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On the Equivalence of the Loss Ratio and Pure Premium Methods of Determining Property and Casualty Rating Relativities

Robert L. Brown*

Abstract

There are two distinct stages in the property and casualty ratemaking process. First, there is the portfolio average rate change. Second, there is the adjustment of classification relativities. It is well known that the loss ratio and pure premium (also called the loss cost) methods are algebraically equivalent in the stage called the portfolio average rate change. This paper reviews the proof of this equivalence. Further, it is proved algebraically that the loss ratio and pure premium methods are also equivalent in calculating classification relativities (or differentials) if certain data requirements can be met. A short numerical example of this equivalence is included.

Key words: loss cost, ratemaking, relativities

1 Introduction

In property and casualty ratemaking, there are two distinct steps in the process:

a) The portfolio average rate change.
b) A change in classification relativities.

One is able to use either a loss ratio approach or a pure premium (or loss cost) approach in these two distinct ratemaking stages. This paper first reviews the well-documented fact that the loss ratio and the pure premium approaches are algebraically equivalent when portfolio average rate changes are being calculated. The paper then

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proves that these methods are also equivalent when changes in classification relativities (differentials) are being calculated.\(^1\) For these methods to be applied, the data must be in the appropriate form.

## 2 The Portfolio Average Rate Change

The process to be followed in developing the portfolio average rate change (also known as the *statewide or provincewide rate change*) is well known (see Brown, 1993, pp. 70-77) and will not be discussed directly here. There are two methods that can be used to develop rates: the loss ratio method and the pure premium method. It is relatively easy to provide mathematical formulas for these methods and to show algebraically that they are mathematically equivalent. The proof of their equivalence is well known; see, for example, Stern (1965, p. 182) and McClenahan (1990). For convenience, the proof will be repeated here. To this end, the following definitions are needed:

- \(L_{ijk}\) = Dollars of incurred losses for rate cell \((i, j, k)\);
- \(E_{ijk}\) = Units of earned exposure for rate cell \((i, j, k)\);
- \(CR_{ijk}\) = Current manual rate for rate cell \((i, j, k)\);
- \(i, j, k\) = Rating variable indicators such as \(i\) classes, \(j\) territories (There can be any number of such variables.);
- \(PLR\) = Permissible loss ratio = 1 – expense ratio;
- \(ILR\) = Indicated loss ratio;
- \(NAR\) = New average rate;
- \(CAR\) = Current average rate.

It now will be proven that the new average rate is the same for the pure premium method and the loss ratio method.

### 2.1 Pure Premium Method

The new average rate is determined under the pure premium method as:

\[
NAR = \frac{\sum_{ijk} L_{ijk}}{\sum_{ijk} E_{ijk}} \times \frac{1}{PLR}.
\]

---

\(^1\) Throughout this paper, the terms *relativities* and *differentials* are used interchangeably.
2.2 Loss Ratio Method

Under the loss ratio method, the new average rate is given by:

\[ NAR = CAR \times \frac{ILR}{PLR} \]

But the current average rate is determined as

\[ CAR = \frac{\sum_{ijk} CR_{ijk} \times E_{ijk}}{\sum_{ijk} E_{ijk}} \]

and the indicated loss ratio is:

\[ ILR = \frac{\sum_{ijk} L_{ijk}}{\sum_{ijk} CR_{ijk} \times E_{ijk}} \]

Thus, the new average rate is:

\[
NAR = \frac{\sum_{ijk} CR_{ijk} \times E_{ijk}}{\sum_{ijk} E_{ijk}} \times \frac{\sum_{ijk} L_{ijk}}{\sum_{ijk} CR_{ijk} \times E_{ijk}} \times \frac{1}{PLR}
\]

\[
= \frac{\sum_{ijk} L_{ijk}}{\sum_{ijk} E_{ijk}} \times \frac{1}{PLR}
\]

which is the same as the new average rate derived by the pure premium method.

