}}{dy} &= \cos\left(t + \frac{\pi}{3}\right), \\ z\text{-direction} : \quad \frac{d\dot{z}}{dz} &= \left(t + \frac{\pi}{3}\right). \end{aligned} \quad (30)$$

The imposed deformation traverses a plane normal to the pressure axis in principle stress space (known as the π -plane). No pressure or shear stress are induced beyond numerical integration error. The yield surface exponent was 20, and the constant yield stress was set to 480 MPa. Deviatoric stresses in the x , y and z -directions were recorded and projected to a 2-dimensional representation of the π -plane according to

$$x = \frac{\sigma'_x - \sigma'_z}{\sqrt{2}} \quad y = \frac{-\sigma'_x + 2\sigma'_y - \sigma'_z}{\sqrt{6}}. \quad (31)$$

The resulting yield surface is shown in Fig. 2 for three different integration schemes: (1) traditional normal return using the rate equations; (2) radial return, which results in a non-associative flow rule; and (3) the new algorithm described in Section 3. All reproduce the same yield surface shape to within a 0.05% at this time step. J2-Flow theory would create a circle with this projection and the Tresca yield criterion would create a hexagon.

4.2. Comparison with radial return for J2-Flow theory

The accuracy of the new integration algorithm is assessed by comparison with the radial return method applied to standard J2-Flow theory. This is achieved in the proposed algorithm by setting $q = 2$ in Eq. (29). A single element is run through an abrupt change in loading path by prescribing the displacements of all of the nodes. The loading rates are defined to maintain an effective plastic strain rate of approximately 0.001 s^{-1} in both loading stages. The initial deformation resembles uniaxial tension with the lateral strain rates prescribed equal to negative one-half of the extension strain rate in the x -direction. The extension rate is 0.001 s^{-1} . At a time of 40 s, the deformation is abruptly changed to pure shear in the xy -plane with a shearing rate of $0.0005\sqrt{3} \text{ s}^{-1}$. All of the degrees of freedom of the element are again specified.

The shear modulus is 40 GPa and the constant yield strength is set to 480 MPa. Initial yield occurs at 4 s when the deviatoric stress in the x -direction is 320 MPa. The deformation is isochoric, so the bulk modulus is inconsequential. Since all of the degrees of freedom are prescribed, inertial contributions do not affect the solution.

Fig. 2. Yield surface in the π -plane for a yield surface exponent of $q = 20$. Results are plotted for normal return, radial return and the proposed algorithm. They are nearly coincident.

Fig. 3 shows the x -direction stress from the radial return algorithm and the proposed method for runs at several fixed time step sizes. The solid curves are for a time step of 0.002 s. The results for the two methods at this time step are the same to 0.01 MPa, and this is considered to be the converged solution. Time steps of 2 s, 8 s and 40 s are indicated by symbols. These correspond to strain increments of half, twice and ten times the initial yield strain. Fig. 3a displays the radial return results and Fig. 3b shows the results from the new integration scheme. Following the loading path change, the radial return method displays increased error with increasing time step size. The new algorithm matches the baseline solution much more closely with larger time steps. This is expected since method is based on the exact incremental solution for a J2-Flow theory material. Similar plots of shear stress show less time step error for the radial return algorithm and similar time step insensitive results for the new algorithm.

4.3. Evaluation for non-quadratic surfaces

The same loading path change test is run for $q = 8$ in Eq. (29). The baseline result is established using the traditional forward integration method based on the rate equations run at a time step of 0.0002 s. Results are shown in Fig. 4 for the radial return method and the proposed algorithm at two time steps: 0.0002 s and 8.0 s. The latter corresponds to twice the yield strain. The forward time step integration method does not work at this large time step. As indicated in the discussion following Eq. (14), the current algorithm has an integration path error at small strain increments resulting in deviation from the baseline solution. However, this error does not increase with time step, and it is of the same order as the time step errors shown for the J2-Flow theory radial return algorithm in Fig. 3a. The radial return results shown in Fig. 4 appear close to the baseline solutions even though using a radial return for a non-quadratic model creates a non-associative flow rule. One may wonder why not just use the radial return method?

4.4. Evaluation in off-axis loading

The examples reported above involved axial and shear loading which accesses a rather simple path along the yield surface with no rotation. The final unit tests are performed by a path change between two more general loadings. As before, the locations of all of the nodes on the single finite element are prescribed. Path $\mathcal{A}$ is applied for 40 s, and the deformation is abruptly switched to path $\mathcal{B}$. The isochoric velocity gradients associated with these two paths are

$$\mathcal{A} = \begin{bmatrix} 0.5 & 1.0 & 0.4 \\ 0.0 & -0.3 & 0.0 \\ 0.4 & 0.0 & -0.2 \end{bmatrix} \quad \mathcal{B} = \begin{bmatrix} -0.4 & 0.0 & -0.2 \\ 0.5 & 0.7 & 0.4 \\ 0.0 & -0.3 & -0.3 \end{bmatrix}. \quad (32)$$

Similar to Fig. 4, the baseline solution was obtained by using the forward gradient method and a time step of 0.0002 s. Solutions with the radial return method and the new algorithm are run at time steps of 0.0002 s and 8 s.

