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Research Article

Oscillation of Certain Emden-Fowler Dynamic Equations on Time Scales

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We deal with the oscillation of a generalized Emden-Fowler dynamic equation in the form

\[ r(t) \left| Z^\Delta(t) \right|^{\alpha-1} Z^\Delta(t) + f(t, x(\sigma(t))) = 0, \]

where \( t \in [t_0, \infty)_T \).

1. Introduction

The theory of time scales has attracted a great deal of attention since it was first introduced by Hilger [1] in order to unify continuous and discrete analysis. For completeness, we recall the following concepts related to the notion of time scales; see [2, 3] for more details. A time scale \( \mathbb{T} \) is an arbitrary nonempty closed subset of the real numbers \( \mathbb{R} \). In this paper, since we shall be concerned with the oscillatory behavior of solutions, we shall also assume that \( \sup \mathbb{T} = \infty \). We define the time scale interval \( [t_0, \infty)_T \) by \( [t_0, \infty)_T := [t_0, \infty) \cap \mathbb{T} \). The forward and backward jump operators are defined by

\[ \sigma(t) := \inf \{ s \in \mathbb{T} : s > t \}, \]
\[ \rho(t) := \sup \{ s \in \mathbb{T} : s < t \}, \]

where \( \inf \emptyset := \sup \mathbb{T} \) and \( \sup \emptyset := \inf \mathbb{T} \); here \( \emptyset \) denotes the empty set. A point \( t \in \mathbb{T} \) and \( t > \inf \mathbb{T} \) is said to be left-dense if \( \rho(t) = t \), right-dense if \( t < \sup \mathbb{T} \) and \( \sigma(t) = t \), left-scattered if \( \rho(t) < t \), and right-scattered if \( \sigma(t) > t \). The graininess function \( \mu \) for the time scale \( \mathbb{T} \) is defined by \( \mu(t) := \sigma(t) - t \), and for any function \( f : \mathbb{T} \to \mathbb{R} \), the notation \( f^\Delta(t) \) denotes \( f(\sigma(t)) \). A function \( g : \mathbb{T} \to \mathbb{R} \) is said to be rd-continuous provided \( g \) is continuous at right-dense points and at left-dense points in \( \mathbb{T} \) and left-hand limits exist and are finite. The set of all such rd-continuous functions is denoted by \( C_{\text{rd}}(\mathbb{T}) \). We say that \( x : \mathbb{T} \to \mathbb{R} \) is differentiable at \( t \in \mathbb{T} \) provided

\[ x^\Delta(t) := \lim_{s \to t} \frac{x(t) - x(s)}{t - s}, \]

exists when \( \sigma(t) = t \) (here by \( s \to t \) it is understood that \( s \) approaches \( t \) in the time scale) and when \( x \) is continuous at \( t \) and \( \sigma(t) > t \)

\[ x^\Delta(t) := \frac{x(\sigma(t)) - x(t)}{\mu(t)}. \]

Note that if \( \mathbb{T} = \mathbb{R} \), then the delta derivative is just the standard derivative and when \( \mathbb{T} = \mathbb{Z} \) the delta derivative is just the forward difference operator. The set of functions \( f : \mathbb{T} \to \mathbb{R} \) which are differentiable and whose derivative is rd-continuous is denoted by \( C_{\text{rd}}^1(\mathbb{T}, \mathbb{R}) \).

