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Class Notes for Math 901/902: Abstract Algebra, Instructor Tom Marley

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Topics include: Free groups and presentations; Automorphism groups; Semidirect products; Classification of groups of small order; Normal series: composition, derived, and solvable series; Algebraic field extensions, splitting fields, algebraic closures; Separable algebraic extensions, the Primitive Element Theorem; Inseparability, purely inseparable extensions; Finite fields; Cyclotomic field extensions; Galois theory; Norm and trace maps of an algebraic field extension; Solvability by radicals, Galois' theorem; Transcendence degree; Rings and modules: Examples and basic properties; Exact sequences, split short exact sequences; Free modules, projective modules; Localization of (commutative) rings and modules; The prime spectrum of a ring; Nakayama's lemma; Basic category theory; The Hom functors; Tensor products, adjointness; Left/right Noetherian and Artinian modules; Composition series, the Jordan-Holder Theorem; Semisimple rings; The Artin-Wedderburn Theorem; The Density Theorem; The Jacobson radical; Artinian rings; von Neumann regular rings; Wedderburn's theorem on finite division rings; Group representations, character theory; Integral ring extensions; Burnside's $p^a q^b$ Theorem; Injective modules.

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1 Chapter 1: Groups

1.1 Free Groups and Presentations

Definition 1.1. *Let S be a set. Then a* ***free group*** *on S is a group F together with a map $i : S \to F$, usually referred to as (F, i), with the following "universal" property: If G is any group and $j : S \to G$ is any map, then $\exists!$ group homomorphism $f : F \to G$ such that $fi = j$, i.e., the following diagram commutes:*

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[d, "j"'] & F \arrow[dl, dashed, "\exists! f"] \\ G & \end{tikzcd}$$

Theorem 1.2. *Let S be any set. Then a free group on S exists.*

Proof. See Lang. □

Proposition 1.3. *Let S and T be sets of the same cardinality. Then any free group on S is isomorphic to any free group on T.*

Proof. Let $\ell : S \to T$ be a bijection. Let (F, i) and (G, j) be free groups on S and T, respectively.

$$\begin{array}{ccc} S & \xrightarrow{i} & F \\ \ell\downarrow & & \downarrow \\ T & \xrightarrow{j} & G \\ \ell^{-1}\downarrow & & \downarrow \\ S & \xrightarrow{i} & F \end{array}$$

Then, by the universal property $\exists! f : F \to G$ and $\exists! g : G \to F$. Compacting the above commutative diagram, we see

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[d, "i=i\ell^{-1}\ell"'] & F \arrow[dl, dashed, "gf"] \\ F & \end{tikzcd}$$

by the uniqueness of the universal property, as we have the homomorphism $gf : F \to F$ and the identity homomorphism $1_F : F \to F$, that $gf = 1_F$. Similarly, by swapping the S and T in the diagrams above, we see $fg = 1_G$. Thus f and g are bijective homomorphisms and thus f is an isomorphism. □

Corollary 1.4. *Let S be a set and (F_1, i_1) and (F_2, i_2) free groups on S. Then $\exists!$ isomorphism $f : F_1 \to F_2$ such that $fi_1 = i_2$.*

Thus we can now talk about the unique (up to isomorphism) free group on a set.

Proposition 1.5. *Let S be a set and (F, i) the free group on S. Then i is injective.*

Proof. Suppose not, that is, $i(x) = i(y)$ for $x \neq y \in S$. Consider the homomorphism $j : S \to \mathbb{Z}_2$ defined by $s \mapsto 0$ for $s \neq x$ and $x \mapsto 1$. Then we have the commutative diagram

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[d, "j"'] & F \arrow[dl, "f"] \\ \mathbb{Z}_2 & \end{tikzcd}$$

where f is the unique homomorphism given by the universal property of free groups. Now

$$0 = j(y) = fi(y) = fi(x) = j(x) = 1,$$

which is clearly a contradiction. $\square$

Thus, we can now identify S with its image $i(S) \subseteq F$. For simplicity we will simply say $S \subseteq F$. Also, we will now simply say $F(S)$ is the free group for S.

Proposition 1.6. *The set S generates $F(S)$.*

Proof. Let F' be the subgroup of $F = F(S)$ generated by S.

$$\begin{tikzcd} S \arrow[r, "i_1"] \arrow[d, "i_2"'] & F \arrow[dl, dashed, "\exists f"] \\ F' \arrow[d, "j"'] & \\ F & \end{tikzcd} \qquad \Rightarrow \qquad \begin{tikzcd} S \arrow[r, "i_1"] \arrow[d, "ji_2"'] & F \arrow[dl, "jf"] \\ F & \end{tikzcd}$$

By the uniqueness of the universal property, $jf = 1_F$. Thus jf is a surjection, which implies j is surjective. Thus $F' = F$. $\square$

If $|S| = n$, call $F(S)$ the free group on n generators. So $F(S) = \{s_1^{e_1} \cdots s_k^{e_k} | s_i \in S, e_i = \pm 1\}$. Note that since homomorphisms preserve order and commutativity we can not have any conditions like $s^n = 1$ or $s_1 s_2 = s_2 s_1$ as these conditions do not hold in all groups. Thus there are no relations on the elements of S, which is why we say $F(S)$ is the *free* group. [For example, say $s^n = 1$ and consider $j : S \to \mathbb{Z}$ where $j(s) = 2$. Then, there exists a homomorphism $f : F(S) \to \mathbb{Z}$. Then $2 = j(s) = f(i(s)) = f(s)$. If $s^n = 1$, then $2^n = 1$, a contradiction].

Example. What is the free group on one element, i.e., $S = \{x\}$?

Since S generates $F(S)$, we know $F(S) =< x >$. By above, x does not have finite order. Thus $F(S)$ is infinite cyclic, which says $F(S) \cong \mathbb{Z}$. Note: This is the only abelian free group.

Definition 1.7. *Let F be the free group on a set S and R any subset of F. Let N be the intersection of all normal subgroups of F containing R (i.e., N is the smallest normal subgroup containing R). Then F/N is called the group generated by S with relations $R = 1$. Write $F/N =< S|R = 1 >$ and call it a* ***presentation*** *for F/N.*

Definition 1.8. *Say a group G has the presentation $< S|R = 1 >$ if $G \cong F(S)/N$ where N is the smallest normal subgroup of $F(S)$ containing R. Here G is defined by the generators S and relations R.*

Example. What group G is defined by the presentation $< x, y|x^2 = 1, y^3 = 1, xyxy = 1 >$?

Here, $G =< x, y >$ where $x^2 = 1, y^3 = 1, yx = xy^2$. Thus $G = \{x^i y^j | i = 0, 1, \ j = 0, 1, 2\}$. Clearly, G could be the trivial group, but let's see if there is a nontrivial group for this presentation.

Define $j : \{x, y\} \to S_3$ by $x \mapsto (12)$ and $y \mapsto (123)$. By the universal property of the free group, $\exists!$ group homomorphism $f : F(\{x, y\}) \to S_3$ such that $f(x) = (12)$ and $f(y) = (123)$. Note that since (12) and (123) generate S_3, f is surjective.

With a little work, we see $x^2, y^3, xyxy \in \ker f$ and since $\ker f \lhd F(\{x, y\})$ and N is the smallest normal subgroup containing $x^2, y^3, xyxy$, we have $N \subseteq \ker f$. Thus we have

$$G \cong F(S)/N \twoheadrightarrow F(S)/\ker f \twoheadrightarrow S_3$$

by the First Isomorphism Theorem. Therefore we have the surjective homomorphism $\psi : G \twoheadrightarrow S_3$. Of course, as $|G| \leq 6$ we see $G \cong S_3$.

Here, we saw that the trivial group could be presented by any given presentation. However, in practice we want to find the largest group that satisfies the relations.

Claim: Let D_{2n} be the group of symmetries of a regular $n-gon$. Let f be any reflection and r a rotation by $2\pi/n$ radians. Then D_{2n} has the presentation $G =< x, y|x^2 = 1, y^n = 1, xyxy = 1 >$.

Proof. By the same argument as above, $|G| \leq 2n$. Now, define a homomorphism $f : F(\{x, y\}) \to D_{2n}$ by $x \mapsto f$ and $y \mapsto r$. As above, $x^2, y^n, xyxy \in \ker f$ which gives us the surjective mapping $F/N \twoheadrightarrow F/\ker f \twoheadrightarrow D_{2n}$. Thus we find $F/N \cong D_{2n}$. □

1.2 Automorphisms

Definition 1.9. *Let G be a group. An **automorphism** of G is an isomorphism $f : G \to G$. Let **Aut(G)** denote the group of all automorphisms of G. Let $g \in G$. An **inner automorphism** of G is an isomorphism of the form $\psi_g : G \to G$ such that $x \mapsto gxg^{-1}$. Clearly $(\psi_g)^{-1} = \psi_{g^{-1}}$ and $\psi_g\psi_h = \psi_{gh}$. Thus the set of inner automorphisms forms a group, which we will denote **Inn(G)**. In fact, $Inn(G) \lhd Aut(G)$. Thus, we can define $Aut(G)/Inn(G)$ as the group of **outer automorphisms**.*

Notation. Let R be a ring with 1. Let $R^* = \{u \in R|u \text{ is a unit in } R\}$. This is a group under multiplication.

Theorem 1.10. *Let $C_n =< a >$ denote the cyclic group of order n. Then $Aut(C_n) \cong \mathbb{Z}_n^*$.*

Proof. Define $\phi : \mathbb{Z}_n^* \to \mathrm{Aut}(C_n)$ by $\overline{k} \mapsto \psi_{\overline{k}}$ where $\psi_{\overline{k}} : C_n \to C_n$ is such that $a \mapsto a^k$. Since if $\gcd(k, n) = 1$, then $|a^k| = n$, we know $< a^k >=< a >= C_n$. Thus ψ is surjective and therefore injective (as the image has the same order). Therefore ψ is an isomorphism and ϕ is well-defined. Clearly, ϕ defines a homomorphism. Thus it remains to show it is injective and surjective. Notice if $\overline{k} \in \ker \phi$, then $\psi_{\overline{k}} = 1_{C_n}$ which implies $\psi_{\overline{k}} = a^k = a$. Thus $n|k - 1$, that is, $\overline{k} = 1$ which implies $\ker \phi = \{1\}$ and ϕ is injective. Also $\psi_{\overline{k}} \in \mathrm{Aut}(G)$ if and only if $< a^k >=< a >$ which happens if and only if $\gcd(k, n) = 1$. Thus ϕ is surjective and therefore an isomorphism. □

Example. $\mathrm{Aut}(C_{15}) \cong \mathbb{Z}_{15}^* = \{1, 2, 4, 7, 8, 11, 13, 14\}$. As none of those elements have order 8, the group is not cyclic. Thus $\mathrm{Aut}(G)$ is not always cyclic. In general, let $n = pq$ where p, q are odd primes. By the Chinese Remainder Theorem, $\mathbb{Z}_n \cong \mathbb{Z}_p \times \mathbb{Z}_q$. Thus $\mathbb{Z}_n^* \cong (\mathbb{Z}_p \times \mathbb{Z}_q)^* \cong \mathbb{Z}_p^* \times \mathbb{Z}_q^* \cong \mathbb{Z}_{p-1} \times \mathbb{Z}_{q-1}$. Since $p-1, q-1$ are not relatively prime (they are both even), this is not cyclic.

Theorem 1.11. *Let F be a field and H a finite subgroup of F^*. Then H is cyclic.*

Proof. Since a field is commutative, H is a finite abelian group. Thus all subgroups of H are normal and, in particular, the Sylow subgroups are unique by the Second Sylow Theorem. Therefore H is the internal direct product of its Sylow subgroups, that is, $H \cong P_1 \times \cdots \times P_l$ where P_i are the Sylow subgroups. If we show all of the P_i are cyclic, we will be done. WLOG, assume $|H| = p^n$, that is, there is only one Sylow subgroup. By the Fundamental Structure Theorem for finitely generated groups, $H \cong C_{p^{n_1}} \times \cdots \times C_{p^{n_k}}$ where $n_1 \geq n_2 \geq \cdots \geq n_k$. Since $p^{n_i}|p^{n_1}$ for all i, $h^{p^{n_1}} = 1$ for all $h \in H$. Since F is a field, every element of H is therefore a root of $x^{p^{n_1}} - 1$. This polynomial has $\leq p^{n_1}$ roots, which implies $|H| \leq p^{n_1}$. Then $H \cong C_{p^{n_1}}$ and thus H is cyclic. □

Corollary 1.12. *For a prime p, $\mathbb{Z}_p^*$ is cyclic, as $\mathbb{Z}_p$ is a field.*

Corollary 1.13. *$Aut(C_p) \cong \mathbb{Z}_p^* \cong C_{p-1}$.*

Example. Find an automorphism of C_{13} of order 6.

By above, $\mathrm{Aut}(C_{13}) \cong C_{12}$, which has an element of order 6. By brute force, we see $o(4) = 6$. Thus, if $C_{13} =< a >$, then the automorphism $a \mapsto a^4$ has degree 6.

Example. Find an automorphism of C_{55} of order 20.

By the Chinese Remainder Thm, $\text{Aut}(C_{55}) \cong \mathbb{Z}_{55}^* \cong \mathbb{Z}_5^* \times \mathbb{Z}_{11}^* \cong C_4 \times C_{10}$. We know 2 is an element of C_4 of order 4 and 4 is an element of C_{10} of order 5. Thus we want $x \in \mathbb{Z}_{55}^*$ such that $x \equiv 2 \mod 5$ and $x \equiv 4 \mod 11$. Brute force tells us $x = 37$ works. Thus $\phi : C_{55} \to C_{55}$ defined by $a \mapsto a^{37}$ is an automorphism of order 20.

Theorem 1.14. *Let p be an odd prime, $n \geq 1$. Then $Aut(C_{p^n})$ is cyclic of order $p^n - p^{n-1}$.*

Proof. We know $|\text{Aut}(C_{p^n})| = |\mathbb{Z}_{p^n}^*| = p^n - p^{n-1}$.

Claim: Let p be prime, $n \geq 1$. Let $1 \leq i \leq p^n$. Write $i = p^j x$ where $p \nmid x$. Then $p^{n-j} | \binom{p^n}{i}$ but $p^{n-j+1} \nmid \binom{p^n}{i}$.

Claim: Let p be prime. Then $(1+p)^{p^{n-1}} \equiv 1 \mod p^n$.

Proof: By the Binomial Theorem, $(1+p)^{p^{n-1}} = \displaystyle\sum_0^{p^n} \binom{p^{n-1}}{i} p^i$. Let $1 \leq i \leq p^n, i = p^j x$ as above. Note that $i \geq p^j \geq j+1$. Thus $p^{j+1} | p^i$. Also $p^{n-j-1} | \binom{p^{n-1}}{i}$. Multiplying these together gives us $p^n | \binom{p^{n-1}}{i} p^i$, which implies $(1+p)^{p^{n-1}} \equiv 1 \mod p^n$.

Claim: Let $p > 2$. Then $(1+p)^{p^{n-2}} \not\equiv 1 \mod p^n$.

Proof: Let $1 \leq i \leq p^n, i = p^j x$ as above. If $j = 0$, then $p^{n-2} | \binom{p^{n-2}}{i}$. Since, for $i \geq 2$, we have $p^2 | p^i$ we know $p^n | \binom{p^{n-2}}{i} p^i$. If $j \geq 1$, $i \geq p^j \geq j+2$ and so $p^{j+2} | p^i$. Also $p^{n-j-2} | \binom{p^{n-2}}{i}$. Combining these, we see $p^n | \binom{p^{n-2}}{i} p^i$. Thus the only nonzero terms are $i = 0, 1$. Thus $(1+p)^{p^{n-2}} \equiv 1 + p^{n-1} \not\equiv 1 \mod p^n$.

Thus $1+p$ is an element of order p^{n-1} in $\mathbb{Z}_{p^n}^*$. As $\mathbb{Z}_{p^n}^*$ is abelian, all its subgroups are normal, which implies the Sylow subgroups are unique and $\mathbb{Z}_{p^n}^*$ is the internal direct product of its Sylow subgroups. Thus it is enough to show every Sylow subgroup is cyclic. Note $|\mathbb{Z}_{p^n}^*| = p^{n-1}(p-1)$. Consider the Sylow $p-$subgroup. Since $1+p$ has order p^{n-1}, it is a generator for the Sylow subgroup and thus the Sylow $p-$subgroup is cyclic. Let q be any other prime such that $q | p^{n-1}(p-1)$. Let Q be the Sylow $q-$subgroup of $\mathbb{Z}_{p^n}^*$. Define the homomorphism $\psi : \mathbb{Z}_{p^n}^* \to \mathbb{Z}_p^*$ by $[a]_{p^n} \mapsto [a]_p$, that is, send an element to its corresponding residue class. Since $\gcd(a, p^n) = 1$ if and only if $\gcd(a, p) = 1$, the map is well-defined. Clearly the map is surjective and $|\ker \psi| = p^{n-1}$. Thus $Q \bigcap \ker \psi = 1$ and $\psi|_Q$ is injective. So Q is isomorphic to a subgroup of $\mathbb{Z}_p^*$, a cyclic group. Since subgroups of cyclic groups are cyclic, Q is cyclic. Thus all Sylow subgroups are cyclic and therefore $\mathbb{Z}_{p^n}^*$ is cyclic. □

Note. If $p = 2$, then $\mathbb{Z}_{2^n}^*$ is not cyclic for $n > 2$. For example, in $\mathbb{Z}_8^* = \{1, 3, 5, 7\}$, all nontrivial elements have order 2.

Example. If F is a field, then $\text{GL}_n(F) = \{\phi : F^n \to F^n | \phi \text{ is a vector space isomorphism}\}$.

Remark 1.15. *Suppose $|F| = q$. Then $|GL_n(F)| = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$.*

Proof. Fix a basis $e_1, ..., e_n$ for F^n. Then ϕ is determined by the values $\phi(e_1), ..., \phi(e_n)$, which must be a basis for F^n. Then $|\text{GL}_n(F)| =$ the number of distinct ordered bases for F^n. There are $q^n - 1$ choices for e_1, $q^n - q$ for e_2, etc. □

Proposition 1.16. *Let $G = \underbrace{C_p \times \cdots \times C_p}_{n \text{ times}}$. Then $Aut(G) \cong GL_n\mathbb{Z}_p$. Thus $|Aut(G)| = (p^n - 1)(p^n - p) \cdots (p^n - p^{n-1})$.*

Proof. Using additive notation, $G \cong \underbrace{\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p}_{n \text{ times}}$. This is a $\mathbb{Z}_p$ vector space. Thus any group homomorphism $\phi : G \to G$ is actually a $\mathbb{Z}_p$ linear transformation as $\phi(\overline{a}(\overline{h_1}, ..., \overline{h_n})) = \overline{a}\phi(\overline{h_1}, ..., \overline{h_n})$. So every bijective linear transformation of G is a group homomorphism and vice versa. Thus $\text{Aut}(G) \cong \text{GL}_n(\mathbb{Z}_p)$. □

1.3 Semi Direct Products

Let H, K be groups and $\phi : K \to \mathrm{Aut}(H)$, a group homomorphism. Define

$$H \rtimes_\phi K = \{(h,k) | h \in H, k \in K\}$$

and

$$(h_1,k_1)(h_2,k_2) = (h_1\phi(k_1)(h_2), k_1k_2).$$

Claim. $H \rtimes_\phi K$ is a group.
Proof: Clearly, $(1,1)$ is the identity. Also $(h,k)^{-1} = (\phi(k^{-1})(h^{-1}), k^{-1})$ as

$$\begin{aligned}(h,k)(\phi(k^{-1})(h^{-1}), k^{-1}) &= (h\phi(k)(\phi(k^{-1})(h^{-1})), kk^{-1}) \\ &= (h(\phi(k)\phi(k^{-1}))(h^{-1}), 1) \\ &= (h\phi(kk^{-1})(h^{-1}), 1) \\ &= (hh^{-1}, 1) \\ &= (1,1)\end{aligned}$$

and

$$\begin{aligned}(\phi(k^{-1})(h^{-1}), k^{-1})(h,k) &= (\phi(k^{-1})(h^{-1})\phi(k^{-1})h, k^{-1}k) \\ &= (\phi(k^{-1})(h^{-1}h), 1) \\ &= (\phi(k^{-1})(1), 1) \\ &= (1,1).\end{aligned}$$

Lastly, associativity holds. □

Definition 1.17. *Say $H \rtimes_\phi K$ is the* ***(external) semidirect product*** *of H and K (and ϕ). (Note: If $\phi(k) = 1$ for all $k \in K$, then the semidirect product is the usual direct product.*

Example. Find a nonabelian group of order 21.

Take $K = C_3 =< a >$ and $H = C_7 =< b >$. To find ϕ we want to send a to an element of order $o(a)$ in $\mathrm{Aut}(C_7)$. So let $\phi : C_3 \to \mathrm{Aut}(C_7)$ be defined by $a \mapsto \psi$ where $\psi : C_7 \to C_7$ is such that $b \mapsto b^2$. Thus we can now define $G = C_7 \rtimes_\phi C_3$. We know G is nonabelian as

$$(b,1)(1,a) = (b\phi(1)(1), a) = (b,a)$$

and

$$(1,a)(b,1) = (\phi(a)(b), a) = (b^2, a).$$

For simplicity, let's say $\overline{a} = (1,a)$ and $\overline{b} = (b,1)$. Notice $(b^i, a^j) = (b^i,1)(1,a^j) = (b,1)^i(1,a)^j = \overline{b}^i\overline{a}^j$. Then we see that $\overline{a}^3 = 1, \overline{b}^7 = 1$, and $\overline{a}\overline{b} = \overline{b^2}\overline{a}$.

What's a presentation for G? Let $H =< x, y | x^3 = 1, y^7 = 1, xy = y^2x >$. As before, we can show $|H| \leq 21$ and map it onto G, so the map is bijective and thus G is isomorphic to H.

Let $G = H \rtimes_\phi K$. There are the natural injective homomorphisms $i_1 : H \to G$ such that $h \mapsto (h,1)$ and $i_2 : K \to G$ such that $k \mapsto (1,k)$. Let $H' = i_1(H)$ and $K' = i_2(K)$.

Remarks.

1. $G = H'K'$ as $(h,k) = (h,1)(1,k) \in H'K'$
2. $H' \cap K' = \{(1,1)\}$
3. $H' \lhd G$ since $(h',k)(h,1)(h',k)^{-1} = (h',k)(h,1)(\phi(k^{-1})(h'^{-1}), k^{-1}) = (*,1) \in H'$.

Proposition 1.18. *$K' \lhd H \rtimes_\phi K$ if and only if ϕ is trivial. In this case, the semidirect product is exactly the direct product.*

Proof. $(\Leftarrow)$: Easy
$(\Rightarrow)$: Let $h \in H, k \in K$. Want to show $\phi(k)(h) = h$. Since $H', K' \lhd G$ and $H' \cap K' = \{(1,1)\}$, we know that $h'k' = k'h'$ for all $h' \in H', k' \in K'$. Thus $(h,k) = (h,1)(1,k) = (1,k)(h,1) = (\phi(k)h, k)$. Thus $\phi(k)h = h$. $\square$

Corollary 1.19. *$H \rtimes_\phi K$ is abelian if and only if ϕ is trivial and H, K are abelian.*

Definition 1.20. *Let G be an abelian group. Then $f : G \to G$ such that $g \mapsto g^{-1}$ is an automorphism of the group, called the* ***inversion map****. Note $o(f) = 2$, except when every element is its own inverse.*

Example. Let $n > 2$. Define $\phi : C_2 \to \mathrm{Aut}(C_n)$ where $C_2 =< x >$ and $C_n =< y >$ such that $x \mapsto$ the inversion map. Then $C_n \rtimes_\phi C_2$ is a nonabelian group of order $2n$. (In fact, its the dihedral group.) Notice

$$(1,x)(y,1)(1,x)^{-1} = (\phi(x)y, x)(1, x^{-1}) = (\phi(x)y\phi(x)(1), 1) = (\phi(x)y, 1) = (y^{n-1}, 1).$$

Thus we get the presentation

$$< x, y | x^2 = 1, y^n = 1, xyx^{-1} = y^{n-1}\}.$$

Theorem 1.21. *Let G be a group and H, K subgroups such that*
(1) $G = HK$ *(2) $H \cap K = \{1\}$* *(3) $H \lhd G$*
Then $\phi : K \to Aut(H)$ defined by $k \mapsto \psi_k(h) = khk^{-1}$ is a group homomorphism and $G \cong H \rtimes_\phi K$. In this case, we say G is the ***internal semidirect product*** *of H and K.*

Proof. Define $f : H \rtimes_\phi K \to G$ by $(h,k) \mapsto hk$. Then f is a group homomorphism as

$$f((h_1,k_1)(h_2,k_2)) = f((h_1\phi(k_1)h_2, k_1k_2)) = f((h_1k_1h_2k_1^{-1}, k_1k_2)) = h_1k_1h_2k_2 = f((h_1,k_1))f((h_2,k_2)).$$

Also, f is surjective as $G = HK$ implies that for $g \in G$ there exists h, k such that $g = hk$ and thus $(h,k) \mapsto g$. Finally, f is injective as if $(h,k) \mapsto 1$ then $hk = 1$ which implies $k = h^{-1} \in H \cap K = \{1\}$ and so $k = 1$ and similarly $h = 1$ and thus $\ker f = \{(1,1)\}$. $\square$

Theorem 1.22. *Let G be a group of order $2p$ where p is an odd prime. Then $G \cong C_{2p}$ or $G \cong D_{2p}$.*

Proof. Let P be the Sylow $p-$subgroup (By 3ST, there exists only one and it is normal). Let Q be the Sylow $2-$subgroup. Then, since $|P \cap Q| = 1$, we know $G = PQ$. Thus there exists $\phi : Q \to \mathrm{Aut}(P)$ such that $G \cong P \rtimes_\phi Q$. Since $|Q| = 2$, we know $Q \cong C_2 =< x >$. Similarly, $P \cong C_p =< y >$. Now, $\mathrm{Aut}(C_p) \cong \mathbb{Z}_p^*$ and so we have two cases.

Case 1: If $\phi(x) = 1_P$, then $G \cong P \times Q \cong C_{2p}$.

Case 2: If $|\phi| = 2$, there exists a unique element of order 2, as $\mathbb{Z}_p^*$ is cyclic. Clearly, its -1. Then $\phi(x)(y) = y^{-1}$, that is, ϕ is the inversion map. By our previous example, this says $G \cong D_{2p}$. $\square$

Theorem 1.23. *Let K be a cyclic group of order n and H be any group. Suppose $\phi_1, \phi_2 : K \to Aut(H)$ are group homomorphisms. If $\phi_1(K)$ and $\phi_2(K)$ are conjugate in $Aut(H)$ (that is, $\phi_1(k) = \psi\phi_2(K)\psi^{-1}$ for $\psi \in Aut(H)$), then $H \rtimes_{\phi_1} K \cong H \rtimes_{\phi_2} K$.*

Special Cases.

1. If $|\phi_1(K)| = |\phi_2(K)|$ and $\mathrm{Aut}(H)$ is cyclic, since there is only one subgroup of each order, they are equal.

2. If $\phi_1(K), \phi_2(K)$ are Sylow $p-$subgroups for some p, they are conjugate by 2ST.

Example. Classify all groups of order $75 = 3 \cdot 5^2$.

Let $P \in \mathrm{Syl}_3(G)$ and $Q \in \mathrm{Syl}_5(G)$. By 3ST, $Q \lhd G$. So $G = Q \rtimes_\phi P$ for some ϕ. Now $P \cong C_3 =< x >$ and since Q has order 5^2 it is abelian and thus either $Q \cong C_{25}$ or $Q \cong C_5 \times C_5$.

Case 1: $Q \cong C_{25}$. Then $|\mathrm{Aut}(Q)| = |\mathbb{Z}_{25}^*| = 25 - 5 = 20$. Since $3 \nmid 20$, ϕ is trivial. Thus we have $G \cong C_3 \times C_{25} \cong C_{75}$.

Case 2: $Q \cong C_5 \times C_5 =< y, z >$. Then $\mathrm{Aut}(Q) = GL_2(\mathbb{Z}_5)$, which has order $(5^2 - 5)(5^2 - 1) = 20 \cdot 24$. Now if we have $\phi = 1$, then $G \cong C_{15} \times C_5$. Otherwise, $|\phi| = 3$ which implies it is a Sylow 3-subgroup and thus all ϕ of this order yield an isomorphic semidirect product. Now, lets try to find a presentation for this group. We know $x^3 = 1, y^5 = z^5 = 1, yz = zy$, however we need to know what xyx^{-1} and xzx^{-1} are. One can see that $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ has order 3 in $GL_2(\mathbb{Z}_5)$. This corresponds to $\psi : Q \to Q$ such that $y \mapsto yz^2$ and $z \mapsto yz^3$ (take $y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $z = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$). Thus we see that G is presented by $< x, y, z | x^3 = y^5 = z^5 = 1, yz = zy, xy = yz^2x, xz = yz^3x >$.

Example. Classify all groups of order $20 = 2^2 \cdot 5$.

Let $Q \in \mathrm{Syl}_5(G)$ and $P \in \mathrm{Syl}_2(G)$. Then, by the 3ST, $Q \lhd G$ and also we know $Q \cong C_5 =< y >$. Now P has order 2^2 which implies it is abelian and thus $P \cong C_4$ or $P \cong C_2 \times C_2$. Define $G = Q \rtimes_\phi P$ where $\phi : P \to \mathrm{Aut}(Q) \cong \mathbb{Z}_5^*$.

Case 1: $P = C_4 =< x >$.

Case 1a: ϕ is trivial. Then $G = C_5 \times C_4 = C_{20}$.

Case 1b: $|\phi(P)| = 2$. There is only one subgroup of $\mathbb{Z}_5^*$ of order 2, since its cyclic. Since $y \mapsto y^4$ works, we're done. So $xyx^{-1} = y^4$ and this group is presented by $< x, y | x^4 = 1, y^5 = 1, xyx^{-1} = y^{-1} >$.

Case 1c: $|\phi(P)| = 4$. Then $\phi(P)$ is a Sylow subgroup, which says all possible ϕ here will be isomorphic- so we can choose any one. We see $y \mapsto y^2$ works, so $xyx^{-1} = y^2$. This group is presented by $< x, y | x^4 = 1, y^5 = 1, xyx^{-1} = y^2 >$.

We just need to check that these are different. In case 1b, we see $x^2 \in Z(G)$. We will show that $Z(G) = 1$ in case 1c. Let Z be the center of G and suppose $Z \neq \{1\}$. First note that $Z \cap Q = \{1\}$. If not, then (as the order of Q is prime) $Q \subseteq Z$. But this means Q commutes with every element of P, implying that $\phi = \{1\}$. Thus, if $Z \neq \{1\}$, it must contain an element, say z, of order 2. But as z is in some Sylow 2-subgroup and every Sylow 2-subgroup is conjugate to P, we must have $z \in P$ (a conjugate of z is still z!). But then $\phi(z) =$ identity map, contradicting that ϕ is an isomorphism . Hence, $Z = 1$. Thus the groups really are different.

Case 2: $P = C_2 \times C_2$

Case 2a: ϕ is trivial. Then $G \cong C_2 \times C_{10}$.

Case 2b: $|\phi(P)| = 4$. This would say ϕ was an isomorphism, contradiction since P is not cyclic but $\mathbb{Z}_5^*$ is.

Case 2c: $|\phi(P)| = 2$. Then $|\ker\phi|$=2. Let $x \in \ker\phi \setminus \{1\}$ and $z \in P \setminus \{\ker\phi\}$. Then $P =< x, z >$. (P is generated by any 2 nonidentity elements.) Since $x \in \ker\phi$, $x \in Z(G)$. Let $Q' = Z(G)Q =< x, y >\cong C_{10} =< xy >$ and $P' =< z >\cong C_2$. Note $G = P'Q'$ (since G is generated by x, y, z), $P' \cap Q' = \{1\}, Q' \lhd G$. Therefore $G \cong C_{10} \rtimes_{\phi'} C_2$ which implies D_{20}. This is clearly not isomorphic to the other 2 as the Sylow 2 subgroup is $C_2 \times C_2$.

Example. Classify all groups of order 30.

Let G be a group, $|G| = 30$. Let $P \in \mathrm{Syl}_2(G), Q \in \mathrm{Syl}_3(G), R \in \mathrm{Syl}_5(G)$. By Sylows Theorems, $n_3 \in \{1, 10\}, n_5 \in \{1, 6\}$. If $n_3 = 10$, there exists 20 elements of order 3 and if $n_5 = 6$, there exists 24 elements of order 5, but there are only 30 elements total. So either $n_3 = 1$ or $n_5 = 1$. Thus either Q or R is normal. So QR is indeed a subgroup (since one of Q and R are normal). But $[G : QR] = 2$ implies $QR \lhd G$ and further QR is cyclic (since it is of the form pq where $p \nmid q - 1$.) [Note: This shows Q and R are normal: Let $Q' \in \mathrm{Syl}_3(G)$. Then $Q' = xQx^{-1}$ for some $x \in G$. As

$Q \subseteq QR \lhd G, Q' = xQx^{-1} \subseteq xQRx^{-1} = QR$. Since QR is cyclic, it has only 1 subgroup of order 3 which implies $Q' = Q$. Hence $n_3 = 1$ and $Q \lhd G$. Similarly, $R \lhd G$.] Let $QR =< b >$ and $P =< a >$. Since $G = P(QR), QR \cap P = \{1\}$ and $QR \lhd G$, we get $G = QR \rtimes_\phi P$ for $\phi : P \to \mathrm{Aut}(QR)$. Now, $|\phi(P)||2$ and $|\phi(P)|||\mathrm{Aut}(QR)|$. Since $\mathrm{Aut}(QR) \cong \mathbb{Z}_{15}^*$ which has 3 elements of order 2: 4, 11, 14, there are 3 possibilities for a nontrivial ϕ.

Case 1: $\phi_1(a) = \psi_1 : QR \to QR$ defined by $b \mapsto b^{-1}$. Then $G_1 \cong D_{30}$.

Case 2: $\phi_2(a) = \psi_2 : QR \to QR$ defined by $b \mapsto b^4$. Then G_2 is presented by $< x, y | x^2 = y^{15} = 1, xyx^{-1} = y^4 >$.

Case 3: $\phi_3(a) = \psi_3 : QR \to QR$ defined by $b \mapsto b^{11}$. Then G_3 is presented by $< x, y | x^2 = y^{15} = 1, xyx^{-1} = y^{11} >$.

Case 4: ϕ is trivial and $G_4 \cong C_{30}$.

How do we know G_1, G_2, G_3 are different? Since $G_i/Z(G_i)$ cyclic implies G is abelian, $|Z(G_i)| \in \{1, 2, 3, 5\}$. If $|Z(G_i)| = 2$, some Sylow 2 subgroup is in the center which implies all Sylow 2 subgroups are in the center (since the Sylow 2 subgroups are conjugate), which implies $n_2 = 1$, that is $P \lhd G$. Thus G is abelian, a contradiction. So $|Z(G_i)| \in \{1, 3, 5\}$. If $|Z(G_i)| = 3$, then $Z(G_i) = Q =< b^5 >$ (since there is only one Sylow 3 subgroup) and if $|Z(G_i)| = 5$, then $Z(G_i) = R =< b^3 >$. In G_1, $ab^3a^{-1} = b^{-3} = b^{12}$ which implies $b^3 \notin Z(G_1)$. Similarly $b^5 \notin Z(G_1)$. Thus $Z(G_1) = 1$. In G_2, we see $ab^5a^{-1} = b^{20} = b^5$. Thus $Z(G_2) =< b^5 >$. Similarly, $Z(G_3) =< b^3 >$. Thus they are all different. Now, we know $Z(S_3 \times C_5) \geq 5$, so $G_3 \cong S_3 \times C_5$. Similarly, $G_2 \cong D_{10} \times C_3$.

Suppose $m|n$. Then $f : \mathbb{Z}_n \to \mathbb{Z}_m$ defined by $[a]_n \mapsto [a]_m$ is a surjective ring homomorphism.

Lemma 1.24. *Suppose $m|n$. Then the group homomorphism $f^* : \mathbb{Z}_n^* \to \mathbb{Z}_m^*$ is surjective.*

Proof. Suppose $n = p^s$ for some prime p. Then $m = p^r$ where $r \leq s$. If $[a]_{p^r} \in \mathbb{Z}_{p^r}^*$, then $[a]_{p^s} \in \mathbb{Z}_{p^s}^*$. So f^* is surjective. In general, let $n = p_1^{s_1} \cdots p_k^{s_k}$ for $p_1, \dots, p_k$ distinct primes. Then $m = p_1^{r_1} \cdots p_k^{r_k}$ where $r_i \leq s_i$. Using the Chinese Remainder Theorem, we see

$$\begin{array}{ccc} \mathbb{Z}_n^* & \longrightarrow & \mathbb{Z}_m^* \\ \downarrow\cong & & \downarrow\cong \\ \mathbb{Z}_{p_1^{s_1}}^* \times \cdots \times \mathbb{Z}_{p_k^{s_k}}^* & \longrightarrow & \mathbb{Z}_{p_1^{r_1}}^* \times \cdots \times \mathbb{Z}_{p_k^{r_k}}^* \end{array}$$

By the previous case, the bottom map is surjective. Since the bottom three maps are surjective, the top is as well. $\square$

Corollary 1.25. *Suppose $m|n$ and $\gcd(a, m) = 1$. Then there exists $t \in \mathbb{Z}$ such that $\gcd(a + tm, n) = 1$.*

Proof. Let $[a]_m \in \mathbb{Z}_m^*$. As $f^* : \mathbb{Z}_n^* \to \mathbb{Z}_m^*$ is onto, there exists $[c]_n \in \mathbb{Z}_n^*$ such that $f([c]_n) = [a]_m$. Thus $\gcd(c, n) = 1$ and $c \equiv a \mod m$ which implies $c = a + tm$. $\square$

Corollary 1.26. *Let $\phi : C_n \to C_m$ be a surjective group homomorphism (thus $m|n$). Let $C_n =< a >$ and $C_m =< b >$. Then $b = \phi(a)^r$ where $\gcd(r, n) = 1$.*

Proof. Since $< \phi(a) >= C_m =< b >$, $b = \phi(a)^s$ where $\gcd(s, m) = 1$. By the previous corollary, there exists $t \in \mathbb{Z}$ such that $\gcd(s + tm, n) = 1$. Let $r = s + tm$. Then $\phi(a)^r = \phi(a)^{s+tm} = \phi(a)^s \phi(a)^{tm} = \phi(a)^s = b$. $\square$

Theorem 1.27. *Let K be a cyclic group of order n and $\phi_1, \phi_2 : K \to Aut(H)$ be group homomorphisms, where H is some group. Suppose $\phi_1(K)$ and $\phi_2(K)$ are conjugate. Then $H \rtimes_{\phi_1} K \cong H \rtimes_{\phi_2} K$.*

Proof. Let $\sigma \in \mathrm{Aut}(H)$ be such that $\phi_2(K) = \sigma\phi_1(K)\sigma^{-1}$. Let $K =< a >$. Then $\phi_2(K) = \sigma < \phi_1(a) > \sigma^{-1} =< \sigma\phi_1(a)\sigma^{-1} >$. Then $\phi_2 : K \to < \sigma\phi_1(a)\sigma^{-1} >$ is a surjective group homomorphism. By the corollary, there exists $r \in \mathbb{Z}$ with $\gcd(r, n) = 1$ such that $\sigma\phi_1(a)\sigma^{-1} = \phi_2(a)^r$. Let $x \in K$. Then $x = a^s$ for some s. Then

$$\sigma\phi_1(x)\sigma^{-1} = (\sigma\phi_1(a)\sigma^{-1})^s = (\phi_2(a)^r)^s = (\phi_2(a)^s)^r = \phi_2(x)^r.$$

Thus $\sigma\phi_1(x) = \phi_2(x)^r\sigma$. Define $f : H \rtimes_{\phi_1} K \to H \rtimes_{\phi_2} K$ by $(h,k) \mapsto (\sigma(h), k^r)$. Then

$$\begin{aligned}
f((h_1,k_1)(h_2,k_2)) &= f((h_1\phi_1(k_1)(h_2), k_1k_2)) \\
&= (\sigma(h_1\phi_1(k_1)(h_2)), (k_1k_2)^r) \\
&= (\sigma(h_1)\sigma(\phi_1(k_1)(h_2)), k_1^rk_2^r) \\
&= (\sigma(h_1)\phi_2(k_1)^r\sigma(h_2), k_1^rk_2^r) \\
&= (\sigma(h_1), k_1^r)(\sigma(h_2), k_2^r) \\
&= f((h_1,k_1))f((h_2,k_2)).
\end{aligned}$$

Thus f is a homomorphism. Also, we know it is 1-1 and onto as $h \mapsto \sigma(h)$ and $k \mapsto k^r$ are automorphisms (since $\gcd(r,n) = 1$). Thus f is an isomorphism. $\square$

1.4 Characteristic Groups

Definition 1.28. *Let G be a group. A subgroup H of G is called* ***characteristic*** *if $\sigma(H) = H$ for all $\sigma \in Aut(G)$. We denote this as $H\, char\, G$.*

Example. $Z(G)$char G. To see this, let $\sigma \in \mathrm{Aut}(G), x \in Z(G)$ and $y \in G$. Then $y = \sigma(z)$ for some $z \in G$ and

$$\sigma(x)y = \sigma(x)\sigma(z) = \sigma(xz) = \sigma(zx) = \sigma(z)\sigma(x) = y\sigma(x).$$

So $\sigma(x) \in Z(G)$. So $\sigma(Z(G)) \subseteq Z(G)$ for all $\sigma \in \mathrm{Aut}(H)$ which implies $\sigma^{-1}(Z(G)) \subseteq Z(G)$ for all σ and applying σ, we see $Z(G) \subseteq \sigma(Z(G))$. Thus $Z(G) = \sigma(Z(G))$.