The results shown in Fig. 5 again show that the proposed integration algorithm tracks the baseline solution and that there is little time step dependence. A behavior not noted in the prior examples is that the radial return method converges to the wrong stress. With these more complex loading states the stress evolution is dependent on the plastic strain rate direction. A radial return is not consistent with the flow potential, so deviations should be expected.

4.5. Strain localization in an expanding ring

One of the goals of using a yield surface with a higher curvature is to allow strain localization more readily than J2-Flow theory. This possibility was investigated using an expanding ring. The plane strain ring has in outer radius of 25.4 mm and a

Fig. 3. Axial stress for a J2-Flow theory at various time steps during a loading path change: (a) radial return and (b) proposed algorithm. The solid lines correspond to a baseline time increment of 0.0002 s.

Fig. 4. Results from path change test for isotropic, non-quadratic yield surface, $q = 8$. The baseline solution (base) was with the traditional rate equations, and solutions from a radial return (radial) and the proposed algorithm (new) are shown. (a) Axial stress and (b) shear stress. The lines correspond to a time increment of 0.0002 s.

Fig. 5. Stress response during a more general path change: (a) x-direction stress and (b) x–y shear stress. The baseline solution (base) is obtained by forward integration with a time step of 0.0002 s. The radial return (radial) and proposed algorithm (new) are run with $q = 8$ and time steps of 0.0002 s (lines) and 8 s (symbols).

wall thickness of 3 mm. There are 20 constant stress quadrilateral elements across the thickness of the ring and 180 circumferentially in the quarter symmetry model. The density is 7.8 g/cc, the shear modulus 40 GPa and the bulk modulus 66.67 GPa. The yield stress was varied randomly from element to element to provide defects to seed strain localization. The statistics of the strength distribution were Gaussian. The average yield strength was 480 MPa and the standard deviation was 0.96 MPa—representing 0.2% of the initial yield stress. A slight linear strain hardening of 0.005 MPa was used. A constant pressure of 100 MPa was applied to the inner surface from the beginning of the analysis.

Plastic strain rate contours at 60 μ s are shown in Fig. 6 for yield surface exponents of 8 and 2. The latter is J2-Flow theory. For $q = 8$, strain localization patterns emerge before 20 μ s and are continuously refined as the deformation proceeds, Fig. 6a. These patterns do not appear in the $q = 2$ analysis, even at late times, Fig. 6b. A calculation was also run with $q = 8$ and the radial return algorithm. This provides a $q = 8$ yield surface and a $q = 2$ plastic flow potential. The results are very similar to Fig. 6a, indicating that the yield surface shape is facilitating the localization rather than the direction of plastic flow.

4.6. Plugging during plate perforation

A configuration where the plastic flow direction is likely to be more important is in “plugging” of a plate penetrated by a projectile. In recovered samples, shear bands are observed ahead of the projectile, and the material ahead of the penetrator appears to be pushed as a plug through the remaining plate (Murr et al., 2009). Thermal softening and damage within the shear bands can accentuate the bands in a positive feedback mechanism.

For these simulations, the configuration and material properties are idealized to isolate the role of the plastic flow direction. Temperature dependence, strain hardening and fracture are not included in the calculations so that these factors do not have a role in the current strain localization predictions. A 5.0 mm diameter, 10.0 mm long, right circular steel cylinder is impacted against a 5.0 mm thick, 200 mm diameter aluminum plate at 225 m/s. The steel cylinder has a shear modulus of 71.8 GPa and a bulk modulus of 419 GPa. It remains elastic. The aluminum plate has a shear modulus of 28 GPa, a bulk modulus of 80 GPa and a constant flow strength of 150 MPa. The axisymmetric finite element mesh is uniform in the vicinity of the penetration with an element size of 0.05208 mm in both the r and z directions. This provides 48 elements across the penetrator radius and 96 in the plate thickness. The penetration is simulated in Eulerian mode in ALE3D to preclude mesh motion biases that would complicate comparisons among ALE runs with solution dependent mesh motion. Simulations were

Fig. 6. Strain rate contours at 60 μs for an expanding ring. (a) Non-quadratic yield surface, $q = 8$; and (b) quadratic yield surface. Strain rate legend units are μs^{-1} .

run with: J2-Flow theory; the proposed integration scheme with $q = 8$ and an associative flow rule; and with $q = 8$ for the yield surface and radial return ($q = 2$ flow potential, non-associative flow rule).

Fig. 7 shows strain rate contours (μs^{-1} units) for the three runs at 15 μs . The scale is logarithmic to highlight the strain rate fan ahead of the projectile corner. The plots are similar, but the fan ahead of the corner for the new algorithm is somewhat narrower, Fig. 7b. This indicates greater strain localization than for the other two runs. The fan feature is not steady state and the gap in the fan near the free surface is due to wave reflections.

Significant differences among the runs are not evident until the projectile exits at approximately 50 μs . The projectile exit is earlier for the new model than for the other two runs. The projectile velocity at exit for the J2-Flow theory run is 52 m/s. The exit velocity is 70.5 m/s for $q = 8$ and the new algorithm using an associate flow rule and 58 m/s for a $q = 8$ yield surface and radial return. These latter differences indicate that the alignment of the plastic flow direction changes the ease of shear and the dissipated energy.

