TWO-DIMENSIONAL SCALAR DIFFERENTIAL EQUATIONS FOR TRANSVERSELY VARYING THICKNESS MODES IN PIEZOELECTRIC PLATES AND APPLICATIONS IN ACOUSTIC WAVE RESONATOR SENSORS

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by

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TWO-DIMENSIONAL SCALAR DIFFERENTIAL EQUATIONS
FOR TRANSVERSELY VARYING THICKNESS MODES IN QUARTZ PLATES
AND APPLICATIONS IN ACOUSTIC WAVE RESONATORS AND SENSORS

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Generalizations are made for three types of well-known and widely used two-dimensional scalar differential equations in the literature describing transversely varying thickness modes in piezoelectric plate resonators. They are for singly-rotated quartz plates, doubly-rotated quartz plates, and plates of crystals of class 6mm with the c-axis along the plate thickness, respectively. The purpose of the generalizations is to include the effects of surface mechanical loads such as mass layers or fluids for resonator-based acoustic wave sensor applications. Surface acoustic impedance is introduced to take into account various surface loads in a general manner for time-harmonic motions. Both unelectroded and electroded plates are treated. For electroded plates, both free and electrically-forced vibrations under a time-harmonic driving voltage are considered. Simple two-dimensional scalar differential equations are constructed from the asymptotic dispersion relations quadratic in the small wave numbers of transversely varying thickness waves at long wavelengths. The equations obtained can be reduced to the equations in the literature in the special case when the surface acoustic impedance is set to zero. As illustrations of the usefulness and effectiveness of the equations obtained, simple examples of pure thickness vibrations of unbounded plates with surface loads, propagation of long thickness waves in unbounded plates with surface loads, and
vibrations of finite plates with surface loads are presented. It is expected that a lot of theoretical results can be obtained in the future using the equations derived in this dissertation for piezoelectric plate acoustic wave sensors.
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Chapter 1
Introduction

Due to their high sensitivity and stability, piezoelectric resonators have found broad applications in biomedical and chemical sensing. Theoretical modeling is of fundamental importance to the design and analysis of resonators and sensors. Most traditional analytical methods of resonator sensors are based on one-dimensional approximation of the real vibrations. It has been shown that they are too crude to predict frequency shifts and many other important characteristics and parameters such as quality factors and mode coupling. Therefore, developing new theoretical approaches accounting for the three-dimensional variation of resonant vibrations is of fundamental importance to the design and analysis of high performance sensors. This is exactly the main motivation for this dissertation. In this chapter, we first give a basic description of piezoelectric crystal acoustic wave resonators and sensors. Then the classical methods along with a brief literature review for the theoretical analysis methods of crystal resonators are presented. At the end of this chapter, we take an overlook of the works done in this dissertation.

1.1 Acoustic Wave Resonators and Sensors

(a) Piezoelectric effects, piezoelectric materials and piezoelectric devices

In the years before 1880, people had already known that certain materials become electrically polarized when experiencing temperature changes. This phenomenon is known as the pyroelectric effect. In 1880, Pierre and Jacques Curie [1] discovered that some crystals are capable of generating positive and negative electric charges on certain
surfaces when subjected to external pressure along specific directions while no temperature change is presented. The polarization charge is proportional to the applied loads and changes signs when subjected to extensional loads. This phenomenon is called direct piezoelectric effect. In the next year, Lippmann predicted that the inverse effect, i.e., the same materials become mechanically strained when placed in external electric fields, might exist based on the theory of thermodynamics. This phenomenon was verified by the Curie brothers in the same year and now it is known as the converse piezoelectric effect. After that, the scientific community paid little attention to this new effect. The situation changed when World War I came. In 1917, Langevin created the world’s first ultrasonic emitter and receiver using quartz plates that were excited by an electric voltage. Soon after, quartz plates were used as transducers in sonar to detect submarines. Since then, scientists all around the world have carried out extensive and in-depth studies on piezoelectric effects and piezoelectric materials. Many new piezoelectric materials have been discovered such as barium titanate (BaTiO₃), lead zirconate titanate (PZT), lithium tantalite (LiTaO₃) and lithium niobate (LiNbO₃) [2, 3]. Meanwhile, piezoelectric materials have been used in fabricating various electromechanical devices. For instance, quartz crystals have been made into frequency control and signal processing units such as resonators, oscillators, filters and delay lines [4], which are the key components in communication and navigation systems. Piezoelectric materials with strong electromechanical coupling, like PZT, have been used to fabricate transducers and ultrasonic probes for ultrasonic imaging [5] and nondestructive evaluation [6]. By utilizing the converse piezoelectric effect, piezoelectric actuators and micromotors [7, 8] have been created and been used as high precision movement controllers in
microelectromechanical systems (MEMS). Resonator-based sensors also have been produced by properly modifying the surface conditions of piezoelectric structures. They are widely used in the fields of chemistry, biomedicine and biology [9, 10]. This dissertation is mainly focused on resonator-based acoustic wave sensors, so in the sequel we give a detailed discussion of the operating principle and evolution of resonant sensors

(b) Quartz crystal resonator sensors
Quartz crystals have many superior properties such as outstanding physical and chemical stability and very high quality factors. Moreover, quartz is the second most abundant mineral in the Earth's continental crust and quartz crystal can be mass produced in autoclaves. Consequently, quartz crystals become the most favorable piezoelectric material for manufacturing resonators. It is shown that quartz plates with certain cut angles have a zero frequency-temperature coefficient at a temperature of about 80°C. Hence, they are called temperature-compensated cuts. AT-cut was the first temperature-compensated cut discovered. Due to its high temperature stability, a large number of quartz crystal resonators (QCRs) are made of AT-cut quartz plates. Meanwhile, a variety of other cut types have also been discovered. For example, the SC-cut (Stress Compensated cut) quartz plate not only has a zero frequency-temperature coefficient, but also has a zero frequency-stress coefficient. Overall, an SC-cut quartz plate has much better stability than an AT-cut plate, so it becomes one of the most frequently used cut type. Since SC-cut belongs to doubly-rotated cuts while AT-cut belongs to singly-rotated cuts, the anisotropy of the SC-cut is much stronger than that of the AT-cut. As will be shown in chapter 3, the SC-cut quartz behaves like a general anisotropic material in the
plate coordinate system. In this dissertation, we only consider these two cuts because they constitute ninety percent of the currently used QCMs.

Quartz crystal resonators have a number of different configurations, such as plano-plano, plano-convex and convex-convex. The electrode can either be directly deposited on the plate surfaces or be separated from the plate surface by an air gap. When the electrodes are deposited on the opposite major surfaces of the plate, electric field along the thickness direction is excited and this type of resonators are called thickness field excited (TFE) resonators. Piezoelectric resonance can also be activated by a lateral electric field via electrodes deposited either on the opposite side faces or on the same major surface. This type of QCRs is called the lateral field excited (LFE) resonator. Most QCRs have a sandwiched structure in which a quartz plate is covered by two surface electrodes, so in this work we only consider this type (TFE) of resonators. The quartz plate is mounted to a holder and the electrodes are connected to external circuits. A schematic diagram of a quartz crystal resonator is shown in Fig. 1.1.

![Fig. 1.1](image)

Fig. 1.1. (a) A quartz plate resonator with driving electrodes, mounting, and packaging.

(b) Thickness-shear deformation of a crystal plate.

The resonant frequency of a crystal resonator can be affected by many factors, e.g., temperature change, initial stress, acceleration, magnetic field and electric field, surface
mass layers, or contacting with a fluid [11]. Therefore, detection of frequency shifts in a resonator can be used as the basis for making various acoustic wave sensors [9-10]. Specifically, when a thin layer of another material is added to the surface of a quartz resonator, see Fig. 1.2(a), the resonant frequency becomes lower due to the inertia of the mass layer. This phenomenon has been used to make mass sensors for measuring the density and/or thickness of a mass layer.

![Diagram](image)

(a)                                                        (b)

Fig. 1.2. (a) A crystal plate with a thin mass layer as a mass sensor.

(b) A crystal plate in contact with a fluid for viscosity/density measurement.

Similarly, when a crystal plate resonator is in contact with a fluid, see Fig. 1.2(b), the inertia and viscosity of the fluid lower the resonant frequency of the resonator. In this case, the frequency shift can be used to determine the density/viscosity of the fluid. Mass and fluid sensors are called quartz crystal microbalances (QCMs).

As for sensor applications, the surface electrodes of the quartz plate are connected to properly designed oscillating circuits by which an oscillating voltage/current is obtained. The signals of both the perturbed and reference quartz crystal resonators are sent into a
differential frequency circuit. After a series of signal processing such as shaping and amplification, the frequency shift is read out through a frequency counter. Once this quantity is obtained, the properties of the measurand can be calculated via carefully designed theoretical models. According to the theory of Newtonian fluids, we can only obtain the product of density and viscosity of the fluid. To obtain the specific value of each of them, we need additional measurements to fix one of them. For example, we often connect the QCM to an acoustic impedance analyzer to fix the damping coefficient.

The operating principle and realization of a typical QCM is shown in Fig. 1.3.

![Diagram of operating principle of resonator sensors](image)

Figure 1.3 Schematic diagram of operating principle of resonator sensors

Since Quimby [12] carried out the first measurement of the viscosity of solids using a piece of quartz crystal, quartz crystal resonators have found broad applications in many research fields [13-22]. For mass sensing applications in vacuum deposition techniques [13], quartz crystal deposition monitors and controllers have been manufactured to regulate the deposition rate and total thickness of thin films. The sensitivity of areal mass density is on the order of $10^{-8}$ g/cm$^2$. Conventional QCMs work properly with relative
frequency changes less than 2%. Nevertheless, the frequency shifts can easily exceed this limit when depositing heavy metals like gold, silver or lead. To this end, Lu and Lewis [14] proposed a mass sensor capable of measuring large mass loadings that cause relative frequency shift up to 15%. QCMs are also used in determining viscosity for Newtonian fluid and rheological properties for non-Newtonian fluids [15-20]. Saluja and coworkers [15] developed a fast and accurate viscometer using QCMs and carried out viscosity measurement of aqueous solutions of sucrose, urea, glucose and ethylene glycol. Traditional fluid viscometers employ either rotational or tube technologies. The volume of test liquid ranges from a few hundred microliters to milliliters. In other cases, this requirement could not be satisfied. For example, fluid samples in chemical or pharmaceutical industries are often expensive or only available in small quantities. To overcome this difficulty, droplet QCMs have been developed which are capable of detecting fluid droplets with a volume of only 2-10 microliters. With the droplet QCMs, Zhuang [16] measured the viscosity of silicone oils and discovered positive frequency shifts. Ash and coworkers [17] carried out viscosity measurements of a wide variety of industrial oils such as commercial automotive lubricants, heavy fuel oil and calibration oils. Kim [18-19] explored the possibility of measuring the fluid’s density, viscosity and acoustic wave velocity simultaneously by using an SC-cut QCM. QCMs can also be made into biological sensors by depositing active or selective films on the surfaces [20-21]. Fredriksson [20] reported a cell biological sensor used to characterize the adhesion process of living cells. It is very sensitive and able to detect less than 100 cells. Perrot’s research group [21] reported a high sensitivity DNA sequence detector based on a 50 MHz QCM. The sensor was shown to be able to detect DNA targets in a solution with
concentration as low as 50 ng/mL. The sensor also has a good resolution of 7.1 ng/cm².

**(c) Thin-Film bulk acoustic wave resonator (FBAR) sensors**

![FBAR schematic](image)

**Figure 1.4** Schematic diagrams of three types of configurations of FBARs:

(a) membrane type FBAR, (b) air-gap type FBAR, (c) solidly mounted FBAR

The thickness of typical quartz plate resonators ranges from half a millimeter to a few millimeters and the lateral dimension is an order of magnitude larger. Their resonant frequencies range from a few to a couple hundred megahertz. These technical features could not meet the needs of modern telecommunication and microelectronic technologies.
In modern communication systems, resonators operating with 500 MHz-10GHz and able to provide more frequency bands along with larger bandwidths are required. Moreover, the manufacturing process of QCRs is incompatible with CMOS technology, the mainstream technology to produce integrated circuits. The emergence of thin-film bulk acoustic wave resonators (FBAR) overcome these difficulties [23-38]. Due to the invention of magnetron reactive iron deposition systems [23] and the development of surface and bulk micromachining technologies [24], thin piezoelectric film and metallic electrodes can now be deposited and be patterned on silicon wafers with a very high precision. In the past few years, FBARs with many different configurations have been developed. Based on the strategies used to confine the acoustic energy in the piezoelectric film, FBARs can be classified into three categories [24]: membrane type, air-gap type and solidly mounted type, as shown in Figure 1.4. The membrane type FBAR is composed of a multilayered membrane hanging in the air. This configuration provides the best acoustic isolation since the acoustic impedance of air is much less than that of solids. The air-gap type FBAR adopts an air gap to separate the active film from the substrate. The manufacturing process is somewhat complicated since a sacrificial layer is in need to release the film. As for the solidly mounted FBAR (SMR), acoustic isolation is realized via the so-called Bragg acoustic reflector. This special structure placed between the piezoelectric film and the substrate consists of a series of quarter wavelength layers with alternating low and high acoustic impedance. The manufacturing process of SMR is a little bit easier than the other two since it can be deposited layer by layer with no etching process. In this dissertation we only consider the membrane type FBARs since it has the simplest configuration. The thickness of most commercial FBAR resonators is only about
several nanometers to a few hundred micrometers and the lateral dimension ranges from tens of micrometers to several hundred micrometers. FBAR resonators operate in the frequency range of 0.5 GHz to 10 GHz, ten to a thousand times higher than QCRs. Nowadays FBARs have been widely used in communication systems. For example, high performance monolithic filters with 30 MHz bandwidth and very low insertion loss have been used in satellite communication systems [25]. Piezoelectric films are the key part of FBAR sensors. AlN and ZnO [23] are the most frequently used materials since they have high quality factor, large electromechanical coupling, low energy loss and stable physical and chemical properties. Different from quartz plates, which are cut from bulk single crystals, ZnO and AlN films are deposited on a silicon substrate or an electrode via a RF magnetron reactive sputtering system. This is compatible with the manufacturing process of modern IC technologies, thus FBARs can be integrated onto silicon wafers and mass produced at a very low price.

Similar to QCRs, FBARs can also be used as sensors. As a natural result of their high frequency, good quality factor and small size, FBAR sensors have a very high mass sensitivity and a low power consumption. Moreover, a large sensor array with each sensing pixel modified by different chemically or biochemically active films can be integrated onto a single silicon chip [39]. This is quite attractive to experimentalists working in the fields of biochemistry and chemistry sensing since it enables them to measure a number of analytes simultaneously. One further advantage of FBAR sensors is that they can be used to make wireless sensors working in remote locations and harsh environments. This is due to their ultrahigh operating frequency which lies in the low microwave range. In recent years, a variety of types of FBAR sensors have been
developed [41-47]. Zhang [41] reported a micromachined longitudinal wave FBAR mass sensor operating near 1 GHz in a liquid environment. The mass sensitivity of the FBAR microbalance is shown to be 782.7 cm$^2$/g, 50 times larger than that of a conventional bulk quartz crystal microbalance. The minimum detectable mass of the FBAR is estimated to be 2.8 ng/cm$^2$. Zhang [42] reported a mercuric ion sensor made of a ZnO FBAR. The FBAR sensor detected as low as $10^{-9}$ M Hg$^{2+}$ (0.2 ppb Hg$^{2+}$) in water. The sensor also has a good selectivity. The ions such as K$^+$, Ca$^{2+}$, Mg$^{2+}$, Zn$^{2+}$ and Ni$^{2+}$ had little or no effects on the resonant frequency. J. Weber [43] investigated applications of FBAR sensors in biosensing. Dynamic measurements in liquids were carried out and the adsorption of an antibody–antigen system was observed. Although this is the very first FBAR biosensor system operating in a liquid environment, it was found that the smallest detectable mass attachment is already better (2.3 ng/cm$^2$) than that of QCMs. An FBAR immunosensor used in detection of drug molecules was manufactured by G. Wingqvist [44]. FBAR sensors used as localized-mass sensors [45] and DNA [46] sensors have also been fabricated.

Figure 1.5 An FBAR mass sensor array worked in liquid environment [43]
1.2 Analytical and Numerical Modeling of Resonator Sensors

Developing accurate theoretical models that are able to describe the relation between the frequency changes and the material properties of the analytes is the key point of QCM applications. It is advantageous for us to take a brief historical review of the modeling of resonators and sensors. Generally, modeling resonator sensors is a 3D problem, so at first we present the three dimensional theory of linear piezoelectricity.

(a) Three-dimensional governing equations [2, 3, 48-50]:

The wave propagation and vibration of quartz are governed by the stress equation of motion:

\[ T_{ij,j} = \rho \ddot{u}_i, \]  

where \( T_{ij} \) is the stress tensor, \( u_i \) with \( i=1, 2, \) and 3 is the mechanical displacement vector, \( \rho \) is the mass density. The Cartesian tensor notation, the summation convention for repeated tensor indices, and the convention that a comma followed by an index denotes partial differentiation with respect to the coordinate associated with the index are used. A superimposed dot represents a time derivative.
To accurately describe the electromagnetic fields in resonators we need Maxwell’s equations. The operating frequency of most quartz resonators ranges from several to hundreds of megahertz and the wavelength of electromagnetic waves in this frequency domain is about 10 to 300 meters. However, the dimension of the resonator is only several millimeters to a few centimeters which is far less than the wavelength of the electromagnetic waves in resonators. For this reason, the quasi-static approximation for the electromagnetic fields is frequently adopted. Under this circumstance, we can neglect the coupling of the electric field and the magnetic field and only consider the quasi-static electric field. Quartz and other piezoelectric materials are perfect dielectrics, so Gauss’s law for electric charges takes the form:

$$D_{i,j} = 0,$$

(1.2)

where $D_i$ is the electric displacement.

The constitutive relations coupling the electrical and mechanical quantities are given by:

$$T_{ij} = c_{ijkl}S_{kl} - e_{ijkl}D_k, \quad D_i = \varepsilon_{ij}E_j - e_{ijkl}S_{ij},$$

(1.3)

where $c_{ijkl}$, $e_{ijkl}$ and $\varepsilon_{ij}$ are the elastic, piezoelectric and dielectric constants, and $S_{ij}$ and $E_i$ are the strain and electric fields, respectively.

The relation between the strain and mechanical displacements is:

$$S_{kl} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

(1.4)

The electric field-electric potential relationship is:

$$E_i = -\varphi_j,$$

(1.5)

where $\varphi$ is the electric potential.
Upon substitution of (1.3) into (1.1) and (1.2) and then applying (1.4) and (1.5), we obtain the equation of motion for piezoelectric crystals [2, 48]:

\[
\begin{align*}
    c_{ijk\ell} u_{k\ell,i} + \epsilon_{ikj} \phi_{jk}^\prime &= \rho \ddot{u}_i, \\
    \epsilon_{ikj} u_{k,i} - \epsilon_{ijk} \phi_{ij}^\prime &= 0,
\end{align*}
\]

(1.6)

Equation (1.6) represents a system of four coupled partial differential equations. When supplemented with proper boundary and initial conditions equations (1.1)-(1.6) constitute a standard boundary value problem. Classical mathematical physics methods such as the method of separation of variables, Fourier series and integrals, integral transformation and the theory of integral equations can be used to solve these boundary value problems. He and coworkers [51-57] have carried out a series of analytical studies on the resonant behavior and electrical representation of a variety of types of resonators based on 3D equations. Closed-form analytical solutions to resonators with different cross sections are obtained using Fourier analysis and special functions. However, exact solutions are available for a few relatively simple problems only. These are mostly vibrations of bodies with regular geometries or wave propagation in unbounded plates [58-60]. Due to the anisotropy of piezoelectric crystals and their electromechanical coupling, analyzing piezoelectric devices with general and irregular geometry using the above three-dimensional equations usually presents considerable mathematical challenges.

(b) Finite element modeling of resonators and sensors [61-73]

The most powerful and efficient tool to solve these complex problems is the finite element method (FEM). The major commercial FEM platforms such as ANSYS, ABAQUS and COMSOL have developed multiphysical modules to enable the capability
of analyzing complex multifield (electromagnetic, thermal, fluid-solid and elastic fields) coupling problems. Many researchers have already utilized these softwares to analyze electromechanical coupling problems of piezoelectric devices with complex configurations. For instance, Hempel et. al [61] performed FEM simulation to study of a LFE QCM using both ANSYS and COMSOL. They found that the electric permittivity strongly affect the sensor response. Campanella and collaborators [45, 62] investigated the nonlinear dependence of frequency shifts on the location and size of a surface localized mass on an FBAR sensor. Finite lateral dimensions of sensors induce spurious modes that degrade the quality factor and cause ripples in the admittance response curves. This phenomenon can not be explained in the framework of 1D approximation. Therefore many FEM simulations are done to investigate the effects of spurious modes [63-65]. Some specific configurations of FBAR sensors like columnar array type FBAR [64] and piezoelectric film with alternating poling domains [65] were suggested to suppress the spurious modes and were verified through FEM models.

Nevertheless, analyzing a real resonator often involves complicated modeling procedures and large amounts of running time to obtain meaningful results. Indeed, simple but accurate analytical models are still fascinating to both the industry and academic communities. Before getting involved in developing the novel analytical models, it is advantageous for us to have a historical review of the development of classical theoretical models for resonant sensors.

(c) Lumped parameter models and one-dimensional approximation [74-82]
For most QCMs and FBAR sensors, the lateral dimensions are much smaller than the thickness. This provides the physical basis of one-dimensional approximation. In this case both the electric fields and the mechanical displacement are dependent only on the thickness coordinate. The system of partial differential equations degenerates into ordinary differential equations for which exact solution can be obtained. The displacement distributions of the first few pure thickness-shear modes of an infinite quartz plate are shown in Fig. 1.7.

![Figure 1.7 Displacement of the first few pure thickness modes](image)

Classical analytical methods like the transform matrix method [59-60], the transmission line method [74-77] and the equivalent circuit method [78-81] are all based on one-dimensional approximation. The equivalent circuit method is probably the most widely used method due to its simplicity. In order to represent the electric response of resonators and sensors, different types of equivalent circuits have been proposed, for instance, Mason’s model [78], the Butterworth-Van Dyke (BVD) model [23] and the Krimholtz-Leedom-Matthaei (KLM) model [79]. These models are composed of different compositions of electrical parameters, i.e. resistance, static capacitance, motional capacitance and inductance. The values of these parameters can be extracted from the one-dimensional solutions. Engineers often use equivalent electric circuits to evaluate resonant frequencies and layer thicknesses of QCRs and FBARs in the design process. By
properly adding electric elements into equivalent circuits, the influence of surface mass layers or adjacent fluid can be included. For instance, modified BVD models have been developed for analysis of QCMs and FBAR sensors. Zhang and his collaborators [47] fabricated a prototype of an FBAR mass sensor which is able to work either in air or in liquid environment. An MBVD equivalent circuit model was proposed as shown in Figure 1.8. In this model, the mass loading, i.e. adsorbed particles, are represented by an additional inductance $L_3$. $L_2$ and $R_2$ represent the acoustic energy loss and mass loading by the liquid. The expressions of these parameters are obtained from the 1D analysis.

![Figure 1.8 Modified BVD circuits for a mass sensor working in air or fluid environment.][47]

However, no information concerning the relation between frequency shifts and physical properties of the determinand can be obtained from these models. To fill this gap, some practical formulae have been developed. In 1959, Sauerbrey [82] carried out the first theoretical analysis of quartz crystal mass sensors and obtained an approximation formula to evaluate the frequency shift caused by surface mass loads:

$$
\Delta f = -\frac{2f^2}{\rho v} \Delta m, 
$$

(1.7)
in which $\Delta m$ is the areal mass density of the surface layer, $f$ is the unperturbed frequency, $\rho$ is the mass density of the piezoelectric film, $v$ is the velocity of the acoustic wave in the film and $\Delta f$ the frequency shift. In 1985, Kanazawa [83] analyzed a quartz crystal liquid sensor and gave an approximate relation between the frequency shift and the viscosity of the fluid:

$$\Delta f = -f \sqrt{\frac{\eta_i \rho_i}{\mu Q \rho Q}}.$$ (1.8)

where $f$ is the unperturbed frequency of the QCM sensor, $\eta_i$ and $\rho_i$ are the viscosity and mass density of the liquid, $\mu Q$ and $\rho Q$ are the elastic modulus and mass density of the quartz crystal. These formulas are shown to be very accurate when relative frequency shift is less than 2%. Due to their simplicity, they have been extensively used in the design and analysis of QCMs [16-17].

All of the methods discussed so far are based on one-dimensional approximation of the vibration. In these models, the particle displacement and the electric field are functions of the thickness coordinate only. The pure thickness modes can exist only in unbounded layered structure. On the other hand, many other important effects observed in practical devices could not be explained in the framework of 1D approximation. For instance, experimental studies reveal that spurious modes exist in nearly all resonators and they cause activity dips in resonator’s admittance. Leaky waves and reduction of Q-factors are other performance deteriorations that are very important to resonator design. It can be seen the vibration is largely confined in the electroded region. This phenomenon is called energy trapping. This is of fundamental importance to device mounting since the resonator can be mounted to supporting structures on its edge without affecting its
resonant frequency. In-plane variations of operating modes are also important in the design of monolithic arrays of FBARs where sufficient energy trapping is necessary to avoid undesirable interactions among neighboring FBARs. These phenomena can only be identified by 2D or 3D analysis. Nevertheless, theoretical studies of in-plane variations of modes in bounded plates are relatively few because of the mathematical complications arising from material anisotropy and electromechanical coupling.

In order to describe the in-plane variations of resonators, two-dimensional theories for piezoelectric plate have been developed, which can be classified into two groups. One group is represented by Mindlin’s first-order, two-dimensional theory for piezoelectric plates. The other group is represented by scalar differential equations developed by Tiersten.

(d) Mindlin’s plate theory [84-104]

Conventional plate theories adopted different simplified approximations, i.e. Love assumption, Kirchhoff assumption, can be used to analyze low-frequency deformation mechanisms like deflection and bending. In order to analyze high frequency vibration of resonators, Mindlin [84-86] developed a new type of plate theory based on power series expansion of the displacement and electric potential with respect to the thickness coordinate where no such assumptions are employed. The series expression takes the form:

\[ u_i = \sum_{n=0}^{\infty} u_{i}^{(n)}(x_1, x_2) x_2^n, \quad \varphi = \sum_{n=0}^{\infty} \varphi^{(n)}(x_1, x_2) x_2^n. \]  

(1.9)
The coefficient functions in the series correspond to different orders of displacement and electric potential that represent fundamental deformation mechanisms such as flexile, thickness extension and thickness shear. They are dependent on in-plane coordinates only. By substituting the power series into the variational principle of linear piezoelectric body and then using the independence of each dependent variable variation, Mindlin first obtained the governing equations for each order of the displacement/potential. The governing equations for the lowest order displacement and electric potential are given by [86]:

\[
\begin{align*}
\bar{c}_1u_{1,11}^{(0)} + c_{55}u_{1,33}^{(0)} + (\kappa_1c_{56} + \kappa_3c_{14})u_{2,13}^{(0)} + (\bar{c}_{15} + c_{35})u_{3,13}^{(0)} \\
+ \kappa_1c_{56}u_{1,3}^{(1)} + \kappa_3c_{14}u_{3,1}^{(1)} + e_{35}\phi_{1,1}^{(0)} + e_{33}\phi_{3,3}^{(0)} + T_{11}^{(0)} = \rho\ddot{u}_1^{(0)}, \\
(\kappa_1c_{56} + \kappa_3c_{14})u_{1,13}^{(0)} + \kappa_3^2c_{56}u_{1,1}^{(0)} + \kappa_3^2c_{14}u_{3,1}^{(0)} + \kappa_3^2c_{34}u_{2,3}^{(0)} + \kappa_3^2c_{34}u_{3,3}^{(0)} \\
+ \kappa_3^2c_{56}u_{1,1}^{(1)} + \kappa_3^2c_{34}u_{3,1}^{(1)} + (\kappa_1c_{36} + \kappa_3c_{14})\phi_{1,1}^{(0)} + T_2^{(0)} = \rho\ddot{u}_2^{(0)}, \\
(\bar{c}_{13} + c_{55})u_{1,13}^{(0)} + \kappa_1c_{56}u_{2,1}^{(0)} + \kappa_3c_{34}u_{2,3}^{(0)} + c_{35}u_{3,11}^{(0)} + \bar{c}_{33}u_{3,33}^{(0)} \\
+ \kappa_1c_{56}u_{1,1}^{(1)} + \kappa_3c_{34}u_{3,3}^{(1)} + (c_{13} + e_{33})\phi_{1,1}^{(0)} + T_3^{(0)} = \rho\ddot{u}_3^{(0)}, \\
\bar{e}_{11}u_{1,11}^{(0)} + e_{35}u_{1,35}^{(0)} + (\kappa_1e_{36} + \kappa_3e_{14})u_{2,13}^{(0)} + (\bar{e}_{13} + e_{33})u_{3,13}^{(0)} \\
+ \kappa_1e_{36}u_{1,3}^{(1)} + \kappa_3e_{14}u_{3,1}^{(1)} - e_{11}\phi_{1,1}^{(0)} - e_{33}\phi_{3,3}^{(0)} + D^{(0)} = 0.
\end{align*}
\]

Equation (1.10) is a system of partial differential equations coupling the zero and first order of displacement and the zero order electric potential, which is still very complicated and difficult to solve. In practical applications, Equation (1.10) are usually simplified based on specific conditions. For example, some material constants of AT-cut quartz plates are very small and related displacement components are of less importance, thus they all can be neglected. Moreover, stress relaxation are often assumed since the normal stress component in thin plates can be neglected. When these simplifying assumptions are introduced, the system of equations takes a much simpler form and become sovable. As a result, many theoretical results can be obtained. When these
equations were derived, Mindlin first used them to obtain a series of frequency spectra of piezoelectric plates with different configurations. These equations have been employed in studying the effects of mode coupling and activity dip on the admittance of a rectangular resonator, see He and Yang [89]. Following the guidelines of Mindlin’s theory, Peter Lee [90] derived a system of two-dimensional equations to take into account the effect of initial stress on frequency shift of resonators. By combining Mindlin’s theory with the nonlinear thermoelectroelasticity theory [91], Tiersten [92-93] analyzed the temperature stability of frequencies of both singly- and doubly-rotated quartz resonators. With the aid of Mindlin’s plate theory, Tiersten also developed a systematic theory for evaluating the acceleration sensitivity based on the nonlinear electroelastic theory for the case of small biasing fields superposed on a finite initial deformation [94-96]. With this theory, he investigated the normal and in-plane acceleration sensitivities for a variety of types of resonators [97-100]. Recently, He and Yang [101] carried out detailed analysis of the second order in-plane acceleration sensitivity using these equations. Furthermore, Mindlin’s first-order plate equations were generalized to include surface acoustic impedances, which can be used to model piezoelectric plates with surface mechanical loads for acoustic wave sensor applications [102-103]. More applications of Mindlin’s plate theory can be found in [104] and references therein.

Despite its great success in analyzing fundamental behaviors of plate vibration, there are some obvious limitations. Mindlin’s plate equations were developed mainly for piezoelectric plates in coupled vibrations of the fundamental thickness-shear mode and the flexural mode. The equations are valid for the frequency range from zero to slightly above the fundamental thickness-shear frequency. When dealing with resonators
operating in the third, the fifth, and the higher-order overtone thickness-shear modes, Mindlin’s higher-order equations become increasingly complicated and inconvenient to use.

(e) Two-dimensional scalar differential equations for resonators [105-140]

The isotropic approximation, in which the anisotropy and electromechanical coupling are neglected, is probably the simplest approximation of quartz crystals. However, it is still possible to obtain some useful and interesting results. In this case, the governing equation degenerates into the standard wave equation. Based on this model, Martin and Hager [105] obtained the analytical solution for the case of an AT-cut QCM working in a liquid environment. They used the least squares method to fix the parameters needed and then analyzed the frequency shift and velocity amplitude of surface particles. Similarly, Cumpson and Seah [106] analyzed a circular QCM mass sensor and obtained the analytical expressions for radial and polar dependence of mass sensitivity.

A better approximate model of AT-cut QCM operating in the thickness-shear mode is given by McSkimin [107]. He noticed that the only large displacement is the one parallel to the $x_1$ axis, which is denoted by $u_1$ here. By neglecting the piezoelectric coupling and the coupling with other displacement components, he obtained the following equation:

$$c_{11}u_{1,11} + c_{66}u_{1,22} + c_{55}u_{1,33} = \rho u_1. \quad (1.11)$$

This equation takes into account the anisotropy of quartz crystals, so the accuracy is much better than the isotropic wave equation. Wilson [108], He [109] and Josse [110] calculated the frequencies of several AT-cut QCRs using this equation and found the
relative error of eigenfrequencies is on the order of $10^{-3}$. However, errors on this order is unacceptable for the design of high performance QCRs, so improving the accuracy of analytical models is still in need.

The next fundamental breakthrough concerning high accuracy analytical modeling of resonators is attributed to Tiersten. It is widely acknowledged that most QCRs work with the so-called essentially thickness mode, i.e. the in-plane variations of the electromechanical field are very slow and the frequency is still near the cutoff frequency of a pure thickness mode [108, 111]. H. F Tiersten and his collaborators first made use of these characteristics and carried out a comprehensive analysis of the essentially thickness vibrations [112-114] of the AT-cut resonators. After adopting several approximations based on the characteristics of these modes, they obtained scalar equations for a series of anharmonic modes near the fundamental and odd overtones of pure thickness modes. The scalar differential equation for singly-rotated quartz crystal resonators reads:

$$
M_n \frac{\partial^2 u_1}{\partial x_1^2} + c_{55} \frac{\partial^2 u_1}{\partial x_3^2} - \bar{c}_{66} \left( \frac{n\pi}{2h} \right)^2 u_1 + \rho \omega^2 u_1 = 0, \quad (1.12)
$$

where $M_n$ is a complicated function of $c_{ijkl}$, $e_{ijkl}$ and $e_{ij}$, $c_{55}$ is the elastic stiffness and $\bar{c}_{66}$ is the piezoelectric stiffened elastic constant, $u_1^n$ is the $n$th-order ($n=1, 3, 5, \ldots$) thickness-shear displacement, $x_1$ and $x_3$ are the in-plane coordinates of the plate.

From a theoretical point of view, the accuracy of this equation should be much higher since the influences of anisotropy, coupling with other displacements and piezoelectric effects have all been properly included in its coefficients. This judgment is conformed via a large number of concrete examples [115-128]. For instance, He and Yang [115] evaluated the first few eigenfrequencies of an AT-cut circular plate and found
that the relative error is on the order of $10^{-4}$ or better. Moreover, the predicted mode shapes are in perfect agreement with that observed using diffraction X-ray (XRD) spectroscope, as shown in Fig. 1.6.

Figure 1.9 (a) XRD topographs of a vibrating circular AT-cut quartz plate [108]

(b) Numerical evaluation of normal modes of a circular quartz crystal plate using the scalar differential equations [115]

As pointed out before, this equation is able to describe the essentially thickness-shear (TSh) vibrations near both the fundamental and higher overtones of pure TSh modes. However, it should be noticed that they cannot describe mode couplings among, e.g. thickness-shear and flexure, which is different from Mindlin’s plate theory.
The approach of deriving a single scalar equation for the resonator operating in thickness-shear modes only was later generalized to doubly-rotated quartz plates with general material anisotropy [129], and used in analysis of SC-cut quartz resonators which are fully anisotropic [131-137]. Tiersten originally considered the case of slow thickness-shear mode (C mode) only. Later, scalar differential equations for the other two thickness modes: thickness-stretch mode (A mode) and fast thickness-shear mode (B mode), in quartz resonators were also obtained [138-139].

As discussed above, very thin plates of crystals of class 6mm including ZnO and AlN have been made into thin-film resonators of which the dimensions are an order of magnitude smaller than quartz resonators with resonant frequencies an order of magnitude higher. Most thin film resonators operate with thickness-stretch modes. Scalar differential equations were also derived for these modes in thin plates of crystals of class 6mm in [140].

Due to their simplicity and high accuracy, the SDEs have been extensively used in theoretical analysis of QCRs. The scalar equation has only one dependent variable and two independent variables, so the standard method of separation of variables can be used to obtain exact solutions. Resonant frequencies and modes of resonators with different configurations can be calculated through free vibration analysis while the impedance/admittance and motional capacitance can be obtained via a scalar differential equation adapted for electrically forced vibrations.

1.3 Objectives of This Dissertation

In the derivation of the scalar equations, the plate surface is either free or carrying thin
electrodes and, as a result, they are not applicable to sensors. Compared to crystal resonators, resonator-based acoustic wave sensors are structurally more complicated. In addition to the electrodes on the crystal plates, the crystal plates always have additional surface mechanical loads such as mass layers or fluids. In this dissertation, we shall generalize these equations to include the effects of the surface loading and give some concrete applications of the new scalar equations in analyzing mass and liquid sensors.

The generalized two-dimensional scalar differential equations for the following three types of resonator sensors are obtained: (i) QCMs made of singly-rotated quartz plates; (ii) QCMs made of doubly-rotated quartz plates; and (iii) composite membrane-type FBAR sensors with piezoelectric film of class 6mm symmetry. The analysis of these three types of resonators are presented in chapter 2, chapter 3 and chapter 4, respectively. For each type of these sensors, three cases are considered: free vibration of unelectroded plates, free vibration of electroded plates and forced vibration of electroded plates. To show the possible applications in analyzing practical sensors, a few numerical examples are given at the end of each chapter.
Chapter 2
Scalar Differential Equations for Singly-rotated Quartz Plates

Singly-rotated quartz plates have relatively high symmetry and essentially thickness-shear modes can exist in these plates, so we first carry out comprehensive analysis for this type of QCMs in this chapter. In section 2.1, we present the three dimensional governing equations for AT-cut quartz plate which serve as the basis for following sections. It is advantageous for us to first obtain the exact analytical solutions to unbounded quartz plates since we can focus on the effects of surface conditions without obstruct of size effects. The analysis for free vibrations of both the unelecroded and the electroded unbounded plates and forced vibration of unbounded electrode plates are presented in Sections. 2.2-2.4. Then the closed-form asymptotic dispersion relations for bounded quartz plates with surface impedances are detailed in sections 2.5-2.7. Based on the correspondence of wavenumber-spatial partial derivatives of the series solutions, scalar differential equations are deduced by proper inversion of the obtained dispersion relations. Finally, a few numerical examples are carried out to show the application and accuracy of the newly developed scalar differential equations.

2.1 Three Dimensional Governing Equations

In this section, we present the three-dimensional equations for singly rotated quartz plates. The constitutive equations of singly-rotated cut plate take the following form:
\[
\begin{bmatrix}
T_{11} \\
T_{22} \\
T_{33} \\
T_{23} \\
T_{31} \\
T_{12}
\end{bmatrix}
= 
\begin{bmatrix}
c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\
c_{12} & c_{22} & c_{23} & c_{24} & 0 & 0 \\
c_{13} & c_{23} & c_{33} & c_{34} & 0 & 0 \\
c_{14} & c_{24} & c_{34} & c_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{55} & c_{56} \\
0 & 0 & 0 & 0 & c_{56} & c_{66}
\end{bmatrix}
\begin{bmatrix}
S_{11} \\
S_{22} \\
S_{33} \\
2S_{23} \\
2S_{13} \\
2S_{12}
\end{bmatrix}
- 
\begin{bmatrix}
e_{11} & 0 & 0 \\
e_{12} & 0 & 0 \\
e_{13} & 0 & 0 \\
e_{14} & 0 & 0 \\
0 & e_{25} & e_{35} \\
0 & e_{26} & e_{36}
\end{bmatrix}
\begin{bmatrix}
E_1 \\
E_2 \\
E_3
\end{bmatrix},
\] (2.1.1)

\[
\begin{bmatrix}
D_1 \\
D_2 \\
D_3
\end{bmatrix}
= 
\begin{bmatrix}
e_{11} & e_{12} & e_{13} & e_{14} & 0 & 0 \\
0 & 0 & 0 & 0 & e_{25} & e_{26} \\
0 & 0 & 0 & 0 & e_{35} & e_{36}
\end{bmatrix}
\begin{bmatrix}
S_{11} \\
S_{22} \\
S_{33} \\
2S_{23} \\
2S_{13} \\
2S_{12}
\end{bmatrix}
+ 
\begin{bmatrix}
0 & 0 & 0 & 0 & e_{22} & e_{23} \\
0 & e_{23} & e_{33} & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
E_1 \\
E_2 \\
E_3
\end{bmatrix},
\] (2.1.2)

The strain-displacement and electric field-electric potential relations are given by:

\[
\begin{bmatrix}
S_{11} \\
S_{22} \\
S_{33} \\
2S_{23} \\
2S_{13} \\
2S_{12}
\end{bmatrix}
= 
\begin{bmatrix}
u_{1,1} \\
u_{2,2} \\
u_{3,3} \\
u_{2,3} + u_{3,2} \\
u_{1,3} + u_{3,1} \\
u_{1,2} + u_{2,1}
\end{bmatrix}
\begin{bmatrix}
E_1 \\
E_2 \\
E_3
\end{bmatrix}
\quad \Rightarrow 
\begin{bmatrix}
\varphi_1 \\
\varphi_2 \\
\varphi_3
\end{bmatrix}.
\] (2.1.3)

Substituting from (2.1.3) into (2.1.1)-(2.1.2), we obtain the stress components and electric displacements expressed in terms of displacements and electric potential:
\[
\begin{align*}
\begin{bmatrix}
T_{11} \\
T_{12} \\
T_{13}
\end{bmatrix}
&= 
\begin{bmatrix}
0 & 0 & c_{11} \\
0 & c_{66} & c_{56} \\
0 & c_{56} & c_{55}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u_{1,1} \\
u_{2,1} \\
u_{3,1}
\end{bmatrix}
+ 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u_{1,2} \\
u_{2,2} \\
u_{3,2}
\end{bmatrix}
+ 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u_{1,3} \\
u_{2,3} \\
u_{3,3}
\end{bmatrix}
+ 
\begin{bmatrix}
e_{11} & 0 & 0 \\
e_{26} & e_{36} & 0 \\
e_{25} & e_{35} & 0
\end{bmatrix}
\begin{bmatrix}
\varphi_{1} \\
\varphi_{2} \\
\varphi_{3}
\end{bmatrix},
\end{align*}
\]
\[
\begin{align*}
\begin{bmatrix}
T_{21} \\
T_{22} \\
T_{23}
\end{bmatrix}
&= 
\begin{bmatrix}
0 & 0 & c_{12} \\
0 & c_{22} & c_{24} \\
0 & c_{24} & c_{32}
\end{bmatrix}
\begin{bmatrix}
u_{1,1} \\
u_{2,1} \\
u_{3,1}
\end{bmatrix}
+ 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u_{1,2} \\
u_{2,2} \\
u_{3,2}
\end{bmatrix}
+ 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u_{1,3} \\
u_{2,3} \\
u_{3,3}
\end{bmatrix}
+ 
\begin{bmatrix}
e_{26} & e_{36} & 0 \\
e_{12} & 0 & 0 \\
e_{14} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\varphi_{1} \\
\varphi_{2} \\
\varphi_{3}
\end{bmatrix},
\end{align*}
\]
\[
\begin{align*}
\begin{bmatrix}
T_{31} \\
T_{32} \\
T_{33}
\end{bmatrix}
&= 
\begin{bmatrix}
0 & 0 & c_{14} \\
0 & c_{24} & c_{34} \\
0 & c_{34} & c_{33}
\end{bmatrix}
\begin{bmatrix}
u_{1,1} \\
u_{2,1} \\
u_{3,1}
\end{bmatrix}
+ 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u_{1,2} \\
u_{2,2} \\
u_{3,2}
\end{bmatrix}
+ 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u_{1,3} \\
u_{2,3} \\
u_{3,3}
\end{bmatrix}
+ 
\begin{bmatrix}
e_{25} & e_{35} & 0 \\
e_{14} & 0 & 0 \\
e_{13} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\varphi_{1} \\
\varphi_{2} \\
\varphi_{3}
\end{bmatrix},
\end{align*}
\]
\[
\begin{align*}
\begin{bmatrix}
D_{1} \\
D_{2} \\
D_{3}
\end{bmatrix}
&= 
\begin{bmatrix}
e_{11} & 0 & 0 \\
e_{26} & e_{25} & 0 \\
e_{36} & e_{35} & 0
\end{bmatrix}
\begin{bmatrix}
u_{1,1} \\
u_{2,1} \\
u_{3,1}
\end{bmatrix}
+ 
\begin{bmatrix}
e_{12} & e_{14} & 0 \\
e_{26} & 0 & 0 \\
e_{36} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u_{1,2} \\
u_{2,2} \\
u_{3,2}
\end{bmatrix}
+ 
\begin{bmatrix}
e_{14} & e_{13} & 0 \\
e_{25} & 0 & 0 \\
e_{35} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u_{1,3} \\
u_{2,3} \\
u_{3,3}
\end{bmatrix}
- 
\begin{bmatrix}
e_{11} & 0 & 0 \\
e_{22} & e_{23} & 0 \\
e_{23} & e_{33} & 0
\end{bmatrix}
\begin{bmatrix}
\varphi_{1} \\
\varphi_{2} \\
\varphi_{3}
\end{bmatrix},
\end{align*}
\]

Substitution of (2.1.4)-(2.1.7) in (1.1) yields the displacement-electric potential differential equations governing the motion of singly-rotated quartz plates:
For essentially thickness-shear vibration, the in-plane variations of the field variables are much less than their variations along the thickness direction. Moreover, the piezoelectric coupling is small in quartz, so we can make two approximations about the electric field. First, neglect the in-plane components of electric displacements. Second, omit the dependence of the electric variables (\( \varphi \) and \( D_2 \)) on the in-plane coordinates \( x_1 \) and \( x_3 \), i.e., \( \partial/\partial x_1 = 0 \), \( \partial/\partial x_3 = 0 \), thus we get:

\[
D_1 = 0, \quad D_3 = 0, \tag{2.1.9}
\]

and

\[
D_2 = \varepsilon_{26} u_{1,2} - \varepsilon_{22} \varphi_{2}. \tag{2.1.10}
\]

Under these assumptions, the Gauss’s divergence relation for electric displacement becomes:

\[
D_{2,2} = 0. \tag{2.1.11}
\]
Substitution of (2.1.10) in (2.1.11) yields:

$$\varphi_{22} = \frac{\epsilon_{26}}{\epsilon_{22}} u_{1,22}.$$  \hspace{1cm} (2.1.12)

(2.1.8) can be simplified using (2.1.9)-(2.1.12):

$$\begin{bmatrix}
c_{11} & 0 & 0 & u_{1,11} \\
0 & c_{56} & c_{66} & u_{2,11} \\
0 & c_{55} & c_{56} & u_{3,11}
\end{bmatrix} + \begin{bmatrix}
\epsilon_{66} & 0 & 0 & u_{1,22} \\
0 & c_{22} & c_{24} & u_{2,22} \\
0 & c_{24} & c_{44} & u_{3,22}
\end{bmatrix} = \begin{bmatrix}
c_{11} & 0 & 0 & u_{1,13} \\
0 & c_{22} & c_{24} & u_{2,13} \\
0 & c_{24} & c_{44} & u_{3,13}
\end{bmatrix} \begin{bmatrix}
0 & 0 & 0 & u_{1,23} \\
0 & 2c_{24} & c_{23} + c_{44} & u_{2,23} \\
0 & c_{23} + c_{44} & 2c_{44} & u_{3,23}
\end{bmatrix} = \rho \begin{bmatrix}
\ddot{u}_1 \\
\ddot{u}_2 \\
\ddot{u}_3
\end{bmatrix}.$$  \hspace{1cm} (2.1.13)

where

$$\tilde{\epsilon}_{66} = \epsilon_{66} + \frac{\epsilon_{26}^2}{\epsilon_{22}}.$$  \hspace{1cm} (2.1.14)

Next we neglect the relatively small unimportant elastic constants \(c_{14}, c_{24}\) and \(c_{56}\), and obtain:
\[
\begin{bmatrix}
\sigma_{11} & 0 & 0 & u_{1,11} \\
\sigma_{66} & 0 & 0 & u_{1,22} \\
\sigma_{55} & c_{22} & 0 & u_{2,23} \\
0 & 0 & c_{13} + c_{55} & 0 \\
\end{bmatrix} + \begin{bmatrix}
0 & c_{12} & 0 & u_{1,12} \\
0 & c_{22} & 0 & u_{2,22} \\
0 & c_{23} & c_{44} & u_{2,23} \\
0 & 0 & c_{33} & 0 \\
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & u_{1,33} \\
0 & 0 & c_{44} & 0 \\
0 & c_{34} & c_{33} & 0 \\
0 & 0 & 0 & u_{3,33} \\
\end{bmatrix} = \begin{bmatrix}
0 & c_{12} + c_{66} & 0 & u_{1,12} \\
0 & 0 & c_{22} & u_{2,22} \\
0 & c_{23} + c_{44} & u_{2,23} \\
0 & 0 & 0 & u_{3,12} \\
\end{bmatrix}
\]

(2.1.15)

When the small elastic constants are neglected, the constitutive equations become:

\[
\begin{bmatrix}
\tilde{T}_{11} \\
\tilde{T}_{12} \\
\tilde{T}_{13} \\
\tilde{T}_{21} \\
\tilde{T}_{22} \\
\tilde{T}_{23} \\
\tilde{T}_{31} \\
\tilde{T}_{32} \\
\tilde{T}_{33} \\
\end{bmatrix} = \begin{bmatrix}
c_{11} & 0 & 0 & 0 \\
c_{66} & 0 & 0 & 0 \\
c_{55} & c_{22} & 0 & 0 \\
c_{13} & c_{55} & 0 & 0 \\
c_{22} & c_{66} & 0 & 0 \\
c_{23} & c_{23} & c_{44} & 0 \\
c_{13} & c_{55} & c_{33} & 0 \\
c_{12} & c_{55} & c_{33} & 0 \\
c_{13} & c_{55} & c_{33} & 0 \\
\end{bmatrix} \begin{bmatrix}
\tilde{u}_{1} \\
\tilde{u}_{2} \\
\tilde{u}_{3} \\
\tilde{u}_{1} \\
\tilde{u}_{2} \\
\tilde{u}_{3} \\
\tilde{u}_{1} \\
\tilde{u}_{2} \\
\tilde{u}_{3} \\
\end{bmatrix} = \begin{bmatrix}
\tilde{\rho} \tilde{u} \\
\tilde{\rho} \tilde{u} \\
\tilde{\rho} \tilde{u} \\
\tilde{\rho} \tilde{u} \\
\tilde{\rho} \tilde{u} \\
\tilde{\rho} \tilde{u} \\
\tilde{\rho} \tilde{u} \\
\tilde{\rho} \tilde{u} \\
\tilde{\rho} \tilde{u} \\
\end{bmatrix}
\]

(2.1.16)

(2.1.17)

(2.1.18)


2.2 Thickness Vibrations of Unbounded Plates

(a) Free vibrations of an unelectroded plate

Before performing the analysis for plates with finite lateral dimensions, we first carry out detailed analysis for infinite plates. As pointed out before, pure thickness-shear vibration can exist in unbounded plates only. The governing equations take a very simple form to which the exact solution can be obtained using the standard method of separation of variables. These solutions provide insights to the behaviors of bounded plates and the results will be used in future analysis. Three cases are considered: free vibration of unelectroded plate, free vibration of electrode plates and forced vibration of electrode plates driving by an alternative voltage.

For unbounded plates, the three dimensional problem degenerates into one-dimensional and the governing equations (2.1.8) are simplified as:

\[ c_{60} u_{1,22} + e_{26} \varphi_{22} = \rho \ddot{u}_1, \]
\[ e_{26} u_{1,22} - \kappa_{22} \varphi_{22} = 0. \] (2.2.1)

The constitutive equations are:

\[ T_{12} = c_{60} u_{12} + e_{26} \varphi_{2}, \]
\[ D_2 = e_{26} u_{12} - \kappa_{22} \varphi_{2}. \] (2.2.2)

In this work, the substances being examined is assumed to be perfect insulators so the electric boundary condition is open circuit. Their mechanical characteristics are represented by surface acoustic impedance. The concrete expressions of different types of surface loading can be derived from the corresponding governing equations. A few frequently used acoustic impedances are given in App. C. For an unelectroded plate, the boundary conditions are:

\[ T_{21} = \mp Z_t(\omega) \dot{u}_1, \quad D_2 = 0, \text{ at } x_2 = \pm h. \] (2.2.3)
A general solution of the following form is assumed:

\[ u_1 = A_n \sin(\eta_n x_2) e^{i\omega t}, \]
\[ \varphi = \frac{e_2}{e_{22}} A_n \sin(\eta_n x_2) e^{i\omega t}. \] (2.2.4)

One can check that equation (2.2.1), (2.2.2) and boundary condition \( D_2 = 0 \) are automatically satisfied. Substitution of (2.2.4) into equation (2.2.1) and (2.2.2) yields:

\[ \bar{c}_{66} u_{1,22} = -\rho \omega^2 u_1, \] (2.2.5)
\[ T_{21} = \bar{c}_{66} \eta_n A_n \cos(\eta_n x_2). \] (2.2.6)

where

\[ \eta_n = \sqrt{\frac{\rho \omega^2}{\bar{c}_{66}}}. \] (2.2.7)

Substituting (2.2.6) into the stress boundary condition in (2.2.3), we get

\[ \bar{c}_{66} \eta_n A_n \cos(\eta_n h) = -Z_t(\omega) i \omega A_n \sin(\eta_n h). \] (2.2.8)

Nontrivial solution of (2.2.8) requires:

\[ \tan(\eta_n h) = -\frac{\bar{c}_{66} \eta_n h}{Z_t(\omega) i \omega h}. \] (2.2.9)

Eq. (2.2.9) is the frequency-wavenumber equation for unbounded plate.

At this point, it is ready to obtain explicit approximate expressions of the roots of (2.2.9) that will be used later. We notice the fact that the acoustic impedances of most surface loads, i.e., viscous fluid, thin mass layer and small particles, are much less than that of the quartz plate. Consequently, solutions to (2.2.9) can be expressed as a summation of a small quantity and the solution of corresponding pure thickness shear mode, so

\[ \eta_n h = \frac{n\pi}{2} - \Delta_n. \] (2.2.10)
Substituting (2.2.10) into (2.2.9) and using the addition theorems for trigonometric functions, expanding the trigonometric functions as a power series in $\Delta_n$ and retaining terms linear in $\Delta_n$ and $Z_t(\omega)$, we obtain:

$$\Delta_n = -\frac{2hi\omega}{n\pi\varepsilon_{66}} Z_t(\omega),$$  \hspace{1cm} (2.2.11)

then we get the approximate expressions for the roots and frequencies:

$$\eta_n h = \frac{n\pi}{2} + \frac{2hi\omega}{n\pi\varepsilon_{66}} Z_t(\omega),$$  \hspace{1cm} (2.2.12)

$$\omega_n = \sqrt{\varepsilon_{66} \frac{n\pi}{\rho} h} \left[ 1 + \frac{4hi\omega_n}{n^2\pi^2\varepsilon_{66}} Z_t(\omega_n) \right].$$  \hspace{1cm} (2.2.13)

(b) General equations of unbounded electroded plates with surface impedance

In this case, the governing equations and constitutive equations are the same as that of unelectroded plate and given in (2.2.1) and (2.2.2). The top and bottom surfaces of the plate are covered by electrodes, so short-circuit electric boundary condition is used. Since the thickness of the surface electrodes are much less than that of quartz plate, the elastic effect can be neglected. For here and in the sequel, we only consider the inertial effects of the surface electrodes. The mechanical and electrical boundary conditions are:

$$T_{21} = \mp2\rho' h' \dddot{u}_1 \mp Z_t(\omega) \dot{u}_1, \ \varphi = 0, \ \text{at} \ x_2 = \pm h.$$  \hspace{1cm} (2.2.14)

Assume a solution to (2.2.1) of the following form:

$$u_1 = A_n \sin(\eta_n x_2) \exp(i\omega_n t),$$  \hspace{1cm} (2.2.15)

$$\varphi = \left[ \frac{e_{26}}{e_{22}} A_n \sin(\eta_n x_2) - \frac{e_{26}}{e_{22}} \frac{\sin(\eta_n h)}{h} A_n x_2 \right] \exp(i\omega_n t).$$  \hspace{1cm} (2.2.16)
It can be verified that (2.2.1)_2 and the electrical boundary condition are automatically satisfied.

Substituting from (2.2.15) and (2.2.16) into equation (2.2.1)_1 and (2.2.2)_1, we get:

\[
\bar{c}_{66}u_{1,22} = -\rho \omega_n^2 u_1, \quad (2.2.17)
\]

\[
T_{21} = \bar{c}_{66} \eta_n A_n \cos(\eta_n x_2) - \frac{e_{26}^2}{\epsilon_{22}} \sin(\eta_n h) A_n, \quad (2.2.18)
\]

where

\[
\eta_n = \sqrt{\rho \omega_n^2 / \bar{c}_{66}}. \quad (2.2.19)
\]

Substitution of (2.2.18) into the mechanical boundary condition in (2.2.14) yields:

\[
\bar{c}_{66} \eta_n A_n \cos(\eta_n h) - \frac{e_{26}^2}{\epsilon_{22}} \frac{\sin(\eta_n h)}{h} A_n = 2\rho' h' \omega_n^2 A_n \sin(\eta_n h) - Z_i(\omega_n)i \omega_n A_n \sin(\eta_n h). \quad (2.2.20)
\]

Nontrivial solutions of (2.2.20) require:

\[
\tan(\eta_n h) = \frac{\eta_n h}{k_{26}^2 + R \eta_n^2 h^2 - \frac{Z_i(\omega_n)i \omega_n h}{\bar{c}_{66}}}, \quad (2.2.21)
\]

where

\[
k_{26}^2 = \frac{e_{26}^2}{\bar{c}_{66} \epsilon_{22}}, \quad R = \frac{2\rho' h'}{\rho h}. \quad (2.2.22)
\]

Both the mass ratio \( R \) and the electromechanical coupling factor \( k_{26}^2 \) of quartz are very small and can be considered as a small quantities of the same order as \( Z_i \). Similar to the former case, we seek for a solution near the thickness-shear mode of the form:

\[
\eta_n h = \frac{n \pi}{2} - \Delta_n. \quad (2.2.23)
\]
Substituting from (2.2.23) into (2.2.21) and using the addition theorems for trigonometric functions, expanding the trigonometric functions as a power series in $\Delta_n$ and retaining terms linear in $\Delta_n, k_{26}^2, R$ and $Z_i(\omega)$, we obtain:

$$\Delta_n = \frac{n\pi}{2} R + \frac{2k_{26}^2}{n\pi} - \frac{2hi\omega_n}{n\pi c_{66}} Z_i(\omega_n). \quad (2.2.24)$$

Applying (2.2.24) in (2.2.23), we get:

$$\eta_n h = \frac{n\pi}{2} - \frac{n\pi}{2} R - \frac{2k_{26}^2}{n\pi} + \frac{2hi\omega_n}{n\pi c_{66}} Z_i(\omega_n). \quad (2.2.25)$$

$$\omega_n = \sqrt{\frac{c_{66}}{\rho}} \left[ 1 - R - \frac{4k_{26}^2}{n^2 \pi^2} + \frac{4hi\omega_n}{n^2 \pi^2 c_{66}} Z_i(\omega_n) \right]. \quad (2.2.26)$$

(c) Forced vibrations of an unbounded plate with surface impedances

QCM sensors are driving by an alternative voltage applied on the surface electrodes which are connected to readout circuits. Therefore, the solution for forced vibration is of great importance to practical design and analysis of sensors. For example, to facilitate the design of oscillating circuit, we first need to extract the electric parameters of equivalent circuits, such as admittance, impedance, motional capacitance and inductance. These parameters can only be predicted by forced vibration analysis.