In this paper, we consider the oscillatory behavior of the nontrivial solutions of the second-order Emden-Fowler dynamic equation of the form

\[ \left( r(t) \left| Z^\Delta(t) \right|^{\alpha-1} Z^\Delta(t) \right)^\Delta + f(t, x(\sigma(t))) = 0, \]

where \( t \in [t_0, \infty)_T \).
on an arbitrary time scale $\mathbb{T}$, with $\sup \mathbb{T} = \infty$, where $Z(t) = x(t) + p(t)x(\tau(t))$, and $\alpha > 0$ is a constant. Throughout this paper, we always assume that

- (A1) $r \in C_{rd}(I_{0,\infty})_{\mathbb{T}}, (0, \infty) \cup \{0\}$ with $\int_{t_0}^{\infty} r^{-1/\alpha}(s) \Delta s = \infty$;
- (A2) $p \in C_{rd}(I_{0,\infty})_{\mathbb{T}}$ with $0 \leq p(t) < 1$;
- (A3) $r, \delta \in C_{rd}(I_{0,\infty})_{\mathbb{T}}$, $\tau(t) \leq t$, $\delta(t) \leq t$, and $\lim_{t \to \infty} \tau(t) = \lim_{t \to \infty} \delta(t) = \infty$;
- (A4) $f(t, u) \in C(I_{[t_0,\infty)}_{\mathbb{T}} \times \mathbb{R}, \mathbb{R})$ is a continuous function such that $uf(t, u) > 0$, for all $u \neq 0$ and there exists a positive right-dense continuous function $q(t)$ defined on $I_{[t_0,\infty)}_{\mathbb{T}}$ such that $|f(t, u)| \geq q(t)|u|^{\beta}$ for all $t \in [t_0, \infty)$ and for all $u \in \mathbb{R}$, where $\beta > 0$ is a constant. Throughout this paper, we assume that $r(t)$ is a continuous function such that $r(t) = 0$ for all $t \geq t_0$.

We begin with the following lemmas.

**Lemma 1.** Assume that (4) has a positive solution $x(t)$ on $[t_0, \infty)$, Then for sufficiently large $T$, one has

$$Z(t) > 0, \quad Z^\Delta(t) > 0, \quad \left(r(t) \left|Z^\Delta(t)\right|^{\alpha-1}Z^\Delta(t)\right)^\Delta \leq 0, \quad t \in [T, \infty)_{\mathbb{T}}. \tag{8}$$

**Proof.** Assume that (4) has a nonoscillatory solution on $[t_0, \infty)$. Without loss of generality, we assume that there exists a $T \in I_{[t_0,\infty)}_{\mathbb{T}}$ such that $x(t), x(\tau(t)), x(\delta(t)) > 0$ for all $t \in [t_0, \infty)$ and for all $u \in \mathbb{R}$, where $\beta > 0$ is a constant.

By a solution of (4), we mean a nontrivial real-valued function $x \in C^1_{rd}(I_{[\tau(t),\infty)}_{\mathbb{T}})$, $\tau(t) \geq t_0$ which has the property that $r(t)(Z^\Delta(t))^{\alpha} \in C_{rd}(I_{[\tau(t),\infty)}_{\mathbb{T}})$ and satisfies (4) that holds on $(\tau(t), \infty)$.

2. Main Results

For notational simplicity, define

$$R(t) := \int_{t_0}^{t} r^{-1/\alpha}(s) \Delta s; \tag{12}$$

$$\theta(t, u) := \left(\int_{u}^{t} r^{-1/\alpha}(s) \Delta s\right)^{-1} \int_{u}^{t} r^{-1/\alpha}(s) \Delta s, \tag{13}$$

$$t > u \geq t_0.$$
Since \(r(t)(Z^\Delta(t))^\alpha\) is decreasing on \([t_1, \infty)_T\), we can choose \(t_2 > t_1\) so that \(\delta(t) \geq t_1\), for \(t \geq t_2\). Then
\[
Z(t) - Z(\delta(t)) = \int_{\delta(t)}^t \frac{1}{r^{1/\alpha}(s)} \left[r(s)(Z^\Delta(s))^\alpha\right]^{1/\alpha} \Delta s \leq \left[r(\delta(t))(Z^\Delta(\delta(t)))^\alpha\right]^{1/\alpha} \int_{\delta(t)}^t \frac{1}{r^{1/\alpha}(s)} \Delta s.
\]
(15)

consequently,
\[
\frac{Z(t)}{Z(\delta(t))} \leq 1 + \left[r(\delta(t))(Z^\Delta(\delta(t)))^\alpha\right]^{1/\alpha} \int_{\delta(t)}^t \frac{1}{r^{1/\alpha}(s)} \Delta s.
\]
(16)

