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Funding Methods and Pension Plan Amendments

Keith P. Sharp*

Abstract**

This paper considers the treatment of plan amendments under the individual entry age normal and projected unit credit methods. Alternative treatments are considered, and comments are made about their acceptability.

Key words: nonretroactive amendment, normal cost, entry age normal, projected unit credit

1 Introduction

It is common for a pension plan to be amended to improve benefits in respect of service after the date of amendment. This will be referred to as a nonretroactive amendment. The application of the entry age normal and projected unit credit cost methods to this situation requires that a decision be made about the way to handle such an amendment. This paper considers these two cost methods and their application to such an amendment. A retroactive improvement can be treated in a more straightforward manner and is not considered in this paper.

The discussion of the entry age normal method is relevant to funding calculations under the Pensions Benefits Acts in Canada and under the Employee Retirement Income Security Act of 1974 (ERISA) and Internal Revenue Code (IRC) Regulation Section 1.412 in the United States. The discussion of the projected unit credit method is relevant to funding calculations and pension expense calculations.

Before developing the main results of this paper, it is important to introduce the notation used in the sequel. As there is no internationally accepted standard pension notation, we will follow, to a large extent, the notation used by Anderson (1992).

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\( j \) = Label of an individual member of the plan;

\( NCl(t) \) = Normal cost for individual \( j \) at time \( t \), paid at the beginning of each year and expressed in dollars;

\( v_j \) = Age on the first valuation date coinciding with or next following the date of participation assuming current participation requirements always had been in effect;

\( w_j \) = Age from which credited pensionable service is calculated, i.e., the entry age for individual \( j \) that determines the start of the period to which the benefit formula applies. In some cases the individual may join the plan after age \( w_j \) and be given retroactive pensionable service;

\( x_j(t) \) = Age at time \( t \) of individual \( j \);

\( y_j \) = Retirement age of individual \( j \);

\( Bi(y_j) \) = Projected annual pension benefit of individual \( j \) from retirement at age \( y_j \);

\( S_{ij} \) = Projected measure of final pay for individual \( j \); and

\( s_{Xj} \) = Salary scale for individual \( j \) at age \( x_j \).

2 Plan Amendments Under Individual Entry Age Normal

2.1 Individual Entry Age Normal

The individual entry age normal pension cost method is used in both the United States and Canada. There are two common forms of the method (Anderson, 1992, pp. 13-19; Trowbridge and Farr, 1976, pp. 47-54; and Berin, 1989, p. 14). Under one form, the normal cost is expressed as a level dollar annual amount. This method is alternatively known as the projected benefit cost method (with supplemental liability, constant amount) (Winklevoss, 1977), the entry age actuarial cost method, and the level dollar cost method (entry age, with supplemental liability) (McGill and Grubbs, 1989, p. 27). Under another form, the normal cost is expressed as a level percentage of salary. The latter method also is known as the projected benefit cost method (with supplemental liability, constant percentage) (Winklevoss, 1977), the entry age actuarial cost method, and the level percentage cost method (entry age, with supplemental liability), (McGill and Grubbs, 1989, p. 327).

Under the individual entry age normal method, the normal cost is found by taking an equation of value. The equation usually is taken on the first valuation date coinciding with or next following a member’s participation date, assuming current participation requirements always had been in effect. This age could be that at a date before
plan inception. The normal cost under the level dollar method, equation (1), is given from this equation of value by dividing both sides of the equation by the service-based annuity (Anderson, 1992, p. 13):

\[ NC(t) = B_l(y_j) \frac{D_{y_j}}{a_{y_j}} \frac{1}{a_{v_j: y_j - v_j}} \]

\[ = B_l(y_j) \frac{D_{y_j}}{N_{v_j} - N_{y_j}}. \]  

(1)

Under the level percentage of salary method (Anderson, 1992, p. 18), the annuity in the denominator of equation (1) takes the salary scale into account. The dollar normal cost is found by multiplying by the ratio of the salary scale factors:

\[ NC(t) = B_l(y_j) \frac{D_{y_j}}{a_{y_j}} \frac{sD_{y_j}}{(sN_{v_j} - sN_{y_j})} \frac{s_{x_j(t)}}{s_{y_j}}. \]  

(2)

The focus of this paper is the choice of a cost method variant that is acceptable and makes sense to a client on a plan amendment; this amounts to a discussion about the method of calculating \( B_l(y_j) \). For simplicity it is assumed that all retirements occur at age \( y_j \) and that the only benefit is a retirement annuity.