3 Change in Classification Relativities

Again, there are two methods that can be used to change classification relativities: the pure premium (or loss cost) method and the loss ratio method. Some confusion exists, however, about which method is better and why. Also, the classical ratemaking papers found in the Casualty Actuarial Society’s associateship syllabus may not make clear what data must be used to guarantee a correct analysis.
For example, Stern (1965, p. 170) states in the section on classification relativities:

The pure premium indices above measure the relationship of the loss cost per car for each class to the base class. Consequently, they also indicate how the rate for each class should relate to the rate for the base class, if it is accepted that the expense portion of the rate is obtained by a uniform expense loading ... However, pure premiums obtained from a consolidation of widely divergent bodies of experience must be used with great caution since they may contain distortions. The above model may contain in Class 11 a proportionally larger share of experience coming from low loss cost territories than is contained in the experience for Class 12. Consequently, a part of the indicated rate differential is purely due to distribution; this distortion due to distribution would have to be corrected for, prior to accepting pure premium indices as true indications of classification relativities.

Stern (1965) continues:

There are, however, many advantages in favor of using collected loss ratios. These loss ratios can be obtained with relative ease directly from the experience; unlike pure premiums, they are less likely to be distorted by the influence of divergent distributions, since the premiums reflect the different rate and loss levels of the component territories; and finally, loss ratios based on the actual experience have an air of reality, reflecting the over-all underwriting record for each class.

Finger (1990, Chapter 5, p. 259) states:

When earned premium is used, the method is usually a "loss ratio" method; when earned exposures are used, the method is usually a "pure premium" approach. The loss ratio method can produce equivalent results if "earned premiums at current rates" are calculated.

Finger (1990) adds:

There are advantages and disadvantages of using the loss ratio and pure premium methods. The loss ratio method may be applicable when there is less detailed data available or when there are many different sets of relativities; earned premiums will reflect the various charges made for different classes, territories, and coverages. If earned premiums correspond to historical rate levels, however, it may be difficult to make adjustments for intervening changes in rate relativities. The pure premium approach is usually more accurate, because it requires more information. It also has the advantage of producing frequency and severity relativities, as well as pure premium relativities; the loss ratio method only produces loss ratio and severity relativities.
ties. Severity relativities, however, will not be meaningful if the underlying coverage is not consistent (e.g., there are differing deductibles or insured limits).

Finger then provides an arithmetic illustration of an actual calculation of some classification relativities using both the loss ratio and pure premium techniques. In his solved example using the pure premium method, Finger does not use just earned exposures for the denominator of each respective loss cost. Rather, he calculates and uses what he calls base exposures. He explains base exposures (p. 266):

It should be noted that “base exposures” are used in this exhibit in place of earned exposures. “Base exposures” are calculated using the current rate relativities for all relevant rating variables.

Finger argues that the reason for using base exposures instead of actual exposures is to correct for varying exposure levels in the non-reviewed relativities. For example, Territory A and Territory B may differ in the distribution of insureds by class.

Finger corrects for the distortion caused by the heterogeneity of exposure distributions across the variables not now under review, as previously alluded to in Stern’s paper (e.g., varying exposure levels by class in the different territories) and for which Stern suggests corrections must be made. This is illustrated in the example in section 5 below.

Finger provides a one line arithmetic illustration of how the base exposure adjustment is made. It is difficult to conclude, however, that an average reader could reproduce the solution with only the information available.

Some questions remain: Which is superior, the loss ratio method or the pure premium method? What does Finger mean when he says that “The loss ratio method can produce equivalent results if ‘earned premiums at current rates’ are calculated?” Unfortunately, Finger does not elaborate further on this comment.

To deal with these questions, an algebraic description of this aspect of the ratemaking process must be developed. Without loss of generality, consideration is limited to cases where there are only two classification parameters. Define two vectors of differentials:

\[ x_i \text{ for } i = 1, 2, \ldots, n \text{ (e.g., class)} \]

\[ y_j \text{ for } j = 1, 2, \ldots, m \text{ (e.g., territory)} \]
Assume there is a base cell, $B$, for any variable, such that for that cell:

$$x_B = y_B = 1.000$$

The current rate for the base cell will be denoted $CR_B$. Otherwise, the notation used is as defined previously.

