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STABILITY OF POISEUILLE-COUETTE  
FLOW BETWEEN CONCENTRIC CYLINDERS

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SYNOPSIS

The stability of laminar flow between concentric cylinders is analyzed. Combined Poiseuille and Couette flows are considered. It is shown that instabilities exist if the laminar flow is symmetric or near-symmetric on part of the annulus. Instabilities are reported for aspect ratios as small as  $\eta = 0.001$ .

KEYWORDS

Poiseuille-Couette Flow, Stability

SLEUTELWOORDE

Poiseuille-Couette-Vloei, Stabiliteit

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## ENCLATURE

- complex wave velocity of perturbation,  $c = c_R + ic_i$   
 linear operator defined by equation (4)  
 wave number of azimuthal operations  
 density of fluid,  $\text{kg/m}^3$
- dimensionless constant axial pressure gradients  $-\frac{dP_0'}{dz'}(pr_2'u_{max}^2)$
- constant axial pressure gradient, Pa/m
- dimensionless radial variable,  $r'/r_2'$   
 outer radius of annulus, m  
 inner radius of annulus, m  
 Reynolds number  $pr_2'u_{max}/\mu$   
 dimensionless time  $t'/(pr_2'^2/\mu)$   
 dimensionless laminar velocity  $u'/u'_{max}$   
 laminar velocity, m/s  
 maximum velocity, m/s  
 dimensionless velocity of inner cylinder,  $V'/u'_{max}$   
 velocity of inner cylinder, m/s  
 dimensionless perturbation of radial velocity component,  $v_r'/(\mu/pr_2')$   
 perturbation of radial velocity component, m/s  
 dimensionless axial distance  $z'/r_2'$   
 axial distance, m

k symbols

- wave-number
- $\frac{dP_0'}{dz}$
- $V'/r_2'$
- dynamic viscosity of fluid,  $\text{Ns/m}^2$

## INTRODUCTION

Since the time of Osborne Reynolds, the stability of laminar flow in pipes, has been the subject of theoretical and experimental studies. Experimental results indicated that the laminar flow became unstable at Re numbers of 2 000 for rough surfaces, to 100 000 for smooth surfaces<sup>1</sup>. It is clear that the surface finish, which acts as a source for disturbances of finite amplitude, plays a very important role in determining the critical Reynolds number,  $Re_c$ .

In this work we are particularly interested in the stability of axial laminar flow in an annulus to axisymmetric perturbations of infinitesimal amplitude. Thus we are considering the case of near-zero roughness, of the walls. Two concentric cylinders of aspect ratio  $r_1'/r_2' = \eta < 1$ , is considered. In the limit  $\eta \rightarrow 1$  the flow pattern is similar to flow between two parallel plates. The other limit is  $\eta \rightarrow 0$ ; although the transition in this limit to Hagen-Poiseuille flow is not smooth (one must still assume non-slip condition at cylinder surface of infinitesimally small radius) the  $Re_c$  goes to infinity, indicating that the laminar flow becomes unconditionally stable to axisymmetric infinitesimal perturbations. Davey and Drazin<sup>2</sup> reported numerical results for Hagen-Poiseuille which are in support of the asymptotic results. Graebel<sup>3</sup> also did a linear stability analysis of Hagen-Poiseuille flow, but he considered azimuthal perturbations and found instability for wavenumbers  $n \geq 2$ . Furthermore, experimental observations by Fox et al<sup>4</sup> showed that laminar flow became unstable to azimuthal perturbations. All these results are in compliance with the fact that there is no cylindrical equivalent of Squire's theorem, which states that in the planar case, two-dimensional perturbations are most unstable.

The laminar flow in the annulus originates from a constant axial pressure gradient (Poiseuille flow) and by moving the centre cylinder at constant velocity  $V$  (Couette flow). If the Couette and Poiseuille flows are in the same direction, the line of maximum velocity gets closer to the centre and for  $V$  large enough, the maximum is at the walls of the center cylinder. If the

cylinder is pulled anti-parallel to the Poiseuille flow direction, a strong boundary-layer develops adjacent to the center cylinder. Under conditions where the Couette flow dominates, the profile is almost inverted.