2.2 Plan Amendment

We focus attention on \( B_l(y_j) \). For the purpose of illustration, we will assume that the benefit is a fraction \( r_0 \) (e.g., \( r_0 = 0.01 \) or 0.02) of a projected measure \( S_f \) of final pay for each year of credited pensionable service. The measure \( S_f \) will depend on the plan document definition of the pension benefit; \( S_f \) may be, for example, the average of the earnings in the final three years of service. Thus, prior (subscript \( p \)) to any possible plan amendments, we have:

\[ B_p^l(y_j) = r_0(y_j - w_j)S_f. \]  

(3)

From equations (1) and (2) we can see that two persons with the same entry age \( w \) and the same retirement age \( y \) will have the same normal cost as a fraction of the measure of final salary.

Now consider a situation where at a certain date, the benefit fraction \( r_0 \) is changed nonretroactively from \( r_0 \) to \( r_1 \). Usually \( r_1 \) will exceed \( r_0 \), although the funding methods discussed here apply math-
ematically, if not in the view of regulators, also to the case where \( r_0 \)
exceeds \( r_1 \). One method is to spread the funding of the increase over the
period from amendment to retirement, with no change in the
amendment date actuarial liability; this is the individual level
premium method as described by Anderson (1992, p. 25). Two of the
possible methods of handling this situation under entry age normal
are described below.

2.3 Variant 1: EAN Total Service Spread

For an individual \( j \) with pensionable service credited from age \( w_j \)
and age at plan amendment \( x_j \), one initially might assume that the
projected benefit should be given by:

\[
B_{A1}^j(y_j) = [r_0(x_j(t_A) - w_j) + r_1(y_j - x_j(t_A))] S_j^j
\]  

(4)

where the subscript \( A \) indicates that the situation after the plan
amendment is being considered and \( t_A \) is the date of the amendment.
This indicates that the normal cost for individual \( j \), by equation (1)
and (2), would increase under this EAN—total service spread method
in the ratio

\[
\frac{EANNC_{A1}^j}{EANNC_{p}^j} = \frac{[r_0(x_j(t_A) - w_j) + r_1(y_j - x_j(t_A))]}{r_0(y - w_j)}
\]  

(5)

This ratio depends on the values of \( x_j(t_A) \) and \( w_j \). For example, for
two members \( i \) and \( k \) with the same pensionable service commence­
tment dates \( w_i = w_k \) but differing ages at amendment \( x_i(t_A) \not= x_k(t_A) \),
the normal cost as a fraction of salary no longer will be the same as
the fraction of the measure of final salary. Also, the increase in the
normal cost is not the same ratio \( r_1/r_0 \) as the increase in the benefit
accumulation rate.

It is instructive also to consider the effect on the actuarial liability \( AL \).
At age \( x_j(t) \) prior to the plan amendment, the actuarial liability is the difference between the present values of future benefits and future normal costs:

\[
EANAL_p^j(x_j(t)) = PVFB_p^j - EANPVFNC_p^j(x_j(t))
\]  

(6)

Immediately after the plan amendment at age \( x_j(t_A) \) we have (noting
that the future benefits should be those actually projected to be paid
for both the constant dollar and constant percentage methods):
Thus, the plan actuarial liability at the date of the amendment increases because of the amendment, although the benefit rate change is not retroactive. The proportionate increase in the actuarial liability equals the proportionate increase in the projected benefit. This aspect may be difficult to explain to a client who is not an actuary. The increase in accrued liability results because the normal cost increases only by the same proportionate amount as the increase in projected benefit. If \( r_1 < r_0 \) then the accrued liability is reduced, which may be unacceptable to regulators.

2.4 Variant 2: EAN Retroactive NC Mimic

An alternative method of handling normal costs under a plan amendment is described in this section. It is used by some pension consultants and gives results that are more acceptable than those described in the previous section.