Consider a rate manual produced by the base rate $CR_B$ and the two vectors of relativities $x_i$ and $y_j$. This produces a matrix of $m \times n$ rates. Consider (without loss of generality) that the new differential for class $k$, $x_k$ is to be calculated. One can think of class (or territory) $k$ as occupying the $k^{th}$ row of our rate manual matrix.

The calculations that follow assume that the various rate relativities are calculated independently (as opposed to interactively, as in Brown’s (1988) minimum bias approach) and applied multiplicatively. While the latter assumption is not essential in practice (i.e., additive differentials are possible), multiplicative differentials are the norm. The algebraic proofs that follow assume a multiplicative relationship. The proofs also assume that all expenses are expressed as a percentage of the gross premium (i.e., there are no flat-loaded expenses). This means that the loss relativities and rate relativities are the same.

The papers by Stern and Finger indicate that the calculation of multiplicative rating relativities can be expressed algebraically as follows:

### 3.1 Pure Premium Method

The loss cost for variable $k$, $LC_k$, adjusted for heterogeneity under the pure premium method is:

$$LC_k = \frac{\sum_{j} L_{kj}}{\sum_j E_{kj} y_j}.$$

The loss cost for the base cell, $B$, adjusted for heterogeneity under the pure premium method is denoted by $LC_B$ where:

$$LC_B = \frac{\sum_j L_{Bj}}{\sum_j E_{Bj} y_j}.$$

Thus, the new differential is:
3.2 Loss Ratio Method

The loss ratio for variable \( k \), \( LR_k \), is determined as:

\[
LR_k = \frac{\sum L_{kj}}{\sum E_{kj} CR_{kj}}.
\]

The loss ratio for the base cell, \( B \) is:

\[
LR_B = \frac{\sum L_{Bj}}{\sum E_{Bj} CR_{Bj}}.
\]

Thus, the adjustment factor for cell \( k \), \( AF_k \), is:

\[
AF_k = \frac{\sum L_{kj}}{\sum E_{kj} CR_{kj}} \times \frac{\sum E_{Bj} CR_{Bj}}{\sum L_{Bj}},
\]

and the new differential is determined as:

\[
x_k AF_k = x_k \frac{\sum L_{kj}}{\sum E_{kj} CR_{kj}} \times \frac{\sum E_{Bj} CR_{Bj}}{\sum L_{Bj}}.
\]

But \( \sum E_{kj} CR_{kj} = \sum E_{kj} CR_B x_k y_j = CR_B x_k \sum E_{kj} y_j \) (existing \( x_i, y_j \)). Thus,

\[
x_{k}^{new} = x_k \frac{\sum L_{kj}}{CR_B x_k \sum E_{kj} y_j} \times \frac{CR_B x_B \sum E_{Bj} y_j}{\sum L_{Bj}}.
\]

and \( x_B = 1 \). Therefore
Thus it can be seen that when the correct data are used, the pure premium method and the loss ratio method are algebraically equivalent.

4 Comments

Now that we have proved algebraically the equivalency of the loss ratio and the pure premium methods in the entire ratemaking process (i.e., both the overall rate change and also the change in relativities) if the appropriate data are used, a number of issues surrounding the calculation of risk classification relativities disappear or are resolved.

First, if the data requirements can be satisfied, then the loss ratio method and the pure premium method provide equivalent results. Therefore, there should be no need to discuss the advantages of one method over the other. They are equivalent given the appropriate data are available and used. To the extent that one cannot attain the data requirements, then one can see clearly what inadequacies will result because of the particular data one often is forced to use.

For example, if in the loss ratio approach one uses collected earned premiums (or collected loss ratios, as Stern suggests, because they are readily available), this will result in an error to the extent that the collected earned premiums are not equal to earned premiums at the current rate level. If there have been some sizable changes in relativities in recent rate changes, then this will be a problem. If the relativities have not changed drastically over the last few rate changes, however, then there may not be much of a difference between collected earned premiums and earned premiums at current rate levels. (Note that overall rate changes are not of any consequence at this stage; only the changes in classification relativities matter.)

