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A Simple Upper Bound of the Gaussian ?-Function with Closed-Form Error Bound

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***Abstract*—We present a simple upper bound of the Gaussian Q-function with its closed-form error bound. The proposed approximation can be applied to the evaluation of the average error probability of digital modulations with its error bound. Our bit error rate (BER) bound can be easily applied to fading channels.**

***Index Terms*—Gaussian Q-function, upper bound, relative error bound.**

I. Introduction

In this letter, we propose a simple upper bound of the Gaussian Q-function with its error bound in a closed-form. Error bounds for the Q-function approximation are not generally suitable for a closed-form solution because of the complex expression of the approximation. Compared to other known Q-function approximations [1]–[6], our approximation provides the upper bound of Q-function in a simple form. We also provide the error bound introduced by the approximation in a closed-form solution. The proposed BER bound is applied to Nakagami-m fading channels. In this letter, we compare the probability of a bit error with other known approximations.

II. A Simple Approximation of the Gaussian Q-Function

Let us consider the Gaussian Q-function, $Q(\alpha) = (2\pi)^{-1/2}\int_\alpha^\infty e^{-y^2/2}\mathrm{d}y$. A well known upper bound of the Q-function is

$$Q(\alpha) \leq \frac{1}{\sqrt{2\pi}\alpha}\exp\{-\alpha^2/2\} = P_o(\alpha). \quad (1)$$

To find a tighter upper bound of the bit error rate (BER), we modify the approximation BER as

$$P(\alpha) = \frac{1}{\sqrt{2\pi}\alpha}(1-\exp\{-c\alpha\})\exp\{-\alpha^2/2\} \quad (2)$$

with $c = \sqrt{\pi/2}$. Applying L'Hôpital's rule, we can see that $\lim_{\alpha\to 0} P(\alpha) = c/\sqrt{2\pi}$ = 0.5. Let us consider the approximation error $e(\alpha) = P(\alpha) - Q(\alpha)$. Then,

$$\frac{\mathrm{d}e(\alpha)}{\mathrm{d}\alpha} = -\frac{1}{\sqrt{2\pi}}\exp\{-\alpha^2/2\}$$
$$\left\{\frac{1}{\alpha^2} - \frac{\exp\{-c\alpha\}}{\alpha^2} - \frac{c}{\alpha}\exp\{-c\alpha\} - \exp\{-c\alpha\}\right\}. \quad (3)$$

Lemma 1: $P(\alpha)$ is an upper bound of $Q(\alpha)$.

Proof: We can evaluate that $\mathrm{d}e(\alpha)/\mathrm{d}\alpha > 0$ for $\alpha < \alpha^*$ and $\mathrm{d}e(\alpha)/\mathrm{d}\alpha < 0$ for $\alpha > \alpha^*$ with $\alpha^* \approx 0.5461$. From Eq. (2) we can also observe that $\lim_{\alpha\to 0} e(\alpha) = 0$ and $\lim_{\alpha\to\infty} e(\alpha) = 0$. Since $e(\alpha)$ is a continuous function, $P(\alpha)$ is an upper bound of $Q(\alpha)$, $\forall\alpha \geq 0$.

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Eq. (2) has a similar form with Karagiannidis approximation given in [1, Eq. (6)] as

$$\mathrm{erfc}(\alpha) \approx f(\alpha, A, B) = \frac{(1-\exp\{-A\alpha\})\exp\{-\alpha^2\}}{B\sqrt{\pi}\alpha} \quad (4)$$

where A and B are constant real numbers. However, our approximation has a number of nice properties that the approximation in [1] does not have, either in general, or using the suggested parameter values given in the paper. First, Karagiannidis approximation is neither an upper bound nor a lower bound with the suggested parameter values. Another nice property of our bound is that $\lim_{\alpha\to\infty} P(\alpha)/Q(\alpha) = 1$, i.e., the approximation is asymptotically tight in the sense that the relative error goes to zero in the tail. On the other hand, the relative error of the approximation in Eq. (4) goes to $|B-1|/B$, which is nonzero for $B \neq 1$. These properties are nicely pointed out by the reviewer.