Remarks.

1. If H is a unique subgroup of G of order $|H|$, then Hchar G. Therefore, every subgroup of a cyclic group is characteristic.
   Example. Let $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then $\phi : G \to G$ defined by $(a,b) \mapsto (b,a)$ is an automorphism but $\phi(< (1,0) >) =< (0,1) >$. So $< (1,0) >$ is not characteristic in G.

2. Characteristic subgroups are always normal.
   Proof: Let $g \in G$. Then $\psi_g : G \to G$ defined by $x \mapsto gxg^{-1}$ is an automorphism. If Hchar G, then $gHg^{-1} = \psi_g(H) = H$. Thus $H \lhd G$.
   Note: The converse is not true (see previous example).

3. Let $P \in \mathrm{Syl}_p(G)$. Then P char G if and only if $P \lhd G$.
   Proof: $(\Leftarrow)$: If $P \lhd G$, then P is the only Sylow $p-$subgroup. Done by Remark 1.

Note. If $K \lhd H$ and $H \lhd G$ does NOT imply $K \lhd G$.

Example. $D_8 =< x, y | x^2 = y^4 = 1, xy = y^3x >$. We see $< xy > \lhd \{1, xy, xy^3, y^2\} \lhd D_8$ (the first because a group of order 4 is abelian and the second because its index 2). However, $< xy > \ntriangleleft D_8$.

Remarks.

1. K char H and H char G implies K char G.
   Proof: Let $\phi \in \mathrm{Aut}(G)$. As H char G, $\phi(H) = H$ which implies $\phi|_H \in \mathrm{Aut}(H)$ and thus $\phi(K) = \phi|_H(K) = K$ as K char H. So K char G.

2. K char H and $H \lhd G$ implies $K \lhd G$.
   Proof: Let $g \in G$ and consider $\psi_g \in \mathrm{Aut}(G)$ where $\psi_g(x) = gxg^{-1}$. As $H \lhd G$, $\psi_g(H) = H$. In particular, $\psi_g|_H \in \mathrm{Aut}(H)$. Since K char H, $\psi_g(K) = \psi_g|_H(K) = K$. Thus $K \lhd G$.

Example. (Old Comp Problem) Let $P \in \mathrm{Syl}_p(G)$. Then $N_G(N_G(P)) = N_G(P)$, where $N_G(H) = \{g \in G | gHg^{-1} = H\}$.

Proof. Clearly, $P \lhd N_G(P)$ implies P char $N_G(P)$ (since Sylow $p-$subgroups are normal if and only if they are characteristic). But $N_G(P) \lhd N_G(N_G(P))$. By Remarks 2, $P \lhd N_G(N_G(P))$. Thus $N_G(N_G(P)) \subseteq N_G(P)$ and since the other containment is obvious, they are equal.

1.5 Solvable Groups

Definition 1.29. *Let G be a group and $x, y \in G$. Define the* ***commutator*** *of x and y by*

$$[x, y] := xyx^{-1}y^{-1}.$$

The commutator subgroup of G, denoted $[G, G]$ or G', is the subgroup of G generated by all its commutators.

Remarks.

1. x, y commute if and only if $[x, y] = 1$.
2. G is abelian if and only if $G' = \{1\}$.
3. G' char G
   Proof: Let $\phi \in \mathrm{Aut}(G)$, $x, y \in G$. Then $\phi([x, y]) = \phi(xyx^{-1}y^{-1}) = \phi(x)\phi(y)\phi(x)^{-1}\phi(y)^{-1} = [\phi(x), \phi(y)]$. So $\phi(G') \subseteq G'$. If $[x, y]$ is a generator of G', then there exists $a, b \in G$ such that $\phi(a) = x, \phi(b) = y$ which implies $\phi([a, b]) = [x, y]$. Thus $G' \subseteq \phi(G')$ and so they are equal.

Lemma 1.30. *Let G be a group. Then*

1. *$G' \lhd G$ and G/G' is abelian.*
2. *If $H \supseteq G'$, then $H \lhd G$ and G/H is abelian.*
3. *If $H \lhd G$ and G/H is abelian, then $H \supseteq G'$.*

Proof.
1. As G' char G, $G' \lhd G$. Let $\overline{x}, \overline{y} \in G/G'$. Then $\overline{xyx}^{-1}\overline{y}^{-1} = \overline{1}$ which implies $\overline{xy} = \overline{yx}$ which implies G/G' is abelian.
2. If $H \supseteq G'$, $H/G' < G/G'$ which is abelian. Thus $H/G' \lhd G/G'$ which implies $H \lhd G$. Note $G/H \cong \dfrac{G/G'}{H/G'}$ is abelian as G/G' was.
3. Let $[x, y]$ be a commutator. Then, as G/H is abelian, $\overline{[x, y]} = [\overline{x}, \overline{y}] = \overline{xyx}^{-1}\overline{y}^{-1} = \overline{1}$. Thus $[x, y] \in H$ and therefore $H \supseteq G'$.

□

Definition 1.31. *A sequence of subgroups $\cdots G_i \lhd G_{i-1} \lhd \cdots G_0 = G$ is called a* ***normal series****. The* ***derived normal series*** *is $\cdots G''' \lhd G'' \lhd G' \lhd G$. For simplicity, we will take $G^{(0)} = G$, $G^{(1)} = G'$, and $G^{(i)} = (G^{(i-1)})'$ for $i \geq 2$.*

Example. Let $G = S_3$. Then, $G' =< (123) >$.

Proof. As $< (123) > \lhd S_3$ (index 2) and $G/ < (123) >$ is abelian (its cyclic), the above lemma says $< (123) > \supseteq S_3'$. As S_3 is nonabelian, $S_3' \neq \{1\}$. So $S_3' =< (123) >$.
Now $< (123) >$ is abelian, so $(S_3)'' = \{1\}$. Thus

$$\{1\} \lhd < (123) > \lhd S_3$$

is the derived normal series for S_3.

Definition 1.32. *A group is* ***solvable*** *if $G^{(n)} = \{1\}$ for some n.*

Remark. Suppose $\phi : A \to B$ is a surjective group homomorphism. Then $\phi(A^{(i)}) = B^{(i)}$ for all i.

Proof. Induct on i. If $i = 0$, clear. Suppose true for $i-1$. Want to show $\phi((A^{(i-1)})') = (B^{(i-1)})'$. For simplicity, we can take $i = 1$. Know $\phi([a,b]) = [\phi(a), \phi(b)]$. Thus $\phi(A') \subseteq B'$. On the other hand, as ϕ is surjective, any commutator of B is the image of a commutator of A.

Special Case. Suppose $H \lhd G$ and $\phi : G \to G/H$ is the natural homomorphism. Then $\overline{G^{(i)}} = \overline{G}^{(i)}$.

Proposition 1.33. *Let G be a group and $H \leq G$.*

1. *If G is solvable, then so is H. Furthermore, if $H \lhd G$, the G/H is solvable.*

2. *If $H \lhd G$ and H and G/H are solvable, then so is G.*

Proof. 1. For some n, $G^{(n)} = \{1\}$. But $H^{(i)} \subseteq G^{(i)}$ for all i. Thus $H^{(n)} = \{1\}$. Also, if $H \lhd G$, then $(G/H)^{(n)} = \overline{G^{(n)}} = \overline{\{1\}}$.

2. Since G/H is solvable, there exists n such that $\overline{G^{(n)}} = (G/H)^{(n)} = \{1\}$. Thus $G^{(n)} \subseteq H$. Since H is solvable, there exists m such that $H^{(m)} = \{1\}$. Then $G^{(n+m)} \subseteq H^{(m)} = \{1\}$. Thus G is solvable. □

Proposition 1.34. *Let G be a group of order p^n, p prime. Then G is solvable.*

Proof. Induct on n. If $n = 0, 1, 2$, then G is abelian and thus $G' = \{1\}$. So suppose $n \geq 3$. Recall that $p-$groups have nontrivial center. Since $Z(G)$ is abelian, it is solvable. Now $|G/Z(G)| = p^r$ for some $r < n$. Thus $G/Z(G)$ is solvable by induction and by Proposition 1.33, G is solvable. □

Fact. A_n is not solvable for $n \geq 5$. We know A_n is simple and nonabelian for $n \geq 5$. Since the commutator is a normal subgroup, $(A_n)^{(i)} = A_n$ for all $i \geq 1$. Thus A_n is not solvable. By Prop 1.33, we see S_n is therefore not solvable for $n \geq 5$ as then its subgroup A_n would be. Note: A_4 is solvable (see later)

Note. Since G' char G and $G^{(2)}$ char G', we know that $G^{(2)}$ char G and by induction, $G^{(n)}$ char G. In particular, this says $G^{(n)} \lhd G$.

Definition 1.35. *A **solvable series** for a group G is a normal series*

$$\{1\} = G_n \lhd G_{n-1} \lhd \cdots \lhd G_0 = G$$

such that G_i/G_{i-1} is abelian for all i.

Proposition 1.36. *G is solvable if and only if G has a solvable series.*

Proof. ($\Rightarrow$:) The derived normal series is a solvable series for G.

($\Leftarrow$:) Let $\{1\} = G_n \lhd \cdots \lhd G_0 = G$ be a solvable series for G. Induct on n. If $n = 0$, then $G = \{1\}$ and we are done. Let $n > 0$. Then G_1 has a solvable series of length $n-1$. So G_1 is solvable by induction. Also G/G_1 is abelian, which implies it is solvable. Then, since G_1 and G/G_1 is solvable, G is solvable by Prop 1.33. □

Fact. A_4 is solvable. We see it has the solvable series

$$\{1\} \lhd \{(1), (12)(34), (13)(24), (14)(23)\} \lhd A_4 \lhd S_4.$$

Thus A_4 and S_4 are solvable.

Lemma 1.37. *If $|G| = pq$ for primes p, q, then G is solvable.*

Proof. If $p = q$, then G is abelian and thus solvable. Say $p < q$. By ST, the Sylow $q-$subgroup is normal and solvable (since abelian). Of course $|G/Q| = p$ implies G/Q is abelian and thus solvable. Thus by Prop 1.33, G is solvable. □

Proposition 1.38. *Every group of order pqr for primes p, q, r is solvable.*

Proof. Case 1: $p = q = r$. Then done by Prop 1.36.

Case 2: $p < q < r$. By counting arguments, at least one of the Sylow subgroups is normal and hence solvable, say H. Then $|G/H| = p'q'$ for primes $p'q'$ and is thus solvable by the lemma. Thus by Prop 1.33, G is solvable.

Case 3: $|G| = p^2q, p < q$. Similar.

Case 4: $|G| = pq^2, p < q$. Similar. $\square$

2 Fields

Definition 2.1. *A **field** is a commutative ring with identity such that every nonzero element has a multiplicative inverse. Let R be a ring with identity. Consider the ring homomorphism $\phi : \mathbb{Z} \to R$ defined by $n \mapsto n \cdot 1_R$. Say R has **characteristic 0** if ϕ is injective. Otherwise, if $\ker\phi = (n)$, then R has **characteristic** n. In this case $\mathbb{Z}/(n) \hookrightarrow R$. If R is a domain, then so is $\mathbb{Z}/(n)$ which says (n) is prime. In particular, if R is a field, then char $R = 0$ or (p) for some prime p. Let R be a commutative domain. Then the **fraction field** or **quotient field** of R is $Q(R) = \{\frac{a}{b} | a, b \in R, b \neq 0\}$.*

Note. Instead of saying a field F has characteristic 0, it is often said that F contains the rationals. This is because if $\mathbb{Z} \to F$ defined by $n \mapsto n \cdot 1$ is injective, then $\mathbb{Z} \subseteq F$ which implies its quotient field $Q(F) = \mathbb{Q} \subseteq F$.

Remark. If R is a domain, then $R[x]$ is a domain. In this case, $Q(R[x]) = Q(R)(x) = \left\{\frac{f}{g} | f, g \in Q(R)[x], g \neq 0\right\}$.

Notation. Let $F \subseteq E$ be fields. Usually, we will say E/F is a field extension.

2.1 Algebraic Extensions

Definition 2.2. *Let E/F be a field extension, $\alpha \in E$. Then α is **algebraic** over F if there exists $f(x) \in F[x] \setminus \{0\}$ such that $f(\alpha) = 0$. If α is not algebraic, we say it is **transcendental**. The **degree** of E/F, denoted $[E : F]$, is the dimension of E as an $F-$vector space. We say $[E : F]$ is finite if $[E : F] < \infty$.*

Examples.

1. If x is an indeterminant, then $F(x)/F$ is a field extension and $[F(x) : F] = \infty$ as $\{1, x, x^2, ...\}$ is an $F-$basis for $F(x)$.

2. $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ as $\{1, \sqrt{2}\}$ is a $\mathbb{Q}-$basis.

Lemma 2.3. *Let $L \subseteq F \subseteq E$ be fields. Then $[E : L] = [E : F][F : L]$.*

Proposition 2.4. *Let $\alpha \in E$, E/F a field extension. TFAE*

1. *α is algebraic over F.*

2. *$F[\alpha] = F(\alpha)$.*

3. *$[F(\alpha) : F] < \infty$.*

Proof. $(1) \Rightarrow (2)$: Define $\phi : F[x] \to F[\alpha]$ by $f(x) \mapsto f(\alpha)$. Then ϕ is a surjective ring homomorphism. Thus $F[\alpha] \cong F[x]/(\ker\phi)$. Since $F[x]$ is a PID, we know $\ker\phi = (h(x))$ for some $h(x) \in F[x]$. Since α is algebraic over f, we know $\ker\phi \neq 0$. So $h(x) \neq 0$. Since $F[\alpha] \subseteq F(\alpha)$, its an integral domain. Thus $\ker\phi$ is prime and $h(x)$ is irreducible over F (as if it factored, the factors would be zero divisors). So $(h(x))$ is a maximal ideal which implies $F[x]/(h(x))$ is a field. Thus $F[\alpha] = F(\alpha)$.

$(2) \Rightarrow (3):$ If $\alpha = 0$, trivial. So let $\alpha \neq 0$. Then $\frac{1}{\alpha} \in F(\alpha) = F[\alpha]$. So $\frac{1}{\alpha} = c_0 + c_1\alpha + \ldots + c_n\alpha^n$ for $c_n \neq 0$. Multiplying by $\frac{\alpha}{c_n}$, we see $\alpha^{n+1} \in \mathrm{Span}_F\{1, \alpha, ..., \alpha^n\}$ which implies $\alpha^i \in \mathrm{Span}_F\{1, \alpha, ..., \alpha^n\}$ for all i. Then $\dim_F F[\alpha] \leq n+1$ which implies $[F(\alpha) : F] \leq n+1$.

$(3) \Rightarrow (1):$ Say $[F(\alpha) : F] = n$. Then $\{1, \alpha, ..., \alpha^n\}$ is a linearly dependent set over F. Thus there exists $c_0, ..., c_n \in F$ (not all zero) such that $c_0 \cdot 1 + ... + c_n\alpha^n = 0$ which implies α is a root of $f(x) = c_0 + ... + c_nx^n$. Thus α is algebraic. □

Corollary 2.5. *Let $\{\alpha_1, ..., \alpha_n\} \in E$, E/F a field extension. TFAE*

1. *$\alpha_1, ..., \alpha_n$ is algebraic over F.*
2. *$F[\alpha_1, ..., \alpha_n] = F(\alpha_1, ..., \alpha_n)$.*
3. *$[F(\alpha_1, ..., \alpha_n) : F] < \infty$.*

Proposition 2.6. *If $[E : F] < \infty$, then E/F is algebraic.*

Proof. Let $\alpha \in E$. Then $[F(\alpha) : F] \leq [E : F] < \infty$. By Prop 2.4, α is algebraic. □

Note. The converse is not true. Consider $\mathbb{Q} \subseteq \mathbb{C}$ and let $\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} | \alpha \text{ is algebraic over } \mathbb{Q}\}$. Clearly, $[\overline{\mathbb{Q}} : \mathbb{Q}] = \infty$ and $[\mathbb{Q}(\sqrt[n]{2}) : \mathbb{Q}] = n$.

Proposition 2.7. *Suppose E/F and F/L are algebraic. Then E/L is also algebraic.*

Proof. Let $\alpha \in E$. Then α is algebraic over F which implies $[F(\alpha) : F] < \infty$. Say $f(\alpha) = 0$ where $f(x) = c_nx^n + ...c_0 \in F[x] \setminus \{0\}$. Let $K = L(c_0, ..., c_n)$. Then K/L is finite and α is algebraic over K. Then $[K(\alpha) : L] = [K(\alpha) : K][K : L] < \infty$. Thus α is algebraic over L. □

Proposition 2.8. *Let E/F be a field extension, $\alpha \in E$ algebraic over F. Say $h \in F[x] \setminus \{0\}$ such that $h(\alpha) = 0$. TFAE*

1. *$h(x)$ is irreducible over F.*
2. *$h | f$ for all $f(x) \in F[x]$ such that $f(\alpha) = 0$.*
3. *$h(x) = \ker \phi$ for $\phi : F[x] \to F[\alpha]$.*

*If h is monic and satisfies the above, say h is the **minimal polynomial** for α over F and denote it by Irred(α, F) or Min$(\alpha, F$.)*

Proposition 2.9. *Suppose α is algebraic over F. Then $[F(\alpha) : F] = \deg \mathit{Irred}(\alpha, F)$.*

Definition 2.10. *Let F be a field and $f(x) \in F[x] \setminus F$. Then a **splitting field** for $f(x)$ over F is a field $L \supseteq F$ such that $f(x)$ factors into linear factors in $L[x]$ and $f(x)$ does not split in $E[x]$ for all $F \subseteq E \subsetneq L$.*

Remark. Let $f(x) \in F[x]$ and $E \supseteq F$ such that $f(x) = c(x - \alpha_1) \cdots (x - \alpha_n)$ in $E[x]$. Then a splitting field for $f(x)$ over F is $F[\alpha_1, ..., \alpha_n]$.

Examples.

1. Find the splitting field of $x^4 - 2$ over $\mathbb{Q}$.

   The roots of $x^4 - 2$ are $\pm\sqrt[4]{2}, \pm i\sqrt[4]{2}$. So $\mathbb{Q}(\sqrt[4]{2}, i)$ is the splitting field.

$$\begin{array}{cl} \mathbb{Q}(\sqrt[4]{2}, i) & \\ \mid 2 & \quad \text{since } x^2+1 \text{ is irreducible as } i \notin (\sqrt[4]{2}). \\ \mathbb{Q}(\sqrt[4]{2}) & \\ \mid 4 & \quad \text{since } x^4-2 \text{ is irreducible (by Eisenstein).} \\ \mathbb{Q} & \end{array}$$

   Thus $[\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}] = 8$.

2. Find the splitting field for x^6+3 over $\mathbb{Q}$.

   First, lets find the roots. In polar coordinates, $z^6 = -3 = 3e^{i\pi} = r^6e^{i6\theta}$. Thus $r^6 = 3$ and $6\theta = \pi + 2\pi k$ which implies $\theta = \frac{\pi}{6} + \frac{\pi k}{3}$. Thus the roots are $\sqrt[6]{3}e^{\frac{\pi i}{6}}(e^{\frac{\pi i}{3}})^k$ for $k = 0, ..., 5$.

$$\begin{array}{cl}
\mathbb{Q}(\sqrt[6]{3}e^{\frac{\pi i}{6}}, e^{\frac{\pi i}{3}}) & \\
\mid m \leq 2 & \text{since the cyclotomic polynomial works.} \\
\mathbb{Q}(\sqrt[6]{3}e^{\frac{\pi i}{6}}) & \\
\mid 6 & \text{since } x^6+3 \text{ is irreducible (by Eisenstein).} \\
\mathbb{Q} &
\end{array}$$

   In fact, $m = 1$ as $(\sqrt[6]{3}e^{\frac{i\pi}{6}})^3 = \sqrt{3}i$ implies $\frac{1 \pm \sqrt{3}i}{2} \in \mathbb{Q}(\sqrt[6]{3}e^{\frac{i\pi}{6}})$ which is the roots of the cyclotomic polynomial.

3. Find the splitting field of x^5-2 and its degree.

   We see the roots are $\omega^i\sqrt[5]{2}$ for $i = 0, ..., 4$ where $\omega = e^{\frac{2\pi i}{5}}$. So the splitting field is $\mathbb{Q}(\sqrt[5]{2}, \omega)$.

$$\begin{array}{ccccccl}
 & & \mathbb{Q}(\sqrt[5]{2}, \omega) & & & & \\
 & {\scriptstyle \leq 4}\diagup & & \diagdown & & & \\
\mathbb{Q}(\sqrt[5]{2}) & & & & \mathbb{Q}(\omega) & & \text{since } x^5-2 \text{ and the cyclotomic} \\
 & {\scriptstyle 5}\diagdown & & \diagup{\scriptstyle 4} & & \leftarrow & \text{polynomials are irreducible} \\
 & & \mathbb{Q} & & & & \text{over } \mathbb{Q} \text{ by Eisenstein.}
\end{array}$$

   Then $D = [\mathbb{Q}(\sqrt[5]{2}, \omega) : \mathbb{Q}] \leq 20$. Of course $4|D$ and $5|D$ implies $20|D$. Thus $D = 20$.

   Note. This says $x^4 + x^3 + x^2 + x + 1$ is irreducible over $\mathbb{Q}(\sqrt[5]{2}, \omega)$.

Lemma 2.11. *Let K be a field and $f(x) \in K[x]$ a nonconstant polynomial. Then there exists a field extension $E \supseteq K$ such that $[E : K] \leq \deg f$ and $f(x)$ has a root in E.*

Proof. Let $p(x)$ be an irreducible factor of $f(x)$. It is enough to show true for $p(x)$. Let t be an indeterminant over K and $E = K[t]/(p(t))$, a field as $p(t)$ is irreducible in $K[t]$. Let $\alpha = t + (p(t)) = \bar{t}$. Then $\{1, \alpha, ..., \alpha^{n-1}\}$ is a $K-$basis for E where $n = \deg p(t)$. Define $\sigma : K \to E$ by $a \mapsto a + (p(t))$. Since there does not exist constants in $(p(t))$ we see $\ker \phi = \{0\}$ and σ is an injective field map. So by identifying K with $\sigma(K)$, we can assume $K \subseteq E = K(\alpha)$. Note $p(\alpha) = p(t) + (p(t)) = \bar{0}$. So $\alpha \in E$ is a root of $p(x)$ and $[E : K] = n$. ☐

Lemma 2.12. *Let K be a field. Then there exists a field $E \supseteq K$ such that every nonconstant polynomial $f \in K[x]$ has a root in E.*

Proof. For each nonconstant $f \in K[x]$, let t_f be an indeterminant. Let $R = K[\{t_f\}_{f \in K[x] \setminus K}]$ and I an ideal of R generated by $\{f(t_f)\}_{f \in K[x] \setminus K}$.

> **Claim.** $I \neq R$.
>
> *Proof:* Suppose $I = R$. Then $1 \in I$ which implies
>
> $$1 = r_1 f_1(t_{f_1}) + \ldots + r_s f_s(t_{f_s}) \tag{1}$$
>
> for $f_1, ..., f_s \in K[x] \setminus K$ and $r_1, ..., r_s \in R$. For ease of notation, let $t_i := t_{f_i}$. Let $t_1, ..., t_s, ..., t_n$ be all the indeterminants involved in $r_1, ..., r_s$ along with $t_1, ..., t_s$. Now, define $F_1 \supseteq K$ such that $f_1(t_1)$ has a root in F_1. Iteratively define $F_i \supseteq F_{i-1}$ such that $f_i(t_i)$ has a root in F_i. Then $F_s \supseteq K$ is such that $f_i(t_i)$ has a root α_i in F_s for all $i = 1, ..., s$. Plug in $\alpha = (\alpha_1, .., \alpha_s)$ into Equation (1) to get $1 = 0$, a contradiction. Thus $I \neq R$.

Let M be a maximal ideal of R containing I (this exists by Zorn's Lemma) and let $E = R/M$, a field. Define $\sigma : K \hookrightarrow R \to R/M$ by $a \mapsto a \mapsto a + M$. Here, we see $\ker \sigma = \{0\}$ as if $a + M = 0$ then $a \in M$ which implies M contains a unit. Thus σ is injective and so we can identify K with its image $\sigma(K)$ and conclude $K \subseteq E$. Let $f(x) \in K[x]$ be a nonconstant polynomial and $\alpha_f = t_f + M$. Then $f(\alpha_f) = f(t_f) + M = 0$ since $f(t_f) \in I \subseteq M$. So α_f is a root of f in E. $\square$

Definition 2.13. *A field F is* ***algebraically closed*** *if every nonconstant polynomial $f(x) \in F[x]$ has a root in F. Equivalently, $f(x)$ splits completely in $F[x]$. An* ***algebraic closure*** *of a field F is a field $\overline{F} \supseteq F$ such that $\overline{F}$ is algebraically closed and $\overline{F}/F$ is algebraic.*

Proposition 2.14. *If $F \subseteq L$ and L is algebraically closed, then $\overline{F} = \{\alpha \in L | \alpha \text{ is algebraic over } F\}$ is an algebraic closure of F.*

Proof. First, we want to show that this is a field. Given $\alpha, \beta \in \overline{F}$, we want to show $\alpha\beta, \alpha \pm \beta, \frac{\alpha}{\beta}$ are algebraic over F. Since α, β are algebraic, $[F(\alpha, \beta) : F] < \infty$ by the Corollary. But $\alpha\beta, \alpha \pm \beta, \frac{\alpha}{\beta} \in F(\alpha, \beta)$ where every element is algebraic over F (since the degree is finite). Thus they are algebraic over F and thus in $\overline{F}$. Now, we show $\overline{F}$ is algebraically closed. Let $f(x) \in \overline{F}[x] \setminus \overline{F}$. Then $f(x)$ has a root $\alpha \in L$. So $\overline{F}(\alpha)/\overline{F}$ is algebraic and $\overline{F}/F$ is algebraic which implies $\overline{F}(\alpha)/F$ is algebraic. Thus α is algebraic over F which implies $\alpha \in \overline{F}$. $\square$

Theorem 2.15. *Let F be a field. Then there exists an algebraic closure of F.*

Proof. Let $E_0 = F$. For $n \geq 1$, define $E_n \supseteq E_{n-1}$ to be a field such that every nonconstant polynomial in $E_{n-1}[x]$ has a root in E_n. Let $L = \cup_{i=1}^{\infty} E_i$. This is a field as the E_i's are nested. L is also algebraically closed as for $f(x) \in L[x] \setminus L$, there exists n such that $f(x) \in E_n[x]$. Then $f(x)$ has a root in $E_{n+1} \subseteq L$. Now, let $\overline{F} = \{\alpha \in L | \alpha \text{ is algebraic over } F\}$. Then by the above proposition, $\overline{F}$ is an algebraic closure for F. $\square$

Corollary 2.16. *Let $f(x) \in F[x] \setminus F$. Then there exists a splitting field for $f(x)$.*

Proof. Let $\overline{F}$ be an algebraic closure of F. Then $f(x) = c(x - \alpha_1) \cdots (c - \alpha_n)$ in $\overline{F}[x]$. Then $F(\alpha_1, ..., \alpha_n)$ is a splitting field for $f(x)$ over F. $\square$

Definition 2.17. *Let E/F and E'/F' be field extensions. Let $\sigma : F \to F'$ and $\tau : E \to E'$ be field homomorphisms. Say τ* ***extends*** *σ if $\tau|_F = \sigma$. As a special case, if $F = F'$ and $\sigma = 1_F$, then τ extends σ if and only if τ fixes F.*

Remarks. Suppose τ extends σ.

1. σ extends to a ring homomorphism $\widetilde{\sigma} : F[x] \to F'[x]$ by $a_0 + a_1 x + \ldots + a_n x^n \mapsto \sigma(a_0) + \sigma(a_1)x + \ldots + \sigma(a_n)x^n$. Write this as $p(x) \mapsto p^\sigma(x)$. Check: $(fg)^\sigma = f^\sigma g^\sigma$ and $(f+g)^\sigma = f^\sigma + g^\sigma$.

2. Suppose $\alpha \in E$ is a root of $p(x)$ in $F[x]$. Then $\tau(\alpha)$ is a root of $p^\sigma(x)$:

$$p^\sigma(\tau(\alpha)) = \sigma(a_0) + \sigma(a_1)\tau(\alpha) + \ldots \sigma(a_n)\tau(\alpha)^n = \tau(a_0) + \tau(a_1\alpha) + \ldots \tau(a_n\alpha^n) = \tau(p(\alpha)) = \tau(0) = 0.$$

   Note that in general $p^\sigma(\tau(\alpha)) = \tau(p(\alpha))$ for all α(i.e., not just roots).

3. If $F = F'$, $\sigma = 1_F$. If $\alpha \in E$ is a root of $p(x)$, then $\tau(\alpha)$ is also a root of $p(x)$.

Proposition 2.18. *Let E/F be an algebraic extension and $\tau : E \to E$ a field homomorphism fixing F. Then τ is an isomorphism.*

Proof. Clearly τ is 1-1. So its enough to show τ is surjective. Let $\alpha \in E$. As α is algebraic over F, there exists some $p(x) \in F[x] \setminus F$ such that $p(\alpha) = 0$. Let $R = \{\alpha = \alpha_1, \alpha_2, ..., \alpha_n\}$ be all the roots of $p(x)$ in E. Then $\tau(\alpha_i) \in R$ for all i. We know $\tau|_R$ is 1-1 and since finite it is also onto. Thus $\tau(\alpha_j) = \alpha$ for some j. $\square$

Theorem 2.19. *Let $\sigma : F \to K$ be a nonzero field homomorphism where $K = \overline{K}$. Suppose E/F is an algebraic extension. Then there exists $\tau : E \to K$ extending σ.*

Proof. Let $\Lambda = \{(T, \phi) | F \subseteq T \subseteq E, T \text{ is a field}, \phi : T \to K \text{ extends } \sigma\}$. Note that $\Lambda \neq \emptyset$ as $(F, \sigma) \in \Lambda$. Define a partial order on Λ by $(T_1, \phi_1) \subseteq (T_2, \phi_2)$ if and only if $T_1 \subseteq T_2$ and $\phi_2|_{T_1} = \phi_1$. Let C be a totally ordered subset of Λ (i.e., a chain). Let $T_0 = \cup T$ such that $(T, \phi) \in C$, a field (since the T's are nested), and $F \subseteq T_0 \subseteq E$. Define $\psi : T_0 \to K$ by $t \mapsto \phi(t)$ if $t \in T$ for some $(T, \phi) \in C$. Check this is well-defined and ψ is a field homomorphism. Clearly $\psi|_T = \phi$ for all $(T, \phi) \in C$. Then $(T_0, \psi) \in \Lambda$ is an upper bound for C. By Zorn's Lemma, there exists a maximal element $(M, \delta) \in \Lambda$. Want to show $M = E$. Let $N \cong \delta(M) \subseteq K$. We can extend δ to $\overline{\delta} : M[x] \to N[x]$ by $p(x) \mapsto p^\delta(x)$. This is an isomorphism as δ is. Suppose there exists $\alpha \in E \setminus M$. Let $f(x) = \text{Irred}(\alpha, M)$. Then $f^\delta(x)$ is irreducible in $N[x] \subseteq K[x]$. As K is algebraically closed, $f^\delta(x)$ has a root $\beta \in K$. Of course $\text{Irred}(\beta, N) = f^\delta(x)$. Then

$$\delta' : M(\alpha) \to M[x]/(f) \to N[x]/(f^\delta) \to N(\beta) \subseteq K$$

defined by $g(\alpha) \mapsto \overline{g(x)} \mapsto \overline{g^\delta(x)} \mapsto g^\delta(\beta)$. So $\delta' : M(\alpha) \to K$. We can see $\delta'|_M = \delta$. So $(M, \delta) < (M(\alpha), \delta')$, a contradiction. Thus $M = E$. $\square$

Corollary 2.20. *Using the notation of the above theorem, suppose E is algebraically closed and K is algebraic over $\sigma(F)$. Then τ is an isomorphism.*

Proof. Since $\ker \tau$ is an ideal, it is either (0) or E. Since σ is nonzero, $\ker \tau \neq E$. Thus τ is injective. So it is enough to show τ is surjective. Note $\tau(E) \cong E$ and since E is algebraically closed, $\tau(E)$ is. Since $K/\sigma(F)$ is algebraic, so is $K/\tau(E)$ since $\sigma(F) \subseteq \tau(E) \subseteq K$. But $\tau(E)$ is algebraically closed, so $K = \tau(E)$. $\square$

Corollary 2.21. *Let F be a field. Then any two algebraic closures of F are isomorphic via an isomorphism fixing F.*

Proof. Let L_1, L_2 be algebraic closures of F. Consider $\sigma : F \to L_2$. We can extend σ to $\tau : L_1 \to L_2$. By previous corollary, τ is an isomorphism fixing F. $\square$

Definition 2.22. *Let F be a field and $S \subset F[x] \setminus F$. A* ***splitting field*** *for S over F is a field $L \supseteq F$ such that every $f \in S$ splits in $L[x]$ and L is minimal with respect to this property.*

Remark. Let F, S be as above and fix an algebraic closure $\overline{F}$ of F. Then there exists a unique splitting field $L \subseteq \overline{F}$ of S over F. Namely $L = F(T)$ where $T = \{\alpha \in \overline{F} | f(\alpha) = 0 \text{ for some } f \in S\}$.

Proposition 2.23. *Let F be a field and $S \subseteq F[x] \setminus F$. Any two splitting fields for S over F are isomorphic via an isomorphism fixing F.*

Proof. Let L_1, L_2 be splitting fields for S over F and $\overline{L_1}, \overline{L_2}$ their algebraic closures. Since L_1, L_2 are algebraic over F, $\overline{L_1}, \overline{L_2}$ are also algebraic closures for F. Define $T_i = \{\alpha \in \overline{L_i} | f(\alpha) = 0 \text{ for some } f \in S\}$. Then $L_i = F(T_i)$. Extend 1_F to $\tau : \overline{L_1} \to \overline{L_2}$. By the corollary, τ is an isomorphism. Since τ fixes F, $\tau(T_1) = T_2$. Thus $\tau(L_1) = \tau(F(T_1)) = F(\tau(T_1)) = F(T_2) = L_2$. So $\tau|_{L_1} : L_1 \to L_2$ is an isomorphism. $\square$

Remark. With the above notation, $\rho : L_1 \to \overline{L_2}$ which fixes F is an isomorphism from L_1 to L_2.

Proposition 2.24. *Let F be a field, $S \subseteq F[x] \setminus F$ and $\overline{F}$ an algebraic closure of F. Let $L \subset \overline{F}$ be a splitting field for S over F. Then any field map $\sigma : L \to \overline{F}$ which fixes F is an automorphism of L.*

Proof. Apply previous proposition with $L_1 = L_2 = L$. $\square$

2.2 Normality

Theorem 2.25. *Let F be a field and $\overline{F}$ an algebraic closure of F. Let $F \subseteq E \subseteq \overline{F}$ be a field. Then TFAE*

1. *E is a splitting field for some $S \in F[x] \setminus F$.*

2. *Any embedding $\sigma : E \to \overline{F}$ which fixes F is an automorphism of E.*

3. *Any irreducible polynomial in $F[x]$ with a root in E splits in E.*

If E/F satisfies the above, we say E/F is ***normal****.*

Proof. **(1)⇒(2)** Previous Proposition

(2)⇒(3) Let $f(x) \in F[x] \setminus F$ be irreducible and have a root $\alpha \in E$. Let β be another root of $f(x)$ in $\overline{F}$. Consider $F(\alpha) \to F(\beta) \hookrightarrow \overline{F}$ defined by $p(\alpha) \mapsto p(\beta)$. Extend σ to $\tau : E \to \overline{F}$. Then τ fixes F and by (2), $\tau(E) = E$. So $\beta = \tau(\alpha) \in E$. Thus all the roots of f are in E which implies $f(x)$ splits.

(3)⇒(1) Let $S = \{f(x) \in F | f(x)$ is irreducible and has a root in $E\}$. Let L be the splitting field in $\overline{F}$ for S over F. Want to show $E = L$. By (3), every polynomial in S splits in E so $L \subseteq E$. Let $\alpha \in E \subseteq \overline{F}$. Let $f(x) = \text{Irred}(\alpha, F)$. Then $f(x) \in S$ implies $\alpha \in L$. Thus $L = E$. □

Remarks.

1. If $[E : F] = 2$, then E/F is normal as (3) is true.

2. $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not normal since $x^3 - 2$ is irreducible in $\mathbb{Q}[x]$ and has a root in $\mathbb{Q}(\sqrt[3]{2})$ but the other two roots are not in $\mathbb{Q}(\sqrt[3]{2})$ as they are complex.

3. If $K \subseteq F \subseteq E$ and E/K is normal, so is E/F. If E is a splitting field for S over K then it is also the splitting field for S over F. Note that F/K need not be normal. For example $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[3]{2}) \subseteq \mathbb{Q}(\sqrt[3]{2}, \omega)$.

4. If F/K and E/F are normal, then E/K need not be normal. For example $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt[4]{2})$ as $x^4 - 2$ does not split in $\mathbb{Q}(\sqrt[4]{2})$.

Note. If we say normal, we imply algebraic.

Proposition 2.26. *Let F be a field, $\overline{F}$ an algebraic closure of F, and $\{E_\lambda\}$ a family of subfields of $\overline{F}$ containing F. If each E_λ/F is normal, then $\cap E_\lambda/F$ is normal.*

Proof. Let $f(x)$ be an irreducible polynomial in $F[x]$ with a root in $\cap E_\lambda$. Then it has a root in each E_λ which implies it splits in each E_λ as they are normal. Thus f splits in $\cap E_\lambda$. □

Definition 2.27. *Let E/F be an algebraic extension. The* ***normal closure*** *of E/F in $\overline{F}$ is*

$$\bigcap_{E \subset L \subset \overline{F}, L/F \text{ normal}} L,$$

the smallest normal extension of F containing E.

Remark. Suppose $E = F(\alpha_1, ..., \alpha_n)$ is algebraic over F. Let L be the splitting field for

$$\{\text{Irred}(\alpha_1, F), ..., \text{Irred}(\alpha_n, F)\}$$

over F. Then L is the normal closure of E/F.

Example. Let $E = \mathbb{Q}(\sqrt[3]{2})$. The normal closure of $E/\mathbb{Q}$ is $\mathbb{Q}(\sqrt[3]{2}, \omega)$.

Definition 2.28. *Let E_1, E_2 be subfields of a field L. The **compositum** (or **join**) of E_1 and E_2 is*

$$E_1E_2 = \bigcap_{E_1 \cup E_2 \subset F \subset L, F \text{ a field}} F,$$

the smallest subfield of L containing E_1 and E_2.

Remarks. Let $E_1, E_2 \subseteq L$.

1. $E_1E_2 = E_1(E_2) = E_2(E_1) = \{\frac{\sum \alpha_i\beta_i}{\sum \gamma_j\delta_j} | \alpha_i, \gamma_j \in E_1, \beta_i, \delta_j \in E_2\}$.
2. If E_1, E_2 are algebraic over F then $E_1E_2 = \{\sum \alpha_i\beta_i | \alpha_i \in E_1, \beta_i \in E_2\}$ since if α is algebraic over F then the smallest field containing it is the smallest ring containing it.
3. $E_1 = K(\alpha_1, ..., \alpha_n), E_2 = K(\beta_1, ..., \beta_n)$. Then $E_1E_2 = K(\alpha_1, ...\alpha_n, \beta_1, ..., \beta_n)$.

Proposition 2.29. *Suppose E_1/F and E_2/F are normal. Then E_1E_2/F is normal.*

Proof. Suppose $\sigma : E_1E_2 \to \overline{F}$ is an embedding which fixes F. Now $\sigma|_{E_1}, \sigma|_{E_2}$ are embeddings of E_1, E_2 into $\overline{F}$ which fix F. Thus $\sigma(E_1) = E_1$ and $\sigma(E_2) = E_2$. Now $\underbrace{\sigma(E_1E_2) = \sigma(E_1)\sigma(E_2)}_{\text{this requires work}} = E_1E_2$. So E_1E_2/F is normal.

The work: Let $\alpha \in E_1E_2$. Then $\alpha = e_1\ell_1 + ... + e_n\ell_n$. Then $\sigma(\alpha) = \sigma(e_1)\sigma(\ell_1) + ... + \sigma(e_n)\sigma(\ell_n) \in \sigma(E_1)(\sigma(E_n))$. Similarly, $\sigma(E_1)\sigma(E_2) \subseteq \sigma(E_1E_2)$. □

2.3 Separability

Definition 2.30. *Let $f(x) \in F[x] \setminus F$. A root $\alpha \in \overline{F}$ of $f(x)$ is called a **multiple root** of $f(x)$ if $(x-\alpha)^2 | f(x)$ in $\overline{F}[x]$. Otherwise, α is a **simple root**.*

Definition 2.31. *Let $f(x) \in F[x]$ and say $f(x) = a_nx^n + ... + a_1x + a_0$. The **derivative** f' of $f(x)$ is $f'(x) = na_nx^{n-1} + ... + a_1$ where $ka_k = \underbrace{a_k + ... + a_k}_{k \text{ times}}$.*

Note. One can check $(f+g)' = f' + g', (cf)' = cf', (fg)' = fg' + f'g, (f(g))' = f'(g)g'$

Example. Consider $f(x) = x^6 + 2x^5 + x^3 + 2 \in \mathbb{Z}_3[x]$. Then $f' = 6x^5 + 10x^4 + 3x^2 = 10x^4$.