The simulations were run in parallel on eight processors on a dedicated compute node. The time increment was set by the Courant limit and advection accuracy criteria, so the number of time steps is very near the same in each run. The wall clock time for the new algorithm with the penalty approach to solve Eq. (19) was 18% higher than for J2-Flow theory with radial return. The time was 50% longer when using singular valued decomposition to solve the matrix. The timing reflects modest impact to the computational analyst for using a more complex material model. It is not a statement of numerical efficiency of the integration algorithm since J2-Flow theory is a much simpler model than the non-quadratic. An implementation of a conventional integration approach for a non-quadratic model would be necessary for a meaningful timing comparison. The CPU time for the non-quadratic model will be much greater than the J2-Flow solution, but the expense of advection and other physics reduces the overall impact on turn-around time. Some fraction of this can be recovered by weighting the domain decomposition so more processors are allocated to work on the aluminum plate rather than the steel and surrounding

Fig. 7. Plastic strain rate contours (μs^{-1} units) for plugging simulation using: (a) J2-Flow theory; (b) new algorithm with $q = 8$ and associative flow rule; and (c) $q = 8$ yield surface and radial return ($q = 2$ flow potential, non-associative flow).

air. Gains for the proposed approach would also come from a more efficient algorithm to calculate the stress given the flow potential normal.

5. Summary and conclusions

A new approach for integrating continuum plasticity relations has been introduced based on observations from closed form integration of a generalized quadratic yield function over a time step. The normal to the flow potential is computed from a relatively simple expression, and the new stress is calculated to be consistent with the flow potential normal. This introduces a different set of approximations than traditional approaches, and further evaluation will be necessary using a variety of yield surfaces and flow potentials. A consistent tangent was not derived in this work. It should be straight forward, albeit tedious.

The preliminary results presented here are promising. The method may be somewhat less accurate than traditional approaches at small strain increments, but it can be significantly more accurate at large strain increments. The method is also very robust numerically.

The new approach has additional attractive features from a numerical implementation perspective. The physics approximations in the time integration scheme are decoupled from the expensive computations. Thus, the approximations are well defined, and work to improve numerical efficiency can proceed without concerns of impacting the solution quality. Gains may be realized over the current implementation by using a different approach to find the stress tensor given the flow potential normal. No attempts have yet been made to improve the numerical efficiency of the new algorithm. However, a hybrid approach using a forward integration algorithm at small strain increments improves the overall computational efficiency while also providing a more accurate solution for small strain increments.

Although not the focus of this work, it was observed that simulations using the associative and non-associative flow rules give similar results in some circumstances but not others. Depending on the application, one may be able to use a radial return method and get a satisfactory solution at appreciable computational savings. Further investigation along these lines could be fruitful.

Appendix A. Integration of generalized quadratic yield function

Eq. (5b) is integrated over a time step assuming that that corrections for spin have already been incorporated and that $\dot{\lambda}$ and $\bar{\sigma}$ are constant over the time step.

$$\dot{\sigma}' = \mathbf{L} : \mathbf{d}' - \frac{3}{2} \frac{\dot{\lambda}}{\bar{\sigma}} \mathbf{L} : \mathbf{K} : \sigma' \quad (\text{A1})$$

The time increment is divided into N steps giving the recursion relation at the M 'th sub-step

$$\left[\sigma'_{t+\frac{M}{N}\Delta t} \right] = \left\{ \mathbf{I}' - \frac{3}{2} \frac{\dot{\lambda}}{\bar{\sigma}} \frac{\Delta t}{N} \mathbf{L} : \mathbf{K} \right\} : \left[\sigma'_{t+\frac{M-1}{N}\Delta t} \right] + \mathbf{L} : \mathbf{d}' \frac{\Delta t}{N}. \quad (\text{A2})$$

$\mathbf{I}'$ is defined in Eq. (7). Applying this formula to successive sub-steps and noting the trend:

$$\sigma'_{t+\frac{1}{N}\Delta t} = \left\{ \mathbf{I}' - \frac{3}{2} \frac{\dot{\lambda}}{\bar{\sigma}} \frac{\Delta t}{N} \mathbf{L} : \mathbf{K} \right\} : \sigma'_t + \mathbf{L} : \mathbf{d}' \frac{\Delta t}{N}, \quad (\text{A3})$$

$$\sigma'_{t+\frac{2}{N}\Delta t} = \left\{ \mathbf{I}' - \frac{3}{2} \frac{\dot{\lambda}}{\bar{\sigma}} \frac{\Delta t}{N} \mathbf{L} : \mathbf{K} \right\}^2 : \sigma'_t + \left\{ \mathbf{I}' - \frac{3}{2} \frac{\dot{\lambda}}{\bar{\sigma}} \frac{\Delta t}{N} \mathbf{L} : \mathbf{K} \right\} : \mathbf{L} : \mathbf{d}' \frac{\Delta t}{N} + \mathbf{L} : \mathbf{d}' \frac{\Delta t}{N}, \quad (\text{A4})$$

$$\begin{aligned} \sigma'_{t+\frac{3}{N}\Delta t} = & \left\{ \mathbf{I}' - \frac{3}{2} \frac{\dot{\lambda}}{\bar{\sigma}} \frac{\Delta t}{N} \mathbf{L} : \mathbf{K} \right\}^3 : \sigma'_t + \left\{ \mathbf{I}' - \frac{3}{2} \frac{\dot{\lambda}}{\bar{\sigma}} \frac{\Delta t}{N} \mathbf{L} : \mathbf{K} \right\}^2 : \mathbf{L} : \mathbf{d}' \frac{\Delta t}{N} \\ & + \left\{ \mathbf{I}' - \frac{3}{2} \frac{\dot{\lambda}}{\bar{\sigma}} \frac{\Delta t}{N} \mathbf{L} : \mathbf{K} \right\} : \mathbf{L} : \mathbf{d}' \frac{\Delta t}{N} + \mathbf{L} : \mathbf{d}' \frac{\Delta t}{N}, \end{aligned} \quad (\text{A5})$$

