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## ALTERNATING-CURRENT CIRCUITS

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## ALTERNATING-CURRENT CIRCUITS

### INTRODUCTION

The electric clock on the wall, radio and television, the incredibly rapid handling of information by computers, and the transmission of signals by our own nerves are among countless devices and effects that depend on circuits in which currents or voltages vary with time. Alternating-current (ac) circuits, in which charges oscillate back and forth in a wire in such a way that the average current is zero, are among the simpler time-varying circuits. In this module you will study the behavior of simple ac circuits containing resistors, inductors, and capacitors.

### PREREQUISITES

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Before you begin this module, you should be able to:	Location of Prerequisite Content
*Convert degrees to radians and radians to degrees (needed for Objective 1 of this module)	Trigonometry Review
*Differentiate and integrate sine and cosine functions (needed for Objectives 1 and 2 of this module)	Calculus Review
*Relate emf, current, and resistance in dc circuits (needed for Objectives 1 through 3 of this module)	Direct-Current Circuits Module
*Relate capacitance, charge, and potential difference (needed for Objectives 1 through 3 of this module)	Capacitors Module
*Relate inductance, current, and emf (needed for Objectives 1 through 3 of this module)	Inductance Module
*Calculate energy in the electric field of a capacitor (needed for Objective 4 of this module)	Capacitors Module
*Calculate energy in the magnetic field of an inductor (needed for Objective 4 of this module)	Inductance Module

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### LEARNING OBJECTIVES

After you have mastered the content of this module, you will be able to:

1. Rotating vectors - Illustrate phase relations among currents and potential differences in different portions of a given ac circuit, using vector and/or graphical techniques.
2. Alternating-current analysis - Solve for unknown quantities in an ac circuit, using vector techniques, given a suitable set of currents, potential differences, emfs, resistances, capacitances, inductances, and/or frequencies.
3. Averages - Calculate root-mean-square (rms) potential differences, currents, and/or average power in a given ac circuit.
4. Energy - Describe the energy flow in a given ac circuit.

TEXT: Frederick J. Bueche, Introduction to Physics for Scientists and Engineers (McGraw-Hill, New York, 1975), second edition

### SUGGESTED STUDY PROCEDURE

You have already studied simple examples in which currents or potential differences varied with time. For Objective 1, it will be useful to review quickly Sections 21.12 in Chapter 21 and Section 25.4 in Chapter 25. Then study Section 27.1 in Chapter 27 for Objective 2. Usually angular frequency,  $\omega = 2\pi f$  (radians per second), is used to avoid carrying factors of  $2\pi$  along in all the equations.

Study General Comments 1 through 3 and Chapter 27, Sections 27.2 through 27.4 for Objective 3. Keep in mind the text's convention on notation:  $v$ ,  $i$  are time-varying quantities;  $v_0$ ,  $i_0$  are the maximum values, or amplitudes of time-varying quantities; and  $V$ ,  $I$  are rms quantities. To make the text on p. 524 consistent with Figure 27.3, change "C" and "D" to "A" and "B," respectively, in the second paragraph and in the footnote.

#### BUECHE

Objective Number	Readings	Problems with Solutions		Assigned Problems		Additional Problems (Chap. 27)
		Study Guide	Text	Study Guide	Text (Chap. 27)	
1	Sec. 27.4, General Comments 3, 5	A	Illus. <sup>a</sup> 27.5, 27.6, 27.7	D	7, 8, 14	9, 10, 15, 16, 19
2	Secs. 27.2, 27.3, 27.5, General Comments 1, 5	B	Illus. 27.2, 27.3, 27.4	E	3, 5, 7, 8, 14	Quest. <sup>a</sup> 9 to 13, Probs. 4, 6, 9, 10 to 13, 15, 16, 19
3	Sec. 27.1, General Comments 2, 5	C		F		Quest. 5, Probs. 1, 2, 9, 10
4	Sec. 27.6, General Comment 4	B, C		E, F		Quest. 3, 4

<sup>a</sup>Illus. = Illustration(s). Quest. = Question(s).

The vector treatment used to represent rms potential differences and currents (e.g., Fig. 27.8) can be extended to give instantaneous values,  $v(t)$  and  $i(t)$ . Suppose the potential difference between two points in a circuit is given by the expression

$$v_1 = v_{10} \sin \omega t = \sqrt{2}V \sin \omega t,$$

where  $V = [v(t)]_{\text{rms}}$ . Let us draw a vector representing the amplitude,  $v_{10} = \sqrt{2}V$ . If the vector is horizontal at  $t = 0$  and is allowed to rotate counterclockwise with angular velocity  $\omega$ , it will look like the sketches in Figure B1 at the stated times. The instantaneous value of  $v(t)$  at time  $t$  is then simply the vertical component of the rotating vector.

If a potential difference is the sum of two potential differences, with a difference in phase, as in Figure 27.8, the same analysis can be applied. The entire vector triangle rotates with angular velocity  $\omega$ . Thus, if  $i = i_0 \sin \omega t$  as in Figure 27.8, the vector triangle looks, at different times, like Figure B2. In summary, to find the instantaneous potential difference or current at any instant  $t$ , algebraically add the component instantaneous values. To find the amplitude or the rms potential difference, add the component values vectorially. In comparing the triangles drawn here with those in Figure 27.8, notice that Bueche has drawn the triangle for rms values, but we have plotted maximum values. This does not affect the geometry in any way, since we multiply the length of each rms quantity by the same factor  $\sqrt{2}$  to get the amplitude. Solve Problems A and B. Compare your solutions with those in the study guide. Then solve Problems D and E, and Problems 3, 5, 7, 8 and 14 in Chapter 27.

Study Sections 27.5 and 27.6. Resonance is an important special case of the general RLC circuit. If you can handle the general case, resonance is easy! In Section 27.6 the author suddenly changes his notation. To avoid confusion, add the subscript "av" to the left-hand side of Eq. (27.14b) and the unnumbered equation following it. Read General Comments 4 and 5. Solve Problem C and compare your answer to that in the study guide. Then solve Problem F. Take the Practice Test, and work some Additional Problems if necessary, before trying a Mastery Test.

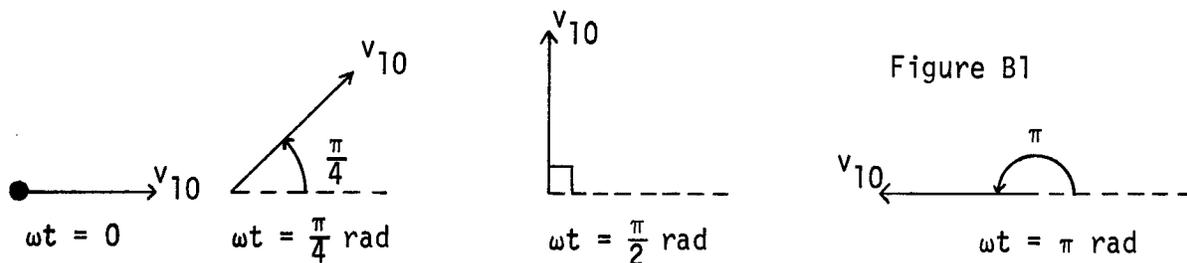


Figure B1

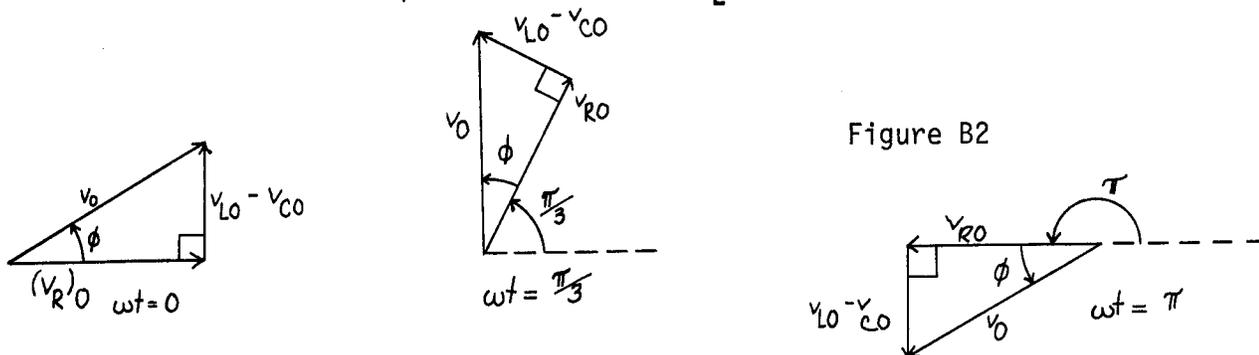


Figure B2

STUDY GUIDE: Alternating-Current Circuits

3(HR 1)

TEXT: David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974)

SUGGESTED STUDY PROCEDURE

Your text does not treat the subject of ac circuits. If you have available any of the books listed below, your instructor can supply you with a study guide for this module:

Frederick J. Bueche, Introduction to Physics for Scientists and Engineers (McGraw-Hill, New York, 1975), second edition.

Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Mass., 1970), fourth edition.

Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol. 2.

TEXT: Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Mass., 1970), fourth edition

### SUGGESTED STUDY PROCEDURE

You have already studied simple examples in which currents or potential differences varied with time. It will be useful to review quickly Chapter 29, Section 29-7, Chapter 33, Sections 33-11, 33-12, and 33-13. Study General Comments 1 and 2. Then work Problems 35-1(a), (b), (c), and 35-3(a), (b), (c).

Study Sections 35-1 and 35-2. Notice the conventions used by your text: lower-case letters imply instantaneous values, upper-case letters imply amplitudes, or maximum values. The use of rotors was first described in Section 11-4. Review that section if the use of rotor diagrams seems unfamiliar. Rotors are also called rotating vectors or phasors. Solve Problem A and compare your solution with that given. Then work Problem D. The word "voltage," derived from the volt unit, is commonly used for either potential difference or emf. Its usage is so common that you will have to learn to live with it.

Study Section 35-3. Impress on your mind the last paragraph on p. 507. Kirchhoff's rules for circuit analysis depend ultimately on two of the great conservation laws of nature: the conservation of charge and the conservation of energy. We sometimes get so accustomed to applying "rules" that we forget what physics lies behind them.

### SEARS AND ZEMANSKY

Objective Number	Readings	Problems with Solutions	Assigned Problems		Additional Problems
			Study Guide	Text	
1	Secs. 11-4, 35-3, General Comments 3, 5	A	D	35-6 to 35-9, 35-11, 35-13	35-10, 35-12, 35-14, 35-15, 35-16, 35-18, 35-19, 35-20
2	Secs. 35-1, 35-2, 35-3, 35-6, General Comments 1, 5	B	E	35-1(a), (b), (c), 35-3(a), (b), (c)	35-10, 35-12, 35-14 to 35-20
3	Secs. 35-4, 35-5, General Comments 2, 5	C	F		35-12, 35-14 to 35-17, 35-19
4	General Comment 4	B, C	E, F		

STUDY GUIDE: Alternating-Current Circuits

3(SZ 2)

Study Sections 35-4 and 35-5. Study General Comment 3. Then solve Problem B and compare your solution with the one provided. Solve Problem E and Problems 35-6 to 35-9, 35-11, and 35-13. Study Section 35-6. Resonance is an important special case of the general RLC circuit. If you can handle the general case, resonance is easy! Study General Comment 5. Solve Problem C and compare your solution with the one in the study guide. Then solve Problem F.