Proposition 2.32. *Let $f(x) \in F[x] \setminus F$ and $\alpha \in \overline{F}$. Then α is a multiple root in $f(x)$ if and only if $f(\alpha) = f'(\alpha) = 0$.*

Proof. $\Rightarrow$ Say $f(x) = (x-\alpha)^2 g(x)$ for $g(x) \in \overline{F}[x]$. Then $f'(x) = 2(x-\alpha)g(x) + (x-\alpha)^2 g'(x)$. Clearly $f'(\alpha) = f(\alpha) = 0$.

$\Leftarrow$ As $f(\alpha) = 0$, we can say $f = (x-\alpha)g(x)$ for some $g(x) \in \overline{F}[x]$. Taking the derivative, we see $f'(x) = g(x) + (x-\alpha)g'(x)$ and plugging in α we see $0 = g(\alpha)$. Thus $g(x) = (x-\alpha)h(x)$ for some $h(x) \in \overline{F}[x]$. Then $f(x) = (x-\alpha)^2 h(x)$. □

Proposition 2.33. *Let $f \in F[x]$. Then $f(x)$ has no multiple roots in $\overline{F}$ if and only if $\gcd(f, f') = 1$.*

Proof. Suppose $\gcd(f, f') = h \neq 1$. Let α be a root of h in $\overline{F}$. Then α is a root of f and f' which implies α is a multiple root. Now suppose f has a multiple root $\alpha \in \overline{F}$. Let $h = \text{Irred}(\alpha, F)$. Since $f(\alpha) = f'(\alpha) = 0$, we see $h|f$ and $h|f'$. Thus $h| \gcd_F(f, f')$ which implies $\gcd(f, f') > 1$. □

Proposition 2.34. *Let F be a field and $f(x)$ an irreducible polynomial in $F[x]$.*

1. *If char $F = 0$, then f has no multiple roots.*
2. *If char $F = p > 0$, then $f(x)$ has a multiple root if and only if $f(x) = g(x^p)$ for some $g(x) \in F[x]$.*

Proof. 1. Let $f = a_nx^n + \cdots + a_1x + a_0$. Then $f' = na_nx^{n-1} + \cdots + a_1 \neq 0$. Since f is irreducible and $\deg f' < \deg f$, we see $\gcd(f, f') = 1$. Thus f has no multiple roots by the previous proposition.

2. By the same argument, f has multiple roots if and only if $f' = 0$. Of course, $f' = 0$ if and only if $ia_i = 0$ for all i which occurs if and only if $i = 0 \mod p$ or $a_i = 0 \mod p$ for all i as F is an integral domain. This is if and only if $f(x) = a_{pm}x^{pm} + a_{p(m-1)}x^{p(m-1)} + \cdots + a_0 = g(x^p)$ for some $g(x) \in F[x]$. $\square$

Theorem 2.35. *Let E/F be an algebraic extension and let $\sigma : F \to L_1$ and $\tau : F \to L_2$ be embeddings of F into algebraically closed fields L_1 and L_2. Let $S_\sigma = \{\pi : E \to L_1 | \pi|_F = \sigma\}$ and $S_\tau = \{\pi : E \to L_2 | \pi|_F = \tau\}$. Then $|S_\sigma| = |S_\tau|$.*

Proof. Consider the isomorphism $\tau\sigma^{-1} : \sigma(F) \to F \to \tau(F) \hookrightarrow L_2$. Then there exists an extension $\lambda : \overline{\sigma(F)} \to L_2$ such that $\lambda|_{\sigma(F)} = \tau\sigma^{-1}$ where $\overline{\sigma(F)}$ is the algebraic closure of $\sigma(F)$ in L_1. In fact, if $\overline{\tau(F)}$ is the algebraic closure of $\tau(F)$ in L_2, then $\lambda : \overline{\sigma(F)} \to \overline{\tau(F)}$ is an isomorphism. Let $\pi \in S_\sigma$. Since E/F is algebraic and π extends σ, we see $\pi(E)$ is algebraic over $\sigma(F)$. So $\pi(E) \subseteq \overline{\sigma(F)}$. Then $\lambda\pi : E \to \overline{\sigma(F)} \to L_2$ and $\lambda\pi|_F = \lambda\sigma = \tau$. Thus $\lambda\pi \in S_\tau$. Thus we have a map $\widetilde{\lambda} : S_\sigma \to S_\tau$ defined by $\pi \mapsto \lambda\pi$. This is injective as λ is. Similarly, we can define $\widetilde{\lambda^{-1}} : S_\tau \to S_\sigma$ which is again injective. Clearly $\widetilde{\lambda}\widetilde{\lambda^{-1}}(\pi) = \widetilde{\lambda}\lambda^{-1}(\pi) = \lambda\lambda^{-1}(\pi) = \pi$ and $\widetilde{\lambda}\widetilde{\lambda^{-1}}(\pi) = \pi$. Thus λ is bijective which implies $|S_\tau| = |S_\sigma|$. $\square$

Definition 2.36. *Let E/F be an algebraic extension. The* ***separable degree*** *of E/F, denoted $[E : F]_S$, is $|S_\sigma| = |\{\pi : E \to \overline{F} | \pi|_F = 1_F\}|$.*

Proposition 2.37. *Let $E = F(\alpha)$ where α is algebraic over F. Then $[E : F]_S =$ the number of distinct roots of Irred(α, F) in $\overline{F} \leq \deg$ Irred$(\alpha, F) = [E : F]$.*

Proof. Let $f(x) = \text{Irred}(\alpha, F) \in F[x]$. Let $\pi : F(\alpha) \to \overline{F}$ such that π fixes F. Clearly π is determined by $\pi(\alpha)$. Also $\pi(\alpha)$ is a root of $f(x)$ as π fixes F. So $[F(\alpha) : F]_S \leq$ the number of distinct roots of $f(x)$ in $\overline{F}$. Let β be any root of $f(x)$. Then $\pi : F(\alpha) \to F[x]/(f(x)) \to F(\beta) \subseteq \overline{F}$ is an embedding of $f(\alpha)$ into $\overline{F}$ taking $\alpha \mapsto \beta$. So $[F(\alpha) : F]_S \geq$ the number of distinct roots of $f(x)$ in $\overline{F}$. $\square$

Theorem 2.38. *Let $K \subseteq F \subseteq E$ be fields and E/K algebraic. Then $[E : K]_S = [E : F]_S[F : K]_S$. Moreover, if E/K is finite, then $[E : K]_S \leq [E : K]$.*

Proof. Let $\overline{E}$ be a fixed algebraic closure of E. Let $S = \{\pi : F \to \overline{E} | \pi \text{ fixes } K\}$. Then $|S| = [F : K]_S$. Let $T_\pi = \{\tau : E \to \overline{E} | \tau|_F = \pi\}$ for all $\pi \in S$. By the Theorem, $|T_\pi| = [E : F]_S$. If $\pi_1 \neq \pi_2 \in S$, then $T_{\pi_1} \cap T_{\pi_2} = \emptyset$. If $\tau \in T_\pi$, then $\tau|_K = 1_K$. Therefore $\cup_{\pi\in S} T_\pi \subseteq \{\sigma : E \to \overline{E} | \sigma \text{ fixes } K\}$. On the other hand, if $\sigma : E \to \overline{E}$ fixes K, then $\sigma|_F : F \to \overline{E}$ fixes K which implies $\sigma|_F \in S$. Say $\sigma|_F = \pi$. Then $\sigma \in T_\pi$. So $\cup_{\pi\in S} T_\pi = \{\sigma : E \to \overline{E} | \sigma \text{ fixes } K\}$. Now

$$[E : K]_S = |\{\sigma : E \to \overline{E} | \sigma \text{ fixes } K\}| = | \cup_{\pi\in S} T_\pi| = \cup_{\pi\in S}|T_\pi| = |S||T_\pi| = [F : K]_S[E : F]_S.$$

Moreover, suppose $[E : K] < \infty$. Then $E = K(\alpha_1, ..., \alpha_n)$ for some n. If $n = 1$, done by previous proposition. If $n > 1$, let $L = K(\alpha_1, ..., \alpha_{n-1})$. By induction on n, $[L : K]_S \leq [L : K]$. Now $E = L(\alpha)$ implies $[E : L]_S \leq [E : L]$ by proposition. Thus, by the multiplicative property, $[E : K]_S \leq [E : K]$. $\square$

Definition 2.39. *A polynomial $f(x) \in F[x]$ is called* ***separable*** *if $f(x)$ has no multiple roots in an algebraic closure. Let α be algebraic over F. Then α* ***is separable*** *over F if Irred(α, F) is separable. Let E/F be an algebraic extension. Then E/F is separable if and only if $\alpha \in E$ is separable over F for all α.*

Remarks.

1. Suppose α is algebraic over F. Then α is separable over F if and only if $[F(\alpha) : F]_S = [F(\alpha) : F]$.

2. Let $K \subseteq F \subseteq E$ be algebraic extensions. If E/K is separable, then E/F and F/K are separable.

   Proof. Let $\alpha \in E$. Know $\text{Irred}(\alpha, F) | \text{Irred}(\alpha, K)$ in $F[x]$. If α is separable over K, then $\text{Irred}(\alpha, K)$ has no multiple roots which implies $\text{Irred}(\alpha, F)$ has no multiple roots. Thus α is separable over F. $\square$

Theorem 2.40. *Suppose E/F is finite. Then E/F is separable if and only if $[E:F]_S = [E:F]$.*

Proof. $(\Leftarrow)$ Let $\alpha \in E$. Consider $F \subseteq F(\alpha) \subseteq E$. We know

$$[E:F(\alpha)]_S[F(\alpha):F]_S = [E:F]_S = [E:F] = [E:F(\alpha)][F(\alpha):F].$$

Since $[F(\alpha):F]_S \leq [F(\alpha):F]$, they are equal and thus α is separable by remark 1.

$(\Rightarrow)$ Assume $E = F(\alpha_1, ..., \alpha_n)$. Induct on n. If $n = 1$, done by remark. Let $L = F(\alpha_1, ..., \alpha_{n-1})$. Then L/F is separable by remark 2 and by the induction hypothesis $[L:F]_S = [L:F]$. Note $E = L(\alpha_n)$. Since E/F is separable, so is E/L. So $[E:L]_S = [E:L]$ by the $n = 1$ case. Thus

$$[E:F]_S = [E:F]_S[L:F]_S = [E:F][L:F] = [E:F].$$

$\square$

Corollary 2.41. *Suppose $E = F(\alpha_1, ..., \alpha_n)$. Then E/F is separable if and only if each α_i is separable over F.*

Proof. $(\Rightarrow)$ Clear

$(\Leftarrow)$ Induct on n. If $n = 1$, done by remark and theorem. Let $L = F(\alpha_1, ..., \alpha_{n-1})$. Then L/F is separable and thus $[L:F]_S = [L:F]$. Also E/L is separable by the $n = 1$ case which implies $[E:L]_S = [E:L]$. Multiplying, we see $[E:F]_S = [E:F]$ which implies E/F is separable. $\square$

Definition 2.42. *Let E be an arbitrary algebraic extension of F. Then E is* ***separable*** *over F is every finitely generated subextension is separable.*

Corollary 2.43. *Suppose $E = F(S)$. Then E/F is separable if and only if α is separable over F for all $\alpha \in S$.*

Proof. $(\Rightarrow)$ Clear

$(\Leftarrow)$ Note that $F(S) = \{\sum_{finite} a_i s_i | a_i \in F, s_i \in S\}$. Thus, for all $\alpha \in E$, there exists a finitely generated subfield such that $\alpha \in F(s_1, ..., s_n)$. By the finite case, each of these finitely generated subfields are separable. Thus, by definition, E is separable. $\square$

Proposition 2.44. *Suppose $K \subseteq F \subseteq E$ are fields. Then E/K is separable if and only if E/F and F/K are separable.*

Proof. $(\Rightarrow)$ Done (Remark 2 above)

$(\Leftarrow)$ Let $\alpha \in E$ and $f(x) = \text{Irred}(\alpha, F) = x^n + c_{n-1}x^{n-1} + ... + c_1 x + c_0$. Since α is separable over F, f is a separable polynomial. Let $L = K(c_0, ..., c_{n-1})$. Then $f(x) \in L[x]$ and $f(x) = \text{Irred}(\alpha, L)$. So f is separable, which implies α is separable over L. Thus $[L(\alpha):L]_S = [L(\alpha):L]$. Since F/K is separable, each c_i is separable over K. So $L = K(c_0, ..., c_{n-1})$ is separable over K. Thus $[L(\alpha):K]_S = [L(\alpha):K]$. Thus $L(\alpha)/K$ is separable, which implies α is separable over K. $\square$

Proposition 2.45. *Suppose E/F is separable and let L be the normal closure of E/F. Then L/F is separable.*

Proof. Let $S = \{\text{Irred}(\alpha, F) | \alpha \in E\} \subseteq F[x]$. Then L is the splitting field for S over F. Let

$$R = \{\alpha \in \overline{F} | \alpha \text{ is a root of } f(x) \text{ for some } f \in S\}.$$

Then $L = F(R)$. Since S is a set of separable polynomial, for all $r \in R$ we see $\text{Irred}(r, F) \in S$ which implies r is separable. Thus L is separable. $\square$

Definition 2.46. *A field F is called* ***separably closed*** *if whenever $\alpha \in \overline{F}$ is separable over F we have $\alpha \in F$. Equivalently, every separable irreducible polynomial in $F[x]$ is degree 1. A* ***separable closure*** *of a field F is a field $E \supseteq F$ such that E is separably closed and E/F is separable.*

Proposition 2.47. *Separable closures exist.*

Proof. Let F be a field, $\overline{F}$ an algebraic closure of F, and $E = \{\alpha \in \overline{F} | \alpha \text{ is separable over } F\}$. This is a field as for $\alpha, \beta \in E$, $F(\alpha, \beta)$ is separable over F which implies $\alpha \pm \beta, \alpha\beta, \frac{\alpha}{\beta} \in F(\alpha, \beta)$ which implies they are separable and thus in E. Clearly E/F is separable, so we need only to show it is separably closed. Suppose $\alpha \in \overline{F} = \overline{E}$ is separable over E. Then $E(\alpha)/E$ is separable and E/F is separable which implies $E(\alpha)/F$ is separable. Thus α is separable over F and therefore $\alpha \in E$. ☐

Notation. F^{sep} denotes a separable closure of F.

Definition 2.48. *A field F is called* ***perfect*** *if every algebraic extension of F is separable. Equivalently, $\overline{F}/F$ is separable.*

Proposition 2.49. *Every field of characteristic 0 is perfect.*

Proof. Let α be algebraic over F where char $F = 0$ and $f(x) = \text{Irred}(\alpha, F)$. Then f has no multiple roots which implies α is separable. ☐

Suppose char $F = p$. Then $(a+b)^{p^n} = a^{p^n} + b^{p^n}$. Thus there exists a field homomorphism $\phi : F \to F$ defined by $a \mapsto a^p$. This is called the **Frobenius map**. Then $\phi(F) = F^p = \{a^p | a \in F\}$ is a subfield of F.

Proposition 2.50. *Suppose char $F = p$. Then F is perfect if and only if $F = F^p$.*

Proof. ($\Rightarrow$) Let $a \in F$. Consider $f(x) = x^p - a \in F[x]$. Let α be a root of $f(x)$ in some splitting field of $f(x)$ over F. Let $g(x) = \text{Irred}(\alpha, F)$. Then $g(x)|f(x)$. Note $\alpha^p = a$ implies $x^p - a = x^p - \alpha^p = (x-\alpha)^p$. Then $g(x) = (x-\alpha)^m$ for $m < p$ in the splitting field. But α is separable over F as F is perfect. So $m = 1$. Then $g(x) = x - \alpha \in F[x]$ which implies $\alpha \in F$. So $a = \alpha^p \in F^p$. So $F = F^p$.

($\Leftarrow$) Let α be an algebraic element over F. Let $f(x) = \text{Irred}(\alpha, F)$. Suppose α is not separable, i.e., f has multiple roots. This means $f(x) = g(x^p)$ for some $g(x) \in F[x]$. Say $g(x) = x^m + x_{m-1}x^{m-1} + ... + c_0$. As $F = F^p$, let $c_i = d_i^p$. Then

$$f(x) = g(x^p) = (x^m)^p + d_{m-1}^p(x^{m-1})^p + ... + d_1^p x^p + d_0^p = (x^m + d_{m-1}x^{m-1} + ... + d_1 x + d_0)^p.$$

This contradicts the fact that f is irreducible. So α is separable. ☐

Corollary 2.51. *Every finite field is perfect.*

Proof. First note a finite field F has characteristic $p < 0$ where p is prime [Since $\phi : \mathbb{Z} \to F$ defined by $n \mapsto n \cdot 1$ is not injective (as F is finite), say $\ker \phi = (p) \neq 0$. Then $\mathbb{Z}/(p) \hookrightarrow F$ and since F is a domain, p is prime.] Consider the Frobenius map $\phi : F \to F$ defined by $a \mapsto a^p$. This is an injective homomorphism and since $|F| < \infty$ it is surjective as well. Thus $F = F^p$ which implies F is perfect. ☐

Examples.

- Let F be any field of characteristic $p > 0$. Let t be an indeterminant and $E = F(t)$. Then $x^p - t \in E[x]$ is an irreducible nonseparable polynomial. Thus E is not perfect.

  Proof: **Eisenstein:** Let R be a UFD, K its fraction field and $f(x) = a_n x^n + ... + a_0 \in R[x]$. Suppose there exists a prime element $p \in R$ such that $p \nmid a_n, p | a_i$ for $0 \leq i \leq n-1$, and $p^2 \nmid a_0$. Then $f(x)$ is irreducible over $K[x]$.

  Apply Eisenstein with $R = F[t]$, a PID. Note t is a prime. Then $f(x) = x^p - t \in R[x]$ is irreducible in $E[x]$, a quotient field. Note $f'(x) = 0$, so $f(x)$ has multiple roots, which implies nonseparable.

- By the same proof, $F(t)/F(t^p)$ is not separable as $x^p - t^p = \text{Irred}(t, F(t^p))$ has multiple roots.

Definition 2.52. *Let E/F be a field extension. A primitive element for E/F is an element $\alpha \in E$ such that $E = F(\alpha)$.*

Theorem 2.53 (Primitive Element Theorem). *Let $[E : F] < \infty$. Then there exists a primitive element for E/F if and only if there are finitely many intermediate fields of E/F. Furthermore, if E/F is separable, then there exists a primitive element.*

Proof. **(⇒)** Suppose $E = F(\alpha)$. Let $f(x) = \text{Irred}(\alpha, F)$. Let L be a splitting field of $f(x) = (x - \alpha_1)\cdots(x - \alpha_n)$. Define a map

$$\lambda : \{\text{Intermediate fields of } E/F\} \to \{\text{monic factors of } f(x) \text{ in } L\},$$

such that $K \mapsto \text{Irred}(\alpha, K)$. Clearly, there are only finitely many factors of $f(x)$ in L.

Claim: λ is injective.

Proof: It is enough to show K is determined by $\text{Irred}(\alpha, K) = x^n + c_{n-1}x^{n-1} + ... + c_0 = g(x)$. Note $[E : K] = [K(\alpha) : K] = n$. Let $L' = F(c_0, ..., c_{n-1}) \subseteq K$. Then $g(x) \in L'[x]$ and is irreducible over L'. So $g(x) = \text{Irred}(\alpha, L')$. Since $E = L'(\alpha)$, $[E : L'] = [L'(\alpha) : L'] = n$. So $[K : L'] = 1$, that is $K = L'$. Thus λ is injective.

(⇐) Suppose $|F| < \infty$. Since $[E : F] < \infty$ we have $|E| < \infty$. Note that E^* is cyclic, so $E^* =< \alpha >$ for some $\alpha \in E$. Of course, E is a field, so everything but 0 is a unit. Thus $E = F(\alpha)$. Now suppose $|F| = \infty$. Let $E = F(\alpha_1, ..., \alpha_n)$. We will induct on n. If $n = 1$, obvious. So let $L = F(\alpha_1, ..., \alpha_{n-1})$. By induction, $L = F(\gamma)$ for some $\gamma \in L$. Then $E = F(\alpha_1, \gamma)$. So it is enough to prove the result for $E = F(\alpha, \beta)/F$. Let $\Lambda = \{F(\alpha + c\beta) | c \in F\}$. This is a subset of the set of all intermediate fields of E/F. Thus Λ is finite. Since $|F| = \infty$, there exists $c_1 \neq c_2 \in F$ such that $F(\alpha + c_1\beta) = F(\alpha + c_2\beta) =: L$. Then $\alpha + c_1\beta, \alpha + c_2\beta \in L$. Subtracting, we get $(c_1 - c_2)\beta \in L$. But $0 \neq c_1 - c_2 \in F \subseteq L$. Thus $\beta \in L$ which implies $\alpha \in L$. So $F(\alpha + c_1\beta) = F(\alpha, \beta)$. Thus we have found a primitive root.

Finally, let E/F be finite and separable. As above, the finite case has a primitive element equal to the cyclic generator and we can reduce the infinite case to $E = F(\alpha, \beta)/F$. Let $[E : F] = n = [E : F]_S$. Let $\{\sigma_1, ..., \sigma_n\}$ be the distinct embeddings of $E \to \overline{F}$ which fix F. Let $P(x) = \prod_{i<j}(\sigma_i(\alpha) - \sigma_j(\alpha))x + (\sigma_i(\beta) - \sigma_j(\beta)) \in \overline{F}[x]$. Note that $P(x) \neq 0$ as $\sigma_i \neq \sigma_j$ and σ_i are determined by $\sigma_i(\alpha)$ and $\sigma_i(\beta)$. So $P(x)$ has finitely many roots in F. Since $|F| = \infty$, there exists $c \in F$ such that $P(c) \neq 0$. Thus, rearranging the terms of each factor of $P(x)$ we see $\sigma_i(c\alpha + \beta) \neq \sigma_j(c\alpha + \beta)$ for all $i < j$. Now $c\alpha + \beta \in E$ and $\sigma_i|_{F(c\alpha+\beta)}$ are distinct for $i = 1, ..., n$. Thus $[F(c\alpha + \beta) : F]_S \geq n$. Of course, $[F(c\alpha + \beta) : F]_S \leq [E : F]_S = n$. So $[F(c\alpha + \beta) : F] = n$ which implies $E = F(c\alpha + \beta)$.

□

Example. Let F be a field of characteristic p (e.g. $F = \mathbb{Z}_p$). Let t, u be algebraically independent elements over F (that is, t and u are indeterminants with no relations like $u = t^2$). Consider

$$\begin{array}{cl} F(t,u) = L(t) & \\ |p & \text{since } x^p - t^p \text{ is irreducible over } L. \\ F(t^p, u) = L = K(u) & \\ |p & \text{since } x^p - u^p \text{ is irreducible over } K. \\ F(t^p, u^p) = K & \end{array}$$

Then $[F(t, u) : F(t^p, u^p)] = p^2$. We will show there does not exist a primitive element for this extension. Let $g(t, u) \in F(t, u)$ and note that $g(t, u)^p \in F(t^p, u^p)$. So $[F(t^p, u^p, g(t, u)) : F(t^p, u^p)] \leq p$. So $F(t, u) \neq F(t^p, u^p, g(t, u))$. Thus there is no primitive element. Note that this also implies there are infinitely many intermediate fields between the two fields.

2.4 Finite Fields

Often, if char $F = p$, we say that $\mathbb{Z}_p \subseteq F$. We can do this by considering the embedding $\mathbb{Z}_p \to F$ defined by $\overline{1} \mapsto 1$ and identifying $\mathbb{Z}_p$ with its image.

Proposition 2.54. *Let F be a finite field of characteristic p. Then $|F| = p^n$.*

Proof. Note that F is a $\mathbb{Z}_p$ vector space with dimension n, for some n. Then $F \cong \mathbb{Z}_p^n$ as vector spaces. This says $|F| = p^n$. $\square$

Proposition 2.55. *Let p be a prime and $n > 0$ an integer. Then there exists a field F such that $|F| = p^n$. In fact, any field of order p^n is a splitting field for $x^{p^n} - x$ over $\mathbb{Z}_p$. Therefore, any two fields of order p^n are isomorphic and any algebraically closed field of characteristic p contains a unique field of order p^n.*

Proof. First we show existence. Let E be the splitting field for $f(x) = x^{p^n} - x$ over $\mathbb{Z}_p$. Let $F = \{\alpha \in E | \alpha^{p^n} - \alpha = 0\}$. Since $(\alpha+\beta)^{p^n} = \alpha^{p^n} + \beta^{p^n} = \alpha + \beta$ and $(\alpha\beta)^{p^n} = \alpha^{p^n}\beta^{p^n} = \alpha\beta$ for all $\alpha, \beta \in F$, we see that F is a subfield of E. Now $|F| \leq p^n$ as $x^{p^n} - x$ has at most p^n roots. Of course, $\gcd(f, f') = 1$ as $f' = -1$, so $x^{p^n} - x$ has distinct roots, which implies $|F| = p^n$. Thus, we have found a field of order p^n. To show uniqueness, let F be a field of order p^n and note that F^* is a group of order $p^n - 1$. So for all $\alpha \in F \setminus \{0\}$, $\alpha^{p^n-1} = 1$, which implies $\alpha^{p^n} = \alpha$. Thus every element of F is a root of $x^{p^n} - x = 0$. As $|F| = p^n$, all the roots of $x^{p^n} - x$ are in F. So F is a splitting field. $\square$

Proposition 2.56. *Let F be a field of order p^n. Then F is a splitting field for an irreducible polynomial $f(x) \in \mathbb{Z}_p[x]$ of degree n. Moreover, any irreducible polynomial of degree n in $\mathbb{Z}_p[x]$ splits in F. Finally $F \cong \mathbb{Z}_p[x]/(f(x))$ where $f(x)$ is irreducible and* $\deg f = n$.

Proof. Recall (HW Exercise) that F is normal over $\mathbb{Z}_p$. Let $F = \mathbb{Z}_p(\alpha)$. (We can do this by the Primitive Element Theorem as every finite field is separable). Let $f(x) = \text{Irred}(\alpha, \mathbb{Z}_p)$. Since $F/\mathbb{Z}_p$ is normal and $f(x)$ has a root in F, $f(x)$ splits over F. Note that $\mathbb{Z}_p[x]/(f(x))$ is a field of order p^n as f is irreducible of degree n. Let E be a splitting field for $g(x)$ contained in $\overline{F}$ where $\deg g = n$. Then $E = \mathbb{Z}_p(\beta)$ where β is a root of $g(x)$ and $|E| = p^n$. But, there exists a unique field of order p^n in $\overline{F}$. Thus $E = F$. $\square$

2.5 Inseparability

Theorem 2.57. *Let F be a field of characteristic $p > 0$ and $\alpha \in \overline{F}$.*

1. *α is separable over F if and only if $F(\alpha) = F(\alpha^p)$.*
2. *If α is inseparable over F, then $[F(\alpha) : F(\alpha^p)] = p$ and Irred$(\alpha, F(\alpha^p)) = x^p - \alpha^p$.*
3. *For all $n \geq 1$, $[F(\alpha^{p^n}) : F]_S = [F(\alpha) : F]_S$.*
4. *α^{p^n} is separable over F for all $n >> 0$.*
5. *Let n be the smallest $n >> 0$ such that α^{p^n} is separable over F. Then $[F(\alpha) : F] = p^n[F(\alpha) : F]_S$.*

Proof. 1. ($\Rightarrow$) Suppose α is separable over F. Then α is separable over $F(\alpha^p)$. Certainly, α is a root of $x^p - \alpha^p$. So $\text{Irred}(\alpha, F(\alpha^p)) | x^p - \alpha^p = (x - \alpha)^p$. Since α is separable, there are no multiple roots. Thus $\text{Irred}(\alpha, F(\alpha^p)) = x - \alpha$. So $\alpha \in F(\alpha^p)$. Thus $F(\alpha) = F(\alpha^p)$.

($\Leftarrow$) Suppose $F(\alpha) = F(\alpha^p)$. Let $f(x) = \text{Irred}(\alpha, F)$. Suppose $f(x)$ has a multiple root. Then $f(x) = g(x^p)$ for some $g(x) \in F[x]$. Then $g(\alpha^p) = f(\alpha) = 0$ which implies $\text{Irred}(\alpha^p, F) | g(x)$. Then $[F(\alpha^p) : F] \leq \deg g < \deg f = [F(\alpha) : F]$, a contradiction. Thus f has no multiple roots, which implies α is separable.

2. Suppose α is inseparable over F. Consider $\text{Irred}(\alpha, F(\alpha^p)) | (x-\alpha)^p$. This says $\text{Irred}(\alpha, F(\alpha^p)) = (x-\alpha)^m = x^m - m\alpha x^{m-1} - \cdots \in F(\alpha^p)[x]$ where $1 \leq m \leq p$. If $m < p$, then m is a unit. But $-m\alpha \in F(\alpha^p)[x]$. Thus $\alpha \in F(\alpha^p)$. This says $F(\alpha) = F(\alpha^p)$, a contradiction to (1) as α is inseparable. Thus $m = p$ which implies $[F(\alpha) : F(\alpha^p)] = p$.

3. Consider $[F(\alpha) : F(\alpha^p)]_S$. This is the number of distinct roots of $\text{Irred}(\alpha, F(\alpha^p))$. By (1) and (2), $[F(\alpha) : F(\alpha^p)]_S = 1$. By induction (and the $n = 1$ case), $[F(\alpha^{p^n}) : F]_S = [F(\alpha^{p^{n-1}}) : F]_S = [F(\alpha) : F]_S$.

4. Consider the descending chain of fields: $F(\alpha) \supseteq F(\alpha^p) \supseteq F(\alpha^{p^2}) \supseteq \cdots \supseteq F$. This can be viewed as a descending chain of $F-$vector spaces, all of which are subspaces of the finite dimensional vector space $F(\alpha)$. Thus there exists n such that $F(\alpha^{p^n}) = F(\alpha^{p^{n+1}})$ and by (1), α^{p^n} is separable over F.

5. Let n be the least element such that α^{p^n} is separable over F. Then

$$\begin{aligned}
[F(\alpha) : F] &= [F(\alpha) : F(\alpha^{p^n})][F(\alpha^{p^n}) : F] \\
&= p^n [F(\alpha^{p^n}) : F] \text{ by iterative applications of (2)} \\
&= p^n [F(\alpha^{p^n}) : F]_S \text{ as } \alpha^{p^n} \text{ is separable} \\
&= p^n [F(\alpha) : F]_S \text{ by (3).}
\end{aligned}$$

□

Corollary 2.58. *Suppose E/F is finite and char $F = p$. Then $[E : F] = p^n [E : F]_S$ for some n.*

Proof. Say $E = F(\alpha_1, ..., \alpha_k)$. Induct on k. For $k = 1$, done by Theorem. Let $L = F(\alpha_1, ..., \alpha_{n-1})$. By induction, $[L : F] = p^{n_1}[L : F]_S$ and by $k = 1$ case $[E : L] = p^{n_2}[E : L]_S$. By the multiplicative property of separable degrees, letting $n = n_1 + n_2$, done. □

Definition 2.59. *Let E/F be a finite extension. Define the* ***inseparable*** *degree of E/F to be*

$$[E : F]_i = \frac{[E : F]}{[E : F]_S} = \begin{cases} 1 & \textit{if characteristic } 0, \\ p^n & \textit{if characteristic } p. \end{cases}$$

Remark. If $F \subseteq L \subseteq E$ where E/F is finite, $[E : F]_i = [E : L]_i [L : F]_i$.

Definition 2.60. *Let F be a field of characteristic $p > 0$ and $\alpha \in \overline{F}$. Then α is* ***purely inseparably (p.i.)*** *over F is $\alpha^{p^n} \in F$ for some $n >> 1$. An algebraic extension E/F is p.i. if $\alpha \in E$ is p.i. over F for all $\alpha \in E$.*

Lemma 2.61. *Let $\alpha \in \overline{F}$. Then TFAE*

1. *α is p.i. over F*

2. *$[F(\alpha) : F]_S = 1$*

3. *$[F(\alpha) : F]_i = [F(\alpha) : F]$*

Proof. We know (2)⇔(3) by the definition of inseparable degree. So we have

$$\begin{aligned}
\alpha \text{ is p.i. over } F &\Leftrightarrow \alpha^{p^n} \in F \text{ for } n >> 0 \\
&\Leftrightarrow [F(\alpha^{p^n}) : F] = 1 \text{ for } n >> 0 \\
&\Leftrightarrow [F(\alpha^{p^n}) : F]_S = 1 \text{ by (4) of Thm} \\
&\Leftrightarrow [F(\alpha) : F]_S = 1 \text{ by (3) of Thm}
\end{aligned}$$

□

Proposition 2.62. *Let $E = F(\alpha_1, ..., \alpha_n)$ be algebraic over F. TFAE*

1. *E/F is p.i.*

2. *Each α_i is p.i. over F*

3. $[E : F]_S = 1$

4. $[E : F]_i = [E : F]$

Proof. **(3)⇔(4):** By definition of inseparable degree.

(1)⇒(2): Clear

(2)⇒(3): Use induction on n. If $n = 1$, done by Lemma. Let $n > 1$ and $L = F(\alpha_1, ..., \alpha_{n-1})$. Then $E = L(\alpha_n)$. By induction $[L : F]_S = 1$ and by the $n = 1$ case (since α_n p.i. over F implies α_n is p.i. over L) $[E : L]_S = 1$. By multiplicative property, done.

(3)⇒(1): Let $\beta \in E$. By the Lemma, it is enough to show $[F(\beta) : F]_S = 1$. But $[F(\beta) : F]_S \leq [E : F]_S = 1$. Thus $[F(\beta) : F]_S = 1$ and β is p.i. □

Example. Let F be a field of characteristic p and t an indeterminate over F. Then $F(t)/F(t^p)$ is p.i. Note that $\overline{F(t)}/F(t^p)$ is inseparable, but not p.i.

2.6 Cyclotomic Field Extensions

Let $U_n = \{z \in \mathbb{C} | z^n = 1\}$. Note that $U_n =< e^{2\pi i/n} >=< e^{2\pi ik/n} >$ for all k such that $\gcd(k, n) = 1$. Any cyclic generator of U_n is called a **primitive nth root of unity**. There are $\phi(n)$ primitive nth roots of unity.

Definition 2.63. *The **nth cyclotomic polynomial** is*

$$\Phi_n(x) = \prod_{1 \leq i \leq n,\ \gcd(i,n)=1} (x - \omega^i)$$

where ω is any primitive root of unity.

Examples.

- $\Phi_1(x) = x - 1$
- $\Phi_2(x) = x + 1$
- $\Phi_4(x) = (x - i)(x + i) = x^2 + 1$

Facts.

1. $x^n - 1 = \displaystyle\prod_{i=0}^{n-1} (x - \omega^i)$

2. $x^n - 1 = \displaystyle\prod_{d|n,\ d>0} \Phi_d(x)$ since $x^n - 1 = \displaystyle\prod_{d|n} \left(\prod_{\omega \text{ has order } d} (x - \omega^i) \right)$.

3. $\deg \Phi_n(x) = \phi(n)$.

Lemma 2.64. $\Phi_n(x) \in \mathbb{Z}[x]$.

Proof. Induct on n. The $n = 1$ case is trivial. Let $n > 1$ and assume $\Phi_d(x) \in \mathbb{Z}[x]$ for all $d < n$. By Fact 2, $x^n - 1 = \prod_{d|n,\ d>0} \Phi_d(x) = f(x)\Phi_n(x)$ where $f(x) \in \mathbb{Z}[x]$ by induction. Note that $f(x)$ is monic, so by the Division Algorithm, $x^n - 1 = f(x)q(x) + r(x)$ where $q(x), r(x) \in \mathbb{Z}[x]$. Thus it is also true in $\mathbb{C}[x]$, where we know $x^n - 1 = f(x)\Phi_n(x)$. By the uniqueness of quotients and remainders, $r(x) = 0$ and $\Phi_n(x) = q(x) \in \mathbb{Z}[x]$. □

Theorem 2.65. *$\Phi_n(x)$ is irreducible over $\mathbb{Q}$.*

Proof. Suppose not. Then by Gauss's Lemma, since $\Phi_n(x) \in \mathbb{Z}[x]$, there exists $f, g \in \mathbb{Z}[x]$ such that $\Phi_n(x) = fg$ where f, g are monic and f is irreducible over $\mathbb{Q}$ (if not, take an irreducible factor of f and group the other factors into g). Let ω be a root of f (and therefore of $\Phi_n(x)$) and p any prime such that $p \nmid n$. Since $\gcd(p, n) = 1$ we see ω^p is also a primitive nth root of unity and thus is a root of Φ_n.

Claim: ω^p is a root of f.

Proof: If not, then $g(\omega^p) = 0$ which says ω is a root of $g(x^p)$. Since f is monic and irreducible, $f = \text{Irred}(\omega, \mathbb{Q})$. Thus $f | g(x^p)$ in $\mathbb{Q}[x]$ (and thus in $\mathbb{Z}[x]$ as it is monic). So $g(x^p) = fh$ for some $h \in \mathbb{Z}[x]$. In $\mathbb{Z}_p[x]$ we see $(\overline{g}(x))^p = \overline{g}(x^p) = \overline{fh}$. Let β be any root of $\overline{f}(x)$ in $\overline{\mathbb{Z}_p}$. Then $\overline{G}(\beta) = 0$ as we are in an ID. Then $\overline{\Phi_n(x)}$ has multiple roots, which says $\overline{x^n - 1} = x^n - 1$ has multiple roots in $\mathbb{Z}_p[x]$. But $\gcd(x^n - 1, nx^{n-1}) = 1$, a contradiction. Thus ω^p is a root of f.

Thus every primitive nth root of unity is a root of f which is enough to say $f = \Phi_n$ and since f is irreducible, $\Phi_n(x)$ is irreducible. $\square$

Corollary 2.66. *If $\omega \in \mathbb{C}$ is a primitive nth root of unity, then $[\mathbb{Q}(\omega) : \mathbb{Q}] = \phi(n)$ and Irred$(\omega, \mathbb{Q}) = \Phi_n$.*

Note. The above extension is normal as it is the splitting field for $\Phi_n(x)$.

Example. Let ω be a primitive 9th root of unity. Then $[\mathbb{Q}(\omega) : \mathbb{Q}] = \phi(9) = 6$. To find the minimal polynomial, note that $x^9 - 1 = \Phi_1\Phi_3\Phi_9 = (x^3 - 1)\Phi_9$. Thus $\text{Irred}(\omega, \mathbb{Q}) = \Phi_9(x) = x^6 + x^3 + 1$.

Definition 2.67. *An extension $\mathbb{Q}(\omega)/\mathbb{Q}$ where ω is a root of unity is called a* ***cyclotomic extension****.*

2.7 Inseparable Closure

Definition 2.68. *Say the* ***inseparable closure*** *of E/F is $F^{insep} = \{\alpha \in E | \alpha^{p^n} \in F \text{ for } n >> 0\}$. Note that F^{insep}/F is p.i. and F^{insep} is a field by the Frobenius property.*

Proposition 2.69. *Let E/F be normal and inseparable. Then there exists $\alpha \in E \setminus F$ such that α is p.i. over F.*

Proof. By assumption, there exists $\beta \in E$ which is inseparable over F. Let $f(x) = \text{Irred}(\beta, F)$. Then, as E/F is normal, $f(x)$ splits in E. Let $E' \subseteq E$ be the splitting field of f over F. Then $[E' : F] < \infty$, E'/F is normal, and E'/F is inseparable as $\beta \in E'$ is inseparable. So it is enough to show there exists a p.i. element in E'. So, since inseparable, we may suppose the characteristic of F is $p > 0$. Then $f(x) = g(x^p)$ for some $g(x) \in F[x]$. Since f is irreducible, g is. If g is inseparable, then $g(x) = h(x^p)$. So $g(x) = h(x^{p^2})$. Continue until $f(x) = g(x^{p^n})$ where $g(x)$ is irreducible and separable (we must stop as $\deg f < \infty$). Say $\deg g = r$ and let $\alpha_1, ..., \alpha_r \in \overline{F}$ be the roots of g. Then $g(x) = (x - \alpha_1) \cdots (x - \alpha_r)$ (note that it is monic as f is). So $f(x) = (x^{p^n} - \alpha_1) \cdots (x^{p^n} - \alpha_r)$. Let β_i be a root of $x^{p^n} - \alpha_i$. Then $f(x) = ((x - \beta_1) \cdots (x - \beta_r))^{p^n}$. Thus $f(x) = \ell(x)^{p^n}$ where $\ell(x) \in E'[x]$. Say $\ell(x) = x^r + d_{r-1}x^{r-1} + \ldots + d_0 \in E'[x]$ and $g(x) = x^r + c_{r-1}x^{r-1} + \ldots + c_0 \in F[x]$. Then $\ell(x)^{p^n} = x^{p^n r} + d_{r-1}^{p^n} x^{p^n(r-1)} + \ldots d_0^{p^n} = f(x) = g(x^{p^n})$. Thus $d_i^{p^n} = c_i$. Note that if $\ell(x) \in F[x]$, then f would be reducible. So there exists some i such that $d_i \in E' \setminus F$. Then d_i is p.i. over F. $\square$

Theorem 2.70. *Let E/F be normal with $K = F^{sep}$ and $L = F^{insep}$. Then*

1. *K, L are fields*
2. *E/K is p.i. and E/L is separable*
3. *$E = KL$.*

Proof. 1. Easy

2. E/K p.i. is a HW exercise. So we will only show E/L is separable. Know E/L is normal as E/F is. If it were inseparable, then the previous proposition says there exists $\alpha \in E \setminus L$ which is p.i. over L, that is $\alpha^{p^n} \in L$ for some $n >> 0$. But L/F is p.i. so there exists $r >> 0$ such that $(\alpha^{p^n})^{p^r} \in F$ which says α is p.i. over F, that is, $\alpha \in L$, a contradiction. Thus E/L is separable.