For forced vibrations, the governing equations are also given by (2.2.1). In this work, we only consider steady state vibration in which the time variation of each quantity is given by an exponential factor $\exp(i\omega t)$. Neglecting the time factor, the boundary conditions take the form:

$$T_{21} = \pm2\rho h' \omega^2 u_1 \mp i\omega Z_i(\omega)u_1, \quad \varphi = \pm V, \quad \text{at} \quad x_2 = \pm h. \quad (2.2.27)$$

We seek a solution of the form:
\( u_1 = \tilde{u}_1 + Kx_2, \quad \phi = \tilde{\phi} + \frac{x_1}{h} V. \)  

(2.2.28)

\( K \) is introduced so that the inhomogeneous term is transformed from the boundary conditions into the differential equations. Substituting from (2.2.28) into (2.2.1) and (2.2.27), we get:

\[
\begin{align*}
& c_{66}\tilde{u}_{1,22} + e_{26}\tilde{\phi}_{,22} = -\rho\omega^2(\tilde{u}_1 + Kx_2), \\
& e_{26}\tilde{u}_{1,22} - e_{22}\tilde{\phi}_{,22} = 0,
\end{align*}
\]

(2.2.29)

\[
\begin{align*}
& c_{66}\tilde{u}_{1,2} + c_{66}K + e_{26}\tilde{\phi}_{,2} + e_{26}\frac{V}{h} = [2\rho'h'\omega^2 - i\omega Z_i(\omega)](\tilde{u}_1 + K)h, \quad x_2 = h, \\
& c_{66}\tilde{u}_{1,2} + c_{66}K + e_{26}\tilde{\phi}_{,2} + e_{26}\frac{V}{h} = -[2\rho'h'\omega^2 - i\omega Z_i(\omega)](\tilde{u}_1 - K), \quad x_2 = -h,
\end{align*}
\]

(2.2.30)

\( \tilde{\phi} = 0, \quad x_2 = \pm h. \)

The constant \( K \) selected so that the nonhomogeneous terms in (2.2.30) is eliminated:

\[
 c_{66}K + e_{26}\frac{V}{h} = 0.
\]

(2.2.31)

Thus:

\[
 K = -\frac{e_{26}V}{c_{66}h}. \tag{2.2.32}
\]

\( \tilde{u}_1 \) and \( \tilde{\phi} \) satisfy a new set of differential equations and boundary conditions:

\[
\begin{align*}
& c_{66}\tilde{u}_{1,22} + e_{26}\tilde{\phi}_{,22} + \rho\omega^2\tilde{u}_1 = \rho\omega^2\frac{e_{26}}{c_{66}}\frac{V}{h} x_2, \\
& e_{26}\tilde{u}_{1,22} - e_{22}\tilde{\phi}_{,22} = 0,
\end{align*}
\]

(2.2.33)

\[
 c_{66}\tilde{u}_{1,2} + e_{26}\tilde{\phi}_{,2} = \pm[2\rho'h'\omega^2 - i\omega Z_i(\omega)](\tilde{u}_1 \pm K), \quad \tilde{\phi} = 0, \quad \text{at} \quad x_2 = \pm h. \tag{2.2.34}
\]

The homogeneous form of Eqs. (2.2.33) and (2.2.34), i.e. \( V = 0, \quad K = 0, \) is the same as Eqs. (2.2.1) and (2.2.14). The solution to the homogeneous system is given by (2.2.15) and (2.2.16):
\[ \tilde{u}_1^{(n)} = A_n \sin(\eta_n x_2) \exp(i\omega_n t), \]
\[ \tilde{\varphi}^{(n)} = \left[ \frac{e_{26}}{e_{22}} A_n \sin(\eta_n x_2) - \frac{e_{26}}{e_{22}} \frac{\sin(\eta_n h)}{h} A_n x_2 \right] \exp(i\omega_n t), \]

where \( \eta_n \) satisfies Eq. (2.2.19), \( \omega_n \) is the \( n \)th eigenfrequency of the electroded plate. The first order approximations of \( \eta_n \) and \( \omega_n \) are given by (2.2.25) and (2.2.26).

From the theory of ordinary differential equations, we know that solutions to a nonhomogeneous equation can be expressed as an infinite series of the solutions to the corresponding homogeneous problem, that is:

\[ \tilde{u}_1(x_2, t) = \sum_{n=1}^{\infty} A_n \sin(\eta_n x_2) \exp(i\omega_n t), \]
\[ \tilde{\varphi}(x_2, t) = \frac{e_{26}}{e_{22}} \sum_{n=1}^{\infty} A_n \left[ \sin(\eta_n x_2) - \frac{x_2}{h} \sin(\eta_n h) \right] \exp(i\omega_n t). \]

Substituting from (2.2.36) into (2.2.33), we get:

\[ \sum_{n=1}^{\infty} \left( \rho \omega^2 - \tilde{c}_{66} \eta_n^2 \right) A_n \sin(\eta_n x_2) = -\rho \omega^2 K x_2. \]

Combining (2.2.19) in (2.2.37), we obtain:

\[ \sum_{n=1}^{\infty} \rho (\omega^2 - \omega_n^2) A_n \sin(\eta_n x_2) = -\rho \omega^2 K x_2. \]

Substitution of (2.2.36) in (2.2.34) yields:
After a series of rearrangement and simplification, (2.2.39) becomes:

$$
\sum_{n=1}^{\infty} \eta_n h A_n \cos(\eta_n h) = \left( \frac{R \rho h^2 \omega^2}{\bar{c}_{66}} + \frac{e_{26}^2}{\bar{c}_{66} \bar{\epsilon}_{22}} - \frac{i \omega Z_i(\omega)}{\bar{c}_{66}} \right) \sum_{n=1}^{\infty} A_n \sin(\eta_n h) + \left[ \frac{R \rho h^2 \omega^2}{\bar{c}_{66}} - \frac{i \omega Z_i(\omega)}{\bar{c}_{66}} \right] \bar{\kappa}_h, \quad x_2 = \pm h.
$$

(2.2.40)

In order to make use of further simplify (2.2.40), we first need an equivalent expression of (2.2.21), which is given by:

$$
\eta_n h \cos(\eta_n h) = \left[ \frac{e_{26}^2}{\bar{c}_{66} \bar{\epsilon}_{22}} + R(\eta_n h)^2 - \frac{Z_i(\omega_n) i \omega h}{\bar{c}_{66}} \right] \sin(\eta_n h).
$$

(2.2.41)

Substituting (2.2.41) into (2.2.40), we have

$$
\sum_{n=1}^{\infty} \left[ \frac{e_{26}^2}{\bar{c}_{66} \bar{\epsilon}_{22}} + R(\eta_n h)^2 - \frac{Z_i(\omega_n) i \omega h}{\bar{c}_{66}} \right] A_n \sin(\eta_n h) = \left( \frac{R \rho h^2 \omega^2}{\bar{c}_{66}} + \frac{e_{26}^2}{\bar{c}_{66} \bar{\epsilon}_{22}} - \frac{Z_i(\omega) i \omega h}{\bar{c}_{66}} \right) \sum_{n=1}^{\infty} A_n \sin(\eta_n h) + \left[ \frac{R \rho h^2 \omega^2}{\bar{c}_{66}} - \frac{i \omega Z_i(\omega)}{\bar{c}_{66}} \right] \bar{\kappa}_h. \quad (2.2.42)
$$

With the aid of (2.2.19), (2.2.42) can be simplified as:

$$
\sum_{n=1}^{\infty} \left[ R \rho h(\omega^2 - \omega_n^2) - \text{Im}[Z_i(\omega) \omega - Z_i(\omega_n) \omega_n] \right] A_n \sin(\eta_n h) = -[R \rho h \omega^2 - i \omega Z_i(\omega)] \bar{\kappa}_h.
$$

(2.2.43)

The undetermined amplitudes $A_n$ can be obtained by utilizing the orthogonality conditions of eigenfunctions. Due to the existence of nontrivial boundary conditions, the process is somewhat cumbersome here. The detailed procedure is given below. First, multiply both sides of (2.2.38) by $\sin(\eta_n x_2)$ and then integrate over $(-h, h)$, we get:
\[ \sum_{n=1}^{\infty} \rho (\omega^2 - \alpha_n^2) A_n \int_{-h}^{h} \sin(\eta_n x) \sin(\eta_m x) \, dx = -\rho \omega^2 \int_{-h}^{h} K x \sin(\eta_m x) \, dx. \]  \hspace{1cm} (2.2.44)

The explicit results of the integrals involved are given by:

\[ \int_{-h}^{h} \sin(\eta_n x) \sin(\eta_m x) \, dx = \frac{\sin(\eta_n - \eta_m) h}{\eta_n - \eta_m} - \frac{\sin(\eta_n + \eta_m) h}{\eta_n + \eta_m}, \quad \text{if } m \neq n, \]  \hspace{1cm} (2.2.45)

\[ \int_{-h}^{h} \sin(\eta_n x) \sin(\eta_m x) \, dx = h - \frac{\sin(\eta_n + \eta_m) h}{\eta_n + \eta_m} = h - \frac{\sin(\eta_m h \cos \eta_m h)}{\eta_m}, \quad \text{if } m = n, \]  \hspace{1cm} (2.2.46)

\[ \int_{-h}^{h} x \sin(\eta_m x) \, dx = \frac{2 \sin(\eta_m h)}{\eta_m^2} - \frac{2h}{\eta_m} \cos(\eta_m h). \]  \hspace{1cm} (2.2.47)

With the aid of (2.2.41), (2.2.45) can be further simplified as:

\[ \frac{\sin(\eta_n - \eta_m) h}{\eta_n - \eta_m} - \frac{\sin(\eta_n + \eta_m) h}{\eta_n + \eta_m} = \left[ -2Rh + 2i \frac{Z_1(\omega_n)\omega_n - Z_r(\omega_n)\omega_m}{c_{gg}(\eta_n^2 - \eta_m^2)} \right] \sin(\eta_m h) \sin(\eta_n h). \]  \hspace{1cm} (2.2.48)

Substitution of (2.2.46)-(2.2.48) into (2.2.44) yields:

\[ \rho (\omega^2 - \alpha_m^2) A_m \left[ h - \frac{\sin(\eta_m h) \cos(\eta_m h)}{\eta_m} \right] - \sum_{n=1}^{\infty} \frac{\rho (\omega^2 - \alpha_n^2) A_n}{\eta_n} \left[ 2Rh - 2i \frac{Z_1(\omega_n)\omega_n - Z_r(\omega_n)\omega_m}{\rho(\omega_n^2 - \alpha_n^2)} \right] \sin(\eta_m h) \sin(\eta_n h) = -\rho \omega^2 K \left[ \frac{2 \sin(\eta_m h)}{\eta_m^2} - \frac{2h}{\eta_m} \cos(\eta_m h) \right]. \]  \hspace{1cm} (2.2.49)

Multiply both sides of (2.2.43) by \( 2\sin(\eta_m h) \):

...
\[
\sum_{n=1}^{\infty} \left\{ 2R\rho (\omega^2 - \omega_n^2) - 2i[Z_r(\omega)\omega - Z_i(\omega_n)\omega_n] \right\} A_n \sin(\eta_n h)\sin(\eta_m h) = -[2R\rho \omega^2 - 2i\omega Z_r(\omega)]K h \sin(\eta_m h).
\] 

(2.2.50)

Taking sum of (2.2.49) and (2.2.50), we get

\[
\rho (\omega^2 - \omega_m^2) A_m \left( h - \frac{\sin(\eta_m h)\cos(\eta_m h)}{\eta_m} \right) + \left\{ 2R\rho (\omega^2 - \omega_m^2) - 2i[Z_r(\omega)\omega - Z_i(\omega_m)\omega_m] \right\} \sin^2(\eta_m h) A_m
\]

\[
+ \sum_{n=1}^{\infty} 2i \left[ \frac{\omega^2 - \omega_n^2}{\omega_m^2 - \omega_n^2} [Z_r(\omega_n)\omega_n - Z_i(\omega_m)\omega_m] - [Z_i(\omega)\omega - Z_i(\omega_m)\omega_m] \right] A_n \sin(\eta_n h)\sin(\eta_m h)
\]

\[
= -\rho \omega^2 K \left( \frac{2\sin(\eta_m h)}{\eta_m^2} - \frac{2h}{\eta_m} \cos(\eta_m h) \right) - [2R\rho \omega^2 - 2i\omega Z_r(\omega)]K h \sin(\eta_m h).
\] 

(2.2.51)

As can be seen from Appendix C, the acoustic impedance \(Z_r(\omega)\) of a general surface loading is not proportional to \(\omega\). The only exception is thin mass layer whose impedance \(Z_r(\omega)\) is really proportional to \(\omega\). For the former case the third term on the left hand side of (2.2.51) does not vanish, but for the latter case (2.2.51) vanishes. Thanks to the smallness of the surface impedance, neglecting this term only cause little errors to the solution of the undetermined amplitudes. Moreover, since most sensors operate near a resonant frequency, i.e. \(\omega \approx \omega_m\), this assumption cause even less error to the solution. Consequently, this assumption is universally assumed in this work unless otherwise indicated. We only need to keep in mind that the derivation is accurate for pure mass loadings while approximate when viscosity or elastic effect could not neglect. With this approximation, (2.2.51) becomes:
\begin{equation}
(\omega^2 - \omega_m^2)A_m \left[ 1 - \frac{\sin(\eta_m h) \cos(\eta_m h)}{\eta_m h} \right] + \left\{ 2R(\omega^2 - \omega_m^2) - \frac{2i}{\rho h} [Z_i(\omega) + Z_i(\omega_m)] \right\} \sin^2(\eta_m h) A_m \\
= -\omega^2 K_h \left[ \frac{2\sin(\eta_m h)}{\eta_m h^2} - \frac{2}{\eta_m h} \cos(\eta_m h) \right] - \left[ 2R\omega^2 - \frac{2i\omega}{\rho h} Z_i(\omega) \right] K_h \sin(\eta_m h).
\end{equation}

After lengthy derivation and simplification, the amplitude \( A_m \) is obtained:

\begin{equation}
A_m = (-1)^{m-1/2} \frac{\omega^2}{\omega^2 - \omega_m^2} \frac{e_{26}}{c_{66}} \frac{8V}{m^2 \pi^2} (1 + R).
\end{equation}

The procedure is detailed in Appendix D.

Meanwhile, the displacement and electric potential are obtained:

\begin{align}
u_1 &= \left[ -\frac{e_{26} V}{c_{66} h} x_2 + \sum_{n=1}^{\infty} A_n \sin(\eta_n x_2) \right] \exp(i\omega t), \\
\varphi &= \left\{ \frac{V}{h} x_2 + \frac{e_{26}}{\varepsilon_{22}} \sum_{n=1}^{\infty} A_n \left[ \sin(\eta_n x_2) - \frac{x_2}{h} \sin(\eta_n h) \right] \right\} \exp(i\omega t).
\end{align}
2.3 The Scalar Differential Equation of Unelectroded Plates

In real applications, quartz crystal microbalances are always partially covered by electrodes. A typical rectangular QCM is shown in Fig. 2.1. Since the lateral dimension of the crystal plate is much larger than its thickness, the portion of the plate covered by electrode and the rest without electrode can be treated separately. The solution for the two parts will be joint together via properly established continuation conditions. In this section, we perform detailed analysis of free vibration of a bounded unelectroded plate. The cross section of an unelectroded plate is shown in Fig. 2.2.

Due to the existence of side surfaces, pure thickness-shear vibration could not exist. The propagating bulk waves reflect back and forth and form lateral standing waves.
Therefore, we need to start our analysis from the three-dimensional governing equations for AT-cut quartz plates. As discussed in section 2.1, the displacement equation of motion and the constitutive relations are given by:

\begin{align}
\frac{c_{11}}{\rho} \ddot{u}_{1,11} + c_{66} \ddot{u}_{1,22} + c_{55} \ddot{u}_{1,33} + (c_{12} + c_{66}) \ddot{u}_{2,12} + (c_{13} + c_{55}) \ddot{u}_{3,13} + e_{26} \ddot{\varphi}_{22} &= \rho \ddot{u}_1, \\
\frac{c_{12}}{\rho} \ddot{u}_{2,11} + c_{22} \ddot{u}_{2,22} + c_{44} \ddot{u}_{2,33} + (c_{12} + c_{66}) \ddot{u}_{1,12} + (c_{23} + c_{44}) \ddot{u}_{3,23} &= \rho \ddot{u}_2, \\
\frac{c_{13}}{\rho} \ddot{u}_{3,11} + c_{33} \ddot{u}_{3,22} + c_{33} \ddot{u}_{3,33} + (c_{13} + c_{55}) \ddot{u}_{1,13} + (c_{23} + c_{44}) \ddot{u}_{3,23} + 2c_{34} \ddot{u}_{3,23} &= \rho \ddot{u}_3.
\end{align}

(2.3.1)

\begin{align}
T_{11} &= c_{11} u_{1,1} + c_{12} u_{2,2} + c_{13} u_{3,3}, \\
T_{12} &= c_{66} (u_{2,1} + u_{1,2}) + e_{26} \varphi_{22}, \\
T_{13} &= c_{55} u_{3,1} + c_{55} u_{1,3} + e_{25} \varphi_{22}, \\
T_{22} &= c_{12} u_{1,1} + c_{23} u_{2,2} + c_{33} u_{3,3}, \\
T_{23} &= c_{44} u_{3,2} + c_{44} u_{2,3} + c_{34} u_{3,3}, \\
T_{33} &= c_{13} u_{1,1} + c_{23} u_{2,2}.
\end{align}

(2.3.2)

The displacement component \( u_2 \) is shown to be one order of magnitude smaller than the principle displacement component \( u_1 \) [113]. An examination of (2.3.1) reveals that \( u_3 \) is two orders smaller than \( u_1 \). Based on these judgments we can infer the order of magnitude of the displacements and their derivatives involved in (2.3.1) and (2.3.2):

Zero order: \( u_1, u_{1,2}, u_{1,22} \);

First order: \( u_2, u_{1,1}, u_{1,3}, u_{2,2}, u_{1,12}, u_{2,22} \);

Second order: \( u_3, u_{2,1}, u_{2,3}, u_{3,2}, u_{1,11}, u_{1,13}, u_{1,33}, u_{2,12}, u_{2,23}, u_{3,23} \);

Third order: \( u_{3,1}, u_{3,3}, u_{2,11}, u_{2,33}, u_{3,23}, u_{2,23}, u_{3,23} \);

Fourth order: \( u_{3,11}, u_{3,13}, u_{3,33} \).

It can be verified that equation (2.3.1) contains only second order or higher order terms. For analysis of essentially thickness-shear mode, we can completely neglect this equation. Retaining terms up to second order in (2.3.1) and (2.3.2), we obtain the simplified governing equations and constitutive relations as follows:
\[ c_{11}\ddot{u}_{1,1} + (c_{12} + c_{66})\ddot{u}_{2,12} + c_{66}\ddot{u}_{1,22} + c_{26}\ddot{u}_{1,33} + e_{26}\varphi_{,22} = \rho\ddot{u}_{1}, \]
\[ c_{66}\ddot{u}_{2,11} + (c_{12} + c_{66})\ddot{u}_{1,12} + c_{22}\ddot{u}_{2,22} = \rho\ddot{u}_{2}, \]
\[ e_{26}\ddot{u}_{1,22} - \kappa_{22}\varphi_{,22} = 0. \]

The third order term \( c_{66}\ddot{u}_{2,11} \) in (2.3.3)2 is retained just to keep consistent with the analysis given by Tiersten [112].

Different from the case of pure thickness-shear vibrations, two displacement components are involved in (2.3.3) and (2.3.4). As a result, both the normal and the tangential acoustic impedances on the major surfaces must be considered. The electric boundary conditions are the same as that for pure TSh vibrations. Thus, the mechanical and electrical boundary conditions are of the form:

\[ T_{12} = c_{66}(u_{2,1} + u_{1,2}) + e_{26}\varphi_{,2}, \quad T_{22} = c_{12}u_{1,1} + c_{22}u_{2,2}, \]
\[ T_{11} = c_{11}u_{1,1} + c_{12}u_{2,2}, \quad T_{13} = c_{55}u_{1,1} + e_{25}\varphi_{,2}, \]
\[ T_{23} = c_{44}u_{3,2} + c_{44}u_{2,3}, \quad T_{33} = c_{13}u_{1,1} + c_{23}u_{2,2}, \]
\[ D_{2} = e_{26}u_{1,1} + e_{22}\varphi_{,2}. \]

where \( Z_n(\omega) \) and \( Z_t(\omega) \) are the normal and tangential impedances.

The electric potential \( \varphi \) is selected as:

\[ \varphi = \frac{e_{26}}{e_{22}}u_{1}. \]

It is easy to verify that \( \varphi \) satisfies both (2.3.3)3 and the electrical boundary conditions in (2.3.5) and (2.3.6).

Substituting from (2.3.7) into (2.3.3)1,2 and (2.3.4)1,2, we get:

\[ c_{11}\ddot{u}_{1,11} + (c_{12} + c_{66})\ddot{u}_{2,12} + c_{66}\ddot{u}_{1,22} + c_{26}\ddot{u}_{1,33} = \rho\ddot{u}_{1}, \]
\[ c_{66}\ddot{u}_{2,11} + (c_{12} + c_{66})\ddot{u}_{1,12} + c_{22}\ddot{u}_{2,22} = \rho\ddot{u}_{2}, \]
\[ T_{22} = c_{12}u_{1,1} + c_{22}u_{2,2}, \quad T_{12} = c_{66}u_{1,2} + c_{66}u_{2,1}. \]
Search for an approximate solution to (2.3.8) in the form:

\[ u_1 = B_1 \sin(\eta x_2) e^{-\xi x_3} \cos(\nu x_3) e^{i\omega t}, \]
\[ u_2 = B_2 \cos(\eta x_2) e^{-\xi x_3} \cos(\nu x_3) e^{i\omega t}. \] (2.3.10)

A few explanations to (2.3.10) are in order now. As can be seen in Fig. 2.1, the plate is entirely covered by electrodes along the \( x_3 \) axis while it is only partially covered along the \( x_1 \) axis. Accordingly, the lateral standing wave in the unelectroded portions must be oscillating along the \( x_3 \) axis and decay along the \( x_1 \) axis. Under this circumstance, it is reasonable for us to assume a solution of the form given by (2.3.10) in which the oscillating behavior is described using a cosine function and the decaying behavior is described by an exponential function. Experimental observations reveals that the decay- or wavenumber of the lateral standing waves are much smaller than the thickness wavenumber, thus they can be treated as small quantities.

Substituting (2.3.10) into (2.3.8), we get:

\[ (\rho \omega^2 + c_{11} \xi^2 - c_{66} \eta^2 - c_{53} \nu^2)B_1 + (c_{12} + c_{66}) \xi \eta B_2 = 0, \]
\[ -(c_{12} + c_{66}) \xi \eta B_1 + (\rho \omega^2 + c_{66} \xi^2 - c_{22} \eta^2)B_2 = 0. \] (2.3.11)

The system of equations has nontrivial solution if and only if the coefficient determinant vanishes, thus we get the frequency-wavenumber relation. It is a polynomial equation of degree four for both the frequency and the wavenumbers. As pointed out above, the in-plane decay- or wavenumber are one order of magnitude smaller than the thickness wavenumber. It is advantage for us to obtain a few approximated results at first. Zero order approximation of the thickness wavenumber can be obtained by neglecting all the terms containing \( \xi \) and \( \nu \), they are:

\[ \eta_1^2 = \frac{\rho \omega^2}{c_{66}}, \quad \eta_2^2 = \frac{\rho \omega^2}{c_{22}}. \] (2.3.12)
However, the relation between the amplitudes could not be obtained using zero order approximation since one of the coefficients is vanished \[112\]. It can only be fixed using the second order approximation, i.e. retaining the terms up to second order in \(\xi\) and \(\nu\). We get:

\[
B_2^{(1)} = \frac{r\xi}{\eta_1} B_1^{(1)}, \quad B_2^{(2)} = \frac{r\xi}{\eta_2} B_2^{(2)},
\]

where the superscripts “1” and “2” enclosed in a bracket represent the quantities corresponding to the wavenumber \(\eta_1\) and \(\eta_2\), respectively.

The relation between the two thickness wavenumbers can be derived from (2.3.12):

\[
\eta_2 = \kappa \eta_1.
\]

(2.3.14)

The constants appeared in (2.3.13) and (2.3.14) are given by:

\[
r = \frac{c_{12} + c_{66}}{c_{66} - c_{22}}, \quad \kappa = \sqrt{c_{66}/c_{22}}.
\]

(2.3.15)

In order to satisfy the boundary conditions in (2.3.5) and (2.3.6), we take a sum of the two asymptotic solutions in the form:

\[
u_1 = [B_1^{(1)} \sin(\eta_1 x_1) + B_1^{(2)} \sin(\eta_2 x_2)] e^{-\xi x_1} \cos(\nu x_3) e^{iot},
\]

\[
u_2 = [B_2^{(1)} \cos(\eta_1 x_1) + B_2^{(2)} \cos(\eta_2 x_2)] e^{-\xi x_1} \cos(\nu x_3) e^{iot}.
\]

(2.3.16)

Substituting from (2.3.16) into (2.3.9) and then into (2.3.5) and (2.3.6), and simplifying by (2.3.15), we obtain
\[ B_1^{(1)} \left( \tilde{c}_{66} \eta_1 - c_{66} r \frac{\xi^2}{\eta_1} \right) \cos(\eta_1 h) + B_2^{(2)} \xi (r \tilde{c}_{66} - c_{66}) \cos(\eta_2 h) = -i \omega Z_1(\omega) \left[ \sin(\eta_1 h) B_1^{(1)} + \frac{r \xi}{\eta_2} \sin(\eta_2 h) B_2^{(2)} \right], \]
\[ B_1^{(1)} \xi (c_{12} + c_{22} r) \sin(\eta_1 h) + B_2^{(2)} \left( c_{22} \eta_2 + c_{12} r \frac{\xi^2}{\eta_2} \right) \sin(\eta_2 h) = -i \omega Z_2(\omega) \left[ \frac{r \xi}{\eta_1} \cos(\eta_1 h) B_1^{(1)} + \cos(\eta_2 h) B_2^{(2)} \right]. \]

(2.3.17)

Since vibrations only near the pure thickness shear modes are of interest, the product of thickness wavenumber and the thickness can be expressed by

\[ \eta h = n \pi/2 + \alpha_n, \quad n = 1,3,5 \cdots, \]

(2.3.18)

where the first term on the right hand side is the unperturbed value and \( \alpha_n \) is a small quantity representing the perturbation.

Substituting from (2.3.18) into (2.3.17) and expanding the resulting trigonometric functions in powers of \( \alpha_n \), retaining terms linear in \( \alpha_n \), we get:

\[ B_1^{(1)} \left( \tilde{c}_{66} \eta_1 - c_{66} r \frac{\xi^2}{\eta_1} \right) (-1)^{\frac{n+1}{2}} \alpha_n + B_2^{(2)} \xi (r \tilde{c}_{66} - c_{66}) \cos(\eta_2 h) = -Z_1(\omega)(i \omega) \left[ (-1)^{\frac{n+1}{2}} B_1^{(1)} + \frac{r \xi}{\eta_2} \sin(\eta_2 h) B_2^{(2)} \right], \]
\[ B_1^{(1)} \xi (c_{12} + c_{22} r) (-1)^{\frac{n-1}{2}} + B_2^{(2)} \left( c_{22} \eta_2 + c_{12} r \frac{\xi^2}{\eta_2} \right) \sin(\eta_2 h) = -Z_2(\omega)(i \omega) \left[ \frac{r \xi}{\eta_1} (-1)^{\frac{n-1}{2}} \alpha_n B_1^{(1)} + \cos(\eta_2 h) B_2^{(2)} \right]. \]

(2.3.19)

Equation (2.3.19) is a system of linear homogeneous equations in \( B_1^{(1)} \) and \( B_2^{(2)} \). For a nontrivial solution the determinant of coefficients must vanish, thus we obtain
\[
\alpha_n = \frac{(c_{12} + c_{22}r)(c_{66} - r\bar{c}_{66})\cos\left(\frac{\kappa n \pi}{2}\right) - \frac{2c_{22}r^2}{\kappa n \pi} Z_i(\omega)(i\omega)\sin\left(\frac{\kappa n \pi}{2}\right)}{c_{66} \frac{n \pi}{2h} \left[ c_{22} \frac{\kappa n \pi}{2h} \sin\left(\frac{\kappa n \pi}{2}\right) + Z_n(\omega)(i\omega)\cos\left(\frac{\kappa n \pi}{2}\right) \right]} \xi^2 + \frac{\left[ Z_n(\omega)(i\omega)\cos\left(\frac{\kappa n \pi}{2}\right) + c_{22} \frac{\kappa n \pi}{2h} \sin\left(\frac{\kappa n \pi}{2}\right) \right] Z_i(\omega)(i\omega)}{c_{66} \frac{n \pi}{2h} \left[ c_{22} \frac{\kappa n \pi}{2h} \sin\left(\frac{\kappa n \pi}{2}\right) + Z_n(\omega)(i\omega)\cos\left(\frac{\kappa n \pi}{2}\right) \right]} .
\] (2.3.20)

Substituting (2.3.13), (2.3.18) and (2.3.20) into (2.3.11), and retaining terms linear in \(\alpha_n\), we get

\[
M_n(\omega)\xi^2 - c_{33} v^2 - \bar{c}_{66}\left(\frac{n \pi}{2h}\right)^2 + F_n(\omega) = 0 ,
\] (2.3.21)

where

\[
M_n(\omega) = c_{11} + (c_{12} + c_{66})r - \bar{c}_{66} \frac{n \pi}{h^2} \frac{(c_{12} + c_{22}r)(c_{66} - r\bar{c}_{66})\cos\left(\frac{\kappa n \pi}{2}\right) - \frac{2c_{22}r^2}{\kappa n \pi} Z_i(\omega)(i\omega)\sin\left(\frac{\kappa n \pi}{2}\right)}{c_{66} \frac{n \pi}{2h} \left[ c_{22} \frac{\kappa n \pi}{2h} \sin\left(\frac{\kappa n \pi}{2}\right) + Z_n(\omega)(i\omega)\cos\left(\frac{\kappa n \pi}{2}\right) \right]} ,
\] (2.3.22)

\[
F_n(\omega) = \rho \omega^2 - \bar{c}_{66} \frac{n \pi}{h^2} \frac{\left[ Z_n(\omega)(i\omega)\cos\left(\frac{\kappa n \pi}{2}\right) + c_{22} \frac{\kappa n \pi}{2h} \sin\left(\frac{\kappa n \pi}{2}\right) \right] Z_i(\omega)(i\omega)}{c_{66} \frac{n \pi}{2h} \left[ c_{22} \frac{\kappa n \pi}{2h} \sin\left(\frac{\kappa n \pi}{2}\right) + Z_n(\omega)(i\omega)\cos\left(\frac{\kappa n \pi}{2}\right) \right]} \approx \rho \omega^2 - \frac{2i \omega}{h} Z_i(\omega) .
\] (2.3.23)

Neglecting terms of third and higher order in the small in-plane decay- and wavenumbers and retaining terms linear in \(Z_n\), we obtain a closed-form dispersion relation accurate up to the second order of the in-plane wave/decay numbers:

\[
M_n(\omega)\xi^2 - c_{33} v^2 - \bar{c}_{66}\left(\frac{n \pi}{2h}\right)^2 + \rho \omega^2 - \frac{2i \omega}{h} Z_i(\omega) = 0 .
\] (2.3.24)
Numerical examples have shown these relations are very accurate for vibrations near the cutoff frequency of pure thickness-shear vibration. By properly identify the in-plane decay- and wavenumbers with partial derivatives with respect of the in-plane coordinates, we finally obtain the scalar differential equations:

\[ M_0 \frac{\partial^2 u_1}{\partial x_1^2} + c_{55} \frac{\partial^2 u_1}{\partial x_3^2} - \bar{c}_{66} \left( \frac{n \pi}{2h} \right)^2 u_1 + \rho \omega^2 u_1 - \frac{2i \omega}{h} Z_r(\omega) u_1 = 0. \] (2.3.25)

Some researchers further identify the quantity \( n\pi/2h \) with partial derivative with respect to thickness coordinate \( x_2 \) and get:

\[ M_0 \frac{\partial^2 u_1}{\partial x_1^2} + c_{55} \frac{\partial^2 u_1}{\partial x_3^2} + \bar{c}_{66} \frac{\partial^2 u_1}{\partial x_2^2} + \rho \omega^2 u_1 - \frac{2i \omega}{h} Z_r(\omega) u_1 = 0. \] (2.3.26)

(2.3.26) is a three-dimensional scalar differential equation that can be used to investigate resonator sensors with slow varying thickness.
2.4 The Scalar Differential Equation of Electroded Plates

In this section, we analyze quartz plates covered by surface electrodes. The cross section is shown in Fig. 2.3.

![Figure 2.3 cross section of an electrode quartz plate](image)

The governing equations and the constitutive equations are given by (2.3.3) and (2.3.4), respectively. As discussed in Sec. 2.2, the inertia effects of the electrodes and the surface acoustic impedances are included in the mechanical boundary conditions. We consider free vibrations of an electroded plate at first, so the electric boundary condition is short-circuit, and we get:

\[ T_{22} = -2\rho' h' \ddot{u}_2 - Z_n(\omega)u_2, \quad T_{21} = -2\rho' h' \ddot{u}_1 - Z_i(\omega)\dot{u}_1, \quad \phi = 0, \quad \text{at} \quad x_2 = h, \quad (2.4.1) \]

\[ T_{22} = 2\rho' h' \ddot{u}_2 + Z_n(\omega)u_2, \quad T_{21} = 2\rho' h' \ddot{u}_1 + Z_i(\omega)\dot{u}_1, \quad \phi = 0, \quad \text{at} \quad x_2 = -h, \quad (2.4.2) \]

The electric potential \( \phi \) is chosen to be:

\[ \phi = \frac{e_{26}}{e_{22}} u_1 + C x_2, \quad (2.4.3) \]

where \( C \) is a constant to be determined.

In this case, \( \phi \) satisfies (2.3.3) only. In order to satisfy the electrical boundary conditions in (2.4.1) and (2.4.2), the following equations must hold:

\[ \frac{e_{26}}{e_{22}} u_1(h) + Ch = 0, \quad \frac{e_{26}}{e_{22}} u_1(-h) - Ch = 0. \quad (2.4.4) \]
The displacement $u_1$ is antisymmetric about $x_2$, i.e. $u_i(h) = -u_i(-h)$, so there is only one independent equation in (2.4.4). The constant $C$ can be solved:

$$C = -\frac{e_{66} u_i(h)}{\varepsilon_{22} h}.$$  \hspace{1cm} (2.4.5)

Now the electric potential takes the form:

$$\varphi = \frac{e_{66} u_1}{\varepsilon_{22} h} - \frac{e_{66} u_i(h)}{\varepsilon_{22} h} x_2.$$ \hspace{1cm} (2.4.6)

Substituting from (2.4.6) into (2.3.3)$_{1,2}$ and (2.3.4), we get:

$$c_{11} u_{i,11} + (c_{12} + c_{66}) u_{2,12} + \tilde{\varepsilon}_{66} u_{i,22} + c_{55} u_{1,33} = \rho \tilde{u}_1,$$

$$c_{66} u_{2,11} + (c_{12} + c_{66}) u_{1,12} + c_{22} u_{2,22} = \rho \tilde{u}_2.$$ \hspace{1cm} (2.4.7)

$$T_{22} = c_{12} u_{1,1} + c_{22} u_{2,2},$$

$$T_{12} = \tilde{\varepsilon}_{66} u_{1,2} + c_{66} u_{2,2} - \frac{e_{26}^2 u_1(h)}{\varepsilon_{22} h},$$ \hspace{1cm} (2.4.8)

$$D_2 = c_{66} u_1(h).$$

Seek for an approximate solution to (2.4.7) in the form:

$$u_1 = B_1 \sin(\eta x_2) \cos(\xi x_1) \cos(\nu x_3) e^{iat},$$

$$u_2 = B_2 \cos(\eta x_2) \cos(\xi x_1) \cos(\nu x_3) e^{iat}. \hspace{1cm} (2.4.9)$$

Since the major surfaces of an electroded plate is entirely covered by electrodes, the solution is chosen to be oscillating in both $x_1$ and $x_2$ directions.

Substitution of (2.4.9) into (2.4.7) yields

$$(c_{11} \xi^2 + \tilde{\varepsilon}_{66} \eta^2 + c_{55} \nu^2 - \rho \omega^2) B_1 + (c_{12} + c_{66}) \xi \eta B_2 = 0,$$

$$(c_{12} + c_{66}) \xi \eta B_1 + (c_{66} \xi^2 + c_{22} \eta^2 - \rho \omega^2) B_2 = 0.$$ \hspace{1cm} (2.4.10)

The following steps are parallel to the steps given in last section.

From zero order approximation, we get
\[ \eta_i^2 = \frac{\rho \omega^2}{c_{66}}, \quad \eta_2^2 = \frac{\rho \omega^2}{c_{22}}. \]  

(2.4.11)

From the second order approximation, we get

\[ B^{(1)}_2 = \frac{r_2}{\eta_1} B^{(1)}_1, \quad B^{(2)}_1 = \frac{\xi}{\eta_2} B^{(2)}_2. \]  

(2.4.12)

Relations in (2.3.14) and (2.3.15) still hold.

In order to satisfy the boundary conditions in (2.4.1) and (2.4.2), we take a sum of the two asymptotic solutions of the differential equations in the form:

\[
\begin{align*}
\eta_1 &= [B^{(1)}_1 \sin(\eta_1 x_2) + B^{(2)}_1 \sin(\eta_2 x_2) \cos(\xi \lambda_1)] \cos(\xi x_3) e^{i \omega t}, \\
\eta_2 &= [B^{(1)}_2 \cos(\eta_1 x_2) + B^{(2)}_2 \cos(\eta_2 x_2) \sin(\xi \lambda_1)] \cos(\xi x_3) e^{i \omega t}. 
\end{align*}
\]  

(2.4.13)

Substituting from (2.4.13) into (2.4.8) and then into (2.4.1) and (2.4.2), and simplifying by (2.4.12), we obtain

\[
\begin{align*}
B^{(1)}_1 \left\{ \xi c_{66} h + c_{66} r_2 \frac{\xi^2}{\eta_1} \right\} \cos(\eta_1 h) - \left\{ \frac{e_{26}^2 h}{e_{22}} + 2 \rho' \omega^2 - Z_l(\omega) i \omega \right\} \sin(\eta_1 h) + \\
B^{(2)}_2 \left\{ \xi (c_{66} - r_2) \cos(\eta_2 h) + \frac{\xi r}{\eta_2} \frac{e_{26}^2 h}{e_{22}} + 2 \rho' \omega^2 - Z_l(\omega) i \omega \right\} \sin(\eta_2 h) = 0, \\
B^{(1)}_1 \left\{ \xi (c_{12} + c_{22} r) \sin(\eta_1 h) + [2 \rho' \omega^2 - Z_\omega(\omega) i \omega] \frac{r_2}{\eta_1} \cos(\eta_1 h) \right\} + \\
B^{(2)}_2 \left\{ c_{22} \eta_2 - c_{12} r \frac{\xi^2}{\eta_2} \right\} \sin(\eta_2 h) + [2 \rho' \omega^2 - Z_\omega(\omega) i \omega] \cos(\eta_2 h) = 0. 
\end{align*}
\]  

(2.4.14)

(2.4.15)

For vibrations near the pure thickness shear mode, the roots of the characteristic equation can be expressed as:

\[
\eta_l h = \frac{n \pi}{2} - \frac{n \pi}{2} R - \frac{2 k_{26}^2}{n \pi} + \frac{2 h i \omega}{n \pi c_{66}^2} Z_l(\omega) + \theta_n = \frac{n \pi}{2} + \beta_n, \quad n = 1, 3, 5, \ldots, \]  

(2.4.16)

where \( \theta_n \) and \( \beta_n \) are small quantities.
Substituting from (2.4.16) into (2.4.14) and (2.4.15) and expanding the resulting
trigonometric functions in powers of $\beta_n$, retaining terms linear in $\beta_n$, we get

\[ B_1^{(1)} \left[ \left( \bar{c}_{66}\eta_1 + c_{66}r \frac{\xi^2}{\eta_1} \right) (-1)^{\frac{n+1}{2}} \beta_n - L(-1)^{\frac{n-1}{2}} \right] + \]

\[ B_2^{(2)} \left[ (c_{66} - r\bar{c}_{66}) \cos(\eta_2 h) + \frac{r}{\eta_2} L \sin(\eta_2 h) \right] = 0, \]

where

\[ L = \frac{\bar{c}_{66}}{h} (k_{26}^2 + R\eta_1^2 h^2) - i\omega Z_r(\omega), \]

Equations (2.4.17) and (2.4.18) constitute a system of linear homogeneous equations in
$B_1^{(1)}$ and $B_2^{(2)}$. For this system to have nontrivial solutions, its determinant of coefficients
must vanish. This results in an approximate dispersion relation. Retaining the terms linear
in $\beta_n$, we get:

\[ \beta_n = \frac{(c_{12} + c_{22}r)(r\bar{c}_{66} - c_{66}) \cot(\eta_2 h) \xi^2 - k_{26}^2}{\bar{c}_{66}c_{22}\eta_1\eta_2} - \eta_1 h R + \frac{Z_r(\omega) i\omega h}{\bar{c}_{66}\eta_1 h} \]

\[ \equiv \frac{4h^2(c_{12} + c_{22}r)(r\bar{c}_{66} - c_{66}) \cot\left( \frac{k\eta_1}{2} \right) \xi^2 - 2k_{26}^2}{c_{66}c_{22}\kappa(n\pi)^2} \xi^2 - \frac{2k_{26}^2}{n\pi} - \frac{n\pi}{2} R + \frac{2Z_r(\omega) i\omega h}{c_{66}n\pi}, \]

\[ \theta_n = \beta_n + \frac{n\pi}{2} R + \frac{2k_{26}^2}{n\pi} - \frac{2hi\omega}{n\bar{c}_{66}} Z_r(\omega) = \frac{4h^2(c_{12} + c_{22}r)(r\bar{c}_{66} - c_{66}) \cot\left( \frac{k\eta_1}{2} \right)}{c_{66}c_{22}\kappa(n\pi)^2} \xi^2. \]

Substituting from (2.4.12), (2.4.16), (2.4.20) into (2.4.10) and retaining terms linear in $\beta_n$, we get
\[
M_n \dddot{z}^2 + c_{55} \ddot{v}^2 + c_{66} \left( \frac{n \pi}{2h} \right)^2 - \rho \dot{\omega}^2 + \frac{2}{h} i \omega Z_i(\omega) = 0 .
\]

(2.4.22)

where

\[
M_n = c_{11} + (c_{12} + c_{66})r = \frac{4(c_{12} + c_{22})r(c_{66} - r \bar{c}_{66}) e^{\text{col} \left( \frac{kn \pi}{2} \right)}}{c_{22} k \pi r},
\]

(2.4.23)

\[
c_{66} = c_{66} \left( 1 - 2R - \frac{8k_{26}^2}{n^2 \pi^2} \right).
\]

(2.4.24)

Inverting (2.4.22) using the same criteria discussed in last section, we can obtain the scalar equation in the frequency domain:

\[
M_n \frac{\partial^2 u_1}{\partial x_1^2} + c_{55} \frac{\partial^2 u_1}{\partial x_3^2} + c_{66} \frac{\partial^2 u_1}{\partial x_2^2} + \rho \dot{\omega}^2 + \frac{2}{h} i \omega Z_i(\omega) u_1 = 0 .
\]

(2.4.25)

The three dimensional form is:

\[
M_n \frac{\partial^2 u_1}{\partial x_1^2} + c_{55} \frac{\partial^2 u_1}{\partial x_3^2} + c_{66} \frac{\partial^2 u_1}{\partial x_2^2} + \rho \dot{\omega}^2 + \frac{2}{h} i \omega Z_i(\omega) u_1 = 0 .
\]

(2.4.26)

2.5 Forced Vibrations of a Bounded Electroded Plate

In this section, we give detailed derivation for the scalar equation governing the forced vibration of a bounded electrode plate.

Rearranging (2.2.53), we can get:

\[
A_m \omega^2 - A_m \omega^2 = (-1)^m \frac{\beta^2}{2} \frac{\epsilon_{26}}{\epsilon_{66}} \frac{8V}{m^2 \pi^2} (1 + R) .
\]

(2.5.1)

Multiply both sides of (2.5.1) by \( \rho \) and then use the relation (2.2.19), we have

\[
-\epsilon_{66} \eta_m A_m + \rho \omega^2 A_m = (-1)^m \frac{\beta^2}{2} \frac{\rho \epsilon_{26}}{\epsilon_{66}} \frac{8V}{m^2 \pi^2} (1 + R) .
\]

(2.5.2)

If identify the magnitude \( A_m \) with the dependent variable \( u_1 \) in (2.4.25), we get
\[
M_n \frac{\partial^2 u_t}{\partial x_1^2} + c_{55} \frac{\partial^2 u_t}{\partial x_3^2} - c_{66} \left( \frac{n \pi}{2h} \right)^2 u_t + \left[ \rho \omega^2 - \frac{2}{h} i \omega Z_i(\omega) \right] u_t = (-1)^{m/2} \frac{\rho \omega^2 e_{26}}{c_{66}} \frac{8V}{m^2 \pi^2} (1 + R). \tag{2.5.3}
\]

However, this expression will cause inconvenience when we apply the continuity conditions of the dependent variables at the interface between an electroded and an unelectroded region. This is because the inhomogeneous term must be included in the electroded side. However, this inconvenience can be resolved by properly including the influence of the inhomogeneous term in the differential equation. The inhomogeneous displacement field is given by:

\[
\vec{u}_t^0 = -\frac{e_{26} V}{c_{66} h} x_2. \tag{2.5.4}
\]

Expanding (2.5.4) in a series of thickness eigenfunctions:

\[
\vec{u}_t^0 = \sum_{n=1,3,5}^\infty A_n^0 \sin(n \eta x_2), \tag{2.5.5}
\]

where \(A_n^0\) are undetermined amplitudes of the inhomogeneous term.

On the surfaces of the plate, we have:

\[
\sum_{n=1,3,5}^\infty A_n^0 \sin(n \eta h) = -\frac{e_{26} V}{c_{66}}, \quad \sum_{n=1,3,5}^\infty A_n^0 \sin(-n \eta h) = \frac{e_{26} V}{c_{66}}. \tag{2.5.6}
\]

Multiply both sides of (2.5.5) by \(\rho \sin(n \eta x_2)\) and integrate over \([-h, h]\):

\[
\sum_{n=1,3,5}^\infty \int_{-h}^{h} A_n^0 \rho \sin(n \eta x_2) \sin(n \eta x_2) \, dx_2 = -\frac{e_{26} V}{c_{66}} \int_{-h}^{h} \rho \sin(n \eta x_2) \, dx_2. \tag{2.5.7}
\]

In order to implement the orthogonality condition, we need to consider the boundary conditions. Multiplying both sides of the two equations in (2.5.6) by \(2 \rho h' \sin(\pm \eta_m h)\) and adding the results into (2.5.7), we get
\[
\sum_{n=1,3,5} A_n^0 \left[ \int_{-h}^{h} \rho \sin(\eta_m x_2) \sin(\eta_m x_2) dx_2 + 2 \rho h' \sin(\eta_m h) \sin(\eta_m h) + 2 \rho h' \sin(-\eta_m h) \sin(-\eta_m h) \right]
\]
\[
= -\frac{e_26}{c_{66}} V \int_{-h}^{h} \rho x_2 \sin(\eta_m x_2) dx_2 + 2 \rho h' \sin(\eta_m h) - 2 \rho h' \sin(-\eta_m h).
\]

After lengthy simplifying procedure, the approximate expression for \( A_n^0 \) are obtained:

\[
A_n^0 = -(-1)^{m/2} \frac{e_{26}}{c_{66}} \frac{8V}{m^2 \pi^2} \left[ 1 + \left( \frac{12}{m^2 \pi^2} - 1 \right) k_{26}^2 + R \right].
\]

The detailed process are given in Appx. D.

Multiplying both sides of (2.5.9) by \( \rho(\omega^2 - \omega_m^2) \), we get:

\[
\rho\omega^2 A_m^0 - \rho \omega_m A_m^0 = -(-1)^{m/2} \frac{e_{26}}{c_{66}} \rho(\omega^2 - \omega_m^2) \frac{8V}{m^2 \pi^2}(1 + R).
\]

Applying (2.2.19) in (2.5.10), we obtain:

\[
\rho\omega^2 A_m^0 - \tilde{c}_m \eta_m^2 A_m^0 = -(-1)^{m/2} \frac{e_{26}}{c_{66}} \rho(\omega^2 - \omega_m^2) \frac{8V}{m^2 \pi^2}(1 + R).
\]

The influence of the homogeneous term can be taken into account by adding \( A_m^0 \) to \( A_m \) in (2.5.2):

\[
-\tilde{c}_m \eta_m^2 A_m + \rho \omega^2 A_m = -(-1)^{m/2} \frac{e_{26}}{c_{66}} \frac{8V}{m^2 \pi^2}(1 + R),
\]

where

\[
\tilde{A}_m = (A_m + A_m^0).
\]

Finally, we obtain the scalar equation for the forced vibration by identifying \( \tilde{A}_m \) with the transversely varying \( \bar{u}_1^n \) in (2.4.25):

\[
M_n \frac{\partial^2 u_1}{\partial x_1^2} + c_{55} \frac{\partial^2 u_1}{\partial x_1^2} - c_{66} \left( \frac{n \pi}{2h} \right)^2 u_1 + \left[ \rho \omega^2 - \frac{2}{h} i \omega Z_i(\omega) \right] u_1
\]
\[
= -(-1)^{m/2} \rho \omega_n^2 \frac{e_{26}}{c_{66}} \frac{8V}{n^2 \pi^2}(1 + R) \exp(i\omega t).
\]

\[\text{(2.5.14)}\]
Now the continuity conditions at the interface of an electrode region and an unelectroded region can be directly expressed as the continuity of the displacement in (2.5.14) and (2.3.24), respectively.

The three dimensional scalar differential equation for forced vibration is:

\[
M_n \frac{\partial^2 u_1}{\partial x_1^2} + c_{55} \frac{\partial^2 u_1}{\partial x_3^2} + c_{66} \frac{\partial^2 u_1}{\partial x_2^2} + \left[ \rho \omega^2 - \frac{2}{h} i \omega Z_i(\omega) \right] u_1 = (-1)^{n-1/2} \rho_h \frac{e_{36}}{c_{66}} \frac{8V}{n^2 \pi^2} (1 + R) \exp(i \omega t).
\]

### 2.6 Numerical Examples

**Example 1 Quartz crystal viscosity sensor**

We first use the new scalar equations to calculate frequency shift of a QCM liquid sensor. A circular, partially electroded plate of AT-cut quartz is shown in Fig. 2.4. It has thickness 2\(h\) and mass density \(\rho\). The top and bottom electrodes are identical and are with thickness 2\(h'\) and mass density \(\rho'\). The plate is immersed in a fluid.

![Figure 2.4 Geometry of a circular AT-cut quartz crystal sensor](image-url)

The viscosity and the density of the fluid are \(\mu\) and \(\rho_l\), respectively. As can be found in Appendix C, the characteristic impedance is
\[ Z_i(\omega) = (1+i)\sqrt{\frac{\mu \rho c_{66}}{2}}. \]  

(2.6.1)

Specifically, we consider the case when the elliptical electrode boundary is given by:

\[ \frac{x_1^2}{a^2 M_n / c_{55}} + \frac{x_2^2}{a^2} = 1. \]

(2.6.2)

The semi-major and semi-minor axes are \( a\sqrt{M_n / c_{55}} \) and \( a \), respectively. \( M_n \) and \( c_{55} \) are to be defined later. We need to analyze the unelectroded and electroded regions of the plate separately.

The scalar equations governing the fundamental and overtone TSh modes for electroded and unelectroded portions are given by:

\[
M_n \frac{\partial^2 u_1^n}{\partial x_1^2} + c_{55} \frac{\partial^2 u_1^n}{\partial x_3^2} + \rho(\omega^2 + \frac{1-i}{\rho h} \omega^{3/2} \sqrt{2\mu \rho_1 - \overline{\omega_n}^2})u_1^n = 0, \tag{2.6.3}
\]

\[
M_n \frac{\partial^2 u_1^n}{\partial x_3^2} + c_{55} \frac{\partial^2 u_1^n}{\partial x_3^2} + \rho(\omega^2 + \frac{1-i}{\rho h} \omega^{3/2} \sqrt{2\mu \rho_1 - \overline{\omega_n}^2})u_1^n = 0, \tag{2.6.4}
\]

respectively, where

\[
\overline{\omega_n}^2 = \frac{n^2 \pi^2 c_{66}}{4h^2 \rho} < \omega_n^2 = \frac{n^2 \pi^2 c_{66}}{4h^2 \rho}.
\]

(2.6.5)

In the \((x_1, x_3)\) plane, we introduce a new coordinate system \((\xi_1, \xi_3)\) by

\[
x_1 = \xi_1 \sqrt{M_n / c_{55}}, \quad x_3 = \xi_3. \tag{2.6.6}
\]

In this coordinate system, the elliptical electrodes in (2.6.2) are represented by a circular domain described by

\[
\frac{\xi_1^2}{a^2} + \frac{\xi_3^2}{a^2} = 1. \tag{2.6.7}
\]

(2.6.3) and (2.6.4) become
\[ c_{55} \nabla^2 u_1^n + \rho (\omega^2 + \frac{1-i}{\rho \mu} \frac{\omega^2}{\sqrt{2\mu \rho_l - \omega_n^2}}) u_1^n = 0, \quad (2.6.8) \]

\[ c_{55} \nabla^2 u_1^n + \rho (\omega^2 + \frac{1-i}{\rho \mu} \frac{\omega^2}{\sqrt{2\mu \rho_l - \omega_n^2}}) u_1^n = 0, \quad (2.6.9) \]

where

\[ \nabla^2 = \partial^2 / \partial \xi_1^2 + \partial^2 / \partial \xi_3^2. \quad (2.6.10) \]

We then introduce a polar coordinate system \((r, \theta)\) defined by

\[ \xi_1 = r \cos \theta, \quad \xi_3 = r \sin \theta. \quad (2.6.11) \]

We are only interested in modes that are independent of \(\theta\). Then (2.6.8) and (2.6.9) reduce to

\[ \frac{\partial^2 u_1^n}{\partial r^2} + \frac{1}{r} \frac{\partial u_1^n}{\partial r} + \alpha^2 u_1^n = 0, \quad \text{for } r < a, \quad (2.6.12) \]

\[ \frac{\partial^2 u_1^n}{\partial r^2} + \frac{1}{r} \frac{\partial u_1^n}{\partial r} - \beta^2 u_1^n = 0, \quad \text{for } r > a, \quad (2.6.13) \]

where

\[ \alpha^2 = \rho (\omega^2 + \frac{1-i}{\rho \mu} \frac{\omega^2}{\sqrt{2\mu \rho_l - \omega_n^2}}) / c_{55} > 0, \quad (2.6.14) \]

\[ \beta^2 = \rho (\omega_n^2 - \frac{1-i}{\rho \mu} \frac{\omega^2}{\sqrt{2\mu \rho_l - \omega_n^2}}) / c_{55} > 0. \]

The general solution to (2.6.12) and (2.6.13) can be written as

\[ u_1^n = \begin{cases} A J_0(\alpha r), & r < a, \\ B K_0(\beta r), & r > a, \end{cases} \quad (2.6.15) \]

where \(A\) and \(B\) are undetermined constants. \(J_0\) is the zero order Bessel function of the first kind that is bounded at the origin. \(K_0\) is the zero order modified Bessel function of the second kind which decays exponentially at infinity. At the interface between the
electroded and unelectroded regions, we have the continuity of \( u_1^0 \) and its radial derivative:

\[
AJ_0'(\alpha a) - BK_0' (\beta a) = \beta J_0'(\alpha a)K_0'(\beta a)
\]

(2.6.16)

where a prime represents differentiation with respect to the entire argument, i.e., \( \alpha r \) or \( \beta r \). For nontrivial solutions of \( A \) and/or \( B \), the determinant of the coefficient matrix must vanish, which gives the following frequency equation that determines \( \omega \):

\[
\alpha J_0'(\alpha a)K_0(\beta a) - \beta J_0'(\alpha a)K_0'(\beta a) = 0.
\]

(2.6.17)

With the following identities:

\[
J_0'(\alpha a) = -J_1(\alpha a), \quad K_0'(\beta a) = -K_1(\beta a),
\]

(2.6.18)

(2.6.17) can be written as

\[
\alpha J_1(\alpha a)K_0(\beta a) - \beta J_0(\alpha a)K_1(\beta a) = 0.
\]

(2.6.19)

Consider an AT-cut quartz plate with \( 2h=1 \) mm, \( a=9 \) mm and \( R=0.01 \). The fluid is water, \( \rho_l =1000 \text{ kg/m}^3, \mu=8.90 \times 10^{-4} \text{ Pa·s} \). Only one trapped mode is found. The frequency is \( \omega=10374629.46+i1621.86 \) (rad/s) which is very close to the frequency \( \omega_0 = 10376251.82 \) (rad/s) for the plate without surface impedances. The frequency shift \( \Delta \omega \) is -1622.36 (rad/s), and the relative frequency shift \( [\text{Re}(\omega) - \omega_0]/\omega_0 \) is -0.01564%.

**Example 2 Quartz crystal microbalance**

As pointed out before, QCMs can be used as mass sensors and liquid sensors. For comparison and verification, we examine the dispersion curves for propagating waves in a QCM, which is represented by an unelectroded plate. Consider a wave propagating along \( x_1 \) determined from (2.3.24), which takes the form:
\[ u_i^n(x_1,t) = A \exp(i(\omega t - \xi x_1)). \] (2.6.20)

Substitution from (2.6.20) into (2.3.24), and then by the non-triviality of the solution we get the dispersion equation:

\[-M_n \xi^2 - c_{66} \left( \frac{n \pi}{2h} \right)^2 u_i + \rho \omega^2 - \frac{2i \omega}{h} Z_i(\omega) = 0. \] (2.6.21)

Introducing the dimensionless wave number and frequency:

\[ X = 2h \xi / \pi, \quad \Omega = \sqrt{\rho / c_{66}} 2h \omega / \pi . \] (2.6.22)

(2.6.21) can be simplified as:

\[-M_n X^2 - c_{66} n^2 + c_{66} \Omega^2 - \frac{4i}{\pi} \sqrt{\frac{c_{66}}{\rho}} \Omega Z_i(\omega) = 0. \] (2.6.23)

The surface acoustic impedance can be found in Appendix C:

\[ Z_i(\omega) = 2i \omega \overline{\rho} h , \] (2.6.24)

where \( \overline{2h} \) and \( \overline{\rho} \) denote the thickness and the density of the mass layer, respectively.

Substitute (2.6.24) into (2.6.23), we obtain

\[-M_n X^2 - c_{66} n^2 + c_{66} \Omega^2 + 2R_m c_{66} \Omega^2 = 0 , \] (2.6.25)

where

\[ R_m = 2 \overline{\rho} h / (\rho h) . \] (2.6.26)

Thus we have

\[ \Omega = \sqrt{\frac{c_{66} n^2 + M_n X^2}{c_{66} (1 + 2R_m)}} \quad \text{for } |X| > 0 \] (2.6.27)

and

\[ \Omega = \sqrt{\frac{c_{66} n^2 - M_n X^2}{c_{66} (1 + 2R_m)}} \quad \text{for } |X| < 0 . \] (2.6.28)
The dispersion curves are shown in Fig. 2.5. The solid lines correspond to the dispersion curves when $Z_t = 0$. For small $|X|$, these dispersion curves are approximations of the odd TSh branches of the exact dispersion curves [141] with approximately the same intercepts on the frequency axis, the same slope and the same curvature when $X = 0$. Therefore, (2.3.24) can be used to describe these waves approximately for small $|X|$ or long waves. When there are mass layers on the surfaces of an unelectroded plate, the dispersion curves for the case of $R_m = 0.05$ are shown by the dotted lines in Fig. 2.5(a). The dispersion curves become lower due to the inertia of the mass layers.

Similarly, for an electroded plate, we only need to change $\bar{c}_{66}$ into $\hat{c}_{66}$ and the dispersion curves determined by (2.4.25) are shown in Fig. 2.5(b). The curves in Fig. 2(b) are systematically lower than that in Fig. 2.5(a) due to the electrode inertia.
Figure 2.5 (a) Dispersion curves of a unelectroded plate.

Solid lines: \( Z_t = 0 \). Dotted lines: \( R_m = 0.05 \).

(b) Dispersion curves of an electroded plate.

Solid lines: \( Z_t = 0 \) and \( R = 0.05 \). Dotted lines: \( R_m = 0.05 \) and \( R = 0.05 \).