...

$$\sigma'_{t+\Delta t} = \left\{ \mathbf{I}' - \frac{3}{2} \frac{\dot{\lambda}}{\bar{\sigma}} \frac{\Delta t}{N} \mathbf{L} : \mathbf{K} \right\}^N : \sigma'_t + \left[\sum_{M=0}^{N-1} \left\{ \mathbf{I}' - \frac{3}{2} \frac{\dot{\lambda}}{\bar{\sigma}} \frac{\Delta t}{N} \mathbf{L} : \mathbf{K} \right\}^M \right] : \mathbf{L} : \mathbf{d}' \frac{\Delta t}{N}. \quad (\text{A6})$$

Taking the limit as $N \rightarrow \infty$ allows simplification. Writing the quantity in braces as

$$\left\{ \mathbf{I}' - \frac{3}{2} \frac{\dot{\lambda}}{\bar{\sigma}} \frac{\Delta t}{N} \mathbf{L} : \mathbf{K} \right\} = \mathbf{A} = \mathbf{I}' - \frac{1}{N} \mathbf{B} \quad (\text{A7})$$

to shorten the notation, and expanding the binomial gives

$$\mathbf{A}^N = \mathbf{I}' - \frac{N}{N} \mathbf{B} + \frac{N(N-1)}{N^2 2!} \mathbf{B}^2 - \frac{N(N-1)(N-2)}{N^3 3!} \mathbf{B}^3 \dots \quad (\text{A8})$$

Letting N approach infinity results in

$$\lim_{N \rightarrow \infty} \left\{ \mathbf{I}' - \frac{3}{2} \frac{\dot{\lambda}}{\bar{\sigma}} \frac{\Delta t}{N} \mathbf{L} : \mathbf{K} \right\}^N = \exp \left(-\frac{3}{2} \frac{\dot{\lambda}}{\bar{\sigma}} \Delta t \mathbf{L} : \mathbf{K} \right). \quad (\text{A9})$$

The simplified form for the summation follows the form for a truncated geometric series

$$\frac{1}{N} \sum_{M=0}^{N-1} \mathbf{A}^M = \frac{1}{N} (\mathbf{I}' - \mathbf{A}^N) : (\mathbf{I}' - \mathbf{A})^{-1} = (\mathbf{I}' - \mathbf{A}^N) : \mathbf{B}^{-1}. \quad (\text{A10})$$

This can be verified by multiplying Eq. (A10) by $(\mathbf{I}' - \mathbf{A})$; all but the first and last terms of the resulting summation drop out. Upon substituting the series results, Eq. (A6) becomes

$$\sigma'_{t+\Delta t} = \exp \left(-\frac{3}{2} \frac{\dot{\lambda}}{\bar{\sigma}} \Delta t \mathbf{L} : \mathbf{K} \right) : \sigma'_{t+\Delta t} + \frac{2}{3} \frac{\bar{\sigma}}{\dot{\lambda} \Delta t} \left[\mathbf{I}' - \exp \left(-\frac{3}{2} \frac{\dot{\lambda}}{\bar{\sigma}} \Delta t \mathbf{L} : \mathbf{K} \right) \right] : \mathbf{K}^{-1} : \mathbf{d}' \Delta t. \quad (\text{A11})$$

This becomes Eq. (6) in Section 2.2.

Appendix B. Yield function details

Computation of the yield function

$$\phi = \sqrt[q]{\frac{1}{2} [(\sigma_I - \sigma_{II})^q + (\sigma_{II} - \sigma_{III})^q + (\sigma_{III} - \sigma_I)^q]} - \bar{\sigma} = 0 \quad (\text{B1})$$

is through the invariants of the deviatoric stress tensor

$$J'_1 = \sigma'_{11} + \sigma'_{22} + \sigma'_{33} = 0, \quad (\text{B2})$$

$$J'_2 = -\sigma'_{22}\sigma'_{33} - \sigma'_{33}\sigma'_{11} - \sigma'_{11}\sigma'_{22} + \sigma'_{23}\sigma'_{32} + \sigma'_{31}\sigma'_{13} + \sigma'_{12}\sigma'_{21} \quad (\text{B3})$$

$$= 0.5(\sigma'^2_{11} + \sigma'^2_{22} + \sigma'^2_{33} + 2\sigma'_{23}\sigma'_{32} + 2\sigma'_{31}\sigma'_{13} + 2\sigma'_{12}\sigma'_{21}) = \frac{1}{3}\sigma'^2_{vm}, \quad (\text{B4})$$

$$J'_3 = \sigma'_{11}\sigma'_{22}\sigma'_{33} + \sigma'_{12}\sigma'_{23}\sigma'_{31} + \sigma'_{13}\sigma'_{32}\sigma'_{21} - \sigma'_{11}\sigma'_{23}\sigma'_{32} - \sigma'_{22}\sigma'_{31}\sigma'_{13} - \sigma'_{33}\sigma'_{12}\sigma'_{21} \quad (\text{B5})$$

and the Lode angle

$$\alpha = \frac{1}{3} \cos^{-1} \left(\frac{3\sqrt{3}}{2} J'_3 / J'^3_2 \right) = \frac{1}{3} \cos^{-1}(\lambda) \quad 0 \leq \alpha \leq \frac{1}{3} \pi \quad (\text{B6})$$