3. Certainly $KL \subseteq E$. We see that E/KL is p.i. as E/K was and E/KL is separable as E/L was. Thus E/KL is both p.i. and separable which says $[E : KL] = 1$. Thus $E = KL$. □

Example. Let $F = \mathbb{Z}_2(s,t)$ where s,t are indeterminants. Let $f(x) = x^4 + sx^2 + t$ and β be a root of f in $\overline{F}$. Then $F(\beta)/F$ is inseparable,but there are no p.i. elements in $F(\beta) \setminus F$.

Proof. First, we need to show f is irreducible. Let $D = \mathbb{Z}_2[s,t]$. Then $f(x) \in D[x]$ and, by Gauss' Lemma, if f is reducible over $F[x]$, then $f = gh$ for some $g,h \in D[x]$.

Case 1: $\deg g = 1$. Then $g = x - u$ for $u \in D$. Then $f(u) = 0$, which implies $u^4 + su^2 + t = 0$. If u is not constant, say p is an irreducible factor of u, then $p^2 | t$ by the 2 out of 3 lemma, a contradiction. So u is constant, that is, $u = 0$ or 1. But $f(0), f(1) \neq 0$. So $\deg g \neq 1$.

Case 2: $\deg g = 2$. Then

$$\begin{aligned} f(x) &= (x^2+ux+v)(x^2+ax+b) \\ &= x^4 + (u+a)x^3 + (ua+v+b)x^2 + (ub+va)x + bv. \end{aligned}$$

So we have

$$\begin{aligned} &(1) \quad u + a = 0 \\ &(2) \quad ua + v + b = s \\ &(3) \quad ub + va = 0 \\ &(4) \quad bv = t \end{aligned}$$

From (4) we can say WLOG $b = t$ and $v = 1$. From (2) we can say $u = a$. Plugging these into (3) we get $ut = u$, which implies $u = 0 = a$. Plugging this into (2) gives $s = t + 1$, a contradiction as they are indeterminants.

Thus f is irreducible. This tells us that $[F(\beta) : F] = 4$. We also know that β is inseparable as $f' = 0$. So $[F(\beta) : F]_S = 1$ or 2. On the other hand, $g(x) = \mathrm{Irred}(\beta^2, F) = x^2 + sx + t$ (which is irreducible as $g(x^2) = f(x)$, which is irreducible) and $g(x)$ is separable (as $f' \neq 0$.) So $F(\beta^2)$ is separable. [Note that by HW4 #1, this says $F(\beta^2) = F^{sep}$.] This gives $[F(\beta) : F]_i = 2 = [F(\beta) : F]_S$.

Claim: $x^2 - t$ has no roots in $F(\beta)$.

Proof: Suppose $\gamma \in F(\beta)$ satisfies $\gamma^2 = t$. Then $\gamma = c_0 + c_1\beta + c_2\beta^2 + c_3\beta^3$, $c_i \in F$ which implies $t = \gamma^2 = c_0^2 + c_1^2\beta^2 + c_2^2\beta^4 + c_3^2\beta^6$. For simplicity, define $d_i = c_i^2 \in F^2 = \mathbb{Z}_2(s^2,t^2)$. Then $t = d_0 + d_1\beta^2 + d_2\beta^4 + d_3\beta^6$. Of course, since $f(\beta) = 0$, we know

$$\begin{aligned} &\beta^4 = s\beta^2 + t \\ &\beta^6 = \beta^2(s\beta^2 + t) = s\beta^4 + t\beta^2 = s^2\beta^2 + st + t\beta^2. \end{aligned}$$

So

$$\begin{aligned} t &= d_0 + d_1\beta^2 + d_2(s\beta^2 + t) + d_3(s^2\beta^2 + st + t\beta^2) \\ &= (d_0 + d_2t + d_3st) + (d_1 + d_2s + d_3s^2 + d_3t)\beta^2. \end{aligned}$$

Since $t \in F$ and the β's form a basis for $F(\beta)$, we get

$$\begin{aligned} &(1) \quad t = d_0 + d_2t + d_3st \\ &(2) \quad 0 = d_1 + d_2s + d_3s^2 + d_3t. \end{aligned}$$

Then (1) implies $(1 + d_2 + d_3 s)t = d_0$. So $d_0 = 0$, and $d_2 = 1 + d_3 s$. Plugging this into (2), we see

$$0 = d_1 + (1 + d_3 s)s + d_3 s^2 + d_3 t = d_1 + s + d_3 t$$

as we are in $\mathbb{Z}_2$. But this says $s = d_1 + d_3 t \in \mathbb{Z}_2(t, s^2)$, a contradiction.

□

Suppose $\delta \in F(\beta) \setminus F$ is p.i over F. Then $2 \leq [F(\delta) : F] = [F(\delta) : F]_i \leq [F(\beta) : F]_i = 2$. So we see $\delta^2 \in F$. So $[F(\beta) : F(\delta)] = 2$. Consider $x^2 + sx + t = (x - \alpha_1)(x - \alpha_2)$ in $\overline{F}[x]$. Suppose $\beta^2 = \alpha_1$ and let ρ be a root of $x^2 - \alpha_2$. Then $f(x) = (x-\beta)^2(x-\rho)^2$. Since β is separable over $F(\delta)$, we see $h(x) = \text{Irred}(\beta, F(\delta)) = (x-\beta)(x-\rho) = x^2+(\beta+\rho)x+\beta\rho$. Thus we see $\beta\rho \in F(\delta) \subset F(\beta)$. Also $g(x)^2 = f(x)$, which implies $(\beta\rho)^2 = t$, a contradiction to the above claim.

2.8 Galois Groups

Definition 2.71. *Let E/F be a field extension. Then $Aut(E/F) = \{\phi \in Aut(E) : \phi$ fixes $F\}$.*

Remark. Let E/F be a finite extension.

1. $|\text{Aut}(E/F)| \leq [E : F]_S$ with equality if and only if E/F is normal.
2. $|\text{Aut}(E/F)| = [E : F]$ if and only if the extension is normal and separable.

Proof. 1. By definition of the separable degree and normal.

2. We know $|\text{Aut}(E/F)| \leq [E : F]_S \leq [E : F]$. Then we get equality if and only if the extension is normal and separable by definition of normal and separable. □

Definition 2.72. *Say E/F is **Galois** if E/F is normal and separable. In this case, we say $Aut(E/F)$ is the **Galois Group** and denote it $Gal(E/F)$.*

Example. Let E be the splitting field of $x^3 - 2$ over $\mathbb{Q}$. Find $Gal(E/\mathbb{Q})$.

First note that this is a Galois extension as we are in characteristic 0 (thus every extension is separable) and E is a splitting field (thus normal). Further, since $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ and $[\mathbb{Q}(\omega) : \mathbb{Q}] = 2$, which are relatively prime, we see $[E : \mathbb{Q}] = 6$. So $|\text{Aut}(E/\mathbb{Q})| = 6$. Further, we know that any automorphism of E sends roots of $x^3 - 2$ to other roots and similarly for $x^2 + x + 1$. So let $\sigma : E \to E$ be defined by $\left\{ \begin{matrix} \sqrt[3]{2} \mapsto \omega\sqrt[3]{2} \\ \omega \mapsto \omega \end{matrix} \right\}$ and $\tau : E \to E$ be defined by $\left\{ \begin{matrix} \sqrt[3]{2} \mapsto \sqrt[3]{2} \\ \omega \mapsto \omega^2 \end{matrix} \right\}$. Then $\sigma^3 = 1_E, \tau^2 = 1_E$ and $\sigma\tau \neq \tau\sigma$. Thus (since there is only nonabelian group of order 6),

$$Gal(E/F) =< \sigma, \tau | \sigma^3 = \tau^2 = 1, \tau\sigma\tau = \sigma^2 > .$$

Example. Let E be the splitting field of $x^6 + 3$ over $\mathbb{Q}$. Recall (a test problem) that the splitting field is $E = \mathbb{Q}(\omega\sqrt[6]{3})$ where $\omega = e^{\pi i/6}$ and $[E : \mathbb{Q}] = 6$. Define $\sigma_i : E \to E$ by $\omega\sqrt[6]{3} \mapsto \omega^{2i+1}\sqrt[6]{3}$. Then $G = Gal(E/\mathbb{Q}) = \{\sigma_1, ..., \sigma_6\}$. We just need to decide whether G is C_6 or S_3. First note that $\omega^2 = \frac{1}{2} + \frac{1}{2}(\omega\sqrt[6]{3})^3$ and thus $\sigma_1(\omega^2) = \frac{1}{2} + \frac{1}{2}(\sigma_1(\omega\sqrt[6]{3}))^3 = \frac{1}{2} - \frac{i}{2}\sqrt{3} = \omega^{10}$. Thus we see $\sigma_1^2(\omega\sqrt[6]{3}) = \sigma_1(\omega^3\sqrt[6]{3}) = \sigma_1(\omega^2)\sigma_1(\omega\sqrt[6]{3}) = \omega^{10}\omega^3\sqrt[6]{3} = \omega\sqrt[6]{3}$. Thus $\sigma_1^2 = 1$. Similarly, we can show $\sigma_2^3 = 1$ and $\sigma_1\sigma_2 \neq \sigma_2\sigma_1$. Thus $G = S_3$.

Proposition 2.73. *Let $\omega \in \mathbb{C}$ be a primitive nth root of unity. Then $Gal(\mathbb{Q}(\omega)/\mathbb{Q}) \cong \mathbb{Z}_n^*$.*

Proof. By previous study, we know $[\mathbb{Q}(\omega) : \mathbb{Q}] = \phi(n)$. Thus $Gal(\mathbb{Q}(\omega)/\mathbb{Q}) = \{\phi_i : \mathbb{Q}(\omega) \to \mathbb{Q}(\omega) | \phi_i(\omega) = \omega^i, \text{ where } \gcd(i, n) = 1, 1 \leq i < n\}$. Define $\rho : Gal(\mathbb{Q}(\omega)/\mathbb{Q}) \to \mathbb{Z}_n^*$ by $\phi_i \mapsto [i]_n$. Clearly, ρ is an isomorphism. □

Remarks. Let E/F be Galois and L an intermediate field.

1. Then E/L is Galois and $Gal(E/L) < Gal(E/F)$.

Proof. If E/F is separable and normal, then so is E/L. Also, any automorphism of E which fixes L also fixes F. $\square$

2. Let $H \leq Gal(E/F)$. Then $E_H = \{u \in E | \sigma(u) = u \text{ for all } \sigma \in H\}$ is an intermediate field of E/F. Call it the **fixed field** of H.

Theorem 2.74. *Let E/F be Galois and $G = Gal(E/F)$. Then $E_G = F$.*

Proof. Clearly, $F \subseteq E_G$. Let $\alpha \in E_G$ and $\sigma : F(\alpha) \to \overline{F}$ be an embedding which fixes F. Then we can extend σ to $\tau : E \to \overline{F}$. Since E/F is normal, τ is an automorphism, which implies $\tau \in G$. Then $\alpha \in E_G$ implies $\tau(\alpha) = \alpha$ and thus $\sigma(\alpha) = \alpha$. So $\sigma = 1_{F(\alpha)}$. Then $1 = [F(\alpha) : F]_S = [F(\alpha) : F]$, since separable. Thus $\alpha \in F$ and $E_G = F$. $\square$

Lemma 2.75. *Let E/F be a separable extension such that $[F(\alpha) : F] \leq n$ for all $\alpha \in E$. Then $[E : F] \leq n$.*

Proof. Choose $\alpha \in E$ such that $[F(\alpha) : F] = m$ is as large as possible (can do this as it is bounded above by n.) If $E \neq F(\alpha)$, let $\beta \in E \setminus F(\alpha)$. Then, by the Primitive Element Theorem, there exists $\gamma \in E$ such that $F(\gamma) = F(\alpha, \beta)$. Then $[F(\gamma) : F] > [F(\alpha) : F] = m$, a contradiction. So $E = F(\alpha)$ which says $[E : F] = [F(\alpha) : F] \leq n$. $\square$

Theorem 2.76 (**Artin's Theorem**). *Let E be a field and G a finite subgroup of $Aut(E)$. Let $F = E_G$. Then*

1. *E/F is finite, Galois, and $[E : F] = |G|$*
2. *$G = Gal(E/F)$.*

Proof. Let $\alpha \in E$ and $\{\sigma_1(\alpha), ..., \sigma_r(\alpha)\} \subseteq \{\phi(\alpha) | \phi \in G\}$ be maximal with respect to the property $\sigma_1(\alpha), ..., \sigma_r(\alpha)$ are distinct. Let $\tau \in G$. Since τ is injective, $\tau\sigma_1(\alpha), ..., \tau\sigma_r(\alpha)$ are also distinct. Thus $\tau\sigma_1(\alpha), ..., \tau\sigma_r(\alpha)$ is a permutation of $\sigma_1(\alpha), ..., \sigma_r(\alpha)$. Let $f_\alpha(x) = \prod_{i=1}^{r}(x - \sigma_i(\alpha))$. Then for $\tau \in G, f_\alpha^\tau(x) = f_\alpha(x)$. So $f_\alpha(x) \in F[x]$. Thus $\text{Irred}(\alpha, F) | f_\alpha(x)$ and $f_\alpha(x)$ has distinct roots. Thus α is separable over F. Since α was arbitrary, E/F is separable. Also for all $\alpha \in E$, $f_\alpha(x)$ splits in E so $\text{Irred}(\alpha, F)$ splits in E which says E/F is normal. Thus E/F is Galois. Now $[F(\alpha) : F] \leq \deg f_\alpha(x) = r \leq |G|$. Since E/F is separable, the lemma tells us $[E : F] \leq |G|$. Now $G \leq Gal(E/F)$ thus we have $|G| \leq |Gal(E/F)| = [E : F] \leq |G|$. So $|G| = |Gal(E/F)| = [E : F]$ which implies $G = Gal(E/F)$. $\square$

Theorem 2.77 (**Fundamental Thm of Galois Theory**). *Let E/F be a finite Galois Extension. Then there is a bijective correspondence between the intermediate fields of E/F and the subgroups of $Gal(E/F)$ defined by $L \mapsto Gal(E/L)$ and $H \mapsto E_H$ for an intermediate field L and a subgroup H.*

Proof. By the previous lemma, $E_{Gal(E/L)} = L$. By Artin's Theorem, for $H < Gal(E/F)$, E/E_H is Galois and $Gal(E/E_H) = H$. $\square$

Note. The correspondence is inclusion reversing. That is, for intermediate fields

$$L_1 \subseteq L_2 \text{ we see } Gal(E/L_1) \supseteq Gal(E/L_2)$$

and for subgroups

$$H_1 \supseteq H_2 \text{ we see } E_{H_1} \subseteq E_{H_2}.$$

Recall that Artin's Theorem says $|Gal(E/L)| = [E : L]$ and for $H < G = Gal(E/F)$, $|H| = [E : E_H]$, which implies $[G : H] = [E_H : F]$. Thus we can construct the following diagram:

$$\begin{tikzcd}
E \arrow[r, leftrightarrow] \arrow[d, dash, "{[E:L]}"] & \{1\} \arrow[r, leftrightarrow] \arrow[d, dash, "|H|"] & E \arrow[d, dash] \\
L \arrow[r, leftrightarrow] \arrow[d, dash, "{[L:F]}"] & H = Gal(E/L) \arrow[r, leftrightarrow] \arrow[d, dash, "{[G:H]}"] & E_H \arrow[d, dash] \\
F \arrow[r, leftrightarrow] & G = Gal(E/F) \arrow[r, leftrightarrow] & E_G
\end{tikzcd}$$

Example. Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Find primitive elements for all intermediate fields of $E/\mathbb{Q}$.

1. Compute $G = Gal(E/\mathbb{Q})$. We know $[E : \mathbb{Q}] = 4$ and there are 4 obvious automorphisms: $\sqrt{2} \mapsto \pm\sqrt{2}$ and $\sqrt{3} \mapsto \pm\sqrt{3}$. So that must be all of them. It is easy to check that $G =< \sigma, \tau | \sigma^2 = \tau^2 = 1, \sigma\tau = \tau\sigma >= C_2 \times C_2$ where $\sigma : E \to E$ sends $\sqrt{2} \mapsto -\sqrt{2}$ and $\tau : E \to E$ sends $\sqrt{3} \mapsto -\sqrt{3}$.

2. Create a subgroup lattice:

$$\begin{tikzcd}
 & \{1\} \arrow[dl, dash, "2"'] \arrow[d, dash, "2"] \arrow[dr, dash, "2"] & \\
< \sigma > \arrow[dr, dash, "2"'] & < \sigma\tau > \arrow[d, dash, "2"] & < \tau > \arrow[dl, dash, "2"] \\
 & G &
\end{tikzcd}$$

   This tells us our Intermediate fields are $E_{<\sigma>}, E_{<\sigma\tau>}, E_{<\tau>}$, all of which have degree 2 over $\mathbb{Q}$. Now, $\sqrt{3}$ is fixed by σ, $\sqrt{2}$ by τ, and $\sqrt{6}$ by $\sigma\tau$. So

$$E_{<\sigma>} = \mathbb{Q}(\sqrt{3}), E_{<\sigma\tau>} = \mathbb{Q}(\sqrt{6}), E_{<\tau>} = \mathbb{Q}(\sqrt{2})$$

   and of course $E = \mathbb{Q}(\sqrt{2} + \sqrt{3})$ (this element is not fixed by any of the above automorphisms.)

Example. Let $E = \mathbb{Q}(\omega \sqrt[6]{3})$, $\omega = e^{2\pi i/12}$. Then E is the splitting for $x^6 + 3$. Recall from before that $Gal(E/\mathbb{Q}) = S_3$ and was generated by $\sigma : E \to E$ defined by $\omega\sqrt[6]{3} \mapsto \omega^3 \sqrt[6]{3}$ and $\tau : E \to E$ defined by $\omega\sqrt[6]{3} \mapsto \omega^5 \sqrt[6]{3}$. Now, we can again make our subgroup lattice:

$$\begin{tikzcd}
 & \{1\} \arrow[dl, dash, "2"'] \arrow[d, dash, "2"] \arrow[dr, dash, "2"] \arrow[drr, dash, "3"] & & \\
< \sigma > \arrow[dr, dash] & < \sigma\tau > \arrow[d, dash] & < \sigma\tau^2 > \arrow[dl, dash] & < \tau > \arrow[dll, dash] \\
 & G & &
\end{tikzcd}$$

- We see τ fixes $\omega^2 = e^{\pi i/3}$, an element of degree 2 over $\mathbb{Q}$ (the irreducible polynomial is x^2+x+1.) So $E_{<\tau>} = \mathbb{Q}(\omega^2)$.
- Since $\sigma^2 = 1$, we see $\omega\sqrt[6]{3}\sigma(\omega\sqrt[6]{3}) = \omega\sqrt[6]{3}\omega^3\sqrt[6]{3} = \omega^4\sqrt[3]{3}$ is fixed by σ and not in $\mathbb{Q}$. Thus $E_{<\sigma>} = \mathbb{Q}(\omega^4\sqrt[3]{3})$.
- We expect the other roots of $x^3 - 3$ to be fixed by our other two intermediate fields.
- Since $(\sigma\tau)^2 = 1$, we see $\omega\sqrt[6]{3}\sigma\tau(\omega\sqrt[6]{3}) = \sqrt[3]{3}$ is fixed by $\sigma\tau$. So $E_{<\sigma\tau>} = \mathbb{Q}(\sqrt[3]{3})$.
- Similarly, we see $E_{<\sigma\tau^2>} = \mathbb{Q}(\omega^8\sqrt[3]{3})$.

Definition 2.78. *Let F be a field and $\alpha \in \overline{F}$. Let $\sigma_1, ..., \sigma_s$ be the distinct embeddings of $F(\alpha) \to \overline{F}$ fixing F. Then $\sigma_1(\alpha), ..., \sigma_s(\alpha)$ are called the $F-$**conjugates** of α, that is, the $F-$conjugates of α are the distinct roots of Irred(α, F).*

Remark. Suppose α is separable over F and $\text{Irred}(\alpha, F) = \prod_{i=1}^{s}(x - \sigma_i(\alpha)) = x^n + c_{n-1}x^{n-1} + ... + c_0$. Then $\prod \sigma_i(\alpha) = c_0$ and $\sum \sigma_i(\alpha)) = c_{n-1}$. Thus they are in F.

Proposition 2.79. *Let E/F be a finite Galois extension. Say $E = F(\alpha)$. Then*

1. *Irred$(\alpha, E_H) = \prod_{h \in H}(x - h(\alpha)) = x^n - c_1x^{n-1} + ... + c_n$.*
2. *$E_H = F(c_1, ..., c_n)$.*

Proof. 1. Let $f(x) = \prod(x - h(\alpha))$. If $h' \in H$, then $f^{h'}(x) = \prod(x - h'h(\alpha)) = f(x)$ as $h' \in H$. Thus $f(x) \in E_H[x]$. Note that $\deg f = |H|$ and $\deg \mathrm{Irred}(\alpha, E_H) = [E_H(\alpha) : E_H] = [E : H] = |H| = \deg f$. Since $f(\alpha) = 0$ ($1 \in H$) and f is monic, $f = \mathrm{Irred}(\alpha, E_H)$.

2. Let $L = F(c_1, ..., c_n) \subseteq E_H$ (as the c_i's are fixed by H). Then $f(x) \in L[x]$, f is irreducible, and $f(\alpha) = 0$. Thus $f = \mathrm{Irred}(\alpha, L)$. So $[E : L] = [E : E_H]$ which implies $L = E_H$. $\square$

Example. Let ω be a primitive 11th root of unity and $E = \mathbb{Q}(\omega)$. We've proved $Gal(E/\mathbb{Q}) \cong \mathbb{Z}_{11}^* = C_{10}$. Say $Gal(E/\mathbb{Q}) =< \sigma >$ where $\sigma : E \to E$ is such that $\omega \mapsto \omega^2$.

$$\begin{tikzcd}[column sep=small]
 & \{1\} \arrow[dl, dash, "5"'] \arrow[dr, dash, "2"] & \\
<\sigma^2> \arrow[dr, dash, "2"'] & & <\sigma^5> \arrow[dl, dash, "5"] \\
 & G &
\end{tikzcd}
\quad \leftrightarrow \quad
\begin{tikzcd}[column sep=small]
 & E \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\omega + \omega^3 + \omega^4 + \omega^5 + \omega^9) \arrow[dr, dash] & & \mathbb{Q}(\omega + \omega^{10}) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

- $\omega + \sigma^5(\omega) = \omega + \omega^{10} \notin \mathbb{Q}$ as otherwise ω would be a root of both $x^{10} + x - q$ for some $q \in \mathbb{Q}$ and $x^{10} + x^9 + ... + 1$, a contradiction as the minimal polynomial is unique.
- $\omega + \sigma^2(\omega) + \sigma^4(\omega) + \sigma^6(\omega) + \sigma^8(\omega) = \omega + \omega^4 + \omega^5 + \omega^9 + \omega^3 \notin \mathbb{Q}$ as then $[\mathbb{Q}(\omega) : \mathbb{Q}] \leq 9$, a contradiction.

Theorem 2.80. *Let E/F be a finite Galois extension and $G = Gal(E/F)$. Let L be an intermediate field and $H = Gal(E/L)$. Then*

1. *L/F is normal if and only if H is normal*
2. *If $H \lhd G$, then $Gal(L/F) \cong G/H$.*

Proof. **$\Rightarrow$:** Define $\phi : G \to Gal(L/F)$ by $\sigma \mapsto \sigma|_L$. This is well-defined as L/F is normal. Furthermore, ϕ is surjective as for $\pi \in Gal(L/F)$, we can extend π to an element $\sigma \in G$. Thus $\sigma|_L = \pi$ and thus $Gal(L/F) \cong G/\ker\phi$. Now $\sigma \in \ker\phi$ if and only if $\sigma|_L = 1$ if and only if σ fixes L if and only if $\sigma \in H$. Thus $H \lhd G$ and $Gal(L/F) \cong G/H$.

$\Leftarrow$: Suppose $\sigma : L \to \overline{F}$ fixes F. Need to show $\sigma(L) \subseteq L$. Let $\alpha \in L$. Extend σ to $\tau : E \to \overline{F}$. Then $\tau \in G$ as E/F is normal. It is enough to show $\tau(\alpha) \in L = E_H$. Let $h \in H$. As $H \lhd G$, $\tau^{-1}h\tau \in H$. Therefore $\tau^{-1}h\tau(\alpha) = \alpha$, which implies $h\tau(\alpha) = \tau(\alpha)$. Thus h fixes $\tau(\alpha)$. Since h is arbitrary, $\tau(\alpha) \in E_H = L$. Thus σ is an automorphism of L and L/F is normal. $\square$

Definition 2.81. *Let E/F be a Galois extension. Say E/F is* ***abelian/cyclic/solvable*** *if $Gal(E/F)$ is abelian/cyclic/solvable.*

Example. Cyclotomic Extensions are abelian.

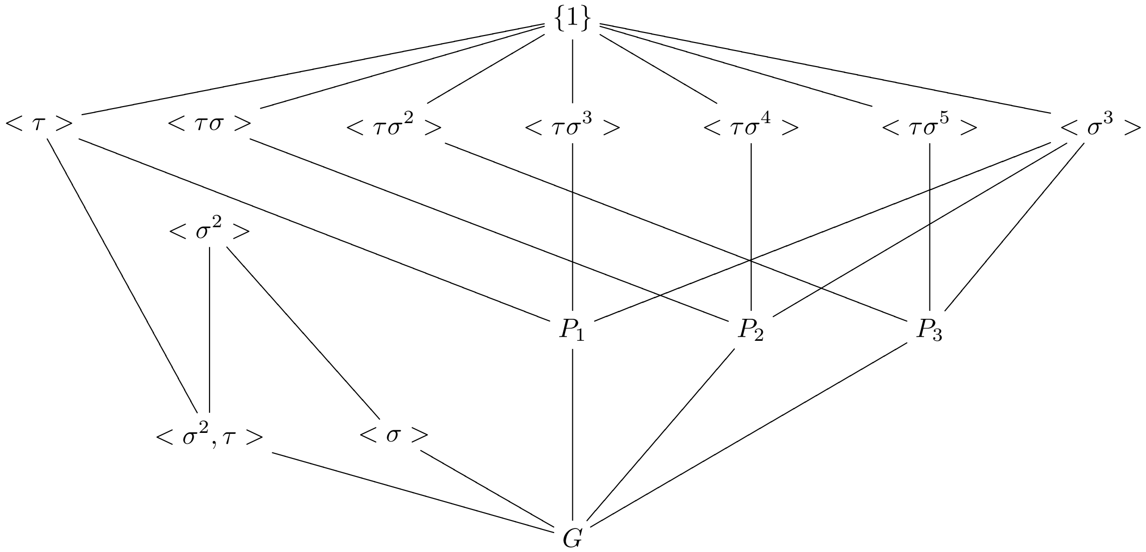
Example. Let E be the splitting field of $x^6 + 5$ over $\mathbb{Q}$. Recall $[E : \mathbb{Q}] = 12$ and $E = \mathbb{Q}(\omega^2, \omega\sqrt[6]{5})$ for $\omega = e^{2\pi i/12}$.

$$\begin{tikzcd}[column sep=small]
 & E \arrow[dl, dash, "2"'] \arrow[dr, dash, "6"] & \\
\mathbb{Q}(\omega\sqrt[6]{5}) \arrow[dr, dash, "6"'] & & \mathbb{Q}(\omega^2) \arrow[dl, dash, "2"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

As $\mathrm{Irred}(\omega\sqrt[6]{5}, \mathbb{Q}(\omega^2)) = x^6 + 5$, we can define $\sigma : E \to E$ such that $\begin{Bmatrix} \omega\sqrt[6]{5} \mapsto \omega^3\sqrt[6]{5} \\ \omega^2 \mapsto \omega^2 \end{Bmatrix}$. Similarly, we can define $\tau : E \to E$ by $\begin{Bmatrix} \omega\sqrt[6]{5} \mapsto \omega\sqrt[6]{5} \\ \omega^2 \mapsto \omega^{10} \end{Bmatrix}$. Note that $\sigma^i : \omega\sqrt[6]{5} \mapsto \omega^{2i+1}\sqrt[6]{5}$ as σ fixes ω^2. So $|\sigma| = 6$ and clearly $|\tau| = 2$. Since $\tau \notin< \sigma >$,

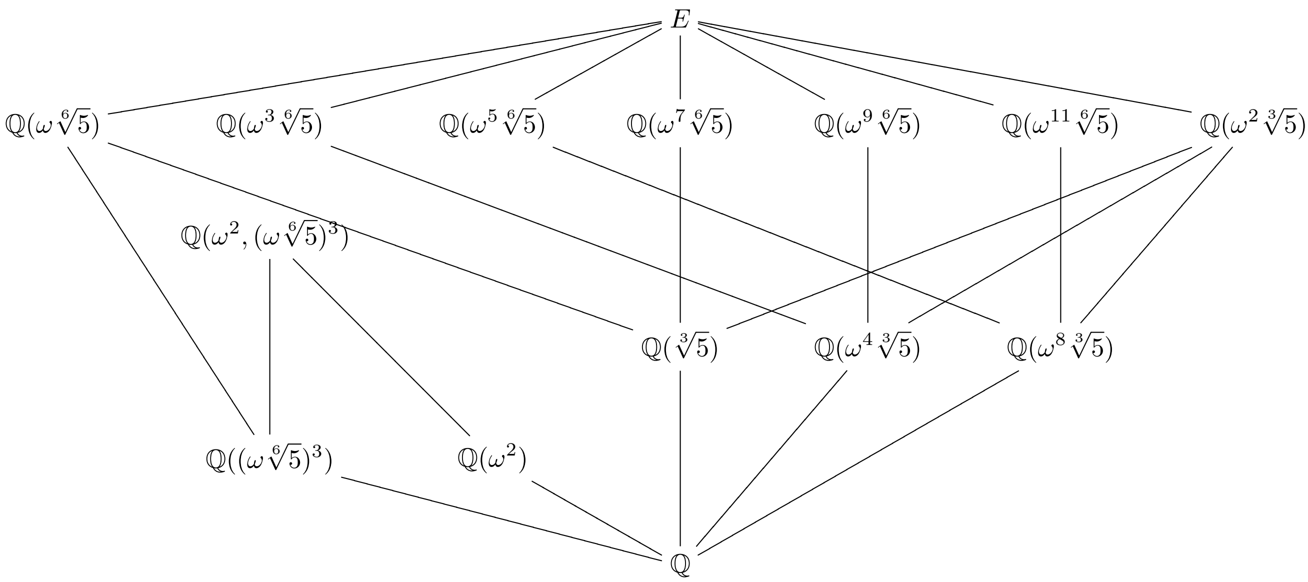
$G =< \sigma, \tau >$. Note $\sigma\tau(\omega\sqrt[6]{5}) = \omega^3\sqrt[6]{5}$ but $\tau\sigma(\omega\sqrt[6]{5}) = \tau(\omega^3\sqrt[6]{5}) = \tau(\omega^2)\tau(\omega\sqrt[6]{5}) = \omega^{10}\omega\sqrt[6]{5} = \omega^{11}\sqrt[6]{5}$. Thus $\tau\sigma \neq \sigma\tau$. Note $\tau\sigma\tau \in < \sigma >$ and by order arguments, $\tau\sigma\tau = \sigma^{-1} = \sigma^5$. So $G = D_{12}$. Now we want to find the subgroups of D_{12}.

- 7 subgroups of order 2: $< \sigma^3 >, < \tau >, < \tau\sigma >, < \tau\sigma^2 >, < \tau\sigma^3 >, < \tau\sigma^4 >, < \tau\sigma^5 >$(All the subgroups generated by the elements of order 2.)
- 1 subgroup of order 3: $< \sigma^2 >$ (since either the Sylow 3 or Sylow 4 subgroup is normal by Sylows Theorems but the Sylow 4 subgroup can not be normal as then we'd only have 3 order 2 elements, not 7)
- 3 subgroups of order 4: $P_1 =< \sigma^3, \tau >, P_2 =< \sigma^3, \tau\sigma >, P_3 =< \sigma^3, \tau\sigma^2 >$ (by Sylow's Theorems)
- 2 subgroups order 6: $< \sigma >, < \sigma^2, \tau >$.



To translate this into field extensions, note:

- Degree 6 extensions: Roots of $x^6 + 5$ correspond to $E_{<\tau\sigma^i>}$ and $E_{<\sigma^3>} = \mathbb{Q}(\omega^2, \sqrt[6]{5})$.
- Degree 4 extension: We've seen this is $\mathbb{Q}(i\sqrt{3}, i\sqrt{5}) = \mathbb{Q}(\omega^2, (\omega\sqrt[6]{5})^3)$.
- Degree 2 extensions: We know one is $E_{<\sigma>} = \mathbb{Q}(\omega^2)$. We expect the other to be $E_{<\sigma^2,\tau>} = \mathbb{Q}((\omega\sqrt[6]{5})^3)$. In fact it is as $\sigma^2((\omega\sqrt[6]{5})^3) = (\sigma^5\sqrt[6]{5})^3 = \omega^3\sqrt{5} = (\sqrt[6]{5})^3$ and $\tau((\omega\sqrt[6]{5})^3) = (\omega\sqrt[6]{5})^3$.
- Degree 3 extensions: Roots of $x^3 + 5$.



Theorem 2.82. *Let F be a finite field and E/F a finite extension. Then E/F is cyclic.*

Proof. Say char $F = p$. Then $\mathbb{Z}_p \subseteq F$. Since $Gal(E/F) \subseteq Gal(E/\mathbb{Z}_p)$, it is enough to show $Gal(E/\mathbb{Z}_p)$ is cyclic. Say $[E : \mathbb{Z}_p] = n$. Then $|E| = p^n$. Let $\sigma : E \to E$ be the Frobenious map. The $\sigma \in Gal(E/\mathbb{Z}_p)$.

Claim: $Gal(E/\mathbb{Z}_p) =< \sigma >$.

Proof: We want to show $|\sigma| = n$. Suppose $\sigma^i = 1$ for some $1 \leq i \leq n$. Then $a = \sigma^i(a) = a^{p^i}$ for all $a \in E$. Then $x^{p^i} - x$ has $|E| = p^n$ roots, contradiction as $p^n > p^i$. Thus $|\sigma| = n$.

□

Corollary 2.83. *Let E be a field with p^n elements. Then E contains a subfield with p^m elements if and only if $m|n$. Equivalently, $x^{p^m} - x$ splits in E if and only if $m|n$.*

Proof. Let $G = Gal(E/\mathbb{Z}_p)$. Then $n = |G| = [E : \mathbb{Z}_p]$. So E contains a subfield F with order p^m if and only if there exists $F \subseteq E$ with $[F : \mathbb{Z}_p] = m$ if and only if there exists $F \subseteq E$ with $[E : F] = \frac{n}{m}$ if and only if there exists a subgroup $H \subseteq G$ such that $|H| = \frac{n}{m}$ if and only $m|n$ as G is cyclic. □

Remark. Let E be the splitting field of a degree n separable irreducible polynomial $f \in F[x]$. Then E/F is Galois and $Gal(E/F) \cong$ a subgroup of S_n.

Proof. Let $E = F(\alpha_1, ..., \alpha_n)$ where $\alpha_1, ..., \alpha_n$ are the roots of $f(x)$. Define $\phi : Gal(E/F) \to Perm(\Gamma)$ such that $\sigma \mapsto \sigma_\Gamma$ where $\Gamma = \{\alpha_1, ..., \alpha_n\}$. Then ϕ is injective as σ is determined by $\sigma(\alpha_1), ..., \sigma(\alpha_n)$. □

Example. Consider $x^3 - 2 \in \mathbb{Q}(\omega)[x]$ where $\omega = e^{2\pi i/3}$. This is irreducible as $[\mathbb{Q}\sqrt[3]{2} : \mathbb{Q}] = 3$ and $[\mathbb{Q}(\omega) : \mathbb{Q}] = 2$ and $\gcd(2, 3) = 1$. Then $|Gal(E/\mathbb{Q}(\omega))| = 3$.

Let $x_1, ..., x_n$ be independent indeterminants over a field F. Let $E = F(x_1, ..., x_n)$. Let $\sigma \in S_n$. Then there exists an automorphism of E induced by σ, say $\widetilde{\sigma} : E \to E$ defined by $\frac{f(x_1,...,x_n)}{g(x_1,...,x_n)} \mapsto \frac{f(\sigma(x_1),...,\sigma(x_n))}{g(\sigma(x_1),...,\sigma(x_n))}$.

Example. Let $n = 3$ and $\sigma = (123)$. Then

$$\widetilde{\sigma}\left(\frac{x_1^2 + 3x_1x_3 + x_2^2}{x_1x_2 - 2x_1^5}\right) = \frac{x_2^2 + 3x_2x_1 + x_3^3}{x_2x_3 - 2x_2^5}.$$

For simplification, we will identify $\widetilde{\sigma}$ with σ.

Let $L = E_{S_n}$. By Artin's Theorem, E/L is Galois and $Gal(E/L) \cong S_n$. We call L the **field of symmetric rational functions**. Now, any finite group is a subgroup of a group of permutations. So $H \leq S_n$ will correspond to an intermediate field of E/L.

Example. Let $n = 3, F \subseteq L$. Let t be an indeterminant over E and consider $f(t) = \prod_1^n(t - x_i) \in E[t]$. For all $\sigma \in S_n$, we see $f^\sigma(t) = f(t)$. Thus $f(t) \in L[t]$. Then, if $f = t^n - s_1t^{n-1} + s_2t^{n-2} - ... + (-1)s_n$, we see $s_i \in L$ for all i. Call $\{s_i\}$ the **elementary symmetric functions in** $x_1, ..., x_n$.

Theorem 2.84. *With the above notation, $L = E_{S_n} = F(s_1, ..., s_n)$.*

Proof. Note $f(t) \in F(s_1, ..., s_n)[t]$. Then $E = F(s_1, ..., s_n)(x_1, ..., x_n)$ is the splitting field of $f(t)$ over $F(s_1, .., s_n)$. But $\deg f = n$, so $[E : F(s_1, ..., s_n)] \leq n!$. But $[E : F(s_1, ..., s_n)] \geq [E : L] = n!$. Thus $E = L$. □

<u>**Inverse Galois Problem:**</u> Is every finite group the Galois group of a Galois extension of $\mathbb{Q}$?

Fact. For all $n \in \mathbb{Z}$ such that $n > 0$, there exist infinitely many primes p such that $p \equiv 1 \mod n$.

Theorem 2.85. *Let G be a finite abelian group. Then there exists a primitive mth root of unity ω and a field $E \subseteq \mathbb{Q}(\omega)$ such that $Gal(E/\mathbb{Q}) \cong G$.*

Proof. Let $G \cong C_{n_1} \times C_{n_2} \times \cdots \times C_{n_k}$. Let $p_1, ..., p_k$ be distinct primes such that $p_i \equiv 1 \mod n_i$. (Note we use the claim here in the case of $n_i = n_j$.) Let $m = p_1 \cdots p_k$. Let ω be a primitive mth root of unity. Then $Gal(\mathbb{Q}(\omega)/\mathbb{Q}) \cong \mathbb{Z}_m^* = \mathbb{Z}_{p_1}^* \times \cdots \times \mathbb{Z}_{p_k}^* \cong C_{p_1-1} \times \cdots \times C_{p_k-1}$. Since $n_i | p_i - 1$, let $H_i \leq C_{p_i-1}$ such that $|H_i| = \frac{p_i-1}{n_i}$. Then $H_1 \times \cdots \times H_k$ is a normal subgroup of $Gal(\mathbb{Q}(\omega)/\mathbb{Q})$. Let E be the fixed field for $H_1 \times \cdots \times H_k$. Then $E/\mathbb{Q}$ is normal and $Gal(E/\mathbb{Q}) \cong \mathbb{Z}_{p_1}^* \times \cdots \times \mathbb{Z}_{p_k}^* / H_1 \times \cdots \times H_k \cong C_{n_1} \times \cdots \times C_{n_k} \cong G$. □

2.9 Norm and Trace

Definition 2.86. *Let E/F be a finite extension. Let $\sigma_1, ..., \sigma_r$ be the distinct embeddings of $E \to \overline{F}$ which fix F. For $\alpha \in E$, define $N_F^E(\alpha) = (\sigma_1(\alpha) \cdots \sigma_r(\alpha))^{[E:F]_i}$ as the **norm** of α and $Tr_F^E(\alpha) = (\sigma_1(\alpha) + \cdots + \sigma_r(\alpha))[E:F]_i$ as the **trace** of α.*

Examples.

