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p-Sidon Sets and a Uniform Property

Gordon S. Woodward

University of Nebraska - Lincoln, gwoodward@unl.edu

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Let \( G \) be a compact abelian group with dual group \( \Gamma \). Denote by \( L^p(E) \) and \( M(E) \) the usual spaces of Haar-measurable functions and bounded regular Borel measures, respectively, which are supported on the subset \( E \) of \( G \) or \( \Gamma \). The Haar measure on \( G \) is normalized and its dual is the Haar measure on \( \Gamma \). Let \( \varphi \) denote the Fourier or Fourier-Stieltjes transform of the function or measure \( \varphi \). A subset \( E \subset \Gamma \) is said to be \( p \)-Sidon for some \( 1 \leq p < 2 \) (not interesting for \( p \geq 2 \)) if there is an \( \alpha > 0 \) such that \( \|\varphi\|_p \leq \alpha \|\varphi\|_\infty \) for all trigonometric polynomials \( \varphi \) on \( G \) with \( \text{supp} \ \varphi \subset E \). This is equivalent to the dual statement: \( E \) is \( p \)-Sidon if and only if \( L^q(E) \subset M(G)|_E \), where \( 1/p + 1/q = 1 \) and \( \|\cdot\|_E \) denotes restriction to \( E \). Hereafter \( p \) and \( q \) will always be as above.

The concept of a \( p \)-Sidon set was independently introduced in [2, 4, and 5] as a natural generalization of the classical Sidon sets (i.e., 1-Sidon sets). In each of these articles, the various equivalent definitions for \( p \)-Sidon sets are given. They correspond to the classical equivalent definitions of a Sidon set as presented in [8, Theorem 5.7.3]. In [5], Hahn extends a theorem of J. P. Kahane to give the best known necessary conditions for a set to be \( p \)-Sidon when \( \Gamma = \mathbb{Z} \), the integers. Edwards and Ross present the most comprehensive treatment of the subject in [4]. It is there that the first non Sidon \( p \)-Sidon set is constructed via an extremely ingenious application of the tensor algebraic techniques of Varapoulos. Their methods are extended in [6] to prove that the classes of all \( 2n/(n+1) \)-Sidon sets are distinct for \( n = 1, 2, \ldots \) One will also find in [6] all known non Sidon \( p \)-Sidon sets to date (except for unions with finite sets). For a somewhat more skillful application of the Varapoulos techniques to this problem, we refer the reader to [1].

In this paper, we adapt an idea of Rider [7] in defining the class of uniformizable \( p \)-Sidon sets. The class is, by design, closed under finite unions. Of course, its members are \( p \)-Sidon sets. Our main result is that Sidon sets are uniformizable \( p \)-Sidon sets for all \( p \). Its proof is a variant of Drury’s famous technique which resembles most closely the approach found in [3]. As a corollary we prove that the union of a Sidon set with any \( p \)-Sidon set is again \( p \)-Sidon, thus enabling one to exhibit many new non Sidon \( p \)-Sidon sets. We conclude with a slight extension of the results in [6], using an argument similar to the one presented there, and a list of open questions.
In what follows, $L^p(\Gamma)_E$ denotes the $L^p(\Gamma)$ functions supported on $E \subset \Gamma$ and $I_E$ denotes the characteristic function of $E$. We begin with a useful technical result of a standard type.

**Lemma 1.** $E \subset \Gamma$ is a $p$-Sidon set if and only if there exist $\beta > 0$ and $0 < \delta < 1$ such that for each $\varphi \in L^p(\Gamma)_E$ there is a $\mu \in M(G)$ satisfying

(i) $||\mu|| \leq \beta ||\varphi||_\varphi$; and

(ii) $||\hat{\mu}I_E - \varphi||_\varphi < \delta ||\varphi||_\varphi$.

**Proof.** Suppose $E$ is a $p$-Sidon set. Define the relation $\sim$ on $M(G)$ by $\mu \sim \nu$ if $\hat{\mu} - \hat{\nu} \equiv 0$ on $E$. Let $M(G)/\sim$ denote the usual Banach quotient space. By definition $L^p(\Gamma)_E$ naturally embeds in $M(G)/\sim$. Moreover, the uniqueness of the Fourier-Stieltjes transform yields that the graph of this map is closed; hence (i). Of course, (ii) holds for any $\delta > 0$.