If the surface of the plate is in contact with water, the surface impedance is:

\[
Z_t = (1 + i) \sqrt{\frac{\mu \rho \omega}{2}}.
\] (2.6.29)

The dispersion relation is:

\[
X = \left[ \frac{\bar{c}_{66}}{M_n} n^2 - \frac{c_{66}}{M_n} \Omega^2 - \frac{2}{\pi} \frac{\bar{c}_{66} 1 - i}{\rho M_n} \Omega \sqrt{\frac{\mu \rho \omega}{h}} \sqrt{\frac{c_{66} \Omega}{\rho}} \right]^{1/2}. \] (2.6.30)

As numerical examples, we consider two fluids, one is water and the other is an imaginary liquid for which the viscosity is ten thousand times larger than that of water. Due to the damping of the fluids, the dimensionless wavenumber has both real and imaginary parts. The dispersion curves for water and the imaginary liquid are shown in Fig. 2.6 and 2.7, respectively. As can be seen, the imaginary part increases with the viscosity of the fluids.
Figure 2.6 The dispersion curves for the case when the plate is in contact with water: $\mu_f = 8.90 \times 10^{-4}$ Pa s: (a) 3D view, (b) plane view
Figure 2.7 The dispersion curves for the case when the plate is in contact with a fluid with viscosity $\mu = 8.90 \text{Pa} \cdot \text{s}$: (a) 3D view, (b) plane view

**Example 3 Thickness vibrations of unbounded plates**

In this section we compare the predictions by (2.3.24) and (2.4.25) with the results of the exact equations of piezoelectricity in the special case of pure TSh modes whose exact solutions have been obtained. For pure thickness modes in unbounded plates without $x_1$ and $x_3$ dependence, (2.3.24) leads to the following frequency equation for unelectroded
plates:

$$\rho \omega^2 = \epsilon_{66} \left( \frac{n\pi}{2h} \right)^2 + \frac{2i\omega}{h} Z_r(\omega).$$ \hspace{1cm} (2.6.31)

For electroded plates (2.4.25) determines a frequency equation similar to (2.6.31), with
\( \epsilon_{66} \) replaced by \( \epsilon_{66} \). In sensor applications, the first term on the right-hand side of
(2.6.31) determines the unperturbed frequency \( \omega_0 \) when \( Z_r = 0 \). In the case of small
impedance which causes small frequency perturbations, the unknown frequency \( \omega \) in the
second term (which is small) on the right-hand side of (2.6.31) can be replaced by the
known unperturbed frequency \( \omega_0 \) and the perturbed frequencies are:

$$\omega = \sqrt{\epsilon_{66} \frac{n\pi}{\rho \ 2h}} \left[ 1 + \frac{4hi\omega_0}{n^2\pi^2 \epsilon_{66}} Z_r(\omega_0) \right].$$ \hspace{1cm} (2.6.32)

Then we can obtained the normalized frequency shifts as follows:

$$\frac{\omega - \omega_0}{\omega_0} \approx \frac{4hi\omega_0}{n^2\pi^2 \epsilon_{66}} Z_r(\omega_0).$$ \hspace{1cm} (2.6.33)

It can be verified that the frequencies of pure thickness modes of an infinite unelectroded
plate given by (2.2.12) is the same as (2.6.32).

Similarly, for electroded plates, the first order approximation of \( \omega \) predicted by
(2.6.31) with \( \epsilon_{66} \) replaced by \( \epsilon_{66} \) is the same as corresponding frequency given by
(2.2.26). The result is

$$\omega = \sqrt{\epsilon_{66} \frac{n\pi}{\rho \ 2h}} \left[ 1 - R - \frac{4k_{66}^2}{n^2\pi^2} + \frac{4hi\omega_0}{n^2\pi^2 \epsilon_{66}} Z_r(\omega_0) \right].$$ \hspace{1cm} (2.6.34)

In real applications, only one surface of sensor is exposed to substances being
examined, i.e., for one mass layer or fluid on one side, so the right hand side of (2.6.33)
should be devided by two, we get:
\[
\frac{\omega - \omega_0}{\omega_0} \approx \frac{2hi\omega_0}{n^2 \pi^2 c_{66}^2} Z_i(\omega_0) .
\] (2.6.35)

As a specific example, consider a mass layer of thickness \(2\hat{h}\) and density \(\bar{\rho}\) on the top surface of an unelectroded plate. The surface acoustic impedance is

\[
Z_i(\omega) = i\omega 2\bar{\rho}h .
\] (2.6.36)

Substitution of (2.6.36) into (2.6.35) yields:

\[
\frac{\omega - \omega_0}{\omega_0} \approx -\frac{\bar{\rho}\hat{h}}{\rho h} ,
\] (2.6.37)

which is the classical result for QCMs in [82].

As another example, consider a semi-infinite fluid of density \(\rho_l\) and viscosity \(\mu\) on the top of the crystal plate. In this case the impedance is

\[
Z_i(\omega) = (1 + i)\sqrt{\mu\rho_l\omega/2} .
\] (2.6.38)

Substitution of (2.6.38) into (2.6.35) yields

\[
\frac{\omega - \omega_0}{\omega_0} \approx -\frac{\sqrt{2(1-i)h}\sqrt{\mu\rho_l}}{c_{66} n^2 \pi^2} \omega_0^{1/2} ,
\] (2.6.39)

which is consistent with the classical result for fluid sensors in [83].

**Example 4 Vibrations of a Rectangular Plate**

More than being able to predict the frequencies of pure TSh modes, the main advantage of (2.3.24) and (2.4.25) is that they can describe the \(x_1\) and \(x_3\) dependence and therefore can be used to analyze finite plates. As an example, consider an unelectroded rectangular plate as shown in Fig. 2.8.

A general solution which is symmetric both in \(x_1\) and \(x_3\) directions is given by:
\[ u_1 = A \sin \left( \frac{n \pi}{2h} x_2 \right) \cos(\xi x_1) \cos(n \pi x_3), \quad (2.60) \]

\[ \begin{align*}
T_{12} &= \bar{c}_{66} u_{1,2} = 0, \quad \text{at } x_1 = \pm a, \\
T_{13} &= c_{35} u_{1,3} = 0, \quad \text{at } x_3 = \pm c.
\end{align*} \quad (2.61) \]

Substituting from (2.60) into (2.61) we find:

\[ u_1 = A_{nm} \cos \left( \frac{l \pi}{c} x_3 \right) \cos \left[ \frac{(2m+1) \pi}{2a} x_1 \right]. \quad (2.62) \]

The frequency-wavenumber relation is obtained by substituting (2.60) into (2.3.24):

\[ \rho \omega^2 = c_{ss} \left( \frac{l \pi}{c} \right)^2 + M \left( \frac{(2m+1) \pi}{2a} \right)^2 + \bar{c}_{66} \left( \frac{n \pi}{2h} \right)^2 + \frac{2i \omega}{h} Z_i(\omega). \quad (2.8.43) \]

For electroded plates \( \bar{c}_{66} \) is replaced by \( \hat{c}_{66} \). (2.63) shows the dependence of the frequencies on the in-plane dimensions \( a \) and \( c \). When the acoustic impedance is set to zero, (2.6.43) reduces to (26) in [119].
Chapter 3
Scalar Differential Equations for Doubly-rotated Quartz Plates

The aim of this chapter is to obtain scalar differential equations for QCMs made of doubly-rotated quartz plates. Three cases are considered, free vibration of unelectroded plates, free vibration of electroded plates and forced vibration of electrode plates. The scalar equations for doubly-rotated still have very simple form. The method presented in this chapter is very general and can be utilized to analyze quartz plates with other cut angles.

3.1 Three dimensional governing equations

A doubly-rotated quartz plate behaves like a general anisotropic material in the plate coordinate system (see Fig. 2.1). Hence, the analysis is much more complicated than that for singly-rotated plates. To simplify the problem, we first obtain the eigenvectors of pure thickness vibration, which form a new orthogonal coordinate system called eigen-coordinate system. Then the displacement vector is decomposed in this system and the principle displacement component of the operating mode is mainly parallel to one of the three coordinate axes. The subsequent steps are similar to that for singly-rotated case. Due to the vectorial nature of the displacement and the tensorial nature of the material properties, i.e., elastic constants and piezoelectric coefficients, the components in different systems have different values. Actually, they are related by a special orthogonal transformation. In order to distinguish these components, we introduce the following
convention: components of displacement vector and material property tensors expressed in the plate coordinate system are denoted by Latin letters with an over bar while the components of the same quantity expressed in the eigen-coordinate system are denoted by the same Latin letter without an over bar. Unless otherwise notified, this convention holds in the subsequent sections. In the plate coordinate system, the equations of motion can be expressed as:

\[
\begin{align*}
\bar{T}_{11,1} + \bar{T}_{21,2} + \bar{T}_{31,3} &= \rho \dddot{u}_1, \\
\bar{T}_{12,1} + \bar{T}_{22,2} + \bar{T}_{32,3} &= \rho \dddot{u}_2, \\
\bar{T}_{13,1} + \bar{T}_{23,2} + \bar{T}_{33,3} &= \rho \dddot{u}_3.
\end{align*}
\] (3.1.1)

The Gauss’s law for electric charges is:

\[
\bar{D}_{1,1} + \bar{D}_{2,2} + \bar{D}_{3,3} = 0.
\] (3.1.2)

The constitutive equations are given in (1.3). We need to keep in mind that all the quantities in (1.3) have an overbar now. In most classical textbooks of elasticity, the Voigt index notation, see Tab. 3.1, is adopted to simplify the indices of elastic stiffness tensor. This notation can perfectly reflect the material symmetry and works well in most cases. However, for the analysis of doubly-rotated quartz plate which will be presented in this work, the material symmetry could not be reflected by the Voigt’s index notation. This is because the orthogonal transformation involved here is not the one obey the tensor transformation rules, i.e. \( \bar{c}_{ijkl} = q_{ip}q_{jq}q_{kr}q_{ls}c_{pqrs} \), where \( q_{ij} \) is the elements of the orthogonal transformation matrix. This will be explained in greater details in Sec. 3.3. Instead of the Voigt’s notation, here we adopt a nine-index notation, which is given in Tab. 3.2. It will be clear later that the material symmetry can be properly described using this notation.
Table 3.1 Voigt’s index notation

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<thead>
<tr>
<th>$ij, kl$</th>
<th>11</th>
<th>22</th>
<th>33</th>
<th>23, 32</th>
<th>13, 31</th>
<th>12, 21</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p, q$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

Table 3.2 Nine-index notation

<table>
<thead>
<tr>
<th>$ij, kl$</th>
<th>11</th>
<th>22</th>
<th>33</th>
<th>23</th>
<th>31</th>
<th>12</th>
<th>32</th>
<th>13</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p, q$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

A few more comments are now in order. First and for most, we should notice that the symmetries of the elastic property and the piezoelectric property are independent of the coordinate system in which they are expressed, but the symmetry of the components of the material tensors are really dependent on the coordinate system used. Second, in the plate coordinate system, the elastic tensor and the piezoelectric tensor can be expressed either in the Voigt’s notation or in the nine-index notation. The symmetry of the indices in the Voigt’s notation can be inferred from the nine-index notation by identifying “7” with “4”, “8” with “5”, “9” with “6” and using the symmetry $c_{pq} = c_{qp}$.
Now the constitutive relations take the form:

\[
\begin{align*}
\begin{bmatrix}
\bar{T}_{11} \\
\bar{T}_{21} \\
\bar{T}_{31}
\end{bmatrix} &= \begin{bmatrix}
\bar{c}_{11} & \bar{c}_{19} & \bar{c}_{15} \\
\bar{c}_{91} & \bar{c}_{99} & \bar{c}_{95} \\
\bar{c}_{51} & \bar{c}_{59} & \bar{c}_{55}
\end{bmatrix} \begin{bmatrix}
\bar{u}_{1,1} \\
\bar{u}_{2,1} \\
\bar{u}_{3,1}
\end{bmatrix} + \begin{bmatrix}
\bar{c}_{16} & \bar{c}_{12} & \bar{c}_{17} \\
\bar{c}_{96} & \bar{c}_{92} & \bar{c}_{97} \\
\bar{c}_{56} & \bar{c}_{52} & \bar{c}_{57}
\end{bmatrix} \begin{bmatrix}
\bar{u}_{1,2} \\
\bar{u}_{2,2} \\
\bar{u}_{3,2}
\end{bmatrix} + \begin{bmatrix}
\bar{c}_{18} & \bar{c}_{14} & \bar{c}_{13} \\
\bar{c}_{98} & \bar{c}_{94} & \bar{c}_{93} \\
\bar{c}_{58} & \bar{c}_{54} & \bar{c}_{53}
\end{bmatrix} \begin{bmatrix}
\bar{u}_{1,3} \\
\bar{u}_{2,3} \\
\bar{u}_{3,3}
\end{bmatrix} + \begin{bmatrix}
\bar{c}_{15} & \bar{c}_{25} & \bar{c}_{35}
\end{bmatrix} \begin{bmatrix}
\phi_{1} \\
\phi_{2} \\
\phi_{3}
\end{bmatrix} \\
\begin{bmatrix}
\bar{T}_{12} \\
\bar{T}_{22} \\
\bar{T}_{32}
\end{bmatrix} &= \begin{bmatrix}
\bar{c}_{61} & \bar{c}_{69} & \bar{c}_{65} \\
\bar{c}_{21} & \bar{c}_{29} & \bar{c}_{25} \\
\bar{c}_{71} & \bar{c}_{79} & \bar{c}_{75}
\end{bmatrix} \begin{bmatrix}
\bar{u}_{1,1} \\
\bar{u}_{2,1} \\
\bar{u}_{3,1}
\end{bmatrix} + \begin{bmatrix}
\bar{c}_{66} & \bar{c}_{62} & \bar{c}_{67} \\
\bar{c}_{26} & \bar{c}_{22} & \bar{c}_{27} \\
\bar{c}_{76} & \bar{c}_{72} & \bar{c}_{77}
\end{bmatrix} \begin{bmatrix}
\bar{u}_{1,2} \\
\bar{u}_{2,2} \\
\bar{u}_{3,2}
\end{bmatrix} + \begin{bmatrix}
\bar{c}_{68} & \bar{c}_{64} & \bar{c}_{63} \\
\bar{c}_{28} & \bar{c}_{24} & \bar{c}_{23} \\
\bar{c}_{78} & \bar{c}_{74} & \bar{c}_{73}
\end{bmatrix} \begin{bmatrix}
\bar{u}_{1,3} \\
\bar{u}_{2,3} \\
\bar{u}_{3,3}
\end{bmatrix} + \begin{bmatrix}
\bar{c}_{19} & \bar{c}_{29} & \bar{c}_{39}
\end{bmatrix} \begin{bmatrix}
\phi_{1} \\
\phi_{2} \\
\phi_{3}
\end{bmatrix} \\
\begin{bmatrix}
\bar{T}_{13} \\
\bar{T}_{23} \\
\bar{T}_{33}
\end{bmatrix} &= \begin{bmatrix}
\bar{c}_{81} & \bar{c}_{89} & \bar{c}_{85} \\
\bar{c}_{41} & \bar{c}_{49} & \bar{c}_{45} \\
\bar{c}_{31} & \bar{c}_{39} & \bar{c}_{35}
\end{bmatrix} \begin{bmatrix}
\bar{u}_{1,1} \\
\bar{u}_{2,1} \\
\bar{u}_{3,1}
\end{bmatrix} + \begin{bmatrix}
\bar{c}_{86} & \bar{c}_{82} & \bar{c}_{87} \\
\bar{c}_{46} & \bar{c}_{42} & \bar{c}_{47} \\
\bar{c}_{36} & \bar{c}_{32} & \bar{c}_{37}
\end{bmatrix} \begin{bmatrix}
\bar{u}_{1,2} \\
\bar{u}_{2,2} \\
\bar{u}_{3,2}
\end{bmatrix} + \begin{bmatrix}
\bar{c}_{88} & \bar{c}_{84} & \bar{c}_{83} \\
\bar{c}_{48} & \bar{c}_{44} & \bar{c}_{43} \\
\bar{c}_{38} & \bar{c}_{34} & \bar{c}_{33}
\end{bmatrix} \begin{bmatrix}
\bar{u}_{1,3} \\
\bar{u}_{2,3} \\
\bar{u}_{3,3}
\end{bmatrix} + \begin{bmatrix}
\bar{c}_{18} & \bar{c}_{28} & \bar{c}_{38}
\end{bmatrix} \begin{bmatrix}
\phi_{1} \\
\phi_{2} \\
\phi_{3}
\end{bmatrix} \\
\begin{bmatrix}
\bar{D}_{1} \\
\bar{D}_{2} \\
\bar{D}_{3}
\end{bmatrix} &= \begin{bmatrix}
\bar{e}_{11} & \bar{e}_{19} & \bar{e}_{15} \\
\bar{e}_{21} & \bar{e}_{29} & \bar{e}_{25} \\
\bar{e}_{31} & \bar{e}_{39} & \bar{e}_{35}
\end{bmatrix} \begin{bmatrix}
\bar{u}_{1,1} \\
\bar{u}_{2,1} \\
\bar{u}_{3,1}
\end{bmatrix} + \begin{bmatrix}
\bar{e}_{16} & \bar{e}_{12} & \bar{e}_{17} \\
\bar{e}_{26} & \bar{e}_{22} & \bar{e}_{27} \\
\bar{e}_{36} & \bar{e}_{32} & \bar{e}_{37}
\end{bmatrix} \begin{bmatrix}
\bar{u}_{1,2} \\
\bar{u}_{2,2} \\
\bar{u}_{3,2}
\end{bmatrix} + \begin{bmatrix}
\bar{e}_{18} & \bar{e}_{14} & \bar{e}_{13} \\
\bar{e}_{28} & \bar{e}_{24} & \bar{e}_{23} \\
\bar{e}_{38} & \bar{e}_{34} & \bar{e}_{33}
\end{bmatrix} \begin{bmatrix}
\bar{u}_{1,3} \\
\bar{u}_{2,3} \\
\bar{u}_{3,3}
\end{bmatrix} + \begin{bmatrix}
\bar{e}_{11} & \bar{e}_{12} & \bar{e}_{13} \\
\bar{e}_{12} & \bar{e}_{22} & \bar{e}_{23} \\
\bar{e}_{13} & \bar{e}_{23} & \bar{e}_{33}
\end{bmatrix} \begin{bmatrix}
\phi_{1} \\
\phi_{2} \\
\phi_{3}
\end{bmatrix}
\end{align*}
\]

(3.1.4)

The governing equations for the displacements and the electric potential are given by:
For the convenience of later applications when the boundary conditions are considered, the stresses can be reorganized as:
The one-dimensional approximation of the electric field discussed in Chap. 2 still holds, i.e.:

\[
\bar{D}_1 = 0, \quad \bar{D}_3 = 0, \quad \frac{\partial}{\partial x_1} = 0, \quad \frac{\partial}{\partial x_3} = 0, \quad (3.1.7)
\]

The only survival electric displacement component \(\bar{D}_2\) is given by:

\[
\bar{D}_2 = \bar{\varepsilon}_{26}\bar{u}_{1,2} + \bar{\varepsilon}_{27}\bar{u}_{2,2} + \bar{\varepsilon}_{23}\bar{u}_{3,2} - \bar{\varepsilon}_{22}\bar{\varphi}_{,2}, \quad (3.1.8)
\]

The Gauss's law for the electric charge degenerates into the form:

\[
\bar{\varepsilon}_{26}\bar{u}_{1,22} + \bar{\varepsilon}_{27}\bar{u}_{2,22} + \bar{\varepsilon}_{23}\bar{u}_{3,22} - \bar{\varepsilon}_{22}\bar{\varphi}_{,22} = 0. \quad (3.1.9)
\]

Consequently, we get:
\[ \varphi_{22} = \frac{1}{\varepsilon_{22}} (\varepsilon_{26}\bar{u}_{1,22} + \varepsilon_{27}\bar{u}_{2,22} + \varepsilon_{27}\bar{u}_{3,22}), \]  

(3.1.10)

Substitution of (3.1.10) into (3.1.5) yields:

\[
\begin{bmatrix}
\bar{c}_{11} & \bar{c}_{19} & \bar{c}_{15} & \bar{u}_{1,11} \\
\bar{c}_{61} & \bar{c}_{69} & \bar{c}_{65} & \bar{u}_{2,11} \\
\bar{c}_{81} & \bar{c}_{89} & \bar{c}_{85} & \bar{u}_{3,11}
\end{bmatrix} + 
\begin{bmatrix}
\bar{c}_{96} & \bar{c}_{92} & \bar{c}_{97} & \bar{u}_{1,22} \\
\bar{c}_{26} & \bar{c}_{22} & \bar{c}_{27} & \bar{u}_{2,22} \\
\bar{c}_{46} & \bar{c}_{42} & \bar{c}_{47} & \bar{u}_{3,22}
\end{bmatrix} + 
\begin{bmatrix}
\bar{c}_{58} & \bar{c}_{54} & \bar{c}_{53} & \bar{u}_{1,33} \\
\bar{c}_{78} & \bar{c}_{74} & \bar{c}_{73} & \bar{u}_{2,33} \\
\bar{c}_{38} & \bar{c}_{34} & \bar{c}_{33} & \bar{u}_{3,33}
\end{bmatrix} + 
\begin{bmatrix}
\bar{c}_{16} + \bar{c}_{91} & \bar{c}_{12} + \bar{c}_{99} & \bar{c}_{17} + \bar{c}_{95} & \bar{u}_{1,12} \\
\bar{c}_{66} + \bar{c}_{21} & \bar{c}_{62} + \bar{c}_{29} & \bar{c}_{67} + \bar{c}_{25} & \bar{u}_{2,12} \\
\bar{c}_{86} + \bar{c}_{41} & \bar{c}_{82} + \bar{c}_{49} & \bar{c}_{87} + \bar{c}_{45} & \bar{u}_{3,12}
\end{bmatrix},
\]  

(3.1.11)
where

\[
\begin{bmatrix}
\hat{c}_{96} & \hat{c}_{92} & \hat{c}_{97} \\
\hat{c}_{26} & \hat{c}_{22} & \hat{c}_{27} \\
\hat{c}_{46} & \hat{c}_{42} & \hat{c}_{47}
\end{bmatrix}
= \begin{bmatrix}
\bar{c}_{96} + \frac{\bar{c}_{29}\bar{c}_{26}}{\bar{c}_{22}} & \bar{c}_{92} + \frac{\bar{c}_{29}\bar{c}_{22}}{\bar{c}_{22}} & \bar{c}_{97} + \frac{\bar{c}_{29}\bar{c}_{27}}{\bar{c}_{22}} \\
\bar{c}_{26} + \frac{\bar{c}_{22}\bar{c}_{26}}{\bar{c}_{22}} & \bar{c}_{22} + \frac{\bar{c}_{24}\bar{c}_{22}}{\bar{c}_{22}} & \bar{c}_{27} + \frac{\bar{c}_{22}\bar{c}_{27}}{\bar{c}_{22}} \\
\bar{c}_{46} + \frac{\bar{c}_{24}\bar{c}_{26}}{\bar{c}_{22}} & \bar{c}_{42} + \frac{\bar{c}_{24}\bar{c}_{22}}{\bar{c}_{22}} & \bar{c}_{47} + \frac{\bar{c}_{24}\bar{c}_{27}}{\bar{c}_{22}}
\end{bmatrix}.
\] (3.1.12)

Recalling the comments given after Tab. 3.2, one can verify that each of the coefficient matrices in (3.1.11) is symmetric.

In (3.1.11) the equation for the electric potential is eliminated and its influence is properly included in the displacement equation of motion. Next, we focus on the elimination of electric potential in the constitutive relations. As can be seen in Chap. 2, the electric potential takes different forms for electroded and unelectroded plates and the constitutive equations are also different in form. The same thing happens to the doubly-rotated plates.

(a) Open-circuit electric boundary conditions for unelectroded plates:

\[
\bar{D}_2 = 0 , \quad \text{at} \quad x_2 = \pm h .
\] (3.1.13)

Combining (3.1.8) and (3.1.13), we get:

\[
\varphi_2 = \frac{1}{\bar{c}_{22}} \left( \bar{e}_{26}\bar{u}_{1,2} + \bar{e}_{22}\bar{u}_{2,2} + \bar{e}_{27}\bar{u}_{3,2} \right) .
\] (3.1.14)

Integrate both sides of (3.1.14) once:

\[
\varphi = \frac{1}{\bar{c}_{22}} \left( \bar{e}_{26}\bar{u}_1 + \bar{e}_{22}\bar{u}_2 + \bar{e}_{27}\bar{u}_3 \right) + L
\] (3.1.15)

We are only interested in vibrations antisymmetric about the middle plane of the plate, so all the dependent variables \( \bar{u}_1 , \bar{u}_2 , \bar{u}_3 \) and \( \varphi \) are odd functions of \( x_2 \), which implies \( L=0 \). Thus, we obtain the electric potential as follows:
\[ \varphi = \frac{1}{\bar{E}^{22}} \left( \bar{e}_{26}\bar{u}_1 + \bar{e}_{22}\bar{u}_2 + \bar{e}_{27}\bar{u}_3 \right), \] (3.1.16)

The constitutive equations are now:

\[
\begin{bmatrix}
T_{11} & T_{12} & T_{13} \\
\bar{c}_{11} & \bar{c}_{19} & \bar{c}_{15} & \bar{u}_{1,1} \\
\bar{c}_{61} & \bar{c}_{69} & \bar{c}_{65} & \bar{u}_{2,1} + \bar{c}_{66} & \bar{c}_{62} & \bar{c}_{67} & \bar{u}_{2,2} \\
\bar{c}_{81} & \bar{c}_{89} & \bar{c}_{85} & \bar{u}_{3,1}
\end{bmatrix}
\begin{bmatrix}
\bar{u}_{1,1} \\
\bar{u}_{2,1} \\
\bar{u}_{3,1}
\end{bmatrix}
+ \begin{bmatrix}
\bar{c}_{16} & \bar{c}_{12} & \bar{c}_{17} \\
\bar{c}_{66} & \bar{c}_{62} & \bar{c}_{67} \\
\bar{c}_{86} & \bar{c}_{82} & \bar{c}_{87}
\end{bmatrix}
\begin{bmatrix}
\bar{u}_{1,2} \\
\bar{u}_{2,2} \\
\bar{u}_{3,2}
\end{bmatrix}
+ \begin{bmatrix}
\bar{c}_{18} & \bar{c}_{14} & \bar{c}_{13} \\
\bar{c}_{68} & \bar{c}_{64} & \bar{c}_{63} \\
\bar{c}_{88} & \bar{c}_{84} & \bar{c}_{83}
\end{bmatrix}
\begin{bmatrix}
\bar{u}_{1,3} \\
\bar{u}_{2,3} \\
\bar{u}_{3,3}
\end{bmatrix}
\]

(3.1.17)

where

\[
\begin{bmatrix}
\hat{\bar{c}}_{16} & \hat{\bar{c}}_{12} & \hat{\bar{c}}_{17} \\
\hat{\bar{c}}_{66} & \hat{\bar{c}}_{62} & \hat{\bar{c}}_{67} \\
\hat{\bar{c}}_{86} & \hat{\bar{c}}_{82} & \hat{\bar{c}}_{87}
\end{bmatrix}
= \begin{bmatrix}
\bar{c}_{16} + \frac{\bar{e}_{26}\bar{E}^{22}}{\bar{E}^{22}} & \bar{c}_{12} + \frac{\bar{e}_{22}\bar{E}^{22}}{\bar{E}^{22}} & \bar{c}_{17} + \frac{\bar{e}_{21}\bar{E}^{22}}{\bar{E}^{22}} \\
-\bar{c}_{66} + \frac{\bar{e}_{26}^2}{\bar{E}^{22}} & \bar{c}_{62} + \frac{\bar{e}_{22}\bar{E}^{22}}{\bar{E}^{22}} & \bar{c}_{67} + \frac{\bar{e}_{27}\bar{E}^{22}}{\bar{E}^{22}} \\
-\bar{c}_{86} + \frac{\bar{e}_{26}\bar{E}^{22}}{\bar{E}^{22}} & \bar{c}_{82} + \frac{\bar{e}_{28}\bar{E}^{22}}{\bar{E}^{22}} & \bar{c}_{87} + \frac{\bar{e}_{29}\bar{E}^{22}}{\bar{E}^{22}}
\end{bmatrix},
\]

(3.1.18)

\[
\begin{bmatrix}
\hat{\bar{c}}_{56} & \hat{\bar{c}}_{52} & \hat{\bar{c}}_{57} \\
\hat{\bar{c}}_{76} & \hat{\bar{c}}_{72} & \hat{\bar{c}}_{77} \\
\hat{\bar{c}}_{36} & \hat{\bar{c}}_{32} & \hat{\bar{c}}_{37}
\end{bmatrix}
= \begin{bmatrix}
\bar{c}_{56} + \frac{\bar{e}_{26}^2\bar{E}^{22}}{\bar{E}^{22}} & \bar{c}_{52} + \frac{\bar{e}_{25}\bar{E}^{22}}{\bar{E}^{22}} & \bar{c}_{57} + \frac{\bar{e}_{24}\bar{E}^{22}}{\bar{E}^{22}} \\
-\bar{c}_{76} + \frac{\bar{e}_{26}\bar{E}^{22}}{\bar{E}^{22}} & \bar{c}_{72} + \frac{\bar{e}_{27}\bar{E}^{22}}{\bar{E}^{22}} & \bar{c}_{77} + \frac{\bar{e}_{28}\bar{E}^{22}}{\bar{E}^{22}} \\
-\bar{c}_{36} + \frac{\bar{e}_{26}\bar{E}^{22}}{\bar{E}^{22}} & \bar{c}_{32} + \frac{\bar{e}_{23}\bar{E}^{22}}{\bar{E}^{22}} & \bar{c}_{37} + \frac{\bar{e}_{29}\bar{E}^{22}}{\bar{E}^{22}}
\end{bmatrix},
\]

(3.1.19)

Among all the coefficient matrices in (3.1.17), only the following three are symmetric:

\[
\begin{bmatrix}
\bar{c}_{11} & \bar{c}_{19} & \bar{c}_{15} \\
\bar{c}_{61} & \bar{c}_{69} & \bar{c}_{65} \\
\bar{c}_{81} & \bar{c}_{89} & \bar{c}_{85}
\end{bmatrix}, \quad
\begin{bmatrix}
\bar{c}_{96} & \hat{\bar{c}}_{92} & \hat{\bar{c}}_{97} \\
\bar{c}_{26} & \hat{\bar{c}}_{22} & \hat{\bar{c}}_{27} \\
\bar{c}_{46} & \hat{\bar{c}}_{42} & \hat{\bar{c}}_{47}
\end{bmatrix}, \quad
\begin{bmatrix}
\bar{c}_{58} & \bar{c}_{54} & \bar{c}_{53} \\
\bar{c}_{78} & \bar{c}_{74} & \bar{c}_{73} \\
\bar{c}_{38} & \bar{c}_{34} & \bar{c}_{33}
\end{bmatrix}.
\]

(3.1.20)

(b) Short circuit electric boundary conditions for electroded plates:
\[ \varphi = 0 \text{, at } x_2 = \pm h. \]  

(3.1.21)

Integrating both sides of (3.1.10) twice, we obtain the electric potential:

\[ \varphi = \frac{1}{\hat{e}_{22}}(\ddot{e}_{26} \ddot{u}_1 + \ddot{e}_{22} \ddot{u}_2 + \ddot{e}_{22} \ddot{u}_3) + L_1 x_2 + L_2. \]  

(3.1.22)

where \( L_1 \) and \( L_2 \) are undetermined constants.

According to the argument given before (3.1.16), the electric potential is an antisymmetric function of \( x_2 \), so

\[ L_2 = 0. \]  

(3.1.23)

Substituting from (3.1.22) with (3.1.23) into (3.1.21), we obtain:

\[ L_1 = -\frac{1}{\hbar \hat{e}_{22}}[\ddot{e}_{26} \ddot{u}_1(h) + \ddot{e}_{22} \ddot{u}_2(h) + \ddot{e}_{22} \ddot{u}_3(h)]. \]  

(3.1.24)

Thus the electric potential is of the form:

\[ \varphi = \frac{\ddot{e}_{26}}{\hat{e}_{22}} \left[ \ddot{u}_1 - \frac{x_2}{\hbar} \ddot{u}_1(h) \right] + \frac{\ddot{e}_{22}}{\hat{e}_{22}} \left[ \ddot{u}_2 - \frac{x_2}{\hbar} \ddot{u}_2(h) \right] + \frac{\ddot{e}_{22}}{\hat{e}_{22}} \left[ \ddot{u}_3 - \frac{x_2}{\hbar} \ddot{u}_3(h) \right]. \]  

(3.1.25)
Substitution of (3.1.25) into (3.1.17) results in:

\[
\begin{bmatrix}
\tilde{T}_{11} \\
\tilde{T}_{12} \\
\tilde{T}_{13}
\end{bmatrix} = \begin{bmatrix}
\bar{c}_{11} & \bar{c}_{19} & \bar{c}_{15} \\
\bar{c}_{61} & \bar{c}_{69} & \bar{c}_{65} \\
\bar{c}_{81} & \bar{c}_{89} & \bar{c}_{85}
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_{1,1} \\
\tilde{u}_{2,2} \\
\tilde{u}_{3,1}
\end{bmatrix} + \begin{bmatrix}
\bar{c}_{16} & \bar{c}_{17} & \bar{c}_{18} \\
\bar{c}_{62} & \bar{c}_{67} & \bar{c}_{68} \\
\bar{c}_{82} & \bar{c}_{87} & \bar{c}_{88}
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_{1,2} \\
\tilde{u}_{2,2} \\
\tilde{u}_{3,2}
\end{bmatrix} - \begin{bmatrix}
\bar{k}_{16}^2 & \bar{k}_{12}^2 & \bar{k}_{17}^2 \\
\bar{k}_{66}^2 & \bar{k}_{26}^2 & \bar{k}_{67}^2 \\
\bar{k}_{88}^2 & \bar{k}_{28}^2 & \bar{k}_{78}^2
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_{1,3} \\
\tilde{u}_{2,3} \\
\tilde{u}_{3,3}
\end{bmatrix},
\]  

(3.1.26)

\[
\begin{bmatrix}
\tilde{T}_{21} \\
\tilde{T}_{22} \\
\tilde{T}_{23}
\end{bmatrix} = \begin{bmatrix}
\bar{c}_{91} & \bar{c}_{92} & \bar{c}_{95} \\
\bar{c}_{26} & \bar{c}_{22} & \bar{c}_{27} \\
\bar{c}_{46} & \bar{c}_{42} & \bar{c}_{47}
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_{1,1} \\
\tilde{u}_{2,1} \\
\tilde{u}_{3,1}
\end{bmatrix} + \begin{bmatrix}
\bar{c}_{96} & \bar{c}_{97} & \bar{c}_{98} \\
\bar{c}_{26} & \bar{c}_{22} & \bar{c}_{28} \\
\bar{c}_{46} & \bar{c}_{42} & \bar{c}_{48}
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_{1,2} \\
\tilde{u}_{2,2} \\
\tilde{u}_{3,2}
\end{bmatrix} - \begin{bmatrix}
\bar{k}_{91}^2 & \bar{k}_{21}^2 & \bar{k}_{95}^2 \\
\bar{k}_{26}^2 & \bar{k}_{22}^2 & \bar{k}_{27}^2 \\
\bar{k}_{46}^2 & \bar{k}_{42}^2 & \bar{k}_{48}^2
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_{1,3} \\
\tilde{u}_{2,3} \\
\tilde{u}_{3,3}
\end{bmatrix},
\]  

(3.1.27)

\[
\begin{bmatrix}
\tilde{T}_{31} \\
\tilde{T}_{32} \\
\tilde{T}_{33}
\end{bmatrix} = \begin{bmatrix}
\bar{c}_{51} & \bar{c}_{52} & \bar{c}_{55} \\
\bar{c}_{76} & \bar{c}_{72} & \bar{c}_{77} \\
\bar{c}_{36} & \bar{c}_{32} & \bar{c}_{37}
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_{1,1} \\
\tilde{u}_{2,1} \\
\tilde{u}_{3,1}
\end{bmatrix} + \begin{bmatrix}
\bar{c}_{56} & \bar{c}_{57} & \bar{c}_{58} \\
\bar{c}_{76} & \bar{c}_{74} & \bar{c}_{73} \\
\bar{c}_{36} & \bar{c}_{34} & \bar{c}_{33}
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_{1,2} \\
\tilde{u}_{2,2} \\
\tilde{u}_{3,2}
\end{bmatrix} - \begin{bmatrix}
\bar{k}_{51}^2 & \bar{k}_{31}^2 & \bar{k}_{55}^2 \\
\bar{k}_{76}^2 & \bar{k}_{26}^2 & \bar{k}_{77}^2 \\
\bar{k}_{36}^2 & \bar{k}_{27}^2 & \bar{k}_{37}^2
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_{1,3} \\
\tilde{u}_{2,3} \\
\tilde{u}_{3,3}
\end{bmatrix},
\]  

(3.1.28)

where

\[
\begin{bmatrix}
\bar{k}_{16}^2 & \bar{k}_{12}^2 & \bar{k}_{17}^2 \\
\bar{k}_{66}^2 & \bar{k}_{26}^2 & \bar{k}_{67}^2 \\
\bar{k}_{88}^2 & \bar{k}_{28}^2 & \bar{k}_{78}^2
\end{bmatrix} = \begin{bmatrix}
\bar{e}_{21}\bar{e}_{26} & \bar{e}_{21}\bar{e}_{22} & \bar{e}_{21}\bar{e}_{27} \\
\bar{e}_{22} & \bar{e}_{22} & \bar{e}_{22} \\
\bar{e}_{26} & \bar{e}_{27} & \bar{e}_{26}
\end{bmatrix},
\]  

(3.1.29)
\[
\begin{bmatrix}
\bar{k}^2_{69} & \bar{k}^2_{29} & \bar{k}^2_{79} \\
\bar{k}^2_{26} & \bar{k}^2_{22} & \bar{k}^2_{27} \\
\bar{k}^2_{46} & \bar{k}^2_{24} & \bar{k}^2_{47}
\end{bmatrix}
= \begin{bmatrix}
\bar{e}_{29}\bar{e}_{26} & \bar{e}_{29}\bar{e}_{22} & \bar{e}_{29}\bar{e}_{27} \\
\bar{e}_{22} & \bar{e}_{22} & \bar{e}_{22} \\
\bar{e}_{22} & \bar{e}_{22} & \bar{e}_{22} \\
\bar{e}_{22} & \bar{e}_{22} & \bar{e}_{22}
\end{bmatrix},
\tag{3.1.30}
\]

\[
\begin{bmatrix}
\bar{k}^2_{56} & \bar{k}^2_{25} & \bar{k}^2_{57} \\
\bar{k}^2_{67} & \bar{k}^2_{27} & \bar{k}^2_{77} \\
\bar{k}^2_{36} & \bar{k}^2_{23} & \bar{k}^2_{37}
\end{bmatrix}
= \begin{bmatrix}
\bar{e}_{25}\bar{e}_{26} & \bar{e}_{25}\bar{e}_{22} & \bar{e}_{25}\bar{e}_{27} \\
\bar{e}_{22} & \bar{e}_{22} & \bar{e}_{22} \\
\bar{e}_{22} & \bar{e}_{22} & \bar{e}_{22} \\
\bar{e}_{22} & \bar{e}_{22} & \bar{e}_{22}
\end{bmatrix}.
\tag{3.1.31}
\]

For the convenience of later comparison, we list the symmetry of components of the elastic and piezoelectric tensor that will be preserved even in the eigen-coordinate system:

\[
\bar{c}_{19} = \bar{c}_{61}, \quad \bar{c}_{15} = \bar{c}_{81}, \quad \bar{c}_{65} = \bar{c}_{89}, \quad \bar{c}_{92} = \bar{c}_{26}, \quad \bar{c}_{97} = \bar{c}_{46}, \quad \bar{c}_{27} = \bar{c}_{42}, \quad \bar{c}_{34} = \bar{c}_{78}, \quad \bar{c}_{33} = \bar{c}_{38}, \quad \bar{c}_{73} = \bar{c}_{34}, \quad \bar{c}_{91} = \bar{c}_{16}, \quad \bar{c}_{62} = \bar{c}_{29}, \quad \bar{c}_{87} = \bar{c}_{45}, \quad \bar{c}_{12} = \bar{c}_{21},
\]

\[
\bar{c}_{99} = \bar{c}_{66}, \quad \bar{c}_{17} = \bar{c}_{41}, \quad \bar{c}_{95} = \bar{c}_{86}, \quad \bar{c}_{67} = \bar{c}_{49}, \quad \bar{c}_{25} = \bar{c}_{82}, \quad \bar{c}_{18} = \bar{c}_{51}, \quad \bar{c}_{64} = \bar{c}_{79}, \quad \bar{c}_{83} = \bar{c}_{35}, \quad \bar{c}_{14} = \bar{c}_{71}, \quad \bar{c}_{59} = \bar{c}_{68}, \quad \bar{c}_{13} = \bar{c}_{11}, \quad \bar{c}_{55} = \bar{c}_{88}, \quad \bar{c}_{63} = \bar{c}_{39},
\]

\[
\bar{c}_{75} = \bar{c}_{84}, \quad \bar{c}_{98} = \bar{c}_{56}, \quad \bar{c}_{24} = \bar{c}_{72}, \quad \bar{c}_{43} = \bar{c}_{37}, \quad \bar{c}_{94} = \bar{c}_{76}, \quad \bar{c}_{52} = \bar{c}_{28}, \quad \bar{c}_{93} = \bar{c}_{36}, \quad \bar{c}_{57} = \bar{c}_{48}, \quad \bar{c}_{23} = \bar{c}_{32}, \quad \bar{c}_{77} = \bar{c}_{44}, \quad \bar{c}_{29} = \bar{e}_{26}, \quad \bar{e}_{24} = \bar{e}_{27}.
\tag{3.1.32}
\]

When the symmetries given above are taken into account, the governing equation (3.1.11) and the constitutive equation (3.1.17) and (3.1.27) can be further simplified as:
\[
\begin{bmatrix}
\ddot{c}_{11} & \ddot{c}_{61} & \ddot{c}_{15} & \ddot{u}_{1,11} \\
\ddot{c}_{61} & \ddot{c}_{69} & \ddot{c}_{65} & \ddot{u}_{2,11} \\
\ddot{c}_{15} & \ddot{c}_{65} & \ddot{c}_{85} & \ddot{u}_{3,11}
\end{bmatrix}
+ \begin{bmatrix}
\ddot{c}_{46} & \ddot{c}_{46} & \ddot{c}_{46} & \ddot{u}_{1,22} \\
\ddot{c}_{26} & \ddot{c}_{22} & \ddot{c}_{27} & \ddot{u}_{2,22} \\
\ddot{c}_{46} & \ddot{c}_{27} & \ddot{c}_{47} & \ddot{u}_{3,22}
\end{bmatrix}
+ \begin{bmatrix}
\ddot{c}_{58} & \ddot{c}_{54} & \ddot{c}_{53} & \ddot{u}_{1,33} \\
\ddot{c}_{54} & \ddot{c}_{24} & \ddot{c}_{34} & \ddot{u}_{2,33} \\
\ddot{c}_{53} & \ddot{c}_{34} & \ddot{c}_{33} & \ddot{u}_{3,33}
\end{bmatrix}
+ \begin{bmatrix}
2\ddot{c}_{16} & \ddot{c}_{12} & \ddot{c}_{66} & \ddot{c}_{17} & \ddot{c}_{86} & \ddot{u}_{1,12} \\
\ddot{c}_{66} & \ddot{c}_{12} & 2\ddot{c}_{29} & \ddot{c}_{67} & \ddot{c}_{25} & \ddot{u}_{2,12} \\
\ddot{c}_{86} & \ddot{c}_{17} & \ddot{c}_{25} & \ddot{c}_{67} & 2\ddot{c}_{45} & \ddot{u}_{3,12}
\end{bmatrix}
\] (3.1.33)

\[T_{21} = \begin{bmatrix}
\ddot{c}_{16} & \ddot{c}_{66} & \ddot{c}_{66} & \ddot{u}_{1,1} \\
\ddot{c}_{66} & \ddot{c}_{29} & \ddot{c}_{25} & \ddot{u}_{2,1} \\
\ddot{c}_{17} & \ddot{c}_{69} & \ddot{c}_{45} & \ddot{u}_{3,1}
\end{bmatrix}
+ \begin{bmatrix}
\ddot{c}_{46} & \ddot{c}_{46} & \ddot{c}_{46} & \ddot{u}_{1,2} \\
\ddot{c}_{26} & \ddot{c}_{22} & \ddot{c}_{27} & \ddot{u}_{2,2} \\
\ddot{c}_{46} & \ddot{c}_{27} & \ddot{c}_{47} & \ddot{u}_{3,2}
\end{bmatrix}
+ \begin{bmatrix}
\ddot{c}_{58} & \ddot{c}_{54} & \ddot{c}_{53} & \ddot{u}_{1,3} \\
\ddot{c}_{54} & \ddot{c}_{24} & \ddot{c}_{34} & \ddot{u}_{2,3} \\
\ddot{c}_{53} & \ddot{c}_{34} & \ddot{c}_{33} & \ddot{u}_{3,3}
\end{bmatrix}
+ \begin{bmatrix}
2\ddot{c}_{16} & \ddot{c}_{12} & \ddot{c}_{66} & \ddot{c}_{17} & \ddot{c}_{86} & \ddot{u}_{1,12} \\
\ddot{c}_{66} & \ddot{c}_{12} & 2\ddot{c}_{29} & \ddot{c}_{67} & \ddot{c}_{25} & \ddot{u}_{2,12} \\
\ddot{c}_{86} & \ddot{c}_{17} & \ddot{c}_{25} & \ddot{c}_{67} & 2\ddot{c}_{45} & \ddot{u}_{3,12}
\end{bmatrix}
\] (3.1.34)

\[T_{21} = \begin{bmatrix}
\ddot{c}_{91} & \ddot{c}_{99} & \ddot{c}_{95} & \ddot{u}_{1,1} \\
\ddot{c}_{91} & \ddot{c}_{99} & \ddot{c}_{95} & \ddot{u}_{2,2} \\
\ddot{c}_{41} & \ddot{c}_{49} & \ddot{c}_{45} & \ddot{u}_{3,1}
\end{bmatrix}
+ \begin{bmatrix}
\ddot{c}_{96} & \ddot{c}_{92} & \ddot{c}_{97} & \ddot{u}_{1,2} \\
\ddot{c}_{26} & \ddot{c}_{22} & \ddot{c}_{27} & \ddot{u}_{2,2} \\
\ddot{c}_{46} & \ddot{c}_{42} & \ddot{c}_{47} & \ddot{u}_{3,2}
\end{bmatrix}
+ \begin{bmatrix}
\ddot{c}_{98} & \ddot{c}_{94} & \ddot{c}_{93} & \ddot{u}_{1,3} \\
\ddot{c}_{94} & \ddot{c}_{24} & \ddot{c}_{23} & \ddot{u}_{2,3} \\
\ddot{c}_{48} & \ddot{c}_{44} & \ddot{c}_{43} & \ddot{u}_{3,3}
\end{bmatrix}
- \begin{bmatrix}
\dddot{k}_{69} & \dddot{k}_{29} & \dddot{k}_{79} & \ddot{u}_{i}(h)/h \\
\dddot{k}_{26} & \dddot{k}_{22} & \dddot{k}_{27} & \ddot{u}_{i}(h)/h \\
\dddot{k}_{46} & \dddot{k}_{24} & \dddot{k}_{47} & \ddot{u}_{i}(h)/h
\end{bmatrix}
\] (3.1.35)
3.2 Thickness Vibrations of Unbounded Plates

In this section, we shall discuss the pure thickness vibration of an unbounded doubly-rotated quartz plate and obtain the aforementioned eigen-coordinate system.

Different from singly-rotated quartz plate, doubly-rotated quartz plate is general anisotropic in the plate coordinate system. For pure thickness vibrations, the displacement equation of motion is:

\[
\begin{bmatrix}
\hat{c}_{96} & \hat{c}_{26} & \hat{c}_{46} \\
\hat{c}_{26} & \hat{c}_{22} & \hat{c}_{27} \\
\hat{c}_{46} & \hat{c}_{27} & \hat{c}_{47}
\end{bmatrix}
\begin{bmatrix}
\ddot{u}_{1,22} \\
\ddot{u}_{2,22} \\
\ddot{u}_{3,22}
\end{bmatrix}
= \rho

\begin{bmatrix}
\dddot{u}_1 \\
\dddot{u}_2 \\
\dddot{u}_3
\end{bmatrix}
\tag{3.2.1}
\]

Search for a solution of the form:

\[
\begin{align*}
\bar{u}_1(x_2, t) &= A_1 \sin(\eta x_2) e^{i\omega t} \\
\bar{u}_2(x_2, t) &= A_2 \sin(\eta x_2) e^{i\omega t} \\
\bar{u}_3(x_2, t) &= A_3 \sin(\eta x_2) e^{i\omega t}
\end{align*}
\tag{3.2.2}
\]

Substituting (3.2.2) into (3.2.1) and simplifying the result, we get:

\[
\begin{bmatrix}
\hat{c}_{96} - \frac{\rho \omega^2}{\eta^2} & \hat{c}_{26} & \hat{c}_{46} \\
\hat{c}_{26} & \hat{c}_{22} - \frac{\rho \omega^2}{\eta^2} & \hat{c}_{27} \\
\hat{c}_{46} & \hat{c}_{27} & \hat{c}_{47} - \frac{\rho \omega^2}{\eta^2}
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_2 \\
A_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\tag{3.2.3}
\]

Inserting the material constants into (3.2.3) and solving the eigenvalue problem, we can obtain three eigenvalues and three normalized eigenvectors:

\[
\begin{bmatrix}
\bar{e}^{(1)} & 0 & 0 \\
0 & \bar{e}^{(2)} & 0 \\
0 & 0 & \bar{e}^{(3)}
\end{bmatrix}
= \begin{bmatrix}
3.4467 & 0 & 0 \\
0 & 12.0814 & 0 \\
0 & 0 & 4.1574
\end{bmatrix}
\times 10^{10} \text{ N/m}^2 ,
\tag{3.2.4}
\]
The components of each normalized eigenvector represent the direction cosines of the corresponding eigen displacement. Due to the strong anisotropy of the plate, the eigenvalues are largely separated and the particle motions associate with each wave are neither perpendicular nor parallel to the direction of propagation. The three eigen-solutions correspond to three bulk waves propagating along the $x_2$ axis, they are:

(i) Quasi-transverse wave: This wave has the lowest velocity among the three waves, which is given by $v_1 = \sqrt{c^{(1)}/\rho} = 3607.107 \text{ m/s}$. The direction of the particle motion is $q^{(1)}$, roughly along the $x_1$ axis. The corresponding mode is called the slow thickness-shear mode, also known as the C mode;

(ii) Quasi-transverse wave: The velocity of this wave lies in the middle of the three: $v_3 = \sqrt{c^{(3)}/\rho} = 3961.616 \text{ m/s}$. The direction of the particle motion is $q^{(3)}$, roughly along the
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The corresponding mode is called fast thickness-shear mode, also known as the B mode;

(iii) Quasi-longitudinal wave: This wave has the highest velocity, which is given by

\[ v_2 = \sqrt{\frac{c^2}{\rho}} = 6753.321 \text{ m/s} \]

The direction of the particle motion is \( q^{(2)} \), roughly along the \( x_2 \) axis. The corresponding mode is called the thickness-extensional mode, also known as the A mode.

In this work we are only interested in the C mode. The three eigenvectors constitute an orthonormal matrix, which is given by:

\[
Q = \begin{bmatrix}
q_1^{(1)} & q_1^{(2)} & q_1^{(3)} \\
q_2^{(1)} & q_2^{(2)} & q_2^{(3)} \\
q_3^{(1)} & q_3^{(2)} & q_3^{(3)}
\end{bmatrix} = \begin{bmatrix}
0.9586 & 0.2259 & -0.1734 \\
-0.2413 & 0.9677 & -0.0730 \\
0.1513 & 0.1118 & 0.9821
\end{bmatrix}.
\]

A general solution to (3.2.1) is:

\[
\begin{bmatrix}
\vec{u}_1(x_2) \\
\vec{u}_2(x_2) \\
\vec{u}_3(x_2)
\end{bmatrix} = \begin{bmatrix}
q_1^{(1)} & q_1^{(2)} & q_1^{(3)} \\
q_2^{(1)} & q_2^{(2)} & q_2^{(3)} \\
q_3^{(1)} & q_3^{(2)} & q_3^{(3)}
\end{bmatrix} \begin{bmatrix}
\sin(\eta_1 x_2) & 0 & 0 \\
0 & \sin(\eta_2 x_2) & 0 \\
0 & 0 & \sin(\eta_3 x_2)
\end{bmatrix} \begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix}
\]

where \( F_1, F_2 \) and \( F_3 \) are constants to be determined.
Substitution of (3.2.7) into (3.1.16) gives the electric potential of the unelectroded plate:

$$\varphi = \frac{1}{\varepsilon_{22}} \begin{bmatrix} e_{26} & e_{22} & e_{27} \end{bmatrix} \begin{bmatrix} q_1^{(1)} & q_1^{(2)} & q_1^{(3)} \\ q_2^{(1)} & q_2^{(2)} & q_2^{(3)} \\ q_3^{(1)} & q_3^{(2)} & q_3^{(3)} \end{bmatrix} \begin{bmatrix} \sin(\eta_1 x_2) & 0 & 0 \\ 0 & \sin(\eta_2 x_2) & 0 \\ 0 & 0 & \sin(\eta_3 x_2) \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}.$$  (3.2.8)

Combing (3.1.25) with (3.2.7), we get the electric potential of the electroded plate:

$$\varphi = \frac{1}{\varepsilon_{22}} \begin{bmatrix} e_{26} & e_{22} & e_{27} \end{bmatrix} \begin{bmatrix} q_1^{(1)} & q_1^{(2)} & q_1^{(3)} \\ q_2^{(1)} & q_2^{(2)} & q_2^{(3)} \\ q_3^{(1)} & q_3^{(2)} & q_3^{(3)} \end{bmatrix} \begin{bmatrix} \sin(\eta_1 x_2) - \sin(\eta_1 h)\frac{x_2}{h} & 0 & 0 \\ 0 & \sin(\eta_2 x_2) - \sin(\eta_2 h)\frac{x_2}{h} & 0 \\ 0 & 0 & \sin(\eta_3 x_2) - \sin(\eta_3 h)\frac{x_2}{h} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}.$$  (3.2.9)

(a) Free vibration of an unelectroded infinite plate

A doubly-rotated quartz plate behaves like a general anisotropic material in the plate coordinate system. In this case, the three displacement components are coupled together. As a result, the particle displacement on the plate surface have both in-plane and out-of-plane components. When immersed in a fluid, the plate generates a damped compressional wave via the out-of-plane component and a damped shear wave through the in-plane component. For the case of a general anisotropic material, a second-rank surface acoustic impedance tensor is introduced [1] to represent the dependence of surface stress components on the surface particle velocities. Thus, the mechanical boundary condition is:
\[
\begin{bmatrix}
\ddot{T}_{11}
\ddot{T}_{22}
\ddot{T}_{23}
\end{bmatrix} = \begin{bmatrix}
\ddot{\bar{Z}}_{11} & \ddot{\bar{Z}}_{12} & \ddot{\bar{Z}}_{13} & \ddot{\bar{u}}_1 \\
\ddot{\bar{Z}}_{21} & \ddot{\bar{Z}}_{22} & \ddot{\bar{Z}}_{23} & \ddot{\bar{u}}_2 \\
\ddot{\bar{Z}}_{31} & \ddot{\bar{Z}}_{32} & \ddot{\bar{Z}}_{33} & \ddot{\bar{u}}_3
\end{bmatrix}, \text{ at } x_3 = \pm h
\] (3.2.10)

Substituting from (3.2.7) into (3.1.17) and then combining the results with (3.2.10), we get:

\[
\begin{bmatrix}
\hat{c}_{96} & \hat{c}_{26} & \hat{c}_{46} \\
\hat{c}_{26} & \hat{c}_{22} & \hat{c}_{27} \\
\hat{c}_{46} & \hat{c}_{27} & \hat{c}_{47}
\end{bmatrix} \begin{bmatrix}
q_1^{(1)} & q_1^{(2)} & q_1^{(3)} \\
q_2^{(1)} & q_2^{(2)} & q_2^{(3)} \\
q_3^{(1)} & q_3^{(2)} & q_3^{(3)}
\end{bmatrix} \begin{bmatrix}
\eta_1 \cos(\eta_1 h) & 0 & 0 \\
0 & \eta_2 \cos(\eta_2 h) & 0 \\
0 & 0 & \eta_3 \cos(\eta_3 h)
\end{bmatrix} \begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix}
\]

\[
= -i \omega \begin{bmatrix}
\bar{Z}_{11} & \bar{Z}_{12} & \bar{Z}_{13} \\
\bar{Z}_{21} & \bar{Z}_{22} & \bar{Z}_{23} \\
\bar{Z}_{31} & \bar{Z}_{32} & \bar{Z}_{33}
\end{bmatrix} \begin{bmatrix}
q_1^{(1)} & q_1^{(2)} & q_1^{(3)} \\
q_2^{(1)} & q_2^{(2)} & q_2^{(3)} \\
q_3^{(1)} & q_3^{(2)} & q_3^{(3)}
\end{bmatrix} \begin{bmatrix}
\sin(\eta_1 x_2) & 0 & 0 \\
0 & \sin(\eta_2 x_2) & 0 \\
0 & 0 & \sin(\eta_3 x_2)
\end{bmatrix} \begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix}
\] (3.2.11)

Premultiply both sides of (3.11) by \(q^T\), we get

\[
\begin{bmatrix}
\bar{c}_1 & 0 & 0 \\
0 & \bar{c}_2 & 0 \\
0 & 0 & \bar{c}_3
\end{bmatrix} \begin{bmatrix}
\eta_1 \cos(\eta_1 h) & 0 & 0 \\
0 & \eta_2 \cos(\eta_2 h) & 0 \\
0 & 0 & \eta_3 \cos(\eta_3 h)
\end{bmatrix} \begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix} = -i \omega \begin{bmatrix}
Z_{11} & Z_{12} & Z_{13} \\
Z_{21} & Z_{22} & Z_{23} \\
Z_{31} & Z_{32} & Z_{33}
\end{bmatrix} \begin{bmatrix}
\sin(\eta_1 x_2) & 0 & 0 \\
0 & \sin(\eta_2 x_2) & 0 \\
0 & 0 & \sin(\eta_3 x_2)
\end{bmatrix} \begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix}
\] (3.2.12)

where

\[
\begin{bmatrix}
Z_{11} & Z_{12} & Z_{13} \\
Z_{21} & Z_{22} & Z_{23} \\
Z_{31} & Z_{32} & Z_{33}
\end{bmatrix} = \begin{bmatrix}
q_1^{(1)} & q_2^{(1)} & q_3^{(1)} \\
q_1^{(2)} & q_2^{(2)} & q_3^{(2)} \\
q_1^{(3)} & q_2^{(3)} & q_3^{(3)}
\end{bmatrix} \begin{bmatrix}
\bar{Z}_{11} & \bar{Z}_{12} & \bar{Z}_{13} \\
\bar{Z}_{21} & \bar{Z}_{22} & \bar{Z}_{23} \\
\bar{Z}_{31} & \bar{Z}_{32} & \bar{Z}_{33}
\end{bmatrix} \begin{bmatrix}
q_1^{(1)} & q_2^{(1)} & q_3^{(1)} \\
q_1^{(2)} & q_2^{(2)} & q_3^{(2)} \\
q_1^{(3)} & q_2^{(3)} & q_3^{(3)}
\end{bmatrix}
\] (3.2.13)
For a nontrivial solution, the determinant of the above equation must vanish, which yields the dispersion relationship:

\[
\begin{vmatrix}
\tilde{c}^{(1)}\eta \cos(\eta h) + i \omega Z_{11} \sin(\eta h) & i \omega Z_{12} \sin(\eta h) & i \omega Z_{13} \sin(\eta h) \\
i \omega Z_{21} \sin(\eta h) & \tilde{c}^{(2)} \eta_2 \cos(\eta_2 h) + i \omega Z_{22} \sin(\eta_2 h) & i \omega Z_{23} \sin(\eta_2 h) \\
i \omega Z_{31} \sin(\eta h) & i \omega Z_{32} \sin(\eta_2 h) & \tilde{c}^{(3)} \eta_3 \cos(\eta_3 h) + i \omega Z_{33} \sin(\eta_3 h)
\end{vmatrix} = 0.
\tag{3.2.14}
\]

As notified before, the impedance components are small quantities. For the first order approximation, the off-diagonal elements can all be neglected. Thus we get:

\[
[\tilde{c}^{(1)}\eta \cos(\eta h) + i \omega Z_{11} \sin(\eta h)][\tilde{c}^{(2)} \eta_2 \cos(\eta_2 h) + i \omega Z_{22} \sin(\eta_2 h)][\tilde{c}^{(3)} \eta_3 \cos(\eta_3 h) + i \omega Z_{33} \sin(\eta_3 h)] = 0
\tag{3.2.15}
\]

(3.2.15) is equivalent to:

\[
\tilde{c}^{(1)}\eta \cos(\eta h) + i \omega Z_{11} \sin(\eta h) = 0, \quad \tilde{c}^{(2)} \eta_2 \cos(\eta_2 h) + i \omega Z_{22} \sin(\eta_2 h) = 0, \quad \tilde{c}^{(3)} \eta_3 \cos(\eta_3 h) + i \omega Z_{33} \sin(\eta_3 h) = 0. \tag{3.2.16}
\]