Under variant 2 (EAN retroactive NC mimic), the hypothetical projected benefit is used in calculating the normal cost under this version of the entry age normal cost method. It is that projected benefit that would be applicable if the amended benefit rate were applied to all service:

\[
B^j_{A2} = r_1 (y_j - w_j) S^j_f. \tag{8}
\]

Under this variant, the normal cost at any post-amendment time \( t \) for individual \( j \) increases under both the level dollar and level percentage methods in the ratio of the benefit rates.
\[
\frac{NC_{A_2}(t)}{NC_{P}(t)} = \frac{r_1 (y_j - w_j)}{r_0 (y_j - w_j)} = \frac{r_1}{r_0}. \tag{9}
\]

Under the individual entry age normal method the normal cost is not interpreted as being the cost of the benefit accrual for the year. Nonetheless, a proportional increase in normal cost equal to the proportional increase in benefit rates is likely to be intuitively appealing to the client.

Let us now consider the actuarial liability under variant 2. Immediately after the plan amendment, variant 2 is given for both the level dollar and level percentage methods by:

\[
EANAL_{A_2}(x_j(t_A)) = PVFB_{A_1}(x_j(t_A)) - EANPVFNC_{A_2}(x_j(t_A)) \tag{10}
\]

Because the actual future benefits are the same for variants 1 and 2, \( PVFB_{A_1}(x_j(t_A)) = PVFB_{A_2}(x_j(t_A)) \). Then we note that \( PVFB_{A_1}(x_j(t_A)) \) is related to \( PVFB_p(x_j(t_A)) \) by the proportionate increase in the projected benefit. Also, the future normal cost increases in the ratio \( r_1/r_0 \). Hence:

\[
EANAL_{A_2}(x_j(t_A)) = \left[ \frac{r_0 (x_j(t_A) - w_j) + r_1 (y_j - x_j(t_A))}{r_0 (y - w_j)} \right] PVFB_p(x_j(t_A)) - \frac{r_1}{r_0} \times EANPVFNC_{p}(x_j(t_A))
\]

\[
= EANAL_p(x_j(t_A)) + \frac{(r_1 - r_0)}{r_0} \times \left[ \left( \frac{y_j - x_j(t_A)}{y_j - w_j} \right) PVFB_p(x_j(t_A)) - EANPVFNC_p(x_j(t_A)) \right] \tag{11}
\]

\[
EANAL_p(x_j(t_A)) + \frac{(r_1 - r_0)}{r_0} \times PVFB_p(x_j(t_A)) \times \left[ \frac{y_j - x_j(t_A)}{y_j - w_j} \frac{sD_{x_j(t_A)}}{sD_{w_j}} - \frac{s\tilde{a}_{x_j(t_A)}: y_j - x_j(t_A)}{s\tilde{a}_{w_j}: y_j - w_j} \right]. \tag{12}
\]
Appendix A shows that if \( r_1 > r_0 \) and if \( sD_2 \) is a decreasing function of \( z \), then:

\[
EANAL_i^{jA2}(x_j(t_A)) \geq EANAL_p^j(x_j(t_A)).
\]  

(13)

For \( r_1 < r_0 \), the actuarial liability is reduced by the amendment.

The last term of equation (11) is likely to be small; the actuarial liability is changed little by the nonretroactive amendment. This is likely to make sense to a client.

In the United States, IRC Regulation Section 1.412(c)(3)-1(c)(2) requires “If each actuarial assumption is exactly realized under a reasonable funding method, no experience gains or losses are produced.” This condition is satisfied by variant 2, as indicated in Appendix B.

2.5 Variant 3: EAN/ILP

A third method of handling the plan amendment under entry age normal is to use the individual level premium (ILP) method. This usually is regarded as a cost method in its own right; here it will be regarded more as a variant of entry age normal. The terminology EAN/ILP will be used.