1. If $E = \mathbb{Q}(\sqrt{2})$. Then $1 : E \to E$ and $\sigma : E \to E$ defined by $\sqrt{2} \mapsto -\sqrt{2}$ are the only 2 embeddings. So $N(a + b\sqrt{2}) = (a + b\sqrt{2})(a - b\sqrt{2}) = a^2 - 2b^2$ and $Tr(a + b\sqrt{2}) = (a + b\sqrt{2}) + (a - b\sqrt{2}) = 2a$.

2. Let $E = \mathbb{Q}(\sqrt[3]{2})$. Then there are three embeddings: $1 : \sqrt[3]{2} \mapsto \sqrt[3]{2}, \sigma : \sqrt[3]{2} \mapsto \omega\sqrt[3]{2}, \tau : \sqrt[3]{2} \mapsto \omega^2\sqrt[3]{2}$, where $\omega = e^{2\pi i/3}$. Then $N_{\mathbb{Q}}^E(a + b\sqrt[3]{2} + c\sqrt[3]{4}) = a^3 + 2b^3 + 4c^3 - 6abc$ and $Tr_{\mathbb{Q}}^E(a + b\sqrt[3]{2} + c\sqrt[3]{4}) = 3a$.

3. Let $F = \mathbb{Z}_p(t)$ and E the splitting field of $f(x) = x^p - t$ over F. Then $E = F(\alpha$ where $\alpha^p = t$. Clearly, α is p.i. over F which implies E/F is p.i. and $[E:F] = p$. So $[E:F]_S = 1$ and $[E:F]_i = p$. Then we have only one embedding-the identity. So $N_F^E(\beta) = \beta^p$ and $N_F^E(\beta) = p\beta = 0$ (since $charF = p$.)

Lemma 2.87. *If E/F is finite and separable, then $N_F^E(\alpha), Tr_F^E(\alpha) \in F$ for all $\alpha \in E$.*

Proof. Let L be the normal closure of E/F. Then L/F is finite and Galois. Let $\sigma_1, ..., \sigma_r$ be the distinct embeddings of $E \to \overline{F}$ which fix F. Let $\phi \in G = Gal(L/f)$. Then $\phi\sigma_i : E \to L$ for all i. Further, $\phi\sigma_i$ are distinct as ϕ is injective. So $\{\phi\sigma_1, ..., \phi\sigma_r\} = \{\sigma_1, ..., \sigma_r\}$ for all $\phi \in F$. Thus $\phi(N_F^E(\alpha)) = \phi(\sigma_1(\alpha) \cdots \sigma_r(\alpha)) = \phi\sigma_1(\alpha) \cdots \phi\sigma_r(\alpha) = N_F^E(\alpha)$. Since this holds for all $\phi \in G$, we see $N_F^E(\alpha) \subseteq L_G = F$. Similarly for $Tr_F^E(\alpha)$. □

Proposition 2.88. *If E/F is finite, then $N_F^E(\alpha), Tr_F^E(\alpha) \in F$ for all $\alpha \in E$.*

Proof. If E/F is inseparable, then $[E:F]_i = p^n$. So $Tr_F^E(\alpha) = p^n(\cdots) = 0$ as char $F = p$. Let L be the separable closure of F in E. Then E/L is p.i. and L/F is separable (by HW4#1). Therefore, $[E:F]_S = [E:L]_S[L:F]_S = [L:F]_S = [L:F]$. Let $\sigma_1, ..., \sigma_r$ be the distinct embeddings of $L \to \overline{F}$ fixing F. Then $r = [L:F]_S$. Extend $\sigma_1, ..., \sigma_r$ to $\tau_1, ..., \tau_r : E \to \overline{F}$. Then $\{\tau_1, ..., \tau_r\}$ is the set of distinct embeddings of $E \to \overline{F}$ fixing F. Let $\alpha \in E$. Then $p^m = [L(\alpha):L]_i \leq [E:L]_i = p^n$. So $\alpha^{p^n} \in L$ since $\alpha^{[L(\alpha):L]_i} \in L$. By the lemma, for all $\beta \in L$, $N_F^L(\beta) \in F$ as L/F is separable. Now $N_F^E(\alpha) = (\tau_1(\alpha) \cdots \tau_r(\alpha))^{[E:F]_i = [E:L]_i} = \tau_1(\alpha^{[E:L]_i}) \cdots \tau_r(\alpha^{[E:L]_i}) = \sigma_1(\alpha^{[E:L]_i}) \cdots \sigma_r(\alpha^{[E:L]_i}) \in F$ by the previous sentence (take $\beta = \alpha^{[E:L]_i}$). □

Proposition 2.89. *Let E/F be a finite extension. Let $\alpha, \beta \in E$. Then*

1. *$N_F^E(\alpha\beta) = N_F^E(\alpha)N_F^E(\beta)$ and $Tr_F^E(\alpha + \beta) = Tr_F^E(\alpha) + Tr_F^E(\beta)$.*

2. *If $\alpha \in F$, then $N_F^E(\alpha) = \alpha^{[E:F]}$ and $Tr_F^E(\alpha) = \alpha[E:F]$.*

3. *If K is an intermediate field, then $N_F^E = N_F^K \circ N_K^E$ and $Tr_F^E = Tr_F^K \circ Tr_K^E$.*

Proof.

1. Follows from the definition as σ_i are homomorphisms.

2. Let $\alpha \in F$. Then $N_F^E(\alpha) = (\sigma_1(\alpha) \cdots \sigma_r(\alpha))^{[E:F]_i} = (\alpha^r)^{[E:F]_i} = \alpha^{[E:F]}$ as $r = [E:F]_S$.

3. Let $\sigma_1, \ldots, \sigma_r$ be the distinct embeddings of $K \to \overline{F}$ fixing F. Extend these to $\tau_1, ..., \tau_r : E \to \overline{F}$. Let $\phi_1, ..., \phi_t$ be the distinct embeddings of $E \to \overline{F}$ fixing K. Then $\{\tau_i\phi_j\}_{i,j}$ are the distinct embeddings of $E \to \overline{F}$ fixing F. Then

$$N_F^K N_K^E(\alpha) = N_F^K\left(\left(\prod_j \phi_j(\alpha)\right)^{[E:K]_i}\right) = \left(\prod_i \tau_i\left(\prod_j \phi_j(\alpha)\right)^{[E:K]_i}\right)^{[K:F]_i} = \left(\prod_{i,j} \tau_i\phi_j(\alpha)\right)^{[E:F]_i} = N_F^E(\alpha).$$

Similarly for the trace. □

Remarks.

1. $N_F^E : E^* \to F^*$ is a group homomorphism and $Tr_F^E : (E, +) \to (F, +)$ is an additive group homomorphism. In fact, $Tr_F^E : E \to F$ is a linear functional of E as an $F-$VS.

   Proof. Let $c \in F, \alpha \in E$. Then

   $$Tr_F^E(c\alpha) = [E:F]_i\left(\sum \sigma_i(c\alpha)\right) = [E:F]_i\left(c\sum \sigma_i(\alpha)\right) = cTr_F^E(\alpha)$$

   as σ_i fixes $c \in F$. We have already seen the trace is additive. □

2. If char $F = 0$, then $Tr_F^E(c) = [E:F]c \neq 0$. If char $F = p$ and $[E:F]_i > 1$, we have already seen $Tr_F^E(\alpha) = [E:F]_i(--) = p^i(--) = 0$. So Tr_F^E degenerates. It's a little harder to see, but if char $F = p$ and $[E:F]_i = 1$, then the trace is non-degenerate. We will prove this.

Lemma 2.90. *Let E/F be a field extension, L a field such that $F \subseteq L$, and $\sigma_1, ..., \sigma_n$ the distinct field embeddings of $E \to L$ which fix F. Then $\sigma_1, ..., \sigma_n$ are linearly independent over F.*

Proof. We will induct on n. Let $n = 1$. Suppose $a\sigma_1 = 0$, where $\sigma_1 \neq 0$. Let $\alpha \in E \setminus \{0\}$. Then $\sigma_1(\alpha) \neq 0$. Since we are in a field, $a\sigma_1(\alpha) = 0$ implies $a = 0$. Let $n > 1$. Suppose $(*) a_1\sigma_1 + \cdots + a_n\sigma_n = 0$ for some $\sigma_1, ..., \sigma_n$ not all zero. If any of these terms are 0, we are done by induction. So assume $a_i \neq 0$ for all i. Let $\beta \in E$ such that $\sigma_1(\beta) \neq \sigma_2(\beta)$. For $\alpha \in E$, we see $a_1\sigma_1(\alpha\beta) + \cdots + a_n\sigma_n(\alpha\beta) = 0$ which implies $a_1\sigma_1(\beta)\sigma_1(\alpha) + \cdots + a_n\sigma_n(\beta)\sigma_n(\alpha) = 0$ for all $\alpha \in E$. This implies $a_1\sigma_1(\beta)\sigma_1 + \cdots + a_n\sigma_n(\beta)\sigma_n = 0$. Now divide by $\sigma_1(\beta)$ and subtract from $(*)$. Then $a_2\left(1 - \frac{\sigma_2(\beta)}{\sigma_1(\beta)}\right)\sigma_2 + \cdots + a_n\left(1 - \frac{\sigma_n(\beta)}{\sigma_1(\beta)}\right)\sigma_n = 0$. By induction, since $a_i \neq 0$, we see $1 = \frac{\sigma_i(\beta)}{\sigma_1(\beta)}$ which implies $\sigma_1(\beta) = \sigma_i(\beta)$, contradiction. □

Corollary 2.91. *If E/F is a finite separable extension, then $Tr_F^E \neq 0$. So Tr_F^E is nondegenerate for separable extensions.*

Theorem 2.92 (**Hilbert's Satz 90**)**.** *Let E/F be a finite cyclic extension. Let $<\sigma>= Gal(E/F)$ and $\beta \in E$. Then $N_F^E(\beta) = 1$ if and only if $\beta = \frac{\alpha}{\sigma(\alpha)}$ for some $\alpha \in E$.*

Proof. Let $|\sigma| = n$.

($\Leftarrow$:) Then $N_F^E(\beta) = \prod_0^{n-1} \sigma^i(\beta) = \prod_0^{n-1} \sigma^i\left(\frac{\alpha}{\sigma(\alpha)}\right) = \prod_0^{n-1} \frac{\sigma^i(\alpha)}{\sigma^{i+1}(\alpha)} = 1$ as $\sigma^n = 1$.

($\Rightarrow$:) Suppose $N(\beta) = 1$. By the lemma, $\{1, \sigma, ..., \sigma^{n-1}\}$ are linearly independent over F. Let

$$g = 1 + \beta\sigma + (\beta\sigma(\beta))\sigma^2 + \ldots + (\beta\sigma(\beta)\cdots\sigma^{n-2}(\beta))\sigma^{n-1} \neq 0.$$

Then there exists $u \in E$ such that $g(u) \neq 0$. Let $\alpha = g(u)$. Then

$$\begin{aligned}
\beta\sigma(\alpha) &= \beta\sigma(g(u)) \\
&= \beta\sigma(u + \beta\sigma(u) + (\beta\sigma(\beta))\sigma^2(u) + \ldots + (\beta\sigma(\beta)\cdots\sigma^{n-2}(\beta))\sigma^{n-1}(u)) \\
&= \beta\sigma(u) + \beta\sigma(\beta)\sigma^2(u) + \beta\sigma(\beta)\sigma^2(\beta)\sigma^3(u) + \ldots + \underbrace{(\beta\sigma(\beta)\cdots\sigma^{n-1}(\beta))}_{=N(\beta)=1}\underbrace{\sigma^n(u)}_{u} \\
&= g(u) = \alpha.
\end{aligned}$$

Thus $\beta = \frac{\alpha}{\sigma(\alpha)}$.

□

Remark. Let F be a field, $n \geq 1$. Then the roots of $x^n - 1$ form a finite subgroup U_n of $(\overline{F})^*$. Thus U_n is a cyclic group, say $U_n =< \omega >$. If char $F \nmid n$, then $x^n - 1$ has n distinct roots. Thus $|U_n| = n$. Any generator for U_n is called a primitive n^{th} root of unity.

Theorem 2.93. *Let F be a field, $n \geq 1$ such that char $F \nmid n$. Assume F contains a primitive n^{th} root of unity. Then E/F is cyclic of* deg $|n$ *if and only if $E = F(\alpha)$ where $\alpha^n \in F$.*

Proof. ($\Rightarrow$:) Let $[E : F] = d$. Then, since $d|n$, there is a primitive d^{th} root of unity, call it $\xi \in F$. Then $\xi^{-1} \in F$ and $N_F^E(\xi^{-1}) = (\xi^{-1})^{[E:F]} = 1$. So there exists $\alpha \in E$ such that $\xi^{-1} = \frac{\alpha}{\sigma(\alpha)}$, where $< \sigma >= Gal(E/F)$. Then $\sigma(\alpha) = \xi\alpha$ which implies $\sigma^i(\alpha) = \xi^i\alpha$ as $\xi \in F$ implies σ fixes ξ. Since $\sigma(\alpha), ..., \sigma^d(\alpha)$ are distinct, we see $[F(\alpha) : F]_S \geq d$. Since $[E : F] = d$ this says $[E : F(\alpha)] = 1$ and thus $E = F(\alpha)$. Now notice $\sigma(\alpha^d) = \sigma(\alpha)^d = (\xi\alpha)^d = \alpha^d$. So $\alpha^d \in E_{<\sigma>} = F$ and since $d|n$, $\alpha^n \in F$.

($\Leftarrow$:) Let $a = \alpha^n \in F$. Then α is a root of $x^n - a \in F[x]$. Let $\omega \in F$ be a primitive n^{th} root of unity. Then

$$x^n - a = \prod_{i=0}^{n-1}(x - \omega^i\alpha) \in E[x].$$

So E is the splitting field of $x^n - a$ which implies E/F is normal. Since char $F \nmid n$, the ω^i are distinct and thus $x^n - a$ is separable. So E/F is Galois. Let $d = [E : F]$. Let $f(x) = \text{Irred}(\alpha, F)$. Then $f(x)|x^n - a$. So $f(x) = \prod_{\ell=0}^{d-1}(x - \omega^{i_\ell}\alpha)$ where $0 \leq i_j \leq n - 1$. Therefore, the d elements of $Gal(E/F)$ are $\sigma_{i_\ell} : E \to E$ defined by $\alpha \mapsto \omega^{i_\ell}\alpha$. Define $\phi : Gal(E/F) \to< \omega >$ by $\sigma_{i_\ell} \mapsto \omega^{i_\ell}$. This is a homomorphisms as $\sigma_{i_\ell}\sigma_{i_j}(\alpha) = \omega^{i_j}\sigma_{i_\ell}(\alpha) = \omega^{i_j + i_\ell}(\alpha)$ and so $\phi(\sigma_{i_\ell}\sigma_{i_j}) = \phi(\sigma_{i_\ell})\phi(\sigma_{i_j})$. This is injective as the ω^{i_j} are distinct. So $Gal(E/F)$ is isomorphic to a subgroup H of $< \omega >$. Clearly, H is cyclic and has order d.

□

2.10 Can we find polynomials whose Galois Group is S_n?

Theorem 2.94. *Let $f(x) \in \mathbb{Z}[x]$ be monic of degree n, with n distinct roots. Let p be prime and $\overline{f}(x) \in \mathbb{Z}_p[x]$ where $\overline{f}(x)$ is obtained by reducing the coefficients of $f(x)$ modulo p. Let $\alpha_1, ..., \alpha_n$ be the roots of $f(x)$ and $u_1, ..., u_n$ the roots of $\overline{f}(x)$ (assume $u_1, ..., u_n$ are also distinct.) After possibly reordering $u_1, ..., u_n$, there exists an injective group homomorphism $Gal_{\mathbb{Z}_p}(\overline{f}) \to Gal_{\mathbb{Q}}(f)$ defined by $\overline{\sigma}(u_i) = u_j$ if and only if $\sigma(\alpha_i) = \alpha_j$.*

Definition 2.95. *A subgroup $H \leq S_n$ is called **transitive** if for all $i \neq j \in [n]$, there exists $\sigma \in H$ such that $\sigma(i) = j$.*

Proposition 2.96. *Suppose $\overline{f}(x)$ is irreducible in $\mathbb{Z}_p[x]$. Then*

1. *$Gal_{\mathbb{Z}_p}(\overline{f})$ is transitive and hence $Gal_{\mathbb{Q}}(f)$ is transitive.*

2. *$Gal_{\mathbb{Q}}(f)$ contains an $n-$cycle.*

Proof. 1. As $\overline{f}(x)$ is irreducible, there exists a map $\phi : \mathbb{Z}_p(\alpha_i) \to \mathbb{Z}_p(\alpha_j)$ sending $\alpha_i \mapsto \alpha_j$. Extend ϕ to the splitting field. Then $\phi : Gal_{\mathbb{Z}_p}(\overline{f}) \to Gal_{\mathbb{Z}_p}(\overline{f})$.

2. As $\mathbb{Z}_p$ is a finite field, $Gal_{\mathbb{Z}_p}(\overline{f})$ is cyclic of order n. Let $<\sigma>= Gal_{\mathbb{Z}_p}(\overline{f})$. Say $\sigma = \pi_1 \cdots \pi_k$, where π_i are disjoint. Of course, $<\sigma>$ is transitive so we must have $\sigma = \pi_1$. Thus π_1 is an $n-$cycle. $\square$

Theorem 2.97. *Let $n \geq 4$ and $f_1, f_2, f_3 \in \mathbb{Z}[x]$ be monic polynomials of degree n such that*

1. *$\overline{f}_1 \in \mathbb{Z}_2[x]$ is irreducible.*
2. *$\overline{f}_2 \in \mathbb{Z}_3[x]$ is such that $\overline{f}_2 = \overline{g}_1\overline{h}_1$ where $\overline{g}_1$ is irreducible of degree $n-1$.*
3. *$\overline{f}_3 = \overline{g}_2\overline{h}_2 \in \mathbb{Z}_5[x]$ where $\overline{g}$ is irreducible of degree 2 and $\overline{h}$ is a product of irreducible factors of odd degree. [Note: we may need that the roots are distinct here...]*

Let $f = -15f_1 + 10f_2 + 6f_3$. Then f is monic of degree n and $Gal_{\mathbb{Q}}(f) \cong S_n$.

Proof. The key here is to note that S_n is generated by an $n-1$ cycle and a transposition. By 2, we see $Gal_{\mathbb{Q}}(f)$ contains an $n-1$ cycle. Now, we will show that the construction in 3 gives us a transposition. Let $f(x) = g(x)h_1(x)\cdots h_k(x)$, where g, h_i are irreducible, $\deg g = 2$ and $\deg h_i$ is odd for all i. Consider $G = Gal_{\mathbb{Z}_p}(f)$ as a subgroup of S_n. Let $\alpha_1, \alpha_2 \in \overline{\mathbb{Z}}_p$ be the roots of $g(x)$ and $\alpha_3, ..., \alpha_n$ the roots of $h_1, ..., h_k$. Let $F = \overline{\mathbb{Z}}_p(\alpha_1, \alpha_2)$ and $L = \overline{\mathbb{Z}}_p(\alpha_3, ..., \alpha_n)$. Then $E = FL$ is the splitting field of f. If we show $[E : L] = 2$, then any nontrivial element of $Gal(E/L)$ corresponds to a transposition (we swap α_1 and α_2 and leave all the other roots fixed). To do this, we need only show $[L : \overline{\mathbb{Z}}_p]$ is odd. Induct on k. If $k = 1$, since every finite extension of a finite field is cyclic, $L = \overline{\mathbb{Z}}_p(\alpha)$ where α is a root of h_1. So $[L : \overline{\mathbb{Z}}_p] = \deg h_1$ which is odd. So suppose $k > 1$. Let T be the splitting field for $h_1, ..., h_{k-1}$ and α be a root of h_k. By the same reasoning as above, h_k splits in $\overline{\mathbb{Z}}_p(\alpha)$. So $[\overline{\mathbb{Z}}_p(\alpha) : \overline{\mathbb{Z}}_p]$ is odd. By induction, $[T : \overline{\mathbb{Z}}_p]$ is odd. Note that L/T and $\overline{\mathbb{Z}}_p(\alpha)/\overline{\mathbb{Z}}_p$ are Galois (they are both splitting fields for h_k. Recall $Gal(L/T) \leq Gal(\overline{\mathbb{Z}}_p(\alpha)/\overline{\mathbb{Z}}_p)$. So $[L : T] \,\big|\, [\overline{\mathbb{Z}}_p(\alpha) : \overline{\mathbb{Z}}_p]$. So $[L : T]$ odd implies $[L : \overline{\mathbb{Z}}_p]$ is odd. Thus $2 = [E : L]$ which says G contains a transposition. Now, by the previous theorem, since there exists an injection $G \to Gal_{\mathbb{Q}}(f)$, we see that $Gal_{\mathbb{Q}}(f)$ contains a transposition. $\square$

Example. Find a polynomial $f(x) \in \mathbb{Q}[x]$ such that $Gal_{\mathbb{Q}}(f) \cong S_4$.

$f_1 = x^4 + x + 1$

$f_2 = (x^3 + 2x + 2)(x) = x^4 + 2x^2 + 2x$

$f_3 = (x^2 + 2)(x)(x + 1) = x^4 + x^3 + 2x^2 + 2x$

Then $f = -15f_1 + 10f_2 + 6f_3 = x^4 + 6x^3 + 32x^2 + 17x - 15$ has Galois group S_4 over $\mathbb{Q}$ by the theorem. Note that f is irreducible as it is modulo 2.

2.11 Solvability by Radicals

<u>Motivation:</u>

- Let $f(x) = ax^2 + bx + c \in F[x], a \neq 0$. Then, if char $F \neq 2$, the roots of $f(x) = \frac{-b \pm \alpha}{2a}$ where α is a root of $x^2 - (b^2 - 4ac)$. Less specifically, we know the roots of $f(x)$ lie in $F(\alpha)$ for some $\alpha \in \overline{F}$ such that $\alpha^2 \in F$.
- Let $f(x) = ax^3 + bx^2 + cx + d \in F[x]$. Then, if char $F \neq 2, 3$, we can reduce f to $f(x) = x^3 + px + q \in F[x]$. Cardano (1500s) found that the roots of $f(x)$ lie in $F(\omega, \delta, y_1, y_2)$ where ω is a primitive 3^{rd} root of unity, δ is a

root of $x^2 - (12p^3 - 81q^2)$, y_1 is a root of $x^2 + (\frac{27}{2}q + \frac{3}{2}\delta)$, and y_2 a root of $x^3 + (\frac{27}{2}q - \frac{3}{2}\delta)$.

$$\begin{array}{c} F(\omega, \delta, y_1, y_2) \\ | \\ y_2^2 \in F(\omega, \delta, y_1) \\ | \\ y_1^2 \in F(\omega, \delta) \\ | \\ \delta^2 \in F(\omega) \\ | \\ \omega^3 \in F \end{array}$$

Definition 2.98. *A finite extension E/F is called **radical** if $E = F(\alpha_1, ..., \alpha_n)$ such that for all $i = 1, .., n$ there exists m_i such that $\alpha_i^{m_i} \in F(\alpha_1, ..., \alpha_{i-1})$. A polynomial $f(x) \in F[x]$ is **solvable by radicals** over F if $f(x)$ splits in some radical extension of F.*

Theorem 2.99. *Let $f(x) \in F[x]$ be a separable polynomial. Let E be the splitting field for $f(x)$ over F. Suppose char $F \nmid [E : F]$. If $Gal(E/F)$ is solvable, then $f(x)$ is solvable by radicals over F.*

Proof. Let $n = [E : F]$ and ω be a primitive n^{th} root of unity. Let $L = F(\omega)$. By HW3#1, EL/L is Galois and $Gal(EL/L)$ is isomorphic to a subgroup of $Gal(E/F)$. Since subgroups of solvable groups are solvable, $Gal(EL/L)$ is solvable. Now EL is the splitting field of $f(x)$ over L. Note that

$$\begin{aligned} f(x) \text{ is solvable by radicals over } L &\Leftrightarrow EL \text{ lives in a radical extension of } L \\ &\Leftrightarrow EL \text{ lives in a radical extension of } F (\text{since } L = F(\omega) \text{ and } \omega^n \in F) \\ &\Leftrightarrow E \text{ lives in a radical extension of } F \\ &\Leftrightarrow f(x) \text{ is solvable by radicals over } F. \end{aligned}$$

So WLOG, we may assume $\omega \in F$. Let $G = Gal(E/F)$. Since G is solvable, there exists a normal series $\{1\} = G_t \lhd G_{t-1} \lhd \cdots \lhd G_0 = G$ such that $G_i/G_{i+1} \cong C_{n_i}$ (we know the factor groups are abelian, if not cyclic then just take smaller subgroups so that they are), where $n_i | n = |G|$. Let E_i be the corresponding intermediate field of G_i with $E = E_t$ and $E_0 = F$. Note that E_{i+1}/E_i is Galois for all i and $Gal(E_{i+1}/E_i) \cong G_i/G_{i+1} \cong C_{n_i}$ and since F contains a primitive n_i^{th} root of unity (it contains a primitive n^{th} root of unity and $n_i | n$). Then by the previous theorem, $E_{i+1} = E_i(\alpha_i)$ where $\alpha_i^{n_i} \in E_i$. Therefore, E is a radical extension of F and so f is solvable by radicals over F. □

Lemma 2.100. *Suppose E/F is a radical extension. Let L be the normal closure of E/F. Then L/F is radical.*

Proof. Let $E = F(u_1, ..., u_n)$ where $u_i^{m_i} \in F(u_1, ..., u_{i-1})$ for $i = 1, ..., n$. Let $\sigma_1, ..., \sigma_s$ be the distinct embeddings of $E \to \overline{F}$ which fix F. Then $L = F(\{\sigma_i(u_j)\}_{i,j})$ (as this gives all of the roots of $\{Irred(u_i, F)\}_i$). Note that $\sigma_i(u_j)^{m_j} = \sigma_i(u_j^{m_j}) \in \sigma_i(F(u_1, ..., u_{j-1})) = F(\sigma_i(u_1), ..., \sigma_i(u_{j-1}))$. □

Lemma 2.101. *Let L/K be a Galois, radical extension. Then $Gal(L/K)$ is solvable.*

Proof. Say $K = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = L$ where $K_i = K_{i-1}(u_i)$ and $u_i^{m_i} \in K_{i-1}$.

Claim: char $K \nmid m_i$ for all i.

Proof: Suppose $m_i = p^t \ell$ where $p =$ char K and $p \nmid \ell$. Then $(u_i^\ell)^{pt} = u_i^{m_i} \in K_{i-1}$. This says u_i^ℓ is p.i. over K_{i-1}. But L/K Galois says L/K_{i-1} is separable. Thus we must have $u_i^\ell \in K_{i-1}$. So we can simply replace m_i with ℓ and since $p \nmid \ell$, done.

Let $m = m_1 \cdots m_n$. Then $u_i^m \in K_{i-1}$ and char $K \nmid m$. Let ω be a primitive m^{th} root of unity.

$$\begin{tikzcd}
 & L(\omega) \arrow[dl, dash] \arrow[dr, dash, "\text{Galois, Radical}"] & \\
L \arrow[dr, dash] & & K(\omega) \arrow[dl, dash, "\text{Galois, Radical}"] \\
 & K &
\end{tikzcd}$$

By the picture, $L(\omega)/K$ is radical and Galois (as $L(\omega) = LK(\omega)$ where $L, K(\omega)$ are Galois). Now, since L/K is normal,

$$Gal(L/K) \cong Gal(L(\omega)/K)/Gal(L(\omega)/L).$$

Since quotient groups of solvable groups are solvable, it is enough to show $Gal(L(\omega)/K)$ is solvable. Also

$$\mathbb{Z}_m^* \cong Gal(K(\omega)/K) \cong Gal(L(\omega)/K)/Gal(L(\omega)/K(\omega)).$$

Recall that $Gal(L(\omega)/K)$ is solvable if and only if $Gal(K(\omega)/K)$ and $Gal(L(\omega)/K(\omega))$ are solvable. Since $Gal(K(\omega)/K)$ is abelian, it is solvable. So we need only show $Gal(L(\omega)/K(\omega))$ is solvable. Note that we have shown that $Gal(L/K)$ is solvable if $Gal(L(\omega)/K(\omega))$ is solvable. Thus, we may assume K contains a primitive m^{th} root of unity. By the theorem on cyclic extensions, K_i/K_{i-1} is cyclic. Let $H_{i-1} = Gal(L/K_{i-1})$ and $H_i = Gal(L/K_i)$. As K_i/K_{i-1} is normal, $H_i \lhd H_{i-1}$ and $H_{i-1}/H_i \cong Gal(K_i/K_{i-1})$ is cyclic. So $\{1\} = H_n \lhd H_{n-1} \lhd \cdots \lhd H_0 = Gal(L/K)$ is a solvable series. Thus G is solvable. □

Theorem 2.102. *Let F be a field and $f(x) \in F[x]$ a separable polynomial. If $f(x)$ is solvable by radicals over F, then $Gal_F(f(x))$ is solvable.*

Proof. Let E be the splitting field for $f(x)$ over F. Then $E \subseteq L$ for some radical extension L over F. WLOG, assume L/F is normal (can do by the first lemma). Define $\phi : Aut(L/F) \to Gal(E/F)$ by $\sigma \mapsto \sigma|_E$. Since E/F is normal, ϕ is well-defined. Also ϕ is surjective as L/F is normal (given $\rho \in Gal(E/F)$, we can extend it to L and it will be an automorphism of L). Hence

$$Gal(E/F) \cong Aut(L/F)/\ker\phi.$$

Since quotients of solvable groups are solvable, it is enough to prove $Aut(L/F)$ is solvable. Note $|Aut(L/F)| = [L : F]_S \leq [L : F] < \infty$ (as radical extensions are by definition finite). Let $G = Aut(L/F)$ and $K = E_G$. By Artin's Theorem, L/K is Galois and $G = Gal(L/K)$. Note that $F \subseteq K$ and L/K is radical. Thus by the second lemma, we're done. □

Definition 2.103. *Let F be a field and $t_1, .., t_n$ indeterminants over F. Then the* ***general equation of degree*** *n over F is $f(x) = x^n - t_1x^{n-1} + t_2x^{n-2} + \ldots + (-1)^n t_n \in F(t_1, ..., t_n)[x]$.*

Theorem 2.104. *Let $L = F(t_1, .., t_n)$ and $f(x)$ as above. Then $Gal_L(f) \cong S_n$.*

Proof. Let E be the splitting field for $f(x)$ over L. Say $f(x) = \prod_{i=1}^n (x - y_i) \in E[x]$. Then $E = L(y_1, ..., y_n) = F(y_1, ..., y_n)$. Thus $t_i = s_i(y_1, ..., y_n)$, where $s_i \in L[x_1, ..., x_n]$ is the i^{th} elementary symmetric function. Define a field homomorphism $\sigma : L \to F(s_1, .., s_n) \subseteq F(x_1, ..., x_n)$ by $t_i \mapsto s_i$ and fixes F. Then σ is clearly surjective.

Claim: σ is an isomorphism

Proof: Define $\tau : F(x_1, ..., x_n) \to E = F(y_1, ..., y_n)$ by $x_i \mapsto y_i$. Then $\tau(s_i) = t_i$ as $t_i = s_i(y_1, ..., y_n)$ and $\tau\sigma\left(\frac{p(t_1,...,t_n)}{q(t_1,...,t_n)}\right) = \tau\left(\frac{p(s_1,...,s_n)}{q(s_1,...,s_n)}\right) = \frac{p(t_1,...,t_n)}{q(t_1,...,t_n)}$. So σ is injective and thus an isomorphism.

Note that $f^\sigma(x) = x^n - s_1x^{n-1} + \ldots + (-1)^n s_n$ and has splitting field $F(x_1, ..., x_n)$ (where the x_i's are such that $f^\sigma(x) = \prod_{i=1}^n (x - x_i)-$ from our definition of the elementary symmetric functions).

$$\text{splitting field of } f(x) \to \begin{array}{ccc} F(y_1, ..., y_n) & \xrightarrow{\phi} & F(x_1, ..., x_n) \\ | & & | \\ F(t_1, ..., t_n) & \xrightarrow{\cong} & F(s_1, ..., s_n) \end{array} \leftarrow \text{splitting field of } f^\sigma(x)$$

By the theorem on the uniqueness of splitting fields, there exists an isomorphism $\phi : F(y_1, ..., y_n) \to F(x_1, ..., x_n)$ where $\phi|_L = \sigma$. Hence $Gal_L(f) \cong Gal_{F(s_1,...,s_n)}(f^\sigma) \cong S_n$, as we saw earlier with the symmetric functions, using Artin's Theorem. □

Recall: S_n is solvable if and only if $n \leq 4$.

Corollary 2.105. *If $n \leq 4$ and char $F \nmid |S_n| = n!$, then the general equation of degree n over F is solvable by radicals.*

Corollary 2.106 (**Abel's Theorem**)**.** *If $n \geq 5$, then the general equation of degree n over F is not solvable by radicals.*

Fact. If p is prime, then S_p is generated by any transposition and any $p-$cycle.

Lemma 2.107. *Let $f(x) \in \mathbb{Q}[x]$ be irreducible of prime degree p and suppose f has exactly $p - 2$ real roots. Then $Gal_{\mathbb{Q}}(f) \cong S_p$.*

Proof. Let $E = \mathbb{Q}(\alpha_1, ..., \alpha_p)$ where $\alpha_1, ..., \alpha_p$ are roots of $f(x)$ with $\alpha_1, \alpha_2 \notin \mathbb{R}$. Let $G = Gal(E/\mathbb{Q}) \subseteq S_p$. Since $f(x)$ is irreducible, $p||G|$. Since p is prime, the only elements of S_p of order p are the $p-$cycles. Thus G contains a $p-$cycle. Let σ be complex conjugation restricted to E. Then σ transposes α_1 and α_2 and fixes $\alpha_3, ..., \alpha_n$. So $\sigma \in G$ is a transposition. Done by fact. □

Example. Let $f(x) = x^5 - 2x^3 - 8x - 2 \in \mathbb{Q}[x]$. This is irreducible by Eisenstein. Using Calculus to find the critical numbers and looking at the end behavior, we see $f(x)$ crosses the $x-$ axis 3 times. Thus $f(x)$ has 3 real roots. By the lemma, $Gal_{\mathbb{Q}}(f) \cong S_5$. Thus f is not solvable by radicals.

2.12 Transcendental Extension

Definition 2.108. *Let E/F be a field extension and $S \subseteq E$. Then S is **algebraically dependent** over F if there exists $s_1, ..., s_n \in S$ and $f(x_1, ..., x_n) \in F[x_1, ..., x_n] \setminus \{0\}$ such that $f(s_1, ..., s_n) = 0$. Otherwise, we say S is **algebraically independent** over F.*

Remarks.

1. $\emptyset$ is algebraically independent over any field.

2. $\{u\}$ is algebraically independent if and only if u is transcendental over F.

3. $\{s_1, ..., s_n\}$ is algebraically independent over F if and only if $F[s_1, ..., s_n] \cong F[x_1, ..., x_n]$, where $x_1, ..., x_n$ are variables.

Lemma 2.109. *Let E/F be a field extension and $S \subseteq E$ an algebraically independent set over F. Let $u \in E$. Then $S \cup \{u\}$ is algebraically independent if and only if u is transcendental over $F(S)$.*

Proof. ($\Leftarrow$) It is enough to show $\{s_1, ..., s_n, u\}$ is algebraically independent for $s_1, ..., s_n \in S$. Suppose $f(x_1, ..., x_{n+1}) \in F[x_1, ..., x_{n+1}]$ and $f(s_1, ..., s_n, u) = 0$. Let $g(x_{n+1}) = f(s_1, ..., s_n, x_{n+1}) \in F(S)[x_{n+1}]$. Note $g(u) = 0$. Since u is transcendental over $F(S)$, we must have $g(x_{n+1}) = 0$. Write

$$f(x_1, ..., x_{n+1}) = h_r(x_1, ..., x_n)x_{n+1}^r + \ldots + h_0(x_1, ..., x_n).$$

Then $0 = g(x_{n+1})$ says $h_i(s_1, ..., s_n) = 0$ for all i. Since $\{s_1, ..., s_n\}$ are algebraically independent, we must have $h_i(x_1, ..., x_n) = 0$ for all i. Thus $f(x_1, ..., x_{n+1}) = 0$.

($\Rightarrow$) Suppose u is algebraic over $F(S)$. Then u is algebraic over a finite subset of S. So WLOG, S is finite. Then there exists $f(x) \in F(S)[x] \setminus \{0\}$ such that $f(u) = 0$. Say

$$f(x) = \frac{g_r(s_1, ..., s_n)}{h_r(s_1, ..., s_n)} x^r + \ldots + \frac{g_0(s_1, ..., s_n)}{h_0(s_1, ..., s_n)},$$

where $g_i(x_1, ..., x_n), h_i(x_1, ..., x_n) \in F[x_1, ..., x_n]$. Multiply f by $h_0 \cdots h_r$ to clear denominators and still get a polynomial that u satisfies. So WLOG, $h_i = 1$. Let $\ell(x_1, .., x_n, x) = g_r(x_1, ..., x_n)x^r + \ldots + g_0(x_1, ..., x_n)$. Note that $\ell(s_1, ..., s_n, u) = 0$. Since $S \cup \{u\}$ is algebraically independent, $\ell(x_1, ..., x_n, x) = 0$, a contradiction as $f(x) \neq 0$. Thus u is transcendental over $F(S)$. $\square$

Definition 2.110. *Let E/F be a field extension. A set $S \subseteq E$ is called a **transcendence base** for E/F if S is algebraically independent over F and $E/F(S)$ is algebraic.*

Theorem 2.111. *Let E/F be a field extension and $L \subseteq E$ an algebraically independent set over F. Then there exists a transcendence base S for E/F such that $L \subseteq S$.*

Proof. Let $\Gamma = \{T | L \subseteq T \subseteq E$ and T is algebraically independent over $F\}$. Note $L \in \Gamma$ so $\Gamma \neq \emptyset$. Let $\mathcal{C}$ be any totally ordered subset of Γ. Then $T_0 = \cup_{t \in \mathcal{C}} T \in \Gamma$ is an upper bound. By Zorn's Lemma, there exists a maximal set $S \in \Gamma$. Then S is algebraically independent by definition of Γ and $E/F(S)$ is algebraic by the lemma and maximality of S. $\square$

Example. Let X, Y be indeterminants over F. Then $\{X, Y\}$ is a transcendence base for $F(X, Y)/F$. Also $\{X^2, Y^2\}$ is a transcendence base.

Theorem 2.112. *Let E/F be a field extension. Then any two transcendence bases for E/F have the same cardinality.*

Proof. We'll prove this in the case that E/F has a finite transcendence base $S = \{s_1, ..., s_n\}$. Let T be a transcendence base for E/F.

Claim: There exists $t_1 \in T$ such that $\{t_1, s_2, ..., s_n\}$ is algebraically independent over E/F.

Proof: Suppose not. Therefore $F(T)$ is algebraic over $F(s_2, ..., s_n)$. But $E/F(T)$ is algebraic, which implies $E/F(s_2, ..., s_n)$ is, so $s_1 \in E$ is algebraic over $F(s_2, ..., s_n)$, a contradiction.

Claim: The set $\{t_1, s_2, ..., s_n\}$ is a transcendence base of E/F.

Proof: Suppose s_1 is transcendental over $F(\{t_1, s_2, ..., s_n\})$. Then $\{t_1, s_1, ..., s_n\}$ is algebraically independent, but t_1 is algebraic over $F(\{s_1, ..., s_n\})$, a contradiction. Thus s_1 is algebraic over $F(\{t_1, s_2, ..., s_n\})$ which implies $F(\{t_1, s_1, ..., s_n\})$ is algebraic over $F(\{t_1, s_2, ..., s_n\})$. But E is algebraic over $F(\{t_1, s_1, ..., s_n\})$ (as it is over $F(\{s_1, ..., s_n\})$) and thus E is algebraic over $F(\{t_1, s_2, ..., s_n\})$.

Repeating this process, replace $s_2, ..., s_n$ by $t_2, ..., t_n \in T$ to obtain a transcendence base $\{t_1, ..., t_n\}$ for E/F. Since T is algebraically independent, $T = \{t_1, ..., t_n\}$. $\square$

Definition 2.113. *The **transcendence degree** of E/F is the cardinality of any transcendence base for E/F.*

Note. The transcendence degree of E/F is 0 if and only if E/F is algebraic.

Theorem 2.114. *Suppose $K \subseteq F \subseteq E$ are fields. The tr deg $E/K =$ tr deg $E/F +$ tr deg F/K.*

Proof. Let S, T be transcendence bases for E/F and F/K respectively. Since $T \subseteq F$ and $S \subseteq E \setminus F$, we see $S \cap T = \emptyset$. Then it is enough to show $S \cup T$ is a transcendence base for E/K.

Claim 1: E is algebraic over $K(S \cup T)$.

Proof: We know that F is algebraic over $K(T)$. So $F(S)$ is algebraic over $K(T)(S) = K(S \cup T)$. As E is algebraic over $F(S)$, E is algebraic over $K(S \cup T)$.