For the converse, let $\varphi \in L^p(\Gamma)_E$. Then (i) and (ii) yields inductively a sequence $\{\mu_n\} \subset M(G)$ with $\mu_1$ satisfying $||\mu_1|| \leq \beta ||\varphi||_\varphi$ and $||\hat{\mu}_1I_E - \varphi||_\varphi \leq \delta ||\varphi||_\varphi$ and continuing

$$||\mu_n|| \leq \beta \delta^{n-1} ||\varphi||_\varphi$$

and

$$||\hat{\mu}_nI_E - \left(\varphi - \sum_{k=0}^{n-1} \hat{\mu}_kI_E\right)||_\varphi \leq \delta^n ||\varphi||_\varphi.$$ 

Since $\sum_{n} ||\mu_n|| \leq \beta ||\varphi||_\varphi (1 - \delta)^{-1}$, the sum $\mu = \sum_{n} \mu_n$ converges in $M(G)$; clearly $\hat{\mu} = \varphi$ on $E$. \qed

**Definition.** $E \subset \Gamma$ is a uniformizable $p$-Sidon set if for each $\delta > 0$ there exists a $\beta > 0$ such that for any $\varphi \in L^p(\Gamma)_E$ there is a $\mu \in M(G)$ satisfying

(i) $||\mu|| \leq \beta ||\varphi||_\varphi$; and

(ii) $||\hat{\mu} - \varphi||_\varphi \leq \delta ||\varphi||_\varphi$.

Denote by $\mathcal{U}_p$ the class of all uniformizable $p$-Sidon sets on $\Gamma$.

It is clear that each element of $\mathcal{U}_p$ is $p$-Sidon. The full strength of the definition is summed up in the following theorem.

**Theorem 1.** $E \in \mathcal{U}_p$ if and only if for each $\delta > 0$ there exists a $\beta > 0$ such that for any $\varphi \in L^p(\Gamma)_E$ there is a $\mu \in M(G)$ satisfying

(i) $||\mu|| \leq \beta ||\varphi||_\varphi$;

(ii) $\hat{\mu} = \varphi$ on $E$; and

(iii) $\left(\sum_{\gamma \in \Gamma} |\hat{\mu}(\gamma)|^p\right)^{1/p} \leq \delta ||\varphi||_\varphi$.

**Proof.** Suppose $E \in \mathcal{U}_p$. Let $\varphi \in L^p(\Gamma)_E$ and choose any $0 < \delta_0 < 1$. Set $\delta = \delta_0/2$. According to the definition of $\mathcal{U}_p$, there is a $\beta > 0$ and a $\mu_1 \in M(G)$ such that $||\mu_1|| \leq \beta ||\varphi||_\varphi$ and $||\hat{\mu}_1 - \varphi||_\varphi \leq \delta ||\varphi||_\varphi$. Apply the definition again to $\varphi - \hat{\mu}_1I_E$ with the same $\delta$ and continue in this manner. This gives rise to a sequence $\{\mu_n\} \subset M(G)$ as in Lemma 1. Thus
\[ \mu = \sum_{n=0}^{\infty} \mu_n \in M(G), \quad ||\mu|| \leq \beta ||\varphi||_\alpha (1 - \delta)^{-1}, \]
and \( \hat{\mu} = \varphi \) on \( E \). But this time
\[ \left| \hat{\mu}_n - \left( \varphi - \sum_{k=1}^{n-1} \hat{\mu}_k \right) \right| q \leq \left| \hat{\mu}_n - \left( \varphi - \sum_{k=1}^{n-1} \hat{\mu}_k I_k \right) \right| q + \left| \sum_{k=1}^{n-1} \hat{\mu}_k (1 - I_k) \right| q \]
\[ \leq \delta^n ||\varphi||_\alpha + \sum_{k=1}^{n-1} \delta^k ||\varphi||_\alpha < \delta_0 ||\varphi||_\alpha. \]

In particular, (iii) is valid. Of course, (i)–(iii) are sufficient to imply \( E \in \mathcal{U}_p \). \( \square \)

Our next theorem is rather trivial at this point, but worth mentioning.