Under variant 3, the nonretroactive benefit increase at \( t_A \) is funded over the period from \( x_j(t_A) \) to \( y_j \). Hence the normal cost after the amendment is given by:

\[
EANNCi^{jA3}(x_j(t))
\]

\[
= EANNCP^j(t) + S_j (r_1 - r_0)(y_j - x_j(t_A))\bar{a}_{y_j}^{(12)}
\]

\[
\times \frac{D_{y_j}}{D_{x_j(t_A)}} \times \frac{sD_{x_j(t_A)}}{sN_{x_j(t_A)} - sN_{y_j}} \times \frac{s_{x_j(t)}}{s_{x_j(t_A)}}
\]

(14)

and the actuarial liability at an age \( x_j(t), t \geq t_A \) is

\[
EANNCi^{jA3}(x_j(t))
\]

\[
= PVFP^j_p(x_j(t)) + S_j (r_1 - r_0)(y_j - x_j(t_A))\bar{a}_{y_j}^{(12)} \frac{D_{y_j}}{D_{x_j(t)}}
\]
Immediately following the amendment, the actuarial liability is found by substituting $t = t_A$ in equation (15):

$$EAN_{A3}^j(x_j(t_A)) = PVFB_p^j(x_j(t_A)) - PVFNC_p^j(x_j(t_A))$$

$$= EAN_{A3}^j(x_j(t_A)).$$

Thus, as is arranged by construction of variant 3, the actuarial liability at the time of the amendment is unchanged by the amendment. Considering equations (9) and (13), it is evident that the normal cost under variant 3 must increase at the amendment by more than the ratio by which it increases for variant 2 for $r_1 > r_0$:

$$\frac{EAN_{NC}_{A3}^j(x_j(t_A))}{EAN_{NC}_p^j(x_j(t_A))} \geq \frac{r_1}{r_0}.$$  

This behavior compares with the $r_1/r_0$ proportionate increase in normal cost under equation (9) (variant 2, the EAN retroactive NC mimic). Which is more acceptable may depend on the perceived relative importance of the behavior of the normal cost and of the actuarial liability.

3 Plan Amendments Under Projected Unit Credit

3.1 Projected Unit Credit

The projected unit credit method commonly is used, partly because the accounting bodies of both Canada and the United States require that it be used in calculating the pension expense to be entered in the employer’s financial statements (CICA, 1986, Section 3460.28; CICA refers to the Canadian Institute of Chartered Accountants.)
Partly as a result, most Canadian and United States pension plans are valued for funding purposes using this method. The method is described under the names projected unit credit (Anderson, 1992, p. 152; Berin, 1989, p. 119), prorate accrued benefit (Trowbridge and Farr, 1976, p. 40), accrued benefit cost method (constant amount) (Winklevoss, 1977, p. 78), or projected accrued benefit cost method (McGill and Grubbs, 1989, p. 291).

Under the service prorate version of the projected unit credit method, the projected retirement age pension is allocated pro rata over years of pensionable service. Thus, \( b_i(x_j(t)) \) is based on pay projected to retirement and service accrued to age \( x_j \). The normal cost is the present value of the current year's benefit allocation. The actuarial liability is the present value of the benefit allocated to the date of valuation at which the age is \( x_j \) nearest \( B_i(x_j) \). It is assumed that the date of valuation corresponds to the beginning of a plan year. Hence, the normal cost for the plan year for individual \( j \) is given by:

\[
PUC\text{NC}_i(t) = (B_i(x_j(t)) + 1 - B_i(x_j(t))) \frac{d_y^{(12)}}{D_{x_j(t)}} D_{x_j(t)}
\]

and the actuarial liability by:

\[
PUC\text{AL}_i(t) = B_i(x_j(t)) \frac{d_y^{(12)}}{D_{x_j(t)}} D_{x_j(t)}
\]

### 3.2 Plan Amendment

Prior to the plan amendment at \( t_A \) but at the attained age \( x_j(t_A) \) of individual \( j \) at the time of the valuation we have:

\[
B_p^l(x_j(t_A)) = r_0(x_j(t_A) - w_j)S_f^j.
\]

Again consider a nonretroactive increase at age \( x_j(t_A) \) of the benefit fraction from \( r_0 \) to \( r_1 \). Two possible methods of handling this situation are described next.

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2 FASB refers to the Financial Accounting Standards Board.

3 SFAS refers to Statement of Financial Accounting Standards.
3.3 Variant 1: PUC Service Weighting

The plan document gives a definition of accrued benefit that may be used in obeying vesting legislation; this accrued benefit may be based on the salary at attained age \( x_j(t) \).