Mott and Joseph<sup>5</sup> studied the stability of laminar flow in annuli, under the action of Poiseuille flow. An interesting result is the importance of the symmetry of the basic laminar profile. If the profile is symmetric around the centre-line of the annulus ( $r' = (r'_1 + r'_2)/2$ ), they found that the laminar flow is the least stable. As the basic profile becomes more asymmetric, the stability increases ( $Re_c \rightarrow \infty$ ). Returning to our problem of the combined Poiseuille-Couette flow, it is possible for certain values of  $\eta$ ,  $V$  and  $Re$ , that the laminar profile is near-symmetric in a part of the annulus, not necessarily for all  $r' \in [r'_1, r'_2]$ . If we now seek infinitesimal perturbations which are non-zero on only this symmetric part of the flow, satisfying the boundary conditions already well-away from the two solid boundaries, it is possible to find  $Re_c$  which are much smaller than the values for perturbations which are non-zero for  $r' \in (r'_1, r'_2)$ . This approach was used with good success by Gatica et al<sup>6</sup> to analyze the stability of non-linearly stratified fluids to thermal instabilities.

Apart from academic interest in this problem, several practical situations exist where this configuration is used. Of specific interest is the coating of yarn or filaments with a protective layer by pulling it through a fluid. If a homogeneous coating is required, it is important to know the maximum stability conditions.

#### LINEAR ANALYSIS

The laminar flow is given by:

$$u = \frac{1}{4} Re \frac{dP_0}{dz} [r^2 - \eta^2 + (1 - \eta^2) \ln(r/\eta) / \ln \eta] + V \ln r / \ln \eta \quad (1)$$

where  $Re$  is the Reynolds number.

Let

$$\Omega = V/Re \frac{dP_0}{dz}$$

When  $\Omega = 0$  (pure Poiseuille flow), the flow is almost symmetric for  $\eta \rightarrow 1$  and becomes more skew as  $\eta \rightarrow 0$ . In the case of  $\Omega = 0,005$ , the annulus flow is distorted, but the important effect is the restoring of symmetry on part of the domain ( $\eta, 1$ ). For axisymmetric perturbations of the form  $v_r(r) \exp(i(\alpha z - \text{Re} c t))$ , the Orr-Sommerfeld equation for cylindrical coordinates takes the form:

$$L^2 v_r - i \alpha \text{Re} [(u-c) L v_r - v_r (u'' - u'/r)] = 0 \quad (2)$$

$$v_r'(r) = v_r(r) = 0, \quad r = \eta, 1 \quad (3)$$

where

$$L \equiv \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} - \alpha^2 \quad (4)$$

A prime denotes differentiation with respect to  $r$ .  $\alpha$  denotes the axial wave number and  $c$  is the wave velocity. Instability sets in when the complex part of  $c$  becomes positive. The associated  $Re$  value will be referred to as the critical Reynolds number.

An inverse shifted version of the Lanczos algorithm is used to solve the eigenvalue problem. This algorithm produces rapidly and efficiently a cluster of eigenvalues in the vicinity of the one with the smallest real part.

#### A. $\Omega = 0$

In Figure 1 the critical values of  $Re$  is shown for different aspect ratios  $\eta$ . These values are in very good agreement with the results of Mott and Joseph (1968) but for comparison reasons it should be remembered that they used  $Re = u'_{\max}(r'_2 - r'_1)/2\mu$ . These results also served as a check for our numerical code.

$$\Omega = 0,005$$

the trough-like behaviour of  $Re$  vs  $\eta$  becomes more pronounced and the minimum value of  $Re$  is less than the minimum value for the previous case. When  $\eta$  approaches 0 and 1, the axial movement of the inner cylinder plays a stabilizing role and values of  $Re$  are higher than the case of a stagnant cylinder, i.e.  $\Omega = 0$ .

To illustrate the concept of perturbations which are more stable in only a part of the annulus-domain, consider the case of  $\eta = 0,6$ . The critical value was  $Re = 47\,750$  and the associated eigenfunction was numerically found to be nonzero only on  $r = [0,6235, 1]$ . This implies that the perturbation did not survive in the part  $[0,6; 0,6235]$ . When the domain  $[0,6; 1]$  was used, the stabilizing role of asymmetric basic velocity profiles gave rise to a critical value of  $Re = 114\,300$ . The sensitivity of  $Re$  towards the form of the basic flow is clearly illustrated. Since the eigenfunction satisfies both Neumann and Dirichlet conditions at both ends, a jump in a higher derivative can be expected at  $r = 0,6235$  for the form:

$$v_r^* = \begin{cases} v_r, & 0,6235 \leq r \leq 1 \\ 0, & 0,6 \leq r \leq 0,6235 \end{cases}$$

One can conclude that the most critical perturbation for the annulus  $r = (r_1, 1)$  is not necessarily associated with this domain, but  $Re(r_2)$  can reach a local minimum for  $(r_2, 1)$ ,  $r_2 > r_1$ . In principle, the same argument holds for the upper limit of the domain, although numerical experiments never showed a decrease in  $Re$  for a decrease in this value. Since the Reynolds stress distribution and energy transfer from the basic flow usually shows sharp changes at the fixed walls, it is expected that at least one solid boundary should remain in this domain.