Claim 2: $S \cup T$ is algebraically independent over K.

Proof: Let $f(x_1, ..., x_m, y_1, ..., y_n) \in K[x_1, ..., x_m, y_1, ..., y_n]$ such that $f(s_1, ..., s_m, t_1, ..., t_n) = 0$. We want to show $f = 0$. Say $f = \sum g_i(y_1, ..., y_n)h_i(x_1, ..., x_m)$ where $g_i \in K[y_1, ..., y_n]$ and the h_i are distinct monomials in the x's. Let $\ell(x_1, ..., x_m) = f(x_1, ..., x_m, t_1, ..., t_n) \in K(T)[x] \subseteq F[x_1, ..., x_m]$. That that $\ell(s_1, ..., s_m) = 0$. As S is algebraically independence over F, we know $\ell = 0$. So $f(x_1, ..., x_m, t_1, ..., t_n) = 0$. Since the $h_i(x_1, ..., x_m)$ are linearly independent over $F[x]$ (as they are distinct monomials), we must have that $g_i(t_1, ..., t_n) = 0$ for all i. Since T is algebraically independent over K, $g_i(y_1, ..., y_n) = 0$. Thus $f = 0$.

□

3 Rings and Modules

We will take all rings to have identity, but not necessarily be commutative.

Definition 3.1. *Let G be a group, k a field. Let B be a $k-$vector space with basis $\{e_g\}_{g \in G}$. Then V is a **group ring** with elements of the form $\sum_{g \in G} c_g e_g$ where all but finitely many terms are zero. Define multiplication in V by $(\sum c_g e_g)(\sum d_g e_g) = \sum c_g d_{g'} e_{gg'}$.*

Remarks. Under this definition, V is a ring with identity element e_1. For convenience, we will write g for e_g and $K[G]$ for the ring V. Note that $K[G]$ is commutative if and only if G is abelian.

Example. Let $G = C_n =< g >$ and K be any field. Then $K[C_n] = \{\sum_{i=0}^{n-1} c_i g^i | c_i \in K\}$. Define a ring homomorphism $K[x] \to K[C_n]$ such that $k \mapsto k$ and $x \mapsto g$. Clearly, this is surjective. As $g^n = 1$, we see $x^n - 1 \in \ker \phi$. So we have an induced map $K[x]/(x^n - 1) \to K[C_n]$. Since both of these have dimension n, we see that they are isomorphic.

Definition 3.2. *A **division ring** is a ring in which every nonzero element is a unit.*

Examples.

1. Any field is a division ring.

2. Consider the ring homomorphism $\mathbb{R} \to M_2(\mathbb{C})$ defined by $r \mapsto rI$. In this way, we can consider $\mathbb{R}$ as a subring of $M_2(\mathbb{C})$. Let $\mathbf{i} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \mathbf{j} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \mathbf{k} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$. Then $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ are linearly independent over $\mathbb{R}$. Let $H = \mathbb{R} \cdot 1 + \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k} \subseteq M_2(\mathbb{C})$. Then H has dimension 4. Note that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{ij} = \mathbf{j} = -\mathbf{ji}, \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \mathbf{ki} = \mathbf{j} = -\mathbf{ik}$. Thus H is closed under multiplication and has identity. Since H is a vector space, its an additive group. Thus H is a noncommutative subring of $M_2(\mathbb{C})$, called the ring of (real) quaternions. Let $\alpha = r_0 + r_1\mathbf{i} + r_2\mathbf{j} + r_3\mathbf{k}$ and $\overline{\alpha} = r_0 - r_1\mathbf{i} - r_2\mathbf{j} - r_3\mathbf{k}$. One can check $\alpha\overline{\alpha} = \overline{\alpha}\alpha = r_0^2 + r_1^2 + r_2^2 + r_3^2 =: |\alpha|^2$. Note $\alpha = 0$ if and only if $|\alpha| = 0$. So if $\alpha \neq 0$, $\alpha^{-1} = \frac{\overline{\alpha}}{|\alpha|^2}$. Thus H is a division ring (but not a field!).

Definition 3.3. *Let R be a ring. A left (respectively, right) R-**module** is an abelian group $(M, +)$ together with a map $R \times M \to M$ defined by $(r, m) \mapsto rm$ such that*

1. *$r(m + n) = rm + rn$*

2. *$(r + s)m = rm + sm$*

3. *$r(sm) = (rs)m$*

4. *$1m = m$*

Notes. Not everyone requires (4). In this case, R is called a **unital module**. Also, we will assume $1 \mapsto 1$ in a ring homomorphism.

Definition 3.4. *Let $f : R \to S$ be a ring homomorphism such that $f(R) \subseteq Z(S)$. Then S is called an $R-$**algebra**.*

Note. The $\ker f$ is a two-sided ideal. Thus $\overline{f} : R/\ker f \to S$ is injective. Thus $R/\ker f$ is commutative and $R/\ker f \subseteq Z(S)$.

Examples. Assume R is a commutative ring.

1. Let $R[x_1, ..., x_n]$ be the polynomial ring in $x_1, ..., x_n$ and I an ideal of $R[x_1, ..., x_n]$. Then $f : R \to R[x_1, ..., x_n]/I$ defined by $r \mapsto \overline{r} = r + I$ is a ring homomorphism. Thus $R[x_1, ..., x_n]/I$ is an $R-$algebra.

2. Define $f : R \to M_n(R)$ by $r \mapsto rI$. This is a ring homomorphism, so $M_n(R)$ is an $R-$algebra.

3. Let G be a group. Define $f : R \to R[G]$ by $r \mapsto re_1$. This is a ring homomorphism, so $R[G]$ is an $R-$algebra.

4. Let $C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} | f \text{ is continuous}\}$. Then $f : \mathbb{R} \to C(\mathbb{R})$ defined by $r \mapsto f_r(x) = r$ is a ring homomorphism. Thus $C(\mathbb{R})$ is an $\mathbb{R}-$algebra.

Definition 3.5. *Let S be a ring, $A \subseteq Z(S)$ a subring, T a subset of S. Say S is **generated** over A by T if every element of S is a finite sum of elements of the form $at_1^{n_1} \cdots t_k^{n_k}$, where $a \in A, t_i \in T, n_i \geq 0$. We write $S = A[T]$. If $S = A[T]$ for some finite subset T of S, then S is **finitely generated over** A **as a ring**. If $f : R \to S$ is a ring homomorphism with $f(R) \subseteq Z(S)$, then S is a **finitely generated** $R-$**algebra** if S is finitely generated over $f(R)$ as a ring.*

Notes.

- If E/K is a finitely generated field extension and F is an intermediate field, then F/K is a finitely generated field extension (HW).
- This is NOT true for algebras. For example, $K[x, y]$ is finitely generated as a $K-$algebra, but $K[x, xy, xy^2, ...]$ is not finitely generated as a $K-$algebra.

Examples. Let R be a commutative ring.

1. $S = R[x_1, ..., x_n]/I$ is a finitely generated $R-$algebra where $T = \{\overline{x_1}, ..., \overline{x_n}\}$. Using the above notation, we can say $S = R[\overline{x_1}, ..., \overline{x_n}]$.

   Claim. Let S be a finitely generated $A-$algebra which is commutative. Say $S = A[T]$ where $T = \{t_1, ..., t_n\}$. Define $\phi : A[x_1, ..., x_n] \to S$ by $f(x_1, ..., x_n) \mapsto f(t_1, ..., t_n)$. Because the t_i's commute, ϕ is an onto ring homomorphism. So $S \cong A[x_1, ..., x_n]/I$.

2. $S = M_n(R)$. Let E_{ij} be the $n \times n$ matrix with a 1 in the i, j^{th} entry and zeros everywhere else. Then for $A = (a_{ij}) \in S$, we see $A = \sum a_{ij}E_{ij}$. Thus S is generated by E_{ij}. So $S = R[\{E_{ij}\}]$.

3. $R[G]$ is a finitely generated $R-$algebra if and only if G is a finitely generated group. For one direction, we see if $G =< g_1, ..., g_n >$, then $R[G] = R[g_1, ..., g_n]$.

4. $C(\mathbb{R})$ is not a finitely generated $R-$algebra.

Let A be a ring. By an $A-$module, we mean a left $A-$module, unless when explicitly stated otherwise.

Remark. Let $f : R \to S$ be a ring homomorphism. Any $S-$module M is an $R-$module via the action $r \cdot m := f(r)m$. In particular, S is an $R-$module.

Definition 3.6. *Let M be an $R-$module and $T \subseteq M$. Say T **generates** M as an $R-$module if every element of M can be expressed as $\sum_1^n r_i t_i$, for $t_i \in T, r_i \in T$, that is, $M = RT = R-$submodule of M generated by T. We say M is **finitely generated as an** $R-$**module** if $M = RT$ for some finite subset T of M. In practice, if $T = \{t_1, ..., t_n\}$, we will write $M = Rt_1 + ... + Rt_n$. Sometimes, this is stated as "M is a finite $R-$module" even though M is not necessarily finite.*

Examples. Let R be a commutative ring.

1. $R[x_1, ..., x_n]/I$ need not be a finitely generated $R-$module. For example $k[x, y]/(xy)$ is not a finitely generated $k-$module.

2. $M_n(R)$ is a finitely generated $R-$module ($M_n(R) = \sum RE_{ij}$).

3. $R[G]$ is a finitely generated $R-$module if and only if $|G| < \infty$.

3.1 Free Modules and Bases

Definition 3.7. *Let M be an $A-$module, $T \subseteq M$. Say T is **linearly independent** over A if whenever $\sum_1^n a_i t_i = 0$ where $t_1, ..., t_n \in T$ are distinct, then $a_i = 0$ for all i.*

Example. Let $R = \mathbb{Z}_6$ and $I = (\overline{2})$. Then $\overline{2}$ is a minimal generating set of I but $\overline{3} \cdot \overline{2} = \overline{0}$. So $\{\overline{2}\}$ is not linearly independent over R.

Definition 3.8. *A **basis** T for an $A-$module M is a generating set for M which is linearly independent over A.*

Proposition 3.9. *Let M be an $A-$module, $S \subseteq M$. TFAE*

1. *S is an $A-$basis for M*

2. *For any $A-$module N and any set map $j : S \to N$, there exists a unique $A-$module homomorphism $\widetilde{j} : M \to N$ such that the following diagram commutes*

$$\begin{tikzcd} S \arrow[r] \arrow[d, "j"'] & M \arrow[dl, dashed, "\exists! \widetilde{j}"] \\ N & \end{tikzcd}$$

Proof. (1) $\Rightarrow$ (2) Given $j : S \to N$, define $\widetilde{j} : M \to N$ by $\widetilde{j} : (\sum_{s \in S} a_s s) = \sum_{s \in S} a_s j(s)$ (where all but finitely many a_s are 0). Since S is a basis for M, every element of M can be written uniquely in the form $\sum_{s \in S} a_s s$. Thus $\widetilde{j}$ is a well-defined homomorphism. Also, $\widetilde{j}$ is clearly unique.

(2) $\Rightarrow$ (1) <u>S is linearly independent:</u> Suppose $\sum_{s \in S} a_s s = 0$. For each $t \in S$, define $j_t : S \to A$ by $t \mapsto 1$ and $s \mapsto 0$ for $s \neq t$. Then $0 = \widetilde{j}_t(0) = \widetilde{j}_t(\sum a_s s) = \sum a_s \widetilde{j}_t(s) = a_t$. Since t was arbitrary, done.

<u>S generates M</u>: Let M' be the $A-$submodule of M generated by S, that is $M' = \{\sum_{s \in S} a_s s | a_s \in A, s \in S\}$. Define $j : S \to M/M'$ by $s \mapsto 0 = s + M'$. Consider $\widetilde{j} : M \to M/M'$ defined by $m \mapsto m + M'$. By the uniqueness of $\widetilde{j}$, since the 0 map also make the diagram commute, $\widetilde{j} = 0$, which implies $m + M' = 0$ for all $m \in M$. Thus $M = M'$. □

Definition 3.10. *An $A-$module is called **free** if M has a basis.*

Remarks.

1. M is a free $A-$module if and only if $M \cong \otimes_{i \in I} A$.

   Proof. $(\Leftarrow):$ For all $j \in I$, let $e_j \in \otimes_{i \in I} A$ where $(e_j)_i = 0$ if $i \neq j$ and 1 if $i = j$. Then $\{e_j\}_{j \in I}$ forms a basis.

   $(\Rightarrow):$ Let S be a basis for M. Define $\phi : \otimes_{s \in S} A \to M$ by $e_s \mapsto s$. Then $\sum a_s e_s \mapsto \sum a_s s$. Since S generates M, its onto. Since S is linearly independent, its injective. □

2. Every $A-$module is the homomorphic image of a free $A-$module.

   Proof. Let M be an $A-$module. Define $\otimes_{m \in M} A \to M$ by $e_m \mapsto m$. Then extend it to $\sum a_m e_m \mapsto \sum a_m m$. Then ϕ is a surjective homomorphism. □

Examples.

1. The $0-$module is always free.

2. Let R be a commutative ring, $I \neq (0)$ an ideal. TFAE

   (a) I is free

   (b) $I \cong R$

   (c) $I = Ra = (a)$ for some non-zero-divisor $a \in R$.

   Proof. (a)$\Rightarrow$(b): Let S be a basis for I. Suppose $|S| > 1$. Let $s \neq t \in S$. Since R is commutative, $st + (-t)s = 0$. Since s and t are linearly independent, the coefficients are 0. Thus $s = t = 0$. So $|S| = 1$ which implies $I \cong R$. $\square$

3. Let $R = \mathbb{Z}[x]$ and $I = (2, x)$. Then I can be shown to be not principal, thus I is not free.

4. Let $R = \mathbb{Z}[\sqrt{-5}]$ and $I = (2, 1 + \sqrt{-5})$. Then I is not principal, so I is not free. However, $I \otimes J \cong R^2$ for some ideal J.

5. Let R be commutative. Then $M_n(R)$ is a free $R-$module with basis $\{E_{ij}\}$.

6. $R[G]$ is a free $R-$module with basis $\{g\}_{g \in G}$.

Remark. Let A be a ring, I a two-sided ideal. Let M be an $A-$module. Then M/IM is an $A/I-$module via $(a + I)(m + IM) = am + IM$.

Lemma 3.11. *Let M be an $A-$module and I a two-sided ideal. If S is a basis for M, then $\overline{S} = \{s + IM | s \in S\}$ is an A/I basis for M/IM.*

Proof. Let $\overline{m} \in M/IM$. Then if $\overline{m} = m + IM$, we know $m = \sum a_s s$, which says $\overline{m} = \sum \overline{a_s s}$. So $\overline{S}$ generates M/IM. Suppose $\sum \overline{a_s s} = \overline{0}$. Then $\overline{\sum a_s s} = \overline{0}$ which implies $\sum a_s s \in IM$. Then $\sum a_s s = \sum_{j=1}^n i_j m_j$ for $i_j \in I, m_j \in M$. Now $m_j = \sum_{s \in S} b_{js} s$. So, $\sum a_s s = \sum_{j,s} i_j b_{js} s = \sum_s (\sum_j i_j b_{js}) s$ which implies $a_s = \sum i_j b_{js} \in I$. Thus $\overline{a_s} = 0$. $\square$

Lemma 3.12. *Let R be a division ring. Any $R-$module M has a basis and any two bases for M have the same cardinality.*

Proposition 3.13. *Let R be a commutative ring and M an $R-$module. Then any two bases have the same cardinality.*

Proof. Let m be a maximal ideal of R (it exists by Zorn's Lemma). Then R/m is a field. Let S_1, S_2 be two $R-$bases for M. By the above two lemmas, $\overline{S_1}, \overline{S_2}$ are $R/m-$bases for M/mM and $\overline{S_1}, \overline{S_2}$ have the same cardinality (as R/m is a field).

Claim: For any basis S of M, S and $\overline{S}$ have the same cardinality.

Proof: We know the map $S \to \overline{S}$ defined by $s \mapsto \overline{s}$ is onto. Suppose $\overline{s} = \overline{t}$ for $s, t \in S$. Then $s - t \in mM$. So $s - t = \sum i_s s$ for $i_s \in m$ by the proof of the first lemma. Comparing coefficients, this says $1 \in m$, a contradiction as $m \neq R$.

Thus S_1 and S_2 have the same cardinality. $\square$

Definition 3.14. *If R is commutative and F is a free $R-$module, then the **rank** of F is defined to be the cardinality of any basis for F. (Note: When R is a field, this is just the dimension).*

Definition 3.15. *Let M be an $A-$module. Define $End_A M = \{f : M \to M | f \text{ is an } A\text{-module homomorphism}\}$.*

Remarks.

1. $End_A M$ is a ring under addition and composition. We call it the **endomorphism ring** of M.

2. If A is commutative, then $\phi : A \to End_A(M)$ defined by $a \mapsto aI$ is a ring homomorphism.

   [Note: If A is not commutative, then for $r \notin Z(A)$, we have $rI \notin End_A(M)$ as $f(r'm) \neq r'f(m)$.]

   Thus if A is commutative, then $End_A(M)$ is an $A-$algebra, and in particular an $A-$module.

3. If A is commutative and F is a free $A-$module of rank n, then $End_A(F) \cong M_n(A)$ (as a homomorphism is determined by where it sends the basis elements).

Example. Let A be a commutative ring, F a free $A-$module with basis $\mathbb{N}$, that is $F \cong \otimes_{i=1}^{\infty} A$. Let $\{e_i | i = 0, 1, ...\}$ be a basis for F and $R = End_A(F)$. Then $R \cong R^n$ for all $n \geq 1$.

Proof. Define $f_1, f_2 : F \to F$ by $f_1(e_{2i}) = e_i, f_1(e_{2i+1}) = 0$ and $f_2(e_{2i}) = 0, f_2(e_{2i+1}) = e_i$ for $i \geq 0$. Then $f_1, f_2 \in End_A(F) = R$.

Claim: $\{f_1, f_2\}$ is an $R-$basis for R.

Proof: Let $g_1, g_2 \in R$. Note that $(g_1 f_1 + g_2 f_2)(e_{2i}) = g_1(e_i)$ and $(g_1 f_1 + g_2 f_2)(e_{2i+1}) = g_2(e_i)$. Now, suppose $g_1 f_1 + g_2 f_2 = 0$. Then, by the note, $g_1(e_i) = g_2(e_i) = 0$ which implies $g_1 = g_2 = 0$ as the set $\{e_i\}$ is a basis. Thus $\{f_1, f_2\}$ is a linearly independent set. To show it is a generating set, let $g \in R$. Define $g_1, g_2 \in R$ by $g_1(e_i) = g(e_{2i})$ and $g_2(e_i) = g(e_{2i+1})$ for all $i \geq 0$. Then $(g_1 f_1 + g_2 f_2)(e_{2i}) = g_1(e_i) = g(e_{2i})$ and $(g_1 f_1 + g_2 f_2)(e_{2i+1}) = g_2(e_i) = g(e_{2i+1})$.

This shows $R \cong R^2$. Now, applying this inductively, we see $R \cong R \oplus R \cong R \oplus R^2 \cong R^3 \cong \cdots \cong R^n$. □

3.2 Exact Sequences

Definition 3.16. *Let L, M, N be $A-$modules and $f : L \to M, g : M \to N$ $A-$module homomorphisms. We say the sequence $L \xrightarrow{f} M \xrightarrow{g} N$ is* ***exact at*** *M if $im f = \ker g$. More generally, if the sequence $M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n$ is exact at each M_i for $1 \leq i \leq n-1$, then we say* ***the sequence is exact****. A* ***short exact sequence*** *is an exact sequence of the form $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$. Equivalently,*

1. *f is injective*

2. *g is surjective*

3. *$imf = kerg$*

Examples.

1. Suppose L is a submodule of M. Then the sequence $0 \to L \to M \to M/L \to 0$ is exact.

2. Let M_1, M_2 be $A-$modules. Then the sequence $0 \to M_1 \to M_1 + M_2 \to M_2 \to 0$ is exact. This is called a **split short exact sequence**.

Definition 3.17. *Let A be a ring and $(*) 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ a short exact sequence of $A-$modules. We say $(*)$* ***splits*** *(or is* ***split exact****) if there exists an $A-$module homomorphism $\phi : M \to L \oplus N$ such that the diagram commutes:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_L} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle 1_N} & & \\ 0 & \longrightarrow & L & \xrightarrow{i} & L \oplus N & \xrightarrow{j} & N & \longrightarrow & 0 \end{array}$$

where $i : \ell \mapsto (\ell, 0)$ and $j : (\ell, n) \mapsto n$.

Proposition 3.18. *Let $(*) 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence. TFAE*

1. *$(*)$ splits*

2. *There exists an $A-$linear map $\sigma : N \to M$ such that $g\sigma = 1_N$*

3. *There exists an $A-$linear map $\pi : M \to L$ such that $\pi f = 1_L$.*

If any of these hold, then $\phi : M \to L \oplus N$ is an isomorphism.

Proof. First, we prove ϕ is an isomorphism. Suppose $\phi(m) = 0$. Then $g \cdot 1_N(m) = j\phi(m) = 0$ implies $m \in \ker g = imf$. So there exists $\ell \in L$ such that $m = f(\ell)$. Then $i \cdot 1_L(\ell) = \phi f(\ell) = \phi(m) = 0$ and since i is injective, we have $\ell = 0$ and thus $m = 0$. So ϕ is injective. Now, let $(\ell, n) \in L \oplus N$. Since g is surjective, find $m \in M$ such that $g(m) = n$. Then $\phi(m) = (\ell', n)$ for some $\ell' \in L$. Consider $\phi(f(\ell - \ell') + m)$. We see $\phi(f(\ell - \ell') + m) = \phi f(\ell - \ell') + \phi(m) = i \cdot 1_L(\ell - \ell') + \phi(m) = (\ell - \ell', 0) + (\ell', n) = (\ell, n)$. Thus ϕ is an isomorphism.

(1) $\Rightarrow$ (2): Define $\sigma : N \to M$ by $n \mapsto \phi^{-1}((0, n))$. Then $g\sigma(n) = g\phi^{-1}((0, n)) = j(0, n) = n$. Thus $g\sigma = 1_N$.

(2) $\Rightarrow$ (3): Let $m \in M$. Note that $g(m - \sigma g(m)) = g(m) - g\sigma g(m) = 0$ as $g\sigma = 1_N$. Thus $m - \sigma g(m) \in \ker g = imf$. As f is injective, there exists a unique $\ell \in L$ such that $f(\ell) = m - \sigma g(m)$. Define $\pi : M \to L$ by $m \mapsto f^{-1}(m - \sigma g(m))$. Then ϕ is a homomorphism and $\pi f(\ell) = f^{-1}(f(\ell) - \sigma \underbrace{gf}_{=0}(\ell)) = f^{-1}(f(\ell)) = 1_L$.

(3) $\Rightarrow$ (1): Define $\phi : M \to L \oplus N$ by $m \mapsto (\pi(m), g(m))$. Then, for $\ell \in L$, we see $\phi(f(\ell)) = (\pi f(\ell), gf(\ell)) = (\ell, 0) = i(\ell)$ and for $m \in M$, we see $j\phi(m) = j(\pi(m), g(m)) = g(m)$. Thus the diagram commutes. ☐

Example. Let $A = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$. Consider $g : A^3 \to A$ by $(a, b, c) \mapsto a\overline{x} + b\overline{y} + c\overline{z}$. Note g is a surjective homomorphism as $g(\overline{x}, \overline{y}, \overline{z}) = \overline{x}^2 + \overline{y}^2 + \overline{z}^2 = \overline{1} \in img$ and since img is an ideal, this says $img = A$. Consider the short exact sequence $0 \to \ker g \hookrightarrow A^3 \xrightarrow{g} A \to 0$. Define $\sigma : A \to A^3$ by $1 \mapsto (\overline{x}, \overline{y}, \overline{z})$. Note $g\sigma(1) = 1$, which implies $g\sigma$ is the identity on the basis for A. Thus $g\sigma = 1_A$. By the proposition, the sequence splits and $A^3 \cong A \oplus \ker g$.

Proposition 3.19. *Let F be a free $A-$module and suppose $0 \to L \xrightarrow{f} M \xrightarrow{g} F \to 0$ is exact. Then the sequence splits.*

Proof. Let S be a basis for F. As g is onto, for all $s \in S$ there exists $m_s \in M$ such that $g(m_s) = s$. Define $\sigma : F \to M$ by $s \mapsto m_s$. This gives a well defined map as S is a basis for F. Then by definition, $g\sigma = 1_S$ and thus $g\sigma = 1_F$. Thus by the proposition, the sequence splits. ☐

Examples.

1. $0 \to (2) \to \mathbb{Z} \to \mathbb{Z}/(2) \to 0$ is a short exact sequence which does not split. Suppose that $\sigma : \mathbb{Z}/(2) \to \mathbb{Z}$ defined by $\overline{1} \mapsto m$ and $\overline{0} \mapsto 0$ for some $m \in \mathbb{Z}$. Then $0 = \sigma(\overline{0}) = \sigma(2 \cdot \overline{1}) = 2\sigma(\overline{1}) = 2m \in \mathbb{Z}$. Thus $m = 0$ and so $\sigma = 0$. But then, $g\sigma = 0 \neq 1$.

2. Let G be a finite group, k a field such that char $k \neq |G|$. Let $A = k[G]$ and V any $A-$module. Let $W = \{u \in V | gu = u \text{ for all } g \in G\}$. Then $W \neq \emptyset$ as $0 \in W$. So W is an $A-$submodule of V. So consider the short exact sequence $0 \to W \hookrightarrow V \to V/W \to 0$. This splits! Define $\rho : V \to W$ by $v \mapsto \frac{1}{|G|} \sum_{g \in G} gv$. Then for $w \in W$, $\rho(w) = \frac{1}{|G|}|G|w = w$. So $\rho i = 1_W$.

3. Let R be a PID and M a finitely generated $R-$module. Recall the torsion submodule of M is $T(M) = \{m \in M | rm = 0 \text{ for some } r \in R \setminus \{0\}\}$. Also, M is called **torsion free** if $T(M) = 0$.

   Remark. $M/T(M)$ is torsion free.

   Fact. Over a PID, finitely generated torsion free modules are free. (If A is a finitely generated abelian group, we know $A \cong \mathbb{Z}^r \oplus \mathbb{Z}/(a_1) \oplus \cdots \oplus \mathbb{Z}/(a_n)$ and if torsion free, then it would just be $A \cong \mathbb{Z}^r$).

   Example. If $R = \mathbb{Z}[x], I = (2, x)$, then I is torsion free but not free (as I is not principal).

   Thus $0 \to T(M) \to M \to M/T(M) \to 0$ splits as $M/T(M)$ is free. Hence $T(M)$ is a direct summand of M.

4. Let $R = k[x, y]$ for a field k (thus not a PID, but it is a UFD). Let $M = R^2/R(x^2, xy)$. Then $T(M) = R\overline{(x, y)} \cong R/(x)$, but $T(M)$ is not a direct summand of M.

Proof. Clearly, $x\overline{(x,y)} = \overline{(x^2, xy)} = \overline{0}$. Thus $\overline{(x,y)} \in T(M)$. Suppose $\overline{(a,b)} \in T(M)$. Then there exists $f \in k[x,y] \setminus \{0\}$ such that $f\overline{(a,b)} = \overline{0}$ which implies $f(a,b) = g(x^2, xy)$ for some $g \in k[x,y]$. WLOG, assume $g \neq 0$ and $\gcd(f,g) = 1$. Then $fa = x^2 g$ and $fb = xyg$ which implies $f|x^2$ and $f|xy$. Thus $f = u$ or $f = ux$ for $u \in k^*$. If $f = u$, then $(a,b) \in R(x^2, xy)$ which says $\overline{(a,b)} = 0$. If $f = ux$, then $(a,b) \in R(x,y)$ (as $a = u^{-1}xy$ and $b = u^{-1}yg$). Thus $T(M) = R\overline{(x,y)}$.

Now suppose $f\overline{(x,y)} = \overline{0}$. Then $f(x,y) = g(x^2, xy)$ which implies $f = gx$. So $f \in (x)$. Define $\phi : R \to R\overline{(x,y)}$ by $r \mapsto r\overline{(x,y)}$. Then ϕ is onto and $\ker\phi = (x)$. Thus $R/(x) \cong R\overline{(x,y)} = T(M)$. □

Now, we show the short exact sequence $0 \to R/(x) \xrightarrow{f} M \xrightarrow{g} M/T(M) \to 0$ where $f : \overline{r} \mapsto r\overline{(x,y)}$ does not split. Suppose it did. Let $\rho : M \to R/(x)$ be a splitting map so that $\rho f = 1$. Let $\overline{r} = \rho\overline{(0,1)}$ and $\overline{s} = \rho\overline{(0,1)}$. Then

$$\overline{1} = \rho(\overline{(x,y)}) = \rho\overline{(x,0)} + \rho\overline{(0,y)} = x\rho\overline{(1,0)} + y\rho\overline{(0,1)} = x\overline{r} + y\overline{s} = \overline{xr + ys}.$$

Thus $1 - xr - ys \in (x)$. So $1 - xr - ys = px$ for some p, a contradiction (just plug in $x = 0$ and $y = 0$ to get $0 = 1$). Thus it doesn't split.

5. Let $R \subseteq S$ be commutative rings and suppose S is an integral domain (thus R is as well), R is a UFD, char $R = 0$ and S is a finitely generated $R-$module (thus $S = Rx_1 + ... + Rx_n$). Then R is a direct summand of S as an $R-$module, that is, $0 \to R \to S \to S/R \to 0$ splits.

   Proof. Let $E = Q(S)$ and $F = Q(R)$. Then E is a finite vector space over F (generated by $x_1, ..., x_n$) and so $[E : F] < \infty$. Since char $R = 0$, we see char $F = 0$ and thus E/F is separable. Define $\rho : S \to R$ by $s \mapsto \frac{1}{[E:F]} Tr_F^E(s)$. There is more work from here, but its beyond the scope of this course. □

6. **Theorem (Miyata)**: If R is a commutative, Noetherian ring and $(*) 0 \to L \to M \to N \to 0$ is a short exact sequence of finitely generated $R-$modules, then $(*)$ splits if and only if $M \cong L \oplus N$.

   This is not true in general. For example, let $R = \mathbb{Z}, F = \oplus_{n=1}^{\infty}\mathbb{Z}, T = \oplus_{n=1}^{\infty}\mathbb{Z}/2\mathbb{Z}$. Note that $F/2F \cong T$. Consider the short exact sequence $0 \to F \oplus T \xrightarrow{\phi} F \oplus T \xrightarrow{\psi} T \to 0$ defined by $\phi : (f,t) \mapsto (2f, t)$ and $\psi : (f,t) \mapsto \overline{f}$. This does not split.

   Proof. Let e_i denote the standard basis for F. Let $\rho : F \oplus T \to F \oplus T$ be a splitting map. Then $\rho\phi = 1$. Now $\phi(e_1) = 2e_1$ implies $e_1 = \rho\phi(e_1) = 2\rho(e_1) = 2\sum a_i e_i$. Setting basis elements equal, we see $e_1 = 0$ for $i \neq 1$ and $a_1 = \frac{1}{2}$, contradiction. □

   Note, however, that $F \oplus T \cong (F \oplus T) \oplus T$ as $T \oplus T = T$ (its a countable sum).

Definition 3.20. *Let P be an $A-$module. Then P is called* ***projective*** *if whenever one has a diagram of the form*

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[r] & 0 \text{ exact} \\ & P \arrow[u, "i"'] \arrow[ul, dashed, "\exists h"] & \end{tikzcd}$$

then there exists $h : P \to M$ such that $i = fh$ (the diagram commutes). Note that this implies f and i are surjective.

Remark. Free modules are projective. Let $F = P$ above, let S be a basis for F. For each $s \in S$, there exists $m_s \in M$ such that $f(m_s) = i(s)$. Define $h : F \to M$ by $h(s) = m_s$. Then the diagram commutes.

Example. Let $R = \mathbb{Z}[\sqrt{-5}], I = (2, 1 + \sqrt{-5})$. Then I is projective, but not free (as it is not principal).

Proposition 3.21. *Let A be a ring and P an $A-$module. TFAE*

1. *P is projective*

2. *there exists an $A-$module Q such that $P \oplus Q$ is free*

3. *Every short exact sequence $0 \to L \to M \to P \to 0$ splits.*

Proof. (1) $\Rightarrow$ (3)**:** Let $0 \to L \xrightarrow{f} M \xrightarrow{g} P \to 0$ be a short exact sequence. Since P is projective and we have $1_P : P \to P$, there exists $\rho : P \to M$ such that the diagram below commutes:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & P \arrow[r] & 0 \text{ exact} \\ & & & P \arrow[u, "1_P"'] \arrow[ul, "\rho"] \end{tikzcd}$$

But then $g\rho = 1_P$ and thus the SES splits.

(3) $\Rightarrow$ (2)**:** Let $\phi : F \to P$ be a surjection, where F is free. Let $Q = \ker \phi$. Then $0 \to Q \to F \to P \to 0$ is exact and splits by (3). Thus $F \cong Q \oplus P$.

(2) $\Rightarrow$ (1):] Consider the diagram

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[r] & 0 \text{ exact} \\ & P \arrow[u, "i"'] & \\ & P \oplus Q \arrow[u, "\pi"'] \arrow[uul, dashed, "h"] & \end{tikzcd}$$

Since free modules are projective, there exists $h : P \oplus Q \to M$ such that $fh = i\pi$. Let $j : P \to P \oplus Q$ be defined by $p \mapsto (p, 0)$. Then $hj : P \to M$. Also, $f(hj) = fhj = i\pi j = i$. Thus the diagram commutes. $\square$

Examples/Remarks.

1. Every free module is projective.

2. Every projective module over $k[x_1, ..., x_n]$ (for a field k) is free. (Quillen-Suslin, 1975).

3. If R is a commutative Noetherian domain, then every non-finitely generated projective $R-$module is free (Bass, 1963).

4. $\mathbb{Z}/2\mathbb{Z}$ is not a projective $\mathbb{Z}-$module. Since the only map from $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ is the 0-map, the diagram below, with $f : 1 \mapsto \overline{1}$, would never commute:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "f"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \text{ exact} \\ & \mathbb{Z}/2\mathbb{Z} \arrow[u, "i"'] \arrow[ul, "0"] & \end{tikzcd}$$

5. $\mathbb{Z}/(6) \cong \mathbb{Z}/(2) \oplus \mathbb{Z}/(3)$. Since $\mathbb{Z}/(6)$ is free (as an $\mathbb{Z}/(6)-$module), we see that $\mathbb{Z}/(2)$ and $\mathbb{Z}/(3)$ are projective $\mathbb{Z}/(6)-$modules. However, they are not free (just count elements...there are too few elements to be a direct sum of copies of $\mathbb{Z}/(6)$.)

6. Let $R = \mathbb{Z}[\sqrt{-5}], I = (2, 1 + \sqrt{-5})$. Then I is not free (it's not principal), but it is projective.

Proof. Define $\phi : R^2 \to I$ by $(a,b) \mapsto 2a + (1+\sqrt{-5})b$. Let $K = \ker\phi$. We'll show $0 \to K \to R^2 \to I \to 0$ splits. Define $\rho : I \to R^2$ by $x \mapsto x\left(\frac{1-3\sqrt{-5}}{2}, \frac{3\sqrt{-5}}{1+\sqrt{-5}}\right)$. We need to show that the image is actually in R^2, but to do that it is enough to show for $x = 2, 1+\sqrt{-5}$:

$$(1+\sqrt{-5})\left(\frac{1-3\sqrt{-5}}{2}\right) = \frac{1-2\sqrt{-5}+15}{2} = 8-\sqrt{-5},\ 2\left(\frac{3\sqrt{-5}}{1+\sqrt{-5}}\right) = \frac{6\sqrt{-5}}{1+\sqrt{-5}} = (1-\sqrt{-5})(\sqrt{-5}) \in R.$$

Since we are just multiplying, this is certainly a homomorphism. Note that $\phi\rho(x) = \phi\left(x\left(\frac{1-3\sqrt{-5}}{2}, \frac{3\sqrt{-5}}{1+\sqrt{-5}}\right)\right) = x(1-3\sqrt{-5}+3\sqrt{-5}) = x$. Thus the SES splits which says that I is a direct summand of a free module, and thus projective. ☐

7. Let G be a finite group and k a field such that char $k \nmid |G|$. Let $R = k[G]$.

   Fact. Let M be any $R-$module and N any $R-$submodule of M. Then N is a direct summand of M.

   Let M be any $R-$module, F any free module. Consider the short exact sequence $0 \to \ker\phi \to F \xrightarrow{\phi} M \to 0$. Since $\ker\phi$ is a summand, we get a splitting map. Thus $F \cong \ker\phi \oplus M$ which implies every module is projective. However, there exist non-free modules. Let $M = R(\sum_{g\in G} g) = k(\sum_{g\in G} g)$. Then $dim_k M = 1$ and $dim_k R = |G|$. Thus M cannot be a free $R-$module as the dimensions do not work out (unless of course $|G| = 1$.)

3.3 Localization

Let R be a ring. A set $S \subseteq Z(R)$ is **multiplicatively closed** (mc) if $ab \in S$ whenever $a, b \in S$.

Definition 3.22. *Let R be a ring and $S \neq \emptyset$ a mcs of R. The **localization of** R **at** S is a ring T together with a ring homomorphism $\phi : R \to T$ such that*

1. *$\phi(s)$ is a unit in T for all $s \in S$.*
2. *If $f : R \to A$ is a ring homomorphism such that $f(s)$ is a unit for all $s \in S$, then there exists a unique ring homomorphism $g : T \to A$ such that*

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[d, "f"'] & T \arrow[dl, dashed, "\exists! g"] \\ A & \end{tikzcd}$$

Proposition 3.23. *If T exists, it is unique up to isomorphism*

Proof. Show 2 maps compose to the identity ☐

Notation. We denote T by $S^{-1}R$ or R_S.

Theorem 3.24. *R_S exists.*

Proof. Define an equivalence relation on $R \times S$ by $(r_1, s_1) \sim (r_2, s_2)$ if and only if $t(s_2 r_1 - s_1 r_2) = 0$ for some $t \in S$.

Claim: This defines an equivalence relation.

Proof: We show transitivity. Suppose $(r_1, s_1) \sim (r_2, s_2)$ and $(r_2, s_2) \sim (r_3, s_3)$. Then there exists $t_1, t_2 \in S$ such that $t_1 s_2 r_1 = t_1 s_1 r_2$ and $t_2 s_3 r_2 = t_2 s_2 r_3$. Then $t_1 s_2 r_1 s_3 = t_1 s_1 r_2 s_3$ and $t_2 s_3 r_2 s_1 = t_2 s_2 r_3 s_1$. Then $t_1 t_2 s_2 (s_3 r_1 - s_1 r_3) = 0$.

Denote the equivalence class of (r, s) by $\frac{r}{s}$. Let $R_S := \{\frac{r}{s} | (r,s) \in R \times S\}$. Define $+, \cdot$ on R_S in the usual manner (this requires a little work to show its well-defined). Thus R_S forms a ring with identity. The identity of R_S is $\frac{s}{s}$ for any $s \in S$. Define $\phi : R \to R_S$ by $r \mapsto \frac{rs}{s}$ for any $s \in S$. This is a ring homomorphism. Let $t \in S$. Then

$\phi(t) = \frac{ts}{s}$ and $\phi(t)^{-1} = \frac{s}{ts}$. Now, suppose $f : R \to A$ is a ring homomorphism such that $f(s)$ is a unit for all $s \in S$. Define $g : R_S \to A$ by $\frac{r}{s} \mapsto f(r)f(s)^{-1}$. To show g is well-defined, suppose $\frac{r_1}{s_1} = \frac{r_2}{s_2}$. Then $t(r_1s_2 - r_2s_1) = 0$ for some $t \in S$. So $f(t)(f(s_2)f(r_1) - f(s_1)f(r_2)) = 0$. This implies $f(s_2)f(r_1) = f(s_1)f(r_2)$ as $f(t)$ is a unit and thus $f(r_1)f(s_1)^{-1} = f(r_2)f(s_2)^{-1}$. To show that g is unique, suppose there exists $g_1 : R_s \to A$ such that $g_1\phi = f$. Then, for some $t \in S$, we see

$$g_1\left(\frac{r}{s}\right) f(s) = g_1\left(\frac{r}{s}\right) g_1\phi(s) = g_1\left(\frac{r}{s}\right) g_1\left(\frac{st}{t}\right) = g_1\left(\frac{rst}{st}\right) = g_1\phi(r) = f(r).$$

Thus $g_1(\frac{r}{s}) = f(r)f(s)^{-1} = g(\frac{r}{s})$. □

Remarks.

1. If S is a mcs of R, so is $S' = S \cup \{1\}$. Furthermore, $R_S \cong R_{S'}$. Thus, WLOG, we may assume $1 \in S$ and the canonical ring homomorphism $\phi : R \to R_S$ is $r \mapsto \frac{r}{1}$.

2. $0 \in S$ if and only if $R_S = \{0\}$ (as $0(s_2r_1 - s_1r_2) = 0$, i.e., there is only one equivalence class).

3. If S consists solely of units of R, then $R_S \cong R$.

4. If S consists solely of non-zero-divisors, then $\frac{r_1}{s_1} = \frac{r_2}{s_2}$ if and only if $s_2r_1 - s_1r_2 = 0$. In particular, $\phi : R \to R_S$ is one-to-one. So we can consider R as a subring of R_S.