Under variant 1 with \( r_1 > r_0 \), we assume instead that the benefit accrued up to age \( x_j(t_A) \) is given by the fractional method:

\[
B^j_{A1}(x_j(t_A)) = \frac{(x_j(t) - w_j)}{(y_j - w_j)} \left[ r_0(x_j(t_A) - w_j) + r_1(y_j - x_j(t_A)) \right] S^j_f. \tag{21}
\]

The normal cost for the year following age \( x_j(t) \), where \( t \geq t_A \), would increase in the ratio:

\[
\frac{PUCN C_{A1}(x_j(t))}{PUCN C_{A1}(x_j(t))} = \frac{r_0(x_j(t_A) - w_j) + r_1(y_j - x_j(t_A))}{r_0(y_j - w_j)}. \tag{22}
\]

This contrasts with the ratio \( r_1 / r_0 \), which is more natural if one regards the benefit as accruing at a rate \( r_0 \), before the effective date of the amendment and at a rate \( r_1 \) afterward instead of using the fractional method.

The accrued liability under the fractional method at age \( x_j(t_A) \) increases, because of the amendment, in the same ratio:

\[
\frac{PUC A^j_{A1}(x_j(t_A))}{PUC A^j_{A1}(x_j(t_A))} = \frac{r_0(x_j(t_A) - w_j) + r_1(y_j - x_j(t_A))}{r_0(y_j - w_j)}. \tag{23}
\]

This increase in actuarial liability is somewhat counterintuitive in a situation where the benefit accrued up to age \( x_j(t_A) \) can be regarded as being unchanged.

In the case \( r_0 < r_1 \), the normal cost and the actuarial liability are both decreased by the amendment.

3.4 Variant 2: PUC Accruals Weighting

Under variant 2, the benefit is assumed accrued at a rate \( r_0 \), for service before the amendment and \( r_1 \) for service afterward. It thus differs from variant 2 of the entry age method. Thus:

\[
B^j_{A2}(x_j(t_A)) = r_0(x_j(t_A) - w_j) S^j_f. \tag{24}
\]
and the actuarial liability is unchanged:

\[
\frac{\text{PUC} A\Gamma A_2(x_j(t_A))}{\text{PUC} A\Gamma p(x_j(t_A))} = 1. \tag{25}
\]

The normal cost at time \( t, t \geq t_A \), increases (or decreases if \( r_0 < r_1 \)) in the expected ratio because the variant 2 accrued benefit increases as:

\[
B_{A2}^j(x_j(t) + 1) = \left[ r_0(x_j(t_A) - w_j) + r_1(x_j(t) + 1 - x_j(t_A)) \right] S_f^j. \tag{26}
\]

Hence

\[
\frac{\text{PUC} N \Gamma A_2(x_j(t))}{\text{PUC} N \Gamma p(x_j(t))} = \frac{B_{A2}^j(x_j(t) + 1) - B_{A2}^j(x_j(t))}{B_p^j(x_j(t) + 1) - B_p^j(x_j(t))} = \frac{r_1}{r_0}. \tag{27}
\]

This variant gives results that might be expected by a client. In the United States, variant 2 usually is required for calculation of pension expense under SFAS 106 and SFAS 87 (Financial Accounting Standards Board, 1990, paragraph 40, footnote 8). Paragraph 40 of SFAS 87 states that "... pension benefits ordinarily should be based on the plan's benefit formula to the extent that the formula states or implies an attribution." Footnote 8 has "... benefit of 1 percent of final pay for each year of service up to 20 years and 1.5 percent of final pay for years of service in excess of 20 ... the attribution ... will not assign the same amount of pension benefit for each year of service." If the plan document defines the benefit accrual on a fractional basis, as in equation (21), then variant 1 is acceptable.

In Canada, the requirements are less clear. The Canadian Institute of Chartered Accountants (1986, paragraph 3460.28) states "the cost of pension benefits ... should be determined using the projected benefit method prorated on services."

The United States IRC Regulation Section 1.412(c)(3)-1(e)(3) discusses the allocation of projected benefits between past and future years. Example (5) of IRC Regulation Section 1.412(c)(3)-1(g) indicates variant 2 (PUC accruals weighting) to be the acceptable method for funding purposes when the plan document defines the accrued benefit according to equation (26) rather than according to the fractional equation (21). This variant also satisfies the zero gain...
condition of IRC Regulation Section 1.412(c)(3)-1(c)(2), as is shown in Appendix C.