Also, we have, for \(t \geq t_2\)
\[
Z(\delta(t)) > Z(\delta(t)) - Z(t_1) = \int_{t_1}^{\delta(t)} \frac{1}{r^{1/\alpha}(s)} \left[r(s)(Z^\Delta(s))^\alpha\right]^{1/\alpha} \Delta s \geq \left[r(\delta(t))(Z^\Delta(\delta(t)))^\alpha\right]^{1/\alpha} \int_{t_1}^{\delta(t)} \frac{1}{r^{1/\alpha}(s)} \Delta s,
\]
(17)

hence,
\[
\frac{r(\delta(t))(Z^\Delta(\delta(t)))^\alpha}{Z(\delta(t))} \leq \left(\int_{t_1}^{\delta(t)} \frac{1}{r^{1/\alpha}(s)} \Delta s\right)^{-1}.
\]
(18)

Therefore, by combining inequalities (16) and (18) we have
\[
\frac{Z(t)}{Z(\delta(t))} \leq \left(\int_{t_1}^{\delta(t)} \frac{1}{r^{1/\alpha}(s)} \Delta s\right)^{-1},
\]
(19)

from which we have
\[
\frac{Z(\delta(t))}{Z(t)} \geq \theta(t, t_1).
\]
(20)

This completes the proof. \(\Box\)

**Lemma 3** (see [11]). Let \(\phi(u) = au - bu^{(\lambda+1)/\lambda}\), where \(a \geq 0\), \(b > 0\), \(\lambda > 0\), and \(u \in [0, \infty)\). Then
\[
\phi(u) \leq \frac{\lambda^\lambda}{(\lambda + 1)^{\lambda+1}} \frac{a^{\lambda+1}}{b^\lambda}.
\]
(21)

For the positive solution \(x(t)\) of (4), it follows from \(Z(t)\) and Lemma 1 that, for \(t \geq T\),
\[
x(t) = Z(t) - p(t)x(\tau(t)) \geq Z(t) - p(t)Z(\tau(t)) \geq (1 - p(t))Z(t),
\]
(22)

which implies
\[
x^\Delta(\delta(t)) \geq (1 - p(\delta(t)))^\alpha Z^\Delta(\delta(t)).
\]
(23)

Combining (23) (A4), (4) one obtains
\[
(r(t)(Z^\Delta(t))^\alpha)^\Delta \leq -q(t)(1 - p(\delta(t)))^\beta Z(\delta(t)) \leq -\overline{p}(t)Z^\beta(\delta(t)),
\]
(24)

where \(\overline{p}(t) := q(t)(1 - p(\delta(t)))^\beta\).

One may now state and prove the main results. In these, one shall consider the two cases \(\alpha \geq \beta\) and \(\alpha < \beta\).

**Theorem 4.** Let \(\alpha \geq \beta\). Assume that there exist a positive rd-continuous differentiable function \(\xi(t)\) and a constant \(M > 0\) such that, for some \(T \in [t_0, \infty)_T\),
\[
\limsup_{t \to \infty} \int_{t_2}^t \left(\xi(s)\overline{p}(s)\theta^\beta(s, T) - \frac{M\alpha r(s)(R(\sigma(s)))^\alpha\beta(\xi_{\alpha}^+(s))^{\alpha+1}}{(\alpha + 1)^\alpha \beta^\alpha\xi_{\alpha}^+(s)}\right) \Delta s = \infty,
\]
(25)

where \(\xi_{\alpha}^+(s) := \max[\xi^+(s), 0]\). Then (4) is oscillatory on \([t_0, \infty)_T\).