**Theorem 2.** \( \mathcal{U}_p \) is closed under finite unions for \( 1 \leq p < 2 \).

**Proof.** Suppose \( E_1, E_2 \in \mathcal{U}_p \) and set \( E = E_1 \cup E_2 \). Since subsets of elements in \( \mathcal{U}_p \) are also in \( \mathcal{U}_p \), we can assume that \( E_1 \cap E_2 = \emptyset \). Let \( \varphi \in L^p(G_E) \), let \( E_i = \varphi I_{E_i} \) for \( i = 1, 2 \), and choose any \( \delta_0 > 0 \). By definition there exist \( \beta > 0 \) and measures \( \mu_1, \mu_2 \) such that \( \delta = \delta_0/2 \), \( \beta, \mu_1, \varphi \) satisfy (i) and (ii) of the definition for a \( \mathcal{U}_p \) set. Thus \( ||\mu_1 + \mu_2|| \leq 2\beta ||\varphi||_\alpha \) and
\[ ||\hat{\mu}_1 + \hat{\mu}_2 - \varphi||_q \leq ||\hat{\mu}_1 - \varphi||_q + ||\hat{\mu}_2 - \varphi||_q < \delta_0 ||\varphi||_\alpha. \] \( \square \)

**Remark.** This author had originally announced [Notices Amer. Math. Soc. 21 (1974), A-163] a somewhat different definition for \( \mathcal{U}_p \). Specifically, replace “for each \( \delta > 0 \)” by “for some \( 0 < \delta < 1 \)” in Theorem 1. Under this change, Theorem 2 would read “the union of any two elements of \( \mathcal{U}_p \) is p-Sidon.” The formally stronger definition that we are now using seems to better reflect the structure of p-Sidon sets.

We now turn to the question of existence of nontrivial uniformizable p-Sidon sets. Fortunately, Drury’s theorem implies that \( \mathcal{U}_1 \) consists of all Sidon sets. But this yields no information about \( \mathcal{U}_p \) for \( p \neq 1 \). In fact, the relationship between \( \mathcal{U}_p \) and \( \mathcal{U}_r \) for \( 1 \leq p \neq r < 2 \) is not at all clear. Our next theorem sheds some light on the matter by showing \( \mathcal{U}_1 \subseteq \mathcal{U}_p \). The key is the observation that \( \mathcal{U}_p \) contains all dissociate sets for \( 1 \leq p < 2 \). We emphasize that many of the techniques used in our next proof parallel those of [3]. A subset \( E \) of an abelian group \( \Lambda \) is **dissociate** if the only solutions to \( \sum \delta_\gamma \gamma = 0 \) (finite sum) with \( \gamma \in E \) and \( \delta_\gamma \in \{-2, -1, 0, 1, 2\} \) are \( \delta_\gamma = 0 \) for all \( \gamma \). As is custom, we denote by \( B(\Gamma) \) the space \( M(G)^{\ast}\Gamma \) with the norm, \( ||\hat{\mu}||_B = ||\mu|| \).

**Theorem 3.** Sidon sets are uniformizable p-Sidon sets for all \( p \).

**Proof.** Let \( E \subseteq \Gamma \) be a Sidon set. Following Drury [3], fix a positive integer \( n \) and let \( \gamma_1, \ldots, \gamma_n \in E \) be any choice of \( n \) distinct nonzero elements. Let \( \Lambda \) be the discrete abelian group generated by \( F = \{\gamma_1, \ldots, \gamma_n\} \) over, say, \( \mathbb{Z} \) mod (3).
where \( \gamma_1, \cdots, \gamma_n \) are simply considered as \( n \) independent symbols. That is 
\[ \Lambda \cong (\mathbb{Z} \mod (3))^{n}. \]
The dual \( H \) of \( \Lambda \) is isomorphic to \( \Lambda \) but it can also be realized 
as the set of all maps \( h : F \to T_3 \) where \( T_3 \) is the set of 3rd roots of unity. The 
group operation, represented by \( + \), is just pointwise multiplication. We insist 
that \( H \) have Haar measure 1. Then the dual Haar measure on \( \Lambda \) is simply 
the counting measure.