The distinction between the ij and ji shear stress components is maintained in Eqs. (B2)–(B5) to facilitate taking derivatives later in this section. The principal stresses are given by

$$\sigma'_I = \frac{2}{\sqrt{3}} J'^{\frac{1}{2}}_2 \cos \left(\alpha - \frac{1}{3} \pi \right) \quad \sigma'_{II} = \frac{2}{\sqrt{3}} J'^{\frac{1}{2}}_2 \cos(\alpha) \quad \sigma'_{III} = \frac{2}{\sqrt{3}} J'^{\frac{1}{2}}_2 \cos \left(\alpha + \frac{1}{3} \pi \right). \quad (\text{B7})$$

After a bit of manipulation, the yield function can be expressed in terms of only α and the von Mises stress.

$$\phi = \frac{2}{\sqrt{3}} \sigma_{vm} \sqrt[q]{\frac{1}{2} \left[\sin^q \left(\alpha - \frac{1}{3} \pi \right) + \sin^q(\alpha) + \sin^q \left(\alpha + \frac{1}{3} \pi \right) \right]} - \dot{\sigma} = \sigma_{eq} - \dot{\sigma} = 0. \quad (\text{B8})$$

σ_{vm} is defined as the vonMises stress by the standard definition, Eq. (B4), and σ_{eq} is defined in Eq. (B8) as the equivalent stress. These are equal if $q = 2$, as in the von Mises yield criterion, but they are not equal in general. The von Mises stress is used in the relations to facilitate ties to J2-Flow theory.

Both first and second derivatives of the yield function are needed. It is convenient to define the quantity under the radical of Eq. (B8) as $\mathcal{F}$, resulting in the first derivative being

$$\frac{d\phi}{d\sigma} = \frac{d\sigma_{eq}}{d\sigma} = \frac{\sigma_{eq}}{\sigma_{vm}} \frac{d\sigma_{vm}}{d\sigma} + \frac{1}{q} \frac{\sigma_{eq}}{\mathcal{F}} \frac{d\mathcal{F}}{d\sigma}. \quad (\text{B9})$$

The second derivative is manipulated by using Eq. (B9) to replace all of the first derivatives of $\mathcal{F}$. This produces a form where the required symmetry is evident

$$\frac{d^2\phi}{d\sigma^2} = \frac{1-q}{\sigma_{eq}} \frac{d\sigma_{eq}}{d\sigma} \frac{d\sigma_{eq}}{d\sigma} - \frac{1+q}{\sigma_{vm}} \frac{\sigma_{eq}}{\sigma_{vm}} \frac{d\sigma_{vm}}{d\sigma} \frac{d\sigma_{vm}}{d\sigma} + \frac{q}{\sigma_{vm}} \left(\frac{d\sigma_{eq}}{d\sigma} \frac{d\sigma_{vm}}{d\sigma} + \frac{d\sigma_{vm}}{d\sigma} \frac{d\sigma_{eq}}{d\sigma} \right) + \frac{\sigma_{eq}}{\sigma_{vm}} \frac{d^2\sigma_{vm}}{d\sigma^2} + \frac{1}{q} \frac{\sigma_{eq}}{\mathcal{F}} \frac{d^2\mathcal{F}}{d\sigma^2}. \quad (\text{B10})$$

The first and second derivatives of the von Mises stress needed in Eq. (B10) are

$$\frac{d\sigma_{vm}}{d\sigma} = \frac{d\sigma_{vm}}{d\sigma'} = \frac{3}{2} \frac{\sigma'}{\sigma_{vm}} \quad \text{and} \quad \frac{d^2\sigma_{vm}}{d\sigma^2} = \frac{3}{2} \frac{1}{\sigma_{vm}} \left(\mathbf{I}' - \frac{3}{2} \frac{\sigma'}{\sigma_{vm}} \frac{\sigma'}{\sigma_{vm}} \right), \quad (\text{B11})$$

where $\mathbf{I}'$ is defined in Eq. (7). The derivatives of $\mathcal{F}$ are written with the chain rule in terms of both α and λ to aid in addressing the singular points.

$$\frac{1}{q} \frac{d\mathcal{F}}{d\sigma} = \frac{1}{q} \frac{d\mathcal{F}}{d\alpha} \frac{d\alpha}{d\lambda} \frac{d\lambda}{d\sigma}, \quad (\text{B12})$$

$$\frac{1}{q} \frac{d^2\mathcal{F}}{d\sigma^2} = \frac{1}{q} \frac{d^2\mathcal{F}}{d\alpha^2} \frac{d\alpha}{d\lambda} \frac{d\lambda}{d\sigma} \frac{d\lambda}{d\sigma} + \frac{1}{q} \frac{d\mathcal{F}}{d\alpha} \frac{d^2\alpha}{d\lambda^2} \frac{d\lambda}{d\sigma} \frac{d\lambda}{d\sigma} + \frac{1}{q} \frac{d\mathcal{F}}{d\alpha} \frac{d\alpha}{d\lambda} \frac{d^2\lambda}{d\sigma^2}, \quad (\text{B13})$$

λ is defined in Eq. (B6).