Take the Practice Test, and work some Additional Problems if necessary, before trying a Mastery Test.

TEXT: Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol. 2

### SUGGESTED STUDY PROCEDURE

You have already studied simple examples in which currents or potential differences varied with time. It will be useful to review quickly Chapter 27, Section 27-7, Chapter 32, Section 32-2, and Chapter 34, Section 34-1. If you have not done so, correct Eq. (34-7); the first term on the right-hand side should read

$$Q_m^2(\cos \omega t)^2/2C.$$

Study carefully Section 34-2. Keep in mind the notation used by your text for instantaneous, rms, and maximum values. Rotating vectors are sometimes abbreviated rotors. Another common term for them is phasors. By convention the vector rotates counterclockwise. Thus  $\omega t$  is the angle measured counterclockwise from the position at time  $t = 0$ . The important point when adding rotating vectors is that instantaneous values add algebraically, amplitudes (maximum values) or rms values add like vectors. Solve Problem A in this study guide, and compare your solution with the one provided. Then solve Problem D. Study General Comments 1 to 3 in the study guide. Then study Sections 34-3 through 34-5. Note the common, but imprecise, use of the word "voltage" to refer either to potential difference or emf. Equation (34-16) states that the average power in the RC circuit is  $P = VI$ , where  $V$  and  $I$  are rms values of  $v$  and  $i$ . This is the average power delivered to the circuit by the source of emf. Where does that power come from? Does the generator create the energy? What happens to the energy when it is delivered to the circuit? Solve Problem 34-4.

### WEIDNER AND SELLS

Objective Number	Readings	Problems with Solutions		Assigned Problems		Additional Problems
		Study Guide	Text	Study Guide	Text	
1	Secs. 34-2, 34-6, General Comments 3, 5	A	Ex. <sup>a</sup> 34-1	D	34-4, 34-8, 34-10 to 34-12	34-3, 34-6, 34-7, 34-9, 34-13, 34-14
2	Secs. 34-2 to 34-6, General Comments 1, 5	B	Ex. 34-1 to 34-3	E	34-8, 34-10 to 34-12	34-3, 34-6, 34-7, 34-13, 34-14
3	Secs. 34-2 to 34-5, General Comments 2, 5	C	Ex. 34-2, 34-3	F	34-8, 34-11, 34-12	34-9, 34-13
4	General Comment 4	B, C		E, F		34-3

<sup>a</sup>Ex. = Example(s).

Study Section 34-6, including especially Example 34-3. Notice that the circuit analysis depends ultimately on two of the great conservation laws; conservation of charge and conservation of energy. We sometimes get so accustomed to applying "rules" that we forget the physics that lies behind them. Study Section 34-7. The resonant circuit is, of course, just a special case of the RLC series circuit. Solve Problems B and E. Then solve Problems 34-8, 34-10, 34-11, and 34-12. In Problem 34-11, draw a graph of  $v_C$ ,  $v_R$ , and  $\mathcal{E}$  as functions of time. Study General Comment 5, and solve Problems C and F.

Take the Practice Test, and work some Additional Problems if necessary, before attempting a Mastery Test.

GENERAL COMMENTS1. Signs

In writing the differential equation representing Kirchoff's loop equation for a circuit, you must be precise and definite in your choice of signs. For example, suppose we have an RC circuit as in Figure 1. We can choose any convention we wish for the signs, but we must make a choice and stick with it. In this case, a common choice is to say that the current is positive clockwise when the top of the source of emf is at a positive potential with respect to the bottom. We still must choose a sign for the charge on C. Let us choose the bottom plate to have positive charge. (That is a reasonable choice. If  $\mathcal{E}$  were missing,  $+q$  on the bottom plate would provide a clockwise current.) Now the circuit diagram looks like Figure 2. Kirchoff's rule tells us to set the emf equal to the potential difference between points a and b:

$$iR - q/C = \mathcal{E}(t).$$

Differentiating with respect to time gives us

$$R\left(\frac{di}{dt}\right) - \left(\frac{1}{C}\right)\left(\frac{dq}{dt}\right) = \frac{d\mathcal{E}(t)}{dt}.$$

Now the temptation is to set  $dq/dt = i$ . However, with the signs chosen here, positive current means that the magnitude of the charge stored in each plate of C will decrease. We therefore must set

$$dq/dt = -i.$$

The circuit equation then becomes

$$R\left(\frac{di}{dt}\right) + \left(\frac{1}{C}\right)i = \frac{d\mathcal{E}(t)}{dt},$$

and the work of solving it can begin.

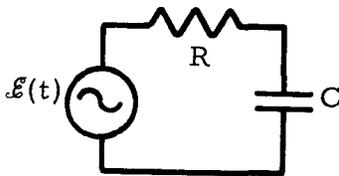


Figure 1

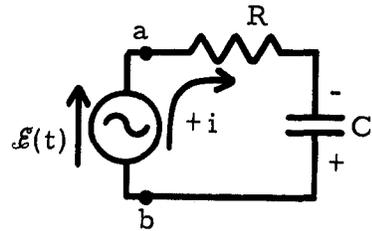


Figure 2

If, on the other hand, we had chosen the reverse convention for the sign of the charge on C, a positive current would cause an increase in the charge stored on the plate. The equation would then read

$$iR + q/C = \mathcal{E}(t) \quad \text{with the condition } dq/dt = i.$$

This gives us, after differentiation,

$$R\left(\frac{di}{dt}\right) + \frac{i}{C} = \frac{d\mathcal{E}(t)}{dt}.$$

We get the same equation, as we must, since it describes the same physical situation. But we get it only if we are consistent in dealing with signs.

In a circuit with an inductor, if the current increases in the direction of positive current ( $di/dt > 0$ ), we know there will be an opposing emf, trying to prevent the increase in current:

$$\mathcal{E}_L = -L(di/dt).$$

See Figure 3. This means that at the time shown in Figure 3, with positive  $di/dt$ , the induced emf in the inductor will tend to drive an opposing current counter-clockwise. The net emf in going around the circuit is  $\mathcal{E} - L(di/dt)$ . (See the redrawn version of the same circuit in Figure 4.)

$$\mathcal{E}(t) - L(di/dt) = iR; \quad \mathcal{E}(t) = L(di/dt) + iR.$$

Try to reach the point where you can analyze a circuit physically to be sure you get the signs correct. Rules memorized without physical understanding are fallible and readily forgotten.

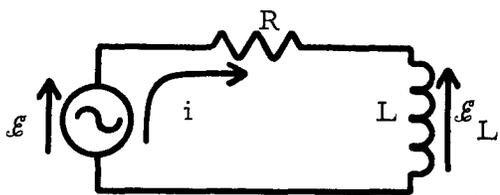


Figure 3

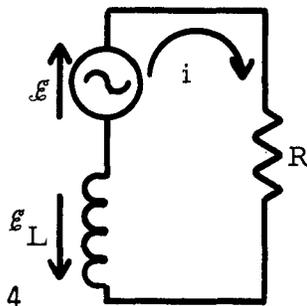


Figure 4

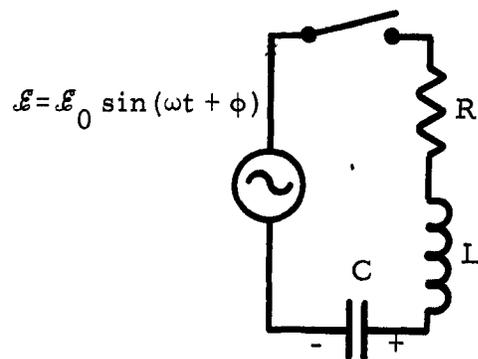


Figure 5

## 2. Approximations

Consider an RLC circuit, with an ac source as in Figure 5. The phase  $\phi$  in the emf is to allow for the fact that the initial phase of the emf may depend on the particular conditions. (The emf might not be zero at the instant you turn on the switch, for example.) The loop equation for the circuit, assuming the signs given, is

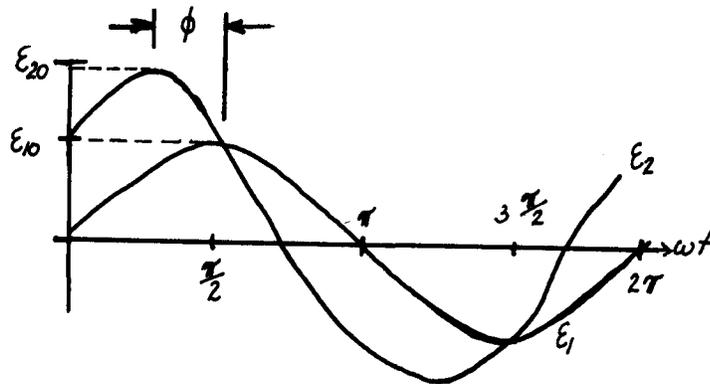
$$iR + q/C = \mathcal{E}_0 \sin(\omega t + \phi) - L(di/dt).$$

Differentiating, rearranging, and using the fact that  $i = dq/dt$  gives us

$$L\left(\frac{d^2i}{dt^2}\right) + R\left(\frac{di}{dt}\right) + \left(\frac{1}{C}\right)i = \frac{d\mathcal{E}(t)}{dt} = \omega\mathcal{E}_0 \cos(\omega t + \phi).$$

This differential equation, along with the initial conditions, determines the complete, exact behavior of the current for all time  $t > 0$ , up to the time when the switch is again opened. There are mathematical techniques for solving this equation. The solution will give you the current, including transient effects. However, since we do not wish to plunge into that much mathematical detail, we can learn a great deal about the circuit by making some assumptions and approximations. In so doing, we lose some of the detailed information contained in the exact solution of the differential equation.

Figure 6



One important assumption we make throughout this module is that we are most interested in the steady-state solution; that is, the solution after the currents and potential differences have had time to reach their maximum final amplitudes. That means we shall not learn how the current builds up from zero to its final value.

Another implicit assumption in the treatment used here is that the current has the same frequency as the applied emf. That is certainly true, but is not proved here.

Because the frequencies are typically so large that we can ignore the details of the oscillation of  $V$  or  $i$ , we frequently are interested only in average or rms values. That is what you read on an ac voltmeter or ammeter. Of course, you are sacrificing a lot of information that would be contained in the exact solution of the differential equation.

### 3. Phase

When two oscillations of the same frequency are superimposed, the result depends on the relative timing of the oscillations. At one extreme, the maxima of both oscillations occur at the same instant, and the oscillations add constructively. At the other extreme, the maximum of one oscillation coincides with the minimum of the other, and the oscillations subtract. There are, of course, infinitely many cases in between the two extremes.