Consider first the group \( \Gamma \times H \) which has dual \( G \times \Lambda \). Since \( E \) is 1-Sidon, 
there exists an \( \alpha > 0 \) such that for each \( h \in H \) there is a \( \mu_h \in M(G) \) satisfying 
\[ ||\mu_h|| \leq \alpha \] 
and \( \tilde{\mu}_h = h \) on \( F \). Set \( g(\gamma, h) = \tilde{\mu}_h(\gamma) \). Then \( g(\gamma, \cdot) \) is a character 
on \( H \). Together with the properties of \( \mu_h \), this yields 
\[ (1') \quad g(\cdot, h) \in B(\Gamma) \quad \text{with} \quad ||g(\cdot, h)||_B \leq \alpha \quad \text{for all} \quad h \in H \]
and 
\[ (2') \quad g(\gamma, \cdot) \in B(H) \quad \text{with} \quad ||g(\gamma, \cdot)||_B = 1 \quad \text{for all} \quad \gamma \in F. \]

We adjust these two statements as follows. Define the function 
\[ r(\gamma, \cdot) = g(\gamma, \cdot) \tilde{\mu}(\gamma, o) \quad \text{(convolution over} \ H). \]

Since \( ||g(\cdot, \cdot)||_B \leq \alpha \), it follows that \( ||r(\gamma, \cdot)||_B \leq \alpha ; \) hence \( ||r(\gamma, \cdot)||_B \leq \alpha^2 \)
for all \( \gamma \in \Gamma \). Since \( r(\cdot, h) \) is a convex linear combination of products of the 
\( \tilde{\mu}, \epsilon \in H \), it follows that \( r(\cdot, h) \in B(\Gamma) \) and \( ||r(\cdot, h)||_B \leq \alpha^2 \) for \( h \in H \). That is,
\begin{enumerate}
  \item \( ||r(\cdot, h)||_B \leq \alpha^2 \) for all \( h \in H \); 
  \item \( ||r(\gamma, \cdot)||_B \leq \alpha^2 \) for all \( \gamma \in \Gamma \); and 
  \item \( r(\gamma, h) = h(\gamma) \) on \( F \) for all \( h \in H \); 
\end{enumerate}
where (3) is immediate from the definition of \( r \).

At this point we fix a real-valued \( \phi \in L^\infty(\Gamma) \) with \( ||\phi||_\infty = 1 \). Let \( 0 < \epsilon \leq 1 \) 
and set \( x_i = (\gamma_i, \gamma_i) \in \Gamma \times \Lambda \) for \( 1 \leq j \leq n \). Define the Riesz polynomials 
\( P_\epsilon \) and \( P_0 \) on \( G \times H \) by 
\[ P_\epsilon(z) = \prod_{i=1}^n [1 + \epsilon/2\phi(\gamma_i)(x_i(z) + \overline{x_i(z)})] \]
and 
\[ P_0(z) = \prod_{i=1}^n [1 + \epsilon/2i\phi(\gamma_i)(x_i(z) - \overline{x_i(z)})]. \]

Since these functions are nonnegative \( ||P_\epsilon||_1 = P_\epsilon(0) \) and \( ||P_0||_1 = P_0(0) \). Their 
formal expansions can be described in the following terms. Set \( \Omega = \{-1, 0, 1\}^n \), 
let \( \delta = (\delta_1, \cdots, \delta_n) \) be a generic point of \( \Omega \), and adopt the convention \( 0^0 = 1 \).
Then, using the additive group notation, we have 
\[ P_\epsilon(z) = \sum_{\delta \in \Omega} \left[ \prod_{i=1}^n (\epsilon/2\phi(\gamma_i))^{\delta_i} \right] (\delta_1 x_1 + \cdots + \delta_n x_n)(z) \]
and 
\[ P_0(z) = \sum_{\delta \in \Omega} \left[ \prod_{i=1}^n (\delta_i \epsilon/2i\phi(\gamma_i))^{\delta_i} \right] (\delta_1 x_1 + \cdots + \delta_n x_n)(z). \]
Note that by definition of \( \Lambda \) the set \( \{x_1, \ldots, x_n\} \) is dissociate; hence distinct \( \delta \in \Omega \) give distinct characters \( \delta \cdot x_1 + \cdots + \delta \cdot x_n \) on \( G \times H \). In particular, \( ||P_\ast||_1 = ||P_0||_1 = 1 \). Moreover, \( P_\ast, P_0 \) are supported on points of the form
\[
y = \sum^n \delta_i x_i \quad \text{with} \quad P_\ast(y) = \prod^n (\epsilon/2\varphi(\gamma_i))^{1/\delta_i}
\]
and
\[
P_0(y) = \prod^n (\delta_i \epsilon/2i\varphi(\gamma_i))^{1/\delta_i}.
\]
Also note, \( P_0(\pm x_i) = \pm \epsilon/2i \varphi(\gamma_i) \).