Following the solution procedure given in (2.2.8)-(2.2.11), we can obtain the approximate solutions to (3.2.16) as follows:

\[
\eta_h = \frac{2l + 1}{2} \pi - \Delta_l = \frac{2l + 1}{2} \pi \left[1 + \frac{4i \omega h Z_{11}}{\tilde{c}^{(1)}(2l + 1)^2 \pi^2}\right], \quad l = 0, 1, 2, \ldots
\]

\[
\eta_2 = \frac{2m + 1}{2} \pi - \Delta_m = \frac{2m + 1}{2} \pi \left[1 + \frac{4i \omega h Z_{22}}{\tilde{c}^{(2)}(2m + 1)^2 \pi^2}\right], \quad m = 0, 1, 2, \ldots
\tag{3.2.17}
\]

\[
\eta_3 = \frac{2n + 1}{2} \pi - \Delta_n = \frac{2n + 1}{2} \pi \left[1 + \frac{4i \omega h Z_{33}}{\tilde{c}^{(3)}(2n + 1)^2 \pi^2}\right], \quad n = 0, 1, 2, \ldots
\]

At the same time, the eigenfrequencies are obtained:
\[
\omega_q^{(1)} = \sqrt{\frac{c^{(1)}}{\rho}} \frac{2l + 1}{2h} \pi \left[ 1 + \frac{4i\omega h Z_{11}^{(1)}}{c^{(1)}(2l + 1)^2 \pi^2} \right],
\]
\[
\omega_m^{(2)} = \sqrt{\frac{c^{(2)}}{\rho}} \frac{2m + 1}{2h} \pi \left[ 1 + \frac{4i\omega h Z_{22}^{(2)}}{c^{(2)}(2m + 1)^2 \pi^2} \right],
\]
\[
\omega_n^{(3)} = \sqrt{\frac{c^{(3)}}{\rho}} \frac{2n + 1}{2h} \pi \left[ 1 + \frac{4i\omega h Z_{33}^{(3)}}{c^{(3)}(2n + 1)^2 \pi^2} \right].
\]

(b) Free vibration of an electroded unbounded plate

Following the lines given above, the solution to the electroded plate can be obtained. In this case, the mechanical boundary conditions are:

\[
\begin{bmatrix}
T_{21} \\
T_{22} \\
T_{23}
\end{bmatrix} = \mp 2\rho' h \begin{bmatrix}
\ddot{u}_1 \\
\ddot{u}_2 \\
\ddot{u}_3
\end{bmatrix} + \begin{bmatrix}
\ddot{Z}_{11} & \ddot{Z}_{12} & \ddot{Z}_{13} \\
\ddot{Z}_{21} & \ddot{Z}_{22} & \ddot{Z}_{23} \\
\ddot{Z}_{31} & \ddot{Z}_{32} & \ddot{Z}_{33}
\end{bmatrix} \begin{bmatrix}
\ddot{u}_1 \\
\ddot{u}_2 \\
\ddot{u}_3
\end{bmatrix}, \text{ at } x_2 = \pm h. \quad (3.2.19)
\]

Substituting (3.1.27) into (3.2.19), we get:

\[
\begin{bmatrix}
\hat{c}_{66} & \hat{c}_{26} & \hat{c}_{46} & \ddot{u}_{1,2} \\
\hat{c}_{26} & \hat{c}_{22} & \hat{c}_{27} & \ddot{u}_{2,2} \\
\hat{c}_{46} & \hat{c}_{27} & \hat{c}_{47} & \ddot{u}_{3,2}
\end{bmatrix} - \begin{bmatrix}
k_{69}^2 & k_{29}^2 & k_{79}^2 \\
k_{26}^2 & k_{22}^2 & k_{27}^2 \\
k_{46}^2 & k_{24}^2 & k_{47}^2
\end{bmatrix} \begin{bmatrix}
\ddot{u}_{1}(h) \\
\ddot{u}_{2}(h) \\
\ddot{u}_{3}(h)
\end{bmatrix} = \begin{bmatrix}
2\rho' h' \omega^2 - ioZ_{11} & -ioZ_{12} & -ioZ_{13} \\
-ioZ_{21} & 2\rho' h' \omega^2 - ioZ_{22} & -ioZ_{23} \\
-ioZ_{31} & -ioZ_{32} & 2\rho' h' \omega^2 - ioZ_{33}
\end{bmatrix} \begin{bmatrix}
\ddot{u}_1 \\
\ddot{u}_2 \\
\ddot{u}_3
\end{bmatrix}. \quad (3.2.20)
\]

Substitute (3.2.7) into (3.2.20),
\[
\begin{bmatrix}
\hat{c}_{24} & \hat{c}_{22} & \hat{c}_{27}
\end{bmatrix}
\begin{bmatrix}
q_1^{(1)} & q_1^{(2)} & q_1^{(3)}
q_2^{(1)} & q_2^{(2)} & q_2^{(3)}
q_3^{(1)} & q_3^{(2)} & q_3^{(3)}
\end{bmatrix}
\begin{bmatrix}
\eta_1 \cos(\eta_1 h) & 0 & 0 \\
0 & \eta_2 \cos(\eta_2 h) & 0 \\
0 & 0 & \eta_3 \cos(\eta_3 h)
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix}
\]

\[
\begin{bmatrix}
q_1^{(1)} & q_1^{(2)} & q_1^{(3)}
q_2^{(1)} & q_2^{(2)} & q_2^{(3)}
q_3^{(1)} & q_3^{(2)} & q_3^{(3)}
\end{bmatrix}
\begin{bmatrix}
\sin(\eta_1 h) & 0 & 0 \\
0 & \sin(\eta_2 h) & 0 \\
0 & 0 & \sin(\eta_3 h)
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix}
\]

(3.2.21)

Premultiply both sides of (3.2.21) by \(Q^T\) and simplify,

\[
\begin{bmatrix}
\left(\frac{e_{26}^2}{c^{(1)} 2^{(1)}} + 2\rho' h' \omega^2 - i\omega Z_{11} - i\omega Z_{12} & -i\omega Z_{13}
\end{bmatrix}
\begin{bmatrix}
\eta_1 \tan(\eta_1 h) & 1 & \left(\frac{e_{26} e_{22} - i\omega Z_{12}}{c^{(2)} 2^{(2)}} \right) & \left(\frac{e_{26} e_{24} - i\omega Z_{13}}{c^{(3)} 2^{(3)}} \right) & \tan(\eta_1 h) & \\
\left(\frac{e_{22} e_{26} - i\omega Z_{21}}{c^{(1)} 2^{(1)}} \right) & \eta_1 \tan(\eta_1 h) & \left(\frac{e_{22} e_{24} - i\omega Z_{23}}{c^{(3)} 2^{(3)}} \right) & \left(\frac{e_{22} e_{24} - i\omega Z_{23}}{c^{(3)} 2^{(3)}} \right) & \tan(\eta_1 h) & \\
\left(\frac{e_{24} e_{26} - i\omega Z_{31}}{c^{(1)} 2^{(1)}} \right) & \eta_1 \tan(\eta_1 h) & \left(\frac{e_{24} e_{24} - i\omega Z_{32}}{c^{(3)} 2^{(3)}} \right) & \left(\frac{e_{24} e_{24} - i\omega Z_{33}}{c^{(3)} 2^{(3)}} \right) & \tan(\eta_1 h) & \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix} = 0
\]

(3.2.22)

where
\[
\begin{bmatrix}
  e_{26} \\
e_{22} \\
e_{24}
\end{bmatrix}
= \begin{bmatrix}
  q_1^{(1)} & q_2^{(1)} & q_3^{(1)} \\
  q_1^{(2)} & q_2^{(2)} & q_3^{(2)} \\
  q_1^{(3)} & q_2^{(3)} & q_3^{(3)}
\end{bmatrix}
\begin{bmatrix}
  e_{26} \\
e_{22} \\
e_{24}
\end{bmatrix}
= \begin{bmatrix}
  -0.0567 \\
  0.0744 \\
  -0.0614
\end{bmatrix}.
\]

The system of equations (3.2.22) has non-trivial solutions if and only if its determinant is zero:

\[
\begin{vmatrix}
  \left( \frac{e_{26}^2}{c^{(1)}e_{22}} + \frac{2\rho' h' \omega^2}{c^{(1)}} - \frac{i \hbar \omega Z_{11}}{c^{(1)}} \right) \tan(\eta_1 h) / h \eta_1 - 1 \\
  \left( \frac{e_{22} e_{26} - i \hbar \omega Z_{21}}{c^{(1)}e_{22}} \right) \tan(\eta_2 h) / h \eta_2 \\
  \left( \frac{e_{24} e_{26} - i \hbar \omega Z_{31}}{c^{(1)}e_{22}} \right) \tan(\eta_3 h) / h \eta_3
\end{vmatrix}
= 0, \quad (3.2.24)
\]

The piezoelectric coupling in quartz is very weak, the terms containing piezoelectric coefficients are very small. Moreover, both the inertial of the surface electrodes and the surface impedances are also very small. In the following analysis, the off-diagonal terms will be neglected. Expanding (3.2.24) and by the first order approximation, we get:

\[
\begin{vmatrix}
  \left( \frac{e_{26}^2}{c^{(1)}e_{22}} + \frac{2\rho' h' \omega^2}{c^{(1)}} - \frac{i \hbar \omega Z_{11}}{c^{(1)}} \right) \tan(\eta_1 h) / h \eta_1 - 1 \\
  \left( \frac{e_{22}^2}{c^{(2)}e_{22}} + \frac{2\rho' h' \omega^2}{c^{(2)}} - \frac{i \hbar \omega Z_{22}}{c^{(2)}} \right) \tan(\eta_2 h) / h \eta_2 - 1 \\
  \left( \frac{e_{24}^2}{c^{(3)}e_{22}} + \frac{2\rho' h' \omega^2}{c^{(3)}} - \frac{i \hbar \omega Z_{32}}{c^{(3)}} \right) \tan(\eta_3 h) / h \eta_3 - 1
\end{vmatrix}
= 0.
\]

(3.2.25) is equivalent to:
The approximate solutions to (3.2.26) are:

\[
\eta_1 h = \frac{2l + 1}{2} \pi \left[ 1 - \frac{4}{(2l + 1)^2 \pi^2} \frac{e_{26}^2}{c_{(1)}^2} - R + \frac{4}{(2l + 1)^2 \pi^2} \frac{i\omega Z_{11}}{c_{(1)}} \right], \quad l = 0, 1, 2, 3 \ldots
\]

\[
\eta_2 h = \frac{2m + 1}{2} \pi \left[ 1 - \frac{4}{(2m + 1)^2 \pi^2} \frac{e_{22}^2}{c_{(2)}^2} - R + \frac{4}{(2m + 1)^2 \pi^2} \frac{i\omega Z_{22}}{c_{(2)}} \right], \quad m = 0, 1, 2, 3 \ldots
\]

\[
\eta_3 h = \frac{2n + 1}{2} \pi \left[ 1 - \frac{4}{(2n + 1)^2 \pi^2} \frac{e_{24}^2}{c_{(3)}^2} - R + \frac{4}{(2n + 1)^2 \pi^2} \frac{i\omega Z_{33}}{c_{(3)}} \right], \quad n = 0, 1, 2, 3 \ldots
\]

We can also get the displacement component \( u_{i}^{(n)} \), which is obtained by projecting the displacement vector onto the eigenvector \( q^{(1)} \):

\[
\begin{align*}
\quad u_{i}^{(n)}(x, t) &= A_i \sin(\eta_i^{(n)} x) \exp(i \omega_n t),
\end{align*}
\]

in which the thickness wavenumber is given by:

\[
(\eta_i^{(n)})^2 = \left[ \frac{2(2n + 1)}{2h} \right]^2 \left[ 1 - \frac{8}{(2n + 1)^2 \pi^2} \frac{e_{26}^2}{c_{(1)}^2} - 2R \right] + \frac{2i \omega_n}{h c_{1}} Z_{11}(\omega_n). \]
The frequency-wavenumber equation for this wave component is:

\[- \bar{c}^{(1)} \eta_i^2 = - \rho \omega^2, \quad (3.2.30)\]

Substituting from (3.2.29) into (3.2.30), we obtain the approximate frequency-wavenumber equation:

\[- \bar{c}^{(1)} \left[ \frac{(2n + 1)\pi}{2h} \right]^2 + \rho \omega_n^2 - \frac{2i\omega_n}{h} Z_{11}(\omega_n) = 0. \quad (3.2.31)\]

where

\[\bar{c}^{(1)} = \bar{c}^{(1)} \left[ 1 - \frac{8}{(2l + 1)^2 \pi^2} \frac{e_{26}^2}{\bar{c}^{(1)} \bar{c}_{22}} - 2R \right]. \quad (3.2.32)\]

### 3.3 Scalar Differential Equations of Bounded Plates

When an AC voltage with arbitrary frequency is applied on the surface electrodes of an unbounded quartz plate, a vibration composed of the three normal modes is activated. In this situation, the displacement are expressed as a sum of the three eigendisplacement vectors. When the frequency of the driving voltage approaches one of the three eigenfrequencies, the normal mode corresponding to that frequency dominates. This phenomenon is call piezoelectric resonance. In real applications, a bounded quartz plate often operates with a frequency in vicinity of one of the three eigenfrequencies, so only the displacement corresponding to that normal mode is large. This is exactly the reason why they are called resonators or resonator sensors. This characteristic of the vibration enlightens us to decompose the displacement vector in the orthogonal eigenvector triad.

To this end, we choose the following orthogonal transformation:
The transformation from plate coordinate system into eigen-coordinate system enables us to make simplifying approximations based on the amplitude order of the three transformed displacement components.

Applying the orthogonal transformation (3.3.1) in (3.1.11), (3.1.17) and (3.1.27), we get:

\[
\begin{bmatrix}
\bar{u}_1 \\
\bar{u}_2 \\
\bar{u}_3 
\end{bmatrix} = \begin{bmatrix}
q_1^{(1)} & q_1^{(2)} & q_1^{(3)} \\
q_2^{(1)} & q_2^{(2)} & q_2^{(3)} \\
q_3^{(1)} & q_3^{(2)} & q_3^{(3)} 
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}. \tag{3.3.1}
\]
Premultiply both sides of (3.3.2) by $Q^T$, we get:

$$
\begin{bmatrix}
\tilde{c}_{11} & \tilde{c}_{19} & \tilde{c}_{15} & u_{1,11} \\
\tilde{c}_{61} & \tilde{c}_{69} & \tilde{c}_{65} & u_{2,11} \\
\tilde{c}_{81} & \tilde{c}_{89} & \tilde{c}_{85} & u_{3,11} \\
\end{bmatrix}
\begin{bmatrix}
\hat{c}_1 \\
\hat{c}_2 \\
\hat{c}_3 \\
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
\tilde{c}_{96} & \tilde{c}_{92} & \tilde{c}_{97} \\
\tilde{c}_{26} & \tilde{c}_{22} & \tilde{c}_{27} \\
\tilde{c}_{46} & \tilde{c}_{42} & \tilde{c}_{47} \\
\end{bmatrix}
\begin{bmatrix}
\tilde{Q} \\
\tilde{u}_{1,22} \\
\tilde{u}_{3,22} \\
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
\tilde{c}_{38} & \tilde{c}_{34} & \tilde{c}_{33} \\
\tilde{c}_{78} & \tilde{c}_{74} & \tilde{c}_{73} \\
\tilde{c}_{38} & \tilde{c}_{34} & \tilde{c}_{33} \\
\end{bmatrix}
\begin{bmatrix}
u_{1,33} \\
u_{2,33} \\
u_{3,33}
\end{bmatrix} + \\
\begin{bmatrix}
\tilde{c}_{16} & \tilde{c}_{12} + \tilde{c}_{99} & \tilde{c}_{17} + \tilde{c}_{95}
\tilde{c}_{66} + \tilde{c}_{21} & \tilde{c}_{62} + \tilde{c}_{29} & \tilde{c}_{67} + \tilde{c}_{25}
\tilde{c}_{86} + \tilde{c}_{41} & \tilde{c}_{82} + \tilde{c}_{39} & \tilde{c}_{87} + \tilde{c}_{45}
\end{bmatrix}
\begin{bmatrix}
u_{1,12} \\
u_{2,12} \\
u_{3,12}
\end{bmatrix}
\end{bmatrix}
$$

\text{(3.3.2)}
Now the elastic stiffness are expressed in the eigen-coordinate system, so the overbars are dropped. It is easy to find that the orthogonal transformation here is completely different from the tensorial orthogonal transformation, i.e. \( \tilde{c}_{ijkl} = q_{ip}q_{jq}q_{kr}q_{ls}c_{pqrs} \).

The orthogonal transformation of the stress components that will be used in the mechanical boundary conditions is given by:

\[
\begin{bmatrix}
  \bar{T}_{21} \\
  \bar{T}_{22} \\
  \bar{T}_{23}
\end{bmatrix} = Q
\begin{bmatrix}
  T_{21} \\
  T_{22} \\
  T_{23}
\end{bmatrix}.
\]

(3.3.6)

Applying (3.3.6) in (3.3.3) and (3.3.4), we obtain the transformed stress components for the unelectroded and the electroded plates:

\[
\begin{bmatrix}
  T_{21} \\
  T_{22} \\
  T_{23}
\end{bmatrix} =
\begin{bmatrix}
  c_{91} & c_{99} & c_{95} & u_{1,1} \\
  c_{21} & c_{29} & c_{25} & u_{2,1} \\
  c_{41} & c_{49} & c_{45} & u_{3,1}
\end{bmatrix}
+ \begin{bmatrix}
  \tilde{c}^{(1)} & 0 & 0 & u_{1,2} \\
  0 & \tilde{c}^{(2)} & 0 & u_{2,2} \\
  0 & 0 & \tilde{c}^{(3)} & u_{3,2}
\end{bmatrix}
+ \begin{bmatrix}
  c_{98} & c_{94} & c_{93} & u_{1,1} \\
  c_{28} & c_{24} & c_{23} & u_{2,1} \\
  c_{48} & c_{44} & c_{43} & u_{3,1}
\end{bmatrix}
,
\]

(3.3.7)

\[
\begin{bmatrix}
  T_{21} \\
  T_{22} \\
  T_{23}
\end{bmatrix} =
\begin{bmatrix}
  c_{91} & c_{99} & c_{95} & u_{1,1} \\
  c_{21} & c_{29} & c_{25} & u_{2,1} \\
  c_{41} & c_{49} & c_{45} & u_{3,1}
\end{bmatrix}
+ \begin{bmatrix}
  \tilde{c}^{(1)} & 0 & 0 & u_{1,2} \\
  0 & \tilde{c}^{(2)} & 0 & u_{2,2} \\
  0 & 0 & \tilde{c}^{(3)} & u_{3,2}
\end{bmatrix}
+ \begin{bmatrix}
  c_{98} & c_{94} & c_{93} & u_{1,1} \\
  c_{28} & c_{24} & c_{23} & u_{2,1} \\
  c_{48} & c_{44} & c_{43} & u_{3,1}
\end{bmatrix}

- \begin{bmatrix}
  k_6^2 & k_2^2 & k_4^2 & u_1(h)/h \\
  k_6^2 & k_2^2 & k_4^2 & u_2(h)/h \\
  k_6^2 & k_2^2 & k_4^2 & u_3(h)/h
\end{bmatrix},
\]

(3.3.8)

respectively.
For the convenience of future calculations, we list the numerical values of the constants:

\[
\begin{bmatrix}
\hat{c}^{(1)} \\
\hat{c}^{(2)} \\
\hat{c}^{(3)}
\end{bmatrix} = Q^T \begin{bmatrix}
c_{06} \\
c_{02} \\
c_{07}
\end{bmatrix} = \begin{bmatrix}
34.4667 \\
0 \\
0
\end{bmatrix} \times 10^9 \text{N/m}^2, \quad (3.3.9)
\]

\[
\begin{bmatrix}
c_{11} \\
c_{19} \\
c_{15}
\end{bmatrix} = Q^T \begin{bmatrix}
c_{11} \\
c_{19} \\
c_{15}
\end{bmatrix} = \begin{bmatrix}
83.1857 \\
1.3709 \\
-18.7085
\end{bmatrix} \times 10^9 \text{N/m}^2, \quad (3.3.10)
\]

\[
\begin{bmatrix}
c_{61} \\
c_{69} \\
c_{65}
\end{bmatrix} = Q^T \begin{bmatrix}
c_{61} \\
c_{69} \\
c_{65}
\end{bmatrix} = \begin{bmatrix}
66.9926 \\
-1.6974 \\
18.5319
\end{bmatrix} \times 10^9 \text{N/m}^2, \quad (3.3.11)
\]

\[
\begin{bmatrix}
c_{38} \\
c_{34} \\
c_{33}
\end{bmatrix} = Q^T \begin{bmatrix}
c_{38} \\
c_{34} \\
c_{33}
\end{bmatrix} = \begin{bmatrix}
18.5319 \\
14.1338 \\
103.0991
\end{bmatrix}
\]

\[
\begin{bmatrix}
c_{68} \\
c_{62} \\
c_{67}
\end{bmatrix} = Q^T \begin{bmatrix}
c_{68} \\
c_{62} \\
c_{67}
\end{bmatrix} = \begin{bmatrix}
30.2397 \\
26.5646 \\
-7.9647
\end{bmatrix} \times 10^9 \text{N/m}^2, \quad (3.3.12)
\]

\[
\begin{bmatrix}
c_{91} \\
c_{99} \\
c_{95}
\end{bmatrix} = Q^T \begin{bmatrix}
c_{91} \\
c_{99} \\
c_{95}
\end{bmatrix} = \begin{bmatrix}
-16.3788 \\
-6.5240 \\
1.1812
\end{bmatrix} \times 10^9 \text{N/m}^2, \quad (3.3.13)
\]

\[
\begin{bmatrix}
c_{18} \\
c_{14} \\
c_{13}
\end{bmatrix} = Q^T \begin{bmatrix}
c_{18} \\
c_{14} \\
c_{13}
\end{bmatrix} = \begin{bmatrix}
-1.8230 \\
-0.9727 \\
21.1719
\end{bmatrix}
\]

\[
\begin{bmatrix}
c_{68} \\
c_{64} \\
c_{63}
\end{bmatrix} = Q^T \begin{bmatrix}
c_{68} \\
c_{64} \\
c_{63}
\end{bmatrix} = \begin{bmatrix}
8.0610 \\
1.2884 \\
-5.7541
\end{bmatrix} \times 10^9 \text{N/m}^2, \quad (3.3.14)
\]

\[
\begin{bmatrix}
c_{51} \\
c_{59} \\
c_{55}
\end{bmatrix} = Q^T \begin{bmatrix}
c_{51} \\
c_{59} \\
c_{55}
\end{bmatrix} = \begin{bmatrix}
-1.8230 \\
8.0610 \\
61.0873
\end{bmatrix}
\]

\[
\begin{bmatrix}
c_{31} \\
c_{39} \\
c_{35}
\end{bmatrix} = Q^T \begin{bmatrix}
c_{31} \\
c_{39} \\
c_{35}
\end{bmatrix} = \begin{bmatrix}
-0.9727 \\
1.2884 \\
5.2877
\end{bmatrix} \times 10^9 \text{N/m}^2, \quad (3.3.15)
\]

\[
\begin{bmatrix}
c_{98} \\
c_{94} \\
c_{93}
\end{bmatrix} = Q^T \begin{bmatrix}
c_{98} \\
c_{94} \\
c_{93}
\end{bmatrix} = \begin{bmatrix}
1.1693 \\
4.9158 \\
-8.9395
\end{bmatrix} \times 10^9 \text{N/m}^2, \quad (3.3.16)
\]

\[
\begin{bmatrix}
c_{88} \\
c_{84} \\
c_{83}
\end{bmatrix} = Q^T \begin{bmatrix}
c_{88} \\
c_{84} \\
c_{83}
\end{bmatrix} = \begin{bmatrix}
-3.0641 \\
12.8167 \\
-6.5585
\end{bmatrix} \times 10^9 \text{N/m}^2, \quad (3.3.17)
\]

\[
\begin{bmatrix}
c_{48} \\
c_{44} \\
c_{43}
\end{bmatrix} = Q^T \begin{bmatrix}
c_{48} \\
c_{44} \\
c_{43}
\end{bmatrix} = \begin{bmatrix}
-19.3394 \\
37.5654 \\
3.8340
\end{bmatrix}
\]

\[
\begin{bmatrix}
c_{56} \\
c_{52} \\
c_{57}
\end{bmatrix} = Q^T \begin{bmatrix}
c_{56} \\
c_{52} \\
c_{57}
\end{bmatrix} = \begin{bmatrix}
1.1693 \\
-3.0641 \\
-19.3394
\end{bmatrix} \times 10^9 \text{N/m}^2, \quad (3.3.18)
\]

\[
\begin{bmatrix}
c_{76} \\
c_{72} \\
c_{77}
\end{bmatrix} = Q^T \begin{bmatrix}
c_{76} \\
c_{72} \\
c_{77}
\end{bmatrix} = \begin{bmatrix}
4.9158 \\
12.8167 \\
37.5654
\end{bmatrix} \times 10^9 \text{N/m}^2, \quad (3.3.19)
\]

\[
\begin{bmatrix}
c_{36} \\
c_{32} \\
c_{37}
\end{bmatrix} = Q^T \begin{bmatrix}
c_{36} \\
c_{32} \\
c_{37}
\end{bmatrix} = \begin{bmatrix}
-8.9395 \\
-6.5585 \\
3.8340
\end{bmatrix}
\]
The symmetry of the transformed elastic and piezoelectric constants can be inferred from the numerical values given above:

\[ c_{19} = c_{61}, \quad c_{91} = c_{16}, \quad c_{15} = c_{81}, \quad c_{18} = c_{51}, \quad c_{65} = c_{89}, \quad c_{56} = c_{98}, \quad c_{92} = c_{26}, \quad c_{29} = c_{62}, \]

\[ c_{97} = c_{46}, \quad c_{64} = c_{79}, \quad c_{27} = c_{42}, \quad c_{24} = c_{72}, \quad c_{54} = c_{78}, \quad c_{87} = c_{45}, \quad c_{53} = c_{38}, \quad c_{83} = c_{35}, \]

\[ c_{73} = c_{34}, \quad c_{43} = c_{37}, \quad c_{62} = c_{29}, \quad c_{26} = c_{92}, \quad c_{12} = c_{21}, \quad c_{99} = c_{66}, \quad c_{17} = c_{41}, c_{14} = c_{71}, \]

\[ c_{95} = c_{86}, \quad c_{59} = c_{68}, \quad c_{67} = c_{49}, \quad c_{94} = c_{76}, \quad c_{25} = c_{82}, \quad c_{52} = c_{28}, \quad c_{13} = c_{31}, c_{23} = c_{32}, \]

\[ c_{77} = c_{44}, \quad c_{55} = c_{88}, \quad c_{63} = c_{39}, \quad c_{93} = c_{36}, \quad c_{75} = c_{84}, \quad c_{57} = c_{48}, \quad c_{98} = c_{56}, c_{89} = c_{65}, \]

\[ c_{33} = c_{38}, \quad c_{35} = c_{83}, \quad e_{29} = e_{26}, \quad e_{24} = e_{27}, \quad e_{25} = e_{28}. \]  \hspace{1cm} (3.3.20)

One can verify that the material symmetry shown in (3.3.20) coincides with that given in (3.1.32). For the convenience of future applications, they are expressed in matrix form:
When the symmetries of the material constants are taken into account, (3.3.5), (3.3.7) and (3.3.8) can be further simplified as:
In this work, only essentially thickness vibrations with dominant displacement \( \vec{u}_1 \) are of interest. Their frequencies lie in the vicinity of the eigenfrequencies of the corresponding pure thickness vibration. For vibrations in such a narrow band, the in-plane decay- or wavenumbers are much smaller than the thickness wavenumbers. Consequently, when taking partial derivative of the displacements with respect to the in-plane coordinates once, the magnitude of the resultant quantity becomes one order smaller. Moreover, Tiersten pointed out that the displacement components \( u_2 \) and \( u_3 \) are one order of magnitude smaller than \( u_1 \). Keeping these features in mind, we can infer the order of magnitude of the quantities appeared in (3.3.23)-(3.3.25):
The orders of the terms in (3.3.23):

The zero order term: $u_{1,22}$;

The first order terms: $u_{1,12}, u_{1,23}, u_{2,22}, u_{3,22}$;

The second order terms: $u_{1,11}, u_{1,33}, u_{1,13}, u_{2,12}, u_{2,23}, u_{3,12}, u_{3,23}$;

The third order terms: $u_{2,11}, u_{2,33}, u_{2,13}, u_{3,11}, u_{3,33}, u_{3,13}$;

The orders of the terms in (3.3.24) and (3.3.25):

The zero order term: $u_{1,2}$;

The first order terms: $u_{1,1}, u_{1,3}, u_{2,2}, u_{3,2}$;

The second order terms: $u_{1,1}, u_{1,3}, u_{3,1}, u_{3,3}$.

At this point, it is ready for us to make certain approximations based on the orders of the terms listed above. Since we aim to obtain an approximate dispersion equation, which is accurate up to second order in the in-plane decay- and wavenumbers, it is appropriate for us to: (I) neglect all the third order terms, (II) neglect second order terms correspond to $u_i$ and $u_3$ in the governing equation for $u_2$, (III) neglect second order terms correspond to $u_i$ and $u_2$ in the governing equation for $u_3$, (IV) neglect the second order terms in the constitutive equations for $T_{22}$ and $T_{23}$. When these simplifications are taken into account, a large number of terms in the governing equations (3.3.23) and in the constitutive relations (3.3.24) and (3.3.25) are ignored, regardless of the magnitude of the associated material constants. Finally, (3.3.23)-(3.3.25) are simplified as follows:
The boundary conditions for the unelectroded plate are:

\[
\begin{bmatrix}
T_{21}^u \\
T_{22}^u \\
T_{23}^u
\end{bmatrix} = \begin{bmatrix}
\mathbf{c}_{16} & c_{66} & c_{86} \\
c_{12} & 0 & 0 \\
c_{17} & 0 & 0
\end{bmatrix} \begin{bmatrix}
u_{1,1} \\
u_{2,1} \\
u_{3,1}
\end{bmatrix} + \begin{bmatrix}
\overline{c}^{(1)} & 0 & 0 \\
0 & \overline{c}^{(2)} & 0 \\
0 & 0 & \overline{c}^{(3)}
\end{bmatrix} \begin{bmatrix}
u_{1,2} \\
u_{2,2} \\
u_{3,2}
\end{bmatrix} + \begin{bmatrix}
\mathbf{c}_{56} & c_{76} & c_{36} \\
\mathbf{c}_{52} & c_{76} & 2c_{43} \\
\mathbf{c}_{57} & c_{36} & 2c_{43}
\end{bmatrix} \begin{bmatrix}
u_{1,3} \\
u_{2,3} \\
u_{3,3}
\end{bmatrix} = \rho \begin{bmatrix}
\ddot{u}_1 \\
\ddot{u}_2 \\
\ddot{u}_3
\end{bmatrix}
\] (3.3.27)

\[
\begin{bmatrix}
T_{21}^e \\
T_{22}^e \\
T_{23}^e
\end{bmatrix} = \begin{bmatrix}
\mathbf{c}_{16} & c_{66} & c_{86} \\
c_{12} & 0 & 0 \\
c_{17} & 0 & 0
\end{bmatrix} \begin{bmatrix}
u_{1,1} \\
u_{2,1} \\
u_{3,1}
\end{bmatrix} + \begin{bmatrix}
\overline{c}^{(1)} & 0 & 0 \\
0 & \overline{c}^{(2)} & 0 \\
0 & 0 & \overline{c}^{(3)}
\end{bmatrix} \begin{bmatrix}
u_{1,2} \\
u_{2,2} \\
u_{3,2}
\end{bmatrix} + \begin{bmatrix}
\mathbf{c}_{56} & c_{76} & c_{36} \\
\mathbf{c}_{52} & c_{76} & 2c_{43} \\
\mathbf{c}_{57} & c_{36} & 2c_{43}
\end{bmatrix} \begin{bmatrix}
u_{1,3} \\
u_{2,3} \\
u_{3,3}
\end{bmatrix} = \frac{k_6^2}{k_6^2 + k_2^2} \begin{bmatrix}
u_{1,1}/h \\
u_{2,1}/h \\
u_{3,1}/h
\end{bmatrix} + \frac{k_6^2}{k_6^2 + k_2^2} \begin{bmatrix}
u_{1,2}/h \\
u_{2,2}/h \\
u_{3,2}/h
\end{bmatrix} + \frac{k_6^2}{k_6^2 + k_2^2} \begin{bmatrix}
u_{1,3}/h \\
u_{2,3}/h \\
u_{3,3}/h
\end{bmatrix}.
\] (3.3.28)

The boundary conditions for the electroded plate are:

\[
\begin{bmatrix}
T_{21} \\
T_{22} \\
T_{23}
\end{bmatrix} = \mp i\omega \begin{bmatrix}
Z_{11} & Z_{12} & Z_{13} \\
Z_{21} & Z_{22} & Z_{23} \\
Z_{31} & Z_{32} & Z_{33}
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}, \text{ at } x_2 = \pm h
\] (3.3.29)
Based on the characteristics of the vibration discussed above, we can assume a solution of the following form:

\[
\begin{align*}
T_{11} & = \pm \omega^2 2 \rho' h' \\
T_{22} & = 2 i \omega \\
T_{23} & = 2 i \omega
\end{align*}
\]

\begin{equation}
\begin{bmatrix}
Z_{11} & Z_{12} & Z_{13} \\
Z_{21} & Z_{22} & Z_{23} \\
Z_{31} & Z_{32} & Z_{33}
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{bmatrix}, \text{ at } x_2 = \pm h \quad (3.3.30)
\end{equation}

where \( \eta \) represents the thickness wavenumber, \( \zeta \) and \( \nu \) are the in-plane decay numbers.

The relations among the magnitudes of these quantities are given by: \( \eta >> \zeta, \eta >> \nu \) and \( \zeta \approx \nu \). Since it decays along both the in-plane directions, solution (3.3.31) is most suitable for the unelectroded portion of a plate which lies in the first quadrant of the plane \( O_{X_1, X_3} \) and has no common boundary with the electrode portion of the plate. For an electroded plate, the solution is of the form:

\[
\begin{align*}
u_1(x_1, x_2, x_3, t) &= [A \sin(\eta x_2) + B \cos(\eta x_2)] e^{-2i \zeta x_3} e^{i \omega t} \\
u_2(x_1, x_2, x_3, t) &= [C \sin(\eta x_2) + D \cos(\eta x_2)] e^{-2i \zeta x_3} e^{i \omega t} \\
u_3(x_1, x_2, x_3, t) &= [E \sin(\eta x_2) + F \cos(\eta x_2)] e^{-2i \zeta x_3} e^{i \omega t}
\end{align*}
\]

(3.3.31)

The differences in the two forms of solutions make very little distinction in the following derivation. Moreover, the final dispersion equations for (3.3.32) readily can be obtained from those for (3.3.31), so we use (3.3.31) in the remainder of this chapter.

Substitution of (3.3.31) into (3.3.26) yields:
\((\rho \omega^2 - \bar{c}^{(1)} \eta^2 + c_{11} \xi^2 + c_{58} \nu^2 + 2 c_{51} \xi \nu)[A \sin(\eta x_2) + B \cos(\eta x_2)]\)

\(-2\eta(c_{16} \xi + c_{58} \nu)[A \cos(\eta x_2) - B \sin(\eta x_2)]\)

\(-[(c_{12} + c_{66}) \xi + (c_{52} + c_{76}) \nu] \eta[D \cos(\eta x_2) - D \sin(\eta x_2)]\)

\(-[(c_{17} + c_{88}) \xi + (c_{57} + c_{36}) \nu] \eta[E \cos(\eta x_2) - E \sin(\eta x_2)] = 0,\)

\((\rho \omega^2 - \bar{c}^{(2)} \eta^2)[C \sin(\eta x_2) + D \cos(\eta x_2)] - 2\eta(c_{29} \xi + c_{24} \nu)[C \cos(\eta x_2) - D \sin(\eta x_2)]\)

\(-[(c_{12} + c_{66}) \xi + (c_{52} + c_{76}) \nu] \eta[D \cos(\eta x_2) - D \sin(\eta x_2)] = 0,\)

\((\rho \omega^2 - \bar{c}^{(3)} \eta^2)[E \sin(\eta x_2) + F \cos(\eta x_2)] - 2\eta(c_{45} \xi + c_{43} \nu)[E \cos(\eta x_2) - F \sin(\eta x_2)]\)

\(-[(c_{17} + c_{88}) \xi + (c_{57} + c_{36}) \nu] \eta[D \cos(\eta x_2) - D \sin(\eta x_2)] = 0.\)

Equations (3.33) hold in the whole interval \(-h \leq x_2 \leq h\). Since \(\sin(\eta x_2)\) and \(\cos(\eta x_2)\) are

linearly independent functions, their coefficients must vanish, that is

\((\rho \omega^2 - \bar{c}^{(1)} \eta^2 + c_{11} \xi^2 + c_{58} \nu^2 + 2 c_{51} \xi \nu)A + 2\eta(c_{16} \xi + c_{58} \nu)B\)

\(+[(c_{12} + c_{66}) \xi + (c_{52} + c_{76}) \nu] \eta[D + [(c_{17} + c_{88}) \xi + (c_{57} + c_{36}) \nu] \eta F = 0,\)

\((\rho \omega^2 - \bar{c}^{(2)} \eta^2 + c_{11} \xi^2 + c_{58} \nu^2 + 2 c_{51} \xi \nu)B - 2\eta(c_{16} \xi + c_{58} \nu)A\)

\(-[(c_{12} + c_{66}) \xi + (c_{52} + c_{76}) \nu] \eta[C - [(c_{17} + c_{88}) \xi + (c_{57} + c_{36}) \nu] \eta E = 0,\)

\((\rho \omega^2 - \bar{c}^{(3)} \eta^2)C + 2\eta(c_{29} \xi + c_{24} \nu)D + [(c_{12} + c_{66}) \xi + (c_{52} + c_{76}) \nu] \eta F = 0,\)

\((\rho \omega^2 - \bar{c}^{(2)} \eta^2)D - 2\eta(c_{29} \xi + c_{24} \nu)C - [(c_{12} + c_{66}) \xi + (c_{52} + c_{76}) \nu] \eta A = 0,\)

\((\rho \omega^2 - \bar{c}^{(3)} \eta^2)E + 2\eta(c_{45} \xi + c_{43} \nu)F + [(c_{17} + c_{88}) \xi + (c_{57} + c_{36}) \nu] \eta B = 0,\)

\((\rho \omega^2 - \bar{c}^{(3)} \eta^2)F - 2\eta(c_{45} \xi + c_{43} \nu)E - [(c_{17} + c_{88}) \xi + (c_{57} + c_{36}) \nu] \eta A = 0.\)

Neglecting the terms quadratic in \(\xi\) and \(\nu\) in (3.34), and writing it in matrix form, we get:

\[
\begin{bmatrix}
\rho \omega^2 - \bar{c}^{(1)} \eta^2 & -2q_1 \eta & 0 & - (q_2 + q_4) \eta & 0 & - (q_5 + q_6) \eta \\
2q_1 \eta & \rho \omega^2 - \bar{c}^{(1)} \eta^2 & (q_2 + q_4) \eta & 0 & (q_5 + q_6) \eta & 0 \\
0 & -(q_2 + q_4) \eta & \rho \omega^2 - \bar{c}^{(2)} \eta^2 & -2q_1 \eta & 0 & 0 \\
(q_2 + q_4) \eta & 0 & 2q_1 \eta & \rho \omega^2 - \bar{c}^{(2)} \eta^2 & 0 & 0 \\
0 & -(q_5 + q_6) \eta & 0 & 0 & \rho \omega^2 - \bar{c}^{(3)} \eta^2 & -2q_1 \eta \\
(q_5 + q_6) \eta & 0 & 0 & 0 & 2q_1 \eta & \rho \omega^2 - \bar{c}^{(3)} \eta^2 \\
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
C \\
D \\
E \\
F \\
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \\
0 \\
0 \\
0 \\
0 \end{bmatrix}, \quad (3.35)
\]

where

\[q_1 = -(c_{16} \xi + c_{58} \nu), \quad q_2 = -(c_{66} \xi + c_{76} \nu), \quad q_3 = -(c_{60} \xi + c_{36} \nu), \quad q_4 = -(c_{12} \xi + c_{52} \nu), \]

\[q_5 = -(c_{17} \xi + c_{57} \nu), \quad q_6 = -(c_{29} \xi + c_{24} \nu), \quad q_7 = -(c_{45} \xi + c_{43} \nu). \quad (3.36)\]

(3.35) is a homogeneous system of equations and it has nontrivial solutions if and only if the determinant vanishes. Thus we obtain a polynomial of degree six in \(\eta^2\). To obtain
the exact solution of (3.3.35) is very challenging and is unnecessary since only the solutions in the vicinity of an $u_1$ thickness resonance are of interest. On the contrary, we can solve (3.3.35) by an iterative procedure. At the first step of the iteration process, we neglect both the displacements $\ddot{u}_2$ and $\ddot{u}_3$ in (3.3.26), and retain the equation for $u_1$ only, so we get:

$$\ddot{c}^{(1)}u_{1,22} + c_{11}u_{1,11} + c_{58}u_{1,33} + 2c_{16}u_{1,12} + 2c_{36}u_{1,23} + 2c_{34}u_{1,13} = \rho \ddot{u}_1.$$  

(3.3.37)

Substituting (3.3.31) into (3.3.37) and following the steps given in (3.3.33)-(3.3.35), we get:

$$\begin{bmatrix} \rho \omega^2 - \ddot{c}^{(1)} \eta^2 & 2\eta(c_{16} \dot{\xi} + c_{36} \nu) \\ -2\eta(c_{16} \dot{\xi} + c_{36} \nu) & \rho \omega^2 - \ddot{c}^{(1)} \eta^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$  

(3.3.38)

It can be verified that the coefficient matrix of (3.3.38) coincides with the 2×2 sub-matrix at the top left corner of the matrix in (3.3.35). For a nontrivial solution, the determinant of coefficient must vanish, which results in the following dispersion relation:

$$\Delta_1 = \begin{vmatrix} \rho \omega^2 - \ddot{c}^{(1)} \eta^2 & 2\eta(c_{16} \dot{\xi} + c_{36} \nu) \\ -2\eta(c_{16} \dot{\xi} + c_{36} \nu) & \rho \omega^2 - \ddot{c}^{(1)} \eta^2 \end{vmatrix} = 0,$$  

(3.3.39)

Expanding (3.3.39) and simplifying the result, we have:

$$\ddot{c}^{(1)} \eta_i^2 \pm 2i(c_{16} \dot{\xi} + c_{36} \nu) \eta_i - \rho \omega^2 = 0,$$  

(3.3.40)

(3.3.40) has four roots:

$$\eta_i = \frac{1}{2\ddot{c}^{(1)}} \left[ \pm 2i(c_{16} \dot{\xi} + c_{36} \nu) \pm \sqrt{4\rho \omega^2 \ddot{c}^{(1)} - 4(c_{16} \dot{\xi} + c_{36} \nu)^2} \right].$$  

(3.3.41)

Only two out of the four roots are independent. Without loss of generality, we can select the two roots with positive real parts:
\[
\eta_1^+ = \frac{1}{2\bar{c}_1^{(1)}} \left[ \sqrt{4\rho\omega^2\bar{c}_1^{(1)} - 4(c_{16\xi} + c_{36\nu})^2 + 2i(c_{16\xi} + c_{36\nu})} \right] \\
\eta_1^- = \frac{1}{2\bar{c}_1^{(1)}} \left[ \sqrt{4\rho\omega^2\bar{c}_1^{(1)} - 4(c_{16\xi} + c_{36\nu})^2 - 2i(c_{16\xi} + c_{36\nu})} \right]
\]

(3.3.42)

where the subscript “1” denotes the first set of roots, the superscripts “+” and “-” denote the signs of the imaginary parts of the roots.

(3.3.42) also implies:

\[
\rho\omega^2 = \bar{c}_1^{(1)}(\eta_1^+)^2 - 2i(c_{16\xi} + c_{36\nu})\eta_1^+ \\
\rho\omega^2 = \bar{c}_1^{(1)}(\eta_1^-)^2 + 2i(c_{16\xi} + c_{36\nu})\eta_1^-.
\]

Subtracting the two equations in (3.3.43), we have:

\[
(\eta_1^+ + \eta_1^-)[\bar{c}_1^{(1)}(\eta_1^- - \eta_1^+) - 2i(c_{16\xi} + c_{36\nu})] = 0
\]

(3.3.44)

Both the real parts of \(\eta_1^+\) and \(\eta_1^-\) are positive, so:

\[
\eta_1^+ - \eta_1^- = \frac{2i}{\bar{c}_1^{(1)}}(c_{16\xi} + c_{36\nu}) .
\]

(3.3.45)

Substituting (3.3.43) into (3.3.38), we can fix the relations between the undetermined constants:

\[
B_1^+ = iA_1^+ , \quad B_1^- = -iA_1^- ,
\]

(3.3.46)

Thus we obtain the primary approximation of \(u_1:\)

\[
u_1^{(1)} = (iA_1^+ e^{-i\tilde{n}_2 x_2} - iA_1^- e^{i\tilde{n}_2 x_2}) e^{-\tilde{\xi}_1 x_1} e^{-v_2 x_2} e^{i\omega t} ,
\]

(3.3.47)

where the superscript “(1)” denotes the primary approximation of the displacement component \(u_1\).

As for the primary approximation, the stress component \(T_{21}\) of unelectroded and electroded plates can be obtained by neglecting the terms containing \(u_2\) and \(u_3\) in (3.3.27)
and (3.3.28), respectively. The stress component $T_{21}$ for unelectroded and electrode plates are:

$$T_{21} = c^{(t)} u_{1,2} + c_{16} u_{1,1} + c_{56} u_{1,3},$$  \hspace{1cm} (3.3.48)

and

$$T_{21} = c^{(t)} u_{1,2} + c_{16} u_{1,1} + c_{56} u_{1,3} - k_{66}^2 u_1(h)/h,$$  \hspace{1cm} (3.3.49)

respectively.

The piezoelectric coupling $k_{66}^2$ in quartz is very small, so for the first step of the iterative procedure, the terms containing $k_{66}^2$ can be ignored and the constitutive equation for the electroded plate degenerates into (3.3.48). Actually we can omit all the perturbations caused by the piezoelectric coupling, the surface impedances and the inertial of the electrodes. The influences of these factors will be considered in the last step of the iterative procedure. The mechanical boundary conditions for this simplified problem are:

$$T_{21} = 0, \text{ at } x_2 = \pm h.$$  \hspace{1cm} (3.3.50)

Substitution of (3.3.47) into (3.3.48) gives:

$$T_{21} = [c^{(t)}(A^+_1 \eta^+_i e^{-i\eta^+_i x_2} + A^-_1 \eta^-_i e^{i\eta^-_i x_2}) - i(c_{16} \xi e^{-i\eta^+_i x_2} + c_{56} \nu) (A^+_1 e^{-i\eta^+_i x_2} - A^-_1 e^{i\eta^-_i x_2})] e^{-\xi x_2} e^{-\nu x_2} e^{i\varphi}.$$  \hspace{1cm} (3.3.51)

Combining (3.3.51) and (3.3.50), we get:

$$\begin{bmatrix} [c^{(t)} \eta^+_i - i(c_{16} \xi + c_{56} \nu)] e^{-i\eta^+_i h} & [c^{(t)} \eta^-_i + i(c_{16} \xi + c_{56} \nu)] e^{i\eta^-_i h} \\ [c^{(t)} \eta^+_i - i(c_{16} \xi + c_{56} \nu)] e^{i\eta^+_i h} & [c^{(t)} \eta^-_i + i(c_{16} \xi + c_{56} \nu)] e^{-i\eta^-_i h} \end{bmatrix} A^+_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$  \hspace{1cm} (3.3.52)

The nontrivial solution to (3.3.52) requires:

$$\begin{bmatrix} [c^{(t)} \eta^+_i - i(c_{16} \xi + c_{56} \nu)] e^{-i\eta^+_i h} & [c^{(t)} \eta^-_i + i(c_{16} \xi + c_{56} \nu)] e^{i\eta^-_i h} \\ [c^{(t)} \eta^+_i - i(c_{16} \xi + c_{56} \nu)] e^{i\eta^+_i h} & [c^{(t)} \eta^-_i + i(c_{16} \xi + c_{56} \nu)] e^{-i\eta^-_i h} \end{bmatrix} = 0.$$  \hspace{1cm} (3.3.53)
Simplifying (3.3.53), we have:

\[ \sin(\eta_i^+ + \eta_i^-)h = 0. \]  \hspace{1cm} (3.3.54)

The roots of (3.3.54) are:

\[ \eta_i^+ + \eta_i^- = \frac{n\pi}{h}, \quad \text{where} \quad n = 1, 2, 3, \ldots. \]  \hspace{1cm} (3.3.55)

Combining (3.3.45) and (3.3.55), we obtain the first trial of \( \eta_i^+ \) and \( \eta_i^- \):

\[ \eta_i^+ = \frac{n\pi}{2h} + \frac{i}{c^{(1)}} (c_{16\xi} + c_{56\nu}), \quad \eta_i^- = \frac{n\pi}{2h} - \frac{i}{c^{(1)}} (c_{16\xi} + c_{56\nu}). \]  \hspace{1cm} (3.3.56)

It can be inferred from (3.3.42) that:

\[ \eta_i^+ + \eta_i^- = \frac{2}{c^{(1)}} \sqrt{\rho \omega_n^2 - c^{(1)^2} - (c_{16\xi} + c_{56\nu})^2}. \]  \hspace{1cm} (3.3.57)

Consequently we get the primary approximation of the eigenfrequencies from (3.3.55) and (3.3.57):

\[ \rho \omega_n^2 = \left( \frac{n\pi}{2h} \right)^2 c^{(1)} + \frac{1}{c^{(1)}} (c_{16\xi} + c_{56\nu})^2, \quad n = 1, 2, 3, \ldots. \]  \hspace{1cm} (3.3.58)

So far we have fixed the values of \( \omega_n^2 \) and \( \eta_i^+ \) together with the relations between \( A_i \) and \( B_i \). Now we are in a position to fix the undetermined constants \( C \), \( D \), \( E \) and \( F \).

The constants \( C \) and \( D \) can be obtained from (3.3.34)3 and (3.3.34)4:

\[ (\rho \omega^2 - c^{(2)}\eta^2)C + 2\eta(c_{29\xi} + c_{24\nu})D = -(c_{12} + c_{66})\xi + (c_{52} + c_{76})\nu \eta B \]

\[ (\rho \omega^2 - c^{(2)}\eta^2)D - 2\eta(c_{29\xi} + c_{24\nu})C = [(c_{12} + c_{66})\xi + (c_{52} + c_{76})\nu] \eta A. \]  \hspace{1cm} (3.3.59)

Solve for \( C \) and \( D \):

\[ C = \frac{[(c_{12} + c_{66})\xi + (c_{52} + c_{76})\nu] \eta B}{\rho \omega^2 - c^{(2)}\eta^2}, \]

\[ D = \frac{[(c_{12} + c_{66})\xi + (c_{52} + c_{76})\nu] \eta A}{\rho \omega^2 - c^{(2)}\eta^2}. \]  \hspace{1cm} (3.3.60)

where the second order terms have been neglected.
Substitution of (3.3.43) into (3.3.60) yields:

\[ C_i^+ = -i \frac{r_2 \xi + r_4 \nu}{\eta_i^+} A_i^+ , \quad C_i^- = i \frac{r_2 \xi + r_4 \nu}{\eta_i^-} A_i^- , \quad D_i^+ = \frac{r_2 \xi + r_4 \nu}{\eta_i^+} A_i^+ , \quad D_i^- = \frac{r_2 \xi + r_4 \nu}{\eta_i^-} A_i^- , \] (3.3.61)

where

\[ r_2 = \frac{c_{12} + c_{66}}{\bar{c}^{-1}(1) - \bar{c}^{-1}(2)} , \quad r_4 = \frac{c_{52} + c_{76}}{\bar{c}^{-1}(1) - \bar{c}^{-1}(2)} . \] (3.3.62)

In a similar way, we can fix \( E \) and \( F \). Rearranging (3.3.34)\(_5\) and (3.3.34)\(_6\), we get:

\[ (\rho \omega^2 - \bar{c}^{-1}(3) \eta^2) E + 2\eta (c_{45} \xi + c_{43} \nu) F = -[(c_{17} + c_{86}) \xi + (c_{57} + c_{36}) \nu] \eta B \]
\[ (\rho \omega^2 - \bar{c}^{-1}(3) \eta^2) F - 2\eta (c_{45} \xi + c_{43} \nu) E = [(c_{17} + c_{86}) \xi + (c_{57} + c_{36}) \nu] \eta A . \] (3.3.63)

Solving for \( E \) and \( F \), we obtain:

\[ E = -\frac{[(c_{17} + c_{86}) \xi + (c_{57} + c_{36}) \nu] \eta}{\rho \omega^2 - \bar{c}^{-1}(3) \eta^2} B , \]
\[ F = \frac{[(c_{17} + c_{86}) \xi + (c_{57} + c_{36}) \nu] \eta}{\rho \omega^2 - \bar{c}^{-1}(3) \eta^2} A . \] (3.3.64)

Substitution of (3.3.43) into (3.3.64) yields:

\[ E_i^+ = -i \frac{r_3 \nu + r_5 \xi}{\eta_i^+} A_i^+ , \quad E_i^- = i \frac{r_3 \nu + r_5 \xi}{\eta_i^-} A_i^- , \quad F_i^+ = \frac{r_3 \nu + r_5 \xi}{\eta_i^+} A_i^+ , \quad F_i^- = \frac{r_3 \nu + r_5 \xi}{\eta_i^-} A_i^- , \] (3.3.65)

where

\[ r_3 = \frac{c_{57} + c_{36}}{\bar{c}^{-1}(1) - \bar{c}^{-1}(3)} , \quad r_5 = \frac{c_{17} + c_{86}}{\bar{c}^{-1}(1) - \bar{c}^{-1}(3)} . \] (3.3.66)

(3.3.46), (3.3.61) and (3.3.65) constitute parts of the complete solution which corresponding to the first set of roots \( \eta_i^+ \) and \( \eta_i^- \).

Proceeding in a similar way, we can get the parts of the complete solution corresponding to large \( u_2 \), i.e. large \( C \) and \( D \) and small decay numbers along the plate in the vicinity of the \( u_i \)-thickness frequency \( \omega_n \). From (3.3.34)\(_3\) and (3.3.34)\(_4\) we have:
\[
\begin{bmatrix}
\rho\omega_n^2 - \bar{c}^{(2)}\eta^2 & 2\eta(c_{29}\xi + c_{24}\nu) \\
-2\eta(c_{29}\xi + c_{24}\nu) & \rho\omega_n^2 - \bar{c}^{(2)}\eta^2
\end{bmatrix}
\begin{bmatrix}
C \\
D
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]  

(3.3.67)

The nontrivial solution to (3.3.67) requires:

\[
\Delta_2 = \begin{vmatrix}
\rho\omega_n^2 - \bar{c}^{(2)}\eta^2 & 2\eta(c_{29}\xi + c_{24}\nu) \\
-2\eta(c_{29}\xi + c_{24}\nu) & \rho\omega_n^2 - \bar{c}^{(2)}\eta^2
\end{vmatrix} = 0,
\]  

(3.3.68)

Similar to the case of \( \Delta_1 \), we can obtain two roots \( \eta_2^+ \) and \( \eta_2^- \):

\[
\eta_2^+ = \frac{1}{\bar{c}^{(2)}} \left[ -i(c_{29}\xi + c_{24}\nu) + \sqrt{\rho\bar{c}^{(2)}\omega_n^2 - (c_{29}\xi + c_{24}\nu)^2} \right],
\]

\[
\eta_2^- = \frac{1}{\bar{c}^{(2)}} \left[ -i(c_{29}\xi + c_{24}\nu) + \sqrt{\rho\bar{c}^{(2)}\omega_n^2 - (c_{29}\xi + c_{24}\nu)^2} \right],
\]  

(3.3.69)

where the subscript “2” denotes quantities corresponding to the second set of roots.

Substituting from (3.3.43) into (3.3.69), we can further simplify the roots as:

\[
\eta_2^+ = \kappa_2\eta_1^+ \left[ 1 - \frac{i(c_{16}\xi + c_{16}\nu)}{\bar{c}^{(1)}\eta_1^+} + \frac{i(c_{29}\xi + c_{24}\nu)}{\bar{c}^{(2)}\kappa_2\eta_1^+} \right],
\]

\[
\eta_2^- = \kappa_2\eta_1^- \left[ 1 + \frac{i(c_{16}\xi + c_{16}\nu)}{\bar{c}^{(1)}\eta_1^-} - \frac{i(c_{29}\xi + c_{24}\nu)}{\bar{c}^{(2)}\kappa_2\eta_1^-} \right],
\]  

(3.3.70)

where

\[
\kappa_2 = \sqrt{\frac{\bar{c}^{(1)}}{\bar{c}^{(2)}}}.
\]  

(3.3.71)

When the first order terms are neglected, (3.3.70) implies the approximate relationship:

\[
\frac{\eta_1^2}{\eta_2^2} \approx \frac{1}{\kappa_2^2} = \frac{\bar{c}^{(2)}}{\bar{c}^{(1)}}.
\]  

(3.3.72)

Following the same procedure as that for obtaining (3.3.46), we can obtain the relations between \( C \) and \( D \) from (3.3.67) and (3.3.69):

\[
D_2^+ = iC_2^+ , \quad D_2^- = -iC_2^- .
\]  

(3.3.73)

The amplitudes \( A \) and \( B \) corresponding to \( \eta_2^\pm \) are obtained from (3.3.34)\(_1\) and (3.3.34)\(_2\):

\[
\]
\[ A = - \frac{(c_{12} + c_{66}) \xi \eta + (c_{42} + c_{76}) \nu \eta}{\rho \omega^2 - c^{(1)} \eta^2} D, \quad B = \frac{(c_{12} + c_{66}) \xi \eta + (c_{42} + c_{76}) \nu \eta}{\rho \omega^2 - c^{(1)} \eta^2} C. \] (3.3.74)

Substituting from (3.3.43) into (3.3.74) and simplifying the results by (3.3.72), we get:

\[ A_3^+ = i \frac{r_5 \xi + r_6 \nu}{\eta^*_3} C_2^+, \quad A_2^- = -i \frac{r_5 \xi + r_6 \nu}{\eta^*_2} C_2^-, \quad B_3^+ = - \frac{r_5 \xi + r_6 \nu}{\eta^*_3} C_2^+, \quad B_2^- = - \frac{r_5 \xi + r_6 \nu}{\eta^*_2} C_2^-. \] (3.3.75)

\( E_2^+ \) and \( F_2^+ \) are one order of magnitude smaller than \( C_2^+ \) and \( D_2^+ \). Detailed analysis shows we do not need to consider them in the subsequent derivation.