The relative timing of the oscillations is determined by the relative phase of the two. Mathematically, the relative timing can be expressed by equations such as

$$\epsilon_1 = \epsilon_{10} \sin \omega t \quad \text{and} \quad \epsilon_2 = \epsilon_{20} \sin(\omega t + \phi),$$

where  $\phi$  is the phase angle of  $\epsilon_2$  with respect to  $\epsilon_1$ . A plot of these two expressions, for  $\phi = \pi/4 \text{ rad} = 45^\circ$  looks like Figure 6. Notice that a crest of  $\epsilon_2$  occurs at an angle  $\omega t = \pi/4$  earlier than a crest of  $\epsilon_1$ . The difference between leading and lagging is sometimes confused because, in our right-handed society,  $\epsilon_2$  appears to the left of  $\epsilon_1$ , and therefore looks as if it's "behind"  $\epsilon_1$ . Remember, the abscissa is proportional to the time, not the position. If one wave precedes the other in time, it leads!

If you simply try to memorize the fact that the potential difference leads the current for an inductor, and the current leads the potential difference in a capacitor, that knowledge probably will not stay with you very long. Even if you resort to mnemonic trickery, it will not help you learn physics. The correct way to learn which quantity leads which is to reason physically.

An inductor is a coil of wire. A current in the wire will produce a magnetic field through the coil, and therefore a magnetic flux. If the current changes, the flux will change, and Faraday's law,  $\mathcal{E} = -d\Phi/dt = -L(di/dt)$ , tells us that the changing flux produces an induced emf to oppose the change. Because of the opposing emf, it is hard to force a change in current in the wire of an inductor. In fact, if you try to abruptly change the current, the emf approaches infinity, since it is proportional to  $di/dt$ . Because the induced emf always opposes a current change, the current change cannot "keep up with" an imposed change in emf. The change in current always lags behind a change in the imposed emf.

To change the potential difference of a capacitor, we must transport charge from one of the plates to the other. To change the potential difference abruptly, we must transport a lot of charge in a very short time. In the limit of very short times, we approach an infinite derivative,

$$dV/dt = 1/C(dq/dt) = (1/C)i.$$

Since we cannot get an infinite current, we cannot change the potential across a capacitor in an infinitesimal time. The potential cannot "keep up with" the current; the potential lags behind the current.

#### 4. Energy

In charging up a capacitor, the source of emf performs work on the charges, which increases the energy stored in the electric field of the capacitor. We know that a capacitor  $C$ , charged to a potential difference  $V$ , will have energy

$$E = (1/2)CV^2$$

stored in the electric field. Similarly, an inductor with a current  $I$  has stored in its magnetic field the energy

$$E = (1/2)LI^2.$$

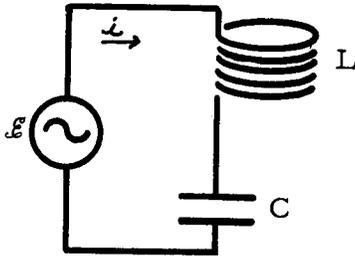


Figure 7

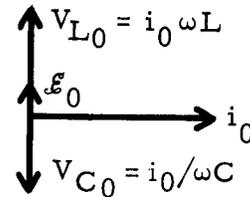


Figure 8

In an ac circuit containing capacitors and inductors, and possibly resistors, there is clearly some kind of energy exchange going on between the electric and magnetic fields, with the source of emf available to do work or have work done on it. To try to understand what is going on, consider a simple LC series circuit, as in Figure 7, in which we assume that the inductor has zero resistance. We construct a vector diagram for the circuit in Figure 8, referring all vectors to the current, since all elements of the circuit have the current in common. The net emf  $\mathcal{E}$  is the difference between  $V_L$  and  $V_C$ , since those two vectors are  $180^\circ$  out of phase. Let us assume that  $\omega L > 1/(\omega C)$ , so that  $\mathcal{E}$  is in phase with  $V_L$ . We can now plot the emf, the current, and the potential differences as functions of the time - see Figure 9. In Figure 10 are shown the configurations of fields at the times  $t_1$ ,  $t_2$ ,  $t_3$ , and  $t_4$  that are indicated on the graphs of Figure 9. These configurations are easy to figure out, if you recall that the magnetic field is proportional to the current, and the electric field is proportional to the potential difference across the capacitor. We see that between  $t_1$  and  $t_2$  the magnetic field is collapsing, and the electric field is building up. The energy in the circuit oscillates between the inductor and the capacitor, as the fields build and collapse in the two elements.

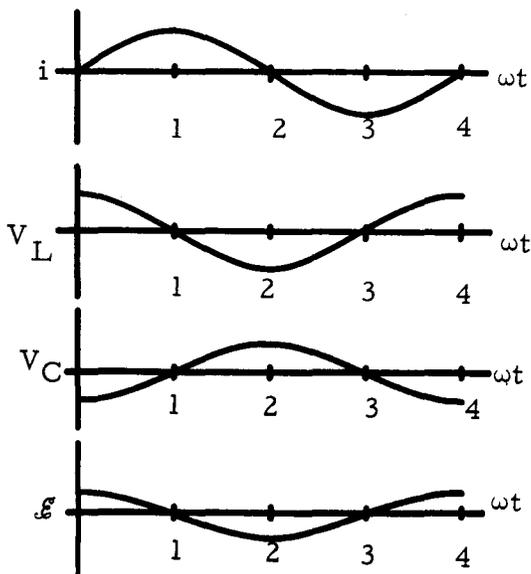


Figure 9

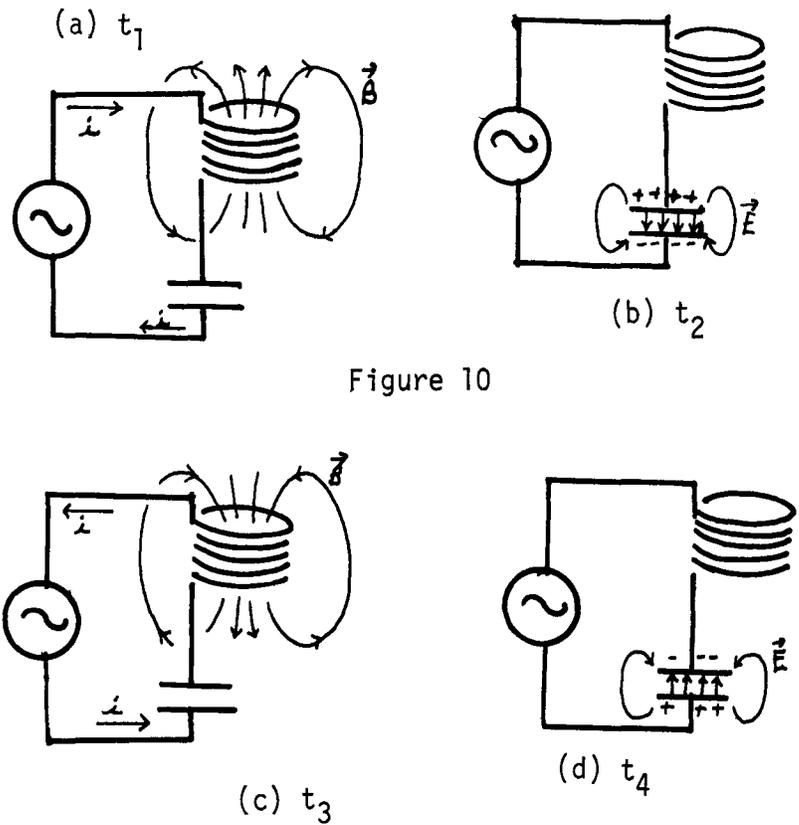


Figure 10

5. More Complicated Circuits

You might by now have the impression that all ac circuits must be hooked up in series. That is not true, and this comment will emphasize a general approach to all ac circuit problems.

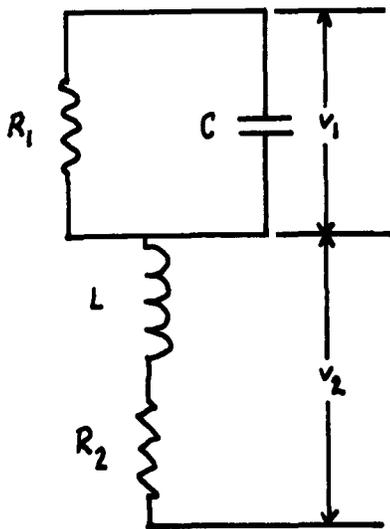


Figure 11

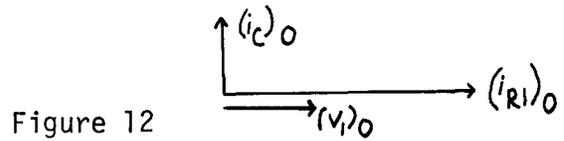


Figure 12

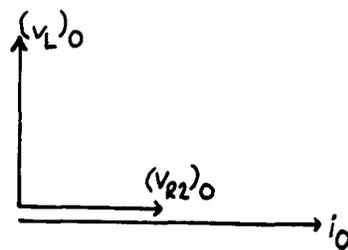


Figure 13

The key is to break a complicated circuit into sections in which all elements have a potential difference or a current in common, and to draw a vector diagram for each section, referring each vector in a diagram to the common factor. For example, suppose we had in a circuit the combination shown in Figure 11. Both  $R_1$  and  $C$  must have, at each instant, the same potential drop. Let us call that  $V_1$ . The inductor  $L$  and resistor  $R_2$  both have, at each instant, the same current  $i$ . For the  $CR_1$  section, refer the two current vectors  $(i_{R_1})_0$  and  $(i_C)_0$ , to the common potential drop. The diagram is shown in Figure 12. The current in a resistor is always in phase with the potential drop across it, and the potential drop across a capacitor always lags behind the ac current in the capacitor by  $\pi/2$  rad.

Refer the two potential drops in the  $LR_2$  section to the common current  $i_0$ . The vector diagram is shown in Figure 13. The potential drop across  $R_2$  is in phase with the current in the resistor. The current in an ideal inductor always lags behind the potential drop across it by  $\pi/2$  rad. The only tricky part is to couple these two vector diagrams together. We do this by noting from the circuit diagram that

$$i(t) = i_{R_1}(t) + i_C(t),$$

or, if we add amplitudes or rms values, that  $i_0$  is the vector sum of  $(i_{R_1})_0$  and  $(i_C)_0$ . Going back to the first vector diagram, Figure 12, we can draw in  $i_0$ . We see from Figure 14 that  $i_0$  leads  $(i_{R_1})_0$  by

$$\phi = \tan^{-1} \left( \frac{(i_C)_0}{(i_{R_1})_0} \right) = \tan^{-1} \left( \frac{(V_1)_0 \omega C}{(V_1)_0 / R_1} \right) = \tan^{-1} (R_1 \omega C).$$

Now, since we have  $i_0$  on the first vector diagram, we can simply add the rest of the second vector diagram, rotated so that all vectors bear the proper relationship to  $i_0$ . See Figure 15.