**Proof.** Let \(x(t)\) be a nonoscillatory solution \(x(t)\) of (4) on \([t_0, \infty)_T\). Without loss of generality, we assume that there exists a \(T \in [t_0, \infty)_T\) (sufficiently large) such that \(x(t), x(\tau(t)), x(\delta(t)) > 0\) on \([T, \infty)_T\), and \(Z(t)\) satisfies the conclusions of Lemmas 1 and 2 on \([T, \infty)_T\). Consider the Riccati substitution
\[
w(t) = \xi(t) \frac{r(t)(Z^\Delta(t))^\alpha}{Z^\beta(t)}, \quad t \geq T.
\]
(26)

Then \(w(t) > 0\). By [2, Theorem 1.20], Lemma 2, and (24), we have
\[
w^\Delta(t) \leq \left(r(t)(Z^\Delta(t))^\alpha\right)^\Delta \frac{\xi(t)}{Z^\beta(t)} + \left(r(t)(Z^\Delta(t))^\alpha\right)^\sigma \left(\frac{\xi(t)}{Z^\beta(t)}\right)^\Delta
\]
\[
\leq -\xi(t)\overline{p}(t) \left(\frac{Z(\delta(t))}{Z(t)}\right)^\beta + \frac{\xi^\alpha(t)(Z^\beta(t))^{\alpha}}{Z^\beta(t)}\overline{p}(t) \theta^\beta(t, T) + \frac{\xi^\alpha(t)}{\xi(\sigma(t))\theta^\sigma(t)} w^\sigma(t)
\]
\[
= -\xi(t)\overline{p}(t) \theta^\beta(t, T) + \frac{\xi^\alpha(t)}{\xi(\sigma(t))\theta^\sigma(t)} w^\sigma(t)
\]
(27)
By the Pötzsche chain rule [2, Theorem 1.87],
\[
(Z^\beta(t))^\Delta = \beta \left\{ \int_0^1 [(1-h)Z(t) + hZ(\sigma(t))]^{\beta-1} \, dh \right\} Z^\Delta(t)
\]
(28)
Thus,
\[
\frac{(Z^\beta(t))^\Delta}{Z^\beta(t)} \geq \begin{cases} \beta Z^\Delta(t), & \beta > 1, \\ \left(\frac{Z(\sigma(t))}{Z^\beta(t)}\right)^{\beta-1} Z^\Delta(t), & 0 < \beta \leq 1. \end{cases}
\]
Noting that \(Z(t)\) is increasing on \([T, \infty)_T\), we get \(Z(t) \leq Z(\sigma(t))\) for \(t \in [T, \infty)_T\). Thus,
\[
\frac{(Z^\beta(t))^\Delta}{Z^\beta(t)} \geq \beta \frac{Z^\Delta(t)}{Z(\sigma(t))}.
\]
Substituting (30) into (27), we obtain
\[
\omega^\Delta(t) \leq -\xi(t) \overline{\rho}(t) \theta^\beta(t, T) + \frac{\xi^\Delta(t)}{\xi(\sigma(t))} w^\sigma(t) + \frac{\beta \xi(t) w^{(\alpha+1)/\alpha}(\sigma(t))}{M_2^1 r^{1/\alpha}(\xi(\sigma(t))) (R(\sigma(t)))^{(\alpha-\beta)/\alpha}}
\]
(31)
Noting that \(r^{1/\alpha}(t)Z^\Delta(t)\) is decreasing, we have \(r^{1/\alpha}(t)Z^\Delta(t) \geq (r^{1/\alpha}(t)Z^\Delta(t))^\sigma\). It follows from the definition of \(w(t)\) that
\[
Z^\Delta(t) \geq \frac{1}{(r(t) \xi(t))^{1/\alpha}} w^{1/\alpha}(\sigma(t)) Z^{\beta/\alpha}(\sigma(t)).
\]
Substituting (32) into (31), we obtain
\[
\omega^\Delta(t) \leq -\xi(t) \overline{\rho}(t) \theta^\beta(t, T) + \frac{\xi^\Delta(t)}{\xi(\sigma(t))} w^\sigma(t) - \frac{\beta \xi(t) w^{(\alpha+1)/\alpha}(\sigma(t))}{r^{1/\alpha}(t) \xi^{(\alpha+1)/\alpha}(\sigma(t)) Z^{\alpha-\beta/\alpha}(\sigma(t))},