The large \( \bar{\omega}_3 \) solution, i.e., for large \( E \) and \( F \) and small decay numbers along the plate in the vicinity of the \( \bar{\omega}_1 \)-thickness frequency can be obtained through a similar procedure as that for the large \( \bar{\omega}_3 \) solution. We list the major results below:

\[ \eta_3^+ = \kappa_3 \eta_1^+ \left[ 1 - \frac{i(c_{10} \xi + c_{56} \nu)}{c^{(1)} \eta_1^+} + \frac{i(c_{45} \xi + c_{43} \nu)}{c^{(3)} \kappa_3 \eta_1^+} \right], \]

\[ \eta_3^- = \kappa_3 \eta_1^+ \left[ 1 + \frac{i(c_{10} \xi + c_{56} \nu)}{c^{(1)} \eta_1^-} - \frac{i(c_{45} \xi + c_{43} \nu)}{c^{(3)} \kappa_3 \eta_1^-} \right], \] (3.3.76)

where

\[ \kappa_3 = \sqrt{\frac{c^{(1)}}{c^{(3)}}}, \] (3.3.77)

\[ \frac{\eta_1^2}{\eta_3^2} \approx \frac{1}{\kappa_3^2} = \frac{c^{(3)}}{c^{(1)}}, \] (3.3.78)

\[ F_3^+ = i F_3^-, \quad F_3^- = -i F_3^+, \] (3.3.79)

\[ A_3^+ = i \frac{r_5 \xi + r_6 \nu}{\eta^*_3} E_3^+, \quad A_3^- = -i \frac{r_5 \xi + r_6 \nu}{\eta^*_3} E_3^- \quad B_3^+ = - \frac{r_5 \xi + r_6 \nu}{\eta^*_3} E_3^+, \quad B_3^- = - \frac{r_5 \xi + r_6 \nu}{\eta^*_3} E_3^- \] (3.3.80)
The complete solution to (3.3.26) can be expressed as a summation of the six asymptotic solutions obtained above:

\[
\begin{align*}
    u_1 &= A_1^+ [\sin(\eta^+_1 x_2) + i \cos(\eta^+_1 x_2)] + A_1^- [\sin(\eta^-_1 x_2) - i \cos(\eta^-_1 x_2)] \\
    &\quad + i \frac{r_2 \xi + r_4 \nu}{\eta^+_2} [\sin(\eta^+_2 x_2) + i \cos(\eta^+_2 x_2)]C^+_2 - i \frac{r_2 \xi + r_4 \nu}{\eta^-_2} [\sin(\eta^-_2 x_2) - i \cos(\eta^-_2 x_2)]C^-_2 \\
    &\quad + i \frac{r_3 \xi + r_4 \nu}{\eta^+_3} [\sin(\eta^+_3 x_2) + i \cos(\eta^+_3 x_2)]E^+_3 - i \frac{r_3 \xi + r_4 \nu}{\eta^-_3} [\sin(\eta^-_3 x_2) - i \cos(\eta^-_3 x_2)]E^-_3 \\
    u_2 &= -i \frac{r_2 \xi + r_4 \nu}{\eta^+_1} [\sin(\eta^+_1 x_2) + i \cos(\eta^+_1 x_2)]A^+_1 + i \frac{r_2 \xi + r_4 \nu}{\eta^-_1} [\sin(\eta^-_1 x_2) - i \cos(\eta^-_1 x_2)]A^-_1 \\
    &\quad + [\sin(\eta^+_1 x_2) + i \cos(\eta^+_1 x_2)]C^+_2 + [\sin(\eta^-_1 x_2) - i \cos(\eta^-_1 x_2)]C^-_2 \\
    u_3 &= -i \frac{r_3 \xi + r_4 \nu}{\eta^+_1} [\sin(\eta^+_1 x_2) + i \cos(\eta^+_1 x_2)]A^+_1 + i \frac{r_3 \xi + r_4 \nu}{\eta^-_1} [\sin(\eta^-_1 x_2) - i \cos(\eta^-_1 x_2)]A^-_1 \\
    &\quad + [\sin(\eta^+_1 x_2) + i \cos(\eta^+_1 x_2)]E^+_3 + [\sin(\eta^-_1 x_2) - i \cos(\eta^-_1 x_2)]E^-_3
\end{align*}
\]

(3.3.81)

Substituting (3.3.81) into the mechanical boundary conditions (3.3.29), we can obtain a system of homogeneous equations for the undetermined constants. The system of equations has nontrivial solutions if and only if the coefficient determinant vanishes. This results in the dispersion relation for the vibration of the bounded plate. The details are as follows:

The boundary conditions for the stress component \( T_{21} \) are:

\[
\begin{align*}
    \bar{c}_1 u_1(h) + q_1 u_1(h) + q_2 u_2(h) + q_3 u_3(h) - k_{26}^2 \frac{u_1(h)}{h} - k_{26}^2 \frac{u_2(h)}{h} - k_{36}^2 \frac{u_3(h)}{h} \\
    = 2 \rho' h \omega^2 u_1(h) - i \omega(Z_1 u_1(h) + Z_{21} u_2(h) + Z_{31} u_3(h)), \quad \text{at } x_2 = h,
\end{align*}
\]

(3.3.82)
\[ \bar{c}\mu_{1,2}(-h) + q_{1}\mu_{1}(-h) + q_{2}\mu_{2}(-h) + q_{3}\mu_{3}(-h) - k_{66}^2 \frac{u_{1}(h)}{h} - k_{26}^2 \frac{u_{2}(h)}{h} - k_{46}^2 \frac{u_{3}(h)}{h} = -2 \rho'h'\omega^2 u_{1}(-h) + i\alpha[Z_{1}\mu_{1}(-h) + Z_{2}\mu_{2}(-h) + Z_{3}\mu_{3}(-h)], \quad \text{at } x_{2} = -h. \]  

Substituting from (3.3.81) into (3.3.82) and (3.3.83), adding and subtracting the resultant equations, we obtain:

\[ \left[ \frac{c^{(1)}_1}{\eta_1^+} + iq_1 - i\frac{k_{66}^2}{h} + q_2 \frac{p_2}{\eta_1^+} + q_3 \frac{p_3}{\eta_1^+} - k_{26}^2 \frac{p_2}{h} \frac{1}{\eta_1^+} - k_{46}^2 \frac{p_3}{h} \frac{1}{\eta_1^+} \right] \cos(\eta_1^+ h) \]

\[ + \left[ i\omega Z_{11} - 2 \rho'h'\omega^2 - \frac{k_{66}^2}{h} + \omega Z_{12} \frac{p_2}{\eta_1^+} + \omega Z_{13} \frac{p_3}{\eta_1^+} + i\frac{k_{26}^2}{h} \frac{p_2}{\eta_1^+} + i\frac{k_{46}^2}{h} \frac{p_3}{\eta_1^+} \right] \sin(\eta_1^+ h) \]

\[ A_i^+ \]

\[ \left[ \frac{c^{(1)}_1}{\eta_1^-} - iq_1 + i\frac{k_{66}^2}{h} + q_2 \frac{p_2}{\eta_1^-} + q_3 \frac{p_3}{\eta_1^-} - k_{26}^2 \frac{p_2}{h} \frac{1}{\eta_1^-} - k_{46}^2 \frac{p_3}{h} \frac{1}{\eta_1^-} \right] \cos(\eta_1^- h) \]

\[ + \left[ i\omega Z_{11} - 2 \rho'h'\omega^2 - \frac{k_{66}^2}{h} - \omega Z_{12} \frac{p_2}{\eta_1^-} - \omega Z_{13} \eta_1^- p_3 - i\frac{k_{26}^2}{h} \frac{p_2}{\eta_1^-} - i\frac{k_{46}^2}{h} \frac{p_3}{\eta_1^-} \right] \sin(\eta_1^- h) \]

\[ A_i^- \]

\[ \left[ \left[ iq_2 - i\frac{k_{66}^2}{h} + i\frac{k_{66}^2}{h} + \frac{k_{66}^2}{h} \frac{p_2}{h} \right] \cos(\eta_2^- h) \right] \]

\[ \left[ i\omega Z_{12} - i\frac{k_{66}^2}{h} \frac{p_2}{\eta_2^-} + \omega Z_{11} \frac{p_2}{\eta_2^-} - 2i\rho'h'\omega^2 \frac{p_2}{\eta_2^-} \sin(\eta_2^- h) \right] \]

\[ C_2^+ \]

\[ - \left[ \left[ iq_3 + i\frac{k_{66}^2}{h} + i\frac{k_{66}^2}{h} + \frac{k_{66}^2}{h} \frac{p_3}{h} \right] \cos(\eta_3^- h) \right] \]

\[ \left[ i\omega Z_{13} - i\frac{k_{66}^2}{h} \frac{p_3}{\eta_3^-} + \omega Z_{11} \frac{p_3}{\eta_3^-} - 2i\rho'h'\omega^2 \frac{p_3}{\eta_3^-} \sin(\eta_3^- h) \right] \]

\[ E_3^+ \]

\[ - \left[ \left[ iq_3 + i\frac{k_{66}^2}{h} + i\frac{k_{66}^2}{h} + \frac{k_{66}^2}{h} \frac{p_3}{h} \right] \cos(\eta_3^- h) \right] \]

\[ \left[ i\omega Z_{13} - i\frac{k_{66}^2}{h} \frac{p_3}{\eta_3^-} + \omega Z_{11} \frac{p_3}{\eta_3^-} + 2i\rho'h'\omega^2 \frac{p_3}{\eta_3^-} \sin(\eta_3^- h) \right] \]

\[ E_3^- = 0 \]

where
\[ p_2 = r_2 \xi + r_4 \nu, \quad p_3 = r_5 \xi + \eta \nu, \quad (3.3.85) \]

\[
\begin{align*}
&\left[ -i\tilde{c}^{(1)} \eta_i + q_i - \frac{p_2}{\eta_i} - i q_3 \frac{p_3}{\eta_i} \right] \sin(\eta_i^+ h) + \left[ -\omega Z_{11} - 2i \rho h' \omega^2 + i \omega Z_{12} \frac{p_2}{\eta_i} + i \omega Z_{13} \frac{p_3}{\eta_i} \right] \cos(\eta_i^+ h) \right] A_i^+ + \\
&\left[ i \tilde{c}^{(1)} \eta_i + q_i + \frac{p_2}{\eta_i} + i q_3 \frac{p_3}{\eta_i} \right] \sin(\eta_i^- h) + \left[ \omega Z_{11} + 2i \rho h' \omega^2 + i \omega Z_{12} \frac{p_2}{\eta_i} + i \omega Z_{13} \frac{p_3}{\eta_i} \right] \cos(\eta_i^- h) \right] A_i^- + \\
&\left[ q_2 + \tilde{c}^{(1)} \frac{p_2}{\eta_2} - i q_1 \frac{p_2}{\eta_2} \right] \sin(\eta_2^+ h) + \left[ -\omega Z_{12} - i \omega Z_{11} \frac{p_2}{\eta_2} + 2 \rho h' \omega^2 \frac{p_2}{\eta_2} \right] \cos(\eta_2^+ h) \right] C_i^+ + \\
&\left[ q_2 + \tilde{c}^{(1)} \frac{p_2}{\eta_2} + i q_1 \frac{p_2}{\eta_2} \right] \sin(\eta_2^- h) + \left[ \omega Z_{12} - i \omega Z_{11} \frac{p_2}{\eta_2} + 2 \rho h' \omega^2 \frac{p_2}{\eta_2} \right] \cos(\eta_2^- h) \right] C_i^- + \\
&\left[ q_3 + \tilde{c}^{(1)} \frac{p_3}{\eta_3} - i q_1 \frac{p_3}{\eta_3} \right] \sin(\eta_3^+ h) + \left[ -\omega Z_{13} - i \omega Z_{11} \frac{p_3}{\eta_3} + 2 \rho h' \omega^2 \frac{p_3}{\eta_3} \right] \cos(\eta_3^+ h) \right] E_i^+ + \\
&\left[ q_3 + \tilde{c}^{(1)} \frac{p_3}{\eta_3} + i q_1 \frac{p_3}{\eta_3} \right] \sin(\eta_3^- h) + \left[ \omega Z_{13} - i \omega Z_{11} \frac{p_3}{\eta_3} + 2 \rho h' \omega^2 \frac{p_3}{\eta_3} \right] \cos(\eta_3^- h) \right] E_i^- = 0
\tag{3.3.86}
\]

The boundary conditions for the stress components \( T_{22} \) are given by:

\[
T_{22}(x_2 = h) : \quad \tilde{c}^{(2)} u_{2,2}(h) + q_4 u_1(h) - k_{26}^2 \frac{u_1(h)}{h} - k_{22}^2 \frac{u_2(h)}{h} = 2 \rho h' \omega^2 u_2(h) - i \omega Z_{21} u_1(h) + Z_{22} u_2(h) \tag{3.3.87}
\]

\[
T_{22}(x_2 = -h) : \quad \tilde{c}^{(2)} u_{2,2}(-h) + q_4 u_1(-h) - k_{26}^2 \frac{u_1(-h)}{h} - k_{22}^2 \frac{u_2(-h)}{h} = 2 \rho h' \omega^2 u_2(-h) - i \omega Z_{21} u_1(-h) + Z_{22} u_2(-h) \tag{3.3.88}
\]

Substituting from (3.3.81) into (3.3.87) and (3.3.88), then adding and subtracting the resultant equations, we obtain:
\[
\begin{align*}
&\left\{i\left(q_4 - c^{(2)} p_4 - \frac{k_{26}^2}{h}\right) \cos(\eta_1^+ h) + \left[i\omega Z_{21} - \frac{k_{26}^2}{h} + (\omega Z_{22} + i2\rho' \omega^2) \frac{p_2}{\eta_1^+}\right] \sin(\eta_1^+ h)\right\}A_1^+ + \\
&\left\{i\left(c^{(2)} p_4 - q_4 + \frac{k_{26}^2}{h}\right) \cos(\eta_1^- h) + \left[i\omega Z_{21} - \frac{k_{26}^2}{h} - (\omega Z_{22} + i2\rho' \omega^2) \frac{p_2}{\eta_1^-}\right] \sin(\eta_1^- h)\right\}A_1^- + \\
&\left\{\left(c^{(2)} \eta_2^+ - q_4 - \frac{k_{26}^2 p_2}{\eta_2^+} + \frac{k_{26}^2}{h} \frac{p_2}{\eta_2^+}\right) \cos(\eta_2^+ h) + \left[i\omega Z_{22} - 2\rho' \omega^2 - \omega Z_{12} \frac{p_2}{\eta_2^+} - i\frac{k_{26}^2 p_2}{\eta_2^+}\right] \sin(\eta_2^+ h)\right\}C_2^+ + \\
&\left\{\left(c^{(2)} \eta_2^- - q_4 - \frac{k_{26}^2 p_2}{\eta_2^-} + \frac{k_{26}^2}{h} \frac{p_2}{\eta_2^-}\right) \cos(\eta_2^- h) + \left[i\omega Z_{22} - 2\rho' \omega^2 + \omega Z_{12} \frac{p_2}{\eta_2^-} + i\frac{k_{26}^2 p_2}{\eta_2^-}\right] \sin(\eta_2^- h)\right\}C_2^- + \\
&\left\{\left(\frac{k_{26}^2}{h} - q_4\right) \cos(\eta_3^+ h) + \left(-\omega Z_{21} - i\frac{k_{26}^2}{h}\right) \sin(\eta_3^+ h)\right\} \left[\frac{p_3}{\eta_3^+} E_3^+ \right] + \left[\left(\frac{k_{26}^2}{h} - q_4\right) \cos(\eta_3^- h) + \left(\omega Z_{21} + i\frac{k_{26}^2}{h}\right) \sin(\eta_3^- h)\right] \left[\frac{p_3}{\eta_3^-} E_3^- \right] = 0 
\end{align*}
\]

\[
\begin{align*}
&\left\{(q_4 - c^{(2)} p_4) \sin(\eta_1^+ h) + \left[-\omega Z_{21} + (i\omega Z_{22} - 2\rho' \omega^2) \frac{p_2}{\eta_1^+}\right] \cos(\eta_1^+ h)\right\}A_1^+ + \\
&\left\{(q_4 - c^{(2)} p_4) \sin(\eta_1^- h) + \left[\omega Z_{21} + (i\omega Z_{22} - 2\rho' \omega^2) \frac{p_2}{\eta_1^-}\right] \cos(\eta_1^- h)\right\}A_1^- + \\
&\left\{\left(-c^{(2)} \eta_2^+ + q_4 \frac{p_2}{\eta_2^+}\right) \sin(\eta_2^+ h) + \left[-i\omega Z_{21} \frac{p_2}{\eta_2^+} - (\omega Z_{22} + 2i\rho' \omega^2)\right] \cos(\eta_2^+ h)\right\}C_2^+ + \\
&\left\{\left(c^{(2)} \eta_2^- - q_4 \frac{p_2}{\eta_2^-}\right) \sin(\eta_2^- h) + \left[-i\omega Z_{21} \frac{p_2}{\eta_2^-} + (\omega Z_{22} + 2i\rho' \omega^2)\right] \cos(\eta_2^- h)\right\}C_2^- + \\
&\left\{iq_4 \frac{p_3}{\eta_3^+} \sin(\eta_3^+ h) - i\omega Z_{21} \frac{p_3}{\eta_3^+} \cos(\eta_3^+ h)\right\}E_3^+ + \left[-iq_4 \frac{p_3}{\eta_3^-} \sin(\eta_3^- h) - i\omega Z_{21} \frac{p_3}{\eta_3^-} \cos(\eta_3^- h)\right]E_3^- = 0 
\end{align*}
\]

The boundary conditions for the stress component \( T_{23} \) are:
Substituting from (3.3.81) into (3.3.91) and (3.3.92), then adding and subtracting the resultant equations, we get:

\[
T_{23}(x_2 = h) : \ c^{(3)}u_{3,2}(h) + q_3u_1(h) - k_{46}^2 \frac{u_1(h)}{h} - k_{44}^2 \frac{u_3(h)}{h} = 2\rho' h' \omega^2 u_3(h) - i\omega[Z_3u_1(h) + Z_{33}u_3(h)] \\
(3.3.91)
\]

\[
T_{23}(x_2 = -h) : \ c^{(3)}u_{3,2}(-h) + q_3u_1(-h) - k_{46}^2 \frac{u_1(h)}{h} - k_{44}^2 \frac{u_3(h)}{h} = -2\rho' h' \omega^2 u_3(-h) + i\omega[Z_3u_1(-h) + Z_{33}u_3(-h)] \\
(3.3.92)
\]

\[
\left\{ \left( q_5 - c^{(3)} p_3 - \frac{k_{46}^2}{h} \right) \cos(\eta_1^+ h) + \left[ i\omega Z_{31} - \frac{k_{46}^2}{h} + (\omega Z_{33} + 2i\rho' h' \omega^2) \frac{p_3}{\eta_1^+} \right] \sin(\eta_1^+ h) \right\} A_1^+ + \\
\left\{ -i \left( q_5 - c^{(3)} p_3 - \frac{k_{46}^2}{h} \right) \cos(\eta_1^- h) + \left[ i\omega Z_{31} - \frac{k_{46}^2}{h} - (\omega Z_{33} + 2i\rho' h' \omega^2) \frac{p_3}{\eta_1^-} \right] \sin(\eta_1^- h) \right\} A_1^- + \\
\left\{ \left( c^{(3)} \eta_3^+ - q_5 p_3 + \frac{k_{46}^2 p_3}{h} \eta_3^+ \right) \cos(\eta_3^+ h) + \left( -\frac{k_{46}^2}{h} \eta_3^+ - \omega Z_{31} \frac{p_3}{\eta_3^+} - i\omega Z_{33} - 2\rho' h' \omega^2 \right) \sin(\eta_3^+ h) \right\} E_3^+ + \\
\left\{ \left( c^{(3)} \eta_3^- - q_5 p_3 + \frac{k_{46}^2 p_3}{h} \eta_3^- \right) \cos(\eta_3^- h) + \left( i\frac{k_{46}^2}{h} \eta_3^- + \omega Z_{31} p_3 \frac{1}{\eta_3^-} + i\omega Z_{33} - 2\rho' h' \omega^2 \right) \sin(\eta_3^- h) \right\} E_3^- + \\
\left\{ \left( -q_5 + \frac{k_{46}^2}{h} \right) \cos(\eta_2^+ h) + \left( -i \frac{k_{46}^2}{h} - \omega Z_{31} \right) \sin(\eta_2^+ h) \right\} \frac{p_2}{\eta_2} C_2^+ + \left\{ -q_5 + \frac{k_{46}^2}{h} \right\} \cos(\eta_2^- h) + \left( i \frac{k_{46}^2}{h} + \omega Z_{31} \right) \sin(\eta_2^- h) \right\} \frac{p_2}{\eta_2} C_2^- = 0 \\
(3.3.93)
\]
(3.3.84), (3.3.86), (3.3.89), (3.3.90), (3.3.93) and (3.3.94) constitute a system of six homogeneous linear algebraic equations in the six amplitudes: $A_1^+$, $A_1^-$, $C_2^+$, $C_2^-$, $E_3^+$ and $E_3^-$. For a nontrivial solution, the determinant of coefficients must vanish. Thus we obtain an implicit dispersion relation for the essentially thickness vibration. Since we are interested in vibrations in the vicinity of the pure $u_1$-thickness modes, for which $A_1^+$ and $A_1^-$ are large and $\eta_1$ is very near the expression given in (3.3.56), we can express the roots $\eta_1^+ h$ and $\eta_1^- h$ in the following form:

\[
\eta_1^+ h = \frac{n\pi}{2} + \frac{1}{c(1)} \frac{c_{60} \xi + c_{50} y}{c_{16 \xi} + c_{50} y} h + \frac{1}{c} \frac{c_{21} \xi + c_{50} y}{c_{16} \xi + c_{50} y} h + \frac{n\pi}{2} + \beta_1^+ , \quad n = 1, 3, 5, \cdots
\]
\[
\eta_1^- h = \frac{n\pi}{2} - \frac{1}{c(1)} \frac{c_{60} \xi + c_{50} y}{c_{16 \xi} + c_{50} y} h + \frac{1}{c} \frac{c_{21} \xi + c_{50} y}{c_{16} \xi + c_{50} y} h + \frac{n\pi}{2} + \beta_1^- , \quad n = 1, 3, 5, \cdots
\]
where $\alpha_n^+\,\alpha_n^-,\,\beta_n^+$ and $\beta_n^-$ are all small quantities which represent the perturbation caused by surface loadings, inertia of electrodes, piezoelectric coupling as well as the effects of finite lateral dimensions.

From (3.3.45) and (3.3.95), we know $\alpha_n^+ = \alpha_n^- = \alpha_n$. Thus

$$\beta_n^+ + \beta_n^- = 2\alpha_n.$$  \hspace{1cm} (3.3.96)

Expanding the trigonometric functions of $\eta h$, we obtain:

$$\cos \left( \frac{n\pi}{2} + \beta_n \right) = \cos \left( \frac{n\pi}{2} \right) \cos(\beta_n) - \sin \left( \frac{n\pi}{2} \right) \sin(\beta_n) \approx (-1)^{\frac{n+1}{2}} \beta_n,$$

$$\sin \left( \frac{n\pi}{2} + \beta_n \right) = \sin \left( \frac{n\pi}{2} \right) \cos(\beta_n) + \cos \left( \frac{n\pi}{2} \right) \sin(\beta_n) \approx (-1)^{\frac{n-1}{2}} .$$ \hspace{1cm} (3.3.97)

Substituting (3.3.97) into the determinant of the coefficient matrix, we get:
\[
\left( iq_3 + i\bar{c}^{(1)} p_3 - i\frac{k_{60}^2}{h} \right) \cos(\eta_3^+ h) + \left( i\alpha Z_{13} - \frac{k_{66}^2}{h} \right) \sin(\eta_3^+ h) - i(q_3 + \bar{c}^{(1)} p_3) \cos(\eta_3^- h) \\
(q_3 + \bar{c}^{(1)} p_3) \sin(\eta_3^+ h) - \alpha Z_{13} \cos(\eta_3^+ h) \\
0 \\
0 \\
\bar{c}^{(3)} \eta_3^+ \cos(\eta_3^+ h) + (i\omega Z_{33} - 2\rho^'h'\omega^2) \sin(\eta_3^+ h) \\
-i\bar{c}^{(3)} \eta_3^+ \sin(\eta_3^+ h) - (\omega Z_{33} + 2i\rho^'h'\omega^2) \cos(\eta_3^+ h) \\
\bar{c}^{(3)} \eta_3^- \cos(\eta_3^- h) + (i\omega Z_{33} - 2\rho^'h'\omega^2) \sin(\eta_3^- h) \\
-i\bar{c}^{(3)} \eta_3^- \sin(\eta_3^- h) + (\omega Z_{33} + 2i\rho^'h'\omega^2) \cos(\eta_3^- h) \\
\right) \right| = 0. \quad (3.3.98)
\]

Dividing the first and second columns by \((-1)^{n-1}\) and then expanding the determinant along the first column, we can obtain a dispersion relation for the bounded plate from which the small quantity \(\alpha_n\) is obtained. We must point out here that we have neglected the third- and higher-order terms of \(p_i\) and \(q_i\). Due to the smallness of the surface impedances, piezoelectric coupling and the mass of the electrodes, we only retain the terms linear in \(Z_{ij}, k_{66}^2,\) and \(\rho^'h'\). The entire process of expansion is given in Appx. D. By adding the six terms together, we get:
\[ i \beta_n^* (\tilde{c}^{(1)})^2 (\tilde{c}^{(2)})^2 (\tilde{c}^{(3)})^2 \eta_1^+ \eta_1^- \eta_2^+ \eta_2^- \eta_3^+ \eta_3^- \sin((\eta_2^+ + \eta_2^-)h) \sin((\eta_3^+ + \eta_3^-)h) \\
+ i \beta_n (\tilde{c}^{(1)})^2 (\tilde{c}^{(2)})^2 (\tilde{c}^{(3)})^2 \eta_1^+ \eta_2^+ \eta_2^- \eta_3^+ \eta_3^- \sin((\eta_2^+ + \eta_2^-)h) \sin((\eta_3^+ + \eta_3^-)h) \\
+ \left( \omega Z_{11} + 2i \rho h' \omega^2 + i \frac{k^2}{h} \right) (\tilde{c}^{(1)})^2 (\tilde{c}^{(2)})^2 (\tilde{c}^{(3)})^2 (\eta_1^+ + \eta_1^-) \eta_2^+ \eta_2^- \eta_3^+ \eta_3^- \sin((\eta_2^+ + \eta_2^-)h) \sin((\eta_3^+ + \eta_3^-)h) \right) \\
- i c^{(1)} \eta_1^- (\tilde{c}^{(2)})^2 \eta_3^+ \eta_3^- (\eta_1^+ + \eta_1^-) (q_4 - \tilde{c}^{(2)} p_2) (q_2 + \tilde{c}^{(1)} p_2) \cos(\eta_2^+ h) \cos(\eta_2^- h) \sin((\eta_2^+ + \eta_2^-)h) \tag{3.3.105} \\
- i c^{(1)} \eta_1^+ (\tilde{c}^{(2)})^2 \eta_3^+ \eta_3^- (\eta_1^+ + \eta_1^-) (q_3 + \tilde{c}^{(1)} p_3) (q_5 - \tilde{c}^{(3)} p_3) \cos(\eta_3^+ h) \cos(\eta_3^- h) \sin((\eta_2^+ + \eta_2^-)h) \\
- i c^{(1)} (\tilde{c}^{(2)})^2 (\eta_1^+ + \eta_1^-) \eta_1^+ \eta_2^+ \eta_2^- (q_4 - \tilde{c}^{(2)} p_2) (q_2 + \tilde{c}^{(1)} p_2) \cos(\eta_2^+ h) \cos(\eta_2^- h) \sin((\eta_2^+ + \eta_2^-)h) \\
- i c^{(1)} (\tilde{c}^{(2)})^2 (\eta_1^+ + \eta_1^-) \eta_2^+ \eta_2^- (q_5 - \tilde{c}^{(3)} p_3) (q_3 + \tilde{c}^{(1)} p_3) \cos(\eta_3^+ h) \cos(\eta_3^- h) \sin((\eta_2^+ + \eta_2^-)h) = 0 \\
\]

Simplification by collecting like terms in (3.3.105) yields:

\[ i(\beta_n^* + \beta_n) (\tilde{c}^{(1)})^2 (\tilde{c}^{(2)})^2 (\tilde{c}^{(3)})^2 \eta_1^+ \eta_1^- \eta_2^+ \eta_2^- \eta_3^+ \eta_3^- \sin((\eta_2^+ + \eta_2^-)h) \sin((\eta_3^+ + \eta_3^-)h) \\
+ \left( \omega Z_{11} + 2i \rho h' \omega^2 + i \frac{k^2}{h} \right) (\tilde{c}^{(1)})^2 (\tilde{c}^{(2)})^2 (\tilde{c}^{(3)})^2 (\eta_1^+ + \eta_1^-) \eta_2^+ \eta_2^- \eta_3^+ \eta_3^- \sin((\eta_2^+ + \eta_2^-)h) \sin((\eta_3^+ + \eta_3^-)h) \\
- i c^{(1)} (\tilde{c}^{(2)})^2 (\eta_1^+ + \eta_1^-) \eta_1^+ \eta_3^+ \eta_3^- (q_4 - \tilde{c}^{(2)} p_2) (q_2 + \tilde{c}^{(1)} p_2) \cos(\eta_2^+ h) \cos(\eta_2^- h) \sin((\eta_2^+ + \eta_2^-)h) \\
- i c^{(1)} (\tilde{c}^{(2)})^2 (\eta_1^+ + \eta_1^-) \eta_1^- (q_3 + \tilde{c}^{(1)} p_3) (q_5 - \tilde{c}^{(3)} p_3) \cos(\eta_3^+ h) \cos(\eta_3^- h) \sin((\eta_2^+ + \eta_2^-)h) = 0 \tag{3.3.106} \\
\]

Notice the following relations:

\[ \eta_1^+ + \eta_1^- \approx \frac{n \pi}{h} , \eta_2^+ + \eta_2^- \approx \kappa_2 \frac{n \pi}{h} , \eta_3^+ + \eta_3^- \approx \kappa_3 \frac{n \pi}{h} , \tag{3.3.107} \]

Substitution of (3.3.96) and (3.3.107) into (3.3.106) yields:
\[
\alpha_n = -\frac{4h^2[((\tilde{c}^{(2)})_2 + c_{12})\xi + (\tilde{c}^{(2)}_4 + c_{52})\nu][((\tilde{c}^{(1)}_2 - c_{68})\xi + (\tilde{c}^{(1)}_4 - c_{76})\nu)\cot\left(\kappa_2\frac{n\pi}{2}\right)]}{\tilde{c}^{(1)}\tilde{c}^{(2)}(n\pi)^2\kappa_2} \]

\[
-\frac{4h^2[((\tilde{c}^{(3)})_2 + c_{37})\nu + (\tilde{c}^{(3)}_5 + c_{17})\tilde{z}][((\tilde{c}^{(1)}_2 - c_{68})\nu + (\tilde{c}^{(1)}_5 - c_{76})\tilde{z})\cot\left(\kappa_3\frac{n\pi}{2}\right)]}{\tilde{c}^{(1)}\tilde{c}^{(3)}(n\pi)^2\kappa_3} + \frac{i\omega Z_{11} - 2\rho'\omega^2 - \frac{k_{66}^2}{h^2}}{\tilde{c}^{(1)}(n\pi)^2h} \].

We can also fix the relations among \( A^+_1, A^-_1, C^+_2, C^-_2, E^+_3 \) and \( E^-_3 \) from (3.3.84), (3.3.86), (3.3.89), (3.3.90), (3.3.93) and (3.3.94).

From (3.3.86) we can find:

\[ A^+_1 = A^-_1. \] (3.3.109)

Substituting from (3.3.109) into (3.3.89), we get

\[ C^+_2 = -C^-_2. \] (3.3.110)

Substitution of (3.3.109) and (3.3.110) into (3.3.90) yields:

\[ C^-_2 = (-1)^{n+1} i \frac{((\tilde{c}^{(2)}_2 + c_{12})\xi + (\tilde{c}^{(2)}_4 + c_{52})\nu)\cot\left(\eta_2\frac{n\pi}{2h}\right)}{\tilde{c}^{(2)}_2\eta_2\sin(\eta_2h)} A^+_1. \] (3.3.111)

Similarly, we can solve for \( E^+_3 \) and \( E^-_3 \) from (3.3.109), (3.3.93) and (3.3.94). The results are:

\[ E^+_3 = -E^-_3. \] (3.3.112)
\[ E_3 = (-1)^{n+1} i \frac{\eta_i (c^{(3)} r_3 + c_{37}) + (c^{(3)} r_5 + c_{17})}{\eta_i \sin(h)} A_i^+. \] (3.3.113)

The undetermined amplitudes \( B_i^+, D_i^+ \) and \( F_i^+ \) have all been expressed in terms of \( A_i^+ \), which can be found in (3.3.46), (3.3.61) and (3.3.65), respectively. Substitution of \( (\eta_i^+, \alpha_n, A_i^+, B_i^+, D_i^+, F_i^+) \) into (3.3.34) yields:

\[
\begin{align*}
[\rho \omega^2 - \bar{c}(\eta_i^+)^2 + c_1 \bar{\zeta}^2 + c_{39} \nu^2 + 2c_{31} \bar{\zeta} \nu] A_i^+ + (2c_{16} \bar{\xi} + 2c_{56} \nu) \eta_i^+ i A_i^+ + \\
[(c_{12} + c_{66}) \bar{\xi} + (c_{52} + c_{76}) \nu] (r_5 \bar{\xi} + r_3 \nu) A_i^+ + [(c_{17} + c_{66}) \bar{\xi} + (c_{57} + c_{36}) \nu] (r_5 \bar{\xi} + r_3 \nu) A_i^+ = 0.
\end{align*}
\] (3.3.114)

Since \( A_i^+ \neq 0 \), the coefficient of \( A_i^+ \) in (3.3.114) must vanish.

An approximate expression of \( \eta_i^+ \) can be found from (3.95):

\[ (\eta_i^+)^2 = \left[ \frac{n \pi}{2h} + i \frac{c_{16} \bar{\xi} + c_{36} \nu}{c^{(1)}} + \frac{\alpha_n}{h} \right] \approx \left( \frac{n \pi}{2h} \right)^2 + i \frac{n \pi}{h} \frac{c_{16} \bar{\xi} + c_{36} \nu}{c^{(1)}} + \frac{n \pi}{h^2} \alpha_n - \left( \frac{c_{16} \bar{\xi} + c_{36} \nu}{c^{(1)}} \right)^2, \] (3.3.115)

Substituting from (3.3.115) into (3.3.114) and eliminating \( A_i^+ \), we get:

\[
\begin{align*}
\rho \omega^2 - \bar{c}(\eta_i^+)^2 & \left[ \left( \frac{n \pi}{2h} \right)^2 + i \frac{n \pi}{h} \frac{c_{16} \bar{\xi} + c_{36} \nu}{c^{(1)}} + \frac{n \pi}{h^2} \alpha_n - \left( \frac{c_{16} \bar{\xi} + c_{36} \nu}{c^{(1)}} \right)^2 \right] + c_1 \bar{\zeta}^2 + c_{39} \nu^2 + 2c_{31} \bar{\zeta} \nu \\
& + 2i(c_{16} \bar{\xi} + c_{36} \nu) \left[ \frac{n \pi}{2h} + i \frac{c_{16} \bar{\xi} + c_{36} \nu}{c^{(1)}} + \frac{\alpha_n}{h} \right] + [(c_{12} + c_{66}) \bar{\xi} + (c_{52} + c_{76}) \nu] (r_5 \bar{\xi} + r_3 \nu) + [(c_{17} + c_{66}) \bar{\xi} + (c_{57} + c_{36}) \nu] (r_5 \bar{\xi} + r_3 \nu) = 0
\end{align*}
\] (3.3.116)

Inserting \( \alpha_n \) into (3.3.116), we obtain the dispersion equation for a bounded electroded plate:
\[
\rho \omega^2 - c^{(1)} \left( \frac{n\pi}{2h} \right)^2 + \frac{4[(\tilde{c}^{(2)} r_2 + c_{12})\xi + (\tilde{c}^{(2)} r_4 + c_{32})\nu][(\tilde{c}^{(1)} r_2 - c_{66})\xi + (\tilde{c}^{(1)} r_4 - c_{76})\nu)] \cot \left( \frac{\kappa_2 n\pi}{2} \right)}{c^{(2)} n\pi \kappa_2} - \frac{2}{h} \left( i\omega Z_{11} - 2\rho h' \omega^2 - \frac{k_{66}^2}{h} \right)
\]
\[
+ \frac{4[(\tilde{c}^{(3)} r_3 + c_{37})\nu + (\tilde{c}^{(3)} r_5 + c_{17})\xi][(\tilde{c}^{(1)} r_3 - c_{36})\nu + (\tilde{c}^{(1)} r_5 - c_{86})\xi)] \cot \left( \frac{\kappa_3 n\pi}{2} \right)}{c^{(3)} n\pi \kappa_3} - \frac{(c_{16}\xi + c_{56}\nu)^2}{c^{(1)}} + c_{11}\xi^2 + c_{58}\nu^2 + 2c_{31}\xi \nu + 0.
\]

(3.3.117) can be expressed in a more compact form:

\[
M_n \xi^2 + Q_n \xi \nu + P_n \nu^2 - c^{(1)} \left( \frac{n\pi}{2h} \right)^2 + \rho \omega^2 - \frac{2}{h} \left( i\omega Z_{11} - 2\rho h' \omega^2 - \frac{k_{66}^2}{h} \right) = 0.
\]

(3.3.118)

where:

\[
M_n = \frac{4 \cot \left( \frac{\kappa_2 n\pi}{2} \right)}{c^{(2)} n\pi \kappa_2} (r_2 \tilde{c}^{(1)} - c_{66})(r_2 \tilde{c}^{(2)} + c_{12}) + \frac{4 \cot \left( \frac{\kappa_3 n\pi}{2} \right)}{c^{(3)} n\pi \kappa_3} (r_5 \tilde{c}^{(1)} - c_{86})(r_5 \tilde{c}^{(3)} + c_{17}),
\]

(3.3.119)
\[ \begin{align*}
Q_n &= 2c_{51} \frac{2c_{16}c_{36}}{c^{(1)}} + r_2(c_{52} + c_{76}) + r_4(c_{12} + c_{66}) + r_5(c_{17} + c_{86}) + r_3(c_{36} + c_{57}) + \\
&\quad 4\cot \left( \frac{k_n \pi}{2} \right) \left[ (r_2 \overline{c}^{(1)} - c_{66})(r_4 \overline{c}^{(2)} + c_{52}) + (r_4 \overline{c}^{(1)} - c_{76})(r_2 \overline{c}^{(2)} + c_{12}) \right] + \\
&\quad 4\cot \left( \frac{k_n \pi}{2} \right) \left[ (r_3 \overline{c}^{(1)} - c_{86})(r_5 \overline{c}^{(3)} + c_{57}) + (r_5 \overline{c}^{(1)} - c_{36})(r_3 \overline{c}^{(3)} + c_{17}) \right],
\end{align*} \]

(3.3.120)

\[ \begin{align*}
P_n &= c_{58} - \frac{c_{36}}{c^{(1)}} + r_4(c_{52} + c_{76}) + r_5(c_{36} + c_{57}) + \\
&\quad 4\cot \left( \frac{k_n \pi}{2} \right) \left( r_2 \overline{c}^{(1)} - c_{76} \right) \left( r_2 \overline{c}^{(2)} + c_{52} \right) + \\
&\quad 4\cot \left( \frac{k_n \pi}{2} \right) \left( r_3 \overline{c}^{(1)} - c_{86} \right) \left( r_3 \overline{c}^{(3)} + c_{57} \right),
\end{align*} \]

(3.3.121)

\[ \begin{align*}
\dot{c}^{(1)} &= \overline{c}^{(1)} \left( 1 - \frac{8}{n^2 \pi^2} k_{26}^2 - 2R \right), \quad k_{26}^2 = \frac{e_{26}^2}{\varepsilon_{12}^2}, \quad R = \frac{2\rho' h'}{\rho h}.
\end{align*} \]

(3.3.122)

The displacement \( \overline{u}_i'' \) can be assumed to have the following form:

\[ u_i'' = 2A^+ \sin \left( \frac{n \pi}{2h} x_2 \right) \exp(-\xi x_1) \exp(-\nu x_3) \exp(i \omega t). \]

(3.3.133)

By identifying the minus in-plane decay number “-\( \xi \)” and “-\( \nu \)” with the partial derivatives with respect to \( x_1 \) and \( x_3 \), respectively, we can recover the scalar differential equation for the free vibration of an electroded plate:
\begin{equation}
M_n \frac{\partial^2 u^n}{\partial x_1^2} + \frac{\partial^2 u^n}{\partial x_i \partial x_3} + \frac{\partial^2 u^n}{\partial x_3^2} - \tilde{c}^{(i)} \left( \frac{n\pi}{2h} \right)^2 u_i^n + \left( \rho \omega^2 - \frac{2i\omega}{h} Z_{11} \right) u_i^n = 0. \tag{3.3.134}
\end{equation}

When the piezoelectric coupling coefficient $k_{26}^2$ and the mass ratio of the electrodes $R$ in $\tilde{c}^{(i)}$ are neglected, (3.3.134) degenerates into the scalar equation of an unelectroded plate:

\begin{equation}
M_n \frac{\partial^2 u^n}{\partial x_1^2} + \frac{\partial^2 u^n}{\partial x_i \partial x_3} + \frac{\partial^2 u^n}{\partial x_3^2} \left( \frac{n\pi}{2h} \right)^2 u_i^n + \left( \rho \omega^2 - \frac{2i\omega}{h} Z_{11} \right) u_i^n = 0. \tag{3.3.135}
\end{equation}
3.4 Forced Vibrations of an Unbounded Plate

We start the analysis for forced vibration of an unbounded plate from equations in the plate coordinate system. The governing equation for the forced vibration of an electroded plate is given by (3.2.1). For the sake of completeness, we rewrite it here:

\[
\begin{bmatrix}
\dot{c}_{96} & \dot{c}_{26} & \dot{c}_{46} \\
\dot{c}_{26} & \dot{c}_{22} & \dot{c}_{27} \\
\dot{c}_{46} & \dot{c}_{27} & \dot{c}_{47}
\end{bmatrix}
\begin{bmatrix}
\ddot{u}_{1,22} \\
\ddot{u}_{2,22} \\
\ddot{u}_{3,22}
\end{bmatrix}
= \rho
\begin{bmatrix}
\dddot{u}_1 \\
\dddot{u}_2 \\
\dddot{u}_3
\end{bmatrix},
\] (3.4.1)

The constitutive equations are:

\[
\begin{bmatrix}
\bar{T}_{21} \\
\bar{T}_{22} \\
\bar{T}_{23}
\end{bmatrix}
= \begin{bmatrix}
\bar{c}_{96} & \bar{c}_{92} & \bar{c}_{97} \\
\bar{c}_{26} & \bar{c}_{22} & \bar{c}_{27} \\
\bar{c}_{46} & \bar{c}_{42} & \bar{c}_{47}
\end{bmatrix}
\begin{bmatrix}
\dddot{u}_1 \\
\dddot{u}_2 \\
\dddot{u}_3
\end{bmatrix}
+ \begin{bmatrix}
\dddot{e}_{29} \\
\dddot{e}_{22} \\
\dddot{e}_{24}
\end{bmatrix}
\phi_2,
\] (3.4.2)

The mechanical boundary conditions are:

\[
\begin{bmatrix}
\bar{T}_{21} \\
\bar{T}_{22} \\
\bar{T}_{23}
\end{bmatrix}
= \pm \omega^2 2\rho' h \begin{bmatrix}
\dddot{u}_1 \\
\dddot{u}_2 \\
\dddot{u}_3
\end{bmatrix}
\mp i\omega
\begin{bmatrix}
\bar{Z}_{11} \\
\bar{Z}_{21} \\
\bar{Z}_{31}
\end{bmatrix}
\begin{bmatrix}
\dddot{u}_1 \\
\dddot{u}_2 \\
\dddot{u}_3
\end{bmatrix}
+ \begin{bmatrix}
\bar{Z}_{13} \\
\bar{Z}_{23} \\
\bar{Z}_{33}
\end{bmatrix}
\dddot{u}_3,
\text{ at } x_2 = \pm h,
\] (3.4.3)

The electrical boundary conditions are:

\[
\phi = \pm \frac{V}{2} e^{i\omega x}, \text{ at } x_2 = \pm h.
\] (3.4.4)

We can transform the inhomogeneous driving voltage from the electrical boundary conditions (3.4.4) into the system of equations (3.4.1) and the mechanical boundary conditions (3.4.3). First we assume that the electric potential have the following form:

\[
\phi = \frac{1}{\bar{e}_{22}} (\bar{e}_{26} \dddot{u}_1 + \bar{e}_{22} \dddot{u}_2 + \bar{e}_{27} \dddot{u}_3) + L x_2 + \frac{V}{2} \frac{x_2}{h}.
\] (3.4.5)

Substituting from (3.4.5) into (3.4.4), we get:

\[
L = \frac{1}{h \bar{e}_{22}} [\bar{e}_{26} \dddot{u}_1 (h) + \bar{e}_{22} \dddot{u}_2 (h) + \bar{e}_{27} \dddot{u}_3 (h)].
\] (3.4.6)
Thus

\[ \phi = \frac{V}{2h} \frac{\bar{c}_{26}}{\bar{e}_{22}} \begin{bmatrix} \bar{u}_1 - \frac{x_1}{h} \bar{u}_1(h) \\ \bar{u}_2 - \frac{x_2}{h} \bar{u}_2(h) \\ \bar{u}_3 - \frac{x_3}{h} \bar{u}_3(h) \end{bmatrix} + \frac{\bar{e}_{22}}{\bar{e}_{22}} \begin{bmatrix} \bar{u}_2 - \frac{x_2}{h} \bar{u}_2(h) \\ \bar{u}_3 - \frac{x_3}{h} \bar{u}_3(h) \end{bmatrix} + \frac{\bar{e}_{27}}{\bar{e}_{22}} \begin{bmatrix} \bar{u}_3 \end{bmatrix}. \] (3.4.7)

Combining (3.4.2), (3.4.3) and (3.4.7), we obtain:

\[
\begin{bmatrix}
\bar{T}_{21} \\
\bar{T}_{22} \\
\bar{T}_{23}
\end{bmatrix} = 
\begin{bmatrix}
\bar{c}_{96} & \bar{c}_{26} & \bar{c}_{46} \\
\bar{c}_{26} & \bar{c}_{22} & \bar{c}_{27} \\
\bar{c}_{46} & \bar{c}_{27} & \bar{c}_{47}
\end{bmatrix}
\begin{bmatrix}
\bar{u}_{1,2} \\
\bar{u}_{2,2} \\
\bar{u}_{3,2}
\end{bmatrix}
- 
\begin{bmatrix}
\bar{e}_{20} \bar{e}_{26} & \bar{e}_{29} \bar{e}_{22} & \bar{e}_{26} \bar{e}_{27} \\
\bar{e}_{22} & \bar{e}_{22} & \bar{e}_{22} \\
\bar{e}_{26} \bar{e}_{26} & \bar{e}_{22} & \bar{e}_{22}
\end{bmatrix}
\begin{bmatrix}
\bar{u}_1(h) \\
\bar{u}_2(h) \\
\bar{u}_3(h)
\end{bmatrix}
+ 
\begin{bmatrix}
\bar{e}_{29} \\
\bar{e}_{22} \\
\bar{e}_{24}
\end{bmatrix}
\begin{bmatrix}
V \\
2h \\
2h
\end{bmatrix}.
\] (3.4.8)

Applying the orthogonal transformation given by (3.3.1) and (3.3.6) to (3.4.8), we get

\[
\begin{bmatrix}
\bar{T}_{21} \\
\bar{T}_{22} \\
\bar{T}_{23}
\end{bmatrix} = 
\begin{bmatrix}
\bar{c}^{(1)} & 0 & 0 \\
0 & \bar{c}^{(2)} & 0 \\
0 & 0 & \bar{c}^{(3)}
\end{bmatrix}
\begin{bmatrix}
\bar{u}_{1,2} \\
\bar{u}_{2,2} \\
\bar{u}_{3,2}
\end{bmatrix}
- 
\begin{bmatrix}
k_{26}^2 & k_{26}^2 & k_{26}^2 \\
k_{22}^2 & k_{22}^2 & k_{24}^2 \\
k_{46}^2 & k_{46}^2 & k_{44}^2
\end{bmatrix}
\begin{bmatrix}
\bar{u}_1(h) \\
\bar{u}_2(h) \\
\bar{u}_3(h)
\end{bmatrix}
+ 
\begin{bmatrix}
\bar{e}_{29} \\
\bar{e}_{22} \\
\bar{e}_{24}
\end{bmatrix}
\begin{bmatrix}
V \\
2h \\
2h
\end{bmatrix}.
\] (3.4.9)

The equations of motion now become:

\[
\begin{bmatrix}
\bar{c}^{(1)} & 0 & 0 \\
0 & \bar{c}^{(2)} & 0 \\
0 & 0 & \bar{c}^{(3)}
\end{bmatrix}
\begin{bmatrix}
\bar{u}_{1,2} \\
\bar{u}_{2,2} \\
\bar{u}_{3,2}
\end{bmatrix}
= \rho
\begin{bmatrix}
\ddot{u}_1 \\
\ddot{u}_2 \\
\ddot{u}_3
\end{bmatrix}.
\] (3.4.10)

Due to the smallness of the piezoelectric coupling, the inertial of the surface electrodes and the surface impedances, the off-diagonal elements in (3.4.9) can be neglected, then the mechanical boundary conditions are simplified as:
In order to eliminate the inhomogeneous terms in (3.4.11), the displacements are assumed to be:

\[
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3 
\end{bmatrix} =
\begin{bmatrix}
  \tilde{u}_1 \\
  \tilde{u}_2 \\
  \tilde{u}_3 
\end{bmatrix} +
\begin{bmatrix}
  K_1 x_2 \\
  K_2 x_2 \\
  K_3 x_2 
\end{bmatrix},
\]

(3.4.12)

where \( K_1, K_2 \) and \( K_3 \) are constants to be determined.

Substitution of (3.4.12) into (3.4.11) yields:

\[
\begin{bmatrix}
  \tilde{c}^{(1)} & 0 & 0 \\
  0 & \tilde{c}^{(2)} & 0 \\
  0 & 0 & \tilde{c}^{(3)} 
\end{bmatrix}
\begin{bmatrix}
  \tilde{u}_{1,2} \\
  \tilde{u}_{2,2} \\
  \tilde{u}_{3,2} 
\end{bmatrix}
= \begin{bmatrix}
  K_1 \\
  K_2 \\
  K_3 
\end{bmatrix}
\begin{bmatrix}
  0 \\
  0 \\
  0 
\end{bmatrix}
\begin{bmatrix}
  \tilde{u}_{1,2} \\
  \tilde{u}_{2,2} \\
  \tilde{u}_{3,2} 
\end{bmatrix}
+ \begin{bmatrix}
  K_1 \\
  K_2 \\
  K_3 
\end{bmatrix}
\begin{bmatrix}
  \tilde{u}_{1,(h)/h} \\
  \tilde{u}_{2,(h)/h} \\
  \tilde{u}_{3,(h)/h} 
\end{bmatrix}
+ \begin{bmatrix}
  e_{29} \\
  e_{22} \\
  e_{24} 
\end{bmatrix}
\frac{V}{2h}.
\]

(3.4.13)

The inhomogeneous terms in (3.4.13) are eliminated if we choose:

\[
\begin{bmatrix}
  \tilde{c}^{(1)} - k_{66}^2 & 0 & 0 \\
  0 & \tilde{c}^{(2)} - k_{22}^2 & 0 \\
  0 & 0 & \tilde{c}^{(3)} - k_{44}^2 
\end{bmatrix}
\begin{bmatrix}
  K_1 \\
  K_2 \\
  K_3 
\end{bmatrix}
= \begin{bmatrix}
  e_{29} \\
  e_{22} \\
  e_{24} 
\end{bmatrix}
\frac{V}{2h}.
\]

(3.4.14)

The undetermined constants are obtained:

\[
\begin{bmatrix}
  K_1 \\
  K_2 \\
  K_3 
\end{bmatrix}
= \frac{V}{2h}
\begin{bmatrix}
  e_{29}/(\tilde{c}^{(1)} - k_{66}^2) \\
  e_{22}/(\tilde{c}^{(2)} - k_{22}^2) \\
  e_{24}/(\tilde{c}^{(3)} - k_{44}^2) 
\end{bmatrix}.
\]

(3.4.15)

The mechanical boundary conditions (3.4.13) now become homogeneous:
The wavenumbers \( n \) -thickness resonance, we only need to consider the following problem:

\[
\tilde{c}^{(1)} \tilde{u}_{1,2} = -\rho \omega^2 \tilde{u}_1 - \rho \omega^2 x_2 K_1 , \tag{3.4.18}
\]

\[
\tilde{c}^{(1)} \tilde{u}_{1,2} - k_{66}^2 \tilde{u}_1 (h)/h = (2 \rho h' \omega^2 - i \omega Z_{11}) (\tilde{u}_1 + K_1 h) , \text{ at } x_2 = \pm h , \tag{3.4.19}
\]

(3.4.18) and (3.4.19) constitute a nonhomogeneous boundary value problem. The eigensolutions to the homogeneous form of (3.4.18) and (3.4.19) (i.e. \( V=0, K_1=0 \)) are exactly the same as the solutions for the free vibration of an unbounded electroded plate.

They are:

\[
\tilde{u}_1^{(n)}(x_2,t) = A_n \sin(\eta_n x_2) \exp(i \omega_i t) , \tag{3.4.20}
\]

The wavenumbers \( \eta_n \) are the roots of the dispersion equation:

\[
\left( \frac{k_{66}^2}{\tilde{c}^{(1)}} + R \eta_n^2 h^2 - \frac{i h \omega Z_{11} (\omega_n)}{\tilde{c}^{(1)}} \right) \tan(\eta_n h) - 1 = 0 , \tag{3.4.21}
\]

The first order approximation of \( \eta_n h \) can be expressed as:
\[
\eta_n h = \frac{n\pi}{2} \left[ 1 - \frac{4}{n^2 \pi^2} \frac{k_{66}^2}{\epsilon^{(1)}} - R + \frac{4}{n^2 \pi^2} \frac{i\omega_n Z_{11}(\omega_n)}{\epsilon^{(1)}} \right], \quad n = 1, 3, 5, \ldots \quad (3.4.22)
\]

and \( \eta_n \) is given by:

\[
\eta_n^2 = \left( \frac{n\pi}{2h} \right)^2 \left( 1 - \frac{8}{n^2 \pi^2} \frac{k_{66}^2}{\epsilon^{(1)}} - 2R \right) + \frac{2i\omega_n}{h\epsilon^{(1)}} Z_{11}(\omega_n). \quad (3.4.23)
\]

The solution to the steady-state forced vibration can be expanded as a series of the homogeneous solutions:

\[
\tilde{u}_1(x_2,t) = \sum_{n=1,3}^{\infty} A_n \sin(\eta_n x_2) \exp(i\omega t), \quad (3.4.24)
\]

Substituting from (3.4.24) into (3.4.18) and (3.4.19), we obtain:

\[
-\epsilon^{(1)} \sum_{n=1,3}^{\infty} \eta_n^2 A_n \sin(\eta_n x_2) = -\rho \omega^2 \sum_{n=1,3}^{\infty} A_n \sin(\eta_n x_2) - \rho \omega^2 x_2 K_1, \quad (3.4.25)
\]
\[
\tilde{c}^{(1)}(\sum_{n=1,3}^{\infty} \eta_n A_n \cos(\eta_n x_2) - \frac{k_{06}^2}{h} \sum_{n=1,3}^{\infty} A_n \sin(\eta_n h) = \left[ 2 \rho' h' \omega^2 - i \omega Z_{11}(\omega) \right] \sum_{n=1,3}^{\infty} A_n \sin(\eta_n h) + K_1 h \right].
\]

(3.4.26)

Applying (3.2.30) in (3.4.25), we get

\[
\sum_{n=1,3}^{\infty} \rho(\omega^2 - \omega_n^2) A_n \sin(\eta_n x_2) = -\rho \omega^2 x_2 K_1,
\]

(3.4.27)

Rearranging (3.4.21), we have:

\[
\tilde{c}^{(1)} \eta_n \cos(\eta_n h) = \left[ k_{06}^2 + R \tilde{c}^{(1)} \eta_n^2 h^2 - i h \omega Z_{11}(\omega) \right] \frac{\sin(\eta_n h)}{h},
\]

(3.4.28)

Substituting from (3.4.28) into (3.4.26), we get:

\[
\sum_{n=1,3}^{\infty} \left[ R \rho \omega_n^2 h - i \omega Z_{11}(\omega_n) \right] A_n \sin(\eta_n h) = \left[ R \rho h \omega^2 - i \omega Z_{11}(\omega) \right] \sum_{n=1,3}^{\infty} A_n \sin(\eta_n h) + K_1 h
\]

(3.4.29)

Collection of like terms in (3.4.29) results in:

\[
\sum_{n=1,3}^{\infty} \left[ R \rho h (\omega_n^2 - \omega^2) - i [\omega Z_{11}(\omega_n) - \omega Z_{11}(\omega)] \right] A_n \sin(\eta_n h) = \left[ R \rho h \omega^2 - i \omega Z_{11}(\omega) \right] K_1 h.
\]

(3.4.30)

In order to implement the orthogonality, we multiply both sides of (3.4.27) by \(\sin(\eta_m x_2)\) and integrate the result over \([-h, h] \):

\[
\sum_{n=1,3}^{\infty} \rho(\omega^2 - \omega_n^2) A_n \int_{-h}^{h} \sin(\eta_n x_2) \sin(\eta_m x_2) dx_2 = -\rho \omega^2 K_1 \int_{-h}^{h} x_2 \sin(\eta_m x_2) dx_2.
\]

(3.4.31)
\[
\rho (\omega^2 - \omega_n^2) A_m \left[ h - \frac{\sin(2\eta_n h)}{2\eta_n} \right] + \sum_{n=1,3, n \neq m}^{\infty} \rho (\omega^2 - \omega_n^2) A_n \int_{-h}^{h} \sin(\eta_n x_2) \sin(\eta_n x_2) dx_2 = -\rho \omega^2 K_1 \left[ \frac{2\sin(\eta_n h)}{\eta_n^2} - \frac{2h \cos(\eta_n h)}{\eta_n} \right].
\] (3.4.32)

Multiply both sides of (3.4.30) by \(-2\sin(\eta_n h)\), we have
\[
\sum_{n=1,3, n \neq m}^{\infty} \{2R\rho h (\omega^2 - \omega_n^2) + 2i[\omega_n Z_{11}(\omega_n) - \omega Z_{11}(\omega)]\} A_n \sin(\eta_n h) \sin(\eta_n h) = -[2R\rho h \omega^2 - 2i \omega Z_{11}(\omega)] K_1 h \sin(\eta_n h)
\] (3.4.33)

Combining (3.4.32) and (3.4.33), we obtain:
\[
\rho (\omega^2 - \omega_n^2) A_m \left[ h - \frac{\sin(2\eta_n h)}{2\eta_n} \right] + \sum_{n=1,3, n \neq m}^{\infty} \rho (\omega^2 - \omega_n^2) A_n \int_{-h}^{h} \sin(\eta_n x_2) \sin(\eta_n x_2) dx_2 +
\sum_{n=1,3, n \neq m}^{\infty} \{2R\rho h (\omega^2 - \omega_n^2) + 2i[\omega_n Z_{11}(\omega_n) - \omega Z_{11}(\omega)]\} A_n \sin(\eta_n h) \sin(\eta_n h)
\] (3.4.34)

\[
= -\rho \omega^2 K_1 \left[ \frac{2\sin(\eta_n h)}{\eta_n^2} - \frac{2h \cos(\eta_n h)}{\eta_n} \right] -[2R\rho h \omega^2 - 2i \omega Z_{11}(\omega)] K_1 h \sin(\eta_n h)
\]

From the orthogonality condition given in Appx. B, we know:
\[
\int_{-h}^{h} \rho \sin(\eta_n x_2) \sin(\eta_n x_2) dx_2 + 2 \left[ 2\rho h - \frac{i\omega_m Z_{11}(\omega_m) - i\omega_n Z_{11}(\omega_n)}{\omega_m^2 - \omega_n^2} \right] \sin(\eta_n h) \sin(\eta_n h) = 0, \ m \neq n
\] (3.4.35)
Substitution of (3.4.35) into (3.4.34) yields:

\[
\rho(\omega^2 - \omega_m^2)A_m\left[h - \frac{\sin(2\eta_m h)}{2\eta_m}\right] + \{2\rho\eta \omega^2 - \omega_m^2\} + 2i[\omega_m Z_{11}(\omega_m) - \omega Z_{11}(\omega)]A_m \sin^2(\eta_m h)
\]

\[
+ \sum_{n=1}^{\infty} 2i\left[\omega_n Z_{11}(\omega_n) - \omega Z_{11}(\omega)\right] \omega_m^2 - \omega_n^2 \omega_m - \omega_n + [\omega_n Z_{11}(\omega_n) - \omega Z_{11}(\omega)]A_m \sin(\eta_n h) \sin(\eta_m h)
\]

\[
= -\rho \omega^2 K_1 \left[\frac{2\sin(\eta_m h)}{\eta_m^2} - \frac{2h}{\eta_m} \cos(\eta_m h)\right] - [2\rho\eta \omega^2 - 2i\omega Z_{11}(\omega)]K_1 h \sin(\eta_m h)
\]

(3.4.36)

For a resonator sensor work near the \( m \)th resonance, i.e. \( \omega \approx \omega_m \), we have:

\[
\rho(\omega^2 - \omega_m^2)A_m\left[h - \frac{\sin(2\eta_m h)}{2\eta_m}\right] + \{2\rho\eta \omega^2 - \omega_m^2\} + 2i[\omega_m Z_{11}(\omega_m) - \omega Z_{11}(\omega)]A_m \sin^2(\eta_m h)
\]

\[
+ \sum_{n=1}^{\infty} 2i\left[\omega_n Z_{11}(\omega_n) - \omega Z_{11}(\omega)\right] \omega_m^2 - \omega_n^2 \omega_m - \omega_n + [\omega_n Z_{11}(\omega_n) - \omega Z_{11}(\omega)]A_m \sin(\eta_n h) \sin(\eta_m h)
\]

\[
= -\rho \omega^2 K_1 \left[\frac{2\sin(\eta_m h)}{\eta_m^2} - \frac{2h}{\eta_m} \cos(\eta_m h)\right] - [2\rho\eta \omega^2 - 2i\omega Z_{11}(\omega)]K_1 h \sin(\eta_m h)
\]

(3.4.37)

The amplitude \( A_m \) can be solved from (3.4.37). The detailed procedure can be found in Appx. D. The result is:

\[
A_m \approx \frac{Ve_{29}}{c^{(1)}} \left(\frac{-1}{k_{66} m^{1/2}}\right) \frac{4}{(1 - \omega_m^2/\omega^2) m^2 \pi^2} (1 + R)
\]

(3.4.38)

Actually, in the forced thickness vibration analysis presented here, \( R \) is negligible compared to 1, as well as \( k_{66}^2 \) and \( Z_{11} \). However, \( R \) must be included in the forced thickness vibration analysis in principle because \( R \) is a result of the inertia of the electrodes and one of
the major purpose of this work is the proper inclusion of the inertia of the electrodes arising in the boundary conditions. When the final results given by (3.5.14) is obtained, R is clearly may be neglected with negligible loss in accuracy.