Figure 14

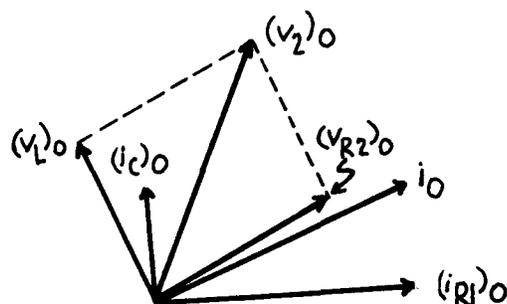
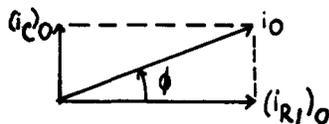


Figure 15

If the problem demands it, we can get  $(V_2)_0$ , the vector sum of  $(V_L)_0$  and  $(V_{R_2})_0$ . We can even get the total amplitude of the potential drop,

$$\vec{V}_0 = (\vec{V}_1)_0 + (\vec{V}_2)_0.$$

Since the vector diagram gets messy by now, let us summarize by drawing two vector diagrams, one for currents and one for potential drops, Figures 16 and 17, respectively. By studying the diagram for potentials, you can see that

$$\theta = \tan^{-1} \frac{(V_2)_0}{(V_{R_2})_0} \quad \text{and} \quad \psi = \tan^{-1} \frac{(V_2)_0 \sin(\theta + \phi)}{(V_1)_0 + (V_2)_0 \cos(\theta + \phi)}.$$

Using these relationships, and equations such as

$$(V_L)_0 = (i_L)_0 \omega L, \quad (V_C)_0 = (i_C)_0 / \omega C, \quad \text{and} \quad (V_R)_0 = (i_R)_0 R,$$

one can learn everything about the circuit, if the algebra does not get too messy. Remember, the instantaneous value of any of these quantities is the vertical component of the vector at time  $t$ , if the diagram is rotated counterclockwise with angular velocity  $\omega$ .

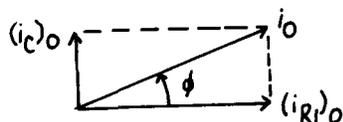


Figure 16

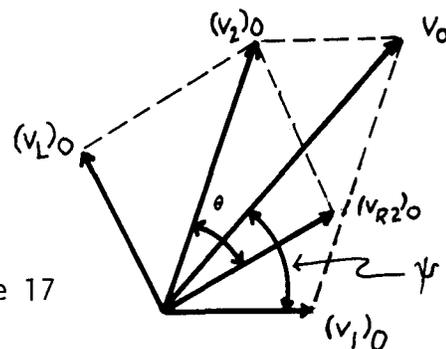


Figure 17

### PROBLEM SET WITH SOLUTIONS

- A(1). Figure 18 represents part of an ac circuit. The potential differences  $V_1$  and  $V_2$  are:  $v_1(t) = v_{10} \sin(\omega t + \delta_1)$ ,  $v_2(t) = v_{20} \sin(\omega t + \delta_2)$ , where  $v_{10} = 24.0$  V,  $v_{20} = 16.0$  V,  $\omega = 3.00 \times 10^3$  rad/s,  $\delta_1 = -30^\circ$ , and  $\delta_2 = 45^\circ$ .
- (a) On a rotating vector diagram, show vectors for  $v_{10}$  and  $v_{20}$  at times  $t = 0$ ;  $t = (\pi/1.80) \times 10^{-4}$  s; and  $t = (\pi/6) \times 10^{-3}$  s.
- (b) Calculate the instantaneous values of  $v_1(t)$ ,  $v_2(t)$ , and the total potential drop  $v_1(t) + v_2(t)$  at the times specified in part (a).

- (c) Determine the maximum value of  $v(t)$ , and its phase with respect to either  $v_{10}$  or  $v_{20}$ .
- (d) Plot  $v_1$ ,  $v_2$ , and  $v$  as functions of time.

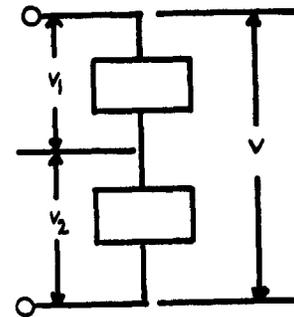


Figure 18

Solution

(a) At  $t = 0$ : see Figure 19(a). At  $t = (\pi/1.80) \times 10^{-4}$ ,

$$\omega t = (3.00 \times 10^3)(\pi/1.80) \times 10^{-4} = \pi/6 \text{ rad} = 30^\circ$$

See Figure 19(b). At  $t = (\pi/6) \times 10^{-3}$ ,

$$\omega t = (3.00 \times 10^3)(\pi/6) \times 10^{-3} = \pi/2 \text{ rad} = 90^\circ.$$

See Figure 19(c).

(b) Component of  $v_0$  along vertical axis  $iv(t)$ . See Figure 20. Instantaneous values add algebraically:

$$t = 0: v_1(0) = v_{10} \sin \delta_1 = 24 \sin(-\pi/6) = -12.0 \text{ V.}$$

$$v_2(0) = v_{20} \sin \delta_2 = 16 \sin(\pi/4) = 16(\sqrt{2}/2) = 11.3 \text{ V.}$$

$$v(0) = v_1(0) + v_2(0) = -0.70 \text{ V.}$$

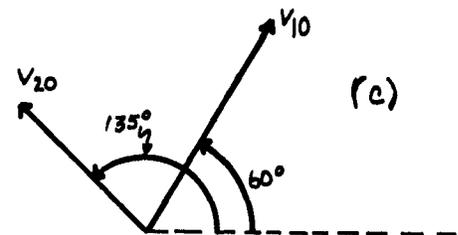
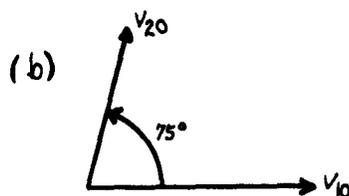
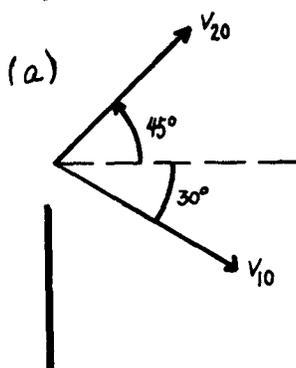


Figure 19

$$t = \pi/1.80 \times 10^{-4}: v_1(\pi/1.80 \times 10^{-4}) = v_{10} \sin(\omega t + \delta_1) = v_{10} \sin(\pi/6 - \pi/6) = 0.$$

This was obvious from the vector diagram. See Figure 21.

Figure 20

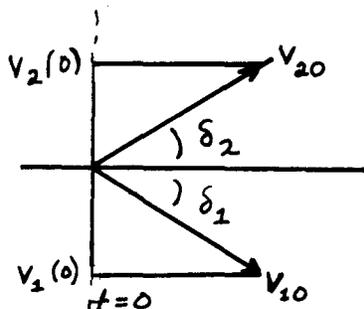
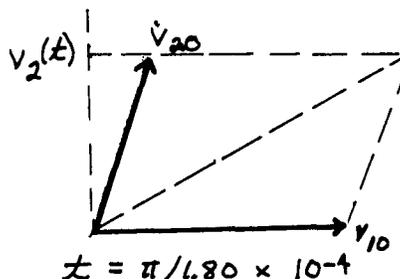


Figure 21



$$v_2(\pi/1.80 \times 10^{-4}) = v_{20} \sin(\omega t + \delta_2) = v_{20} \sin(75^\circ) = 12 \sin(75^\circ) = 11.6 \text{ V},$$

$$v(\pi/1.80 \times 10^{-4}) = 11.6 \text{ V}.$$

$t = \pi/6 \times 10^{-3}$ :  $\omega t = \pi/2$  rad. See Figure 22.

$$v_1(\pi/6 \times 10^{-3}) = v_{10} \sin(\omega t + \delta_1) = v_{10} \sin(\pi/3) = 20.8 \text{ V}.$$

$$v_2(\pi/6 \times 10^{-3}) = v_{20} \sin(\omega t + \delta_2) = v_{20} \sin(3\pi/4) = 11.3 \text{ V}.$$

$$v(\pi/6 \times 10^{-3}) = 32.1 \text{ V}.$$

Figure 22

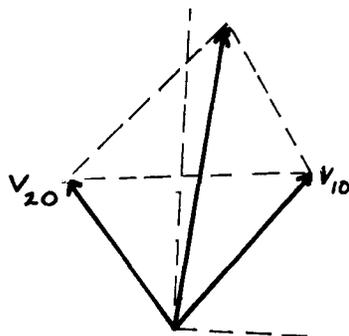
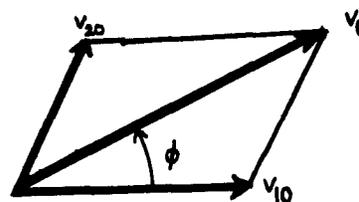


Figure 23



(c) Maximum values add like vectors. We can use any one of the vector diagrams. The easiest to use is Figure 21, since  $v_{10}$  has only one component:

$$v_0^2 = v_{0x}^2 + v_{0y}^2, \quad v_{0y} = v_{20} \sin 75^\circ = 11.6 \text{ V} \quad \text{from part (b).}$$

$$v_{0x} = v_{10} + v_{20} \cos 75^\circ = 28.1 \text{ V}, \quad v_0^2 = 924.17, \quad v_0 = 30.4 \text{ V}.$$

$$\phi = \tan^{-1}(v_{0y}/v_{0x}) = \tan^{-1}(11.6/28.1) = 22.4^\circ,$$

where  $\phi$  is the phase angle by which  $v_0$  leads  $v_{10}$ . See Figure 23.  
 (d) See Figure 24.

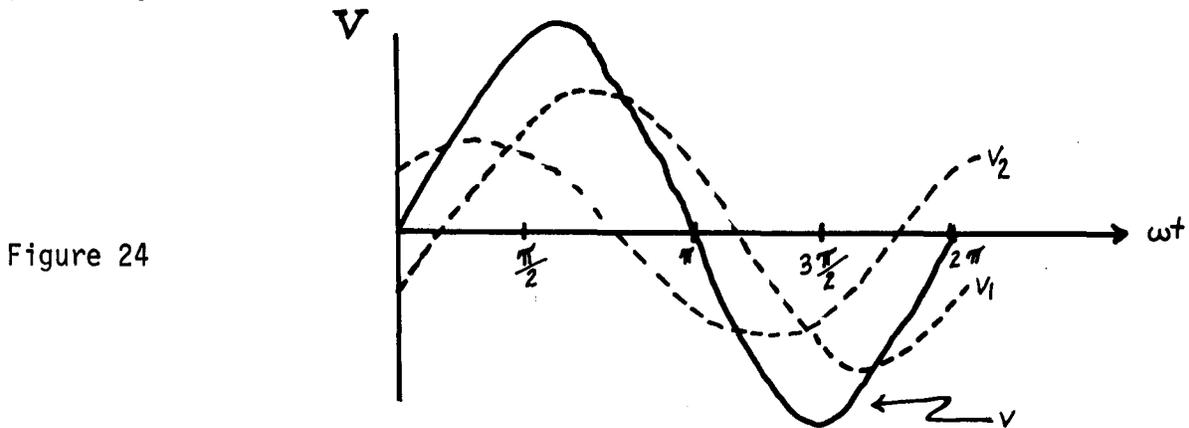


Figure 24

- B(1-3). (a) In the circuit shown in Figure 25, draw a rotating vector diagram showing  $v_{RO}$ ,  $v_{CO}$ ,  $\mathcal{E}_0$ , and  $i_0$ .  
 (b) Calculate the current amplitude  $i_0$ , and  $v_{CO}$  and  $v_{RO}$ , in terms of  $\mathcal{E}_0$ ,  $R$ , and  $C$ .  
 (c) Determine values of  $R$  and  $C$  such that the maximum potential difference across  $C$  is  $(1/2)\mathcal{E}_0$  at a frequency of  $f = \omega/2\pi = 10^6$  Hz.  
 (d) For the values chosen in part (c), determine the phase angle and draw a graph showing  $v_C(t)$  and  $\mathcal{E}(t)$  versus time.  
 (e) Calculate the maximum instantaneous power dissipated in the resistor.  
 (f) Compare the average power dissipated in the resistor with the average power supplied by the source of emf.