The scalar derivatives in Eqs. (B12) and (B13) are

$$\begin{aligned} \frac{1}{q} \frac{d\mathcal{F}}{d\alpha} &= \frac{1}{2} \left[\sin^{q-1} \left(\alpha - \frac{1}{3}\pi \right) \cos \left(\alpha - \frac{1}{3}\pi \right) + \sin^{q-1}(\alpha) \cos(\alpha) \right. \\ &\quad \left. + \sin^{q-1} \left(\alpha - \frac{1}{3}\pi \right) \cos \left(\alpha - \frac{1}{3}\pi \right) \right], \end{aligned} \quad (\text{B14})$$

$$\frac{1}{q} \frac{d^2\mathcal{F}}{d\alpha^2} = \frac{1}{2} \left[\sin^{q-2} \left(\alpha - \frac{1}{3}\pi \right) + \sin^{q-2}(\alpha) + \sin^{q-2} \left(\alpha + \frac{1}{3}\pi \right) \right] - q\mathcal{F}, \quad (\text{B15})$$

$$\frac{d\alpha}{d\lambda} = -\frac{1}{3} \frac{1}{\sqrt{1-\lambda^2}} = \frac{-1}{3 \sin(3\alpha)}, \quad (\text{B16})$$

$$\frac{d^2\alpha}{d\lambda^2} = -\frac{1}{3} \lambda (1-\lambda^2)^{-\frac{3}{2}} = 9\lambda \frac{-1}{3 \sin(3\alpha)} \frac{-1}{3 \sin(3\alpha)} \frac{-1}{3 \sin(3\alpha)}. \quad (\text{B17})$$

The reason for writing Eq. (B17) in this manner will be evident shortly. At $\alpha = 0$ and $\alpha = \pi/3$ the sine function in Eqs. (B16) and (B17) is zero, and the values are singular. However, the product

$$\frac{d\alpha}{d\lambda} \frac{d\lambda}{d\sigma} = \frac{d\alpha}{d\sigma} \quad (\text{B18})$$

is bounded. The value of the product at the singular points is determined by an abusive application of l'Hôpital's rule.

$$\mathcal{L}_{ij} = \frac{d\alpha}{d\sigma_{ij}} = \frac{\frac{d\lambda}{d\sigma_{ij}}}{-3 \sin(3\alpha)} \xrightarrow{\text{l'Hôpital's rule}} \frac{\frac{d}{d\sigma_{kl}} \left(\frac{d\lambda}{d\sigma_{ij}} \right)}{-9 \cos(3\alpha) \frac{d\alpha}{d\sigma_{kl}}} = \frac{\frac{d^2\lambda}{d\sigma_{kl} d\sigma_{ij}}}{-9 \cos(3\alpha) \mathcal{L}_{kl}}. \quad (\text{B19})$$

The derivative at the singular angles is determined by solving Eq. (B19) for $\mathcal{L}_{ij}$.

$$\mathcal{L}_{ij} = \frac{d\alpha}{d\sigma_{ij}} = \frac{-1}{3 \sin(3\alpha)} \frac{d\lambda}{d\sigma_{ij}} = \frac{1}{3} \sqrt{\frac{-1}{\cos(3\alpha)}} \frac{d^2\lambda}{d\sigma_{ij} d\sigma_{ij}} \Big|_{\alpha \rightarrow 0 \parallel \pi/3}. \quad (\text{B20})$$

Note that Eq. (20) takes the diagonal of the second derivative matrix. It will be shown later that the radical is non-singular, so Eq. (B20) addresses the singular values in Eq. (B12) and the first term of Eq. (B13). Further, by the expansion given in Eq. (B17), the result in Eq. (B20) also eliminates two of the three orders of singularity in the second term of Eq. (B13).

The remaining singularities in the second term and in the last term of Eq. (B13) are eliminated because Eq. (B14) is zero at these singular points. Specifically, by series expansion of Eq. (B14)

$$\frac{-1}{3 \sin(3\alpha)} \frac{1}{q} \frac{d\mathcal{F}}{d\alpha} \simeq \left\{ \frac{-\text{sign}(\cos(3\alpha))}{9\alpha(1+O\alpha^2)} \right\} \left\{ \alpha \frac{1}{3} \left(\frac{\sqrt{3}}{2} \right)^q [(q-4) + O\alpha^2 + O\alpha^{q-2}] \right\} \simeq -\text{sign}(\cos(3\alpha)) \frac{(q-4)}{27} \left(\frac{\sqrt{3}}{2} \right)^q \Big|_{\alpha \rightarrow 0 \parallel \pi/3}. \quad (\text{B21})$$

The limits near $\alpha = 0$ and $\alpha = \pi/3$ differ by an algebraic sign. The next highest term is quadratic in α so the limit is accurate numerically in 64 bit calculations to approximately $\alpha < 10^{-7}$ of the singularity. A similar expansion could be used to remove the singularities from Eq. (B12). This would be expedient if the second derivatives were not computed for Eq. (B20).