We can evaluate that $\mathrm{d}(e(\alpha)/Q(\alpha))/\mathrm{d}\alpha > 0$ for $\alpha < \alpha^o$, and $\mathrm{d}(e(\alpha)/Q(\alpha))/\mathrm{d}\alpha < 0$ for $\alpha > \alpha^o$ with $\alpha^o \approx 1.444$. From Eq. (3), we can observe that

$$\frac{\mathrm{d}e(\alpha)}{\mathrm{d}Q(\alpha)} = \frac{1}{\alpha^2} - \frac{\exp\{-c\alpha\}}{\alpha^2} - \frac{c}{\alpha}\exp\{-c\alpha\} - \exp\{-c\alpha\}. \quad (5)$$

Conjecture 1: $\mathrm{d}e(\alpha)/\mathrm{d}Q(\alpha)$ is an upper bound of the relative error for $\alpha \geq \alpha^o$. Since $\mathrm{d}(e(\alpha)/Q(\alpha))/\mathrm{d}\alpha \leq 0$ for $\alpha \geq \alpha^o$,

$$\mathrm{d}(e(\alpha)/Q(\alpha))/\mathrm{d}\alpha = \frac{(\mathrm{d}e(\alpha)/\mathrm{d}\alpha)Q(\alpha) - e(\alpha)(\mathrm{d}Q(\alpha)/\mathrm{d}\alpha)}{Q^2(\alpha)} \leq 0 \quad (6)$$

$$\text{or} \quad (\mathrm{d}e(\alpha)/\mathrm{d}\alpha)Q(\alpha) \leq e(\alpha)(\mathrm{d}Q(\alpha)/\mathrm{d}\alpha). \quad (7)$$

$$\text{Thus} \quad \frac{\mathrm{d}e(\alpha)/\mathrm{d}\alpha}{\mathrm{d}Q(\alpha)/\mathrm{d}\alpha} \geq \frac{e(\alpha)}{Q(\alpha)} \quad (8)$$

since $\mathrm{d}Q(\alpha)/\mathrm{d}\alpha < 0$. Therefore, Eq. (5) is the upper bound of the relative error for $\alpha \geq \alpha^o$.

Conjecture 2: The relative error is asymptotically upper bounded by α^{-2}. Let us introduce $g(\alpha) = \alpha^2 e(\alpha)/Q(\alpha)$. Then, from Eqs. (5) and (8), for $\alpha \geq \alpha^o$,

$$g(\alpha) \leq 1 - \exp\{-c\alpha\} - c\alpha\exp\{-c\alpha\} - \alpha^2\exp\{-c\alpha\} \leq 1. \quad (9)$$

Lemma 2: α^{-2} is a relative error bound for both $P_o(\alpha)$ and $P(\alpha)$ for $\alpha \geq 0$.

Proof: Let $\phi(\alpha) = (2\pi)^{-1/2}e^{-\alpha^2/2}$. Then,

$$Q(\alpha) = \int_\alpha^\infty \phi(u)\mathrm{d}u \geq (1+\alpha^{-2})^{-1}\int_\alpha^\infty (1+u^{-2})\phi(u)\mathrm{d}u \quad (10)$$

$$= (1+\alpha^{-2})^{-1}\frac{\phi(\alpha)}{\alpha} = \frac{\alpha^2}{\alpha^2+1}P_o(\alpha), \text{ or } \frac{P_o(\alpha)}{Q(\alpha)} \leq 1+\alpha^{-2} \quad (11)$$

TABLE I
COMPARISON OF ERFC(α), EQS. (22), (23), (4), (24), (25) AND (2)

α	Erfc(α)	Chiani	Börj.	Karag.	Ruskai	Boyd	Eq. (2)
0.001	9.989e-1	6.667e-1	1.003	9.989e-1	1.127	9.989e-1	9.991e-1
0.01	9.887e-1	6.666e-1	9.926e-1	9.887e-1	1.112	9.887e-1	9.911e-1
0.1	8.875e-1	6.584e-1	8.888e-1	8.840e-1	9.704e-1	8.880e-1	9.073e-1
0.5	4.795e-1	4.881e-1	4.799e-1	4.866e-1	4.935e-1	4.823e-1	5.165e-1
1	1.573e-1	1.931e-1	1.579e-1	1.576e-1	1.586e-1	1.587e-1	1.723e-1
2	4.678e-3	5.467e-3	4.697e-3	4.465e-3	4.682e-3	4.711e-3	5.018e-3
3	2.209e-5	2.364e-5	2.215e-5	2.039e-5	2.209e-5	2.219e-5	2.310e-5
4	1.542e-8	1.903e-8	1.545e-8	1.398e-8	1.542e-8	1.546e-8	1.586e-8
5	1.537e-12	2.316e-12	1.540e-12	1.381e-12	1.537e-12	1.541e-12	1.567e-12
10	2.088e-45	6.200e-45	2.089e-45	1.849e-45	2.088e-45	2.090e-45	2.099e-45

