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A Reformulation Strategy for Multi-Dimensional CSPs: The Case Study of the SET Game

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A Reformulation Strategy for Multi-Dimensional CSPs: The Case Study of the SET Game

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Outline

• General reformulation strategy for CSPs
  – Multidimensional CSPs (MD-CSPs)
  – Problem reformulation by value interchangeability
  – A general reformulation strategy for MD-CSPs

• Game of Set: A new toy problem
  – Game, CSP model
  – Problem reformulation
  – Algorithms & Results

• Conclusions
Multi-Dimensional CSPs

- All variables have the same domain
- Domain is multi-dimensional
  - A set of dimensions
  - Each domain value is described by a combination of dimensions values

- In MD-CSPs, a constraint can be
  - One-dimensional: defined over a single dimension
  - Multi-dimensional, otherwise

- Typical applications
  - Scheduling, resource allocation, configuration, etc.
Reformulation by value interchangeability

• Value interchangeability [Freuder 91]
  – Domain abstraction: equivalent values
  – ‘Perfect’ equivalence rare, small domain partitions
  – Ignoring some constraints yields larger domain partitions, smaller CSPs, smaller search space [Haselboeck 93, Choueiry+ 94]

• Abstraction in MD-CSPs [Freuder+ 95,97]
  – Abstract domains based on a dimension, $P_r$
  – Solve reformulated CSP
  – Use solution of $P_r$ to guide solving original CSP, $P_o$

• How to “use solution of $P_r$ to solve $P_o$”? Hard to automate
Reformulation Strategy for MD-CSPs

• Process
  For each one-dimensional constraint
  Abstract domains using interchangeability
  Enforce one-dimensional constraint
  Solve remaining CSPs with some solver

• Questions
  – Which 1-dim constraint to use first?
  – How to process reformulated problems?

• Case study of the Set game

\[ P_0: \text{Original MD-CSP} \]
- One-dimensional constraints:
  \( \{C_1, C_2, C_3, \ldots, C_n\} \)
- Other constraints

Exploit approximate symmetries to enforce \( C_1 \)

\[ P_i: \text{A set of reformulated CSPs} \]
- One-dim constraints: \( \{C_2, C_3, \ldots, C_n\} \)

Exploit approximate symmetries to enforce \( C_2 \)

\[ P_n: \text{A set of reformulated CSPs} \]
- One-dim constraints: \( \emptyset \)

Enforce remaining constraints
Outline

• General reformulation strategy for CSPs
  – Multidimensional CSPs (MD-CSPs)
  – Problem reformulation by value interchangeability
  – A general reformulation strategy for MD-CSPs

• Game of Set: A new toy problem
  – Game, CSP model
  – Problem reformulation
  – Algorithms & Results

• Conclusions
Game of Set

- Deck of 81 (=3^4) cards, each card with a unique combination of 4 attributes values
  1. Number ∈ {1,2,3}
  2. Color ∈ {green, purple, red}
  3. Filling ∈ {empty, stripes, full}
  4. Shape ∈ {diamond, squiggle, oval}

- Solution set: 3 cards
  ∀ attribute, the 3 cards have either the same value or all different values

- 12 cards are dealt, on table [3,21]
- Recreational game, favorite of children & CS/math students
- New toy problem for AI: a typical multi-dimensional CSP
Set as an MD-CSP

- Model
  - Three variables
  - Same domain (12 cards)
  - One ‘physical’ constraints
  - Four 1-dimensional constraints

Same domain for all 3 variables

Domain Table

<table>
<thead>
<tr>
<th>Number</th>
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</table>
Reformulation by Value Interchangeability

- **Filling**=empty, **Color**=red
- **Number** yields 3 domain partitions by neighborhood interchangeability (meta), replacing $6^3$ solutions by $3^3$ subproblems
- Enforcing $N=\oplus N^\#$ replaces $3^3$ by 4 subproblems

$D_1 = D_2 = D_3$

$D_1 \neq D_2 \neq D_3$
Reformulation Strategy for Set

- Which dimension to choose first?
  - For Set, heuristics based on data in ‘Domain Table:’ Fewest subproblems first (infamously, Fail First Principle)

\[ P_0: \text{Original MD-CSP} \]
- One-dim constraints: \{C_1, C_2, C_3, \ldots, C_n\}

Exploit approximate symmetries to enforce \( C_1 \)

\[ P_1: \text{A set of reformulated CSPs} \]
- One-dim constraints: \{C_2, C_3, \ldots, C_n\}

Exploit approximate symmetries to enforce \( C_2 \)
Selecting Domain Dimension

• Goal: Reduce branching factor
  – In example below, no card in domain has a shaded filling, thus, subproblems for $Filling=s$ and $F\neq$ do not exist

<table>
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<tr>
<th>Domain</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$c_4$</th>
<th>$c_5$</th>
<th>$c_6$</th>
<th>$c_7$</th>
<th>$c_8$</th>
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<td>4</td>
</tr>
</tbody>
</table>

$\mathcal{C} = \{F,S,C,N\}$

$D_1 = D_2 = D_3$

$F\neq$

$\mathcal{C} = \{S,C,N\}$, $F \equiv e$

$\mathcal{C} = \{S,C,N\}$, $F \equiv f$

• Our reformulation algorithm for Set
  – Uses ‘Domain Table’ & ‘Summary of Domain Table’
  – Has 4 tests & 5 heuristics
Algorithms: Finding all Solutions

1. Brute-force search (BF)
   - 3-nested for-loops generate all combinations, then test for solutions
   - Contradicts 40+ years of CP research & experience 😞
   - Does not scale ($12^3 \sim d^n$)

2. Backtrack search (Basic Solver)
   - Symmetry breaking (lexicographic ordering)
   - Both forward-checking (equality) & back-checking (All-diff constraints)

3. Reformulation-based algorithm
   - Uses 2 data structures: ‘Domain Table’ & ‘Summary of Domain Table’
   - Includes 5 selection heuristics
   - Open subproblems maintained in an agenda: room for heuristics (1Sol)
   - Empirical tests: randomly selected ‘hands’ of 3 to 81 cards, results averaged over of 1,000 runs
Results

- **#CC,#NV**: Reformulation dramatically reduces # of combinations tested
- **CPU time** reflects the cost of setting up the data structures for the CSP & search

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>#Cards</th>
<th>#Sol</th>
<th>#CC</th>
<th>#NV</th>
<th>Time [msec]</th>
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</thead>
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<td>2.77</td>
<td>1956.8</td>
<td>220</td>
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<td>39.44</td>
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</table>

• #CC,#NV: Reformulation dramatically reduces # of combinations tested
• CPU time reflects the cost of setting up the data structures for the CSP & search
Online Game gameofset.unl.edu

- Game running online
- Interface explaining the reformulation still in development
- Advertzmt: minesweeper.unl.edu & sudoku.unl.edu (CP-based)
Conclusions

• Contributions
  – A systematic approach to reformulation and ‘conditional’ symmetries
  – Applicability to real-world problems highly promising
  – A new toy problem for AI research & education 😊

• Technical issues
  – Generalize heuristics for dimension selection and problem decomposition
  – Explore other types of interchangeability/symmetries
  – Extend definition of MD-CSP to allow unequal/all-diff domains

• Modeling lesson
  – CSP variables and values are often ‘objects’ with attributes
  – So far, we have integrated those attributes in the constraint definitions
  – Let’s rethink CSP modeling: Maybe multi-dimensional CSPs are more common than we thought they are..