For a continuous \( P \) on \( G \times H \), denote its transform with respect to the \( j \)-th variable by \( P^j \) \((j = 1, 2)\). It follows that \( (P^j)^{-2} = \hat{P} \) and that \( ||P^j(\gamma, \cdot)||_1 \leq ||P||_1 \). In particular, the functions
\[
s_\ast(\gamma) = (P_\ast(\gamma), -1) \hat{r}(\gamma, \cdot)(0)
\]
and
\[
s_0(\gamma) = (iP_0(\gamma), -i) \hat{r}(\gamma, \cdot)(0)
\]
are convex linear combinations of \( B(\Gamma) \) functions with norm bounded by \( 2\alpha^2 \).

Thus
\[
(4) \quad s = s_\ast + s_0 \in B(\Gamma) \quad \text{and} \quad ||s||_B \leq 4\alpha^2.
\]
Moreover, since \( r(\gamma_j, h) = h(\gamma_j) \) for \( 1 \leq j \leq n \),
\[
(5) \quad s(\gamma_j) = P_\ast(x_j) + iP_0(x_j) = \epsilon \varphi(\gamma_j).
\]

We now want to estimate \( ||s - \epsilon \varphi||_\| \). To this end, denote the Dirac point measure at \( 0 \in \Lambda \) by \( \delta_0 \). Then applying Parseval’s formula (relative to \( H \)) to the definition of \( s(\gamma) \) yields
\[
|s(\gamma)| = \left| \int_H [P_\ast(\gamma, h) + iP_0(\gamma, h) - (1 + i)] \hat{r}(\gamma, -h) \, dh \right| = \left| \int_\Lambda [P_\ast(\gamma, \lambda) + iP_0(\gamma, \lambda) - (1 + i)\delta_0] \hat{r}(\gamma, \lambda) \, d\lambda \right| \leq ||P_\ast(\gamma, \cdot) + iP_0(\gamma, \cdot) - (1 + i)\delta_0||_\infty \alpha^2 \quad \text{for all} \quad \gamma \in \Gamma.
\]
by (2). Set \( R = P_\ast + iP_0 - (1 + i)\delta_0 \). The preceding inequalities and (5) yield
\[
(6) \quad ||s - \epsilon \varphi||_\| = \sum_\gamma |s(\gamma)|^2 \leq \alpha^2 \sum_{\gamma \in \Lambda} |R(\gamma)|^2 = \alpha^2 \epsilon ||R||_\|^2 - \epsilon^2 ||\varphi||_\|^2.
\]
To estimate \( ||R||_\| \), partition \( \Omega \) by the equivalence relation \( \delta \sim \sigma \) if and only if \( |\delta_i| = |\sigma_j| \) for \( 1 \leq j \leq n \). Call this partition \( \mathcal{E} \). Given \( u \in \mathcal{E} \) and any \( \delta \in u \), define
\[
|u| = \sum^n |\delta_i| \quad \text{and} \quad A_u = \prod^n |\varphi(\gamma_i)|^{1/\delta_i}.
\]
Both symbols are well defined. Let \( z = (x_1, \ldots, x_n) \). Then the expansions obtained earlier for \( P_* \) and \( P_0 \) yield

\[
| R(\delta \cdot z) | = (\epsilon/2)^{|u|} A_u | 1 + \beta_u | \quad \text{for} \quad \delta \in u \in \mathcal{E},
\]

where \( \delta \cdot z \) denotes the usual vector inner product and \( \beta_u \in \{ \pm 1, \pm i \} \). It is important to note that \( R(x_j) = \epsilon \varphi(\gamma_j) \) and \( R(0) = R(-x_j) = 0 \) for \( 1 \leq j \leq n \). Since the cardinality of each \( u \in \mathcal{E} \) is \( 2^{|u|} \), it follows that

\[
\sum_{\delta \in u} | R(\delta \cdot z) |^q \leq 2^{|u|} (\epsilon/2)^{|u|} A_u^q 2^q \quad \text{if} \quad |u| > 1,
\]

\[
\sum_{\delta \in u} | R(\delta \cdot z) |^q = (\epsilon/2)^{|u|} A_u^q 2^q \quad \text{if} \quad |u| = 1,
\]

and

\[
\sum_{\delta \in u} | R(\delta \cdot z) |^q = 0 \quad \text{if} \quad |u| = 0.
\]