Table B1

Second derivative of the third invariant.

$\frac{d^2 J_3}{d\sigma_{ij} d\sigma_{kl}}$	11	22	33	23	31	12
11	$\frac{2}{3}\sigma'_{11}$	$\frac{2}{3}\sigma'_{33}$	$\frac{2}{3}\sigma'_{22}$	$-\frac{2}{3}\sigma'_{23}$	$\frac{1}{3}\sigma'_{31}$	$\frac{1}{3}\sigma'_{12}$
22	$\frac{2}{3}\sigma'_{33}$	$\frac{2}{3}\sigma'_{22}$	$\frac{2}{3}\sigma'_{11}$	$\frac{1}{3}\sigma'_{23}$	$-\frac{2}{3}\sigma'_{31}$	$\frac{1}{3}\sigma'_{12}$
33	$\frac{2}{3}\sigma'_{22}$	$\frac{2}{3}\sigma'_{11}$	$\frac{2}{3}\sigma'_{33}$	$\frac{1}{3}\sigma'_{23}$	$\frac{1}{3}\sigma'_{31}$	$-\frac{2}{3}\sigma'_{12}$
23	$-\frac{2}{3}\sigma'_{23}$	$\frac{1}{3}\sigma'_{23}$	$\frac{1}{3}\sigma'_{23}$	$-\frac{1}{2}\sigma'_{11}$	$\frac{1}{2}\sigma'_{12}$	$\frac{1}{2}\sigma'_{31}$
31	$\frac{1}{3}\sigma'_{31}$	$-\frac{2}{3}\sigma'_{31}$	$\frac{1}{3}\sigma'_{31}$	$\frac{1}{2}\sigma'_{12}$	$-\frac{1}{2}\sigma'_{22}$	$\frac{1}{2}\sigma'_{23}$
12	$\frac{1}{3}\sigma'_{12}$	$\frac{1}{3}\sigma'_{12}$	$-\frac{2}{3}\sigma'_{12}$	$\frac{1}{2}\sigma'_{31}$	$\frac{1}{2}\sigma'_{23}$	$-\frac{1}{2}\sigma'_{33}$

Considering Eqs. (B16) and (B17), Eq. (B13) can be rewritten as

$$\frac{1}{q} \frac{d^2 \mathcal{F}}{d\sigma^2} = \left[\frac{1}{q} \frac{d^2 \mathcal{F}}{d\alpha^2} + 9 \cos(3\alpha) \left(\frac{1}{q} \frac{d\mathcal{F}}{d\alpha} \frac{-1}{3 \sin(3\alpha)} \right) \right] \frac{d\alpha}{d\sigma} \frac{d\alpha}{d\sigma} + \left(\frac{1}{q} \frac{d\mathcal{F}}{d\alpha} \frac{-1}{3 \sin(3\alpha)} \right) \frac{d^2 \lambda}{d\sigma^2}. \quad (\text{B22})$$

It is not difficult to demonstrate that the argument of the square brackets is zero at $\alpha = 0$ and $\alpha = \pi/3$.

What remains is to specify the tensor derivatives in Eqs. (B12) and (B13). Using

$$\frac{dJ'_2}{d\sigma} = \sigma' \quad \text{and} \quad \frac{d^2 J'_2}{d\sigma^2} = \mathbf{I}', \quad (\text{B23})$$

these become

$$\frac{d\lambda}{d\sigma} = \frac{27}{2} \frac{1}{\sigma_{vm}^3} \left[\frac{dJ'_3}{d\sigma} - \frac{9}{2} \frac{J'_3}{\sigma_{vm}^2} \sigma' \right], \quad (\text{B24})$$

$$\frac{d^2 \lambda}{d\sigma^2} = \frac{27}{2} \frac{1}{\sigma_{vm}^3} \left[-\frac{9}{2} \frac{1}{\sigma_{vm}^2} \left(J'_3 \mathbf{I}' + \frac{dJ'_3}{d\sigma} \sigma' + \sigma' \frac{dJ'_3}{d\sigma} \right) + \frac{135}{4} \frac{J'_3}{\sigma_{vm}^4} \sigma' \sigma' + \frac{d^2 J'_3}{d\sigma^2} \right]. \quad (\text{B25})$$

Calculating the derivatives of the third invariant is accomplished by utilizing the derivatives with respect to the deviatoric stress

$$\frac{dJ'_3}{d\sigma} = \frac{dJ'_3}{d\sigma'} + \frac{1}{9} \sigma_{vm}^2 \delta. \quad (\text{B26})$$

In component form, the first term on the right hand side of Eq. (B26) is

$$\begin{aligned} \frac{dJ'_3}{d\sigma'_{11}} &= \sigma'_{22}\sigma'_{33} - \sigma'_{23}\sigma'_{32} & \frac{dJ'_3}{d\sigma'_{22}} &= \sigma'_{11}\sigma'_{33} - \sigma'_{31}\sigma'_{13} & \frac{dJ'_3}{d\sigma'_{33}} &= \sigma'_{11}\sigma'_{22} - \sigma'_{12}\sigma'_{21} \\ \frac{dJ'_3}{d\sigma'_{23}} &= \sigma'_{12}\sigma'_{31} - \sigma'_{11}\sigma'_{32} & \frac{dJ'_3}{d\sigma'_{31}} &= \sigma'_{12}\sigma'_{23} - \sigma'_{22}\sigma'_{13} & \frac{dJ'_3}{d\sigma'_{12}} &= \sigma'_{23}\sigma'_{31} - \sigma'_{33}\sigma'_{21}. \end{aligned} \quad (\text{B27})$$

Note that not distinguishing between the ij and ji shears in taking the derivatives would have increased the corresponding derivatives by a factor of 2. The second derivative of J'_3 is given in Table B1.