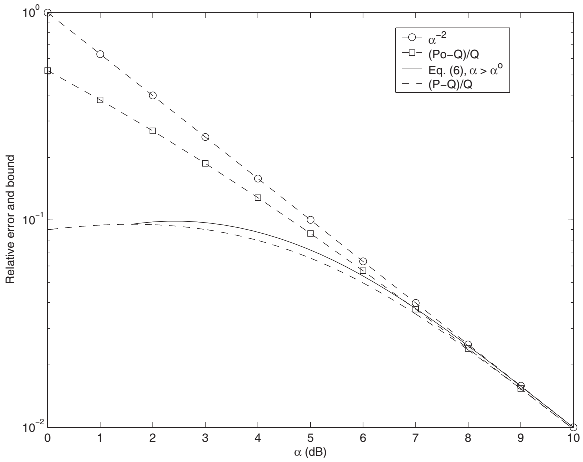

Fig. 1. Relative error and bound.

since $\{-u^{-1}\phi(u)\}$ is an antiderivative of $(1+u^{-2})\phi(u)$. The reviewer graciously derived the above result. Therefore,

$$\frac{e(\alpha)}{Q(\alpha)} = \frac{P(\alpha)-Q(\alpha)}{Q(\alpha)} \leq \frac{P_o(\alpha)-Q(\alpha)}{Q(\alpha)} \leq \alpha^{-2},\ \forall\alpha \geq 0. \tag{12}$$

The relative error and bound are shown in Fig. 1. The bound in Eq. (5) is plotted for $\alpha \geq \alpha^o$.

III. APPLICATION TO FADING CHANNELS

We can easily extend Eq. (2) to Nakagami-m fading channels using the power series representation of $\exp\{-c\alpha\}$. With the binary phase-shift keying (BPSK) modulation, we can express the probability of a bit error in fading channels as

$$P_b = \int_0^\infty Q\left(\sqrt{\frac{2E_b}{N_o}}\,x\right) f_x(x)\mathrm{d}x \tag{13}$$

where $f_x(x)$ is the probability density function (pdf) of the fading. E_b and N_o are the average bit energy and the one-sided noise power spectral density, respectively. Replacing $\sqrt{2E_b/N_o}$ with K and using Eq. (2),

$$P_b \leq \int_0^\infty \frac{1}{\sqrt{2\pi}Kx}\exp\{-K^2x^2/2\}f_x(x)\mathrm{d}x$$
$$-\int_0^\infty \frac{1}{\sqrt{2\pi}Kx}\exp\{-cKx\}\exp\{-K^2x^2/2\}f_x(x)\mathrm{d}x \tag{14}$$
$$= P_{b_1} - P_{b_2} \tag{15}$$

where P_{b_1} and P_{b_2} indicate the first and second integration, respectively. With Nakagami-m fading channels,

$$f_x(x) = \frac{2}{\Gamma(m)}m^m x^{2m-1}\exp\{-mx^2\} \tag{16}$$

TABLE II
RELATIVE ERROR BOUND OF EQ. (2)

α	$Q(\alpha)$	Eq. (2)	Relative Error	Bound Eq. (5)	$\frac{1}{\alpha^2}$
1.5	0.0668	0.0732	0.0952	0.0965	0.4444
2	0.0228	0.0248	0.0898	0.0970	0.2500
3	0.0013	0.0014	0.0689	0.0755	0.1111
4	3.1671e-5	3.3235e-5	0.0494	0.0534	0.0625
5	2.8665e-7	2.9678e-7	0.0353	0.0375	0.0400
10	7.6199e-24	7.6946e-24	0.0098	0.0100	0.0100
20	2.7536e-89	2.7605e-89	0.0025	0.0025	0.0025