Thus

\[
||R||_q^q = \sum_{u \in \mathcal{E}} \sum_{\delta \in u} | R(\delta \cdot z) |^q \leq \sum_{u \in \mathcal{E}} 2^{|u|} (\epsilon/2)^{|u|} A_u^q 2^q - \sum_{|u| = 1} (\epsilon/2)^q A_u^q 2^q - 2^q = 2^q \left( \sum_{u \in \mathcal{E}} 2^{|u|} (\epsilon/2)^{|u|} A_u^q - (\epsilon/2)^q - 1 \right),
\]

where the second line of the inequality reflects, via subtraction, the differences between the cases \(|u| > 1, |u| = 1, \) and \(|u| = 0\). We have also used

\[
\sum_{|u| = 1} (A_u)^q = (||\varphi||_q)^q = 1.
\]

This can be further simplified with the aid of the equation

\[
\sum_{u \in \mathcal{E}} 2^{|u|} (\epsilon/2)^{|u|} A_u^q = \prod_{j=1}^n \left( 1 + 2 |\epsilon/2\varphi(\gamma_j)|^q \right)
\]

and the inequality

\[
\ln \prod_{j=1}^n \left( 1 + 2 |\epsilon/2\varphi(\gamma_j)|^q \right) = \sum_{j=1}^n \ln \left( 1 + 2 |\epsilon/2\varphi(\gamma_j)|^q \right) \leq \sum_{j=1}^n 2 |\epsilon/2\varphi(\gamma_j)|^q = 2(\epsilon/2)^q ||\varphi||_q^q = 2(\epsilon/2)^q.
\]

In fact a slight computation yields

\[
||R||_q^q \leq 2q[\exp \left( (\epsilon/2)^q \right) - 1 - (\epsilon/2)^q].
\]

Together with (6), this yields

\[
||s - \epsilon \varphi||_q \leq 2\alpha^2[\exp \left( (\epsilon/2)^q \right) - (1 + 2(\epsilon/2)^q)]^{1/q} \leq \alpha^2 \epsilon^q.
\]
Now apply (4) and (5). We conclude: (i) \( \epsilon^{-1}s \in B(\Gamma) \) and \( \| \epsilon^{-1}s \|_n \leq 4\epsilon^{-1}\alpha^2 \); (ii) \( \epsilon^{-1}s = \varphi \) on \( F \); (iii) \( \| \epsilon^{-1}s - \varphi \|_n \leq \alpha \epsilon^2 \). In particular, given any \( \psi \in L'(\Gamma)_F \), we can apply (i)-(iii) to its normalized real and imaginary parts. It follows that there is a \( \mu \in M(G) \) satisfying

\[
\begin{align*}
(a) \quad & \| \mu | \| \leq 8\epsilon^{-1}\alpha^2 \| \psi \|_o, \\
(b) \quad & \text{\( \mu = \psi \) on \( F \), and} \\
(c) \quad & \| \mu - \psi \|_o \leq 2 \epsilon \alpha^2 \| \psi \|_o.
\end{align*}
\]

The argument extends from finite sets \( F \) to \( E \) via a standard weak* compactness argument. \( \square \)

We can now describe a large variety of new \( p \)-Sidon sets. Just consider the sets in [6] together with the following corollary.

**Corollary.** Suppose \( S \subseteq \Gamma \) is Sidon and \( E \subseteq \Gamma \) is \( p \)-Sidon. Then \( S \cup E \) is \( p \)-Sidon.

**Proof.** We can assume \( S \cap E = \emptyset \). The \( p \)-Sidon property and Theorem 3 imply that there exists \( \beta > 0 \) such that for any \( \varphi \in L'(\Gamma)_{S \cup E} \) there are measures \( \mu, \mu_1, \mu_2 \in M(G) \) satisfying

\[
\begin{align*}
(1) \quad & \| \mu | \| \leq \beta, \mu = 1 \text{ on } S, |\mu| < 1/4 \text{ off } S; \\
(2) \quad & \| \mu_1 | \| \leq \beta \| \varphi I_S \|_o, \mu_1 = \varphi \text{ on } S, |\mu_1 - \varphi I_S \|_o \leq 1/4 \| \varphi I_S \|_o; \\
(3) \quad & \| \mu_2 | \| \leq \beta \| \varphi I_E \|_o, \mu_2 = \varphi \text{ on } E.
\end{align*}
\]