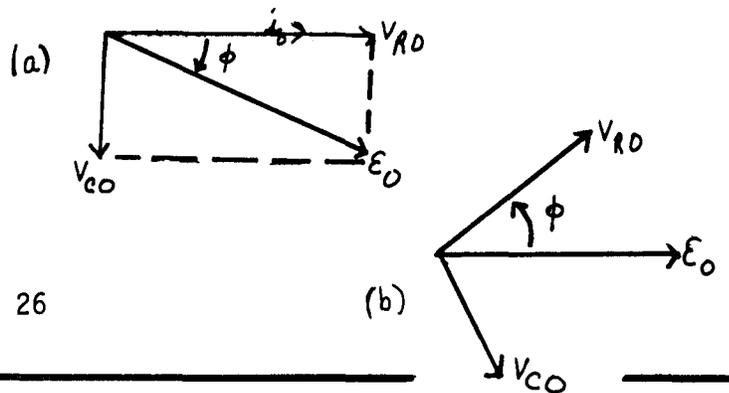
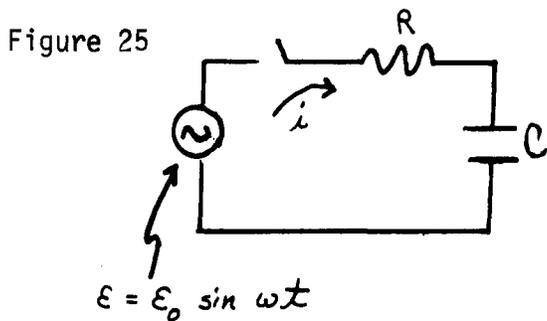


Figure 26

Solution

(a) Since  $i(t)$  is common to both  $R$  and  $C$ , refer the vectors to the current  $i$ .  $v_R = iR$  is in phase with  $i$ , thus  $v_{RO}$  is parallel to  $i$ ;  $v_C$  lags behind the current by  $\pi/2$  rad (why?). See Figure 26. If you wish to rotate the diagram to time,  $t = 0$ , get  $\mathcal{E}_0$  horizontal, as in Figure 26(b).

$$(b) \quad v_{RO} = i_0 R, \quad v_{CO} = i_0 / \omega C,$$

$$\varepsilon_0^2 = i_0^2 [R^2 + (1/\omega C)^2] \quad \text{or} \quad i_0 = \varepsilon_0 / Z,$$

where

$$Z = [R^2 + (1/\omega C)^2]^{1/2}.$$

Does that make sense? Are dimensions right? As the plates of the capacitor come together, the capacitance increases. If they touch, the capacitor is shorted. See Figure 27.

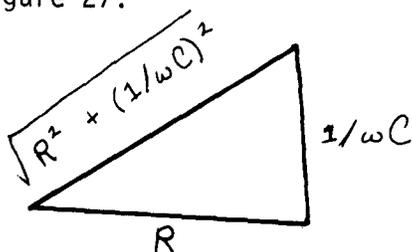


Figure 27

$$\lim_{C \rightarrow \infty} i_0 = \varepsilon_0 / R. \quad \text{That looks OK.}$$

$$\lim_{R \rightarrow 0} i_0 = \varepsilon_0 \omega C = \varepsilon_0 / X_C, \quad \text{which also looks OK. Then } v_{CO} = i_0 X_C = \varepsilon_0 \sin \phi,$$

where  $\phi = \tan^{-1}(v_{CO}/v_{RO}) = 1/R\omega C$  from the diagram. Then

$$v_{CO} = \varepsilon_0 \sin \phi = \varepsilon_0 (1/\omega C) (1/\sqrt{R^2 + (1/\omega C)^2}) = \varepsilon_0 (1/\sqrt{1 + (R\omega C)^2}),$$

where  $v_{CO}$  is a function of frequency. The higher  $\omega$  is, the smaller the potential drop across  $C$ . This circuit is called a "low-pass filter" if  $v_C(t)$  is taken as the output. It "passes" low frequencies.

$$v_R = i_0 R = \varepsilon_0 \cos \phi = \varepsilon_0 \frac{R}{Z} = \varepsilon_0 C \left[ 1 + C \left( \frac{1}{R\omega C} \right)^2 \right]^{1/2}.$$

$v_R$  gets bigger as  $\omega$  increases. If  $v_R$  is used as the output, it is called a "high-pass filter." As a high-pass filter, the circuit is usually written as shown in Figure 28.

$$(c) \quad v_{CO} = \varepsilon_0 / \sqrt{1 + (R\omega C)^2} = \varepsilon_0 / 2, \quad \text{therefore}$$

$$1 + (R\omega C)^2 = 4, \quad R = \sqrt{3}/\omega C = \sqrt{3}/(2\pi \times 10^6) C.$$

If we choose  $C = 1 \text{ nF} = 10^{-9} \text{ F}$ , then

$$R = \sqrt{3}/2\pi \times 10^3 = 276 \Omega.$$

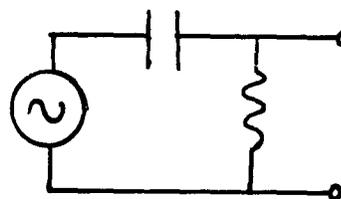


Figure 28

(d) See Figures 29 and 30.

$$\tan \phi = \frac{X_C}{R} = \frac{1}{R\omega C} = \frac{1}{(\sqrt{3}/2\pi \times 10^3)(2\pi \times 10^6)(10^{-9})} = \frac{1}{\sqrt{3}} = 0.577$$

$$\phi = 0.523 \text{ rad} \approx 30^\circ.$$

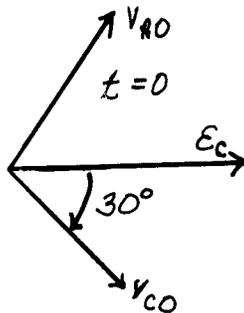


Figure 29

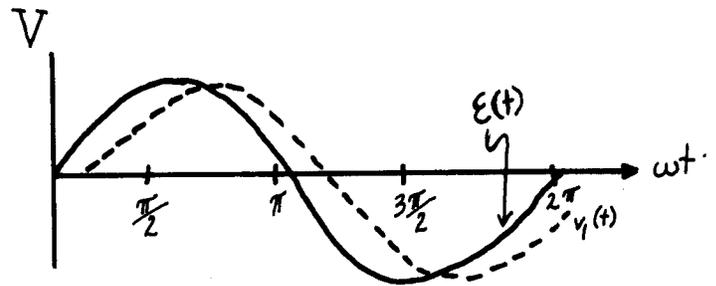


Figure 30

(e)  $p_R(t) = [i(t)]^2 R$ . But  $i(t) = i_0 \sin(\omega t + \theta)$ , where  $\theta = 90 - \phi = 60^\circ$ . The maximum power is

$$[p_R(t)]_{\max} = [i_0^2 R \sin^2(\omega t + \theta)]_{\max} = i_0^2 R = \frac{\epsilon_0^2 R}{R^2 + (1/\omega C)^2}.$$

For  $\omega = 2\pi \times 10^6$ ,  $R = 276$ ,  $C = 10^{-9}$ , and

$$[p_R(t)]_{\max} = \epsilon_0^2 (2.72 \times 10^{-3}).$$

C(1, 2, 3, 4). In the circuit shown in Figure 31,

$$R = 3.00 \times 10^4 \Omega, C = 10^{-8} \text{ F},$$

$$L = 2.00 \text{ H}, \omega = 10^4 \text{ rad/s}, \text{ and}$$

$$\epsilon_0 = 25.0 \text{ V}.$$

- Calculate  $i(t)$ .
- Calculate the average power supplied to the circuit.
- Plot  $\epsilon(t)$ ,  $i(t)$ , and  $p(t)$  versus  $\omega t$ .

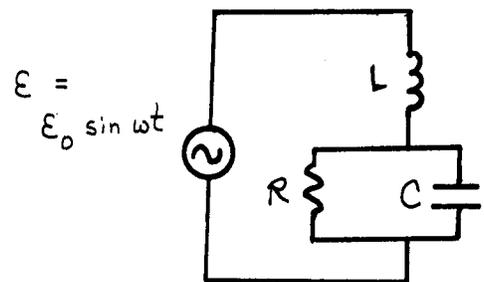


Figure 31

Solution

Begin by dividing the circuit into sections in which all elements have a potential drop or a current in common. See Figure 32. The RC combination has a common potential difference. Call it  $v_1$ .

$$v_1 = i_R R, \tag{1}$$

$$v_{10} = i_{C0} (1/\omega C) \tag{2}$$

where  $i_R$  is in phase with  $v_1$ , and  $i_C$  leads  $v_1$  by  $\pi/2$  rad (why?). The total current  $i$  is the sum of  $i_R$  and  $i_C$ . The amplitude  $i_0$  is the vector sum of  $i_{R0}$  and  $i_{C0}$ . (Of course, so far we do not know any of the currents.) The angle  $\phi$  in Figure 33 is given by

$$\tan \phi = i_{C0}/i_{R0} = v_{i0} \omega C R / v_{i0} = \omega R C = 10^4 (3.00 \times 10^4) (10^{-8}) = 3.00; \quad \phi = 71^\circ 34'. \tag{3}$$