where $\Gamma(m)$ represents the gamma function defined as $\Gamma(m) = \int_0^\infty x^{m-1}\exp\{-x\}\mathrm{d}x$, and m is the fading parameter. With the power series representation of $\exp\{-cKx\}$, the P_{b_2} in Eq. (15) can be written as

$$P_{b_2} = \frac{m^m}{K\Gamma(m)}\sum_{k=0}^{n}(-1)^k c^k \sigma\frac{K^k}{k!}$$
$$\int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}}x^{k+2m-2}\exp\{-x^2/(2\sigma^2)\}\mathrm{d}x$$
$$+\frac{m^m}{K\Gamma(m)}(-1)^{n+1}c^{n+1}\exp\{-cKz\}\sigma\frac{K^{n+1}}{(n+1)!}$$
$$\int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}}x^{n+2m-1}\exp\{-x^2/(2\sigma^2)\}\mathrm{d}x \tag{17}$$

where z is between 0 and x, and $\sigma^2 = 1/(K^2+2m)$.

Lemma 3: We maintain the upper bound of the BER in fading channels as long as we expand to an odd degree of the power series term.

Proof: The second term in Eq. (17) is the remainder, R_n, when we expand the power series to n-th degree polynomial at zero. If n is an odd number, then $n+1$ is an even number. Therefore the remainder is a positive number. Therefore $P^*_{b_2}$ which is without the remainder is less than P_{b_2}. Thus, from Eq. (15),

$$P_b \leq P_{b_1} - P_{b_2} \leq P_{b_1} - P^*_{b_2}. \tag{18}$$

To maintain the upper bound of the BER, we will assume n is an odd integer in this paper. Now, the BER in fading channels can be easily obtained using the moment of the Gaussian random variable [7, pp. 148] as

$$P_b \leq \frac{m^m}{K\Gamma(m)}\left\{1\cdot 3\cdots(2m-3)\sigma^{2m-1}\right\}$$
$$-\frac{m^m}{K\Gamma(m)}\sum_{\substack{k=0\\ k\ even}}^{n-1} c^k K^k \frac{1}{k!}\left\{1\cdot 3\cdots(k+2m-3)\sigma^{k+2m-1}\right\}$$
$$+\frac{m^m}{K\Gamma(m)}\sum_{\substack{k=1\\ k\ odd}}^{n} c^k K^k \frac{1}{k!}\left\{2^{k_1}k_1!\,\sigma^{2k_1+2}\sqrt{2/\pi}\right\} = P_b^u \tag{19}$$

TABLE III
COMPARISON OF BER IN NAKAGAMI-m FADING CHANNELS

E_b/N_o (dB)	Exact (m=2)	Karag. (m=2)	Eq. (19) (m=2)	Exact (m=4)	Karag. (m=4)	Eq. (19) (m=4)	Exact (m=6)	Karag. (m=6)	Eq. (19) (m=6)
-10	3.389e-1	3.416e-1	3.564e-1	3.333e-1	3.364e-1	3.515e-1	3.313e-1	3.346e-1	3.498e-1
0	1.151e-1	1.161e-1	1.248e-1	9.751e-2	9.820e-2	1.063e-1	9.136e-2	9.190e-2	9.990e-2
10	5.528e-3	5.500e-3	6.000e-3	1.039e-3	1.000e-3	1.100e-3	3.814e-4	3.674e-4	4.110e-4
20	7.257e-5	7.192e-5	7.890e-5	3.039e-7	2.933e-7	3.276e-7	3.803e-9	3.603e-9	4.054e-9
30	7.475e-7	7.405e-7	8.127e-7	3.450e-11	3.325e-11	3.716e-11	5.090e-15	4.812e-15	5.417e-15
40	7.497e-9	7.427e-9	8.151e-9	3.495e-15	3.368e-15	3.764e-15	5.242e-21	4.957e-21	5.581e-21

where $k_1 = (k-1)/2 + m - 1$, and P_b^u denotes the upper bound.

Lemma 4: The error involved in truncating the infinite series to n terms is bounded by

$$R_n \leq \frac{m^m c^{n+1}}{K\Gamma(m)} \frac{K^{n+1}}{(n+1)!} \left\{1 \cdot 3 \cdots (n+2m-2)\sigma^{n+2m}\right\} \quad (20)$$

where the order of the bound of R_n is $O(\exp\{-0.5n\log(n)\})$ since $2\log(1 \cdot 3 \cdots (n-2)) \leq \int_1^n \log(x)dx \leq \log(n!)$ for an odd integer n.