Substitution of (3.4.24) into (3.4.12) results in the displacement of the forced vibration:

\[ u_1 = \left[ -\frac{e_2 V}{c^{(1)} - k_{66}} \frac{x_2}{2h} + \sum_{n=1,3,5} A_n \sin \left( \frac{n \pi x_2}{2h} \right) \right] \exp(i \omega t). \] (3.4.41)

In the vicinity of the Nth resonance, only one term in the sum dominates and the solution may be written in the form:

\[ u_1 = \left[ -\frac{e_2 V}{c^{(1)} - k_{66}} \frac{x_2}{2h} + A_N \sin \left( \frac{N \pi x_2}{2h} \right) \right] \exp(i \omega t). \] (3.4.42)
3.5 The Scalar Differential Equation for Forced Vibrations

Rearranging (3.4.38), we can get:

\[ A_m \omega^2 - A_m \omega_m^2 = (-1)^{m-1/2} \omega^2 \frac{Ve_{29}}{c^{(1)} - k_{e6}^2 m^2 \pi^2} (1 + R). \]  

(3.5.1)

Multiplying both sides of (3.5.1) by \( \rho \) and then use the relation \( \rho \omega_m^2 = c^{(1)} \eta_m^2 \), we have:

\[ -c^{(1)} \eta_m^2 A_m + \rho \omega^2 A_m = (-1)^{m-1/2} \rho \omega^2 \frac{Ve_{29}}{c^{(1)} - k_{e6}^2 m^2 \pi^2} (1 + R) \]  

(3.5.2)

If we identify the magnitude \( A_m \) with the displacement \( u_i^n \) in (3.3.134), we get

\[ M_n \frac{\partial^2 u_i^n}{\partial x_1^2} + Q_n \frac{\partial^2 u_i^n}{\partial x_2^2} + P_n \frac{\partial^2 u_i^n}{\partial x_3^2} - c^{(1)} \left( \frac{n \pi}{2h} \right)^2 u_i^n + \left( \rho \omega^2 - \frac{2i \omega}{h} Z_{11} \right) u_i^n \]

\[ = (-1)^{n-1/2} \rho \omega^2 \frac{e_{26}}{c^{(1)} - k_{e6}^2 n^2 \pi^2} (1 + R) \exp(i \omega t) \]  

(3.5.3)

However, this expression will cause inconvenience when we apply the continuity conditions of the dependent variables at an interface between an electroded and an unelectroded region. This is because the inhomogeneous term in (3.4.41) must be included in the electroded region. However, this inhomogeneous term can be removed by properly transforming the inhomogeneous term into the differential equation. The inhomogeneous displacement field is given by:

\[ \bar{u}_0^i = -\frac{e_{29} V}{c^{(1)} - k_{e6}^2} \frac{x_2}{2h}. \]  

(3.5.4)

Expanding (3.5.4) as a series of the thickness solutions, we have:

\[ \sum_{n=1,3,5} A_n^0 \sin(\eta_n x_2) = -\frac{e_{29} V}{c^{(1)} - k_{e6}^2} \frac{x_2}{2h}. \]  

(3.5.5)

In order to implement the orthogonality condition, we need the boundary values of the functions in (3.5.5), which are given by:
\[
\sum_{n=1,3,5}^{\infty} A_n^0 \sin(\eta_n h) = -\frac{e_{29} V/2}{c^{(1)} - k_{66}^2}, \quad \sum_{n=1,3,5}^{\infty} A_n^0 \sin(-\eta_n h) = \frac{e_{29} V/2}{c^{(1)} - k_{66}^2}
\]  

(3.5.6)

Multiplying both sides of (3.5.5) by \( \rho \sin(\eta_m x_2) \) and integrating over \([-h, h]\), we have:

\[
\sum_{n=1,3,5}^{\infty} \int_{-h}^{h} A_n^0 \rho \sin(\eta_m x_2) \sin(\eta_m x_2) \, dx_2 = -\frac{e_{29}}{c^{(1)} - k_{66}^2} \frac{V}{2h} \int_{-h}^{h} \rho x_2 \sin(\eta_m x_2) \, dx_2.
\]  

(3.5.7)

Multiplying both sides of the two equations in (3.5.6) by \( 2\rho' \sin(\pm \eta_m h) \) and adding the results into (3.5.7), we get

\[
\sum_{n=1,3,5}^{\infty} A_n^0 \left[ \int_{-h}^{h} \rho \sin(\eta_n x_2) \sin(\eta_m x_2) \, dx_2 + 2\rho' \sin(\eta_n h) \sin(\eta_m h) + 2\rho' \sin(-\eta_n h) \sin(-\eta_m h) \right]
\]

\[
= -\frac{e_{29}}{c^{(1)} - k_{66}^2} \frac{V}{2h} \left[ \int_{-h}^{h} \rho x_2 \sin(\eta_m x_2) \, dx_2 + 2\rho' h \sin(\eta_m h) - 2\rho' h \sin(-\eta_m h) \right]
\]

(3.5.8)

Applying the orthogonality conditions of the eigensolutions in (3.5.8), the amplitudes \( A_n^0 \) can be solved:

\[
A_n^0 = -(-1)^{m-1/2} \frac{e_{29} V}{c^{(1)} - k_{66}^2} \frac{4}{m^2 \pi^2} \left[ 1 + \left( \frac{12}{m^2 \pi^2} - 1 \right) \frac{k_{66}^2}{c^{(1)} + R} \right]
\]

(3.5.9)

The detailed process is given in Appx. D.

Multiplying both sides of (3.5.9) by \( \rho(\omega^2 - \omega_m^2) \), we have:

\[
\rho \omega^2 A_m^0 - \rho \omega_m^2 A_m^0 = -(-1)^{m-1/2} \frac{e_{29} V}{c^{(1)} - k_{66}^2} \rho(\omega^2 - \omega_m^2) \frac{4V}{m^2 \pi^2} (1 + R)
\]

(3.5.10)

Application of the frequency-wavenumber relation \( \rho \omega_m^2 = c^{(1)} \eta_m^2 \) in (3.5.10) results in:

\[
\rho \omega^2 A_m^0 - c^{(1)} \eta_m^2 A_m^0 = -(-1)^{m-1/2} \frac{e_{29} V}{c^{(1)} - k_{66}^2} \rho(\omega^2 - \omega_m^2) \frac{4V}{m^2 \pi^2} (1 + R)
\]

(3.5.11)

In order that the continuity conditions at the interface contain only homogeneous terms, we must add \( A_m^0 \) and \( A_m \) in (3.5.2), thus we obtain:
\[-c^{(1)} \eta_m^2 \bar{A}_m + \rho \omega^2 \bar{A}_m = (-1)^{m-1/2} \rho \omega_m^2 \frac{e_{29}}{c^{(1)}} - k_{66}^2 \frac{4V}{m^2 \pi^2} (1 + R), \tag{3.5.12}\]

where \( \bar{A}_m = A_m + A_m^0 \).

Finally, we get the scalar equation for the forced vibration by identifying \( \bar{A}_m \) with the transversely varying displacement \( u^m_1 \) in (3.3.134):

\[ M_n \frac{\partial^2 u^n_1}{\partial x_1^2} + Q_n \frac{\partial^2 u^n_1}{\partial x_1 \partial x_3} + P_n \frac{\partial^2 u^n_1}{\partial x_3^2} - c^{(1)} \left( \frac{n \pi}{2h} \right)^2 u^n_1 + \left( \rho \omega^2 - \frac{2i \omega}{h} \right) Z_{11} u^n_1 = (-1)^{m-1/2} \rho \omega_m^2 \frac{e_{29}}{c^{(1)}} - k_{66}^2 \frac{4V}{m^2 \pi^2} (1 + R) \exp(i\omega t) \tag{3.5.13}\]

where

\[ c^{(1)} = c^{(1)} - k_{66}^2. \tag{3.5.14} \]

As can be seen, the second order differential operator, i.e. \( M_n u^n_{1,11} + Q_n u^n_{1,13} + P_n u^n_{1,33} \) in the scalar differential equations contains a mixed partial derivative term. Therefore, the method of separation of variables could not be used to solve the equation. According to the theory of partial differential equations, this term can be eliminated by an orthogonal transformation of the coordinate system.

Figure 3.2 Transformation of coordinate systems to eliminate the mixed derivative
The new coordinate system $OX_1X_3$ is obtained by a rotation of the old coordinate system $Ox_1x_3$ about the $x_2$ axis by an angle $\theta$. The orthogonal transformation is given by

$$
\begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix} = \begin{bmatrix}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
$$

(3.5.15)

where

$$\theta_n = -\frac{1}{2} \arctan \frac{Q_n}{M_n - P_n}.$$  

(3.5.16)

The scalar differential equations are now:

$$
M_n \frac{\partial^2 u_1^n}{\partial t^2} + P_n \frac{\partial^2 u_1^n}{\partial Z_1^2} - c(1)^2 \left( \frac{n \pi}{2h} \right) u_1^n + \left( \rho \omega^2 - \frac{2i \omega}{h} Z_{11} \right) u_1^n = 0
$$

(3.5.17)

$$
M_n \frac{\partial^2 u_2^n}{\partial t^2} + P_n \frac{\partial^2 u_2^n}{\partial Z_1^2} - c(1)^2 \left( \frac{n \pi}{2h} \right) u_2^n + \left( \rho \omega^2 - \frac{2i \omega}{h} Z_{11} \right) u_2^n = 0
$$

(3.5.18)

$$
M_n \frac{\partial^2 u_3^n}{\partial t^2} + P_n \frac{\partial^2 u_3^n}{\partial Z_1^2} - c(1)^2 \left( \frac{n \pi}{2h} \right) u_3^n + \left( \rho \omega^2 - \frac{2i \omega}{h} Z_{11} \right) u_3^n
\begin{aligned}
= (-1)^{n-\frac{1}{2}} \rho \omega_n^2 \frac{e_{26}}{c_{(1)^2}} \frac{4V}{n^2 \pi^2} (1 + R) \exp(i \omega t)
\end{aligned}
$$

(3.5.19)

where

$$
M_n' = M_n \cos^2(\theta_n) - Q_n \sin(\theta_n) \cos(\theta_n) + P_n \sin^2(\theta_n)
$$

$$
P_n' = M_n \sin^2(\theta_n) + Q_n \sin(\theta_n) \cos(\theta_n) + P_n \cos^2(\theta_n)
$$

(3.5.20)

These equations now can be solved using the method of separation of variables.

### 3.6 Numerical Examples

In this section, we present two specific examples to show the possible applications of the new scalar equations.

**Example 1 An SC-cut QCM Liquid Sensor**
The schematic diagram is essentially the same as that for the AT-cut QCM sensor, see Fig. 2.4. The characteristic impedance matrix of viscous fluid can be found in Appx. C, which is given by:

\[
Z_{ij}(\omega) = \begin{bmatrix}
(1+i)\sqrt{\frac{\mu \rho \omega}{2}} & 0 & 0 \\
0 & \rho c_{ij} & 0 \\
0 & 0 & (1+i)\sqrt{\frac{\mu \rho \omega}{2}}
\end{bmatrix},
\]  

(3.6.1)

The governing equations for the electroded and unelectroded portions of the plate are given by:

\[
M_n' \frac{\partial^2 u_i^n}{\partial x_1^2} + P_n' \frac{\partial^2 u_i^n}{\partial x_3^2} - \frac{c(t)}{2} \left( \frac{n \pi}{2h} \right)^2 u_i^n + \left( \rho \omega^2 - \frac{2i \omega}{\rho h} Z_{i1} \right) u_i^n = 0,
\]  

(3.6.2)

\[
M_n' \frac{\partial^2 u_i^n}{\partial x_1^2} + P_n' \frac{\partial^2 u_i^n}{\partial x_3^2} - \frac{c(t)}{2} \left( \frac{n \pi}{2h} \right)^2 u_i^n + \left( \rho \omega^2 - \frac{2i \omega}{\rho h} Z_{i1} \right) u_i^n = 0.
\]  

(3.6.3)

Consider an elliptic electrode of the form:

\[
\frac{x_1^2}{a^2 M_n' / P_n'} + \frac{x_3^2}{a^2} = 1,
\]  

(3.6.4)

In the \((X_1, X_3)\) plane, we introduce a new coordinate system \((x_1, x_3)\) by

\[
x_1 = X_1 \sqrt{M_n' / P_n'}, \quad x_3 = X_3.
\]  

(3.6.5)

In this coordinate system, the elliptical electrode is represented by a circular domain described by

\[
\frac{x_1^2}{a^2} + \frac{x_3^2}{a^2} = 1.
\]  

(3.6.6)

(3.6.2) and (3.6.3) become

\[
P_n' \nabla^2 u_i^n + \rho (\omega^2 - \frac{2i \omega}{\rho h} Z_{i1} - \bar{\omega}_c^2) u_i^n = 0,
\]  

(3.6.7)
\[ p_n^2 \nabla^2 u_i^n + \rho (\omega^2 - \frac{2i\omega}{\rho h} Z_{11} - \omega_n^2) u_i^n = 0, \quad (3.6.8) \]

where

\[ \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \quad \omega_n^2 = c_n^2 \left( \frac{n\pi}{2h} \right)^2, \quad (3.6.9) \]

We then introduce a polar coordinate system \((r, \theta)\) defined by

\[ x_1 = r \cos \theta, \quad x_3 = r \sin \theta, \quad (3.6.10) \]

We are interested only in modes independent of \(\theta\). Then (3.6.7) and (3.6.8) reduce to

\[ \frac{\partial^2 u_i^n}{\partial r^2} + \frac{1}{r} \frac{\partial u_i^n}{\partial r} + \alpha^2 u_i^n = 0, \quad r < a, \quad (3.6.11) \]

\[ \frac{\partial^2 u_i^n}{\partial r^2} + \frac{1}{r} \frac{\partial u_i^n}{\partial r} - \beta^2 u_i^n = 0, \quad r > a, \quad (3.6.12) \]

where

\[ \alpha^2 = \rho (\omega^2 - \frac{2i\omega}{\rho h} Z_{11} - \omega_n^2)/P_n^2 > 0, \]

\[ \beta^2 = \rho (\omega_n^2 + \frac{2i\omega}{\rho h} Z_{11} - \omega_n^2)/P_n^2 > 0. \quad (3.6.13) \]

The general solutions to (3.6.11) and (3.6.12) can be written as

\[ u_i^n = \begin{cases} AJ_0(\alpha r), & r < a, \\ BK_0(\beta r), & r > a, \end{cases} \quad (3.6.14) \]

where \(A\) and \(B\) are undetermined constants, \(J_0\) is the zero order Bessel function of the first kind which is bounded at the origin, \(K_0\) is the zero order modified Bessel function of the second kind which decays exponentially at infinity. \(u_i^n\) and its radial derivative are continuous across the interface between the electroded and unelectroded regions, so we have:
where a prime represents differentiation with respect to the entire argument, i.e. \( \alpha r \) or \( \beta r \). For nontrivial solutions of \( A \) and/or \( B \), the determinant of the coefficient matrix must vanish, which gives the following frequency equation that determines \( \omega \):

\[
\alpha J_0'(\alpha a)K_0(\beta a) - \beta J_0(\alpha a)K_0'(\beta a) = 0. \quad (3.6.16)
\]

With the following identities:

\[
J_0'(\alpha a) = -J_1(\alpha a), \quad K_0'(\beta a) = -K_1(\beta a), \quad (3.6.17)
\]

(3.6.16) can be written as

\[
\alpha J_1(\alpha a)K_0(\beta a) - \beta J_0(\alpha a)K_1(\beta a) = 0. \quad (3.6.18)
\]

The thickness of the plate is \( h=0.5 \) mm, and \( a=9 \) mm. the fluid assumed to be water with viscosity \( \eta = 8.90 \times 10^{-3} \) Pa·s, \( c_i = 1500 \) m/s. The complex frequency for the SC-cut plate contacting with water is \( 11244054.63 + i \times 67516.97 \) (rad/s). The frequency for the plate without surface impedance is \( 11245852.59 \) (rad/s). There is a drop in frequency \( \Delta \omega = 1797.96 \) rad/s.

**Example 2 Dispersion curves**

Similar to the discussion for the AT-cut plate, we can obtain the dispersion curves for the propagating waves along the \( X_1 \) axis in an unelectroded SC-cut plate as follows:

\[
\Omega = \sqrt{\frac{c^{(1)}n^2 + M_n'X^2}{c^{(1)}(1+2R_m)}}, \text{ for } |X| > 0 \quad (3.6.19)
\]

and
\[ \Omega = \sqrt{\frac{\tilde{\sigma}^{(i)} n^2 - M_n' X^2}{\tilde{c}^{(i)} (1 + 2 R_m^n)}} \], for \( |X| < 0 \). \hspace{1cm} (3.6.20)

As for an electroded plate, \( \tilde{\sigma}^{(i)} \) in the above equations need to be changed into \( \tilde{c}^{(i)} \). The dispersion curves are given below:

![Dispersion curves](image)

Figure 3.3. (a) Dispersion curves of an unelectroded plate. Solid lines: \( Z_{11} = 0 \). Dotted lines: \( R_m = 0.05 \). (b) Dispersion curves of an electroded plate.

Solid lines: \( Z_{11} = 0 \) and \( R = 0.05 \). Dotted lines: \( R_m = 0.05 \) and \( R = 0.05 \).
Chapter 4
Scalar Differential Equations for Plates of Crystals of Class 6mm

In this chapter, we consider a membrane type FBAR sensor which is composed of a piezoelectric film and a silicon substrate. As pointed out in chapter 1, AlN and ZnO are most frequently used materials to fabricate the piezoelectric film. Both of them are hexagonal crystals. According to the theory of linear piezoelectricity, the operating mode of thickness excited FBARs is directly determined by the orientation of the c-axis. When the c-axis is perpendicular to the surface of the film, thickness-extensional (TE) mode can be excited. To excite thickness-shear (TS) mode with thickness applied electric field, the c-axis is must be oriented parallel to the surface of the film. If the c-axis is generally inclined, TS and TE coupled mode is excited. Several experimentalists have studied the factors in the sputtering deposition process which affect the precision control of the c-axis. In this chapter we only consider FBAR sensors operating in TE modes.

4.1 Three-dimensional Governing Equations
Consider an FBAR composed of a ZnO film deposited on a silicon layer as shown in Fig. 4.1. The thickness of the film and the silicon layer are $h^f$ and $h^s$, respectively. The polarization direction of the ZnO film is parallel to the $x_3$ axis. The displacement equation of motion and the electrostatic charge equation are given by:
Figure 4.1 Schematic diagram of a cross section of a composite trapped energy resonator consisting of a thin piezoelectric film on a silicon layer

\[
\begin{align*}
\frac{\partial u_{1,11}}{\partial t} &+ (c_{12} + c_{66}) \frac{\partial u_{2,12}}{\partial t} + (c_{13} + c_{44}) \frac{\partial u_{3,13}}{\partial t} + c_{66} \frac{\partial u_{1,22}}{\partial t} + c_{44} \frac{\partial u_{1,33}}{\partial t} + (e_{31} + e_{15}) \frac{\partial \varphi_{1,13}}{\partial t} = \rho \ddot{u}_1 \\
\frac{\partial u_{2,11}}{\partial t} &+ (c_{12} + c_{66}) \frac{\partial u_{1,12}}{\partial t} + (c_{13} + c_{44}) \frac{\partial u_{3,23}}{\partial t} + c_{66} \frac{\partial u_{2,22}}{\partial t} + c_{44} \frac{\partial u_{2,33}}{\partial t} + (e_{31} + e_{15}) \frac{\partial \varphi_{2,23}}{\partial t} = \rho \ddot{u}_2 \\
\frac{\partial u_{3,11}}{\partial t} &+ (c_{13} + c_{44}) \frac{\partial u_{2,23}}{\partial t} + (c_{13} + c_{44}) \frac{\partial u_{1,13}}{\partial t} + c_{44} \frac{\partial u_{3,32}}{\partial t} + e_{33} \frac{\partial \varphi_{3,33}}{\partial t} + e_{15} \frac{\partial \varphi_{1,11}}{\partial t} + e_{15} \frac{\partial \varphi_{1,22}}{\partial t} + e_{33} \frac{\partial \varphi_{3,33}}{\partial t} = \rho \ddot{u}_3 \\
e_{15}u_{3,11} + (e_{15} + e_{31})u_{1,13} + e_{15}u_{3,22} + (e_{15} + e_{31})u_{2,23} + e_{33}u_{3,33} - \varepsilon_{11}\varphi_{1,11} - \varepsilon_{11}\varphi_{1,22} - \varepsilon_{33}\varphi_{3,33} = 0
\end{align*}
\]

The linear piezoelectric constitutive equations for the ZnO film are:

\[
\begin{align*}
T_{11} &= c_{11}u_{1,11} + c_{12}u_{2,12} + c_{13}u_{3,13} + e_{31}\varphi_{3,1} \\
T_{22} &= c_{11}u_{1,11} + c_{12}u_{2,12} + c_{13}u_{3,13} + e_{31}\varphi_{3,1} \\
T_{33} &= c_{13}u_{1,11} + c_{13}u_{2,12} + e_{33}\varphi_{3,3} \\
T_{23} &= c_{44}(u_{2,11} + u_{3,12}) + e_{15}\varphi_{2,1} \\
T_{31} &= c_{44}(u_{1,13} + u_{3,13}) + e_{15}\varphi_{3,1} \\
T_{12} &= c_{66}(u_{1,21} + u_{2,12}) \\
D_1 &= e_{15}(u_{3,11} + u_{1,13}) - \varepsilon_{11}\varphi_{1,1} \\
D_2 &= e_{15}(u_{3,21} + u_{2,12}) - \varepsilon_{11}\varphi_{1,2} \\
D_3 &= e_{31}(u_{3,11} + u_{1,22}) + e_{33}u_{3,33} - \varepsilon_{11}\varphi_{3,3}
\end{align*}
\]

Single-crystal silicon is an anisotropic crystalline material with cubic symmetry. In this work, the cubic axis of the silicon film is supposed to be parallel to the $x_3$ axis. The displacement equation of motion and the linear elastic constitutive equation for silicon take the following form:
\begin{align}
&c_{11}u_{1,1} + (c_{12} + c_{44})u_{2,12} + (c_{12} + c_{44})u_{3,13} + c_{44}u_{1,22} + c_{44}u_{1,33} = \rho \ddot{u}_1,
&c_{44}u_{2,11} + (c_{44} + c_{12})u_{1,12} + (c_{12} + c_{44})u_{3,23} + c_{44}u_{2,22} + c_{44}u_{2,33} = \rho \ddot{u}_2, \\
&c_{44}u_{3,11} + (c_{44} + c_{12})u_{1,13} + (c_{12} + c_{44})u_{2,23} + c_{44}u_{3,33} + c_{44}u_{3,22} = \rho \ddot{u}_3,
\end{align}

\begin{align}
T_{11} &= c_{11}u_{1,1} + c_{12}u_{2,2} + c_{12}u_{3,3}, \\
T_{22} &= c_{13}u_{1,1} + c_{11}u_{2,2} + c_{12}u_{3,3}, \\
T_{33} &= c_{13}u_{1,1} + c_{13}u_{2,2} + c_{13}u_{3,3}, \\
T_{23} &= c_{44}(u_{2,3} + u_{3,2}), \\
T_{13} &= c_{44}(u_{1,3} + u_{3,1}), \\
T_{12} &= c_{44}(u_{2,1} + u_{1,2}).
\end{align}

We are interested in the essentially thickness-extensional trapped energy modes. In this case, the decay- or wavenumbers in both the \(x_1\) and \(x_2\) directions are much smaller than the thickness wavenumber. As a consequence of the small in-plane wave/decay numbers and the small piezoelectric coupling in zinc-oxide, we can ignore the \(x_1\) and \(x_2\) dependence of all electrical variables. Now the charge equation and the only survival electric displacement component \(D_3\) can be written as:

\begin{align}
&\varepsilon_{33}u_{3,3} - \varepsilon_{13}\varphi_{3,3} = 0, \\
&D_3 = \varepsilon_{33}u_{3,3} - \varepsilon_{13}\varphi_{3}.
\end{align}

Next we consider the interfacial and boundary conditions. The layers are supposed to be perfectly bonded at the interfaces, so the displacements of the ZnO film and the silicon substrate are continuous across the interface. Part of the top surface and the entire bottom surface of the ZnO film are coated with very thin electrodes. The material and the geometry of the top and bottom electrodes are supposed to be different. The thickness of the top electrode and the bottom electrode are \(2h'\) and \(2h''\), respectively, while their mass density are \(\rho'\) and \(\rho''\), respectively. The thickness of the electrodes is far more less than that of the piezofilm, so the stiffness of the electrodes are neglected. The only mechanical effect considered here is their mass inertia. In practical applications, the top electrode is
used as the driving electrode while the bottom electrode is always grounded. Similar to
the analysis presented in the above chapters, the portions with and without a driving
electrode are treated separately. With these features in mind, we can write the boundary
and continuity conditions of the portion without a driving electrode as follows:

\[
\begin{align*}
    T_{3j}^f &= -Z_j \ddot{u}_j^f, \quad D_j^f = 0, \quad \text{at } x_3 = h^f, \\
    T_{3j}^f - T_{3j}^s &= \rho^s h^s \ddot{u}_j^s, \quad \ddot{u}_j^f = u_j^s, \quad \phi_j^f = 0, \quad \text{at } x_3 = 0, \\
    T_{3j}^s &= 0, \quad \text{at } x_3 = -h^s.
\end{align*}
\]  

As for the portion covered by the driving electrode, an alternating voltage \( V \exp(i \omega t) \) is
applied on the driving electrode, so only the boundary conditions at the top surface need
to be modified, which are given by:

\[
T_{3j}^f = -\rho h^s \ddot{u}_j^f - Z_j \ddot{u}_j^f, \quad \phi_j^f = V \exp(i \omega t), \quad \text{at } x_3 = h^f,
\]  

\[\text{(4.1.7)}\]

\[\text{(4.1.8)}\]

4.2 Thickness Vibrations of Unbounded Plates

(a) A composite plate without a driving electrode

The asymptotic dispersion relation of the essentially thickness-extensional vibration of a
bounded plate can be viewed as a perturbation of the dispersion relation of the pure
thickness-extensional vibration of an unbounded plate, so it is advantageous for us to first
obtain this dispersion relation. In this section, we treat the composite plate without
surface driving electrode. The schematic diagram of the composite plate is shown in
Figure 4.2.
The governing equations of the pure thickness-extensional vibration are:

\[ \begin{align*}
    c_{33}^{f}u_{3,33}^{f} + e_{33}^{f} \varphi_{33}^{f} &= \rho^{f} \ddot{u}_{3}^{f} \\
    e_{33}^{f}u_{3,33}^{f} - e_{33}^{f} \varphi_{33}^{f} &= 0 \\
    c_{11}^{f}u_{3,33}^{f} &= \rho^{f} \ddot{u}_{3}^{f}
\end{align*} \] (4.2.1)

The boundary conditions are:

\[ \begin{align*}
    c_{33}^{f}u_{3,33}^{f} + e_{33}^{f} \varphi_{33}^{f} &= -Z_{33} \ddot{u}_{3}^{f}, \quad e_{33}^{f}u_{3,33}^{f} - e_{33}^{f} \varphi_{33}^{f} = 0, \quad \text{at } x_{3} = h^{f}, \\
    c_{33}^{f}u_{3,33}^{f} + e_{33}^{f} \varphi_{33}^{f} - c_{11}^{f}u_{3,33}^{f} &= \rho^{f} h^{*} \ddot{u}_{3}^{f}, \quad u_{3}^{f} = \ddot{u}_{3}^{f}, \quad \varphi^{f} = 0, \quad \text{at } x_{3} = 0, \\
    T_{33} &= 0, \quad \text{at } x_{3} = -h^{*}.
\end{align*} \] (4.2.2)

The electric potential takes the form:

\[ \varphi^{f} = \frac{e_{33}^{f}}{e_{33}^{f}} [u_{3}^{f}(x_{3}, t) - u_{3}^{f}(0, t)]. \] (4.2.3)

It can be verified that (4.2.3) satisfies (4.2.1)$_{2}$, (4.2.2)$_{2}$ and (4.2.2)$_{5}$.

Substitution of (4.2.3) into (4.2.1)$_{1}$, (4.2.2)$_{1}$ and (4.2.2)$_{3}$ yields:

\[ \begin{align*}
    \bar{c}_{33}^{f}u_{3,33}^{f} &= \rho^{f} \ddot{u}_{3}^{f}, \\
    \bar{c}_{33}^{f}u_{3,33}^{f} &= -Z_{33} \ddot{u}_{3}^{f}, \quad \text{at } x_{3} = h^{f} \\
    \bar{c}_{33}^{f}u_{3,33}^{f} - c_{11}^{f}u_{3,33}^{f} &= \rho^{f} h^{*} \ddot{u}_{3}^{f}, \quad \text{at } x_{3} = 0
\end{align*} \] (4.2.4)
where \( \varepsilon_{33}^f = c_{33}^f + (e_{33}^f)^2 / \varepsilon_{33}^f \).

For steady-state vibrations of the plate, the displacements of the ZnO film and the silicon layer are given by:

\[
\begin{align*}
    u_j^f &= [A^f \cos(\eta_j x_3) + B^f \sin(\eta_j x_3)] \exp(i \omega t), \\
    u_3^s &= [A^s \cos(\eta_s x_3) + B^s \sin(\eta_s x_3)] \exp(i \omega t).
\end{align*}
\]  

Substituting from (4.2.7) into (4.2.13) and (4.2.4), we obtain the wavenumber-circular frequency relations:

\[
\begin{align*}
    \varepsilon_{33}^f \eta_j^2 &= \rho^f \omega^2, \\
    \varepsilon_{11}^s \eta_s^2 &= \rho^s \omega^2.
\end{align*}
\]  

Substituting from (4.2.7) into (4.2.14), (4.2.25) and (4.2.6), we get

\[
\begin{align*}
    [i \omega Z_{33} \cos(\eta_j h^f) - \varepsilon_{33}^f \eta_j \sin(\eta_j h^f)] A^f + [\varepsilon_{33}^f \eta_j \cos(\eta_j h^f) + i \omega Z_{33} \sin(\eta_j h^f)] B^f &= 0, \\
    A^s \sin(\eta_j h^s) + B^s \cos(\eta_j h^s) &= 0, \\
    \varepsilon_{33}^f \eta_j B^f - \varepsilon_{11}^s \eta_s B^s &= -\omega^2 \rho^s h^s A^s, \\
    A^f &= A^s.
\end{align*}
\]  

(4.2.10) can be written in matrix form:

\[
\begin{pmatrix}
    0 \\
    -\varepsilon_{33}^f \eta_j \sin(\eta_j h^f) \\
    \sin(\eta_j h^f) \\
    -\omega^2 \rho^s h^s \\
    \cos(\eta_j h^f) \\
    -\varepsilon_{33}^f \eta_j
\end{pmatrix}
\begin{pmatrix}
    A^f \\
    B^f
\end{pmatrix} = 0. \quad (4.2.11)
\]

(4.2.11) has nontrivial solutions if and only if the determinant of coefficients vanishes, thus we get:

\[
\begin{pmatrix}
    0 \\
    -\varepsilon_{33}^f \eta_j \sin(\eta_j h^f) \\
    \sin(\mu \eta_j h^f) \\
    -\omega^2 \rho^s h^s \\
    \cos(\mu \eta_j h^f) \\
    -\varepsilon_{33}^s \mu \eta_j
\end{pmatrix}
\begin{pmatrix}
    0 \\
    A^f \\
    B^f
\end{pmatrix} = 0. \quad (4.2.12)
\]

With the aid of (4.2.8) and (4.2.9), (4.2.12) can be simplified as:
\[ c' \mu \tan(\mu \eta_f \sigma h^f) + \eta_f h R^e + \frac{c_{33} \eta_f \sin(\eta_f h^f) - i \omega Z_{33} \cos(\eta_f h^f)}{c_{33} \eta_f \cos(\eta_f h^f) + i \omega Z_{33} \sin(\eta_f h^f)} = 0. \]  \hspace{1cm} (4.2.13)

where

\[ \eta_i = \sqrt{\frac{\rho \rho_{33}}{\rho' c_{11}}} \eta_f = \mu \eta_f, \quad \mu = \sqrt{\frac{\rho \rho_{33}}{\rho' c_{11}}} h' = \sigma h^f, \quad c' = \frac{c_{11}'}{c_{33}}, \quad R^e = \frac{\rho' h^e}{\rho' h^f}. \]  \hspace{1cm} (4.2.14)

(4.2.13) can be written in an equivalent form that will be useful for subsequent analysis:

\[ c' \mu \tan(\mu \sigma \eta_f h^f) + \eta_f h R^e + \frac{\tan(\eta_f h^f) - \frac{i \omega Z_{33}}{c_{33} \eta_f}}{1 + \frac{i \omega Z_{33}}{c_{33} \eta_f} \tan(\eta_f h^f)} = 0. \]  \hspace{1cm} (4.2.15)

(4.2.15) has an infinite number of roots \( \eta_f^n h^f, n=1,2,3,\ldots \). Once the roots are solved, the eigenfrequencies can also be obtained from (4.2.8).

Since the mass of the electrodes and the surface impedance are all small quantities, we can find approximate solutions to (4.2.15) using the perturbation method.

The equation for the unperturbed problem can be obtained by setting \( R^e = 0, Z_{33} = 0 \) in (4.2.15):

\[ c' \mu \tan(\mu \sigma \eta_f h^f) + \tan(\eta_f h^f) = 0. \]  \hspace{1cm} (4.2.16)

The roots of (4.2.16) is denoted by \( \eta_f^0 h^f \).

Since \( R^e \ll 1, |Z_{33}| \ll 1 \), it is convenient to write the roots \( \eta_f h^f \) as a sum of the unperturbed roots \( \eta_f^0 h^f \) and a perturbation \( \Delta^f \):

\[ \eta_f h^f = \eta_f^0 h^f + \Delta^f, \]  \hspace{1cm} (4.2.17)

where

\[ c' \mu \tan(\mu \sigma \eta_f^0 h^f) + \tan(\eta_f^0 h^f) = 0. \]  \hspace{1cm} (4.2.18)
Substituting from (4.2.17) into (4.2.15) and simplifying the result by (4.2.18), we get the first order approximation of $\Delta f$ as:

$$
\Delta f = -\frac{\eta_f^0 R^* - \frac{i\omega Z_{33}}{c_3} \sec^2(\eta_f^0 h^f)}{c' \mu^2 \sigma \sec^2(\mu \sigma \eta_f^0 h^f) + \sec^2(\eta_f^0 h^f)}. 
$$

(4.2.19)

Thus the thickness wavenumber is:

$$
\bar{\eta}_f = \eta_f^0 - \frac{\eta_f^0 R^* - \frac{i\omega Z_{33}}{c_3} \sec^2(\eta_f^0 h^f)}{c' \mu^2 \sigma \sec^2(\mu \sigma \eta_f^0 h^f) + \sec^2(\eta_f^0 h^f)}. 
$$

(4.2.20)

At the same time, we get the resonant frequency with the aid of (4.2.8):

$$
\bar{\omega}_e = \sqrt{\frac{c_f^3}{\rho_f^4 \eta_f^0}} \left[ 1 - \frac{R^* - \frac{i\omega Z_{33}}{c_3} \eta_f^0 h^f}{c' \mu^2 \sigma \sec^2(\mu \sigma \eta_f^0 h^f) + \sec^2(\eta_f^0 h^f)} \right]. 
$$

(4.2.21)

If we neglect the influence of the inertia of the ground electrode and the surface impedance, i.e. $R^* = 0$, $Z_{33} = 0$, we get the unperturbed resonant frequency:

$$
\omega_e^0 = \sqrt{\frac{c_f^3}{\rho_f^4 \eta_f^0}}. 
$$

(4.2.22)

(b) A composite plate with a driving electrode

Figure 4.3 Cross section of an unbounded electroded composite plate
First, we analyze free vibrations of the composite plate with a driving electrode. Except for the boundary conditions of the top surface, the governing equations and the boundary conditions are the same as that for plates without a driving electrode. For the case of plates with a driving electrode, the boundary conditions on the top surface become:

\[ e_{33}^f u_{33}^f + e_{33}^f \varphi_{33}^f = -\rho' h' \ddot{u}_{33}^f - Z_{33} \ddot{\varphi}_{33}^f , \quad \varphi^f = 0 \text{, at } x_3 = h^f . \]  \hfill (4.2.23)

We take a solution of the electric potential of the following form:

\[ \varphi^f = \frac{e_{33}^f}{e_{33}^f} u_{33}^f + Cx_3 + K . \]  \hfill (4.2.24)

The displacements of the ZnO film and the silicon layer are given by (4.2.7). Substituting from (4.2.24) into (4.2.2) and (4.2.23)\textsubscript{2}, we get:

\[ C = \frac{e_{33}^f}{e_{33}^f} [u_{33}^f (0) - u_{33}^f (h^f)] \frac{1}{h^f} , \quad K = -\frac{e_{33}^f}{e_{33}^f} u_{33}^f (0) . \]  \hfill (4.2.25)

Thus the electric potential can be written as:

\[ \varphi^f = \frac{e_{33}^f}{e_{33}^f} \left[ u_{33}^f (x_3) - \frac{x_3}{h^f} u_{33}^f (h^f) \right] + \frac{e_{33}^f}{e_{33}^f} \left( \frac{x_3}{h^f} - 1 \right) u_{33}^f (0) . \]  \hfill (4.2.26)

Substituting of (4.2.7) and (4.2.26) into the boundary conditions (4.2.2)\textsubscript{3,4,6} and (4.2.23) yields:

\[ - \left\{ \eta_j h^f \sin(\eta_j h^f) + \left[ k^2 + R'(\eta_j h^f)^2 \right] \cos(\eta_j h^f) - k^2 \right\} A^f \]

\[ + \left\{ \eta_j h^f \cos(\eta_j h^f) - \left[ k^2 + R'(\eta_j h^f)^2 \right] \sin(\eta_j h^f) \right\} B^f = 0 , \]

\[ [k^2 - k^2 \cos(\eta_j h^f)] A^f + R'(\eta_j h^f)^2 A^f + [\eta_j h^f - k^2 \sin(\eta_j h^f)] B^f - \frac{e_{33}^f}{e_{33}^f} \eta_j h^f B^f = 0 , \]  \hfill (4.2.27)

\[ A^f = A' , \]

\[ A' \sin(\eta_j h^s) + B' \cos(\eta_j h^s) = 0 . \]
For this system to have nontrivial solutions, its determinant of coefficients must vanish.

Expanding the determinant and neglecting the terms of second and higher order in $k^2$, $R'$, $R''$ and $Z_{33}$, we get

$$\tan(\eta_j h^f) + c' \mu \tan(\mu \sigma\eta_j h^f) - \frac{k^2}{\eta_j h^f} \left[ \frac{2}{\cos(\eta_j h^f)} - 2 + c' \mu \tan(\eta_j h^f) \tan(\mu \sigma\eta_j h^f) \right]$$

$$+ R'\eta_j h^f [1 - c' \mu \tan(\eta_j h^f) \tan(\mu \sigma\eta_j h^f)]$$

$$+ R''\eta_j h^f \left[ 1 - c' \mu \tan(\eta_j h^f) \tan(\mu \sigma\eta_j h^f) \right] = 0.$$  \hspace{1cm} (4.2.28)

Similar to the case for the plate without a driving electrode, we can express the roots $\eta_j h^f$ of (4.2.28) as a sum of the root $\eta^0_j h^f$ and a small perturbation parameter $\Delta^f$:

$$\eta_j h^f = \eta^0_j h^f + \Delta^f$$  \hspace{1cm} (4.2.29)

Substituting from (4.2.29) into (4.2.28) and employing (4.2.14), we obtain

$$\Delta^f = \eta^0_j h^f P^0 / G^0,$$  \hspace{1cm} (4.2.30)

where

$$P^0 = \frac{k^2}{(\eta_j^0 h^f)^2} \left[ \frac{2}{\cos(\eta_j^0 h^f)} - 2 + c' \mu \tan(\eta_j^0 h^f) \tan(\mu \sigma\eta_j^0 h^f) \right] - R''$$

$$- R' [1 - c' \mu \tan(\eta_j^0 h^f) \tan(\mu \sigma\eta_j^0 h^f)]$$

$$+ i \omega h^f Z_{33} \left[ 1 - c' \mu \tan(\eta_j^0 h^f) \tan(\mu \sigma\eta_j^0 h^f) \right],$$

$$G^0 = \sec^2(\eta_j^0 h^f) + c' \mu^2 \sigma \sec^2(\mu \sigma\eta_j^0 h^f).$$

Thus the wavenumber $\tilde{h}_j$ is given by:

$$\tilde{h}_j = \eta_j^0 (1 + P^0 / G^0).$$  \hspace{1cm} (4.2.32)

Meanwhile, the resonant frequency is obtained:

$$\omega = \sqrt{\frac{\epsilon}/\rho \eta_j^0 \left( 1 + P^0 / G^0 \right)}.$$  \hspace{1cm} (4.2.33)
4.3 Scalar Differential Equations of Bounded Composite Plates

At first, we consider a wave propagating along the $x_1$ axis. Thus all the dependent variables are independent of $x_2$, i.e. $\partial / \partial x_2 = 0$. Under these circumstances, $u_2$ is decoupled from $u_1$, $u_3$ and $\varphi$. Similar to the argument before (4.1.6), we neglect the $x_1$ and $x_2$ dependence of all the electric variables. In the following derivation, we only consider the case of plates without a driving electrode. It will be clear later that the scalar differential equations for plates with a driving electrode can be obtained by minor modifications of the existing scalar equations. The governing equations are:

\begin{align*}
    c_{11}^f u_{11}^{f} + (c_{13}^f + c_{44}^f) u_{313}^f + c_{44}^f u_{333}^f &= \rho^f \dddot{u}_1^f, \\
    c_{44}^f u_{313}^f + (c_{13}^f + c_{44}^f) u_{113}^f + c_{13}^f u_{333}^f + e_{153}^f \phi_{33}^f &= \rho^f \dddot{u}_3^f, \\
    e_{13}^f \dddot{u}_3^f - e_{153}^f \phi_{33}^f &= 0. \tag{4.3.1}
\end{align*}

The boundary conditions are:

\begin{align*}
    c_{13}^f u_{11}^f + c_{33}^f u_{33}^f + e_{33}^f \phi_{3}^f &= -Z_{13} \dddot{u}_1^f - Z_{33} \dddot{u}_3^f, \\
    c_{44}^f (u_{113}^f + u_{313}^f) &= -Z_{13} \dddot{u}_1^f - Z_{33} \dddot{u}_3^f, \quad \text{at } x_3 = h^f. \tag{4.3.3}
\end{align*}

\begin{align*}
    c_{13}^f u_{11}^f + c_{33}^f u_{33}^f + e_{33}^f \phi_{3}^f - c_{13}^f u_{113}^f - c_{33}^f u_{333}^f &= \rho^* h^* \dddot{u}_3^*, \\
    u_1^f &= u_1^*, \\
    c_{44}^f (u_{113}^f + u_{313}^f) - c_{44}^f (u_{113}^f + u_{313}^f) &= \rho^* h^* \dddot{u}_1^*, \quad \text{at } x_3 = 0. \tag{4.3.4}
\end{align*}

\begin{align*}
    \phi^f &= 0. \\
    c_{13}^f u_{11}^f + c_{33}^f u_{33}^f &= 0, \quad \text{at } x_3 = -h^f. \tag{4.3.5}
\end{align*}
The electric potential \( \varphi^f \) is chosen to be the same as that for the pure thickness-extensional vibration, which is given by (4.2.3). Thus (4.3.1)_3, (4.3.3)_3 and (4.3.4)_3 are satisfied. Substituting from (4.2.3) into (4.3.1)_2, (4.3.3)_1 and (4.3.4)_1, we obtain

\[
c_{44}^f u_{111}^f + (c_{13}^f + c_{44}^f) u_{113}^f + \tilde{c}_{33}^f u_{33}^f = \rho^f \bar{u}_3^f, \tag{4.3.6}
\]

\[
c_{33}^f u_{111}^f + \tilde{c}_{33}^f u_{33}^f = -Z_3 \bar{u}_3^f - Z_{33} \bar{u}_3^f, \tag{4.3.7}
\]

\[
c_{33}^f u_{111}^f + \tilde{c}_{33}^f u_{33}^f - c_{13}^f u_{113}^f - c_{33}^f u_{333}^f = \rho^f h^f \bar{u}_3^f, \tag{4.3.8}
\]

We take a trial solution of the form:

\[
u_i^f = [A_i^f \cos(\eta_i x_i) + B_i^f \sin(\eta_i x_i)] \exp(-\bar{\xi} x_i) \exp(i \omega t),
\]

\[
u_i^s = [A_i^s \cos(\eta_i x_i) + B_i^s \sin(\eta_i x_i)] \exp(-\bar{\xi} x_i) \exp(i \omega t),
\]

\[
u_i^s = [A_i^s \cos(\eta_i x_i) + B_i^s \sin(\eta_i x_i)] \exp(-\bar{\xi} x_i) \exp(i \omega t),
\]

\[
u_i^s = [A_i^s \sin(\eta_i x_i) + B_i^s \cos(\eta_i x_i)] \exp(-\bar{\xi} x_i) \exp(i \omega t).
\]

Substituting from (4.3.9) into (4.3.1)_1 and (4.3.6), and using the linear independence of \( \cos(\eta_i x_i) \) and \( \sin(\eta_i x_i) \), we get:

\[
s_{i1}^f A_i^f + s_{i3}^f A_i^f = 0, \quad -s_{i3}^f A_i^f + s_{i5}^f A_i^f = 0 \tag{4.3.11}
\]

\[
s_{i1}^f B_i^f - s_{i3}^f B_i^f = 0, \quad s_{i1}^f B_i^f + s_{i3}^f B_i^f = 0 \tag{4.3.12}
\]

where

\[
s_{i1}^f = c_{13}^f \bar{\xi}^2 - c_{44}^f \bar{\eta}_3^2 + \rho^f \omega^2, \quad s_{i3}^f = (c_{13}^f + c_{44}^f) \bar{\xi} \bar{\eta}_3, \quad s_{i5}^f = c_{44}^f \bar{\xi}^2 - \tilde{c}_{33}^f \bar{\eta}_3^2 + \rho^f \omega^2. \tag{4.3.13}
\]

Substitution of (4.3.10) into (4.3.2) and using the linear independence of \( \cos(\eta_i x_i) \) and \( \sin(\eta_i x_i) \), we obtain:

\[
s_{i1}^f A_i^f + s_{i3}^f A_i^f = 0, \quad -s_{i3}^f A_i^f + s_{i5}^f A_i^f = 0 \tag{4.3.14}
\]

\[
s_{i1}^f B_i^f - s_{i3}^f B_i^f = 0, \quad s_{i1}^f B_i^f + s_{i3}^f B_i^f = 0 \tag{4.3.15}
\]

where
\[ \sigma_{11}^f = c_{11}^f \xi^2 - c_{44}^f \eta_f^2 + \rho \omega^2, \quad \sigma_{13}^f = (c_{13}^f + c_{44}^f) \xi \eta_f, \quad \sigma_{33}^f = c_{44}^f \xi^2 - c_{33}^f \eta_f^2 + \rho \omega^2. \] (4.3.16)

The four sets of linear homogeneous algebraic equations (4.3.11), (4.3.12), (4.3.14) and (4.3.15) have nontrivial solutions if and only if the determinants of the coefficients vanish. It can be seen that the determinants of (4.3.11) and (4.3.12) are the same, and so do the determinants of (4.3.14) and (4.3.15). Thus we get two independent determinants, both of which are quadratic in \( \xi^2, \eta^2 \) and \( \omega^2 \):

\[
(c_{11}^f \xi^2 - c_{44}^f \eta_f^2 + \rho \omega^2)(c_{13}^f \xi^2 - c_{33}^f \eta_f^2 + \rho \omega^2) + (c_{13}^f + c_{44}^f) \xi^2 \eta_f^2 = 0, \\
(c_{11}^f \xi^2 - c_{44}^f \eta_f^2 + \rho \omega^2)(c_{33}^f \xi^2 - c_{13}^f \eta_f^2 + \rho \omega^2) - (c_{13}^f + c_{44}^f) \xi^2 \eta_f^2 = 0.
\] (4.3.17)

Hence, for given \( \xi \) and \( \omega \), each determinant yields two independent solutions: \((\eta_1^f, \eta_1^f)\) and \((\eta_3^f, \eta_3^f)\), respectively, and each \( \eta^f \) yields an independent amplitude ratio:

\[
A_1^f = \frac{\sigma_{33}^f}{\sigma_{13}^f} A_3^f = \mu^f A_3^f, \quad B_1^f = \frac{\sigma_{13}^f}{\sigma_{11}^f} B_3^f = \nu^f B_3^f, \quad A_1^s = \frac{\sigma_{33}^s}{\sigma_{13}^s} A_3^s = \mu^s A_3^s, \quad B_1^s = \frac{\sigma_{13}^s}{\sigma_{11}^s} B_3^s = \nu^s B_3^s.
\] (4.3.18)

Consequently, the solution to the boundary value problem have the following form:

\[
u^f_1 = [A_3^{f1} \cos(\eta_1 x_3) + A_3^{f2} \sin(\eta_2 x_3)] \exp(-\xi x_1), \quad u_1^f = [A_3^{f1} \sin(\eta_1 x_3) + A_3^{f2} \cos(\eta_2 x_3)] \exp(-\xi x_1),
\]

\[
u^s_1 = [A_3^{s1} \cos(\eta_1 x_3) + A_3^{s2} \sin(\eta_2 x_3)] \exp(-\xi x_1), \quad u_1^s = [A_3^{s1} \sin(\eta_1 x_3) + A_3^{s2} \cos(\eta_2 x_3)] \exp(-\xi x_1),
\] (4.3.19)

where the time harmonic factor has been omitted.

There are only eight unknowns in (4.3.19) because of the relations in (4.3.18). Substituting from (4.3.19) into the boundary conditions (4.3.3)2, (4.3.4)2-4, (4.3.5), (4.3.7) and (4.3.8), employing (4.3.18), and neglecting the terms containing \( Z_0 \) and \( \rho \eta \), in
other words, only the perturbation induced by finite lateral dimensions of the plate are considered, we obtain:

\[
\sum_{i=1}^{2} \left[ A_i^f (\eta_i \mu - \xi) \cos(\eta_i h^f) - B_i^f (\eta_i v^f \xi + \xi) \sin(\eta_i h^f) \right] = 0,
\]

\[
\sum_{i=1}^{2} \left[ A_i^f (c_{135}^f \eta_i - c_{135}^f v^f \xi + \xi) \sin(\eta_i h^f) + B_i^f (c_{135}^f \eta_i - c_{135}^f v^f \xi - \xi) \cos(\eta_i h^f) \right] = 0,
\]

\[
\sum_{i=1}^{2} \left[ B_i^f (c_{35}^f \eta_i - c_{35}^f v^f \xi) - B_i^s (c_{35}^f \eta_i - c_{35}^f v^f \xi) \right] = 0,
\]

\[
\sum_{i=1}^{2} (A_i^f - A_i^s) = 0,
\]

\[
\sum_{i=1}^{2} \left[ A_i^f (\mu \eta_i \xi - \xi) - A_i^s (\mu \eta_i \xi - \xi) \right] = 0,
\]

\[
\sum_{i=1}^{2} (B_i^f v^f - B_i^s v^s) = 0,
\]

\[
\sum_{i=1}^{2} \left[ A_i^s (c_{135}^f \eta_i - c_{135}^f v^f \xi) \sin(\eta_i h^f) + B_i^s (c_{35}^f \eta_i - c_{35}^f v^f \xi) \cos(\eta_i h^f) \right] = 0,
\]

\[
\sum_{i=1}^{2} \left[ A_i^s (\eta_i \mu \xi - \xi) \cos(\eta_i h^f) + B_i^s (\mu \eta_i v^f \xi + \xi) \sin(\eta_i h^f) \right] = 0.
\]

(4.3.20) constitutes a system of eight linear homogeneous algebraic equations in \( A_i^f \), \( B_i^f \), \( A_i^s \) and \( B_i^s \), which has nontrivial solutions when the determinant of coefficients vanishes. We can obtain the dispersion spectrum by solving for the roots \( \xi \) when the frequency \( \omega \) is given. These roots are the exact solutions of the problem, even when \( \xi \) is large. We will obtain an approximate dispersion relation for small \( \xi \) later in this section. The results given by these two dispersion relations will be compared.

We now look for the asymptotic form of the solution (4.3.19) in the limit of small in-plane decay numbers. The method is similar to that used in obtaining the asymptotic dispersion relation of the essentially thickness-shear vibration of an AT-cut plate.

Now return to (4.3.17). Since the decaynumber \( \xi \) is very small, as zero order approximation, we have:
(\eta_f^0)^2 = \frac{\rho_f \omega_f^2}{c_{33}^f}, \quad (\eta_f^2)^2 = \frac{\rho_f \omega_f^2}{c_{44}^f}, \quad (\eta_s^0)^2 = \frac{\rho_s \omega_s^2}{c_{33}^s}, \quad (\eta_s^2)^2 = \frac{\rho_s \omega_s^2}{c_{44}^s}. \quad (4.3.21)

From (4.3.21) we can see:

\eta_f^0 = \eta_f^0, \quad \eta_s^0 = \eta_s^0, \quad \eta_f^2 = \kappa^f \eta_f^0, \quad \eta_s^2 = \kappa^s \eta_s^0, \quad (4.3.22)

where

\kappa^f = \sqrt{\frac{c_{33}^f}{c_{44}^f}}, \quad \kappa^s = \sqrt{\frac{c_{33}^s}{c_{44}^s}}. \quad (4.3.23)

For the second order approximation, we have:

A_1^1 = -r^f A_1^1 \frac{\xi}{\eta_f^0}, \quad A_3^1 = -r^f A_3^1 \frac{\xi}{\eta_f^0}, \quad B_1^1 = r^f B_1^1 \frac{\xi}{\eta_f^0}, \quad B_3^1 = r^f B_3^1 \frac{\xi}{\eta_f^0},

A_1^2 = -r^s A_1^2 \frac{\xi}{\eta_s^0}, \quad A_3^2 = -r^s A_3^2 \frac{\xi}{\eta_s^0}, \quad B_1^2 = r^s B_1^2 \frac{\xi}{\eta_s^0}, \quad B_3^2 = r^s B_3^2 \frac{\xi}{\eta_s^2}, \quad (4.3.24)

where

r^f = \frac{c_{13}^f + c_{44}^f}{c_{33}^f - c_{44}^f}, \quad r^s = \frac{c_{13}^s + c_{44}^s}{c_{33}^s - c_{44}^s} \quad (4.3.25)

Since we are interested in essentially thickness-extensional vibration, small deviations of \eta_f h^f and \eta_s h^s from \eta_f^0 h^f and \eta_s^0 h^s are assumed:

\eta_f h^f = \eta_f^0 h^f + \delta_f, \quad \eta_s h^s = \eta_s^0 h^s + \delta_s, \quad (4.3.26)

where \delta_f and \delta_s are small perturbations.