3 Important Examples of Localizations

1. Let $x \in Z(R)$ and $S = \{x^n\}$. The localization R_S is denoted by R_x. **Example.** $\mathbb{Z}_2 = \mathbb{Z}[\frac{1}{2}]$. (Don't confuse this with $\mathbb{Z}_2 = \{0, 1\}$.

2. Let R be a commutative ring and $S = \{x \in R | x \text{ is a non zero divisor}\}$. Then R_S is called the **total quotient ring** of R, denoted $Q(R)$. If R is a domain, $Q(R)$ is the field of fractions of R.

3. Let R be a commutative ring, $p \neq R$ a prime ideal. Let $S = R - p$. Then S is mc. In this case, we denote R_S by R_p. **Example.** $\mathbb{Z}_{(2)} = \{\frac{a}{b} | a, b \in \mathbb{Z}, 2 \nmid b\}$.

Definition 3.25. *Let R be a commutative ring. The* ***(prime) spectrum*** *of R is $SpecR = \{p | p \neq R$ is a prime ideal of $R\}$.*

Examples.

1. If K is a field, then $SpecK = \{0\}$.

2. $Spec\mathbb{Z} = \{(0), (p) | p \text{ is prime}\}$.

3. $Spec\mathbb{C}[x] = \{(0), (x - a) | a \in \mathbb{C}\}$.

Proposition 3.26. *Let R be commutative, I an ideal of R. Let $V(I) = \{p \in SpecR | p \supseteq I\}$. Then there exists a bijective inclusion preserving correspondence $V(I) \leftrightarrow Spec(R/I)$ defined by $p \in V(I) \mapsto p/I$ and $q \in Spec(R/I) \mapsto \phi^{-1}(q)$ where $\phi : R \to R/I$ is the canonical map $r \mapsto \overline{r}$.*

Remarks.

1. If $\phi : R \to S$ is a ring homomorphism and $q \in SpecS$, then $\phi^{-1}(q) = \{r \in R | \phi(r) \in q\}$ is a prime ideal of R.

2. If $p \in V(I)$, then $p/I \in Spec(R/I)$ as $R/I/p/I \cong R/p$, a domain.

Examples.

1. $Spec\mathbb{Z}/(30) = \{(\overline{2}), (\overline{3}), (\overline{5})\}$.

2. $Spec\mathbb{C}/(x^2+1) = \{(\overline{x+i}), (\overline{x-i})\}$.

3. $Spec\mathbb{R}[x]/(x^2+1) = \{(0)\}$.

Proposition 3.27. *Let R be a commutative ring, S a mcs of R. Then there exists a bijective inclusion preserving correspondence $\{p \in SpecR | p \cap S = \emptyset\} \leftrightarrow SpecR_S$ defined by $p \mapsto p_s = pR_s = \{\frac{a}{s} | a \in p, s \in S\}$ and $q \in SpecR_S \mapsto \phi^{-1}(q)$ where $\phi : R \to R_S$ is the canonical map $r \mapsto \frac{r}{1}$.*

Proof. We will prove several claims.

Claim: p_S is a proper prime ideal of R_S.

Proof: Suppose $\frac{a}{s} \cdot \frac{b}{t} \in p_S$. Then $\frac{ab}{st} = \frac{x}{s'}$ for some $x \in p, s' \in S$. Then there exists $t' \in S$ such that $t's'ab = t'stx \in p$. As $t', s' \in S, t's' \notin p$. So $ab \in p$ which implies $a \in p$ or $b \in p$. Thus $\frac{a}{s} \in p_S$ or $\frac{b}{t} \in p_S$. Thus, its a prime ideal. To show its proper, suppose $p_s = R_S$. Then $\frac{1}{1} \in p_S$ which implies $\frac{1}{1} = \frac{a}{s}$ for $a \in p, s \in S$. Then there exists $t \in S$ such that $t(s-a) = 0$ which implies $ts = ta \in p$, but $t, s \in S$ implies $ts \notin p$, a contradiction.

Claim: $\phi^{-1}(p) \in SpecR$ for $q \in SpecR_S$.

Proof: Since $\phi(1) = 1$, if $1 \in \phi^{-1}(q), 1 \in q$. So $\phi^{-1}(q)$ is proper. It's a prime ideal by the remark.

Claim: $\phi^{-1}(p_s) = p$.

Proof: We know $p \subseteq \phi^{-1}(p_S)$. Suppose $\phi(r) \in p_S$. Then $\frac{r}{1} = \frac{a}{s}, a \in p, s \in S$. Then there exists $t \in S$ such that $tsr = ta \in p$. Since $t, s \in S, ts \notin p$ and so $r \in p$.

Claim: $\phi^{-1}(q)_S = q$.

Proof: Let $\frac{a}{s} \in \phi^{-1}(q)_S$, that is, $a \in \phi^{-1}(q), s \in S$. Then $\frac{a}{1} = \phi(a) \in q$. Thus $\frac{a}{s} = \frac{a}{1} \cdot \frac{1}{s} \in q$ as it is an ideal. Let $x \in q$. Then $x = \frac{r}{s}, r \in R, s \in S$. Then $sx = \frac{r}{1} \in q$. So $r \in \phi^{-1}(q)$ which implies $x = \frac{r}{s} \in \phi^{-1}(q)_S$. □

Examples.

1. $Spec\mathbb{Z}_2 = \{(p)\mathbb{Z}_2 | p > 2 \text{ is prime}\}$.

2. $Spec\mathbb{Z}_{30} = \{p\mathbb{Z}_{30} | p > 5 \text{ is prime}\}$.

3. $Spec\mathbb{Z}_{(2)} = \{(0)\mathbb{Z}_{(2)}, (2)\mathbb{Z}_{(2)}\}$ as $p \cap S = \emptyset$ if and only if $(p) \subseteq (2)$ where $S = R - (2)$.

Remark. If $P \in SpecR$, then $SpecR_P = \{q_p | q \in SpecR, q \subseteq P\}$. Thus R_P has a unique maximal ideal, namely $PR_P = P_P$.

Definition 3.28. *A commutative ring which has a unique maximal ideal is called a* ***local (or quasilocal) ring****. Note: For some, local means Noetherian and has a unique maximal ideal.*

Remark. Let (R, m) be the local ring where m denotes the unique maximal ideal. Then $x \in R$ is a unit if and only if $x \notin m$.

Proof. x is a unit if and only if $(x) = R$ if and only if (x) is not contained in any maximal ideal of R which is if and only if $x \notin m$ as m is the unique maximal ideal. □

Note. $R_m \cong R$. This is because $R_m = R_S$ where $S = R - m$ and everything outside m is already a unit.

Examples.

1. $\mathbb{Z}/(8)$. The only prime ideal is $(\overline{2})$.

2. $\mathbb{C}[[x]]$. $\sum a_i x^i$ is a unit if and only if $a_0 \neq 0$.

Proposition 3.29. *Let S and T be mcs of R. WLOG, assume $1 \in S \cap T$. Then*

1. *$ST = \{st | s \in S, t \in T\}$ is a mcs of R (containing both S and T).*
2. *$\frac{T}{1} = \{\frac{t}{1} \in R_S | t \in T\}$ is a mcs of R_S.*
3. *$\frac{T}{S} = \{\frac{t}{s} \in R_S | t \in T, s \in S\}$ is a mcs of R_S.*

Furthermore, $R_{ST} \cong (R_S)_{\frac{T}{1}} \cong (R_S)_{\frac{T}{S}}$.

Proof. Note that 1,2,3 are trivial. For the last statement, we will use the fact (without proof) that if S consists of units of R, then $R_{ST} \cong R_T$. Note $\frac{T}{S} = \frac{T}{1} \cdot \frac{1}{S}$ and $\frac{1}{S}$ consists of units of R_S. Thus by the fact, $(R_S)_{\frac{T}{S}} \cong (R_S)_{\frac{T}{1}}$. So it is enough to show $R_{ST} \cong (R_S)_{\frac{T}{1}}$. Consider the canonical map $i : R \to R_{ST}$ where $r \mapsto \frac{r}{1}$. Note $i(s)$ is a unit for all $s \in S$ as $S \subseteq ST$. By the universal property, there exists a unique ring homomorphism $g : R_S \to R_{ST}$ defined by $\frac{r}{s} \mapsto \frac{r}{1} \cdot \left(\frac{s}{1}\right)^{-1} = \frac{r}{s}$. Note that $g(\frac{t}{1}) = \frac{t}{1}$ is a unit in R_{ST} for all $t \in T$ as $T \subseteq ST$. Thus, we can again use the universal property to obtain the ring homomorphism $\phi : (R_S)_{\frac{T}{1}} \to R_{ST}$ defined by $\frac{\frac{r}{s}}{\frac{t}{1}} \mapsto \frac{r}{s}\left(\frac{t}{1}\right)^{-1} = \frac{r}{st}$. Now, consider the composition of canonical maps $\psi : R \to R_S \to (R_S)_{\frac{T}{1}}$. Then $\psi(st) = \frac{\frac{st}{1}}{\frac{1}{1}}$, with inverse $\frac{\frac{1}{s}}{\frac{t}{1}}$. Thus $\psi(st)$ is a unit for all $s \in S, t \in T$ and so by the universal property there exists a ring homomorphism $\psi : R_{ST} \to (R_S)_{\frac{T}{1}}$ defined by $\frac{r}{st} \mapsto \frac{\frac{r}{s}}{\frac{t}{1}}$. It is obvious that $\phi\psi = \psi\phi = 1$. $\square$

Corollary 3.30. *Suppose $S \subseteq T$ are mcs of R. Then $(R_S)_{\frac{T}{S}} \cong (R_S)_{\frac{T}{1}} \cong R_{ST} \cong R_T$ as $ST = T$.*

Corollary 3.31. *Let S be a mcs and $P \in SpecR$ such that $P \cap S \neq \emptyset$. Then $P_S \in SpecR_S$ and $(R_S)_{P_S} \cong R_P$.*

Proof. Recall $R_P = R_T$ where $T = R - P$. Also, $(R_S)_{P_S} = (R_S)_{\frac{T}{S}} \cong R_T$ as $P \cap S \neq \emptyset$ implies $S \subseteq T$. $\square$

Corollary 3.32. *Let $P \subseteq Q$ be prime ideals of R. Then $P \cap (R - Q) = \emptyset$. Thus $P_Q \in SpecR_Q$ and $(R_Q)_{P_Q} \cong R_P$.*

Example. $(\mathbb{Z}_{(2)})_{\frac{2}{1}}$. Let $S = \mathbb{Z} - (2) = \{a \in \mathbb{Z} | 2 \nmid a\}, T = \{2^n | n \geq 0\}$. Then, $(\mathbb{Z}_{(2)})_{\frac{2}{1}} \cong (\mathbb{Z}_S)_{\frac{T}{1}} \cong \mathbb{Z}_{ST} \cong \mathbb{Q}$ as $ST = \mathbb{Z} \setminus \{0\}$.

Definition 3.33. *Let R be a commutative ring, I an ideal of R. The* ***radical*** *of I is $\sqrt{I} = \{r \in R | r^n \in I,$ for some $n \geq 0\}$. When $I = (0)$, we call $\sqrt{(0)} = nilradR = \{a \in R | a$ is nilpotent$\}$ the* ***nilradical****.*

Proposition 3.34. *Let I be an ideal of R. Then $\sqrt{I} = \cap_{P \in V(I)} P$ where $V(I) = \{P \in SpecR | P \supseteq I\}$. In particular, $nilradR = \cap_{P \in SpecR} P$.*

Proof. Let $r \in \sqrt{I}$ and $P \in V(I)$. Then $r^n \in I$ for some n. As $I \subseteq P$, $r^n \in P$. Thus $r \in P$ as P is prime. Suppose $r \notin \sqrt{I}$. Then we will show there exists $P \in V(I)$ such that $r \notin P$. Note $I_r \neq R_r$ as otherwise $\frac{1}{1} \in I_r$ which implies $\frac{1}{1} \equiv \frac{i}{r^n}$, that is $r^m(r^n - i) = 0$ which implies $r^{m+n} = r^m i \in I$, a contradiction as that says $r \in \sqrt{I}$. Therefore, there exists a prime (maximal) ideal of R_r containing I_r, that is, there exists $P \in SpecR$ with $r \notin \sqrt{P} = P$ such that $P_r \supseteq I_r$. Let $\phi : R \to R_r$ be the canonical map. Then $P = \phi^{-1}(P_r) \supseteq \phi^{-1}(I_r) \supseteq I$. So $P \in V(I)$ and $r \notin P$. $\square$

Proposition 3.35. *Let R be a commutative ring, I an ideal of R and S a mcs. Then $\overline{S} = \{\overline{s} = s + I | s \in S\}$ is a mcs of R/I. Then $(R/I)_{\overline{S}} \cong R_S/I_S$.*

Proof. Consider the canonical maps $\phi : R \to R/I \to (R/I)_{\overline{S}}$. Note that $\phi(S) = \frac{\overline{S}}{1}$ is a unit for all $x \in X$. Thus there exists a ring homomorphism $f : R_S \to (R/I)_{\overline{S}}$ defined by $\frac{r}{s} \mapsto \frac{\overline{r}}{1} \cdot \left(\frac{\overline{s}}{1}\right)^{-1} = \frac{\overline{r}}{\overline{s}}$. Clearly, f is surjective. Notice $\ker f = I_S$ as $\frac{r}{s} \in \ker f$ if and only if $\frac{\overline{r}}{\overline{s}} = \frac{\overline{0}}{1}$ if and only if there exists $t \in S$ such that $\overline{t}\overline{r} = 0$ if and only if $tr \in I$ for some $t \in S$ if and only if $\frac{r}{s} \in I_S$. Thus, by the First Isomorphism Theorem, done. $\square$

Localization of Modules

Let R be a ring, S a mcs, M a left $R-$module. Define an equivalence relation on $M \times S$ by $(m_1, s_1) \sim (m_2, s_2)$ if and only if there exists $t \in S$ such that $t(s_2 m_1 - s_1 m_2) = 0$. This defines an equivalence relation. Denote the equivalence class of (m, s) by $\frac{m}{s}$. Let $M_S = \{\frac{m}{s} | m \in M, s \in S\}$. Define $\frac{m_1}{s_1} + \frac{m_2}{s_2} := \frac{s_2 m_1 + s_1 m_s}{s_1 s_2}$ and $\frac{r}{s_1} \cdot \frac{m}{s_2} = \frac{rm}{s_1 s_2}$. These are well-defined and make M_S an R_S-module.

Proposition 3.36. *Let R be a commutative ring, M an $R-$module. TFAE*

1. $M = 0$

2. $M_p = 0$ *for all* $p \in SpecR$.

3. $M_m = 0$ *for all maximal ideals* m.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is trivial. So we will only prove (3) $\Rightarrow$ (1). Let $x \in M$ and $I = ann_R x = \{r \in R | rx = 0\}$. Let m be a maximal ideal of R. By (3), $\frac{x}{1} \in M_m = 0$. Thus there exists t not in m such that $tx = 0$. So $t \in I$ and $I \not\supseteq m$. As m is arbitrary, we must have $I = R$. Thus $x = 0$ as $1 \in I$ which implies $M = 0$. $\square$

Let $f : M \to N$ be an $R-$module homomorphism. Let S be a mcs. For $s \in S$, define $\frac{f}{s} : M_S \to N_S$ by $\frac{m}{s'} \mapsto \frac{f(m)}{ss'}$. This is a well-define R_S-module homomorphism.

Proposition 3.37. *Let* $(*) 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *be a short exact sequence of $R-$modules. Then* $(**) 0 \to L_S \xrightarrow{\frac{f}{1}} M_S \xrightarrow{\frac{g}{1}} N_S \to 0$ *is a short exact sequence of R_S-modules for any mcs S of R. Furthermore, if $(*)$ splits, then $(**)$ does.*

Proof. $\frac{f}{1}$ is 1-1: Suppose $\frac{f}{1}(\frac{\ell}{s}) = 0$. Then $\frac{f(\ell)}{s} = \frac{0}{1}$. Thus there exists $t \in S$ such that $tf(\ell) = 0$, which implies $f(t\ell) = 0$ and thus $t\ell = 0$. Therefore $\frac{\ell}{s} \equiv \frac{0}{1}$ in L_S.

$\frac{g}{1}$ is onto: Clear

$im\frac{f}{1} = \ker \frac{g}{1}$: Since $imf \subseteq \ker g$, $gf = 0$. Then $\frac{g}{1} \cdot \frac{f}{1} = 0$. Hence, $im\frac{f}{1} \subseteq \ker \frac{g}{1}$. Now, let $\frac{m}{s} \in \ker \frac{g}{1}$. Then there exists $t \in S$ such that $g(tm) = 0$. So $tm \in \ker g = imf$. So $tm = f(\ell)$. Thus $\frac{tm}{1} = \frac{f(\ell)}{1}$ which implies $\frac{m}{s} = \frac{f(\ell)}{st} = \frac{f}{1}(\frac{\ell}{st}) \in im\frac{f}{1}$.

Thus $(**)$ is exact. If $(*)$ splits, there exists $h : N \to M$ such that $gh = 1_N$. Then $\frac{g}{1} \cdot \frac{h}{1} = 1_{N_S}$. Thus $\frac{h}{1}$ is the splitting map for $(**)$. $\square$

Corollary 3.38. *Suppose $N \subseteq M$ are $R-$modules. Then $(M/N)_S \cong M_S/N_S$.*

Proof. Since $0 \to N \to M \to M/N \to 0$ is exact, the above says $0 \to N_S \to M_S \to (M/N)_S \to 0$ is exact. Thus $M_S/N_S \cong (M/N)_S$. $\square$

Corollary 3.39. $(A \oplus B)_S \cong A_S \oplus B_S$

Proof. Since $0 \to A \to A \oplus B \to B \to 0$ is split exact, so is $0 \to A_S \to (A \oplus B)_S \to B_S \to 0$ is split exact. Thus $(A \oplus B)_S \cong A_S \oplus B_S$. $\square$

Exercise: $(\oplus_{i \in I} A_i)_S \cong \oplus_{i \in I}(A_i)_S$.

Corollary 3.40. *If F is a free $R-$module, then F_S is a free R_S-module.*

Proof. Since $F \cong \oplus_{i \in I} R$, we see $F_S \cong \oplus_{i \in I} R_S$. $\square$

Corollary 3.41. *If P is a projective $R-$module, then P_S is a projective R_S-module.*

Proof. There exists Q such that $P \oplus Q \cong F$, a free module. Therefore $P_S \oplus Q_S \cong F_S$ which is also free. So P_S is a projective R_S-module. $\square$

Definition 3.42. *Let R be a commutative ring. The **Jacobson radical**, denoted $J(R)$, is defined to be the intersection of all maximal ideals of R.*

Examples. $J(\mathbb{Z}) = 0$, $J(k[x]) = 0$, and $J(\mathbb{Z}/(12)) = (\overline{2}) \cap (\overline{3}) = (\overline{6})$.

Remark. If $x \in J(R)$, then $1 - x$ is a unit.

Proof. If $1 - x \in m$, then $1 \in m$, a contradiction. So $1 - x \notin m$ for all maximal ideals m. Thus $1 - x$ is a unit. $\square$

Lemma 3.43 (**Nakayama's Lemma**). *Let R be a commutative ring and M a finitely generated $R-$module. Suppose $M = JM$ where $J = J(R)$. Then $M = 0$.*

Proof. Choose a least n such that M is generated by n elements, say $x_1, ..., x_n$. We will show $n = 0$ (and so $M = 0$). Let $x_n \in JM$, so $x_n = j_1x_1 + ... + j_nx_n, j_i \in J$. Then $(1 - j_n)x_n = j_1x_1 + ... + j_{n-1}x_{n-1}$. Then, since $1 - j_n$ is a unit, $x_n = (1 - j_n)^{-1}j_1x_1 + ... + (1 - j_n)^{-1}j_{n-1}x_{n-1} \in Rx_1 + ... + Rx_{n-1}$, a contradiction to the minimality of n. □

Corollary 3.44. *Suppose $N \subseteq M$ are $R-$modules and M is finitely generated. Suppose $M = N + JM$ where $J = J(R)$. Then $M = N$.*

Proof. Note that $M/N = (N + JM)/N = J(M/N)$. Since M is finitely generated, so is M/N. By Nakayama's Lemma, $M/N = 0$. □

Corollary 3.45. *Let M be a finitely generated $R-$module. Let $x_1, ..., x_n \in M$. Then $x_1, ..., x_n$ generate M if and only if $\overline{x_1}, ..., \overline{x_n}$ generate M/JM where $J = J(R)$.*

Proof. Note that $(\Rightarrow)$ is trivial. To show $(\Leftarrow)$, let $N = Rx_1 + ... + Rx_n$. Since $\overline{x_1}, ..., \overline{x_n}$ generate M/JM, we have $(N + JM)/JM = M/JM$ which implies $M = N + JM$ which implies $M = N$. □

Notation. If M is an $R-$module, let $\mu_R(M) = \inf\{n | M = Rx_1 + ... + Rx_n \text{ for some } x_1, ..., x_n \in M\}$ = the minimal number of generators for M.

Corollary 3.46. *Let M be a finitely generated $R-$module, $J = J(R)$. Then $\mu_R(M) = \mu_{R/J}(M/JM)$.*

Corollary 3.47. *Suppose (R, m) is local. For any finitely generated $R-$module M, $\mu_R(M) = dim_{R/m}M/mM$. In particular, any two minimal generating sets for M have the same number of elements.*

Proof. Since R/m is a field, $\mu_R(M) = \mu_{R/m}(M/mM) = dim_{R/m}M/mM$. □

Proposition 3.48. *Let (R, m) be a local ring and P a finitely generated projective $R-$module. Then P is free.*

Proof. We will use the fact (without proof) that $\oplus M_i/I(\oplus M_i) \cong \oplus(M_i/IM_i)$. Let $n = \mu_R(P) = dim_{R/m}(P/mP)$. Let $x_1, ..., x_n$ be a minimal generating set for P. Define $\phi : R^n \to P$ by $e_i \mapsto x_i$. Then ϕ is surjective. Let $K = \ker\phi$. Then we have the short exact sequence $0 \to K \to R^n \xrightarrow{\phi} P \to 0$. This splits as it ends with a projective module. So $R^n \cong P \oplus K$ and K is finitely generated (as R^n is finitely generated and $R^n \to P \oplus K \to K$ is onto). Then $R^n/mR^n \cong (P \oplus K)/m(P \oplus K)$ which implies $(R/m)^n \cong (P/mP) \oplus (K/mK)$ by our fact. This is an isomorphism as R/m vector spaces. Taking the dimensions of both sides, since $dim(R/m)^n = n = dimP/mP$, we have $dimK/mK = 0$, that is, $K/mK = 0$ and thus $K = mK$. Since K is finitely generated, $K = 0$ by Nakayama's Lemma and thus ϕ is an isomorphism. Thus $R^n \cong P$. □

3.4 Category Theory and the Hom Functor

Definition 3.49. *A* ***category*** *$\mathcal{C}$ consists of a class of objects (denoted by Obj $\mathcal{C}$) and a set of morphisms $Hom_{\mathcal{C}}(A, B)$ for every pair of objects A, B of $\mathcal{C}$ such that*

1. *(Composition) there exists a function $Hom_{\mathcal{C}}(B, C) \times Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{C}}(A, C)$ sending $(f, g) \mapsto f \circ g$ for all objects A, B, C.*

2. *(Associativity) $(fg)h = f(gh)$ for all morphisms f, g, h where $(fg)h$ is defined.*

3. *(Identity) For all objects A of $\mathcal{C}$ there exists $1_A \in Hom_{\mathcal{C}}(A, A)$ such that for all objects B of $\mathcal{C}$ we have $1_Af = f$ for all $f \in Hom_{\mathcal{C}}(B, A)$ and $f1_A = f$ for all $f \in Hom_{\mathcal{C}}(A, B)$.*

Examples.

1. The category of sets: <<Sets>> has sets as objects and functions as morphisms.
2. The category of groups: <<Groups>> has groups as objects and group homomorphisms as morphisms. This category has the **subcategory** <<Abel>> of abelian groups. Note that a subcategory is called a **full subcategory** if it retains all of the morphisms.
3. For a commutative ring R, the category of $R-$algebras: <<$R-$algebra>> has $R-$algebras as objects and $R-$algebra homomorphisms ($\phi : S \to T$ where S, T are $R-$algebras such that ϕ is a ring homomorphism where $\phi(rs) = r\phi(s)$ for all $r \in R$) as the set of morphisms.

   Note. Every ring is a $\mathbb{Z}-$algebra. Thus <<$\mathbb{Z}-$algebras>> = <<Rings>>.
4. For a commutative ring R, the category of left $R-$modules is written << $R-$mod>> and the category of right $R-$modules is written << mod$-R$ >>.

   Special Cases

   (a) << $\mathbb{Z}-$mod>> = << Abel>>

   (b) If k is a field, << $k-$mod>> = << $k-$vector spaces>>

Definition 3.50. *Let $\mathcal{C}$ and $\mathcal{D}$ by categories. A* ***(covariant) functor*** *$F : \mathcal{C} \to \mathcal{D}$ is a rule which associates to each object A of $\mathcal{C}$ an object $F(A)$ of $\mathcal{D}$ and for each morphism $f \in Hom_{\mathcal{C}}(A, B)$ a morphism $F(f) \in Hom_{\mathcal{D}}(F(A), F(B))$ with the following properties:*

1. *$F(fg) = F(f)F(g)$ for all morphisms f, g of $\mathcal{C}$ where fg is defined.*
2. *$F(1_A) = 1_{F(A)}$ for all objects A of $\mathcal{C}$.*

Examples.

1. The **forgetful functor** F :<<Groups>>$\to$<<Sets>> defined by sending a group G to the set G and the group homomorphism g to the function g. Another forgetful functor is F' :<< $R-$mod>>$\to$<<Abel>> .
2. The Localization functor: F :<< $R-$mod>>$\to$<< R_S-mod>> where $F(M) = M_S$ and $F(f) = \frac{f}{1}$.
3. The Modding Out functor: Let I be a 2-sided ideal of R. Then we can define F :<< $R-$mod>>$\to$<< $R/I-$mod>> by $F(M) = M/IM$ and for an $R-$homomorphism $f : M \to M$, $F(f) : M/IM \to N/IN$ where $m + IM \mapsto f(m) + IN$.
   Note. You can mod out by a left ideal, however the functor would then be << $R-$mod>>$\to$<< $R-$mod>> .

Definition 3.51. *Let M, N be left $R-$modules. Then $Hom_R(M, N)$ denotes the set of left $R-$module homomorphisms from $M \to N$.*

Remarks.

1. $Hom_R(M, N)$ is an abelian group.
2. Generally, $Hom_R(M, N)$ is not a left $R-$module, unless R is commutative.
3. Let M be a left $R-$module. Define a functor $Hom_R(M, -)$:<< $R-$mod >>$\to$<< Abel >> by $Hom_R(M, -)(N) = Hom_R(M, N)$ and if $f : N_1 \to N_2$ is an $R-$module homomorphism, then $f_* := Hom_R(M, -)(f) : Hom_R(M, N_1) \to Hom_R(M, N_2)$ defined by $g \mapsto fg$. Note that $(fg)_* = f_* g_*$ and $(1_N)_* = 1_{Hom_R(M,N)}$ (and thus it really is a functor).

Definition 3.52. *A* ***contravariant functor*** *$F : \mathcal{C} \to \mathcal{D}$ is a rule which associates to each object A of $\mathcal{C}$ an object $F(A)$ of $\mathcal{D}$ and for every pair of objects A, B of $\mathcal{C}$ a map $Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{D}}(F(B), F(A))$ defined by $f \mapsto F(f)$ such that $F(fg) = F(g)F(f)$ and $F(1_A) = 1_{F(A)}$.*

Example. Let N be a left $R-$module. Define the contravariant functor $Hom_R(-, N) :<< R-\text{mod} >>\to<< \text{Abel} >>$ by $M \mapsto Hom_R(M, N)$ and $(f : M_1 \to M_2) \mapsto (f^* : Hom_R(M_2, N) \to Hom_R(M_1, N))$ where $g \mapsto gf$. One can check that $(fg)^* = g^* f^*$.

Definition 3.53. *Let F be a functor (of either variance) on module categories. We say F is **additive** if for every pair of objects A, B of the initial category, the map $F : Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{D}}(F(A), F(B))$ (or $F : Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{D}}(F(B), F(A))$) is a group homomorphism, that is, $F(f + g) = F(f) + F(g)$ for all $f, g \in Hom_{\mathcal{C}}(A, B)$.*

Remarks.

1. Localization, Modding Out, and the Hom functors are all additive.

2. Suppose $A \xrightarrow{f} B \xrightarrow{g} C$ is exact and let F be an additive covariant functor. Consider $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$. In general, this is not exact - but we do still get $imF(f) \subseteq kerF(g)$.

   Proof. This is equivalent to showing $F(g)F(f) = 0$. Of course, $F(g)F(f) = F(gf) = F(0) = 0$ as F is additive ($F(0) = F(0) + F(0)$ implies $F(0) = 0$). $\square$

Definition 3.54. *As additive functor on module categories is **exact** if whenever $A \xrightarrow{f} B \xrightarrow{g} C$ is exact in the initial category, then $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ is exact (or in the contravariant case $F(C) \to F(B) \to F(A)$ is exact). Suppose F is covariant. Say F is **left exact** if*

$$0 \to A \to B \to C \text{ exact implies } 0 \to F(A) \to F(B) \to F(C) \text{ is exact}$$

*and F is **right exact** if*

$$A \to B \to C \to 0 \text{ exact implies } F(A) \to F(B) \to F(C) \to 0 \text{ is exact.}$$

*Suppose F is contravariant. Say F is **left exact** if*

$$A \to B \to C \to 0 \text{ exact implies } 0 \to F(C) \to F(B) \to F(A) \text{ is exact}$$

*and F is **right exact** if*

$$0 \to A \to B \to C \text{ exact implies } F(C) \to F(B) \to F(A) \to 0 \text{ is exact.}$$

Proposition 3.55. *Let F be an additive functor. TFAE*

1. *F is exact*

2. *F takes short exact sequences to short exact sequences*

3. *F is both left and right exact.*

Remark. We've shown localization is an exact covariant functor.

Proposition 3.56. *The modding out functor is right exact, but not generally exact.*

Proof. Let I be a left ideal of R, $L \xrightarrow{f} M \xrightarrow{g} N \to 0$ an exact sequence of $R-$modules. Consider $L/IL \xrightarrow{\overline{f}} M/IM \xrightarrow{\overline{g}} N/IN \to 0$ where $\overline{f}(\ell + IL) = f(\ell) + IM$ and $\overline{g}(m + IM) = g(m) + IN$. As g is onto, so is $\overline{g}$. Also, $im\overline{f} \subseteq \ker \overline{g}$ as modding out is an additive functor. So we need only show $im\overline{f} \supseteq \ker \overline{g}$. Let $x \in \ker \overline{g}$. Then $\overline{g(x)} = \overline{g}(\overline{x}) = \overline{0}$ which implies $g(x) \in IN$. Thus there exists $i_j \in I, n_j \in N$ such that $g(x) = \sum_{j=1}^{k} i_j n_j$. Let $u_j \in M$ such that $g(u_j) = n_j$. Then $g(x) = \sum u_j g(u_j) = g(\sum i_j u_j)$. Thus $g(x - \sum i_j u_j) = 0$ which implies $x - \sum i_j u_j \in \ker g = imf$. Let $\ell \in L$ such that $f(\ell) = x - \sum i_j u_j$. Then $\overline{f}(\overline{\ell}) = \overline{x} \in im\overline{f}$.

To show it is not always left exact, consider $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z}$ where $n \mapsto 2n$. Modding out by (2) gives us $0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\overline{2}} \mathbb{Z}/2\mathbb{Z}$ where $\overline{n} \mapsto 2\overline{n} = 0$. Thus the map is not injective. $\square$

Proposition 3.57. *Let M be a left $R-$module. Then $Hom_R(M, -)$ and $Hom_R(-, M)$ are both left exact, but not generally exact.*

Proof. We will prove only for $Hom_R(M, -)$. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ be exact and consider $0 \to Hom_R(M, A) \xrightarrow{f_*} Hom_R(M, B) \xrightarrow{g_*} Hom_R(M, C)$. As f is 1-1, we have $fh = f_*(h) = 0$ which implies $h = 0$. Thus f_* is 1-1. By additivity, $imf_* \subseteq \ker g_*$. Thus we need only show $imf_* \supseteq \ker g_*$. Let $h \in \ker g_*$ where $h : M \to B$. So $g_*(h) = gh = 0$. This says $imh \subseteq kerg = imf$. Thus for all $m \in M$ there exists a unique $a_m \in A$ such that $f(a_m) = h(m)$. Define $k : M \to A$ by $k(m) = a_m$. Then $k \in Hom_R(M, A)$ and $f_*(k) = h \in imf_*$.

To show it is not always right exact, consider $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$. This gives us $Hom_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to Hom_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to Hom_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to 0$. Now, the first two modules are 0 and the last is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Thus it does not preserve surjectivity. $\square$

Proposition 3.58. *Let R be a ring and P a left $R-$module. Then P is projective if and only if $Hom_R(P, -)$ is exact.*

Proof. We will only prove the forward direction. The backward direction is similar. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be exact and apply the Hom functor:

$$0 \to Hom(P, A) \xrightarrow{f_*} Hom(P, B) \xrightarrow{g_*} Hom(P, C) \to 0.$$

By the previous proposition, it is enough to show g_* is onto. Let $h \in Hom_R(P, C)$. By the definition of projective, there exists $k : P \to B$ such that $gk = h$ which implies $g_*(k) = h$. Thus $h \in img_*$ and is thus onto. $\square$

3.5 Tensor Products

Definition 3.59. *Let R, S be rings. An $R - S$ **bimodule** is a left $R-$module M which is also a right $S-$module such that $(rm)s = r(ms)$ for all $r \in R, s \in S, m \in M$.*

Examples.

1. Any ring R is an $R - R$ bimodule.

2. Let S be an $R-$algebra ($\rho : R \to S, \rho(R) \subseteq Z(S), R$ commutative). Any left S module is an $S - R$ bimodule via $m \cdot r = \rho(r)m$ for all $r \in R, m \in M$ (in general, we will just say $m \cdot r = rm$ for simplicity).
   Check: $(sm)r = r(sm) = (rs)m = (sr)m = s(rm) = s(mr)$.

Special Case.

1. If R is a commutative ring, every left $R-$module is an $R - R$ bimodule (R is an $R-$algebra)

2. Any ring is a $\mathbb{Z}-$algebra (as every ring is an abelian group). Thus every left $R-$module is an $R - \mathbb{Z}$ bimodule.

3. $S = M_n(k), k$ a field. Any left $S-$module is an $S - k$ bimodule (i.e., every left $S-$module is a $k-$vector space).

Remark. Let M be an $R - S$ bimodule and N a left $R-$module. Then $Hom_R(M, N)$ is a left S module via $(sf)(m) := f(ms)$. Check: $(sf)(rm) = f((rm)s) = f(r(ms)) = rf(ms) = r(sf)(m)$.
If M is an $R - S$ bimodule, then $Hom_R(M, -) :<< R - mod >>\to<< S - mod >>$. Check: Suppose $f : N_1 \to N_2$ is an $R-$module homomorphism. Then we have $f_* : Hom_R(M, N_1) \to Hom_R(M, N_2)$ defined by $g \mapsto fg$ and we see $f_*(sg)(m) = f \circ (sg)(m) = f(g(ms)) = sfg(m)$. Thus $f_*(sg) = sf_*(g)$.

Similarly, if S is an $R - S$ bimodule, then $Hom_R(M, N)$ is a right $S-$module via $fs(m) = f(m)s$.

Definition 3.60. *Let A be a right $R-$module and B a left $R-$module. An $R-$**biadditive map** on $A \times B$ is a function $f : A \times B \to G$ where G is an abelian group such that for $a_i \in A, b_i \in B, r \in R$*

1. $f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2)$

2. $f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b)$

3. $f(ar, b) = f(a, rb)$

Definition 3.61. *Let A be a right $R-$module, B a left $R-$module. The* ***tensor product*** *of A, B is an abelian group $A \otimes_R B$ and an $R-$biadditive map $\phi : A \times B \to A \otimes_R B$ such that given any $R-$biadditive map $f : A \times B \to T$ (an abelian group), there exists a unique group homomorphism $\widetilde{f} : A \otimes_R B \to T$ such that $\widetilde{f}\phi = f$.*

Note. Hom and $\otimes$ are in some sense adjoints of each other.

Exercise. If it exists, $A \otimes_R B$ is unique up to isomorphism.

Theorem 3.62. *$A \otimes_R B$ exists.*

Proof. Let $F = \oplus_{(a,b) \in A \times B} \mathbb{Z}$ (a free $\mathbb{Z}-$module). Let $[a, b]$ be the standard basis element with 1 in the $[a, b]^{th}$ coordinate and 0's elsewhere. Thus every element of F is uniquely expressed as $\sum_{i=1}^{n} m_i [a_i, b_i]$. Let S be the subgroup of F generated by all the elements of the form

$$[a, b_1 + b_2] - [a, b_1] - [a, b_2], \ [a_1 + a_2, b] - [a_1, b] - [a_2, b], \ [ar, b] - [a, rb].$$

Define $A \otimes_R B = F/S$, with generating elements $a \otimes b = [a, b] + S$. (Note: For $m \in \mathbb{Z}, m > 0$, we have $m(a \otimes b) = (ma) \otimes b$. So every element looks like $\sum a_i \otimes b_i$, but is non uniquely represented).

Claim: The tensor product is biadditive, that is,

1. $a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$
2. $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$
3. $(ar) \otimes b = a \otimes (rb)$.

Proof: Since $[a, b_1 + b_2] - [a, b_1] - [a, b_2] \in S$, we know $[a, b_1 + b_2] + S = [a, b_1] + S + [a, b_2] + S$. Thus (1) holds. Similarly, (2) and (3) are true.

Define $\phi : A \times B \to A \otimes_R B$ by $(a, b) \mapsto a \otimes b$. By the remarks above, ϕ is clearly biadditive.

Now, let $f : A \times B \to T$ be a biadditive map. Define $f' : F \to T$ by $[a, b] \mapsto f(a, b)$. As f is biadditive, $S \subseteq \ker f'$. Thus there exists an induced homomorphism $\widetilde{f} : F/S \to T$ defined by $\overline{[a, b]} \mapsto f(a, b)$, that is $\widetilde{f} : A \otimes_R B \to T$ with $a \otimes b \mapsto f(a, b)$. This makes the diagram commute. Clearly, $\widetilde{f}$ is unique since $A \otimes_R B$ is generated by $\{a \otimes b | a \in A, b \in B\}$. □

Example. $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \cong 0$. A typical generator looks like $\overline{a} \otimes \overline{b}$. Since 2, 3 are relatively prime, there exists $r, s, p, q \in \mathbb{Z}$ such that $a = 2r + 3s, b = 2p + 3q$. Thus $\overline{a} \otimes \overline{b} = 3\overline{s} \otimes 2\overline{p} = 2\overline{s} \otimes 3\overline{p} = 0 \otimes 0 = 0$.

Proposition 3.63. *Let R be a ring, $f : A_1 \to A_2$ an $R-$homomorphism of right $R-$modules and $g : B_1 \to B_2$ an $R-$homomorphism of left $R-$modules. Then there exists a unique group homomorphism $f \otimes g : A_1 \otimes_R B_1 \to A_2 \otimes_R B_2$ defined by $a \otimes b \mapsto f(a) \otimes g(b)$.*

Proof. Define $f \times g : A_1 \times B_1 \to A_2 \otimes_R B_2$ by $(a, b) \mapsto f(a) \otimes g(b)$. Clearly this is $R-$biadditive. Thus we get the unique homomorphism $f \times g$. □

Remarks. $(f_1 + f_2) \otimes g = f_1 \otimes g + f_2 \otimes g$ and $(f \otimes g)(h \otimes \ell) = fh \otimes g\ell$.

Corollary 3.64. *Let R be a ring and A a right $R-$module. Define $A \otimes_R - : << R - mod >> \to << Abel >>$ by $B \mapsto A \otimes_R B$ and $(f : B_1 \to B_2) \mapsto (1_A \otimes f : A \otimes_R B_1 \to A \otimes_R B_2)$. Then $A \otimes_R -$ is an additive covariant functor.*

Note. If A is a left $R-$module, we get $- \otimes_R B : << \text{mod} - R >> \to << \text{Abel} >>$.

Theorem 3.65. *Let A be a right $R-$module. Then $A \otimes_R -$ is right exact.*

Proof. Let $L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be an exact sequence of left $R-$modules. We want to show $A \otimes_R L \xrightarrow{1\otimes f} A \otimes_R M \xrightarrow{1\otimes g} A \otimes_R N \to 0$ is exact.

$1 \otimes g$ is onto: Since $A \otimes N$ is generated by $a \otimes n$, it is enough to show $a \otimes n \in im(1 \otimes g)$. For $n \in N$, there exists $m \in M$ such that $g(m) = n$ as g is onto. Then $(1 \otimes g)(a \otimes m) = a \otimes g(m) = a \otimes n$.

$im(1 \otimes f) \subseteq \ker(1 \otimes g)$: Notice $(1 \otimes g)(1 \otimes f) = 1 \otimes gf = 1 \otimes 0 = 0$.