\]
(33)
Since \(r^{1/\alpha}(t)Z^\Delta(t)\) is decreasing, there exists a constant \(M_1 > 0\) such that \(r^{1/\alpha}(t)Z^\Delta(t) \leq M_1\) for \(t \geq T\), which implies
\[
Z^\Delta(t) \leq \frac{M_1}{r^{1/\alpha}(t)}, \quad t \geq T.
\]
Integrating both sides of (34) from \(T\) to \(t\), we get
\[
Z(t) \leq Z(T) + M_1 (R(t) - R(T)) = R(t) \left( M_1 + \frac{Z(T) - M_1 R(T)}{R(t)} \right).
\]
(35)
Hence, there exists a \(T_1 \geq T\) such that \(Z(t) \leq (M_1 + 1)R(t)\) for \(t \geq T_1\). Then,
\[
Z^{\alpha-\beta/\alpha}(\sigma(t)) \leq (M_1 + 1)^{\alpha-\beta/\alpha}(R(\sigma(t)))^{(\alpha-\beta)/\alpha}
\]
(36)
where \(M_2 = (M_1 + 1)^{\alpha-\beta/\alpha}\). Substituting (36) into (33), we get
\[
\omega^\Delta(t) \leq -\xi(t) \overline{\rho}(t) \theta^\beta(t, T) + \frac{\xi^\Delta(t)}{\xi(\sigma(t))} w^\sigma(t) - \frac{\beta \xi(t) w^{(\alpha+1)/\alpha}(\sigma(t))}{M_2^1 r^{1/\alpha}(\xi(\sigma(t))) (R(\sigma(t)))^{(\alpha-\beta)/\alpha}}
\]
(37)
where
\[
\Psi(t) := \frac{\beta \xi(t) w^{(\alpha+1)/\alpha}(\sigma(t))}{M_2^1 r^{1/\alpha}(\xi(\sigma(t))) (R(\sigma(t)))^{(\alpha-\beta)/\alpha}}.
\]
Taking \(a = \xi^\Delta(t)/\xi(\sigma(t))\), \(b = \Psi(t)\), from Lemma 3 and (37), we obtain
\[
\omega^\Delta(t) \leq -\xi(t) \overline{\rho}(t) \theta^\beta(t, T) + \frac{\xi^\Delta(t)}{\xi(\sigma(t))} w^\sigma(t) - \frac{\beta \xi(t) w^{(\alpha+1)/\alpha}(\sigma(t))}{(\alpha + 1)^{\alpha+1} \beta^{\alpha+1} \xi^\alpha(t)},
\]
(39)
where \(M = M_2^\alpha\). Integrating both sides of (39) from \(T_1\) to \(t\), we have
\[
\int_{T_1}^t \left( \xi(s) \overline{\rho}(s) \theta^\beta(s, T) + \frac{Ma^\alpha r(s) (R(\sigma(s)))^{\alpha-\beta} \xi^\alpha(s)}{(\alpha + 1)^{\alpha+1} \beta^{\alpha+1} \xi^\alpha(s)} \right) ds 
\]
(40)
Taking lim sup of both sides of this last inequality as \(t \to \infty\), we get a contradiction to (25). This completes the proof. □
Theorem 5. Let $\alpha < \beta$. Assume that there exist a positive rd-continuous differentiable function $\xi(t)$ and a constant $K > 0$ such that, for some $T \geq t_0$,

$$\limsup_{t \to \infty} \int_{t_0}^{t} \left( \xi(s) \bar{p}(s) \theta^\beta(s,T) - \frac{\alpha^r \tau(s) \xi^{\alpha+1}(s)}{K(\alpha + 1)^{\alpha+1} \theta^\alpha(s)} \right) \Delta s = \infty,$$

where $\xi^{\alpha}(s)$ is defined as Theorem 4. Then (4) is oscillatory on $[t_0, \infty)$. 