Also, this algebraic illustration shows exactly what is meant by Finger's base exposures. These are effective exposure units that are adjusted because of the heterogeneity of exposure distributions across the different rating parameters. The following illustration makes this clear.
5 Illustration

Given the following information, and assuming the revised rates take effect July 1, 1993 for one year on one year policies, determine new rates for each of Class 1 and Class 2 and for each of Territory 1 and Territory 2. (Class differentials will not change.) Use both the loss ratio and pure premium methods. The permissible loss ratio is 0.600, and all data are fully credible.

<table>
<thead>
<tr>
<th>Present Rates:</th>
<th>Territory 1</th>
<th>Territory 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class 1 (Relativity)</td>
<td>100 (1.00)</td>
<td>200 (2.00)</td>
</tr>
<tr>
<td>Class 2 (Relativity)</td>
<td>300 (3.00)</td>
<td>600 (6.00)</td>
</tr>
<tr>
<td>Collected Earned Premium</td>
<td>1,000,000</td>
<td>1,000,000</td>
</tr>
<tr>
<td>Policy Year 1991 Incurred Losses</td>
<td>360,000</td>
<td>240,000</td>
</tr>
<tr>
<td>Expected Effective Period Incurred Losses (Trended and Developed)</td>
<td>612,000</td>
<td>408,000</td>
</tr>
<tr>
<td>Earned Exposure Units:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Class 1</td>
<td>5,000</td>
<td>2,000</td>
</tr>
<tr>
<td>Class 2</td>
<td>1,000</td>
<td>500</td>
</tr>
</tbody>
</table>

The solution is given below for the loss ratio and pure premium methods. For each of the two methods, the rate change involves three stages: overall average rate change, change in relativities, and balance back.

5.1 Loss Ratio Method

5.1.1 Overall Average Rate Change

For the loss ratio method, the actuary must calculate the earned premium at current rates. The accounting entry for collected earned premium is not the correct denominator, because it could contain earned premiums based on the rates in out-of-date rate manuals.

The earned premium at current rates is calculated as

\[ \sum_{ij} CR_{ij} \times E_{ij} \]

\[ = (100)(5,000) + (300)(1,000) + (200)(2,000) + (600)(500) \]

\[ = 1,500,000. \]
This produces an expected effective period loss ratio at current rates of:

\[
\frac{1,020,000}{1,500,000} = 0.680,
\]

which, with a permissible loss ratio of 0.600, leads to an indicated rate change of

\[
\frac{0.680}{0.600} - 1 = + 13.3 \text{ percent.}
\]

5.1.2 Change in Relativities

The given data allow for a territorial relativity change analysis but not a class relativity change analysis because loss data by class are not given. We are told that class relativities will remain the same and are asked to determine the indicated new relativities for Territories 1 and 2.

For Territory 1 the earned premium at current rates equals:

\[
(100)(5000) + (300)(1000) = 800,000
\]

For Territory 2 the earned premium at current rates equals:

\[
(200)(2000) + (600)(500) = 700,000
\]

<table>
<thead>
<tr>
<th>Territory</th>
<th>Existing Differential</th>
<th>Loss Ratio at Current Rates</th>
<th>Indicated Differential</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
<td>( \frac{360,000}{500,000} = 0.720 )</td>
<td>( \frac{0.720}{0.600} = 1.2 )</td>
</tr>
<tr>
<td>2</td>
<td>2.00</td>
<td>( \frac{240,000}{700,000} = 0.3429 )</td>
<td>( \frac{0.3429}{0.4500} (2.00) = 1.5238 )</td>
</tr>
</tbody>
</table>

Note that as presented, the Territory 1 relativity has been left at 1.00, whereas the Territory 2 relativity has been reduced from 2.00 to 1.5238. This suggests that the actuary could define the new rates as follows:
If this were done, however, the resulting rate increase would be less than the required +13.3 percent due to the off-balance created by the method used to change relativities. This is adjusted in the balance-back step.