Substituting from (4.3.24) and (4.3.25) into (4.3.11) and (4.3.14), respectively, and then substituting from (4.3.26) into (4.3.13) and (4.3.16), combining the two sets of results, retaining the terms linear in \delta_f and \delta_s, we obtain:
\[ g^f \xi^2 - c_{33}^f (\eta_{ij}^0)^2 + 2\eta_{ij}^0 \frac{\delta_{ij}}{h^f} + \rho' \omega^2 = 0, \quad g^s \xi^2 - c_{33}^s (\eta_{ij}^0)^2 + 2\eta_{ij}^0 \frac{\delta_{ij}}{h^s} + \rho' \omega^2 = 0, \]  

(4.3.27)

where

\[ g^f = r^f (c_{13}^f + c_{44}^f), \quad g^s = r^s (c_{13}^s + c_{44}^s). \]  

(4.3.28)

Eliminating the frequency \( \omega \), we get the relation between \( \delta_f \) and \( \delta_s \):

\[ \delta_s = K \xi^2 + \mu \sigma \delta_f, \]  

(4.3.29)

where

\[ K = \frac{\rho^s h^s}{2\eta_{ij}^0 c_{33}^s} \left( \frac{g^s}{\rho^s} - \frac{g^f}{\rho^f} \right). \]  

(4.3.30)

Application of (4.3.24) in (4.3.19) results in the second order approximation of the displacements:

\[
\begin{align*}
    u_3^f &= A_3^{f1} \cos(\eta_{ij} x_3) - r^f \frac{\xi}{\eta_{ij}^2} A_4^{f2} \cos(\eta_{ij} x_3) + B_3^{f1} \sin(\eta_{ij} x_3) + r^f \frac{\xi}{\eta_{ij}^2} B_4^{f2} \sin(\eta_{ij} x_3) \exp(-\xi x_i) \exp(i \omega t) \\
    u_1^f &= A_4^{f1} \sin(\eta_{ij} x_3) + r^f \frac{\xi}{\eta_{ij}^2} B_4^{f1} \cos(\eta_{ij} x_3) + B_1^{f2} \cos(\eta_{ij} x_3) - r^f \frac{\xi}{\eta_{ij}^2} A_3^{f1} \sin(\eta_{ij} x_3) \exp(-\xi x_i) \exp(i \omega t) \\
    u_3^s &= A_3^{s1} \cos(\eta_{ij} x_3) - r^s \frac{\xi}{\eta_{ij}^2} A_4^{s2} \cos(\eta_{ij} x_3) + B_3^{s1} \sin(\eta_{ij} x_3) + r^s \frac{\xi}{\eta_{ij}^2} B_4^{s2} \sin(\eta_{ij} x_3) \exp(-\xi x_i) \exp(i \omega t) \\
    u_1^s &= A_4^{s2} \sin(\eta_{ij} x_3) + r^s \frac{\xi}{\eta_{ij}^2} B_4^{s1} \cos(\eta_{ij} x_3) + B_1^{s2} \cos(\eta_{ij} x_3) - r^s \frac{\xi}{\eta_{ij}^2} A_3^{s1} \sin(\eta_{ij} x_3) \exp(-\xi x_i) \exp(i \omega t)
\end{align*}
\]  

(4.3.31)
Substituting from (4.3.31) into the boundary conditions (4.3.3), (4.3.4), (4.3.5), (4.3.7) and (4.3.8), neglecting $Z_{ij}$ and $\rho^*h^*$, we obtain:

\[
\begin{align*}
&\left(\frac{c_{ij}^f}{\eta_{f1}} - \bar{c}_{ij}^f\eta_{f1}\right) \left[A_{ij}^{f1} \sin(\eta_{f1}h^f) - B_{ij}^{f1} \cos(\eta_{f1}h^f)\right] + (\bar{c}_{ij}^f - c_{ij}^f)\xi\left[A_{ij}^{f2} \sin(\eta_{f2}h^f) + B_{ij}^{f2} \cos(\eta_{f2}h^f)\right] = 0, \\
&- (r^f + 1)\xi\left[A_{ij}^{f1} \cos(\eta_{f1}h^f) + B_{ij}^{f1} \sin(\eta_{f1}h^f)\right] + \left(\frac{\eta_{f2} + r^f}{\eta_{f2}}\right)^2 \left[A_{ij}^{f2} \cos(\eta_{f2}h^f) - B_{ij}^{f2} \sin(\eta_{f2}h^f)\right] = 0, \\
&\left(-\frac{c_{ij}^f r^f}{\eta_{f1}} + \bar{c}_{ij}^f\eta_{f1}\right)B_{ij}^{f1} + (\bar{c}_{ij}^f r^f - c_{ij}^f)\xi B_{ij}^{f2} + \left(\frac{c_{ij}^f r^f}{\eta_{f1}} - c_{ij}^f\eta_{f1}\right)B_{ij}^{f1} - (c_{ij}^f r^f - c_{ij}^f)\xi B_{ij}^{f2} = 0, \\
&A_{ij}^{f1} - r^f \frac{\xi}{\eta_{f2}} A_{ij}^{f2} - A_{ij}^{f1} + r^f \frac{\xi}{\eta_{f2}} A_{ij}^{f2} = 0, \\
&- c_{44}^f (r^f + 1)\xi A_{ij}^{f1} + c_{44}^f \left(\eta_{f2} + r^f \frac{\xi^2}{\eta_{f2}}\right)A_{ij}^{f2} + c_{44}^f (r^f + 1)\xi A_{ij}^{f1} - c_{44}^f \left(\eta_{f2} + r^f \frac{\xi^2}{\eta_{f2}}\right)A_{ij}^{f2} = 0, \\
&r^f \frac{\xi}{\eta_{f1}} B_{ij}^{f1} + B_{ij}^{f2} - r^f \frac{\xi}{\eta_{f1}} B_{ij}^{f1} - B_{ij}^{f2} = 0, \\
&\left(-\frac{c_{ij}^f r^f}{\eta_{f1}} + c_{ij}^f\eta_{f1}\right)\left[A_{ij}^{f1} \sin(\eta_{f1}h^f) + B_{ij}^{f1} \cos(\eta_{f1}h^f)\right] + (c_{ij}^f r^f - c_{ij}^f)\xi\left[-A_{ij}^{f2} \sin(\eta_{f2}h^f) + B_{ij}^{f2} \cos(\eta_{f2}h^f)\right] = 0, \\
&(r^f + 1)\xi\left[-A_{ij}^{f1} \cos(\eta_{f1}h^f) + B_{ij}^{f1} \sin(\eta_{f1}h^f)\right] + \left(\eta_{f2} + r^f \frac{\xi^2}{\eta_{f2}}\right)\left[A_{ij}^{f2} \cos(\eta_{f2}h^f) + B_{ij}^{f2} \sin(\eta_{f2}h^f)\right] = 0.
\end{align*}
\]
The relations between the amplitudes for the silicon substrate and the amplitudes for the piezoelectric film can be obtained from (4.3.32)_{3-6}:

\[ A_4 = D_{33} A_1 \quad \text{and} \quad B_4 = E_{33} B_1, \]

where

\[
A_1 = \frac{1}{c_{44}^2 \eta_{f}^0 \eta_{s}^0} \left[ c_{44} \left( \eta_{f}^0 \right)^2 + \frac{c_{44} \left( \eta_{f}^0 \right)^2}{\eta_{s}^0} \right] r^2 - c_{44} r^f - c_{44} \left( 1 + r^f \right) r^2,
\]

\[
D_{12} = 1 + r^s - \frac{c_{44}^f}{c_{44}^s} (1 + r^f), \quad D_{31} = \frac{c_{44}^f r^s \eta_{f}^0}{c_{44}^s \left( \eta_{f}^0 \right)^2} - \frac{r^f}{\eta_{f}^0}, \quad D_{33} = 1 + \frac{r^s}{c_{44}^s (\eta_{s}^0)^2} \left[ 1 + r^s - \frac{c_{44}^f}{c_{44}^s} (1 + r^f) \right] \xi^2.
\]

For the system (4.3.35) to have nontrivial solutions, the determinant of coefficients must vanish. Thus we obtain the dispersion relation:
\[
\begin{bmatrix}
  l_{11} + m_{11} \xi^2 + p_{11} \delta_f & n_{12} \xi & l_{13} + m_{13} \xi^2 + p_{13} \delta_f & n_{14} \xi \\
  n_{21} \xi & l_{22} + m_{22} \xi^2 & n_{23} \xi & l_{24} + m_{24} \xi^2 \\
  l_{31} + m_{31} \xi^2 + p_{31} \delta_s & n_{32} \xi & l_{33} + m_{33} \xi^2 + p_{33} \delta_f + p_{33} \delta_s & n_{34} \xi \\
  n_{41} \xi & l_{42} + m_{42} \xi^2 & n_{43} \xi & l_{44} + m_{44} \xi^2
\end{bmatrix} = 0 \tag{4.3.36}
\]

where

\[
l_{11} = -\bar{c}_{33}^f \eta_1^0 \sin(\eta_1^0 h_f), \quad l_{13} = \bar{c}_{33}^f \eta_1^0 \cos(\eta_1^0 h_f), \quad l_{22} = \eta_1^0 \cos(\eta_1^0 h_f), \quad l_{24} = -\eta_1^0 \sin(\eta_1^0 h_f)
\]

\[
l_{31} = c_{33}^f \eta_1^0 \sin(\eta_1^0 h_f), \quad l_{33} = c_{33}^f \eta_1^0 \cos(\eta_1^0 h_f), \quad l_{42} = \frac{c_{44}^f}{\eta_2^0} \eta_1^0 \cos(\eta_1^0 h_f), \quad l_{44} = \eta_1^0 \sin(\eta_1^0 h_f)
\]

\[
m_{11} = \frac{c_{13}^f \eta_1^0}{\eta_1^0} r_f \sin(\eta_1^0 h_f), \quad m_{13} = -\frac{c_{13}^f}{\eta_1^0} r_f \cos(\eta_1^0 h_f), \quad m_{22} = \frac{r_f}{\eta_1^0} \sin(\eta_1^0 h_f), \quad m_{24} = -\frac{r_f}{\eta_1^0} \sin(\eta_1^0 h_f)
\]

\[
m_{31} = a_2 \sin(\eta_1^0 h_f) + a_1 \sin(\eta_1^0 h_f), \quad m_{33} = a_2 \cos(\eta_1^0 h_f) + a_6 \cos(\eta_1^0 h_f)
\]

\[
m_{42} = b_2 \cos(\eta_1^0 h_f) + b_4 \cos(\eta_1^0 h_f), \quad m_{44} = b_5 \sin(\eta_1^0 h_f) + b_6 \sin(\eta_1^0 h_f)
\]

\[
n_{12} = (\bar{c}_{33}^f r_f - c_{13}^f) \sin(\eta_1^0 h_f), \quad n_{14} = (\bar{c}_{33}^f r_f - c_{13}^f) \cos(\eta_1^0 h_f)
\]

\[
n_{21} = -(r_f + 1) \cos(\eta_1^0 h_f), \quad n_{23} = -(r_f + 1) \sin(\eta_1^0 h_f)
\]

\[
n_{32} = -a_3 \sin(\eta_1^0 h_f) + a_4 \sin(\eta_1^0 h_f), \quad n_{34} = (c_{33}^f r_f - c_{13}^f) \cos(\eta_1^0 h_f) + a_7 \cos(\eta_1^0 h_f)
\]

\[
n_{41} = -(r_s + 1) \cos(\eta_1^0 h_f) + b_4 \cos(\eta_1^0 h_f), \quad n_{43} = \frac{\bar{c}_{33}^f \eta_1^0}{c_{33}^f \eta_1^0} (r_s + 1) \sin(\eta_1^0 h_f) + b_4 \sin(\eta_1^0 h_f)
\]

\[
p_{11} = -\bar{c}_{33}^f \eta_1^0 \cos(\eta_1^0 h_f) + \frac{1}{h_f} \sin(\eta_1^0 h_f), \quad p_{13} = \bar{c}_{33}^f \left[ \frac{1}{h_f} \cos(\eta_1^0 h_f) - \eta_1^0 \sin(\eta_1^0 h_f) \right]
\]

\[
p_{31} = c_{33}^f \eta_1^0 \cos(\eta_1^0 h_f) + \frac{1}{h_f} \sin(\eta_1^0 h_f), \quad p_{33} = \frac{c_{33}^f}{h_f} \cos(\eta_1^0 h_f), \quad p_{33} = -c_{33}^f \eta_1^0 \sin(\eta_1^0 h_f)
\]
\[ a_1 = \frac{1}{c_{44} \eta_{12}} [c_{44}^r (r^s + 1) - c_{44}^f (r^f + 1)] (c_{43}^r - c_{43}^f) \]
\[ a_2 = -\frac{c_{33}^r}{\eta_f^0} + \frac{c_{33}^r \kappa^r}{c_{44} \eta_{12}^0} [c_{44}^r (r^s + 1) - c_{44}^f (r^f + 1)] \]
\[ a_3 = c_{33}^r \left( r^f \frac{c_{44}^0}{\eta_{12}^0} + \frac{c_{44}^0 \eta_{f2}^0}{c_{44}^0 \eta_{12}^0} r^r \right) \]
\[ a_4 = \frac{c_{44}^0 \eta_{f2}^0}{c_{44}^0 \eta_{12}^0} (c_{13}^r - c_{33}^r) \]
\[ a_5 = \frac{1}{(c_{33}^r \eta_f^0)^2} \left\{ \left[ c_{33}^r c_{33}^s (r^s)^2 \eta_f^0 (r^f) + r^f (c_{13}^s - c_{33}^r r^s) - c_{13}^s \right] \frac{c_{33}^r \eta_f^0}{\eta_f^0} \right\} c_{33}^r \eta_f^0 - c_{33}^r c_{33}^r r^s \eta_f^0 \]
\[ a_6 = \frac{r^f}{\eta_f^0} (c_{33}^r r^s - c_{13}^s) - \frac{c_{13}^s r^s}{c_{33}^r (\eta_f^0)^2} (c_{33}^r r^s - c_{13}^s), \quad a_7 = \frac{c_{33}^r r^f - c_{13}^f + c_{13}^r - c_{33}^r r^f}{r^s} \]
\[ b_1 = r^s + 1 - \frac{c_{44}^r}{c_{44}^s} (r^s + 1), \quad b_2 = (r^s + 1) \left( \frac{r^f}{\eta_{f2}^0} - \frac{c_{44}^0 \eta_{f2}^0}{c_{44}^0 (\eta_f^0)^2} \right) \]
\[ b_3 = \frac{c_{44}^0 \eta_{f2}^0 r^s}{c_{44}^s (\eta_f^0)^2} + \frac{1}{\eta_f^0 c_{44} \eta_{12}^0} [c_{44}^r r^f - c_{44}^r (r^f + 1)] + \frac{c_{44}^0 \eta_{f2}^0 (r^f)^2}{(c_{44}^0 \eta_{12}^0)^2} \]
\[ b_4 = \frac{r^s}{\eta_f^0} - \frac{r^s \eta_f^0}{c_{33}^r \eta_{12}^0} (c_{33}^r r^f - c_{13}^f + c_{13}^r - c_{33}^r r^s) \]
\[ b_5 = \frac{r^s + 1}{c_{33}^r \eta_f^0} (c_{33}^r r^f - c_{13}^f + c_{13}^r - c_{33}^r r^s) \]

Expanding the determinant in (4.3.36) and retaining the terms linear in \( \delta_f \) and \( \delta_s \), and terms up to quadratic in \( \xi \), we obtain:

\[ U + R \delta_f + V \delta_s + P \xi^2 = 0 \tag{4.3.37} \]

where

\[ U = (l_1 l_{33} - l_{13} l_{31}) (l_{24} l_{44} - l_{24} l_{42}) , \quad V = (l_{44} l_{22} - l_{42} l_{24}) (p_{33} l_{13} - p_{33} l_{14}) , \]

\[ P = (l_{24} l_{44} - l_{24} l_{42}) (l_{1} m_{23} - l_{3} m_{31} + m_{1} l_{33} - m_{3} l_{13}) + (l_{3} n_{34} - l_{3} n_{43}) (n_{44} l_{24} - n_{42} l_{24}) +
    (n_{24} l_{34} - n_{24} l_{44}) + (n_{34} l_{22} - n_{34} l_{24}) (n_{44} l_{13} - n_{43} l_{14}) + (n_{24} l_{13} - n_{24} l_{13}) (n_{34} l_{44} - n_{34} l_{42}) , \]

\[ R = (l_{44} l_{22} - l_{42} l_{24}) (p_{33} l_{13} + p_{33} l_{33} - l_{33} p_{33}) . \]

We can also find that (4.2.14) is equivalent to
\[l_3l_{33}-l_{13}l_{31} = 0. \quad (4.3.38)\]

Thus, \( \delta_f \) can be solved from (4.3.37) and (4.3.29):

\[\delta_f = -W\xi^2, \quad (4.3.39)\]

where

\[W = \frac{P+VK}{R+V\mu\sigma}. \quad (4.3.40)\]

Substituting from (4.3.39) into (4.3.27)\(_1\), we obtain:

\[M\xi^2 - \bar{c}_{33}^f(\eta_f^0)^2 + \rho^f \omega^2 = 0, \quad (4.3.41)\]

where

\[M = g^f + 2\bar{c}_{33}^f\eta_f^0\frac{W}{h^f}. \quad (4.3.42)\]

(4.3.41) is the asymptotic form of the straight-crested dispersion relation for small decay number \( \xi \) in the vicinity of the unperturbed thickness-extensional resonant frequency \( \omega_e^0 \).

In the discussions presented in Sec. 4.2, we have seen that the thin ground electrode and the surface impedance serve only to change the resonant frequency from \( \omega_e^0 \) to \( \bar{\omega}_e \). At the same time the thickness wavenumber is changed from \( \eta_f^0 \) to \( \bar{\eta}_f \). For the case of a bounded composite plate, the influence of the ground electrode and the surface impedance can also be properly taken into consideration by changing the thickness wavenumber \( \eta_f^0 \) into \( \bar{\eta}_f \), thus the asymptotic dispersion relation for a bounded plate with a ground electrode and surface impedance is:

\[M\xi^2 - \bar{c}_{33}^f(\bar{\eta}_f)^2 + \rho^f \omega^2 = 0. \quad (4.3.43)\]

For the case of a composite plate with a driving electrode on the top surface, there is a propagating wave instead a decaying one along the \( x_1 \) axis, thus we need to replace the
decay number with $i\bar{\xi}$. Moreover, for the same reason discussed above, we need to change the thickness wavenumber from $\bar{\eta}_f^0$ into $\hat{\eta}_f$, so the asymptotic dispersion relation for a composite plate with a driving electrode is:

$$-M\bar{\xi}^2 - c_{33}^f(\hat{\eta}_f)^2 + \rho' \omega^2 = 0.$$  \hspace{1cm} (4.3.44)

Above we obtained the asymptotic limit of the straight-crested dispersion relations for small wavenumbers along the $x_1$ direction. Since the zinc-oxide is isotropic in the plane of the plate and the silicon is cubic with the coordinate axes along the cube edges, the straight-crested dispersion relation also holds along the $x_2$ axis. Next, we extend the asymptotic dispersion relations for straight-crested waves to that of variable-crested waves. Since the zinc-oxide layer is isotropic in the plane of the plate, we need to consider the silicon layer only. The governing equations and the constitutive relations in the crystallographic coordinate system for the silicon layer are given by (4.1.4) and (4.1.5), respectively. To the approximate order we considered, they can be simplified as:

$$\begin{align*}
(c_{12}^s + c_{44}^s)u_{3,11}^s + c_{13}^s u_{3,33}^s &= \rho \bar{u}_1^s, \\
(c_{12}^s + c_{44}^s)u_{3,22}^s + c_{13}^s u_{3,33}^s &= \rho \bar{u}_2^s, \\
c_{44}^s(u_{1,11}^s + u_{3,22}^s) + (c_{44}^s + c_{12}^s)(u_{1,13}^s + u_{2,23}^s) + c_{33}^s u_{3,33}^s &= \rho \bar{u}_3^s.
\end{align*}$$

$$\begin{align*}
T_{33}^s &= c_{13}^s(u_{1,1}^s + u_{2,2}^s) + c_{33}^s u_{3,3}^s, \\
T_{23}^s &= c_{44}^s(u_{2,3}^s + u_{3,2}^s), \\
T_{13}^s &= c_{44}^s(u_{1,3}^s + u_{3,1}^s). \hspace{1cm} (4.3.45)\hspace{1cm} (4.3.46)
\end{align*}$$

The terms such as $u_{1,11}$, $u_{2,12}$, $u_{1,22}$, $u_{2,11}$, $u_{1,12}$ and $u_{2,22}$ are higher order terms and thus have been omitted. We can see that only the elastic constants $c_{13}^s$, $c_{44}^s$, $c_{33}^s$ are involved in (4.3.45) and (4.3.46). It can be verified that these elastic constants are invariant under the planar coordinate transformation in the $x_1$-$x_2$ plane. Adding (4.3.45)$_1$ and (4.3.45)$_2$, and
reorganizing the terms in (4.3.45) and (4.3.46), we obtain a set of governing equations which is covariant under the planar coordinate transformation:

\[
(c_{13}^s + c_{44}^s) \nabla^p u_{33}^s + c_{44}^s u_{33}^p = \rho^s \ddot{u}_3^p,
\]

\[
c_{44}^s (\nabla^p)^2 u_3^s + (c_{13}^s + c_{44}^s) \nabla^p \cdot u_{33}^p + c_{33}^s u_{33,33}^s = \rho^s \ddot{u}_3^s.
\]

(4.3.47)

\[
T_{33}^s = c_{13}^s \nabla^p \cdot u_{3}^p + c_{33}^s u_{3,33}^s, \quad \mathbf{t}^p = c_{44}^s (\nabla^p u_3^s + u_{33}^p),
\]

(4.3.48)

where

\[
\mathbf{u}^p = e_1 u_1^p + e_2 u_2^p, \quad \mathbf{t}^p = e_1 T_{31} + e_2 T_{32}, \quad \nabla^p = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2}.
\]

(4.3.49)

Consequently, it can be inferred that the asymptotic dispersion relation (4.3.41) is independent of the propagation direction of the straight-crested waves.

![Figure 4.4 Plane view of the crystallographic coordinate system OX₁X₂ and a general coordinate system OX₁X₂ which makes an angle θ with the first one](image)

From the theory of elastodynamics we know that a variable-crested wave can be obtained by superposition of two straight-crested waves propagating in different directions. The direction of the variable-crested wave is midway between the directions of the two straight-crested waves. For this problem, we can obtain a variable-crested
wave $P_3$ propagating along the $X_1$ axis by superposing two straight-crested waves $P_1$ and $P_2$ with wave normal in the direction given by $+\theta$ and $-\theta$, as shown in Figure 4.5.

Figure 4.5 Schematic diagram of two straight-crested waves $P_1$ and $P_2$ propagating in different directions that can result in a variable-crested wave $P_3$.

Due to the planar isotropy of (4.3.47), we can obtain the solution of $P_1$ in the coordinate system $O\tilde{x}_1\tilde{x}_2$, which is given by:

Figure 4.6 Coordinate systems used to obtain the solution of the $P_1$ and $P_2$ waves.
\[ u^f_j = [A^f_j \cos(\eta^0_j x_3) + B^f_j \sin(\eta^0_j x_3)] \cos(\tilde{\xi} x_1) \exp(i \omega t) \]
\[ = [A^l_j \cos(\eta^0_j x_3) + B^l_j \sin(\eta^0_j x_3)] \cos(\tilde{\zeta} x_1 + \tilde{\nu} x_2) \exp(i \omega t) \]
\[ \tilde{u}^f_j = [A^l_j \cos(\eta^0_j x_3) + B^l_j \sin(\eta^0_j x_3)] \cos(\tilde{\xi} x_1) \exp(i \omega t) \]
\[ = [A^l_j \cos(\eta^0_j x_3) + B^l_j \sin(\eta^0_j x_3)] \cos(\tilde{\zeta} x_1 + \tilde{\nu} x_2) \exp(i \omega t) \]
\[ \tilde{u}^s_j = [A^l_j \cos(\eta^0_j x_3) + B^l_j \sin(\eta^0_j x_3)] \cos(\tilde{\xi} x_1) \exp(i \omega t) \]
\[ = [A^l_j \cos(\eta^0_j x_3) + B^l_j \sin(\eta^0_j x_3)] \cos(\tilde{\zeta} x_1 + \tilde{\nu} x_2) \exp(i \omega t) \]

where

\[ \tilde{\xi}^2 = \tilde{\xi}^2 + \tilde{\nu}^2, \quad \cot(\theta) = \frac{\tilde{\xi}}{\tilde{\nu}}, \] (4.3.51)

\[ \tau^f(x_3) = A^l_j \sin(\eta^0_j x_3) + A^l_2 \sin(\kappa^0_j \eta^0_j x_3) + B^l_j \cos(\eta^0_j x_3) + B^l_2 \cos(\kappa^0_j \eta^0_j x_3) \]
\[ \tau^s(x_3) = A^l_j \sin(\eta^0_j x_3) + A^l_2 \sin(\kappa^0_j \eta^0_j x_3) + B^s_j \cos(\eta^0_j x_3) + B^s_2 \cos(\kappa^0_j \eta^0_j x_3) \] (4.3.52)

We need to point out here that according to (4.3.19), the displacement components \( u^f_j \) and \( u^s_j \) should be:

\[ u^f_j = 2[A^l_j \cos(\eta^0_j x_3) + A^l_2 \cos(\kappa^0_j \eta^0_j x_3) + B^l_j \sin(\eta^0_j x_3) + B^l_2 \sin(\kappa^0_j \eta^0_j x_3)] \cos(\tilde{\xi} x_1) \cos(\tilde{\nu} x_2) \exp(i \omega t) \]
\[ u^s_j = 2[A^l_j \cos(\eta^0_j x_3) + A^l_2 \cos(\kappa^0_j \eta^0_j x_3) + B^s_j \sin(\eta^0_j x_3) + B^s_2 \sin(\kappa^0_j \eta^0_j x_3)] \cos(\tilde{\xi} x_1) \cos(\tilde{\nu} x_2) \exp(i \omega t) \] (4.3.53)

Meanwhile, we know from (4.3.24) that the amplitudes \( A^f_j, B^f_j, A^s_j, B^s_j \) can be expressed as:

\[ A^f_j = -\frac{\rho^f}{\eta^f_j} \xi A^l_j, \quad B^f_j = \frac{\rho^f}{\eta^f_j} \xi B^l_j, \quad A^s_j = -\frac{\rho^s}{\eta^s_j} \xi A^l_j, \quad B^s_j = \frac{\rho^s}{\eta^s_j} \xi B^l_j, \] (4.3.54)

Due to the smallness of \( \xi, u^f_j \) and \( u^s_j \), all the terms containing \( A^f_j, B^f_j, A^s_j, B^s_j \) can be neglected. Thus, the solution (4.3.50) is accurate enough for the following analysis.

Similarly, the solution of \( P_2 \) in the coordinate system \( O_3 \xi_3 \tilde{\nu} \) is given by:
\[ 2\hat{u}_f^1 = [A_f^1 \cos(\eta_0^0 x_i) + B_f^1 \sin(\eta_0^0 x_i)] \cos(\zeta \hat{x}_i) \exp(i\omega t) \]
\[ = [A_f^1 \cos(\eta_0^0 x_i) + B_f^1 \sin(\eta_0^0 x_i)] \cos(\zeta \hat{x}_1 - \nu x_2) \exp(i\omega t), \]
\[ 2\hat{u}_s^1 = [A_s^1 \cos(\eta_0^0 x_i) + B_s^1 \sin(\eta_0^0 x_i)] \cos(\zeta \hat{x}_i) \exp(i\omega t), \]
\[ = [A_s^1 \cos(\eta_0^0 x_i) + B_s^1 \sin(\eta_0^0 x_i)] \cos(\zeta \hat{x}_1 - \nu x_2) \exp(i\omega t). \]  
\[ (4.3.55) \]

where the superscripts “1” and “2” on the left hand side of the characters represent the quantities corresponding to \( P_1 \) and \( P_2 \), respectively.

Superposition the two straight-crested waves \( P_1 \) and \( P_2 \) yields the variable-crested wave \( P_3 \):

\[ u_f^1 = \tilde{u}_f^1 + \tilde{u}_s^1, \quad u_s^1 = \tilde{u}_s^1 + \tilde{u}_s^1, \]
\[ u_f^1 = \tilde{u}_f^1 \cos\theta + \tilde{u}_s^1 \cos\theta, \quad u_s^1 = \tilde{u}_s^1 \sin\theta - \tilde{u}_s^1 \sin\theta, \]
\[ u_f^1 = \tilde{u}_f^1 \cos\theta + \tilde{u}_s^1 \cos\theta, \quad u_s^1 = \tilde{u}_s^1 \sin\theta - \tilde{u}_s^1 \sin\theta, \]
\[ (4.3.56) \]

Substituting from (4.3.50) and (4.3.55) into (4.3.56), we obtain

\[ u_f^1 = 2[A_f^1 \cos(\eta_0^0 x_i) + B_f^1 \sin(\eta_0^0 x_i)] \cos(\zeta \hat{x}_1) \cos(\nu x_2) \exp(i\omega t), \]
\[ u_s^1 = 2[A_s^1 \cos(\eta_0^0 x_i) + B_s^1 \sin(\eta_0^0 x_i)] \cos(\zeta \hat{x}_1) \cos(\nu x_2) \exp(i\omega t), \]
\[ u_f^1 = \frac{2\hat{v}}{\zeta} \tilde{\tau}^f (x_3) \sin(\nu x_1) \cos(\zeta \hat{x}_2) \exp(i\omega t), \]
\[ u_s^1 = \frac{2\hat{v}}{\zeta} \tilde{\tau}^s (x_3) \cos(\nu x_1) \sin(\zeta \hat{x}_2) \exp(i\omega t), \]
\[ u_f^1 = \frac{2\hat{v}}{\bar{\nu}} \tilde{\tau}^f (x_3) \sin(\nu x_1) \cos(\bar{\nu} x_2) \exp(i\omega t), \]
\[ u_s^1 = \frac{2\hat{v}}{\bar{\nu}} \tilde{\tau}^s (x_3) \cos(\nu x_1) \sin(\bar{\nu} x_2) \exp(i\omega t). \]
\[ (4.3.57) \]

From (4.3.44) we can see that the dispersion relation of the variable-crested waves takes the form:

\[ -M (\bar{\zeta}^2 + \bar{\nu}^2) - \bar{\zeta}^{P} (\hat{\eta})^2 + \rho^f \omega^2 = 0. \]
\[ (4.3.58) \]
Since the magnitudes of $u_1^f$, $u_1^s$, $u_2^f$ and $u_2^s$ are much smaller than that of $u_3^f$ and $u_3^s$, their influences will be neglected completely.

By properly identifying the in-plane decay- or wavenumber with partial derivatives with respect to the plane coordinates, we can obtain the scalar differential equations for the composite plate with or without a driving electrode. The results are given as follows:

Results for free vibrations of a bounded plate with a driving electrode:

$$ u_1^f = 2[A_1^{f1} \cos(\eta_1^0 x_3) + B_1^{f1} \sin(\eta_1^0 x_3)] \cos(\xi x_1) \cos(\nu x_2) \exp(i\omega t) $$
$$ u_3^f = 2[A_3^{f1} \cos(\eta_3^0 x_3) + B_3^{f1} \sin(\eta_3^0 x_3)] \cos(\xi x_1) \cos(\nu x_2) \exp(i\omega t) 
\tag{4.3.57}_1,2 $$

$$ [-M_n(\xi_n^2 + \nu_n^2) - \tilde{c}_{33}^f(\hat{\nu}_{fn}^2 + \rho^f \omega^2)]u_3^{fn} = 0, \quad 0 < x_3 < h^f, 
\tag{4.3.59} $$

$$ [-M_n(\xi_n^2 + \nu_n^2) - \tilde{c}_{33}^f(\hat{\nu}_{fn}^2 + \rho^f \omega^2)]u_3^{mn} = 0, \quad 0 > x_3 > -h^s, 
\tag{4.3.60} $$

Results for a bounded plate without a driving electrode, which is in contact with the region with a driving electrode along $x_2$:

$$ u_1^f = [A_1^{f1} \cos(\eta_1^0 x_3) + B_1^{f1} \sin(\eta_1^0 x_3)] \exp(-\xi x_1) \cos(\nu x_2) \exp(i\omega t) $$
$$ u_3^f = [A_3^{f1} \cos(\eta_3^0 x_3) + B_3^{f1} \sin(\eta_3^0 x_3)] \exp(-\xi x_1) \cos(\nu x_2) \exp(i\omega t) 
\tag{4.3.61} $$

$$ [M_n(\xi_n^2 - \nu_n^2) - \tilde{c}_{33}^f(\hat{\nu}_{fn}^2 + \rho^f \omega^2)]u_3^{fn} = 0, \quad 0 < x_3 < h^f, 
\tag{4.3.62} $$

Results for a bounded plate without a driving electrode, which is in contact with the region with a driving electrode along $x_1$:

$$ u_1^f = [A_1^{f1} \cos(\eta_1^0 x_3) + B_1^{f1} \sin(\eta_1^0 x_3)] \cos(\xi x_1) \exp(-\nu x_2) \exp(i\omega t) $$
$$ u_3^f = [A_3^{f1} \cos(\eta_3^0 x_3) + B_3^{f1} \sin(\eta_3^0 x_3)] \cos(\xi x_1) \exp(-\nu x_2) \exp(i\omega t) 
\tag{4.3.63} $$
\[ \begin{align*}
[M_n (-v_{\eta}^2 + v_n^2 - c_{33}^f (\tilde{\eta}_{jn})^2) + \rho f \omega^2]u_3^{jn} &= 0, \quad 0 < x_3 < h^f, \\
[M_n (-v_{\eta}^2 + v_n^2 - c_{33}^f (\tilde{\eta}_{jn})^2) + \rho f \omega^2]u_3^{jn} &= 0, \quad 0 > x_3 > -h^s.
\end{align*} \tag{4.3.64} \]

\[ \begin{align*}
M_n \left( \frac{\partial^2 u_3^{jn}}{\partial x_1^2} + \frac{\partial^2 u_3^{jn}}{\partial x_2^2} \right) - c_{33}^f (\tilde{\eta}_{jn})^2 u_3^{jn} - \rho f \ddot{u}_3^{jn} &= 0, \quad 0 < x_3 < h^f, \\
M_n \left( \frac{\partial^2 u_3^{jn}}{\partial x_1^2} + \frac{\partial^2 u_3^{jn}}{\partial x_2^2} \right) - c_{33}^f (\tilde{\eta}_{jn})^2 u_3^{jn} - \rho f \ddot{u}_3^{jn} &= 0, \quad 0 > x_3 > -h^s.
\end{align*} \tag{4.3.65} \]

Generally, the governing equations are of the form:

\[ \begin{align*}
M_n \left( \frac{\partial^2 u_3^{jn}}{\partial x_1^2} + \frac{\partial^2 u_3^{jn}}{\partial x_2^2} \right) - c_{33}^f (\tilde{\eta}_{jn})^2 u_3^{jn} - \rho f \ddot{u}_3^{jn} &= 0, \quad 0 < x_3 < h^f, \\
M_n \left( \frac{\partial^2 u_3^{jn}}{\partial x_1^2} + \frac{\partial^2 u_3^{jn}}{\partial x_2^2} \right) - c_{33}^f (\tilde{\eta}_{jn})^2 u_3^{jn} - \rho f \ddot{u}_3^{jn} &= 0, \quad 0 > x_3 > -h^s,
\end{align*} \tag{4.3.66} \]

in which \( \tilde{\eta}_{jn} \) is understood to be \( \hat{\eta}_{jn} \) in regions with a driving electrode, and \( \tilde{\eta}_{jn} \) in regions without a driving electrode, respectively.

A general form of the solution is:

\[ u_3^{jn} = [A_3^{jn} \cos(\eta_0^0 x_3) + B_3^{jn} \sin(\eta_0^0 x_3)]f^n(x_1, x_2, t), \quad 0 < x_3 < h^f \]

\[ u_3^{jn} = [A_3^{jn} \cos(\eta_0^0 x_3) + B_3^{jn} \sin(\eta_0^0 x_3)]f^n(x_1, x_2, t), \quad 0 > x_3 > -h^s \tag{4.3.67} \]

where

\[ \begin{align*}
B_3^{jn} &= 1, \quad A_3^{jn} = \cot(\eta_0^0 h^f), \quad B_3^{jn} = \frac{c_{33}^f \eta_0^0}{\eta_0^0}, \quad A_3^{jn} = \cot(\eta_0^0 h^f),
\end{align*} \tag{4.3.68} \]

\[ \begin{align*}
M_n \left( \frac{\partial^2 f^n}{\partial x_1^2} + \frac{\partial^2 f^n}{\partial x_2^2} \right) - c_{33}^f (\tilde{\eta}_{jn})^2 f^n - \rho f \ddot{f^n} &= 0,
\end{align*} \tag{4.3.69} \]

where the effects of the small quantities, i.e. \( k^2, R, R^s \) and \( Z_{ij} \), on the magnitudes of the displacements have all been neglected.

The boundary conditions are:

\[ \begin{align*}
\tilde{f^n} &= f^n, \quad \tilde{f^n} = f^n, \quad \text{the junction line is in } x_2, \\
\tilde{f^n} &= f^n, \quad \tilde{f^n} = f^n, \quad \text{the junction line is in } x_1. \tag{4.3.70}
\end{align*} \]
4.4 Forced Vibrations of Composite Plates

The governing equations of pure thickness-extensional vibrations of an unbounded plate are:

\[
\begin{align*}
& c_{33}^{f} u_{3,33}^{f} + e_{33}^{f} \varphi_{3,33}^{f} = \rho^{f} \ddot{u}_{3}^{f}, \\
& e_{33}^{f} u_{3,33}^{f} - c_{33}^{f} \varphi_{3,33}^{f} = 0, \\
& c_{11}^{f} u_{5,33}^{f} = \rho^{f} \ddot{u}_{3}^{f} \tag{4.4.1}
\end{align*}
\]

The boundary conditions are:

\[
\begin{align*}
& c_{33}^{f} u_{3,33}^{f} + e_{33}^{f} \varphi_{3,33}^{f} = -Z_{3} u_{3}^{f} - \rho^{f} h' \ddot{u}_{3}^{f}, \quad \varphi^{f} = V \exp(i\omega t), \quad \text{at } x_{3} = h^{f}, \\
& c_{33}^{f} u_{3,33}^{f} + e_{33}^{f} \varphi_{3,33}^{f} - c_{11}^{f} \dot{u}_{3,33}^{f} = \rho^{f} h' \ddot{u}_{3}^{f}, \quad u_{3}^{f} = u_{3}^{s}, \quad \varphi^{f} = 0, \quad \text{at } x_{3} = 0, \tag{4.4.2} \\
& c_{33}^{f} u_{3,33}^{f} = 0, \quad \text{at } x_{3} = -h^{s}.
\end{align*}
\]

We take a solution of the form:

\[
\begin{align*}
& u_{3}^{f} = \bar{u}_{3}^{f} + K x_{3} \exp(i\omega t), \\
& \varphi^{f} = \varphi^{f} + \frac{V}{h^{f}} x_{3} \exp(i\omega t). \tag{4.4.3}
\end{align*}
\]

Substitution of (4.4.3) into (4.4.1) and (4.4.2) yields:

\[
\begin{align*}
& c_{33}^{f} \ddot{u}_{3,33}^{f} + e_{33}^{f} \ddot{\varphi}_{3,33}^{f} = -\rho^{f} \omega^{2} (\ddot{u}_{3}^{f} + K x_{3}), \\
& e_{33}^{f} \ddot{u}_{3,33}^{f} - c_{33}^{f} \ddot{\varphi}_{3,33}^{f} = 0, \\
& c_{11}^{f} \ddot{u}_{5,33}^{f} = \rho^{f} \ddot{u}_{3}^{f}, \tag{4.4.4}
\end{align*}
\]

\[
\begin{align*}
& c_{33}^{f} \ddot{u}_{3,33}^{f} + e_{33}^{f} \ddot{\varphi}_{3,33}^{f} + c_{33}^{f} K + e_{33}^{f} \frac{V}{h^{f}} = (\omega^{2} \rho^{f} h' - i\omega Z_{33})(\ddot{u}_{3}^{f} + K h^{f}), \quad \varphi^{f} = 0, \quad \text{at } x_{3} = h^{f}, \\
& c_{33}^{f} \ddot{u}_{3,33}^{f} + e_{33}^{f} \ddot{\varphi}_{3,33}^{f} + c_{33}^{f} K + e_{33}^{f} \frac{V}{h^{f}} - c_{11}^{f} \dot{u}_{3,33}^{f} = -\omega^{2} \rho^{f} h' u_{3}^{s}, \quad \ddot{u}_{3}^{f} = u_{3}^{s}, \quad \varphi^{f} = 0, \quad \text{at } x_{3} = 0, \tag{4.4.5} \\
& c_{33}^{f} \ddot{u}_{3,33}^{f} = 0, \quad \text{at } x_{3} = -h^{s}.
\end{align*}
\]

The inhomogeneous terms in (4.4.5) can be eliminated if the following relation holds:

\[
\begin{align*}
& c_{33}^{f} K + e_{33}^{f} \frac{V}{h^{f}} = 0. \tag{4.4.6}
\end{align*}
\]

Now (4.4.5) is simplified as:
\[ \begin{align*}
&c^f_{33} \bar{u}^f_{3,3} + e^f_{33} \bar{\varphi}^f_{3,3} = (\omega^2 \rho h' - i \omega Z_{33}) (\bar{u}^f_{3} + K h'), \quad \bar{\varphi}^f = 0, \quad \text{at } x_3 = h', \\
&c^f_{33} \bar{u}^f_{3,3} + e^f_{33} \bar{\varphi}^f_{3,3} = -\omega^2 \rho h'' \bar{u}^f_{3}, \quad \bar{u}^f_{3} = u_3', \quad \phi^f = 0, \quad \text{at } x_3 = 0, \quad (4.4.7) \\
&c^s_{33} u^s_{3,3} = 0, \quad \text{at } x_3 = -h'.
\end{align*} \]

The homogeneous problem corresponding to (4.4.5) and (4.4.7), i.e. \( V = K = 0 \), is the same as the problem discussed in section 4.2 if we identify \( \bar{u}^f_{3} \) and \( \bar{\varphi}^f \) with \( u^f_{3} \) and \( \phi^f \), respectively. The solutions given there are identified with the \( n \)th eigensolution \( \bar{u}^n_{3} \) of the homogeneous problem here. For the convenience of the following discussion, we list the whole boundary value problem satisfied by \( \bar{u}^n_{3} \):

\[ \begin{align*}
&c^f_{33} u^f_{3,3} + e^f_{33} \frac{1}{h^f} [\bar{u}^f_{3}(0) - \bar{u}^f_{3}(h^f)] = (\omega^2 \rho h' - i \omega_n Z_{33}) \bar{u}^f_{3}, \quad \text{at } x_3 = h', \\
&c^f_{33} u^f_{3,3} - c^s_{11} u^s_{3,3} + e^f_{33} \frac{1}{h^f} [\bar{u}^f_{3}(0) - \bar{u}^f_{3}(h^f)] = -\omega^2 \rho h'' u^m_{3}, \quad \bar{u}^f_{3} = u^m_{3}, \quad \text{at } x_3 = 0, \quad (4.4.9) \\
&c^s_{33} u^s_{3,3} = 0, \quad \text{at } x_3 = -h'.
\end{align*} \]

where

\[ \bar{\varphi}^f(x_3) = \frac{e^f_{33}}{e^f_{33}} \left[ \bar{u}^f_{3}(x_3) - \frac{x_3}{h^f} \bar{u}^f_{3}(h^f) \right] + \frac{e^f_{33}}{e^f_{33}} \left( \frac{x_3}{h^f} - 1 \right) \bar{u}^f_{3}(0). \]

The boundary value problem satisfied by \( \bar{u}^m_{3} \) is:

\[ \begin{align*}
&c^f_{33} u^f_{3,3} = -\rho^f \omega^2 \bar{u}^m_{3}, \\
&c^s_{11} u^s_{3,3} = -\rho^s \omega^2 u^m_{3}, \\
&c^f_{33} u^f_{3,3} + e^f_{33} \frac{1}{h^f} [\bar{u}^m_{3}(0) - \bar{u}^m_{3}(h^f)] = (\omega^2 \rho h' - i \omega_n Z_{33}) \bar{u}^m_{3}, \quad \text{at } x_3 = h', \\
&c^f_{33} u^m_{3,3} - c^s_{11} u^m_{3,3} + e^f_{33} \frac{1}{h^f} [\bar{u}^m_{3}(0) - \bar{u}^m_{3}(h^f)] = -\omega^2 \rho h'' u^m_{3}, \quad \bar{u}^m_{3} = u^m_{3}, \quad \text{at } x_3 = 0, \quad (4.4.12) \\
&c^s_{33} u^m_{3,3} = 0, \quad \text{at } x_3 = -h'.
\end{align*} \]
Next, we study the orthogonality of the eigensolutions. Consider the following integration:

\[ 0 = \int_0^{h_f} (4.5.8)_1 \times \bar{u}_3^{f_m} dx_3 - \int_0^{h_f} (4.5.11)_1 \times \bar{u}_3^{f_n} dx_3 + \int_{-h'}^{0} (4.5.8)_2 \times u_3^{m_n} dx_3 - \int_{-h'}^{0} (4.5.11)_2 \times u_3^{m_n} dx_3 \]

\[ = c_{33}^{-1} \bar{u}_3^{f_m} \left| \begin{array}{c} h_f \hline 0 \end{array} \right. u_3^{m_n} \right|_{-h'} + c_{33}^{-1} u_3^{m_n} \left| \begin{array}{c} 0 \hline h_f \end{array} \right. u_3^{m_n} \right|_{-h'} \]

\[ + \rho^f (\omega_n^2 - \omega_m^2) \int_0^{h_f} \bar{u}_3^{f_m} u_3^{m_n} dx_3 + \rho^f (\omega_n^2 - \omega_m^2) \int_{-h'}^{0} u_3^{m_n} u_3^{m_n} dx_3 \]

\[ = -c_{33}^{-1} \int_0^{h_f} \left[ \bar{u}_3^{f_m}(0) - \bar{u}_3^{f_m}(h_f) \right] \bar{u}_3^{f_m}(h_f) + (\omega_n^2 \rho h' - i \omega_n Z_{33}) \bar{u}_3^{f_m}(h_f) \bar{u}_3^{f_m}(h_f) \]

\[ + c_{33}^{-1} \int_0^{h_f} \left[ \bar{u}_3^{f_m}(0) - \bar{u}_3^{f_m}(h_f) \right] \bar{u}_3^{f_m}(h_f) - (\omega_m^2 \rho h' - i \omega_m Z_{33}) \bar{u}_3^{f_m}(h_f) \bar{u}_3^{f_m}(h_f) \]

\[ + c_{33}^{-1} \int_0^{h_f} \left[ \bar{u}_3^{f_m}(0) - \bar{u}_3^{f_m}(h_f) \right] \bar{u}_3^{f_m}(0) + \omega_n^2 \rho h' u_3^{m_n}(0) u_3^{m_n}(0) \]

\[ - c_{33}^{-1} \int_0^{h_f} \left[ \bar{u}_3^{f_m}(0) - \bar{u}_3^{f_m}(h_f) \right] \bar{u}_3^{f_m}(0) - \omega_m^2 \rho h' u_3^{m_n}(0) u_3^{m_n}(0) \]

\[ + \rho^f (\omega_n^2 - \omega_m^2) \int_0^{h_f} \bar{u}_3^{f_m} u_3^{m_n} dx_3 + \rho^f (\omega_n^2 - \omega_m^2) \int_{-h'}^{0} u_3^{m_n} u_3^{m_n} dx_3 \]

\[ = (\omega_n^2 - \omega_m^2) \left[ \rho h' \bar{u}_3^{f_m}(h_f) \bar{u}_3^{f_m}(h_f) + \rho h' u_3^{m_n}(0) u_3^{m_n}(0) + \rho^f \int_0^{h_f} \bar{u}_3^{f_m} u_3^{m_n} dx_3 + \rho^f \int_{-h'}^{0} u_3^{m_n} u_3^{m_n} dx_3 \right] \]

\[ - \left[ i \omega_n Z_{33}(\omega_n) - \omega_n Z_{33}(\omega_m) \right] \bar{u}_3^{f_m}(h_f) \bar{u}_3^{f_m}(h_f). \]

(4.4.13)

Now we can see that the orthogonality relation holds if and only if the surface impedance \( Z_{33} \) is proportional to \( \omega \). As can be seen from Appx. C, only the impedance of the perfectly bonded mass layer satisfies this requirement. For other surface impedances, \( Z_{33} \) is not proportional to \( \omega \), so the orthogonality relation only holds approximately. Here we consider the former case. Thus, \( Z_{33} \) can be expressed as:

\[ Z_{33}(\omega) = i \omega W_{33}, \]

(4.4.14)

where \( W_{33} \) is independent of \( \omega \).

Consequently, we obtain the orthogonality relation:
\begin{equation}
(\rho' h' + W_{33})\bar{u}_3^{f_n}(h')^2 + \rho' h' u_3^m(0) u_3^m(0) + \rho' \int_0^{h'} \bar{u}_3^{f_n} m dx_3 + \rho' \int_0^{h'} u_3^m m dx_3 = \Lambda_n \delta_{mn},
\end{equation}

where

\begin{equation}
\Lambda_n = (\rho' h' + W_{33})[\bar{u}_3^{f_n}(h')^2] + \rho' h' u_3^m(0)]^2 + \rho' \int_0^{h'} (\bar{u}_3^{f_n})^2 dx_3 + \rho' \int_0^{h'} (u_3^m)^2 dx_3.
\end{equation}

Now we are ready to solve the inhomogeneous problem. The solution to the inhomogeneous problem can be expressed as a series of the solutions for the corresponding homogeneous problem. Thus:

\begin{equation}
\bar{u}_3^f = \sum_{n=1,2} u_3^{f_n}, \quad u_3^s = \sum_{n=1,2} u_3^m, \quad (4.4.17)
\end{equation}

where

\begin{equation}
\bar{u}_3^{f_n} = [A_3^{f_n} \cos(\eta_{fn} x_3)] + B_3^{f_n} \sin(\eta_{fn} x_3)] \exp(i \omega_n t), \quad (4.4.18)
\end{equation}

The relations among the amplitudes in (4.4.18) are given by (4.2.27). There is only one independent amplitude in (4.4.18) and the ratios of other amplitudes to the independent one can be obtained from the dispersion relation. If we choose $B_3^{f_n}$ as the independent amplitude, the other amplitudes are given by:

\begin{equation}
A_3^{f_n} = A_3^m = \frac{\eta_{fn} h' \cos(\eta_{fn} h') - \bar{R} \sin(\eta_{fn} h')}{\eta_{fn} h' \sin(\eta_{fn} h') + \bar{R} \cos(\eta_{fn} h') - k^2} B_3^{f_n},
\end{equation}

\begin{equation}
B_3^{m} = -\tan(\eta_{mn} h') \frac{\eta_{fn} h' \cos(\eta_{fn} h') - \bar{R} \sin(\eta_{fn} h')}{\eta_{fn} h' \sin(\eta_{fn} h') + \bar{R} \cos(\eta_{fn} h') - k^2} B_3^{f_n},
\end{equation}

where

\begin{equation}
\bar{R} = k^2 + R(\eta_{fn} h')^2 + \frac{a_{33}^2 h' W_{33}}{c_{33}}.
\end{equation}
The coefficients in the series can be obtained using the orthogonality conditions of the basis functions. The inhomogeneous boundary value problem is:

\[-\bar{c}_{33}^f \sum_{n=1,2}^{\infty} \eta_{fn}^{2} \bar{u}_3^{fn} = -\rho^f \omega^2 \sum_{n=1,2}^{\infty} \bar{u}_3^{fn} - \rho^f \omega^2 K x_3,\]

\[c_{11}^s \sum_{n=1,2}^{\infty} \eta_{mn}^{2} \mu_3^{mn} = -\rho^s \omega^2 \sum_{n=1,2}^{\infty} \mu_3^{mn},\]  

\[
\sum_{n=1,2}^{\infty} \left[ c_{33}^f \bar{u}_3^{fn} + \frac{e_{33}^f}{h^f} \frac{1}{h^f} \left[ \bar{u}_3^{fn}(0) - \bar{u}_3^{fn}(h^f) \right] \right] = \left( \rho^h + W_{33} \right) \omega^2 \sum_{n=1,2}^{\infty} \bar{u}_3^{fn} + Kh^f, \text{ at } x_3 = h^f,
\]

\[
\sum_{n=1,2}^{\infty} \left[ c_{33}^s \mu_3^{mn} + \frac{e_{33}^s}{h^f} \frac{1}{h^f} \left[ \mu_3^{mn}(0) - \mu_3^{mn}(h^f) \right] \right] = -\omega_n^2 \rho^h \mu_3^{mn}, \text{ at } x_3 = 0,
\]

\[
\sum_{n=1,2}^{\infty} c_{33}^m \mu_3^{mn} = 0, \text{ at } x_3 = -h^s.
\]

Consider the relation between the thickness wavenumbers and the eigenfrequency:

\[
\bar{c}_{33}^f \eta_{fn}^{2} = \rho^f \omega_n^2, \quad c_{33}^s \eta_{mn}^{2} = \rho^s \omega_n^2,
\]

we can simplify (4.4.21) as:

\[
\sum_{n=1,2}^{\infty} \rho^f (\omega^2 - \omega_n^2) \bar{u}_3^{fn} = -\rho^f \omega^2 K x_3,
\]

\[
\sum_{n=1,2}^{\infty} \rho^s (\omega^2 - \omega_n^2) \mu_3^{mn} = 0.
\]

(4.4.22) can be simplified by application of (4.4.9) as:

\[
(\rho^h + W_{33}) \sum_{n=1,2}^{\infty} (\omega^2 - \omega_n^2) \bar{u}_3^{fn} + (\rho^h + W_{33}) \omega^2 K h^f = 0, \text{ at } x_3 = h^f,
\]

\[
\sum_{n=1,2}^{\infty} (\omega^2 - \omega_n^2) \rho^h \mu_3^{mn} = 0, \text{ at } x_3 = 0,
\]

\[
\sum_{n=1,2}^{\infty} e_{33}^s \mu_3^{mn} = 0, \text{ at } x_3 = -h^s.
\]
In order to utilize the orthogonality conditions of the eigenfunctions, we construct the following expression:

$$\sum_{n=1,2}^{\infty} \rho^f (\omega^2 - \omega_n^2) \int_0^{h_f} \bar{u}_3^{m_n} \bar{u}_3^{f_n} \, dx_3 = -\rho^f \omega^2 K \int_0^{h_f} \bar{u}_3^{f_n} x_3 \, dx_3, \quad \sum_{n=1,2}^{\infty} \rho^s (\omega^2 - \omega_n^2) \int_{-h}^{0} u_3^{m_n} u_3^{n_m} \, dx_3 = 0, \quad (4.26)$$

$$(\rho'h' + W_{33}) \sum_{n=1,2}^{\infty} (\omega^2 - \omega_n^2) \bar{u}_3^{m_n} (h_f') \bar{u}_3^{f_n} (h_f') + (\rho'h' + W_{33}) \omega^2 K h_f' \bar{u}_3^{f_n} (h_f') = 0, \quad \sum_{n=1,2}^{\infty} (\omega^2 - \omega_n^2) \rho^h u_3^{m_n} (0) u_3^{n_m} (0) = 0. \quad (4.27)$$

By adding (4.26) and (4.27), we get

$$\sum_{n=1,2}^{\infty} (\omega^2 - \omega_n^2) \left[ \rho^f \int_0^{h_f} \bar{u}_3^{m_n} \bar{u}_3^{f_n} \, dx_3 + \rho^s \int_{-h}^{0} u_3^{m_n} u_3^{n_m} \, dx_3 + (\rho'h' + W_{33}) \bar{u}_3^{m_n} (h_f') \bar{u}_3^{f_n} (h_f') + \rho^h u_3^{m_n} (0) u_3^{n_m} (0) \right]$$

$$= -\rho^f \omega^2 K \int_0^{h_f} \bar{u}_3^{f_n} x_3 \, dx_3 - (\rho'h' + W_{33}) \omega^2 K h_f' \bar{u}_3^{f_n} (h_f'). \quad (4.28)$$

Thus by the orthogonality conditions in (4.15), we can obtain:

$$B^{f_m} = -\frac{\omega^2}{\omega^2 - \omega_n^2} \frac{K}{\Lambda_m} \left[ \rho^f \int_0^{h_f} \bar{u}_3^{m_n} x_3 + (\rho'h' + W_{33}) h_f' \bar{u}_3^{f_n} (h_f') \right]. \quad (4.29)$$

Substitute the solution to the displacement and complete the integrations in (4.29), we get:
\[
B_{nm} = -\frac{\omega^2 K}{\omega^2 - \omega_n^2} \left\{ \rho^f (h^f) \left[ \frac{C_{fn}}{\eta_{fn} h^f} + \frac{1}{(\eta_{fn} h^f)^2} \right] \sin(\eta_{fn} h^f) + \frac{C_{fn}}{(\eta_{fn} h^f)^2} - \frac{1}{\eta_{fn} h^f} \right\} \cos(\eta_{fn} h^f) - \frac{C_{fn}}{(\eta_{fn} h^f)^2} \\
\quad + (\rho'h' + W_{33}) \left[ C_{fn} \cos(\eta_{fn} h^f) + \sin(\eta_{fn} h^f) \right] \right\}, \tag{4.4.30}
\]

where

\[
C_{fn} = \frac{\eta_{fn} h^f \cos(\eta_{fn} h^f) - \tilde{R} \sin(\eta_{fn} h^f)}{\eta_{fn} h^f \sin(\eta_{fn} h^f) + \tilde{R} \cos(\eta_{fn} h^f) - k^2}. \tag{4.4.31}
\]

Since the electromechanical coupling coefficient \( k^2 \), the mechanical inertia of the electrodes \( \rho'h' \) and \( \rho''h'' \) and the surface impedance \( W_{33} \) are all small quantities, the frequencies and the thickness wavenumber are near the unperturbed frequencies and thickness wavenumber given in (4.2.14). The expansion of the roots \( \eta_{fn} h^f \) and \( \eta_{sn} h^s \) about the roots \( \eta_{fn}^0 h^f \) is given by (4.2.29)-(4.2.31). For the completeness of the analysis, we write them again:

\[
\eta_{fn} h^f = \eta_{fn}^0 h^f + \Delta^f, \quad \eta_{sn} h^s = \eta_{sn}^0 h^s + \Delta^s. \tag{4.4.32}
\]

\[
\frac{\tan(\eta_{sn}^0 h^s)}{\tan(\eta_{fn}^0 h^f)} + \frac{\tilde{c}_{33}^f \eta_{fn}^0}{c_{33}^f \eta_{sn}^0} = 0, \quad \eta_{sn}^0 h^s = \sigma \mu \eta_{fn}^0 h^f, \quad \Delta^s = \sigma \mu \Delta^f. \tag{4.4.33}
\]
\[ \Delta^{fn} = \eta_{f, h}^{0} P_{n} / G_{n}, \]

\[ P_{n} = \frac{k^{2}}{(\eta_{f, h}^{0})^{2}} \left[ \frac{2}{\cos(\eta_{f, h}^{0})} - 2 + c^{\prime} \mu \tan(\eta_{f, h}^{0}) \tan(\mu \sigma \eta_{f, h}^{0}) \right] - R^{\prime} - R^{\prime}[1 - c^{\prime} \mu \tan(\eta_{f, h}^{0}) \tan(\mu \sigma \eta_{f, h}^{0})], \]

\[ + \frac{i \omega_{n} h^{0} Z_{33}^{3}}{e_{33}^{0} (\eta_{f, h}^{0})^{2}} [1 - c^{\prime} \mu \tan(\eta_{f, h}^{0}) \tan(\mu \sigma \eta_{f, h}^{0})], \tag{4.4.35} \]

\[ G_{n} = \sec^{2}(\eta_{f, h}^{0}) + c^{\prime} \mu^{2} \sigma \sec^{2}(\mu \sigma \eta_{f, h}^{0}). \]

We can further obtain an approximate expression of the amplitude \( B^{fn} \) which is accurate to the first order in small quantities like \( k^{2} \), \( \rho^{h'} \), \( \rho^{h^{*}} \) and \( Z_{33} \). The calculation is very lengthy but quite straightforward, we only give the result here:

\[ B^{fn} = -\frac{\omega^{2} K h^{f}}{\omega^{2} - \omega_{n}^{2}} A_{1} / (1 + \delta). \tag{4.4.36} \]

where

\[ \delta = \left( R_{1} - \frac{A_{2} R_{2} + A_{3} R_{3}}{A_{2} + A_{3}} \right) \Delta^{fn} + \left[ \frac{D}{A_{1}} + \frac{\rho^{f} h^{f} \cot^{2}(\eta_{f, h}^{0})}{A_{2} + A_{3}} \right] \eta_{f, h}^{0} h^{f} \Delta^{fn} + \tilde{R} - k^{2} \cos(\eta_{f, h}^{0}) \]

\[ + (\rho^{h'} + W_{33}) \left[ \frac{h^{f}}{A_{2} \sin(\eta_{f, h}^{0})} - \frac{1}{(A_{2} + A_{3}) \sin^{2}(\eta_{f, h}^{0})} \right] - \rho^{h^{*}} \frac{\cot^{2}(\eta_{f, h}^{0})}{A_{2} + A_{3}}, \tag{4.4.37} \]

\[ A_{1} = \frac{\rho^{f} h^{f} [1 - \cos(\eta_{f, h}^{0})]}{(\eta_{f, h}^{0})^{2} \sin(\eta_{f, h}^{0})}, \quad A_{2} = \frac{\rho^{f} h^{f}}{4 \eta_{f, h}^{0} \sin^{2}(\eta_{f, h}^{0})} + 2 \eta_{f, h}^{0}, \quad A_{3} = \rho^{h^{*}} \frac{\left( \frac{e_{33}^{0}}{c_{33}^{0}} \right)^{2} 2 \eta_{f, h}^{0} h^{f} + \sin(2 \eta_{f, h}^{0})}{4 \eta_{f, h}^{0} \sin^{2}(\eta_{f, h}^{0})}. \]
Now we consider the inhomogeneous term in the displacement given by (4.4.3)\(_1\). The situation here is the same as the case of quartz plates. It can be expanded as a series of the orthogonal basis functions, thus

\[ u^{f0}_3 = K x_3 = \sum_{n=1,2} A_n^{\eta_{m}^{f} f n}. \]  

We can obtain the amplitudes in (4.4.38) through a procedure similar to the one for \( B^{f m} \). As a result, we get

\[ A_0^f = \frac{K}{\Lambda_m} \left[ \rho' \int_0^{h_f} \tilde{u}_3^{f m} x_3 dx_3 + (\rho' h' + W_3) h_f \tilde{u}_3^{f m} (h_f) \right] = K h_f \frac{A_1}{A_2 + A_3} (1 + \delta). \]  

We can figure out the inhomogeneous term appeared in the scalar differential equation for the forced vibration as follows:

First, rearrange (4.4.36) as:

\[ (\omega^2 - \omega_n^2) B^{f m} = -\omega^2 K h_f \frac{A_1}{A_2 + A_3} (1 + \delta). \]  

Multiplying both sides of (4.4.39) by \( \omega^2 - \omega_n^2 \), we get
\begin{equation}
(\omega^2 - \omega_n^2)A_0^n = (\omega^2 - \omega_n^2)Kh^f \frac{A_1}{A_2 + A_3} (1 + \delta). \tag{4.4.41}
\end{equation}

Adding (4.4.40) and (4.4.41), we obtain:

\begin{equation}
(\omega^2 - \omega_n^2)(A_0^n + B^n) = -\omega_n^2 Kh^f \frac{A_1}{A_2 + A_3} (1 + \delta). \tag{4.4.42}
\end{equation}

Identifying the magnitude $B_i^n + A_0^n$ with the displacement in the scalar differential equation (4.3.69), we can obtain the scalar differential equation for the forced vibration:

\begin{equation}
M_n \left( \frac{\partial^2 f^n}{\partial x_1^2} + \frac{\partial^2 f^n}{\partial x_2^2} \right) - c_{33}(\hat{\eta}_f)^2 f^n - \rho f \ddot{f}^n = -\omega_n^2 Kh^f \frac{A_1}{A_2 + A_3} (1 + \delta) \exp(i\omega t). \tag{4.4.43}
\end{equation}
Chapter 5
Conclusions and Future Work

In conclusion, two-dimensional scalar differential equations have been derived for transversely varying thickness modes in singly- and doubly-rotated quartz plates as well as piezoelectric plates from crystals of class 6mm with surface mechanical loads. The equations lay the foundations for further theoretical work on acoustic wave sensors. They can describe the operating modes of these sensors accurately. They are also relatively simple for theoretical analysis. The equations are shown to be able to lead to simple and useful theoretical results through a few examples.