Proof. Assume that $x(t)$ is a nonoscillatory solution of (4). Proceeding as in the proof of Theorem 4 we get that (33) holds, that is,

$$w^\Delta(t) \leq - \xi(t) \bar{p}(t) \theta^\beta(t,T) + \frac{\xi^{\alpha}(t)}{\xi(\sigma(t))} \theta^\alpha(t) w^\sigma(t)$$

$$- \frac{c_1 \beta \xi(t)}{r^{1/\alpha}(t) \xi^{\alpha+1}(\sigma(t))} Z^{\beta}(\alpha-\alpha)/(\sigma(t)), \quad t \geq T.$$

(42)

Since $\beta > \alpha$ and $Z(t)$ is increasing on $[T, \infty)_T$, then there exist a $T_2 \geq T$ and a positive constant $c_1$ such that $Z^{\beta}(\alpha-\alpha)/(\sigma(t)) \geq c_1$ for $t \geq T_2$. Consequently,

$$w^\Delta(t) \leq - \xi(t) \bar{p}(t) \theta^\beta(t,T) + \frac{\xi^{\alpha}(t)}{\xi(\sigma(t))} \theta^\alpha(t) w^\sigma(t)$$

$$- \frac{c_1 \beta \xi(t)}{r^{1/\alpha}(t) \xi^{\alpha+1}(\sigma(t))} \xi(\sigma(t)), \quad t \geq T_2.$$

(43)

Let

$$\bar{\Psi}(t) := \frac{c_1 \beta \xi(t)}{r^{1/\alpha}(t) \xi^{\alpha+1}(\sigma(t))},$$

then $\bar{\Psi}(t) > 0$, and

$$w^\Delta(t) \leq - \xi(t) \bar{p}(t) \theta^\beta(t,T) + \frac{\xi^{\alpha}(t)}{\xi(\sigma(t))} \theta^\alpha(t) w(\sigma(t))$$

$$- \bar{\Psi}(t) w^{\alpha+1}(\sigma(t)), \quad t \geq T_2.$$

(44)

The remainder of the proof is similar to that of Theorem 4 and is therefore omitted. This completes the proof for the case $\alpha < \beta$. \hfill \Box

Remark 6. Theorems 4 and 5 remove the Conditions (5) and (6). Moreover, the authors in [5] established oscillation theorems for (4) only for the case $\alpha \geq \beta > 0$. Our results here hold without this assumption, so our results improve the main results [5].

Remark 7. The results established here are valid for general time scales, with no additional restrictions, for example, $\mathbb{T} = \mathbb{R}$, $T = \mathbb{Z}$, and $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, $\mathbb{T} = q\mathbb{N}$ with $q > 1$, and $\mathbb{T} = \mathbb{N}_0$; see [2, 3].

3. Some Examples

In this section, we give two examples to illustrate our main results.

Example 1. Let $T = 2^\mathbb{N}_0$ ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$), $\alpha = 3$, $\beta = 2$. Consider the neutral nonlinear dynamic equation

$$\Delta_2\left(\Delta_2 x(t)\right) + \frac{\delta^2 (2^k)}{\delta^2 (2^k, 1)} |x(\delta (2^k))| x(\delta (2^k)) = 0,$$

$$k_0 = 0,$$

(46)

where $r(2^k)$ satisfies (A3), and $Z(2^k) = x(2^k) + (2^k - 1)/2^k x(r(2^k))$.