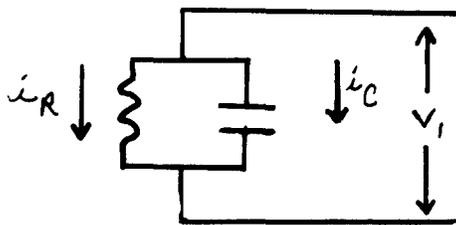


Figure 32

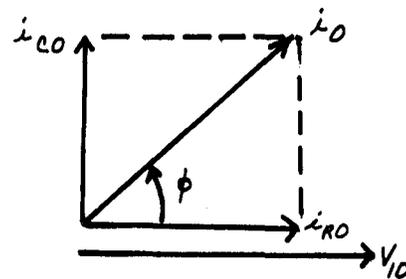


Figure 33

Now we have to hook up the rest of the circuit. The top and bottom elements have the current  $i_0$  in common, therefore we draw a vector diagram, referring  $v_2$  and  $v_1$  to  $i_0$ . See Figures 34 and 35.  $v_2$  leads  $i$  by  $\pi/2$  rad (why?). The amplitude is

$$v_{20} = i_0 X_L = i_0 \omega L. \tag{4}$$

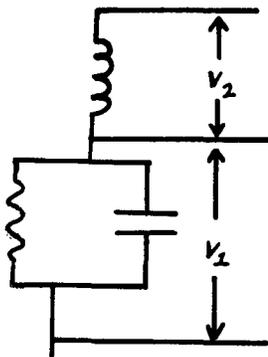


Figure 34

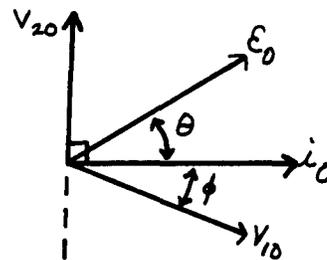


Figure 35

We already know from Eq. (3) that  $v_1$  lags behind  $i_0$  by a phase angle

$$\phi = \tan^{-1} \omega RC = 71^\circ 34'.$$

The vector sum of  $v_{10}$  and  $v_{20}$  is the amplitude of emf  $\mathcal{E}_0$ . We are given that  $\mathcal{E}_0 = 25.0$  V. The angle  $\theta$  is

$$\tan \theta = (v_{20} - v_{10} \sin \phi) / (v_{10} \cos \phi). \quad (5)$$

Now let us take stock. We know  $\mathcal{E}_0$ ,  $R$ ,  $L$ ,  $C$ , and  $\phi$ . We do not know  $i_0$ ,  $v_{10}$ ,  $v_{20}$ , and  $\theta$ . So let us start with  $\mathcal{E}_0$ , and work our way back. From the vector diagram,

$$\mathcal{E}_0^2 = \mathcal{E}_{0x}^2 + \mathcal{E}_{0y}^2 = (v_{10} \cos \phi)^2 + (v_{20} - v_{10} \sin \phi)^2. \quad (6)$$

If we can get  $v_{10}$  and  $v_{20}$  in terms of  $i_0$ , we can solve for  $i_0$ , and also get  $\theta$ :

$$v_{20} = i_0 \omega L, \quad v_{10} = i_0 R = i_0 (\cos \phi) R.$$

The last step comes from the first vector diagram, Figure 33. Equation (6) now is

$$\mathcal{E}_0^2 = i_0^2 (R)^2 \cos^2 \phi + i_0^2 [\omega L - R \cos \phi \sin \phi]^2. \quad (6')$$

$$i_0^2 = \mathcal{E}_0^2 / [(R \cos^2 \phi)^2 + (\omega L - R \cos \phi \sin \phi)^2] = (25)^2 / [(3 \times 10^4 (\cos 71^\circ 34'))^2]^2 + [2 \times 10^4 - 3 \times 10^4 \cos(71^\circ 34') \sin(71^\circ 34')]^2 = 4.81 \times 10^{-6}$$

$$i_0 = 2.19 \times 10^{-3} \text{ A}. \quad (7)$$

We know, from Eq. (4), that  $v_{20} = i_0 \omega L = 43.9$  V and  $v_{10} = i_0 R \cos \phi = 20.8$  V.

Equation (5) then gives us  $\theta$ :

$$\tan \theta = 3.67; \quad \theta = 74.8^\circ.$$

Finally,

$$i = i_0 \sin(\omega t + \theta) = (2.19 \times 10^{-3}) \sin(\omega t + 74.8^\circ) \\ = (2.19 \times 10^{-3}) \sin(\omega t + 1.31) \quad \text{if the angle is in radians.}$$

$$(b) P_{av} = \mathcal{E}_{rms} i_{rms} \cos \theta = (1/2) \mathcal{E}_0 i_0 \cos \theta = 7.20 \times 10^{-3} \text{ W.}$$

(c) See Figure 36.

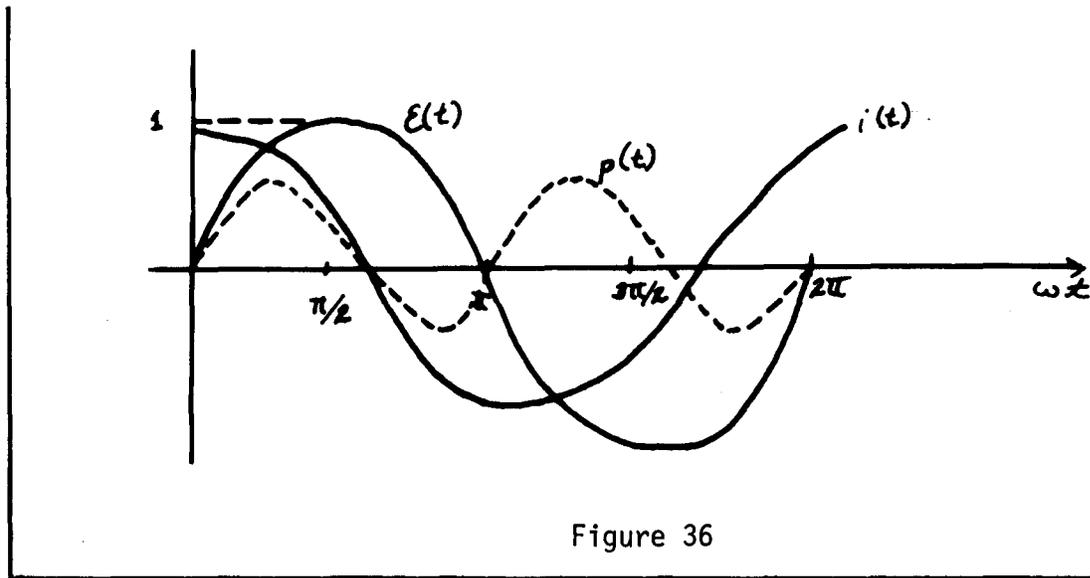


Figure 36

Problems

- D(1). In Figure 37 are sketched two potential differences as functions of time.
- State whether  $v_2$  leads or lags  $v_1$ .
  - Draw a vector diagram for  $v_1$  and  $v_2$  at an instant of time that you choose. State the instant you choose.
  - In terms of  $\omega$  (assumed known) and the other given constants, write algebraic expressions for  $v_1(t)$  and  $v_2(t)$ .

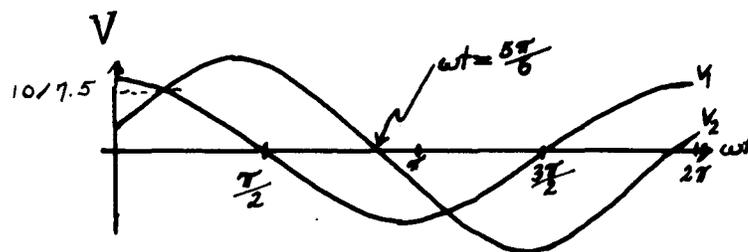


Figure 37

- E(1-4). (a) In the circuit shown in Figure 38, draw a rotating vector diagram showing  $v_{RO}$ ,  $v_{LO}$ ,  $\epsilon_0$ , and  $i_0$ .
- (b) Calculate the current amplitude  $i_0$ , and  $v_{RO}$  and  $v_{LO}$ , in terms of  $\epsilon_0$ ,  $R$ ,  $L$ , and  $\omega$ .
- (c) Determine values of  $R$  and  $L$  such that the maximum potential difference across  $L$  is  $(1/2)\epsilon_0$  at a frequency of  $f = \omega/2\pi = 2.00 \times 10^4$  Hz.

(d) For the values chosen in part (c), determine the phase angle and draw a graph showing  $v_1(t)$  and  $\mathcal{E}(t)$  versus time.

(e) Calculate the maximum instantaneous power dissipated in the resistor.

(f) Compare the average power dissipated in the resistor with the average power supplied by the source of emf.

F(1-4). In the circuit shown in Figure 39,  $i_0 = 10.0$  mA,  $f = 10^3$  Hz,  $L_1 = 10.0$  mH,  $L_2 = 250$  mH,  $C = 0.100$   $\mu$ F, and  $R = 1.00$  k $\Omega$ .

(a) Draw phase diagrams for the  $L_1$ -C segment and the  $L_2$ -R segment of the circuit.

(b) Calculate  $v_{L_1 0}$ ,  $v_{C 0}$ ,  $v_{L_2 0}$ ,  $v_{R 0}$ .

(c) Calculate  $\mathcal{E}_0$ .

(d) At  $\omega t = 2\pi$ , calculate the energy stored in  $L_1$ , C, and  $L_2$ .

(e) At  $\omega t = 3\pi/2$ , calculate the energy stored in  $L_1$ , C, and  $L_2$ .

$$\mathcal{E} = \mathcal{E}_0 \sin \omega t$$

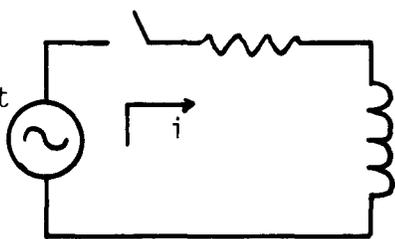


Figure 38

$$i = i_0 \sin \omega t$$

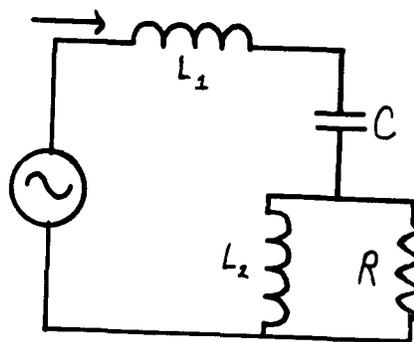


Figure 39

### Solutions

D(1). (a)  $v_2$  lags  $v_1$  by  $\pi/3$ .

(b) See Figure 40, at  $t = 0$ .

(c)  $v_1(t) = 10.0 \sin(\omega t = \pi/2) = 10.0 \cos \omega t$ .

$$v_2(t) = 7.5 \sin(\omega t + \pi/2 - \pi/3) = 7.5 \cos(\omega t - \pi/3).$$

E(1-4). (a). See Figure 41.

$$(b) i_0 = \mathcal{E}_0 / \sqrt{R^2 + (\omega L)^2}. \quad v_{R0} = \mathcal{E}_0 R / \sqrt{R^2 + (\omega L)^2}. \quad v_{L0} = \mathcal{E}_0 \omega L / \sqrt{R^2 + (\omega L)^2}.$$

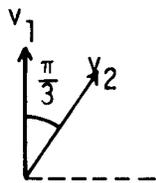


Figure 40

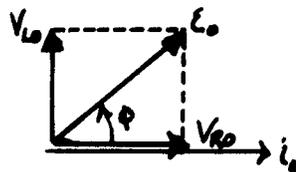


Figure 41

(c)  $R = 21.8 \times 10^4 \Omega$ ; e.g., if  $L = 0.200 \text{ H}$ ,  $R = 43.5 \times 10^3 \Omega$ .