References

- Abaqus, 2009. Abaqus User's Manual. Dassault Systèmes/SIMULIA, Providence, RI, USA.
- Arioli, E., Auricchio, F., Beirão da Veiga, L., 2006. A novel 'optimal' exponential-based integration algorithm for von-Mises plasticity with linear hardening: theoretical analysis on yield consistency, accuracy, convergence and numerical examples. *Int. J. Numer. Methods Eng.* 67, 449–498.
- Auricchio, F., Beirão da Veiga, L., 2003. On a new integration scheme for von-Mises plasticity with linear hardening. *Int. J. Numer. Methods Eng.* 56, 1375–1396.
- Barlat, F., Maeda, Y., Chung, K., Yanagawa, M., Brem, J.C., Hayashida, Y., Lege, D.J., Matsui, K., Murtha, S.J., Hattori, S., Becker, R.C., Makosey, S., 1997. Yield function development for aluminum alloy sheets. *J. Mech. Phys. Solids* 45, 1727–1763.
- Brannon, R.M., Leelavanichkul, S., 2010. A multi-stage return algorithm for solving the classical damage component of constitutive models for rocks, ceramics and other rock-like media. *Int. J. Fract.* 163, 133–149.
- Dunne, F., Petrinic, N., 2005. *Introduction to Computational Plasticity*. Oxford University Press, Oxford, UK.
- Hill, R.J., 1950. *The Mathematical Theory of Plasticity*. Oxford University Press, Oxford, UK.
- Hosford, W.F., 1972. Generalized isotropic yield criterion. *J. Appl. Mech.* 39, 607–609.
- Karafilis, A.P., Boyce, M.C., 1993. A general anisotropic yield criterion using bounds and a transformation weighting tensor. *J. Mech. Phys. Solids* 41, 1859–1886.
- Kojić, M., 2002. Stress integration procedures for inelastic material models within the finite element method. *Appl. Mech. Rev.* 55, 389–414.
- Krieg, R.D., Krieg, D.B., 1977. Accuracies of numerical solution methods for the elastic-perfectly plastic model. *J. Pressure Vessel Technol.* 99, 510–515.
- Krieg, R.D., Xu, S., 1997. Plane stress linear hardening plasticity theory. *Finite Elem. Anal. Des.* 27, 41–67.
- Logan, R.W., Hosford, W.F., 1980. Upper bound anisotropic yield locus calculations assuming <1 1 1> pencil glide. *Int. J. Mech. Sci.* 22, 419–430.
- Maudlin, P.J., Schiffrer, S.K., 1996. Computational anisotropic plasticity for high-rate forming applications. *Comput. Methods Appl. Mech. Eng.* 131, 1–30.

- Mosler, J., Bruhns, O.T., 2010. On the implementation of rate-independent standard dissipative solids at finite strain – variational constitutive updates. *Comput. Methods Appl. Mech. Eng.* 199, 417–429.
- Murr, L.E., Ramirez, A.C., Gaytan, S.M., Lopez, M.I., Martinez, E.Y., Hernandez, D.H., Martinez, E., 2009. Microstructure evolution associated with adiabatic shear bands and shear band failure in ballistic plug formation in Ti-6Al-4V targets. *Mater. Sci. Eng. A* 516, 205–216.
- Nichols, A.L., 2009. Users Manual for ALE3D: An Arbitrary Lagrange/Eulerian 2D and 3D Code System. Lawrence Livermore National Laboratory.
- Press, W.H., Teukolsky, S.A., Vetterling, W.T., Flannery, B.P., 1992. *Numerical Recipes in FORTRAN: The Art of Scientific Computing*, second ed. Cambridge University Press, New York, NY.
- Rezaiee-Pajand, M., Nasirai, C., 2008. On the integration schemes for Durrer–Prager’s elastoplastic models based on exponential maps. *Int. J. Numer. Methods Eng.* 74, 799–826.
- Szabó, L., 2009. A semi-analytical integration method for J2 flow theory of plasticity with linear isotropic hardening. *Comput. Methods Appl. Mech. Eng.* 198, 2151–2166.
- Ulz, M.H., 2009. A Green–Naghdi approach to finite anisotropic rate-independent and rate-dependent thermo-plasticity in logarithmic Lagrangian strain-entropy space. *Comput. Methods Appl. Mech. Eng.* 198, 3262–3277.
- Wilkins, M.L., 1964. Calculation of elastic–plastic flow. In: Alder, B. (Ed.), *Methods in Computational Physics*, vol. 3. Academic Press, New York, pp. 211–263.
- Yu, M.-H., 2002. Advances in strength theories for materials under complex stress state in 20th century. *Appl. Mech. Rev.* 55, 169–218.