4 Conclusion

This paper discusses the use of the individual entry age normal and projected unit credit pension funding methods in the presence of a nonretroactive increase in the benefit accrual rate. In the case of both funding methods, it is recommended that the cost method be handled in such a way that the normal cost increases in the same proportion as the increase in the benefit accrual rate. Alternative methods are discussed, however, that may be more acceptable to some actuaries.

References


*Canadian Institute of Chartered Accountants Handbook*. (Canadian Institute of Chartered Accountants, 1986).


Appendix A—Entry Age Normal, Variant 2 (EAN Retroactive NC Mimic), Proof of Decrease of Actuarial Liability at Amendment

Let \( f(x) = \frac{sN_x - sN_y}{y - x} \)

and assume that \( sN_x \) is a decreasing function of \( x \). Then

\[
f(x + 1) - f(x) = \frac{sN_{x+1} - sN_y}{y - x - 1} - \frac{sN_x - sN_y}{y - x}
\]

\[
= \frac{(y - x)sN_{x+1} - (y - x - 1)(sN_{x+1} + sD_x) - sN_y}{(y - x - 1)(y - x)}
\]

\[
= \frac{sN_{x+1} - (y - x - 1)sD_x - sN_y}{(y - x - 1)(y - x)}
\]

\[
= \frac{(sD_{x+1} - sD_x) + (sD_{x+2} - sD_x) + \cdots + (sD_{y-1} - sD_x)}{(y - x - 1)(y - x)}
\]

\[
\leq 0
\]  

with equality only if \( sD_x = sD_z \) for \( x + 1 \leq z \leq y - 1 \). Because \( w_j < x_j(t_A) \), equation (12) gives us

\[
EANA^{A_2}(x_j(t_A)) = EANA^{L_1}(x_j(t_A))
\]

\[
+ \frac{(r_1 - r_0)}{r_0} PVFB^{A_2}(x_j(t_A)) \times \frac{(y_j - x_j(t_A))}{sN_{w_j} - sN_{y_j}}
\]

\[
\times \left[ \frac{sN_{w_j} - sN_{y_j}}{y_j - w_j} - \frac{sN_{x_j(t_A)} - sN_{y_j}}{y_j - x_j(t_A)} \right]
\]

\[
\geq EANA^{L_1}(x_j(t_A))
\]

using the decreasing nature of \( f(x) \) from equation (1.A) and assuming \( r_1 > r_0 \).
Appendix B—Entry Age Normal, Variant 2 (EAN Retroactive NC Mimic), Proof of Zero Gain

The notation used is

\[ PL \quad = \quad \text{For the whole plan;} \]
\[ A \quad = \quad \text{The set of actives (see Anderson, 1992, p. 9);} \]
\[ G(t) \quad = \quad \text{Gain in year } t \text{ to } t+1; \]
\[ i \quad = \quad \text{Valuation interest rate;} \]
\[ F(t) \quad = \quad \text{Fund value at time } t; \]
\[ UAL_{PL}(t) \quad = \quad \text{Unfunded at time } t, UAL_{PL}(t) = AL_{PL}(t) - F(t) \]
\[ C(t) \quad = \quad \text{Actual contributions in the year } t \text{ to } t+1; \]
\[ I_c(t) \quad = \quad \text{Interest to time } t+1 \text{ at the assumed rate } i \text{ on the contributions } C(t). \]

For simplicity, assume that the membership consists only of actives who will be below retirement age at the end of the year. Assume that the only benefit is on retirement. Use the standard formula for the gain (see, e.g., Anderson, 1992, p. 20). Assume, following IRC Regulation Section 1.412(c)(3)–1(c)(2), that “each actuarial assumption is exactly realized,” so that for example

\[ F(t + 1) - (C(t) + I_c(t) - F(t))(1 + i) = 0. \] (1.B)