Proof: Since $\exp\{-cKz\} \leq 1$, the remainder, R_n can be bounded as

$$R_n \leq \frac{m^m c^{n+1}}{K\Gamma(m)} \frac{K^{n+1}\sigma}{(n+1)!}$$
$$\int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} x^{n+2m-1} \exp\{-x^2/(2\sigma^2)\}\mathrm{d}x. \quad (21)$$

Applying the even moment of the Gaussian random variable, we obtain Eq. (20).

IV. Numerical Results

The complementary error function is defined as $\mathrm{erfc}(\alpha) = 2(\pi)^{-1/2}\int_\alpha^\infty \exp\{-t^2\}\mathrm{d}t = 2Q(\sqrt{2}\alpha)$. We compare the erfc($\alpha$) to Chiani given in [2, Eq. (14)] as

$$\mathrm{erfc}(\alpha) \approx \frac{1}{6}\exp\{-\alpha^2\} + \frac{1}{2}\exp\{-\frac{4}{3}\alpha^2\}, \quad (22)$$

Börjesson given in [3, Eq. (17)] as

$$Q(\alpha) \leq \frac{1}{[-(-\sqrt{x^2+b}+x)a + x]\sqrt{\exp\{x^2\}2\pi}} \quad (23)$$

with a=0.344 and b=5.334, and Karagiannidis in Eq. (4) with the values of A=1.98 and B=1.135 suggested in [1]. We also compare our results to the upper bound of Ruskai [4], [5] in

$$\sqrt{\frac{\pi}{2}} \frac{\exp(-\alpha^2/2)}{(\pi-1)\alpha + \sqrt{\alpha^2+2\pi}} \leq Q(\alpha) \leq \sqrt{\frac{8}{\pi}} \frac{\exp\{-\alpha^2/2\}}{3\alpha + \sqrt{\alpha^2+8}}. \quad (24)$$

and to the upper bound of Boyd [6] in

$$\frac{\gamma_l + 1}{\gamma_l\alpha + \sqrt{\alpha^2 + \frac{2}{\pi}(\gamma_l+1)^2}} \frac{\exp\{-\alpha^2/2\}}{\sqrt{2\pi}} \leq$$
$$Q(\alpha) \leq \frac{\gamma_u + 1}{\gamma_u\alpha + \sqrt{\alpha^2 + \frac{2}{\pi}(\gamma_u+1)^2}} \frac{\exp\{-\alpha^2/2\}}{\sqrt{2\pi}} \quad (25)$$

where $\gamma_l = \pi - 1$ and $\gamma_u = 2/(\pi-2)$. These are the tightest bounds available of the form

$$\frac{c_1}{c_2\alpha + \sqrt{\alpha^2 + c_3}} \frac{\exp\{-\alpha^2/2\}}{\sqrt{2\pi}} \quad (26)$$

such that the bounds are tight both at zero and at infinity as commented by the reviewer. For the complementary error function approximation, our method provides consistently an upper bound compared to other methods, i.e., Eq. (22) (Chiani), and Eq. (4) (Karagiannidis) which exhibit neither an upper bound nor a lower bound as shown in Table I. Our method displays a tighter upper bound than Börjesson and Ruskai at a low signal-to-noise ratio (SNR). Boyd's bounds are tighter both at zero and at infinity than our approximation. However, our method is in a simpler form and provides the relative error bound in Eq. (5) as shown in Table II.

We obtain the BER of Nakagami-m fading channels in Table III using Eq. (19) with n=101. The exact BER is numerically obtained from [8, Eq. (42)]

$$P_b = \int_0^{\pi/2} \left(1 + \frac{\frac{E_b}{N_o}}{m\ \sin^2\phi}\right)^{-m} \mathrm{d}\phi \quad (27)$$

for the fading parameter of m=2, 4 and 6. We observe that our result is consistently an upper bound for all SNR. We can find the corresponding BER of Karagiannidis by replacing c with $A/\sqrt{2}$ in Eq. (19), and dividing P_b^u by B. Karagiannidis BER is also shown in Table III using the suggested values of A=1.98 and B=1.135. Karagiannidis BER displays a larger BER at a low SNR and a smaller BER at moderate or high SNR compared to the exact BER.

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