Set \( \nu = (1 - \mu)\mu_2 + \mu_1 \). Then \( \nu \in M(G) \) and \( \| \nu | \| \leq (1 + \beta)2\beta \| \varphi \|_o \). Moreover

\[
\| \varphi I_S - \varphi I_S \|_o = 0
\]

and

\[
\| \varphi I_E - \varphi I_E \|_o = \| -\mu_2 \mu I_E + \mu I_E \|_o \leq \frac{1}{2} \| \varphi \|_o.
\]

Thus

\[
\| \varphi I_{S \cup E} - \varphi \|_o \leq \frac{1}{2} \| \varphi \|_o.
\]

Now apply Lemma 1. \( \square \)

Our last result exhibits some additional \( p \)-Sidon sets as an extension to the result in [6]. We outline much of the proof and refer the reader to [6] for the details. By \( \pm A \pm B \) we mean \( \{ \delta a + \delta 'b : \delta, \delta ' \in \{-1, 1\} \text{ and } a \in A, b \in B \} \).

**Theorem 4.** Suppose \( A_1, \ldots, A_n \) are mutually disjoint infinite subsets of \( \Gamma \) whose union is dissociate. Then \( E = \pm A_1 \pm A_2 \pm \cdots \pm A_n \) is \( p \)-Sidon if and only if \( p \geq 2n/(n + 1) \).

**Proof.** Lemma 1 in [6] implies that \( p \geq 2n/(n + 1) \) if \( E \) is \( p \)-Sidon. Thus we need only prove that \( E \) is \( p = 2n/(n + 1) \)-Sidon. To begin note that the \( 2^n \) sets of the form \( E_\beta = \sum \beta_j A_j \) where \( \beta = (\beta_1, \ldots, \beta_n) \in \{-1, 1\}^n \) are mutually disjoint since \( \bigcup A_j \) is dissociate. Choose any \( \beta \) and a \( \varphi \in L'(\Gamma)_{E_\beta} \). We shall show
that there is a $\mu_\beta \in M(G)$ such that $\hat{\mu}_\beta = \varphi$ on $E_\beta$ while $\hat{\mu}_\beta \equiv 0$ on $E_\alpha$ for $\alpha \neq \beta$. The theorem then follows by considering sums of the form $\sum \mu_\beta$. It is sufficient to restrict our attention to real-valued $\varphi$ and to $\beta \equiv (-1, 1, \cdots, 1) \in \{-1, 1\}^n$.

Fix such a $\varphi$. As argued in [6], it follows that $\varphi \in C(A_1) \otimes \cdots \otimes C(A_n)$; hence we need only prove the following fact concerning basic tensor elements: there exists a constant $K > 0$ such that for any choice of real-valued functions $\varphi_1, \cdots, \varphi_n$ on $A_1, \cdots, A_n$, respectively, there is a $\mu \in M(G)$ with $||\mu|| \leq K ||\varphi_1||_\infty \cdots ||\varphi_n||_\infty$ satisfying $\hat{\mu} = 0$ on $E_\alpha$ for $\alpha \neq \beta$ and

$$\hat{\mu}(-\gamma_1 + \gamma_2 + \cdots + \gamma_n) = \varphi_1(\gamma_1) \cdots \varphi_n(\gamma_n)$$
on $E_\beta = -A_1 + A_2 + \cdots + A_n$.

To this end, assume for the moment that each $A_j$ is finite and fix a choice of $\varphi_1, \cdots, \varphi_n$. We consider the Riesz polynomials

$$p_j(x) = \prod_{\gamma \in A_j} \left[ 1 + 2(2i \ |\gamma|_\infty)^{-1} \varphi_j(\gamma)(\gamma(x) + \gamma(x)) \right], \quad 1 \leq j \leq n,$$

$$q_j(x) = \prod_{\gamma \in A_j} \left[ 1 + (2i \ |\gamma|_\infty)^{-1} \varphi_j(-\gamma)(\gamma(x) + \gamma(x)) \right],$$

and

$$q_j(x) = \prod_{\gamma \in A_j} \left[ 1 + (2i \ |\gamma|_\infty)^{-1} \varphi_j(\gamma)(\gamma(x) - \gamma(x)) \right], \quad 2 \leq j \leq n.$$ 