(d)  $\phi = 30^\circ$ . See Figure 42.

(e)  $(P_r)_{\max} = i_0^2 R = \epsilon_0^2 R / [R^2 + (\omega L)^2]$ .

(f)  $(P_R)_{\text{av}} = (1/2) i_0^2 R = \epsilon_0^2 R / 2 [R^2 + (\omega L)^2]$ .

$(P_\epsilon)_{\text{av}} = \epsilon_0^2 R / 2 [R^2 + (\omega L)^2]$ .

F(1-4). (a) See Figure 43.  $\phi = 32.5^\circ$

(b)  $v_{L10} = 0.628 \text{ V}$ ;  $v_{C0} = 15.9 \text{ V}$ ;  $v_{L20} = 8.44 \text{ V}$ ;  $v_{R0} = 8.44 \text{ V}$ .

(c)  $\epsilon_0 = 12.9 \text{ V}$ .

(d)  $E(L_1) = 0$ ;  $E(C) = 1.26 \times 10^{-5} \text{ J}$ ;  $E(L_2) = 2.57 \times 10^{-6} \text{ J}$ .

(e)  $E(L_1) = 0.50 \times 10^{-6} \text{ J}$ ;  $E(C) = 0$ ;  $E(L_2) = 1.04 \times 10^{-6} \text{ J}$ .

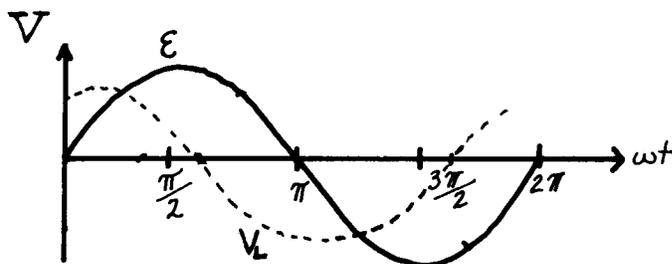


Figure 42

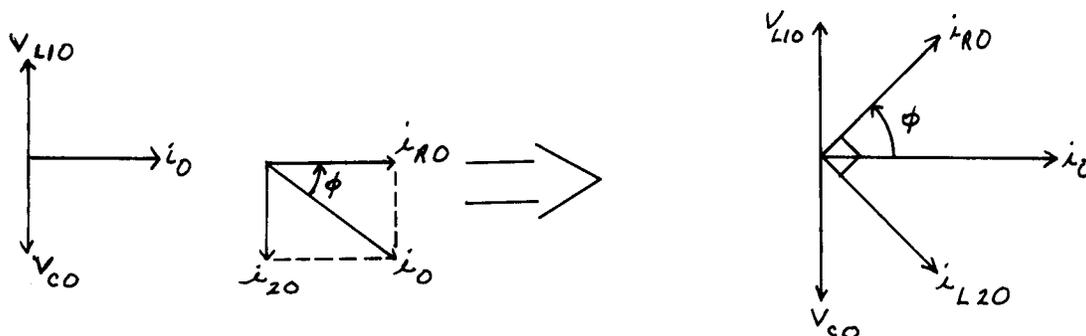


Figure 43

Practice Test Answers

1. See Figure 45.
2.  $i_R(t) = 15.0 \times 10^{-3} \sin(2.00 \times 10^4 t)$ ;  $i_L(t) = 0.300 \sin(2.00 \times 10^4 t - \pi/2)$ .  
 $i_C(t) = 1.20 \sin(2.00 \times 10^4 t + \pi/2)$ ;  $i(t) \approx 0.90 \sin(2.00 \times 10^4 t + 89^\circ)$ .
3.  $(P_R)_{av} = 0.225 \text{ W}$ ;  $(P_L)_{av} = 0$ ;  $(P_C)_{av} = 0$ ;  $(P_\mathcal{E})_{av} = 0.225 \text{ W}$ .
4. Source supplies energy that is dissipated in the resistor, as heat. Some of the remaining energy oscillates back and forth between the electric field of the capacitor and the magnetic field of the inductor. The rest is alternately pumped into the source and is supplied by the source.
5. At the resonance frequency,  $i_0 = i_R = \mathcal{E}/R$ . The phase angle is zero. The total energy in the LC part of the circuit remains constant, simply oscillating back and forth between the capacitor and the inductor.

Figure 45

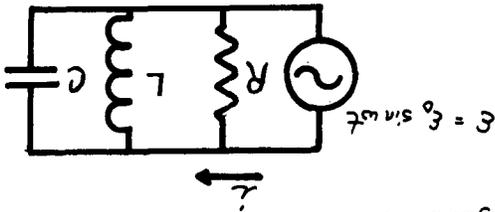
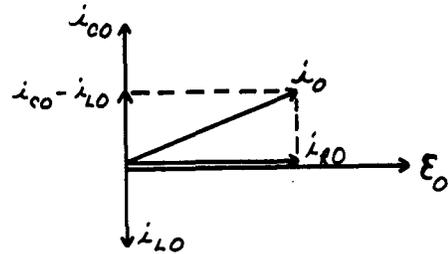


Figure 44

$$\omega_0 = 1/\sqrt{LC} = 1.00 \times 10^4 \text{ rad/s?}$$

1. For the circuit shown in Figure 44, draw a vector diagram illustrating the relative phase relations among the currents,  $i_R$ ,  $i_L$ ,  $i_C$ , and  $i_0$ .
2. Calculate  $i_R$ ,  $i_L$ ,  $i_C$ , and  $i$  as functions of time. Let  $\mathcal{E}_0 = 30.0 \text{ V}$ ,  $R = 2.00 \text{ k}\Omega$ ,  $L = 5.0 \text{ mH}$ ,  $C = 2.00 \mu\text{F}$ ,  $\omega = 2.00 \times 10^4 \text{ rad/s}$ .
3. Calculate the average power input to the resistor, the inductor, and the capacitor. Calculate the average power supplied by the source of emf.
4. Describe qualitatively the energy flow in the circuit.
5. How would your answers change if the angular frequency were changed to

PRACTICE TEST

ALTERNATING-CURRENT CIRCUITS

Date \_\_\_\_\_

Mastery Test Form A

pass recycle

1      2      3      4

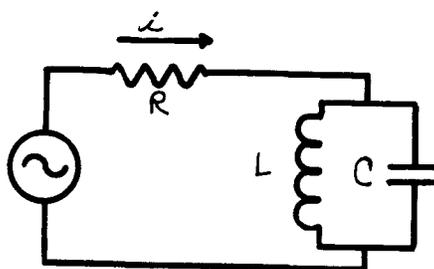
Name \_\_\_\_\_

Tutor \_\_\_\_\_

1. In the circuit shown in Figure 1, calculate the currents as a function of time. Include in your solution a phase diagram showing all relevant phases. Assume that  $\mathcal{E}_0$ ,  $\omega$ ,  $R$ ,  $L$ , and  $C$  are known. Plot  $i(t)$  and  $\mathcal{E}(t)$  as functions of  $\omega t$ .
2. Given that  $\omega$  is  $10^6$  rad/s,  $\mathcal{E}_0 = 15.0$  V,  $R = 10.0$  k $\Omega$ , determine values of  $L$  and  $C$  that will satisfy the resonance condition. What happens to the amplitude of  $i$  under that condition?
3. With the values of  $L$  and  $C$  determined in Problem 2, calculate the maximum energy in the magnetic field of the inductor, and the maximum energy in the electric field of the capacitor. Describe how these energies vary with time.
4. Calculate the average power delivered to the circuit under the conditions of Problem 2.

Figure 1

$$\mathcal{E} = \mathcal{E}_0 \sin \omega t$$



ALTERNATING-CURRENT CIRCUITS

Date \_\_\_\_\_

Mastery Test Form B

pass recycle

1 2 3 4

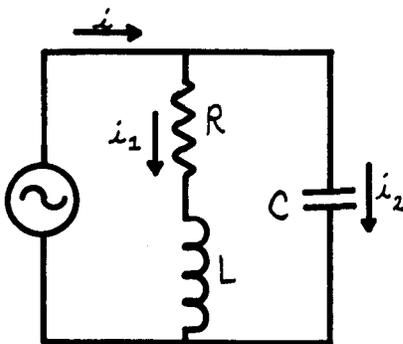
Name \_\_\_\_\_

Tutor \_\_\_\_\_

The following problems apply to the circuit shown in Figure 1. Assume  $\mathcal{E}_0$ ,  $\omega$ ,  $R$ ,  $L$ , and  $C$  are known.

1. Divide the circuit into sections that have a common current or potential. Draw a vector diagram for each section.
2. What is the phase angle between the emf  $\mathcal{E}$  and the potential drop across the resistor  $V_R$ ? Does  $V_R$  lead or lag  $\mathcal{E}$ ? What is the phase angle between the current  $i$  and the emf  $\mathcal{E}$ ? You may express your answer in terms of the (unknown) currents  $i_1$  and  $i_2$  if you wish.
3. Calculate the average power dissipated in the resistor, in terms of  $\mathcal{E}_0$ ,  $R$ ,  $L$ ,  $C$ , and  $\omega$ .
4. Describe the energy flow in the circuit, qualitatively, and as quantitatively as you can.

Figure 1  
 $\mathcal{E} = \mathcal{E}_0 \sin \omega t$



## ALTERNATING-CURRENT CIRCUITS

Date \_\_\_\_\_

Mastery Test Form C

pass recycle

1 2 3 4

Name \_\_\_\_\_

Tutor \_\_\_\_\_

In the circuit shown in Figure 1, draw a vector diagram showing all currents and potential differences. Calculate the current  $i$  as a function of time. Let  $\mathcal{E}_0 = 1.00$  V,  $\omega = 377$  rad/s (or  $f = 60$  Hz),  $R = 500$   $\Omega$ ,  $L = 4.0$  H, and  $C = 2.00$   $\mu$ F.

2. Calculate the power delivered by the source as a function of time. Calculate the power into the field of the capacitor as a function of time.
3. Calculate the average power delivered to the circuit. What would you do to  $L$ , or  $C$ , or the product  $LC$ , to maximize the power delivered?
4. Describe the energy flow as a function of time.

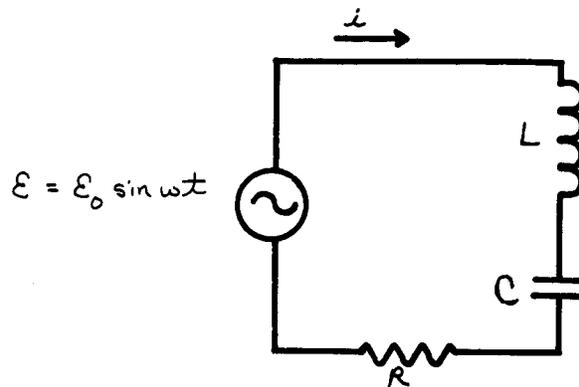


Figure 1

MASTERY TEST GRADING KEY - Form A

1. What To Look For: Diagram may be done in parts, but all phases should be properly indicated. Figure 50 is plotted for positive  $\phi$ . Be sure student knows the difference between positive and negative phase.