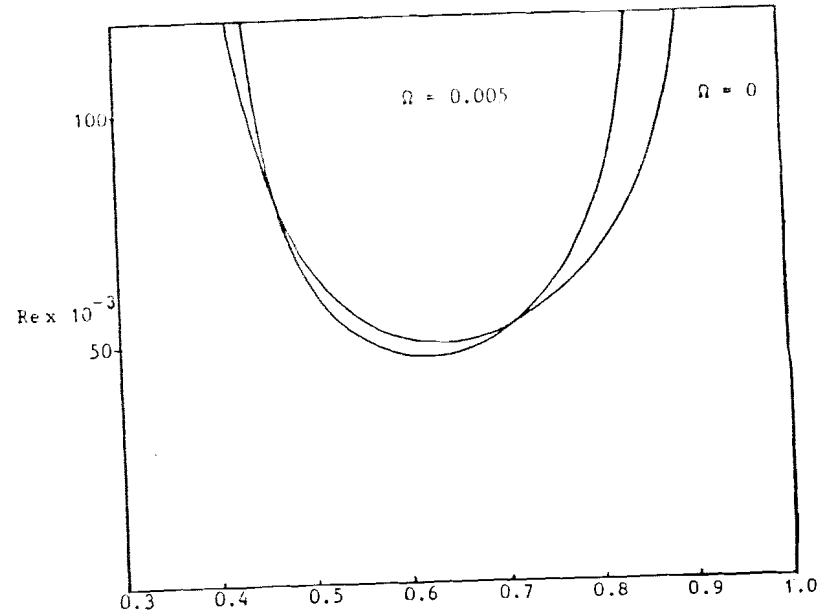


Figure 1: Critical Reynolds Number vs. Aspect Ratio  $\eta$

$$C. \quad \Omega = 1,5 \quad (\eta = 0,001)$$

The basic flow profile associated with this case, is shown in Figure 2. Although the inner cylinder is very thin (and often it is assumed that  $Re_c \rightarrow \infty$  for cylinders this thin), the flow is almost symmetric on the interval  $(0,4596; 1)$  and for this domain  $c_1 = 0,00000$ , at  $Re = 67\,134$ . When considering the whole domain  $(0,001; 1)$  the code was unable to find instability.

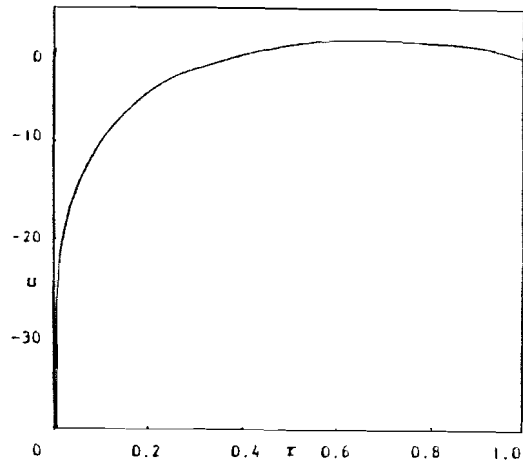


Figure 2: Basic Flow Profile for  $\Omega = 1,5$  ( $\eta = 0,001$ )

4. an infinite perturbation analysis still leads to critical  $Re$  numbers which are considerably higher than experimental values of 2 100 and the key lies in a finite amplitude analysis.

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#### CONCLUSION

the most important points which became apparent from this study are:

laminar flow in an annulus between stagnant cylinders is stabilized when  $\eta \rightarrow 0$ . (In this limit the Reynolds numbers based on gap width and outer radius become identical.)

including Couette flow by axially moving the inner cylinder, the most critical perturbation can be asymptotically zero on a part of the annulus

for very small values of  $\eta$ , one will expect that the laminar flow will become unconditionally stable, but for  $\eta = 0,001$  pulling rates of the centre cylinder exist that will lead to instability at  $Re = 67\ 134$