The discussion of such polynomials in Theorem 3 implies that $\|p_j\|_1 = \|q_j\|_1 = 1$ and that $\hat{p}_j(\pm \gamma) = \varphi_j(\gamma)/(2 \ |\gamma|_\infty)$, $\hat{q}_j(\pm \gamma) = \mp \varphi_j(\gamma)/(2i \ |\gamma|_\infty)$, and $q_j(\pm \gamma) = \pm \varphi_j(\gamma)/(2i \ |\gamma|_\infty)(j \neq 1)$, for all $\gamma$ in the corresponding $A_j$, $1 \leq j \leq n$. In particular, the polynomials

$$P_j = (p_j - 1) \ |\varphi_j|_\infty, \quad Q_j = (q_j - 1) \ |\varphi_j|_\infty$$

and

$$R = \prod_{j=1}^n (P_j + Q_j)$$

satisfy

1. $$(P_j + Q_j)(0) = 0,$$
2. $$(P_j + Q_j)(\gamma) = 0 \text{ and } (P_j + Q_j)(-\gamma) = \varphi_j(\gamma) \text{ for } \gamma \in A_j,$$
3. $$(P_j + Q_j)(\gamma) = \varphi_j(\gamma) \text{ and } (P_j + Q_j)(-\gamma) = 0 \text{ for } \gamma \in A_j,$$

4. $$\|R\|_1 \leq 2^{2n} \sum_{j=1}^n \ |\varphi_j|_\infty.$$ 

Here (1)-(3) are immediate from the definitions and the fact that $\bigcup A_j$ is dissociate. To see (4) observe that $R$ is the sum of $2^n$ terms, each of which has precisely $n$ factors consisting of some combination of $P_j$’s and $Q_j$’s—each
appearing only once. Since $\|P_i\|_1, \|Q_i\|_1 \leq 2 \|\varphi_i\|_\infty$, it follows that each of those terms has $L^1$-norm bounded by $2^n \prod_i \|\varphi_i\|_\infty$; whence (4). Again we use the dissociate property of $\bigcup A_i$, this time in conjunction with (1)-(3) to conclude
\[ \hat{R}(-\gamma_1 + \gamma_2 + \cdots + \gamma_n) = \varphi_1(\gamma_1) \cdots \varphi_n(\gamma_n) \quad \text{for} \quad \gamma_i \in A_i \]
and
\[ \hat{R} = 0 \quad \text{on} \quad E_\alpha \quad \text{for} \quad \alpha \neq \beta. \]
In light of (4), a weak* compactness argument extends (5) to infinite $A_i$ for some $R \in M(G)$. \square

Open questions.

1. Are all $p$-Sidon sets uniformizable $r$-Sidon sets for some $1 \neq p \leq r < 2$? Indeed, do there exists uniformizable $p$-Sidon sets which are not Sidon sets? To be specific, let $A = \{3^{2n}\}_{1}^\infty$ and $B = \{3^{2n+1}\}_{1}^\infty$. Is $A + B$ a uniformizable $p$-Sidon set?

2. Is the union of two $p$-Sidon sets ($p \neq 1$) an $r$-Sidon set for some $p \leq r < 2$? This is open even if one of the sets is assumed to be a uniformizable $p$-Sidon set.

3. There is a form of the Kahane and Salem necessary condition for Sidon sets for $p$-Sidon subsets of $\mathbb{Z}$ (see [5]). It extends immediately to any discrete $\Gamma$ for which every $\gamma \neq 0$ has infinite order and actually improves somewhat for other discrete $\Gamma$'s. The condition appears fairly tight. But what about sufficient conditions? For Sidon sets we at least have the Steckin type conditions (see [7] or [8, Section 5.7.5]). For $p$-Sidon sets ($p \neq 1$) the best result so far in this direction is our Theorem 4. Is there some analogue to the Steckin condition for $p$-Sidon sets?

4. Let $S_p$ be the class of all $p$-Sidon subsets of $\Gamma$. It is immediate that $S_p \subset S_r$ if $p \leq r$. Moreover, if $p_n = 2n/(n + 1)$, then [6] tells us that $S_{p_n} \subsetneq S_{p_{n+1}}$. If $1 \leq p \neq r < 2$ must it follow that $S_p \neq S_r$?

References


University of Nebraska, Lincoln
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