Solution: See Figure 49.  $v_{10} = i_{L0}\omega L = i_{C0}/\omega C$ . (A1)

$$v_{R0} = i_{0}R. \quad (A2)$$

$$i_{0} = i_{C0} - i_{L0}. \quad (A3)$$

From Eq. (A1),  $i_{C0} = i_{L0}\omega^2 LC$ . From Eq. (A3),

$$i_{0} = i_{L0}(\omega^2 LC - 1); \quad (A4)$$

$$\phi = \tan^{-1}(v_{10}/v_{R0}) = \tan^{-1}(i_{L0}\omega L/i_{0}R) = \tan^{-1}[\omega L/R(\omega^2 LC - 1)]. \quad (A5)$$

From Eqs. (A4) and (A1),

$$i_{0} = i_{L0}(\omega^2 LC - 1) = (v_{10}/\omega L)(\omega^2 LC - 1) = (\epsilon_0 \sin \phi/\omega L)(\omega^2 LC - 1); \quad (A6)$$

$i = i_{0} \sin(\omega t + \phi)$ . See Figure 50.

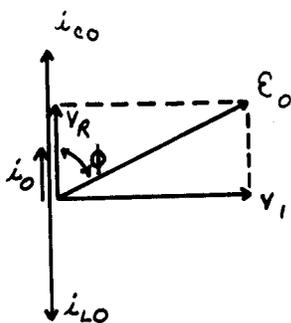


Figure 49

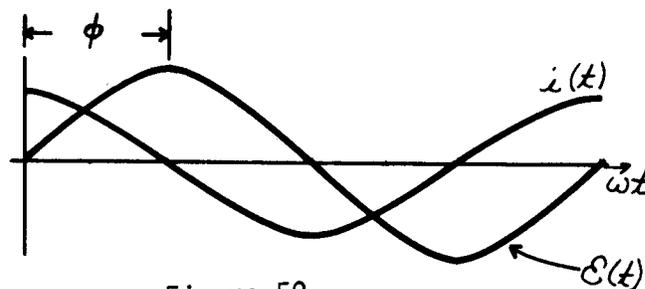


Figure 50

2. What To Look For: Any combination of L and C such that  $LC = 10^{-12}$  is OK.

Solution: Resonance occurs when  $\omega = 1/\sqrt{LC}$ , therefore

$$LC = 1/\omega^2 = 10^{-12} \text{ s/rad.}$$

Let  $L = 1.00 \text{ mH}$ ,  $C = 1 \text{ nF}$ , for example,

$$i_{0} = \epsilon_0 \sin \phi(\omega C - 1/\omega L) = \epsilon_0(\omega C - 1/\omega L)\{\omega L/[R^2(\omega^2 LC - 1)^2 + (\omega L)^2]^{1/2}\}.$$

If  $\omega = 1/\sqrt{LC}$ ,  $i_0 = \epsilon_0 \times 0 = 0$ .

3. What To Look For: There is no current from the source at resonance. Know energy of an inductor and a capacitor. The maximum energies in L and C are the same at resonance.

Solution:  $(E_L)_{\max} = (1/2)Li_{L0}^2$ . Since  $i_0$  is zero, the potential difference across the inductor and capacitor is the potential difference across the source:

$$\epsilon_0 = v_{10} = i_{L0}\omega L, \quad i_{L0} = \epsilon_0/\omega L;$$

$$(E_V)_{\max} = (1/2)Li_{L0}^2 = \epsilon_0^2/2\omega^2 L, \quad \omega^2 = 1/LC.$$

Therefore,

$$(E_L)_{\max} = (1/2)\epsilon_0^2 C = (1/2)(15)^2(10^{-9}) = 1.125 \times 10^{-7} \text{ J},$$

$$(E_C)_{\max} = (1/2)Cv_{10}^2 = (1/2)C\epsilon_0^2 = 1.125 \times 10^{-7} \text{ J}.$$

Since  $\phi = \pi/2$ ,  $E_L$  is a maximum when  $E_C$  is zero, and vice versa. Energy leaves the collapsing magnetic field, and goes into building the electric field, etc.

4. Solution: Since the current  $i_0$  is zero, the power delivered to circuit:

$$P_{av} = (1/2)\epsilon_0 i_0 \cos \phi = 0.$$

Current oscillates in the LC branch, but no power is dissipated, and none is delivered by the source.

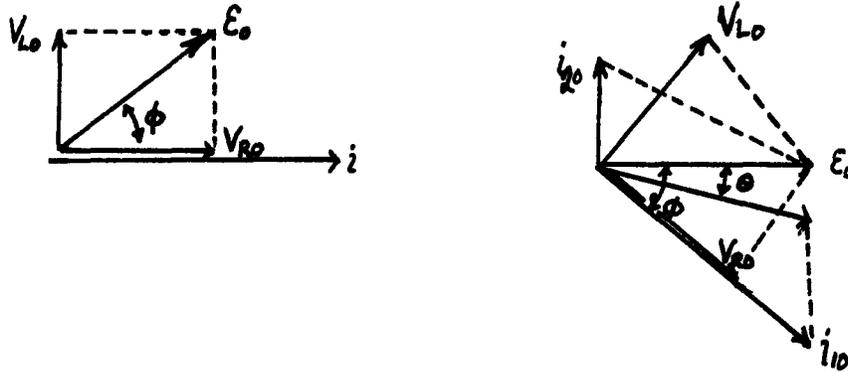
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MASTERY TEST GRADING KEY - Form B

1. What To Look For: Correct phase relations. Ask student about omitted quantities.

Solution: See Figure 51.

Figure 51



2. What To Look For: Any of the relations below are OK. Algebra not as important as beginning with correct relationships.

Solution:  $\phi = \tan^{-1}(v_{L0}/v_{R0}) = \tan^{-1}[i_{10}\omega L/i_{10}R] = \tan^{-1}(\omega L/R)$ .  $v_R$  lags  $\epsilon$ . The angle between  $i$  and  $\epsilon$  is  $\theta$ .

$$\theta = \tan^{-1} \left[ \frac{(i_{20})_y + (i_{10})_y}{(i_{10})_x} \right] = \tan^{-1} \left[ \frac{i_{20} - i_{10} \sin \phi}{i_{10} \cos \phi} \right]$$

$$= \tan^{-1} \left[ \frac{\epsilon_0 \omega C + (\epsilon_0/R) \cos \phi \sin \phi}{(\epsilon_0/R) \cos^2 \phi} \right] = \tan^{-1} \left[ \frac{\omega(CR^2 - \omega^2 L^2) - \omega L}{R} \right].$$

3. What To Look For: Knowledge that rms voltages and currents are  $1/\sqrt{2}$  times amplitudes. Look for correct place to start in  $P_R$  calculation.

Solution:  $i_{10} = v_{R0}/R = (\epsilon_0 \cos \phi)/R = \epsilon_0/(R^2 + \omega^2 L^2)^{1/2}$ .

$$(P_R)_{av} = (v_R)_{rms} (i_1)_{rms} \cos \phi = (1/2)v_{R0} i_{10} \cos \phi = \epsilon_0^2 R^2 / 2(R^2 + \omega^2 L^2)^{3/2}.$$

4. Solution: Energy will oscillate back and forth between magnetic and electric fields. Energy dissipated by the resistor will be supplied by the source. In general, some energy will alternately enter and leave the source from L or C. Quantitatively,

$$\begin{aligned}
 P_L(t) &= i_1 v_L = i_1^2 \omega L \sin(\omega t + \phi) \sin(\omega t + \phi - \frac{\pi}{2}) \\
 &= - \frac{\epsilon_0^2 R^2 \omega L}{(R^2 + \omega^2 L^2)} \sin(\omega t + \phi) \cos(\omega t + \phi) = - \frac{\epsilon_0^2 R^2 \omega L}{2(R^2 + \omega^2 L^2)} \sin[2(\omega t + \phi)].
 \end{aligned}$$

$$P_C(t) = v_C i_2 = \epsilon_0^2 \omega C \sin \omega t \sin(\omega t + \pi/2) = (1/2) \epsilon_0^2 \omega C \sin[2(\omega t)].$$

$$P_R(t) = v_R i_1 = i_1^2 R \sin^2(\omega t + \phi).$$

To plot these would be very messy. It is important that  $P_L$  and  $P_C$  are out of phase, and there is an oscillation in energy.

---

MASTERY TEST GRADING KEY - Form C

1. What To Look For: Correct signs of all phases. All angles correct. Arithmetic not so important. Proper setup is important.

Solution: See Figure 52.

$$i = \frac{\mathcal{E}}{Z} = \frac{\mathcal{E}_0 \sin(\omega t + \phi)}{[R^2 + (\omega L - 1/\omega C)^2]^{1/2}}; \quad \phi = \tan^{-1}\left(\frac{\omega L - 1/\omega C}{R}\right) = 20.0^\circ.$$

$$i = \frac{1.00 \sin(\omega t - 20^\circ)}{\{(500)^2 + [377(4) - 1/377(2 \times 10^{-6})]^2\}^{1/2}} = 1.88 \times 10^{-3} \sin(\omega t - 20^\circ).$$

2. Solution:  $P_{\mathcal{E}}(t) = \mathcal{E}(t)i(t) = \mathcal{E}_0 i_0 \sin \omega t \sin(\omega t - \phi) = 1.88 \times 10^{-3} \sin \omega t \sin(\omega t - \phi) \text{ W}.$

$$P_C(t) = v_C(t)i(t) = (i_0^2/\omega C) \sin(\omega t - \pi/2 - \phi) \sin(\omega t - \phi) = -(i_0^2/\omega C) \cos(\omega t - \phi) \sin(\omega t - \phi) \\ = -4.69 \times 10^{-3} \cos(\omega t - \phi) \sin(\omega t - \phi) \text{ W}$$

$$(P_{\mathcal{E}})_{\text{av}} = (1/2)\mathcal{E}_0 i_0 \cos \phi = (1/2)(1.88 \times 10^{-3}) \cos 20^\circ = 8.8 \times 10^{-4} \text{ W}$$

3. Maximize  $i_C$  and  $\cos \phi$  by setting  $\sqrt{LC} = 1/\omega$ . Maximum energies in the inductor and capacitor are not the same, except in resonance. In this case,

$$(E_L)_{\text{max}} = (1/2)Li_0^2 = 7.07 \times 10^{-6} \text{ J}. \quad (E_C)_{\text{max}} = (1/2)Cv_{C0}^2 = (1/2)C(i_0^2/\omega^2 C^2) = 6.22 \times 10^{-6} \text{ J}$$

The  $0.151 \text{ J}$  is transferred from the electric field of the capacitor to the magnetic field of the inductor. Energy is dissipated in the resistor, with the maximum value of power dissipation being  $(P_R)_{\text{max}} = (1/2)i_0^2 R = 8.84 \times 10^{-4} \text{ W}$  an average value of  $6.25 \times 10^{-4} \text{ W}$ . Some energy ( $0.85 \times 10^{-6} \text{ J max}$ ) is alternatively put back into the source, and delivered by it to the electric field of the capacitor.

Figure 52