Then the gain in a year \( t \), after the amendment, \( t \geq t_A \), is given by

\[
G(t) = \left( EAN UAL_{A2}^{PL}(t) + EAN NC_{A2}^{PL}(t) \right) (1 + i) - (C(t) + I_c(t)) \\
- EAN UAL_{A2}^{PL}(t +1) \\
\quad = \left( EAN AL_{A2}^{PL}(t) - F(t) + EAN NC_{A2}^{PL}(t) \right) (1 + i) - (C(t) + I_c(t)) \\
- \left( EAN AL_{A2}^{PL}(t +1) - F(t + 1) \right) \\
\quad = F(t + 1) - (C(t) + I_c(t)) - F(t)(1 + i) + \\
\quad \left( EAN AL(t)_{A2}^{PL}(t) + EAN NC_{A2}^{PL}(t) \right) (1 + i) - EAN AL_{A2}^{PL}(t + 1)
\]
\[ \begin{align*}
&= \left( EAN_{AL}^{PL}(t)_{A_2(t)} + EAN_{NC}^{PL}_{A_2(t)} \right)(1 + i) - EAN_{AL}^{PL}_{A_2(t + 1)} \\
&= \sum_{A_i} \left[ \left( EAN_{AL}^j_{A_2}(x_j(t)) + EAN_{NC}^j_{A_2}(x_j(t)) \right) \right](1 + i) \\
&\quad - \sum_{A_{t+1}} EAN_{AL}^j_{A_2}(x_j(t + 1)) \\
&= \sum_{A_i} \left[ PVFB_j^j_{A_2}(x_j(t)) - PVF\left( EAN_{NC}^j_{A_2}(x_j(t)) \right) \right] \text{ }(1 + i) \\
&\quad + EAN_{NC}^j_{A_2}(x_j(t))(1 + i) \\
&\quad - \sum_{A_{t+1}} EAN\left[ PVFB_j^j_{A_2}(x_j(t+1)) - PVF\left( EAN_{NC}^j_{A_2}(x_j(t+1)) \right) \right] \\
&= \sum_{A_i} \left[ -PVF\left( EAN_{NC}^j_{A_2}(x_j(t)) \right) + \left( EAN_{NC}^j_{A_2}(x_j(t)) \right) \right](1 + i) \\
&\quad + \sum_{A_{t+1}} \left[ PVF\left( EAN_{NC}^j_{A_2}(x_j(t+1)) \right) \right] \\
&= 0. \tag{2.B}
\end{align*} \]

In the above has been used the assumption that decrements, which reduce \( A_t \) at time \( t \) to \( A_{t+1} \) at time \( t+1 \), give \( A_{t+1} \) as a proportion

\[ 1 - q_{x_j(t)} = \frac{D_{x_j(t+1)}(1 + i)}{D_{x_j(t)}} \]

of \( A_t \).
Appendix C—Projected Unit Credit, Variant 2 (PUC Accruals Weighting), Proof of Zero Gain

Use notation and assumptions as for Appendix B. Then the gain in the year starting at time \( t \) is given (Anderson, 1992, p. 13) using equation (1.2) by

\[
G(t) = \left( \text{PUC}_{A(t+2)}^{PL} + \text{PUC}_{A(t+2)}^{PL} \right)(1 + i) \\
- (C(t) + iC(t)) - \text{PUC}_{A(t+2)}^{PL}(t + 1) \\
= \left( \text{PUC}_{A(t+2)}^{PL} + \text{PUC}_{A(t+2)}^{PL} \right)(1 + i) - \text{PUC}_{A(t+2)}^{PL}(t + 1) \\
= \sum_{A(t)} \left( \text{PUC}_{A(t+2)}^{j} - \text{PUC}_{A(t+2)}^{j} \right)(1 + i) \\
- \sum_{A(t+1)} \text{PUC}_{A(t+2)}^{j}(x_j(t+1)) \\
= \sum_{A(t)} \left[ r_0(x_j(t_A) - w_j) + r_1(x_j(t) - x_j(t_A)) + r_2 \right] S_j^{(1+i)} \frac{D_{y_j}}{D_{x_j(t)}} d_{y_j}^{(12)} \\
- \sum_{A(t+1)} \left[ r_0(x_j(t_A) - w_j) + r_1(x_j(t) + 1 - x_j(t_A)) \right] S_j^{(1+i)} \frac{D_{y_j}}{D_{x_j(t+1)}} d_{y_j}^{(12)} \\
= 0
\]

where again the set \( A(t) \) reduces after a year to \( A(t+1) \) at the assumed proportion \((1 - q_{x_j(t)})\).