For future work, a series of theoretical analyses can be performed using the equations obtained for most practically important sensor structural configurations. These include rectangular, circular, and elliptical plates. The plates can be unelectroded, fully electrode, or partially electroded. They can be made from different materials and are under different surface mechanical loads. The results from these analyses will provide deeper understanding of the operating mechanisms of various acoustic wave sensors and will be useful in their design optimization.
Appendix A Material Constants of Common Piezoelectric Materials

Figure A.1 Schematic diagram of the AT-cut quartz plate:

(a) orientation of the AT-cut plate in the crystallographic coordinate system, $\theta = 35.25^\circ$ (b) the plate coordinate system

Elastic constants:

$$
c = \begin{bmatrix}
86.74 & -8.25 & 27.15 & -3.66 & 0 & 0 \\
-8.25 & 129.77 & -7.42 & 5.7 & 0 & 0 \\
27.15 & -7.42 & 102.83 & 9.92 & 0 & 0 \\
-3.66 & 5.7 & 9.92 & 38.61 & 0 & 0 \\
0 & 0 & 0 & 0 & 68.81 & 2.53 \\
0 & 0 & 0 & 0 & 2.53 & 29.01
\end{bmatrix} \times 10^9 \text{ N/m}^2
$$

Piezoelectric constants:

$$
e = \begin{bmatrix}
0.171 & -0.152 & -0.0187 & 0.067 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.108 & -0.095 \\
0 & 0 & 0 & 0 & -0.0761 & 0.067
\end{bmatrix} \text{ C/m}^2
$$

Dielectric constants

$$
\varepsilon = \begin{bmatrix}
39.21 & 0 & 0 \\
0 & 39.82 & 0.86 \\
0 & 0.86 & 40.42
\end{bmatrix} \times 10^{-12} \text{ C/(V \cdot m)}
$$

The density of quartz is $\rho = 2649 \text{ kg/m}^3$. 
Figure A.2 Schematic diagram of an SC-cut quartz plate

(a) orientation of the SC-cut plate in the crystallographic coordinate system, $\theta = 33.88^\circ, \varphi = 22.4^\circ$ (b) the plate coordinate system

Material constants of SC-cut quartz plate: $\theta = 33.88^\circ, \varphi = 22.4^\circ$

Elastic constants:

\[
\bar{\varepsilon} = \begin{bmatrix}
86.74 & 2.09 & 16.8 & -0.35 & -13.71 & -9.2 \\
2.09 & 115.16 & -3.72 & 8.97 & 0.93 & 19.03 \\
16.8 & -3.72 & 110.05 & 3.12 & 12.78 & -9.83 \\
-0.35 & 8.97 & 3.12 & 42.31 & -9.83 & 0.93 \\
-13.71 & 0.93 & 12.78 & -9.83 & 58.75 & 5.73 \\
-9.2 & 19.03 & -9.83 & 0.93 & 5.73 & 39.07 \\
\end{bmatrix} \times 10^9 \text{ N/m}^2
\]

Piezoelectric constants:

\[
\bar{\varepsilon} = \begin{bmatrix}
0.0663 & -0.0833 & 0.0170 & 0.0153 & 0.0879 & -0.1309 \\
-0.1309 & 0.0902 & 0.0407 & -0.0606 & 0.0587 & -0.0269 \\
0.0879 & -0.0606 & -0.0273 & 0.0407 & -0.0394 & 0.0181 \\
\end{bmatrix} \text{ C/m}^2
\]

Dielectric constants:
\[
\bar{\varepsilon} = \begin{bmatrix}
39.21 & 0 & 0 \\
0 & 39.78 & 0.84 \\
0 & 0.84 & 40.46
\end{bmatrix} \times 10^{-12} \text{ C/(V\cdot m)}
\]

The density of quartz is \( \rho = 2649 \text{ kg/m}^3 \).

Material constants of ZnO (hexagonal crystal)

Elastic stiffness:

\[
\begin{bmatrix}
c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\
c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\
c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & c_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & c_{66}
\end{bmatrix} = \begin{bmatrix}
210 & 121 & 105 & 0 & 0 & 0 \\
121 & 210 & 105 & 0 & 0 & 0 \\
105 & 105 & 211 & 0 & 0 & 0 \\
0 & 0 & 0 & 43 & 0 & 0 \\
0 & 0 & 0 & 0 & 43 & 0 \\
0 & 0 & 0 & 0 & 0 & 44.5
\end{bmatrix} \times 10^9 \text{ N/m}^2
\]

where \( c_{66} = \frac{1}{2}(c_{11} - c_{12}) \), the mass density is \( \rho = 5675 \text{ kg/m}^3 \).

Piezoelectric constants:

\[
\begin{bmatrix}
e_{15} & 0 & 0 & 0 & 0 \\
e_{15} & 0 & 0 & 0 & 0 \\
e_{31} & e_{31} & e_{33} & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 & -0.36 & 0 \\
0 & 0 & 0 & -0.36 & 0 & 0 \\
-0.36 & -0.36 & 1.57 & 0 & 0 & 0
\end{bmatrix} \text{ C/m}^2
\]

\[
\begin{bmatrix}
e_{15} & 0 & 0 & 0 & 0 \\
e_{15} & 0 & 0 & 0 & 0 \\
e_{31} & e_{31} & e_{33} & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 & -0.37 & 0 \\
0 & 0 & 0 & -0.37 & 0 & 0 \\
-0.62 & -0.62 & 0.96 & 0 & 0 & 0
\end{bmatrix} \text{ C/m}^2
\]

Dielectric constants:
Material constants of silicon (cubic crystal, m3m)

Elastic stiffness:

\[
\begin{bmatrix}
\varepsilon_{11} & 0 & 0 \\
0 & \varepsilon_{11} & 0 \\
0 & 0 & \varepsilon_{33}
\end{bmatrix}
= \begin{bmatrix}
8.5 & 0 & 0 \\
0 & 8.5 & 0 \\
0 & 0 & 10.9
\end{bmatrix}
\]

\(\varepsilon_0 = 8.854 \times 10^{-12} \, \text{F/m}\)

\[
\begin{bmatrix}
\varepsilon_{11} & 0 & 0 \\
0 & \varepsilon_{11} & 0 \\
0 & 0 & \varepsilon_{33}
\end{bmatrix}
= \begin{bmatrix}
8.33 & 0 & 0 \\
0 & 8.33 & 0 \\
0 & 0 & 8.31
\end{bmatrix}
\]

Mass density \(\rho = 2332 \, \text{kg/m}^3\)

Dielectric constants:

\[
\begin{bmatrix}
\varepsilon_{11} & 0 \\
0 & \varepsilon_{11} \\
0 & 0
\end{bmatrix}
= \begin{bmatrix}
11.7 & 0 & 0 \\
0 & 11.7 & 0 \\
0 & 0 & 11.7
\end{bmatrix}
\]

Material constants of AlN (hexagonal crystal)

Elastic stiffness:

\[
\begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{12} & \varepsilon_{11} & \varepsilon_{13} \\
\varepsilon_{13} & \varepsilon_{13} & \varepsilon_{33}
\end{bmatrix}
= \begin{bmatrix}
345.0 & 125.0 & 120.0 \\
121 & 345.0 & 120.0 \\
105 & 105 & 395.0
\end{bmatrix}
\]

\(\times 10^9 \, \text{N/m}^2\)
where \( c_{66} = \frac{1}{2}(c_{11} - c_{12}) \), \( \rho = 3260 \text{ kg/m}^3 \).

Piezoelectric constants:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & e_{15} & 0 \\
0 & 0 & 0 & e_{15} & 0 & 0 \\
e_{31} & e_{31} & e_{33} & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 & -0.48 & 0 \\
0 & 0 & 0 & -0.48 & 0 & 0 \\
-0.58 & -0.58 & 1.55 & 0 & 0 & 0
\end{bmatrix} \text{ C/m}^2
\]

Dielectric constants:

\[
\begin{bmatrix}
\varepsilon_{11} & 0 & 0 \\
0 & \varepsilon_{11} & 0 \\
0 & 0 & \varepsilon_{33}
\end{bmatrix} = \begin{bmatrix}
9.04 & 0 & 0 \\
0 & 9.04 & 0 \\
0 & 0 & 10.73
\end{bmatrix} \varepsilon_0,
\]

where \( \varepsilon_0 = 8.854 \times 10^{-12} \text{ F/m} \).
Appendix B Othognality of the Eigensolutions

\[ \bar{c}_1 \bar{u}^{(m)}_{1,22} = -\rho \omega_m^2 \bar{u}^{(m)}_1, \quad \tilde{c}_1 \tilde{u}^{(n)}_{1,22} = -\rho \omega_n^2 \tilde{u}^{(n)}_1 \]  
(A.1)

\[ \tilde{c}_1 \bar{u}^{(m)}_{1,2} (\pm h) - \tilde{k}_{66} \bar{u}^{(m)}_1 (\pm h) / h = [2 \rho' h' \omega_m^2 - i \omega_m \tilde{Z}_{11}(\omega_m)] \bar{u}^{(m)}_1 (\pm h) \]  
(A.2)

\[ \int_{-h}^{h} [\tilde{c}_1 \bar{u}^{(m)}_{1,22} \bar{u}^{(m)}_1 - \tilde{c}_1 \bar{u}^{(n)}_{1,22} \bar{u}^{(n)}_1 + \rho (\omega_m^2 - \omega_n^2) \bar{u}^{(m)}_1 \bar{u}^{(n)}_1] \, dx_2 = 0 \]  
(A.3)

\[ \tilde{c}_1 \bar{u}^{(m)}_{1,22} \bar{u}^{(m)}_1 - \tilde{c}_1 \bar{u}^{(n)}_{1,22} \bar{u}^{(n)}_1 \]  
(A.4)

\[ \int_{-h}^{h} [\tilde{c}_1 \bar{u}^{(m)}_{1,22} \bar{u}^{(m)}_1 - \tilde{c}_1 \bar{u}^{(n)}_{1,22} \bar{u}^{(n)}_1] \, dx_2 = \tilde{c}_1 \bar{u}^{(m)}_{1,22} \bar{u}^{(m)}_1 - \tilde{c}_1 \bar{u}^{(n)}_{1,22} \bar{u}^{(n)}_1 \]  
(A.5)
\[\int_{-h}^{h} \left[ \bar{c}_1 \bar{u}_{1,22} \bar{u}_1^m - c_1 u_{1,22} u_1^m + \rho (\omega_n^2 - \omega_m^2) \bar{u}_1^m \right] dx_2 \]
\[= \int_{-h}^{h} \rho (\omega_n^2 - \omega_m^2) \bar{u}_1^m dx_2 + 2 [2 \rho' h (\omega_m^2 - \omega_n^2) - i \omega_n \bar{Z}_{11}(\omega_m) + i \omega_m \bar{Z}_{11}(\omega_n)] \bar{u}_1^m(h) \bar{u}_1^m(h) \] (A.6)
\[= (\omega_n^2 - \omega_m^2) \int_{-h}^{h} \rho \bar{u}_1^m dx_2 + 2 [2 \rho' h - \frac{i \omega_m \bar{Z}_{11}(\omega_m) - i \omega_n \bar{Z}_{11}(\omega_n)}{\omega_m^2 - \omega_n^2}] \bar{u}_1^m(h) \bar{u}_1^m(h) = 0 \]

When \( m \neq n \),
\[\int_{-h}^{h} \rho \sin(\eta_m x_2) \sin(\eta_n x_2) dx_2 + 2 [2 \rho' h' - \frac{i \omega_m \bar{Z}_{11}(\omega_m) - i \omega_n \bar{Z}_{11}(\omega_n)}{\omega_m^2 - \omega_n^2}] \sin(\eta_m h) \sin(\eta_n h) = 0 \] (A.7')

When \( m = n \),
\[\int_{-h}^{h} \rho \bar{u}_1^m dx_2 + 2 [2 \rho' h - \frac{i \omega_m \bar{Z}_{11}(\omega_m) - i \omega_n \bar{Z}_{11}(\omega_n)}{\omega_m^2 - \omega_n^2}] \bar{u}_1^m(h) \bar{u}_1^m(h) \]
\[= \int_{-h}^{h} \rho \sin^2(\eta_m h) dx_2 + 2 [2 \rho' h' - \frac{i \omega_m \bar{Z}_{11}(\omega_m) - i \omega_n \bar{Z}_{11}(\omega_n)}{\omega_m^2 - \omega_n^2}] \sin^2(\eta_m h) \] (A.8)
\[= \rho \left[ h - \frac{\sin(2 \eta_m h)}{2 \eta_m} \right] + 2 [2 \rho' h' - \lim_{n \to m} \frac{i \omega_n \bar{Z}_{11}(\omega_m) - i \omega_n \bar{Z}_{11}(\omega_n)}{\omega_m^2 - \omega_n^2}] \sin^2(\eta_m h) \]
Appendix C Impedances of Common Mechanical Loads on Plates

1. Imperfectly Bonded Thin Mass Layers

\[ Z_i(\omega) = -\frac{1}{i\omega \left( \frac{1}{2\rho' h' \omega^2} - \frac{1}{k} \right)} \]

Special case: perfectly bonded mass layer

\[ Z_i(\omega) = 2\rho' h' \omega i, \quad Z_{ij}(\omega) = 2\rho' h' \omega i \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

2. Mass Layers with Finite Thickness

\[ Z_i(\omega) = -\frac{1}{i\omega \left( \frac{1}{k} + \frac{\cot(\xi h)}{c_{66} \xi} \right)} \]

\[ \xi = \sqrt{\frac{\rho \omega^2}{c_{66}}} \]

Special case: perfectly bonded mass layer

\[ Z_i(\omega) = i\sqrt{\rho c_{66}} \tan \left( h \frac{\rho \omega^2}{c_{66}} \right) \]

3. Fluid Layers of Finite Thickness

\[ Z_i(\omega) = \mu(1+i)\eta \sqrt{\rho c_{66}} \tanh[(1+i)\eta h], \eta = \sqrt{\frac{\rho \omega}{2\mu}} \]

Special case: semi-infinite fluid

\[ Z_i(\omega) = (1+i)\sqrt{\frac{\mu \rho c_{66} \omega}{2}}, \quad Z_{ij}(\omega) = \begin{bmatrix} (1+i)\sqrt{\frac{\mu \rho c_{66} \omega}{2}} & 0 & 0 \\ 0 & \rho c_i & 0 \\ 0 & 0 & (1+i)\sqrt{\frac{\mu \rho c_{66} \omega}{2}} \end{bmatrix} \]
Appendix D Detailed Derivation of Several Important Results

Detailed derivation of (2.2.48):

\[
\frac{\sin(n_m-n_n)h}{n_n-n_m} - \frac{\sin(n_m+n_n)h}{n_n+n_m} = \frac{(n_n+n_m)\sin(n_n-n_m)h - (n_n-n_m)\sin(n_n+n_m)h}{n_n^2 - n_m^2}
\]

\[
= \frac{1}{n_n^2 - n_m^2} \left[ (n_n+n_m)(\sin n_n h \cos n_m h - \sin n_m h \cos n_n h) \\
- (n_n-n_m)(\sin n_n h \cos n_m h + \sin n_m h \cos n_n h) \right]
\]

\[
= \frac{2}{n_n^2 - n_m^2} [\eta_m \sin n_n h \cos n_m h - \eta_n \sin n_n h \cos n_m h]
\]

\[
= \frac{2}{(n_n^2 - n_m^2)h} [\eta_m^2 \sin n_n h \cos n_m h - \eta_n^2 \sin n_n h \cos n_m h]
\]

\[
= \frac{2}{(n_n^2 - n_m^2)h} \left[ R \eta_m^2 h^2 + \frac{e_{26}^2}{\bar{c}_{66}^2} - \frac{Z_z(\omega_m)io_{\omega_m} h}{\bar{c}_{66}} \sin n_n h \sin n_m h \\
- \left( R \eta_n^2 h^2 + \frac{e_{26}^2}{\bar{c}_{66}^2} - \frac{Z_z(\omega_n)io_{\omega_n} h}{\bar{c}_{66}} \right) \sin n_m h \sin n_n h \right]
\]

\[
= \left[ -2Rh + 2i \frac{Z_z(\omega_m)io_{\omega_n} - Z_z(\omega_n)io_{\omega_m}}{\bar{c}_{66}(n_n^2 - n_m^2)} \right] \sin n_m h \sin n_n h
\]

In the last but one step (2.2.41) is employed.
Detailed derivation of (2.2.53):

\[
A_m = -\omega^2 K_h \left[ \frac{2 \sin(\eta_m h) - 2 \cos(\eta_m h)}{\eta_m h^2} \right] - \left[ 2R \omega^2 - \frac{2i \omega Z_i(\omega)}{\rho h} \right] K_h \sin(\eta_m h)
\]

\[
= -\omega^2 K_h \left[ \frac{2 \sin(\eta_m h) - 2 \cos(\eta_m h)}{\eta_m h^2} \right] - \left[ 2R(\omega^2 - \omega_m^2) - \frac{2i \omega Z_i(\omega)}{\rho h} \right] \sin^2(\eta_m h)
\]

\[
= -\omega^2 \frac{2 \sin(\eta_m h) - 2 \cos(\eta_m h)}{\eta_m h^2} \left[ 1 + 2R \frac{\cos(\eta_m h)}{\eta_m h} \right] - \frac{2i \omega Z_i(\omega)}{\rho h} \frac{\sin(\eta_m h)}{\omega^2 - \omega_m^2}
\]

\[
= -\omega^2 \frac{2 \sin(\eta_m h) - 2 \cos(\eta_m h)}{\eta_m h^2} \left[ 1 + \frac{2R}{\eta_m h} \cos(\eta_m h) \right] - \frac{2i \omega Z_i(\omega)}{\rho h} \frac{\sin(\eta_m h)}{\omega^2 - \omega_m^2}
\]

\[
= (-1)^{m-1/2} \frac{\omega^2}{\omega^2 - \omega_m^2} \frac{e_{26}}{c_{66}} 2V \frac{4}{m^2 \pi^2} \frac{1}{\Delta_m} - \frac{2}{m \pi} \frac{\Delta_m}{1 - \frac{2}{m \pi} \Delta_m} + R - \frac{iZ_i(\omega)}{\rho \omega_h}
\]

\[
= (-1)^{m-1/2} \frac{\omega^2}{\omega^2 - \omega_m^2} \frac{e_{26}}{c_{66}} 2V \frac{1}{\Delta_m} - \frac{2}{m \pi} \frac{\Delta_m}{1 - \frac{2}{m \pi} \Delta_m} + R - \frac{iZ_i(\omega)}{\rho \omega_h}
\]

\[
= (-1)^{m-1/2} \frac{\omega^2}{\omega^2 - \omega_m^2} \frac{e_{26}}{c_{66}} \frac{8V}{m^2 \pi^2} \frac{1}{\Delta_m} - \frac{2}{m \pi} \frac{\Delta_m}{1 - \frac{2}{m \pi} \Delta_m} + R - \frac{iZ_i(\omega)}{\rho \omega_h}
\]
\[
\left( -1 \right)^{-m/2} \frac{\cos^2 - \omega_0^2}{\omega_0^2 - \omega_m^2} e_{26} \frac{8V}{m^2 \pi^2} \left( 1 + \frac{4 m \pi \Delta_m}{2} \right) - \frac{m \pi^2}{4} \frac{R}{\rho \omega h} + \frac{m \pi^2}{4} \frac{i Z_i(\omega)}{2} \eta_m h - \frac{m \pi^2}{4} \frac{2i Z_i(\omega) - Z_i(\omega_m) \omega_m}{\rho \omega h} \left[ 1 - 2R + \frac{2}{m \pi} \Delta_m + \frac{2i Z_i(\omega) - Z_i(\omega_m) \omega_m}{\rho \omega h} \right] \]
Details of (2.5.9):

\[ A_n^0 = \frac{e_{26}}{c_{66} h} \left( \int_{-h}^{h} \rho x_2 \sin(\eta_m x_2) \, dx_2 + 2\rho' h' \sin(\eta_m h) - 2\rho' h' \sin(-\eta_m h) \right) \]
\[ = \frac{e_{26}}{c_{66}} \frac{2\sin(\eta_m h)}{\eta_m^2} - \frac{2h}{\eta_m} \cos(\eta_m h) + 2R h^2 \sin(\eta_m h) \]
\[ = -\frac{e_{26}}{c_{66}} \frac{2\sin(\eta_m h)}{\eta_m^2} - \frac{2h}{\eta_m} \cos(\eta_m h) + 2R h^2 \sin(\eta_m h) \]
\[ = -\frac{e_{26}}{c_{66}} \left( \frac{(-1)^{m/2}}{2} \frac{m\pi - \Delta_m}{2} - \frac{(-1)^{m/2} \Delta_m}{2} + R(-1)^{m/2} \right) \]
\[ = -\frac{e_{26}}{c_{66}} \frac{1}{2} \frac{(-1)^{m-1} \Delta_m}{m\pi - \Delta_m} + 2R(-1)^{m-1} \]
\[ = -(-1)^{-m/2} e_{26} \frac{8V}{c_{66} m^2 \pi^2} \frac{1 + \frac{4}{m\pi} \Delta_m - \frac{4}{m\pi} \Delta_m + \frac{m^2 \pi^2}{4} R}{1 - \frac{2}{m\pi} \Delta_m + 2R} \]
\[ = -(-1)^{-m/2} e_{26} \frac{8V}{c_{66} m^2 \pi^2} \left[ 1 + \frac{12}{m^2 \pi^2} - 1 \right] k_{26}^2 + R \]

where \( \Delta_m = \frac{2}{m\pi} k_{26}^2 + \frac{m\pi}{2} R \), \( \Delta_n = \frac{n\pi}{2} R + \frac{2k_{26}^2}{n\pi} \frac{2\hbar \omega_n}{n\bar{\omega}_{66}} Z_t(\omega_n) \)

The influence of the surface impedance is omitted here.
Expansion and simplification of the determinant (3.3.98)

The first term:

\[-\beta_n^* c^{(1)} \eta_1^* + \left\{ i \omega Z_{11} - 2 \rho h' \omega^2 - \frac{k_{36}^2}{h} \right\} \]

\[
\begin{bmatrix}
(i c^{(1)} \eta_1 + q_1) & (q_2 + c^{(1)} p_2) \sin(\eta_2^* h) - \omega Z_{12} \cos(\eta_2^* h) & (q_2 + \bar{c}^{(1)} p_2) \sin(\eta_2^* h) + \omega Z_{12} \cos(\eta_2^* h) \\
i \omega Z_{21} - \frac{k_{26}^2}{h} & (\bar{c}^{(2)} \eta_2^*) \cos(\eta_2^* h) + i(\omega Z_{22} + 2i \rho h' \omega^2) \sin(\eta_2^* h) & (\bar{c}^{(2)} \eta_2^*) \cos(\eta_2^* h) + (i \omega Z_{22} - 2 \rho h' \omega^2) \sin(\eta_2^* h) \\
(q_4 - \bar{c}^{(2)} p_2) & -i \bar{c}^{(2)} \eta_2^* \sin(\eta_2^* h) - (\omega Z_{22} + 2i \rho h' \omega^2) \cos(\eta_2^* h) & i \bar{c}^{(2)} \eta_2^* \sin(\eta_2^* h) + (\omega Z_{22} + 2i \rho h' \omega^2) \cos(\eta_2^* h) \\
i \omega Z_{31} - \frac{k_{36}^2}{h} & 0 & 0 \\
(q_5 - \bar{c}^{(3)} p_3) & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
(q_5 + c^{(1)} p_3) \sin(\eta_3^* h) - \omega Z_{13} \cos(\eta_3^* h) & 0 & 0 \\
0 & (q_3 + \bar{c}^{(1)} p_3) \sin(\eta_3^* h) + \omega Z_{13} \cos(\eta_3^* h) & 0 \\
\bar{c}^{(3)} \eta_4^* \cos(\eta_4^* h) + (i \omega Z_{33} - 2 \rho h' \omega^2) \sin(\eta_4^* h) & 0 & \bar{c}^{(3)} \eta_5^* \cos(\eta_5^* h) + (i \omega Z_{33} - 2 \rho h' \omega^2) \sin(\eta_5^* h) \\
-i \bar{c}^{(3)} \eta_4^* \sin(\eta_4^* h) - (\omega Z_{33} + 2i \rho h' \omega^2) \cos(\eta_4^* h) & 0 & i \bar{c}^{(3)} \eta_5^* \sin(\eta_5^* h) + (\omega Z_{33} + 2i \rho h' \omega^2) \cos(\eta_5^* h)
\end{bmatrix}
\]

\[
= -c^{(1)} \eta_1^* \beta_n^* + i \left( \omega Z_{11} + 2i \rho h' \omega^2 + i \frac{k_{36}^2}{h} \right) \left( c^{(1)} \eta_1^* \right) i( c^{(2)} \eta_2^* )^2 \eta_2^* \eta_2^- \sin(\eta_2^* h) i( c^{(3)} \eta_3^* )^2 \eta_3^* \eta_3^- \sin(\eta_3^* h)
\]

\[
= i \beta_n^* \left( c^{(1)} \right)^2 \left( c^{(2)} \right)^2 \left( c^{(3)} \right)^2 \eta_1^* \eta_1^* \eta_2^+ \eta_2^- \eta_3^+ \eta_3^- \sin(\eta_2^* h) \sin(\eta_3^* h)
\]

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\[ + \left( \omega Z_{11} + 2i\rho'\omega^2 + i \frac{k_{66}^2}{h} \right) \bar{c}^{(1)} (\bar{c}^{(2)})^2 \eta_1^+ \eta_2^- \eta_3^+ \eta_5^- \sin[(\eta_2^+ + \eta_5^-)h] \sin[(\eta_3^+ + \eta_5^-)h] \]  

(3.3.99)

The second term:

\[-( -ic^{(1)}\eta_1^+ + q_1) \]

\[-c^{(1)}\beta_\eta + i \left( \omega Z_{11} + 2i\rho'\omega^2 + i \frac{k_{66}^2}{h} \right) \right] \left( i \left( q_2 - \frac{k_{26}^2}{h} + c^{(1)}p_2 \right) \cos(\eta_2^-h) + \left( \frac{\omega Z_{12}}{2} - \frac{k_{26}^2}{h} \right) \sin(\eta_2^-h) - i(q_2 + c^{(1)}p_2) \cos(\eta_2^-h) \]

\[
i \omega Z_{21} - \frac{k_{26}^2}{h} \]

\[
c^{(2)}\eta_2^+ \cos(\eta_2^-h) + i(\omega Z_{22} + 2i\rho'\omega^2) \sin(\eta_2^-h) \]

\[
c^{(2)}\eta_2^- \cos(\eta_2^-h) + i(\omega Z_{22} + 2i\rho'\omega^2) \sin(\eta_2^-h) \]

\[
q_4 - c^{(2)}p_2 \]

\[-ic^{(2)}\eta_2^- \sin(\eta_2^-h) - (\omega Z_{22} + 2i\rho'\omega^2) \cos(\eta_2^-h) \]

\[
ic^{(2)}\eta_2^- \sin(\eta_2^-h) + (\omega Z_{22} + 2i\rho'\omega^2) \cos(\eta_2^-h) \]

\[
i \omega Z_{31} - \frac{k_{46}^2}{h} \]

\[
0 \]

\[
0 \]

\[
q_5 - c^{(3)}p_3 \]

\[i \left( q_3 + c^{(1)}p_3 - \frac{k_{46}^2}{h} \right) \cos(\eta_3^-h) + \left( \frac{\omega Z_{33}}{2} - \frac{k_{46}^2}{h} \right) \sin(\eta_3^-h) - i(q_3 + c^{(1)}p_3) \cos(\eta_3^-h) \]

\[
0 \]

\[
0 \]

\[
c^{(3)}\eta_3^+ \cos(\eta_3^-h) + i(\omega Z_{33} + 2i\rho'\omega^2) \sin(\eta_3^-h) \]

\[
c^{(3)}\eta_3^- \cos(\eta_3^-h) + i(\omega Z_{33} + 2i\rho'\omega^2) \sin(\eta_3^-h) \]

\[
-ic^{(3)}\eta_3^- \sin(\eta_3^-h) - (\omega Z_{33} + 2i\rho'\omega^2) \cos(\eta_3^-h) \]

\[
ic^{(3)}\eta_3^- \sin(\eta_3^-h) + (\omega Z_{33} + 2i\rho'\omega^2) \cos(\eta_3^-h) \]
\[
\begin{align*}
= & \, i c^{(1)} \eta_1^+ \left[ -c^{(1)} \eta_1^\prime \beta_n^+ + i \left( \omega Z_{11} + 2i\rho' h' \omega^2 + i \frac{k^2}{h} \right) \right] i(c^{(2)})^2 \eta_2^+ \eta_2 \sin[\eta_2^+ + \eta_2] h i(c^{(3)})^2 \eta_3^+ \eta_3 \sin[\eta_3^+ + \eta_3] h \\
+ & \, (-ic^{(1)} \eta_1^+ + q_1) \left( i\omega Z_{21} - \frac{k^2}{h} \right) \right) \\
\begin{vmatrix}
q_2 - \frac{k^2}{h} + c^{(1)} p_2 & \cos(\eta_2^+ h) + i(\omega Z_{12} - \frac{k^2}{h}) \sin(\eta_2^+ h) - i(q_2 + c^{(1)} p_2) \cos(\eta_2^- h) \\
-i(c^{(2)})^2 \eta_2^+ \sin(\eta_2^+ h) - (\omega Z_{22} + 2i\rho' h' \omega^2) \cos(\eta_2^- h) & ic^{(2)} \eta_2^- \sin(\eta_2^- h) + (\omega Z_{22} + 2i\rho' h' \omega^2) \cos(\eta_2^- h) \\
0 & 0 \\
0 & 0 \\
\end{vmatrix}
\end{align*}
\]

\[
\begin{align*}
& \begin{vmatrix}
q_3 + c^{(1)} p_3 - \frac{k^2}{h} \cos(\eta_3^+ h) + i(\omega Z_{13} - \frac{k^2}{h}) \sin(\eta_3^+ h) - i(q_3 + c^{(1)} p_3) \cos(\eta_3^- h) \\
0 & 0 \\
\end{vmatrix}
\end{align*}
\]

\[
\begin{align*}
& \begin{vmatrix}
c^{(3)} \eta_3^+ \cos(\eta_3^+ h) + i(\omega Z_{33} + 2i\rho' h' \omega^2) \sin(\eta_3^+ h) \\
-c^{(3)} \eta_3^- \sin(\eta_3^- h) - (\omega Z_{33} + 2i\rho' h' \omega^2) \cos(\eta_3^- h) \\
\end{vmatrix}
\end{align*}
\]

\[
\begin{align*}
& \begin{vmatrix}
ic^{(1)} \eta_1^+ \left( i\omega Z_{21} - \frac{k^2}{h} \right) i(c^{(2)})^2 \eta_2^+ \sin(\eta_2^+ h) i(q_2 + c^{(1)} p_2) \cos(\eta_2^- h) \to 0 \\
-(-ic^{(1)} \eta_1^+ + q_4)(q_4 - c^{(2)} p_2) \\
\end{vmatrix}
\end{align*}
\]

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\[
\begin{align*}
&\left| \begin{array}{ccc}
  i \left( q_2 - \frac{k_{26}^2}{h} \right) \cos(\eta_2^+ h) + i \omega Z_{12} - \frac{k_{26}^2}{h} \sin(\eta_2^+ h) & -i(q_2 + \bar{c}(\eta_2) p_2) \cos(\eta_2^+ h) \\
  \bar{c}(\eta_2^+ \cos(\eta_2^+ h) + i(\omega Z_{22} + 2i\rho'h' \omega^2) \sin(\eta_2^+ h) & 0 \\
  0 & 0
  \end{array} \right| \\
&\left| \begin{array}{ccc}
  i \left( q_3 + \bar{c}(\eta_3) p_3 - \frac{k_{46}^2}{h} \right) \cos(\eta_3^+ h) + i \omega Z_{13} - \frac{k_{46}^2}{h} \sin(\eta_3^+ h) & -i(q_3 + \bar{c}(\eta_3) p_3) \cos(\eta_3^+ h) \\
  \bar{c}(\eta_3^+ \cos(\eta_3^+ h) + i(\omega Z_{33} + 2i\rho'h' \omega^2) \sin(\eta_3^+ h) & 0 \\
  -i\bar{c}(\eta_3^+ \sin(\eta_3^+ h) - (\omega Z_{33} + 2i\rho'h' \omega^2) \cos(\eta_3^+ h) & i\bar{c}(\eta_3^+ \sin(\eta_3^+ h) + (\omega Z_{33} + 2i\rho'h' \omega^2) \cos(\eta_3^+ h)
  \end{array} \right| \\
&\left| \begin{array}{ccc}
  0 & 0 & 0 \\
  0 & 0 & 0
  \end{array} \right|
\end{align*}
\]
\[
\begin{array}{lll}
\left( q_3 + \tilde{c}^{(1)} p_3 - \frac{k_{46}^2}{h} \right) \cos(\eta_3^+ h) + \left( i \omega Z_{13} - \frac{k_{46}^2}{h} \right) \sin(\eta_3^+ h) & - i(1 + \tilde{c}^{(1)} p_3) \cos(\eta_3^- h) \\
0 & 0 & 0 \\
- ic^{(3)} \eta_3^+ \sin(\eta_3^+ h) - (\omega Z_{33} + 2i \rho h' \omega^2) \cos(\eta_3^+ h) & ic^{(3)} \eta_3^- \sin(\eta_3^- h) + (\omega Z_{33} + 2i \rho h' \omega^2) \cos(\eta_3^- h) \\
0 & 0 & 0 \\
\end{array}
\]

\[
- (i \tilde{c}^{(1)} \eta_1^- + q_1) (q_5 - \tilde{c}^{(3)} p_3)
\]

\[
\begin{array}{lll}
\left( q_2 - \frac{k_{46}^2}{h} + \tilde{c}^{(1)} p_2 \right) \cos(\eta_2^+ h) + \left( i \omega Z_{12} - \frac{k_{46}^2}{h} \right) \sin(\eta_2^+ h) & - i(1 + \tilde{c}^{(1)} p_2) \cos(\eta_2^- h) \\
\tilde{c}^{(2)} \eta_2^+ \cos(\eta_2^+ h) + i(\omega Z_{22} + 2i \rho h' \omega^2) \sin(\eta_2^+ h) & \tilde{c}^{(2)} \eta_2^- \cos(\eta_2^- h) + i(\omega Z_{22} + 2i \rho h' \omega^2) \sin(\eta_2^- h) \\
- ic^{(2)} \eta_2^+ \sin(\eta_2^+ h) - (\omega Z_{22} + 2i \rho h' \omega^2) \cos(\eta_2^+ h) & ic^{(2)} \eta_2^- \sin(\eta_2^- h) + (\omega Z_{22} + 2i \rho h' \omega^2) \cos(\eta_2^- h) \\
0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{lll}
\left( q_3 + \tilde{c}^{(1)} p_3 - \frac{k_{46}^2}{h} \right) \cos(\eta_3^+ h) + \left( i \omega Z_{13} - \frac{k_{46}^2}{h} \right) \sin(\eta_3^+ h) & - i(1 + \tilde{c}^{(1)} p_3) \cos(\eta_3^- h) \\
0 & 0 & 0 \\
\tilde{c}^{(3)} \eta_3^+ \cos(\eta_3^+ h) + i(\omega Z_{33} + 2i \rho h' \omega^2) \sin(\eta_3^+ h) & \tilde{c}^{(3)} \eta_3^- \cos(\eta_3^- h) + i(\omega Z_{33} + 2i \rho h' \omega^2) \sin(\eta_3^- h) \\
0 & 0 & 0 \\
\end{array}
\]

\[
= ic^{(1)} \eta_1^- \tilde{c}^{(3)} \eta_3^- \beta_n \eta_2^+ \eta_3^+ \eta_3^+ \eta_3^- (\tilde{c}^{(2)})^2 (\tilde{c}^{(3)})^2 \sin[(\eta_2^+ + \eta_3^-)h] \sin[(\eta_3^+ + \eta_3^-)h]
\]

\[
+ \left( \omega Z_{14} + 2i \rho h' \omega^2 + i \frac{k_{46}^2}{h} \right) \eta_1^+ \eta_2^- \eta_3^- \tilde{c}^{(1)} (\tilde{c}^{(2)})^2 (\tilde{c}^{(3)})^2 \sin[(\eta_2^+ + \eta_2^-)h] \sin[(\eta_3^+ + \eta_3^-)h] + 0
\]

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\[-(i\tilde{c}^{(1)}\eta_1^+ + q_1)(q_4 - \tilde{c}^{(2)} p_2)\tilde{c}^{(2)}\eta_2^+ \cos(\eta_2^+ h)i(q_2 + \tilde{c}^{(1)} p_2)\cos(\eta_2^+ h)i(\tilde{c}^{(3)})^2 \eta_3^+ \eta_5^+ \sin[(\eta_3^+ + \eta_5^+) h] \]

\[-(i\tilde{c}^{(1)}\eta_1^+ + q_1)(q_4 - \tilde{c}^{(2)} p_2)\tilde{c}^{(2)}\eta_2^- \cos(\eta_2^- h)i(q_2 + \tilde{c}^{(1)} p_2)\cos(\eta_2^- h)i(\tilde{c}^{(3)})^2 \eta_3^- \eta_5^- \sin[(\eta_3^- + \eta_5^-) h] + 0 \]

\[-(i\tilde{c}^{(1)}\eta_1^+ + q_1)(q_5 - \tilde{c}^{(3)} p_3)\tilde{c}^{(3)}\eta_3^+ \cos(\eta_3^+ h)i(q_3 + \tilde{c}^{(1)} p_3)\cos(\eta_3^+ h)i(\tilde{c}^{(2)})^2 \eta_5^+ \eta_5^- \sin[(\eta_5^+ + \eta_5^-) h] \]

\[-(i\tilde{c}^{(1)}\eta_1^+ + q_1)(q_5 - \tilde{c}^{(3)} p_3)\tilde{c}^{(3)}\eta_3^- \cos(\eta_3^- h)i(q_3 + \tilde{c}^{(1)} p_3)\cos(\eta_3^- h)i(\tilde{c}^{(2)})^2 \eta_5^+ \eta_5^- \sin[(\eta_5^+ + \eta_5^-) h] \]

\[= i\tilde{c}^{(1)}\eta_1^+ \tilde{c}^{(1)}\eta_1^- \beta_{\eta_2}^- \eta_2^+ \eta_5^+ \eta_5^- \eta_3^+ \eta_3^- (\tilde{c}^{(2)})^2 (\tilde{c}^{(3)})^2 \sin[(\eta_3^+ + \eta_3^-) h] \sin[(\eta_3^+ + \eta_3^-) h] \]

\[\quad + \left( \omega Z_{11} + 2i \rho h' \omega^2 + i \frac{\hbar^2}{\hbar} \right) \eta_1^+ \eta_2^+ \eta_3^+ \eta_3^- \tilde{c}^{(1)}(\tilde{c}^{(2)})^2 (\tilde{c}^{(3)})^2 \sin[(\eta_3^+ + \eta_3^-) h] \sin[(\eta_3^+ + \eta_3^-) h] \]

\[+ (-i\tilde{c}^{(1)}\eta_1^+ + q_1)(q_4 - \tilde{c}^{(2)} p_2)(\eta_2^+ + \eta_2^-) \cos(\eta_2^+ h)(q_2 + \tilde{c}^{(1)} p_2)\cos(\eta_2^- h)(\tilde{c}^{(2)})^2 \eta_5^+ \eta_5^- \sin[(\eta_5^+ + \eta_5^-) h] \]

\[+ (-i\tilde{c}^{(1)}\eta_1^+ + q_1)(q_5 - \tilde{c}^{(3)} p_3)(\eta_3^+ + \eta_3^-) \cos(\eta_3^+ h)(q_3 + \tilde{c}^{(1)} p_3)\cos(\eta_3^- h)(\tilde{c}^{(2)})^2 \eta_5^+ \eta_5^- \sin[(\eta_5^+ + \eta_5^-) h] \]  

(3.3.100)

The third term:

\[ \left\{ i\omega Z_{21} - \frac{\hbar^2}{\hbar} \right\} \]
\[
\begin{array}{c|cccc}
-\bar{c}(1)\eta_1 \beta_n^+ + i \left( \omega Z_{13} + 2i\rho'\omega^2 + i \frac{k_{26}^2}{h} \right) & i\left( q_2 - \frac{k_{26}^2}{h} + \bar{c}(1) p_2 \right) \cos(\eta_2^+ h) + i\omega Z_{12} - \frac{k_{26}^2}{h} \right) \sin(\eta_2^+ h) & -i(q_2 + \bar{c}(1) p_2) \cos(\eta_2^+ h) \\
\bar{c}(1)\eta_1^+ + q_1 & (q_2 + \bar{c}(1) p_2) \sin(\eta_2^+ h) - \omega Z_{12} \cos(\eta_2^+ h) & (q_2 + \bar{c}(1) p_2) \sin(\eta_2^- h) + \omega Z_{12} \cos(\eta_2^- h) & \approx 0 \\
q_4 - \bar{c}(2) p_2 & -i\bar{c}(2) \eta_2^- \sin(\eta_2^- h) - (\omega Z_{22} + 2i\rho'\omega^2) \cos(\eta_2^- h) & i\bar{c}(2) \eta_2^- \sin(\eta_2^- h) + (\omega Z_{22} + 2i\rho'\omega^2) \cos(\eta_2^- h) \\
i\omega Z_{31} - \frac{k_{26}^2}{h} & 0 & 0 & \eta_5 - \bar{c}(3) p_3 \\
q_5 - \bar{c}(3) p_3 & 0 & 0 & \\
\end{array}
\]

The fourth term:

\[-(q_4 - \bar{c}(2) p_2)\]
\[ -\bar{c}^{(1)} \eta \beta^- + i \left( oZ_{11} + 2ip' \omega^2 + i \frac{k^2}{h} \right) \]
\[ i\bar{c} (1)^{q_1} - q_1 \]
\[ i\omega Z_{21} - \frac{k^2}{h} \]
\[ i\omega Z_{31} - \frac{k^2}{h} \]
\[ q_5 - \bar{c}^{(3)} p_3 \]
\[ \begin{array}{ccc}
(q_2 - \frac{k^2}{h} + c^{(1)} p_2) \cos(\eta^*_1 h) + \left( i\omega Z_{12} - \frac{k^2}{h} \right) \sin(\eta^*_2 h) & -i(q_2 + \bar{c}^{(1)} p_2) \cos(\eta^*_2 h) \\
(q_2 + c^{(1)} p_2) \sin(\eta^*_1 h) - \omega Z_{12} \cos(\eta^*_1 h) & (q_2 + c^{(1)} p_2) \sin(\eta^*_2 h) + \omega Z_{12} \cos(\eta^*_2 h) \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{array} \]
\[ = i\bar{c}^{(1)} \eta^- (q_4 - \bar{c}^{(2)} p_2) \]
\[ \begin{array}{ccc}
(i\omega Z_{11} + 2ip' \omega^2 + i \frac{k^2}{h}) \cos(\eta^*_1 h) + \left( i\omega Z_{12} - \frac{k^2}{h} \right) \sin(\eta^*_2 h) & -i(q_2 + \bar{c}^{(1)} p_2) \cos(\eta^*_2 h) \\
(q_2 + c^{(1)} p_2) \sin(\eta^*_1 h) - \omega Z_{12} \cos(\eta^*_1 h) & (q_2 + c^{(1)} p_2) \sin(\eta^*_2 h) + \omega Z_{12} \cos(\eta^*_2 h) \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{array} \]
\[i \left( q_3 + \bar{c}^{(1)} p_3 - \frac{k_{16}^2}{h} \right) \cos(\eta_3^* h) + \left( i \omega Z_{13} - \frac{k_{16}^2}{h} \right) \sin(\eta_3^* h) - i (q_3 + \bar{c}^{(1)} p_3) \cos(\eta_3^* h) = 0 \]

\[\bar{c}^{(3)} \eta_3^* \cos(\eta_3^* h) + i (\omega Z_{33} + 2i \rho^h \omega^2) \sin(\eta_3^* h) \]

\[0 \]

\[-i \bar{c}^{(3)} \eta_3^* \sin(\eta_3^* h) - (\omega Z_{33} + 2i \rho^h \omega^2) \cos(\eta_3^* h) \]

\[i \bar{c}^{(3)} \eta_3^* \sin(\eta_3^* h) + (\omega Z_{33} + 2i \rho^h \omega^2) \cos(\eta_3^* h) \]

\[-(q_4 - \bar{c}^{(2)} p_2)(q_5 - \bar{c}^{(3)} p_3) \]

\[\begin{align*}
&\left( q_2 - \frac{k_{26}^2}{h} + \bar{c}^{(1)} p_2 \right) \cos(\eta_2^* h) + \left( i \omega Z_{12} - \frac{k_{26}^2}{h} \right) \sin(\eta_2^* h) - i (q_2 + \bar{c}^{(1)} p_2) \cos(\eta_2^* h) \\
&\left( q_2 + \bar{c}^{(1)} p_2 \right) \sin(\eta_2^* h) - \omega Z_{12} \cos(\eta_2^* h) + \left( q_2 + \bar{c}^{(1)} p_2 \right) \sin(\eta_2^* h) + \omega Z_{12} \cos(\eta_2^* h) \\
&\bar{c}^{(2)} \eta_2^* \cos(\eta_2^* h) + i (\omega Z_{22} + 2i \rho^h \omega^2) \sin(\eta_2^* h) \\
&0
\end{align*} \]

\[\begin{align*}
&\left( q_3 + \bar{c}^{(1)} p_3 - \frac{k_{16}^2}{h} \right) \cos(\eta_3^* h) + \left( i \omega Z_{13} - \frac{k_{16}^2}{h} \right) \sin(\eta_3^* h) - i (q_3 + \bar{c}^{(1)} p_3) \cos(\eta_3^* h) \\
&(q_3 + \bar{c}^{(1)} p_3) \sin(\eta_3^* h) - \omega Z_{13} \cos(\eta_3^* h) + (q_3 + \bar{c}^{(1)} p_3) \sin(\eta_3^* h) + \omega Z_{13} \cos(\eta_3^* h) \\
&0 \\
&\bar{c}^{(3)} \eta_3^* \cos(\eta_3^* h) + i (\omega Z_{33} + 2i \rho^h \omega^2) \sin(\eta_3^* h) \\
&\bar{c}^{(3)} \eta_3^* \cos(\eta_3^* h) + i (\omega Z_{33} + 2i \rho^h \omega^2) \sin(\eta_3^* h) \]

\[= i \bar{c}^{(1)} \eta_3^* (q_4 - \bar{c}^{(2)} p_2) \bar{c}^{(2)} \eta_2^* \cos(\eta_2^* h) i (q_2 + \bar{c}^{(1)} p_2) \cos(\eta_2^* h) i (\bar{c}^{(3)} \eta_3^* \eta_3^* \sin[(\eta_3^* + \eta_3^*) h]) \\
+i \bar{c}^{(1)} \eta_3^* (q_4 - \bar{c}^{(2)} p_2) i (q_2 + \bar{c}^{(1)} p_2) \cos(\eta_2^* h) \bar{c}^{(2)} \eta_2^* \cos(\eta_2^* h) i (\bar{c}^{(3)} \eta_3^* \eta_3^* \sin[(\eta_3^* + \eta_3^*) h]) \]

\[+(q_4 - \bar{c}^{(2)} p_2)(q_5 - \bar{c}^{(3)} p_3) \bar{c}^{(3)} \eta_3^* \cos(\eta_3^* h) \]

\[+(q_4 - \bar{c}^{(2)} p_2)(q_5 - \bar{c}^{(3)} p_3) \bar{c}^{(3)} \eta_3^* \cos(\eta_3^* h) \]
\[
\begin{align*}
&\left( q_2 - \frac{k^2_{16}}{h} + \tilde{c}^{(1)} p_2 \right) \cos(\eta^+_2 h) + \left( i \omega Z_{12} - \frac{k^2_{26}}{h} \right) \sin(\eta^+_2 h) - i(q_2 + \tilde{c}^{(1)} p_2) \cos(\eta^+_2 h) - i(q_3 + \tilde{c}^{(1)} p_3) \cos(\eta^+_3 h) \\
&\quad (q_2 + \tilde{c}^{(1)} p_2) \sin(\eta^+_2 h) - \omega Z_{12} \cos(\eta^+_2 h) \\
&\quad (q_3 + \tilde{c}^{(1)} p_3) \sin(\eta^+_3 h) + \omega Z_{13} \cos(\eta^+_3 h) \\
&\quad - (q_4 - \tilde{c}^{(2)} p_2)(q_3 - \tilde{c}^{(3)} p_3) \tilde{c}^{(3)} \eta^-_3 \cos(\eta^-_3 h) \\
&\left( q_2 - \frac{k^2_{26}}{h} + \tilde{c}^{(1)} p_2 \right) \cos(\eta^+_2 h) + \left( i \omega Z_{12} - \frac{k^2_{26}}{h} \right) \sin(\eta^+_2 h) - i(q_2 + \tilde{c}^{(1)} p_2) \cos(\eta^+_2 h) \\
&\quad (q_2 + \tilde{c}^{(1)} p_2) \sin(\eta^+_2 h) - \omega Z_{12} \cos(\eta^+_2 h) \\
&\quad (q_3 + \tilde{c}^{(1)} p_3) \sin(\eta^+_3 h) + \omega Z_{13} \cos(\eta^+_3 h) \\
&\quad - (q_4 - \tilde{c}^{(2)} p_2)(q_3 - \tilde{c}^{(3)} p_3) \tilde{c}^{(3)} \eta^-_3 \cos(\eta^-_3 h) \\
&\left( q_3 + \tilde{c}^{(1)} p_3 - \frac{k^2_{16}}{h} \right) \cos(\eta^+_3 h) + \left( i \omega Z_{13} - \frac{k^2_{46}}{h} \right) \sin(\eta^+_3 h) \\
&\quad (q_3 + \tilde{c}^{(1)} p_3) \sin(\eta^+_3 h) - \omega Z_{13} \cos(\eta^+_3 h) \\
&\quad 0 \\
&\quad - i(q_4 - \tilde{c}^{(2)} p_2)(q_3 + \tilde{c}^{(1)} p_2)(\eta^+_2 + \eta^-_2)\tilde{c}^{(1)}(\tilde{c}^{(2)})^2 \eta^-_3 \eta^+_3 \cos(\eta^+_3 h) \cos(\eta^-_3 h) \sin[(\eta^+_3 + \eta^-_3) h]
\end{align*}
\]

\(3.3.102\)
\[
\begin{pmatrix}
 i \omega Z_{31} - \frac{k_{46}^2}{h} \\
 -c^{(1)} \eta_1^+ \beta - i \left( \omega Z_{11} + 2i \rho' h' \omega^2 + i k_{46}^2 \right) \nu_q \left( q_2 - \frac{k_{26}^2}{h} + c^{(1)} p_2 \right) \cos(\eta_2^+ h) + \left( i \omega Z_{12} - \frac{k_{26}^2}{h} \right) \sin(\eta_2^+ h) - i(q_2 + c^{(1)} p_2) \cos(\eta_2^- h) \\
 i Z_{21} - \frac{k_{26}^2}{h} \\
 q_4 - c^{(2)} p_2 \\
 q_5 - c^{(3)} p_3 \\
 i \left( q_3 + c^{(1)} p_3 - \frac{k_{46}^2}{h} \right) \cos(\eta_3^+ h) + \left( i \omega Z_{13} - \frac{k_{46}^2}{h} \right) \sin(\eta_3^+ h) - i(q_3 + c^{(1)} p_3) \cos(\eta_3^- h) \\
 (q_3 + c^{(1)} p_3) \sin(\eta_3^+ h) - \omega Z_{31} \cos(\eta_3^+ h) \\
 0 \\
 0 \\
 -ic^{(3)} \eta_3^+ \sin(\eta_3^+ h) - (\omega Z_{31} + 2i \rho' h' \omega^2) \cos(\eta_3^+ h) \\
 i c^{(3)} \eta_3^- \sin(\eta_3^- h) + (\omega Z_{31} + 2i \rho' h' \omega^2) \cos(\eta_3^- h)
\end{pmatrix}
\]

The sixth term:

\[-(q_5 - c^{(3)} p_3)\]
\[
\begin{align*}
-\bar{c}^{(1)}\eta_1 - \beta_\eta^+ c_1 + \left(\omega Z_{11} + 2i\rho' h' \omega^2 + i\frac{k_{16}^2}{h}\right) &\quad \left(i\left(\frac{k_{26}}{h} + c_1\right)\cos(\eta_1^* h) + \left(i\omega Z_{12} - \frac{k_{26}^2}{h}\right)\sin(\eta_2^* h)\right) - i(q_2 + \bar{c}^{(1)} p_2) \cos(\eta_2^* h) \\
&\quad i\bar{c}^{(1)} \eta_1^- + q_1 \\
&\quad i\omega Z_{21} - \frac{k_{26}^2}{h} \\
&\quad q_4 - \bar{c}^{(2)} p_2 \\
i\omega Z_{31} - \frac{k_{46}^2}{h} &\quad 0
\end{align*}
\]
\[
\begin{align*}
&\quad i\left(q_3 + \bar{c}^{(1)} p_3 - \frac{k_{46}^2}{h}\right)\cos(\eta_3^* h) + \left(i\omega Z_{33} - \frac{k_{46}^2}{h}\right)\sin(\eta_3^* h) - i(q_3 + \bar{c}^{(1)} p_3) \cos(\eta_3^* h) \\
&\quad (q_3 + \bar{c}^{(1)} p_3) \sin(\eta_3^* h) - \omega Z_{13} \cos(\eta_3^* h) \\
&\quad 0 \\
&\quad 0 \\
&\quad \bar{c}^{(3)} \eta_4^* \cos(\eta_4^* h) + i(\omega Z_{33} + 2i\rho' h' \omega^2) \sin(\eta_4^* h) \\
&\quad \bar{c}^{(3)} \eta_5^* \cos(\eta_5^* h) + i(\omega Z_{33} + 2i\rho' h' \omega^2) \sin(\eta_5^* h)
\end{align*}
\]

\[= -i\bar{c}^{(1)}\bar{c}^{(3)}(\bar{c}^{(2)})^2 (\eta_3^* + \eta_3^-) \eta_1^+ \eta_2^+ \eta_3^+ (q_5 + \bar{c}^{(1)} p_3) \cos(\eta_1^* h) \cos(\eta_3^* h) \sin[\eta_1^* + \eta_2^- h] \] (3.3.104)
Detailed derivation of (3.4.38)

\[
A_m = \frac{-\rho \omega^2 K_1 \left[ \frac{2\sin(\eta_m h)}{\eta_m} - \frac{2}{\eta_m} \cos(\eta_m h) \right] - [2R \rho h \omega^2 - 2i\omega Z_{11}(\omega)] K_1 h \sin(\eta_m h)}{\rho(\omega^2 - \omega_m^2) \left[ h - \frac{\sin(2\eta_m h)}{2\eta_m} \right] + [2R \rho h (\omega^2 - \omega_m^2) + 2i[\omega Z_{11}(\omega_m) - \omega Z_{11}(\omega)]] \sin^2(\eta_m h)}
\]

\[
= -\omega^2 K_1 \frac{\frac{2\sin(\eta_m h)}{\eta_m^2 h^2} - \frac{2}{\eta_m h} \cos(\eta_m h)}{1 - \frac{\sin(2\eta_m h)}{2\eta_m h}} + \frac{[2R + \frac{2i\omega Z_{11}(\omega_m) - \omega Z_{11}(\omega)}{\rho h(\omega^2 - \omega_m^2)}]}{\rho h(\omega^2 - \omega_m^2)} \sin^2(\eta_m h)
\]

\[
= \frac{Ve_{29}}{\epsilon^{(1)} - k_{66}^2} \frac{\omega^2}{(\omega^2 - \omega_m^2)} \left[ 1 - \frac{\sin(2\eta_m h)}{2\eta_m h} \right] + \frac{[2R + \frac{2i[\omega Z_{11}(\omega_m) - \omega Z_{11}(\omega)]}{\rho h(\omega^2 - \omega_m^2)}]}{\rho h(\omega^2 - \omega_m^2)} \sin^2(\eta_m h)
\]

\[
\approx \frac{Ve_{29}}{\epsilon^{(1)} - k_{66}^2} \frac{\omega^2}{(\omega^2 - \omega_m^2)} \left[ 1 - \frac{(-1)^{m-1} \Delta_m}{m\pi/2 - \Delta_m} \right] + \frac{[2R + \frac{2i[\omega Z_{11}(\omega_m) - \omega Z_{11}(\omega)]}{\rho h(\omega^2 - \omega_m^2)}]}{\rho h(\omega^2 - \omega_m^2)} (-1)^{m-1}
\]

\[
= \frac{Ve_{29}}{\epsilon^{(1)} - k_{66}^2} \frac{(-1)^{m-1/2}}{1 - \frac{m\pi/2 - \Delta_m}{m\pi/2 - \Delta_m}} + \frac{[2R + \frac{2i[\omega Z_{11}(\omega_m) - \omega Z_{11}(\omega)]}{\rho h(\omega^2 - \omega_m^2)}]}{\rho h(\omega^2 - \omega_m^2)} (-1)^{m-1}
\]

\[
= \frac{Ve_{29}}{\epsilon^{(1)} - k_{66}^2} \frac{(-1)^{m-1/2}}{\Delta_m} \left[ 1 - \frac{1}{m\pi/2 - \Delta_m} \right] + \frac{[2R + \frac{2i[\omega Z_{11}(\omega_m) - \omega Z_{11}(\omega)]}{\rho h(\omega^2 - \omega_m^2)}]}{\rho h(\omega^2 - \omega_m^2)} (-1)^{m-1}
\]
\[
\begin{align*}
\mathcal{V}_{29} & \approx \frac{(-1)^{m-\frac{1}{2}}}{c^{(1)} - k_{66}^2 (1 - \omega_m^2/\omega^2) m^2 \pi^2} \left( \frac{1}{\left(1 - \frac{2}{m \pi \Delta_m} \right)^2} \right) \left( \frac{1}{2 - \Delta_m} \right) + \left[ \frac{2i \left[ \omega_m Z_{11}(\omega_n) - \omega Z_{11}(\omega) \right]}{\rho \omega^2 - \omega_m^2} \right] R - \frac{iZ_{11}(\omega)}{\rho \omega} \\
\mathcal{V}_{29} & \approx \frac{(-1)^{m-\frac{1}{2}}}{c^{(1)} - k_{66}^2 (1 - \omega_m^2/\omega^2) m^2 \pi^2} \left( \frac{1}{\left(1 - \frac{2}{m \pi \Delta_m} \right)^2} \right) \left( \frac{1}{2 - \Delta_m} \right) + \left[ \frac{2i \left[ \omega_m Z_{11}(\omega_n) - \omega Z_{11}(\omega) \right]}{\rho \omega^2 - \omega_m^2} \right] \left( \frac{1}{2} + \frac{m \pi}{R} \Delta_m - 2R - \frac{2i \left[ \omega_m Z_{11}(\omega_n) - \omega Z_{11}(\omega) \right]}{\rho \omega^2 - \omega_m^2} \right)
\end{align*}
\]
Detailed derivation of (3.5.9)

\[
A_n^0 = - \frac{e_29}{c^{(I)} - k_{66}^2} \frac{V}{2h} \int_0^h \rho x_2 \sin(\eta_m x_2) dx_2 + 2 \rho' h' \sin(\eta_m h) - 2 \rho' h' \sin(-\eta_m h) \\
\int_0^h \rho \sin^2(\eta_m x_2) dx_2 + 4 \rho' h' \sin^2(\eta_m h)
\]

\[
= - \frac{e_29 V}{c^{(I)} - k_{66}^2} \frac{2 \sin(\eta_m h)}{\eta_m^2} \frac{2}{\eta_m} \frac{2h}{\cos(\eta_m h) + 2Rh^2 \sin(\eta_m h)}
\]

\[
= - \frac{e_29 V/2}{c^{(I)} - k_{66}^2} \frac{2 \sin(\eta_m h)}{\eta_m^2} \frac{2}{\eta_m} \frac{2h}{\cos(\eta_m h) + 2Rh^2 \sin(\eta_m h)}
\]

\[
\frac{(-1)^{m-1/2}}{2} - \frac{(-1)^{m-1/2}\Delta_m + R(-1)^{m-1}}{m\pi - \Delta_m}
\]

\[
= - \frac{e_29 V}{c^{(I)} - k_{66}^2} \frac{4}{m\pi^2 - \Delta_m} \left[ 1 \frac{4}{m\pi^2} \Delta_m - \frac{m\pi^2}{2} + \frac{m^2\pi^2}{4} R \right] \frac{1 - \frac{2}{m\pi} \Delta_m + 2R
\]

\[
= - \frac{e_29 V}{c^{(I)} - k_{66}^2} \frac{4}{m\pi^2 - \Delta_m} \left[ 1 \frac{4}{m\pi^2} \Delta_m - \frac{m\pi^2}{2} + \frac{m^2\pi^2}{4} R + \frac{2}{m\pi} \Delta_m - 2R \right]
\]

\[
\frac{12}{m^2 \pi^2 - 1} \frac{k_{66}^2}{c^{(I)}} + R \]

where

\[
\Delta_n = \frac{2}{n\pi} \frac{k_{66}^2}{c^{(I)}} + \frac{n\pi}{2} R.
\]
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