5.1.3 Balance Back

The existing average differential is equal to:

\[
\frac{(5,000)(1) + (1,000)(3) + (2,000)(2) + (500)(6)}{8,500} = 1.7647.
\]

The proposed average differential is equal to:

\[
\frac{(5,000)(1) + (1,000)(3) + (2,000)(1.5238) + (500)(4.5714)}{8,500} = 1.5686.
\]

The balance-back factor is calculated as:

\[
\frac{\text{Existing average differential}}{\text{Proposed average differential}} = \frac{1.7647}{1.5686} = 1.1250,
\]

leading to the following proposed rates:

<table>
<thead>
<tr>
<th>Territory</th>
<th>Class 1</th>
<th>Class 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Territory 1</td>
<td>127.50</td>
<td>194.28</td>
</tr>
<tr>
<td>Territory 2</td>
<td>382.50</td>
<td>582.85</td>
</tr>
</tbody>
</table>

These proposed rates will result in a 13.3 percent increase in premium income, as required.

5.2 Pure Premium

5.2.1 Overall Average Rate Change

We know that the expected effective period incurred losses (developed and trended) equal 1,020,000, from which we find the indicated loss cost:
and the average rate:

\[
\frac{120}{PLR} = \frac{120}{0.600} = 200.00.
\]

Note that this is the indicated average gross rate. It is not the indicated rate for any particular territory or class that will be determined when we know the new average relativity for the expected book of business.

5.2.2 Change in Relativities

To set the new territorial relativities, the actuary normally calculates the average loss costs for Territory 1 and Territory 2 and compares them as follows:

<table>
<thead>
<tr>
<th>Territory</th>
<th>Existing Relativity</th>
<th>Loss Cost per Unit</th>
<th>Indicated Relativity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
<td>360,000/6,000 = 60</td>
<td>1.00</td>
</tr>
<tr>
<td>2</td>
<td>2.00</td>
<td>240,000/2,500 = 96</td>
<td>1.60</td>
</tr>
</tbody>
</table>

This is not the same answer as we got from the loss ratio method. Remember that the pure premium method will be correct only if the heterogeneity of distributions of exposure units is accounted for. Recall the following earned exposure unit data:

<table>
<thead>
<tr>
<th></th>
<th>Territory 1</th>
<th>Territory 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class 1</td>
<td>5,000</td>
<td>2,000</td>
</tr>
<tr>
<td>Class 2</td>
<td>1,000</td>
<td>500</td>
</tr>
</tbody>
</table>

In Territory 1, 5/6 of drivers are Class 1 and 1/6 are Class 2. In Territory 2, 4/5 of drivers are Class 1 and 1/5 are Class 2. To arrive at the correct answer, this heterogeneity of cross-variable distributions must be reflected. One way to accomplish this is to use exposure units that are weighted by their cross-parameter relativities. That is, Class 1 will count as an exposure unit with weight 1.00, but Class
2 will count as an exposure unit with weight 3.00, because of its class relativity of 3.00. This leads to the following results:

<table>
<thead>
<tr>
<th>Territory</th>
<th>Existing Relativity</th>
<th>Weighted Units of Exposure</th>
<th>Loss Cost per Weighted Unit of Exposure</th>
<th>Indicated Relativity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
<td>8000</td>
<td>360,000</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\frac{360,000}{8,000} = 45.00$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2.00</td>
<td>3500</td>
<td>240,000</td>
<td>1.5238</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\frac{240,000}{3,500} = 68.57$</td>
<td></td>
</tr>
</tbody>
</table>

### 5.2.3 Balance Back

Finally, the actuary determines the rate for Territory 1 and Class 1 that will produce all of the correct manual rates by balancing back for the average indicated relativity. That is:

$$\text{Base rate} = \frac{\text{Average rate}}{\text{Average relativity}}$$

where the average rate is 200 and the average relativity is

$$\frac{(5,000)(1) + (1,000)(3) + (2,000)(1.5238) + (500)(4.5714)}{8,500} = 1.5686.$$ 

This leads to

$$\text{Base rate} = \frac{200}{1.5686} = 127.50.$$ 

The resulting manual rates are the same as with the loss ratio method, as expected. This gives us indicated rates where all calculations are based on existing relativities in the current rate manual and should be treated as a first iteration indicated relativity. These indicated relativities will be used in a second iteration (for example, to recalculate the premium at current rate levels in the loss ratio method) to arrive at a second iteration indication. This process soon converges to the final relativities.

### References


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