Here,

$$r(2^k) = 1, \quad p(2^k) = \frac{2^k - 1}{2^k}, \quad q(2^k) = \frac{\delta^2 (2^k)}{\delta^2 (2^k, 1)}.$$ 

(47)

It is clear that (A1) holds, and $\bar{p}(2^k) = q(2^k)(1 - p(\delta(2^k)))^\beta = 1/\theta^2 (2^k, 1), R(\sigma(2^k)) = 2^{k+1} - 1$.

Let $\xi(2^k) = 2^k$. Noting that $\sum_{k=0}^{\infty} r^{-\alpha}(2^k) = \infty$ implies $\lim_{k \to \infty} \bar{\theta}(2^k, 2^k) = 1$ for $k_T \geq 1$, we get

$$\limsup_{t \to \infty} \int_{t_0}^{t} \left( \xi(s) \bar{p}(s) \theta^\beta(s,T) - \frac{\alpha^r \tau(s) \xi^{\alpha+1}(s)}{K(\alpha + 1)^{\alpha+1} \theta^\alpha(s)} \right) \Delta s = \infty$$

$$= \limsup_{k \to \infty} \sum_{i=0}^{k-1} \left( 2^i \frac{\theta^2 (2^i, 1)}{\theta^2 (2^i, 1) - 2^{i+1} - 2^i} \right) 2^i = \infty.$$ 

(48)

Thus, by Theorem 4, (46) is oscillatory.

Example 2. Consider the neutral dynamic equation

$$\left( \frac{1}{\sigma^\alpha(t)} \left| Z^\Delta(t)^{\alpha-1} Z^\Delta(t) \right|^\Delta \right) + \frac{(1 + \delta(t))^\beta}{\delta^\beta(t) \theta^\beta(T, t_0)} |x(\delta(t))|^{\beta-1} x(\delta(t)) = 0,$$

$$t_0 > 0,$$

where $\beta > \alpha > 0$ are constants, $r(t)$ satisfies (A3), and $Z(t) = x(t) + 1/(t+1)x(r(t))$. 

(49)
For (4), we let
\[ r(t) = \frac{1}{\sigma^{1+\alpha}(t)}, \quad p(t) = \frac{1}{t+1}, \quad q(t) = \frac{(1 + \delta(t))^\beta}{\Theta(t) \theta^\beta(t, t_0)}. \]
Since
\[ \int_{t_0}^\infty \frac{1}{r(s)} \Delta s = \int_{t_0}^\infty \frac{1}{\sigma^{1+\alpha}(s)} \Delta s = \infty, \]
then (A1) holds and \( p(t) = q(t) (1 - p(\delta(t)))^{\frac{\beta}{\alpha}} = 1/\theta^2(t, t_0) \).

Let \( \xi(t) = t \). Noting that \( \int_{t_0}^\infty r^{\frac{1}{\alpha}}(t) \Delta t = \infty \) implies
\[ \lim_{t \to \infty} \frac{\theta(t, T)}{\theta(t, t_0)} = 1 \text{ for } T \geq t_0, \]
we have
\[ \limsup_{t \to \infty} \int_{t_0}^T \left( \xi(s) \Theta(s) \theta^\beta(s, T) \right) \Delta s \]
\[ - \frac{\alpha^\alpha r(s) \left( \xi(s) \right)^{\alpha+1}}{K(\alpha + 1)^{\alpha+1} \beta q(s)} \Delta s \]
\[ = \limsup_{t \to \infty} \int_{t_0}^t \left( \Theta(s) \theta^\beta(s, t_0) \right) \frac{\alpha^\alpha}{K(\alpha + 1)^{\alpha+1} \beta q(s)} \Delta s \]
\[ \geq \limsup_{t \to \infty} \int_{t_0}^t \left( s - \frac{1}{K^2 q(s)} \right) \Delta s \]
\[ \geq \limsup_{t \to \infty} \frac{1}{2} \int_{t_0}^t s \Delta s = \infty. \] (52)

Thus, by Theorem 5, (49) is oscillatory.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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