$im(1 \otimes f) \supseteq \ker(1 \otimes g)$: By the above, we get an induced map $\overline{1 \otimes g} : A \otimes_R M/im(1 \otimes f) \to A \otimes_R N$ defined by $\overline{a \otimes m} \mapsto a \otimes g(m)$. It is enough to show $\overline{1 \otimes g}$ is 1-1. Define $h : A \times N \to A \otimes M/im(1 \otimes f)$ by $(a, n) \mapsto \overline{a \otimes m}$ where $m \in M$ is such that $g(m) = n$.

Claim: h is well-defined.
Proof: Suppose $g(m_1) = g(m_2) = n$. Since $g(m_1 - m_2) = 0$, we have $m_1 - m_2 \in \ker g = imf$. Let $\ell \in L$ such that $f(\ell) = m_1 - m_2$. Then $a \otimes m_1 - a \otimes m_2 = a \otimes (m_1 - m_2) = a \otimes f(\ell) = (1 \otimes f)(a \otimes \ell) \in im(1 \otimes f)$. Thus $\overline{a \otimes m_1} = \overline{a \otimes m_2}$.

It is easy to show h is $R-$biadditive. Thus, there exists a unique group homomorphism $\widetilde{h} : A \otimes_R N \to A \otimes_R M/im(1 \otimes f)$ defined by $a \otimes n \mapsto h(a, m)$. Note that $\widetilde{h}(1 \otimes g)(\overline{a \otimes m}) = \overline{a \otimes m}$. Thus it fixes the generating set, which is enough to say $\widetilde{h}(1 \otimes g) = 1$. Thus $1 \otimes g$ is injective and thus $\ker(1 \otimes g) = im(1 \otimes f)$. ☐

Example. $\mathbb{Z}/2\mathbb{Z} \oplus_{\mathbb{Z}} -$ is not exact. Consider the injection $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z}$ defined by $m \mapsto 2m$. This yields $0 \to \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{1\otimes 2} \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$ defined by $\overline{a} \otimes m \mapsto \overline{a} \otimes 2m = 2\overline{a} \otimes m = 0$, but $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$ is not 0.

Proposition 3.66. *Let M be a left $R-$module. Then there exists a group isomorphism $f : R \otimes M \to M$ defined by $r \otimes m \mapsto rm$.*

Proof. Define $f' : R \times M \to M$ defined by $(r, m) \mapsto rm$. This is $R-$biadditive. Thus we have the unique group homomorphism $f : R \otimes M \to M$. Define $g : M \to R \otimes M$ by $m \mapsto 1 \otimes m$. This is clearly well defined and a group homomorphism. Also $fg = gf = 1$. So f is an isomorphism. ☐

Proposition 3.67. *Let R, S be rings, M an $S-R$ bimodule and N a left $R-$module. Then $M \otimes_R N$ is a left $S-$module under the action $s(\sum m_i \otimes n_i) = \sum (sm_i) \otimes n_i$.*

Proof. The $S-$module axioms are trivial. Thus we just need to show it is well-defined. Let $s \in S$. Define $\mu_S : M \times N \to M \otimes_R N$ by $(m, n) \mapsto (sm, n)$. We see μ_S is $R-$biadditive. Thus we get the group homomorphism $\widetilde{\mu_S} : M \otimes_R N \to M \otimes_R N$ defined by $m \otimes n \mapsto (sm) \otimes n$. Define $s(\sum m_i \otimes n_i) = \widetilde{\mu_S}(\sum m_i \otimes n_i) = \sum \widetilde{\mu_s}(m_i \otimes n_i) = \sum (sm_i) \otimes n_i$. Thus it is well-defined. ☐

Corollary 3.68. *In this situation, $M \otimes_R - :<< R - mod >> \to << S - mod >>$.*

Examples.

1. If R is commutative, every $R-$module M is an $R-R$ bimodule. So $M \otimes_R - :<< R - \text{mod} >> \to << R - \text{mod} >>$.

2. Let k be commutative and R a $k-$algebra. Let M be a right $R-$module. Then M is a $k - R$ bimodule. So $M \otimes_R - :<< R - \text{mod} >> \to << k - \text{mod} >>$.

Theorem 3.69. *Let R, S be rings, A a right $R-$module, B an $R - S$ bimodule, and C a left $S-$module. Then there exists a group isomorphism $g : A \otimes_R (B \otimes_S C) \to (A \otimes_R B) \otimes_S C$ defined by $a \otimes (b \otimes c) \mapsto (a \otimes b) \otimes c$. In addition, if A is an $R - R$ bimodule, then g is a homomorphism of left $R-$modules.*

Proof. Fix $a \in A$. Define $g_a : B \times C \to (A \otimes_R B) \otimes_S C$ by $(b, c) \mapsto (a \otimes b) \otimes c$.

Claim: g_a is $S-$biadditive.

Proof: Let $s \in S$. Then $g_a(bs, c) = (a \otimes (bs)) \otimes c = ((a \otimes b)s) \otimes c = (a \otimes b) \otimes (sc) = g_a(b, sc)$. The other properties follow similarly.

So there exists a unique group homomorphism $\widetilde{g_a} : B \otimes_S C \to (A \otimes_R B) \otimes_S C$. Now, define $f : A \times (B \otimes_S C) \to (A \otimes_R B) \otimes_S C$ by $(a, x) \mapsto \widetilde{g_a}(x)$. A little work shows f is also biadditive. Thus, we get $\widetilde{f} : A \otimes_R (B \otimes_S C) \to (A \otimes_R B) \otimes_S C$ defined by $a \otimes (b \otimes c) \mapsto (a \otimes b) \otimes c$. Analogously, there exists a homomorphism $\widetilde{h} : (A \otimes_R B) \otimes_S C \to A \otimes_R (B \otimes_S C)$ defined by $(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c)$. Then $\widetilde{f}\widetilde{h} = \widetilde{h}\widetilde{f} = 1$ (its clearly true on the generators and thus all elements as they are group homomorphisms). Take $g = \widetilde{f}$.

To show g is a homomorphism of left $R-$modules when A is an $R - R$ bimodule, just need to check the following:

$$\begin{aligned} g(r(a \otimes (b \otimes c))) &= g((ra) \otimes (b \otimes c)) \\ &= (ra) \otimes b) \otimes c \\ &= (r(a \otimes b)) \otimes c \\ &= r((a \otimes b) \otimes c) = rg(a \otimes (b \otimes c)). \end{aligned}$$

□

<u>Change of Rings</u>

Proposition 3.70. *Let $\phi : R \to S$ be a ring homomorphism. Let M be a left $R-$module. Then $S \otimes_R M$ is a left $S-$module. Thus $S \otimes_R - :<< R - mod >> \to << S - mod >>$.*

Proof. Note that S is an $S - R$ bimodule, where $s \cdot r = s\phi(r)$. □

Examples.

1. If I is a 2 sided ideal, then $R/I \otimes_R M$ is a left R/I module. In particular, $R/m \otimes_R M$ is an R/m vector space (as R/m is a field).

2. If S is a multiplicatively closed set, then $R_S \otimes_R M$ $(\cong M_S)$ is an R_S-module.

3. Let $\phi : G_1 \to G_2$ be a group homomorphism. Then there exists an induced ring homomorphism $\widetilde{\phi} : k[G_1] \to k[G_2]$ sending $g \mapsto \phi(g)$ for a field k. Let V be a left $k[G_1]-$module. Then $k[G_2] \otimes_{k[G_1]} V$ is a left $k[G_2]-$module. This is called the **induced representation** of V to G_2.

Proposition 3.71. *If I is a 2 sided ideal, then $R/I \otimes_R M \to M/IM$ defined by $\overline{r} \otimes m \mapsto \overline{rm}$ is an isomorphism.*

Exercise. Let F be an additive functor on module categories. Then F preserves split exact sequences, that is, if F is covariant and $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split exact, then $0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$ is split exact. In particular, F preserves the split exactness of $0 \to A \to A \oplus C \to C \to 0$. Hence, $F(A \oplus C) \cong F(A) \oplus F(C)$.

Corollary 3.72. *$Hom(A \oplus B, C) \cong Hom(A, C) \oplus Hom(B, C)$*
$Hom(A, B \oplus C) \cong Hom(A, B) \oplus Hom(A, C)$
$A \otimes_R (B \oplus C) \cong (A \otimes_R B) \oplus (A \otimes_R C)$

Note. By induction, we can show the Corollary is true for finite sums. In general, this does not apply to infinite sums with the Hom functors, however, it is true for the tensor product.

Proposition 3.73. *Let A be a right $R-$module and $\{B_i\}_{i \in I}$ a family of left R modules. Then $A \otimes_R (\oplus_{i \in I} B_i) \cong \oplus_{i \in I}(A \otimes_R B)$ via $a \otimes (b_i) \mapsto (a \otimes b_i)$.*

Example. $R^m \otimes_R R^n \cong R^m \otimes (\oplus_{i=1}^n R) \cong \oplus_{i=1}^n (R^m \otimes_R R) \cong \oplus_{i=1}^n R^m \cong R^{mn}$.

Corollary 3.74. *Suppose $\phi : R \to S$ is a ring homomorphism. If F is a free left $R-$module, then $S \otimes_R F$ is a free left $S-$module.*

Proof. Recall $F \cong \otimes_{i \in I} R$. So $S \otimes F \cong \oplus_{i \in I}(S \otimes_R R) \cong \oplus_{i \in I} S$, a free left $S-$module. $\square$

Corollary 3.75. *Let P be a projective left $R-$module. Then $S \otimes_R P$ is a projective left $S-$module.*

Proof. Recall that there exists a left $R-$module Q such that $P \oplus Q = F$, a free $R-$module. Then $(S \otimes_R P) \oplus (S \otimes_R Q) \cong S \otimes_R (P \oplus Q) = S \otimes_R F$, a free $S-$module. Thus $S \otimes_R P$ is a direct summand of a free S module and is thus projective. $\square$

Definition 3.76. *Let R be a commutative ring, M an $R-$module. An element $m \in M$ is* ***torsion*** *if there exists a non zero divisor $r \in R$ such that $rm = 0$. Say M is* ***torsion free*** *if the only torsion element is 0.*

Note. Ideals are always torsion free (as if $r \cdot i = 0$, then either r is a zero divisor or $i = 0$.

Example. Let $R = k[[x, y]]/(xy)$, k is a field. This is local. Let $m = (x, y)R$, the maximal ideal. Then m is torsion free.

Claim: $x \otimes y \in m \otimes m$ is torsion.

Proof: We can see $x+y$ is not a zero divisor in R. However, $(x+y)(x \otimes y) = (x+y)x \otimes y = (x+y) \otimes (xy) = (x+y) \otimes 0 = 0$. (In fact, $Ann_R x \otimes y = m$).

Claim: $x \otimes y \neq 0$.

Proof: Recall if (R, m) is local and M is finitely generated, then the minimal number of generators, $\mu_R(M) = dim_{R/m} M/mM$. Consequently, $\mu_R(M \otimes N) = \mu_R(M)\mu_R(N)$ (if M, N are f.g.). Let $h = (x, y)k[[x, y]]$. Clearly, $\mu_{k[[x,y]]}(h) = 2$. Note that $m = n/(xy)$ and $m/m^2 = \frac{n/(xy)}{n^2/(xy)} = n/n^2$. So $\mu_R(m) = \mu_{k[[x,y]]}(n) = 2$. Thus $\mu_R(m \otimes m) = 4$. Every element of $m \otimes m$ is an $R-$linear combination of $x \otimes x, x \otimes y, y \otimes x, y \otimes y$, which implies this is a minimal generating set and thus $x \otimes y \neq 0$.

Example. Let $R = k[[x, y]], m = (x, y)R$. Note R is a domain (so there are no zero divisors). In $m \otimes m$, consider $u = x \otimes y - y \otimes x$. Note that $u \neq 0$ as $x \otimes y, y \otimes x$ are generators and thus basis elements in R/m, a field. Let $r \in m$. Then $ru = r \otimes xy - r \otimes xy = 0$. Thus $Ann_R u = m$.

Theorem 3.77 (Hom - Tensor adjointness). *Let R, S be rings, A a left $R-$module, B an $S-R$ bimodule, C a left $S-$module. Then*

$$Hom_S(B \otimes_R A, C) \cong Hom_R(A, Hom_S(B, C)).$$

Note that this is an isomorphism of abelian groups. However, if A is an $R-S$ bimodule, then it is an isomorphism of left $S-$modules.

Proof. Let $f \in Hom_S(B \otimes_R A, C)$. Fix $a \in A$. Define $f_a : B \to C$ by $b \mapsto f(b \otimes a)$.

Claim: f_a is $S-$linear.

Proof: $f_a(sb) = f((sb) \otimes a) = f(s(b \otimes a)) = sf(b \otimes a) = sf_a(b)$ as f is $S-$linear. Additivity follows similarly. Thus $f_a \in Hom_S(B, C)$.

Define $\widetilde{f} : A \to Hom_S(B, C)$ by $a \mapsto f_a$. This is $R-$linear as $\widetilde{f}(ra) = f_{ra}$ and $r\widetilde{f}(a) = r \cdot f_a$ implies $r \cdot f_a(b) = f_a(br) = f(br) \times a) = f(b \times (ra)) = f_{ra}(b)$. Now, define $\tau : Hom_S(B \otimes_R A, C) \to Hom_R(A, Hom_S(B, C))$ by $f \mapsto \widetilde{f}$. Check that this is additive (and thus a group homomorphism. Let $f \in Hom_R(A, Hom_S(B, C))$. Define $g' : B \times A \to C$ by $(b, a) \mapsto g(a)(b)$.

Claim: g' is $R-$biadditive.

Proof: $g'(br, a) = g(a)(br) = (r \cdot g(a))(b)$ as $Hom_S(B, C)$ is a left $R-$module. Now, $(r \cdot g(a))(b) = g(ra)(b) = g'(b, ra)$ by definition of g.

Thus we get $\overline{g} : B \otimes_R A \to C$ defined by $b \otimes a \mapsto g(a)(b)$. Now, define $\pi : Hom_R(A, Hom_S(B, C)) \to Hom_S(B \otimes_R A, C)$ by $g \mapsto \overline{g}$. Check that π is additive and $\pi\tau = \tau\pi = 1$. $\square$

3.6 Noetherian/Artinian Rings

Definition 3.78. *Let R be a ring and M a left $R-$module. We say M is* ***left Noetherian*** *if every ascending chain of left $R-$submodules of M stabilizes, that is, if $M_0 \subseteq M_1 \subseteq \cdots$ is an ascending chain of left $R-$submodules, then there exists n such that $M_n = M_{n+1} = M_{n+2} = \cdots$. Say M is* ***left Artinian*** *if every descending chain of left $R-$modules of M stabilizes. Say R is a* ***left Noetherian/Artinian ring*** *if R is left Noetherian/Artinian as an $R-$module. Say R is* ***Noetherian/Artinian*** *if it is both left and right Noetherian/Artinian.*

Remarks.

1. Every division ring (and thus every field) is both Noetherian and Artinian (since the only ideals are 0 and 1).
2. Let R be a ring and $D \subseteq R$ a division ring. Suppose R is finite dimensional as a $D-$module. Then R is Noetherian and Artinian. (The length of every proper ascending/descending chain of $D-$submodules over R is bounded by $dim_D R$.)
3. Any PID is Noetherian (but not necessarily Artinian). For example $\mathbb{Z}$ is not Artinian as $(2) \subsetneq (4) \subsetneq (8) \subsetneq \cdots$ does not stabilize.

Example. Let $R = M_n(k)$, where k is a field. Then $dim_k R = n^2$. So R is Noetherian and Artinian.

Theorem 3.79 (Hilbert Basis Theorem). *Let R be a commutative Noetherian ring. Then $R[x]$ is Noetherian.*

Corollary 3.80. *If R is a commutative Noetherian ring, then $R[x_1, ..., x_n]$ is Noetherian.*

Fact. Any left Artinian ring is left Noetherian. (We will prove this later, once we build up more machinery).

The fact is not true for modules. Let $R = \mathbb{Z}_{(2)} \subseteq \mathbb{Q}$. Note that every element of $\mathbb{Q}$ can be expressed uniquely as $u2^\ell$ for some $u \in R$ which is a unit and $\ell \in \mathbb{Z}$.

Claim: The only $R-$submodules of $\mathbb{Q}$ are $N_\ell = R2^\ell$ for $\ell \in \mathbb{Z}$ and $0, \mathbb{Q}$.
Proof: First note

- $\cdots \supsetneq N_{\ell-1} \supsetneq N_\ell \supsetneq N_{\ell+1} \supsetneq \cdots$
- $\cup_\ell N_\ell = \mathbb{Q}$.

Now, let N be an $R-$submodule of $\mathbb{Q}$ such that $N \neq 0, \mathbb{Q}$. Choose smallest ℓ such that $N_\ell \subseteq N$ (such an ℓ exists as $N_\ell \subsetneq N_{\ell-1} \subsetneq \cdots$).

Subclaim: $N = N_\ell$.
Proof: Choose $n \in N$. Then $n = u2^r$. Note that $r \geq \ell$ as otherwise $2^r \in N$ which implies $N_r \subseteq N$. Then $n = u2^r = u2^{r-\ell}2^\ell$ and since $u2^{r-\ell} \in R$, we see $n \in N_\ell$.

Now, let $M = \mathbb{Q}/N_0 = \mathbb{Q}/R$. Then the $R-$submodules of M are

$$M \supsetneq \cdots \supsetneq N_\ell/R \supsetneq N_{\ell-1}/R \supsetneq \cdots \supsetneq N_0/R = 0.$$

Clearly, M satisfies DCC on $R-$submodules, but not ACC.

Proposition 3.81. *Let R be a ring and $0 \to A \to B \to C \to 0$ a short exact sequence of left $R-$modules. Then B is left Noetherian (resp Artinian) if and only if A and C are.*

Proof. We will prove for Noetherian modules. The proof for Artinian is similar. WLOG, we may assume $A \subseteq B$ and $C = B/A$. Now, the forward direction is clear. To prove the backward direction, let $B_1 \subseteq B_2 \subseteq \cdots$ be an ascending chain in B. Consider the chains $(*) B_1 + A \subseteq B_2 + A \subseteq \cdots$ and $(**) B_1 \cap A \subseteq B_2 \cap A \subseteq \cdots$. As A is Noetherian, $(**)$ stabilizes. Since B/A is Noetherian, we can mod $(*)$ by A and that also stabilizes. Thus there exists n such that $B_n + A = B_{n+1} + A = \cdots$ and $B_n \cap A = B_{n+1} \cap A = \cdots$.

Claim: $B_n = B_{n+1} = \cdots$.

Proof: Let $b \in B_{n+1} \subseteq B_{n+1} + A = B_n + A$. Say $b = b_n + a$ for $b_n \in B_n$ and $a \in A$. Now, $b - b_n = a \in A \cap B_{n+1} = A \cap B_n$. So $b - b_n \in B_n$ which implies $b \in B_n$. ☐

Corollary 3.82. *A left $R-$module M is left Noetherian (resp. Artinian) if and only if $M^n = \oplus_{i=1}^n M$ is left Noetherian (resp. Artinian). In particular, if R is a left Noetherian (resp. Artinian) ring, then so is R^n for all $n \geq 1$.*

Proof. The backwards direction is clear. For the forward direction, use induction and the fact that $0 \to M \to M + M \to M \to 0$ is a short exact sequence. ☐

Corollary 3.83. *If R is a left Noetherian (resp. Artinian) ring and M a finitely generated left $R-$module, then M is left Noetherian (resp. Artinian).*

Proof. Since M is finitely generated, $M = Rx_1 + ... + Rx_n$ which induces the short exact sequences $0 \to \ker\phi \to R^n \xrightarrow{\phi} M \to 0$ where $\phi : e_i \mapsto x_i$. Apply previous corollary. ☐

Proposition 3.84. *Let M be a left $R-$module. TFAE*

1. *M is left Noetherian (resp. Artinian).*

2. *Every set of $R-$submodules of M has a maximal (resp. minimal) element.*

For Noetherian only, these are equivalent to

3. *Every $R-$submodule of M is finitely generated.*

Proof. Note that $1 \Leftrightarrow 2$ is clear.

$2 \Rightarrow 3$ Let A be a submodule of M and $\Lambda = \{N | N \text{ is a f.g. } R-\text{submodule of } A\}$. Let M' be maximal in Λ. If $M' \neq A$, choose $x \in A \setminus M'$. Then $M' \subsetneq M' + Rx$, a finitely generated submodule of A, a contradiction. Thus $A = M'$ is finitely generated.

$3 \Rightarrow 1$ Let $M_1 \subseteq M_2 \subseteq \cdots$ be an ascending chain. Let $N = \cup_{i=1}^\infty M_i$. Then N is an $R-$submodule (as the M_i are nested), which implies N is finitely generated. Say $N = Rx_1 + ... + Rx_n$. Choose ℓ large enough so that $x_i \in M_\ell$ for all i. Then $N \subseteq M_\ell \subseteq M_{\ell+1} \subseteq \cdots \subseteq N$. ☐

Corollary 3.85. *Let $\phi : R \to S$ be a ring homomorphism. Suppose S is a finitely generated left $R-$module. If R is left Noetherian (resp. Artinian), then so is S.*

Proof. By the above corollary, S is Noetherian (resp. Artinian) as a left $R-$module. Every left ideal of S is a left $R-$module. Therefore S satisfies ACC (resp. DCC) on left ideals. ☐

Remark. If S is a finite dimensional $k-$algebra (for a division ring k), then S is both Noetherian and Artinian (as it satisfies ACC and DCC).

Example. $k[x]/(x^n)$ (this is Artinian, but not a field) and $M_n(k)$ are Noetherian and Artinian by the above remark.

Remarks.

1. If R is Noetherian (resp. Artinian) and I is an ideal of R, then R/I is Noetherian (resp. Artinian) (as R/I is a finitely generated $R-$module, generated by $\overline{1}$.)

2. Let R be a ring, $S \subseteq Z(R)$ a mcs of R. If R is Noetherian (resp. Artinian), then so is R_S.

3. Let R, S be commutative rings and suppose S is a finitely generated $R-$algebra. Then R Noetherian implies S is Noetherian.

Proof. WLOG, assume $R \subseteq S$. So say $S = R[u_1, ..., u_n]$ for $u_i \in S$. Define a ring homomorphism $\phi : R[x_1, ..., x_n] \to S$ by $x_i \mapsto u_i$. This is surjective and so $S \cong R[x_1, ..., x_n]/\ker\phi$. By the Hilbert Basis Theorem and Remark 1, S is Noetherian. $\square$

Note that this is not true for Artinian rings. For example, the division ring k is Artinian but $k[x]$ is not as $(x) \supsetneq (x^2) \supsetneq \cdots$.

4. Subrings of Noetherian rings are *not* necessarily Noetherian. For example $R = \mathbb{Q}[x, y]$ is Noetherian, but $S = \mathbb{Q}[x, xy, xy^2, ...] \subseteq R$ is not.

Examples.

1. $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a \in \mathbb{Z}, b, c \in \mathbb{Q} \right\}$ is right Noetherian, but not left Noetherian.

2. $S = \left\{ \begin{pmatrix} r & s \\ 0 & t \end{pmatrix} | r \in \mathbb{Q}, s, t \in \mathbb{R} \right\}$ is right Artinian, but not left Artinian.

Definition 3.86. *A left $R-$module M is **simple** or **irreducible** if $M \neq 0$ and has no submodules other than 0 and M.*

Proposition 3.87. *Let M be an $R-$module. TFAE*

1. *M is simple.*
2. *$M = Rx$ for all $x \in M \setminus \{0\}$.*
3. *$M \cong R/I$ where I is a maximal left ideal.*

Proof. $1 \Leftrightarrow 2$ $Rx \neq 0$ is a submodule of M.

$3 \Rightarrow 1$ Any submodule of M corresponds to R/J where $I \subseteq J$. Since I is maximal, done.

$2 \Rightarrow 3$ Define $\phi : R \to Rx = M$ by $r \mapsto rx$. So $M \cong R/\ker\phi$ where $\ker\phi$ is a left ideal. Since M has only 2 submodules, $\ker\phi$ must be maximal. $\square$

Definition 3.88. *Let M be an $R-$module. A **normal series** for M is a finite chain of submodules $(*) M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = (0)$. The **factors** of $(*)$ are M_i/M_{i+1} for $i = 0, ..., n-1$. The **length** of $(*)$ is the number of nonzero factors. We say two normal series are **equivalent** if there exists a bijection between the nonzero factors of the two series such that the corresponding factors and isomorphic. In particular, two equivalent normal series for M have the same length. A **composition series** is a normal series for M such that all nontrivial factors are simple. A **refinement** of $(*)$ is a normal series obtained by inserting additional modules between two links in the chain. A **proper refinement** is a refinement which has length larger than the original normal series.*

Note. A composition series has no proper refinements.

Theorem 3.89 (**Jordan-Hölder Theorem**)**.** *Any two normal series for M have equivalent refinements.*

Corollary 3.90. *Suppose M has a composition series. Then any normal series has a refinement which is equivalent to the given composition series. Therefore, any normal series has length less than the length of a given composition series. In particular, any two composition series are equivalent and have the same length.*

Definition 3.91. *If M has a composition series, define the **length of** M (denoted $\lambda_R(M)$) as the length of any composition series for M. If M does not have a composition series, we say it has infinite length.*

Proposition 3.92. *$\lambda_R(M) < \infty$ if and only if M is both Noetherian and Artinian.*

Proof. $\Rightarrow$ $\lambda_R(M)$ is a bound on the length of any chain. Thus any chain must stabilize.

$\Leftarrow$ Let $M_0 = M$. Let $\Lambda = \{N | N \subsetneq M \text{ is a submodule}\}$. As M is Noetherian, Λ has a maximal element, call it M_1. Then, $M_1 \subsetneq M_0$ and M/M_1 is simple. If $M_1 \neq 0$, repeat. In this way, we get a descending chain $M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots$ which must terminate as M is Artinian, that is, there exists $M_n = 0$. This is a composition series. $\square$

Definition 3.93. *A ring has finite* ***(left) length*** *if* $\lambda_R(R) < \infty$.

Examples.

1. $\lambda_R(k) < \infty$ for a division ring k. (In this case, the length is the dimension).

2. Let $R = M_N(k)$ for a division ring k. Then $\lambda_R(R) < \infty$.

3. Let $R = k[x]$. Then $\lambda_R(R) = \infty$.

Proposition 3.94. *Suppose* $0 \to A \to B \to C \to 0$ *is a short exact sequence. Then* $\lambda_R(B) = \lambda_R(A) + \lambda_R(C)$.

Proof. By the previous proposition, we may assume $\lambda_R(B), \lambda_R(A), \lambda_R(C) < \infty$. Induct on $\lambda_R(B)$. If $\lambda_R(B) = 1$, then B is simple. Since $A \hookrightarrow B$, either $A = B$ (and $C = 0$) or $A = 0$ (and $C = B$). In either case, the equality holds. Otherwise, assume $C = B/A$ and consider the normal series $B \supseteq A \supseteq (0)$. We may refine this series to get a composition series $B \supsetneq B_1 \supsetneq \cdots \supsetneq B_{n-1} \supsetneq B_n = (0)$. Then $A \supseteq B_{n-1}$. Consider $0 \to A/B_{n-1} \to B/B_{n-1} \to B/A \to 0$. By induction, since $\lambda_R(B/B_{n-1}) = \lambda_R(B) - 1$, we see $\lambda_R(B/B_{n-1}) = \lambda_R(A/B_{n-1}) + \lambda_R(B/A)$. Of course, $\lambda_R(A/B_{n-1}) = \lambda_R(A) - 1$ and thus $\lambda_R(B) = \lambda_R(A) + \lambda_R(C)$. $\square$

Definition 3.95. *Let R be a ring and M a left $R-$module. M is* ***completely reducible*** *or semisimple if M is a direct sum of a family of simple submodules. R is left* ***semisimple*** *if it is as an $R-$module.*

Proposition 3.96. *Let M be an $R-$module. TFAE*

1. *M is semisimple.*

2. *M is a sum of a family of simple submodules.*

3. *Every submodule of M is a direct summand of M.*

Proof. $1 \Rightarrow 2$ Trivial, as the direct sum is a sum.

$2 \Rightarrow 3$ Given $M = \sum_{i \in I} M_i$, where M_i is simple, let N be a submodule of M. Let $\Lambda = \{J \subseteq I | N + \sum_{j \in J} M_j = N \oplus (\oplus_{j \in J} M_j)\}$. Since $N \neq M$, there exists M_i such that $M_i \not\subset N$. Then $N \cap M_i \subseteq M_i$ implies $N \cap M_i = \emptyset$. Thus $\Lambda \neq \emptyset$. By Zorn's Lemma, there exists a maximal element $J \in \Lambda$. Let $F = \oplus_{j \in J} M_j$.

Claim: $N \oplus F = M$.
Proof: Note $N \cap F = (0)$ by choice of J. Suppose $N \oplus F \neq M$. Then there exists i such that $M_i \not\subset N \oplus F$. Note $M_i \cap (N \oplus F) = (0)$ or M_i as M_i is simple. Since $M_i \not\subset N \oplus F$, $M_i \cap (N \oplus F) = (0)$. Hence $N + F + M_i = N \oplus F \oplus M_i$, a contradiction to the maximality of J. Thus $N \oplus F = M$.

$3 \Rightarrow 1$ First, we need a claim.

Claim: Assuming M satisfies (3), every nonzero submodule of M contains a simple submodule.
Proof: Let $N \neq 0$ be a submodule of M. WLOG, assume N is cyclic, that is $N = Rx$ for $x \in M \setminus \{0\}$. Then $N \cong R/I$ where $I = Ann(x)$. Note $I \neq R$ as $N \neq 0$. Thus $I \subseteq m$ where m is a maximal left ideal. Then m/I is a maximal proper submodule of $R/I \cong N$. Thus N has a maximal proper submodule N' and so N/N' is simple. By (3), $M = N' \oplus F$ for some $F \subseteq M$. Note $N = N' \oplus (F \cap N)$. Thus $F \cap N \cong N/N'$, which is simple. Thus $F \cap N$ is a simple submodule of M.

Let $T = \{E | E \subseteq M \text{ is simple}\}$. Let $\Lambda = \{J \subseteq T | \sum_{E \in J} E = \oplus_{E \in J} E\}$. By Zorn's Lemma, there exists a maximal element $J \in \Lambda$.

Claim: $M = \oplus_{E \in J} E$.
Proof: If not, let $M' = \oplus_{E \in J} E$. By (3), $M = M' \oplus F$ where $F \subseteq M$. Since $F \neq 0$ as $M \neq M'$, F contains a simple submodule $E' \in T$. Then $J \cup E' \in \Lambda$, a contradiction to maximality. □

Corollary 3.97. *Submodules, quotients, and (direct)sums of semisimple modules are semisimple.*

Proof. • Let M be semisimple and $N \subseteq M$ a submodule. Let N' be the sum of all simple submodules of N.

Claim: $N = N'$.
Proof: By 3 of the proposition, there exists $F \subseteq M$ such that $M = N' \oplus F$. So $N = N' \oplus (F \cap N)$. If $F \cap N \neq (0)$, it contains a simple submodule E. Then $E \subseteq N'$, a contradiction as $M = N' \oplus F$. Thus $F \cap N = 0$ and $N = N'$.

- For quotients, say M/N, we know $M/N \cong F$ where $M = N \oplus F$. Done by previous bullet point.

- Suppose $\{M_i\}_{i \in I}$ is a family of semisimple submodules. Then $M_i = \oplus_{j \in J_i} E_{i_j}$, E_{i_j} is simple. Then $\oplus_{i \in I} M_i = \oplus_{i \in I, j \in J_i} E_{i_j}$ is semisimple. □

Proposition 3.98. *If R is semisimple, every $R-$module is semisimple.*

Proof. R semisimple implies every free module is semisimple which implies quotients of free modules are semisimple which implies all modules are semisimple. □

Examples.

- Division Rings are Semisimple.

- Let $R_1, ..., R_t$ be rings so that $S = R_1 \times \cdots \times R_t$ is a ring. The left ideals of S are of the form $I_1 \times \cdots \times I_t$ where I_i is a left ideal of R_i. Consequently, S is left Noetherian/Artinian/has finite length/is semisimple if and only if each R_i has the corresponding property.

- Let G be a finite group and k a field such that char $k \nmid |G|$. Then $R = k[G]$ is semisimple.

  Proof. Let I be a left ideal of R. So I is a $k-$subspace of R. Let $\Pi : R \to I$ be a projection onto I as $k-$vector spaces, that is, Π is $k-$linear and $\Pi(i) = i$ for all $i \in I$. Define $\widetilde{\Pi} = \frac{1}{|G|} \sum_{g \in G} g \Pi g^{-1}$.

  Claim: $\widetilde{\Pi}$ is $R-$linear.
  Proof: It suffices to show $\widetilde{\Pi}(hr) = h\widetilde{\Pi}(r)$ for all $r \in R, h \in G$. Notice

$$\begin{array}{rcl} \widetilde{\Pi}(hr) & = & \frac{1}{|G|} \sum_{g \in G} g \Pi g^{-1}(hr) \\ & = & \frac{1}{|G|} \sum_{hg \in G} (hg) \Pi (hg)^{-1} hr \\ & = & \frac{1}{|G|} \sum_{g \in G} hg \Pi g^{-1} h^{-1} hr \\ & = & \frac{1}{|G|} \sum_{g \in G} hg \Pi g^{-1}(r) = h\widetilde{\Pi}(r). \end{array}$$

  Note that if $i \in I$, then

$$\widetilde{\Pi}(i) = \frac{1}{|G|} \sum_{g \in G} g \Pi g^{-1}(i) = \frac{1}{|G|} \sum g g^{-1}(i) = i$$

  as $g^{-1}(i) \in I$. This gives rise to the short exact sequence $0 \to I \hookrightarrow R \to R/I \to 0$ with splitting map $\widetilde{\Pi} : R \to I$. Thus $R \cong I \oplus R/I$. Thus every submodule of R is a direct summand of R which implies R is semisimple. □

Let M be a left $R-$module. Let $End_R(M) = Hom_R(M, M)$. Note $End_R(M)$ is a ring under composition. If R is commutative and $F = R^n$, then $End_R(F) \cong M_n(R)$. This is not true if R is noncommutative.

Definition 3.99. *Let R be a ring. Define the **opposite ring** R^{op} by $R^{op} = R$ as abelian groups with multiplication in R^{op} defined by $r \cdot s := sr$.*

Claim. $End_R(R) \cong R^{op}$ as rings.

Proof. Let $a \in R$. Define $f_a : R \to R$ by $r \mapsto ra$. Then $f_a \in End_R(R)$. Furthermore, if $g \in End_R(R)$, then $g = f_a$ where $a = g(1)$. Observe $(f_a \circ f_b)(r) = f_a(rb) = rba = f_{ba}(r)$. Now define $\phi : End_R R \to R^{op}$ by $f_a \mapsto a$. $\square$

Note. If R is a division ring, so is R^{op}. It is easily shown that is $F \cong R^n$ as left $R-$modules, then $End_R(F) \cong M_n(R^{op})$.

Proposition 3.100. *Let D be a division ring, M a finitely generated $D-$module. Then $End_D(M)$ is semisimple.*

Proof. As a $D-$module, $M \cong D^n$ for some n. Thus $End_D(M) \cong M_n(D^{op})$. Since D^{op} is a division ring, it is enough to show $M_n(D)$ is semisimple where D is a division ring. Let e_i be the matrix with a 1 in the $i, i^{th}-$position and zeros elsewhere. Then $M_n(D)e_i$ is the ring with a nonzero i^{th} column and zeros elsewhere. This is simple by Exam 1. Thus $M_n(D) \cong M_n(D)e_1 \oplus \cdots \oplus M_n(D)e_n$, a direct sum of simple modules. Then $M_n(D)$ is semisimple. $\square$

Corollary 3.101. *Let $D_1, ..., D_k$ be division rings, $n_1, ..., n_k \in \mathbb{N}$. Then $M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ is semisimple.*

Note. These rings are left and right Artinian/Noetherian and also right semisimple.

Proposition 3.102. *Let R be a semisimple ring. Then $\lambda_R(R) < \infty$. Thus R is left/right Artinian/Noetherian.*

Proof. As R is semisimple, $R = \oplus_{\alpha \in \Lambda} I_\alpha$, where I_α are simple left ideals. Then $1 = e_{\alpha_1} + ... + e_{\alpha_k}$ where $e_{\alpha_i} \in I_{\alpha_i} \setminus \{0\}$ and $\alpha_1, ..., \alpha_k \in \Lambda$.

Claim: $R = I_{\alpha_1} \oplus \cdots \oplus I_{\alpha_k} \in \Lambda$.

Proof: Suppose there exists $\alpha \in \Lambda$ such that $I_\alpha \neq I_{\alpha_i}$ for $i = 1, ..., k$. Then for $r \in I_\alpha$, $r = re_{\alpha_1} + ... + re_{\alpha_k}$ where $re_{\alpha_i} \in I_{\alpha_i}$ which implies $r \in I_\alpha \cap (\sum I_{\alpha_i})$, a contradiction as R is the direct sum of I'_αs.

Relabel I_{α_i} as I_i for simplicity. Let $M_i = I_1 \oplus \cdots \oplus I_i$. Then $M_i/M_{i-1} = I_i$, which is simple. Thus $0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_k = R$ is a composition series. Thus $\lambda_R(R) = k < \infty$. $\square$

Proposition 3.103. *Let R be a semisimple ring. Then*

1. *Every simple left $R-$module is isomorphic to a simple left ideal.*
2. *There are only finitely many distinct simple left $R-$modules up to isomorphism.*

Proof. Let $R = I_1 \oplus \cdots \oplus I_k$, I_i are simple as in the previous proposition. Let J be a simple left ideal. Then the normal series $0 \subseteq J \subseteq R$ can be refined to a composition series for R. Then J is a factor of the composition series for R which says $J \cong I_i$ for some i by the Jordan-Hölder Theorem. Thus there are only finitely many distinct simple left ideals. Thus it suffices to prove (1). Let M be a simple left $R-$module. Let $x \in M \setminus \{0\}$. Then Rx is a nonzero submodule of M which implies $M = Rx$. Thus M is cyclic and we have the sequence $0 \to \ker\phi \to R \xrightarrow{\phi} M \to 0$ where $\phi(r) = rx$ is exact. As $\ker\phi$ is a left ideal of R, $\ker\phi$ is a direct summand of R by definition of semisimple. Thus the sequence splits and thus there exists a splitting map $\psi : M \to R$ such that $\phi\psi = 1_M$. Then ψ is injective and M is isomorphic to a simple left ideal of R. $\square$

Definition 3.104. *A ring is **simple** if the only two sided ideals of R are (0) and R. Note: Simple rings are not necessarily semisimple (this differs from Lang's definition).*

Note. An Artinian simple ring is semisimple.

Lemma 3.105. *Let R be a ring, I a simple left ideal, M a simple left $R-$module. If $I \ncong M$, then $IM = 0$.*

Proof. Suppose $IM \neq 0$. Then there exists $e \in M$ such that $Ie \neq 0$. Now $Ie \subseteq M$ is a left $R-$module. Since M is simple, $Ie = M$. Define $\phi : I \to M$ by $i \mapsto ie$. This is a left $R-$module homomorphism. Since $Ie = M$, ϕ is surjective. Also, $\ker\phi \neq I$ as $\phi \neq 0$ and so $\ker\phi = \{0\}$ as I is simple. Thus ϕ is an isomorphism. $\square$

Theorem 3.106. *Let R be semisimple, $\{I_1, ..., , I_k\}$ the set of all distinct left $R-$modules. Let $R_i = \sum_{I \text{ left ideal} \cong I_i} I$. Then*

1. *R_i is a ring with identity.*
2. *R_i is semisimple with only 1 distinct simple module.*
3. *R_i is a simple ring.*
4. *$R \cong R_1 \times \cdots \times R_k$ as rings.*

Proof. By the Lemma, $R_iR_j = 0$ for all $i \neq j$. Note $R = R_1 + ... + R_k$ and $R_j \subseteq R_jR = R_j(R_1 + ... + R_k) = R_j^2 \subseteq R_j$. Hence $R_j = R_jR$. Thus R_j is a two sided ideal. Write $1 = e_1 + ... + e_k$ for $e_i \in R_i$. Let $x \in R$. We can write $x = x_1 + ... + x_k$ for $x_i \in R_i$. Note $x_i = x_i \cdot 1 = x_i(e_1 + ... + e_k) = x_ie_i = (x_1 + ... + x_k)e_i = xe_i$ and similarly $x_i = e_ix$. Thus x_i is uniquely determined by x which implies $R = \oplus R_i$. Also, if $x \in R_i$, then $x = xe_i = e_ix$ implies that e_i is the identity on R_i. Thus R_i is a ring with identity. Its easy to show $R \cong R_1 \times \cdots \times R_k$ by mapping $r \mapsto (r_1, ..., r_k)$. Now, note that if J is a left ideal of R_i then $RJ = (R_1 + ... + R_k)J = R_iJ = J$. So J is a left ideal of R contained in R_i. Conversely, if $J \subseteq R_i$ is an ideal of R, then J is an ideal of R_i. Thus the left ideals of R_i are exactly the left ideals of R contained in R_i. Thus $R_i = \sum I$ (where I are in fact simple ideals of R_i) which implies R_i is semisimple. Also, every simple left ideal of R_i is isomorphic to I_i.

Let $J \neq 0$ be a two sided ideal of R_i. Then J is a left ideal of R which implies J contains a simple left ideal I of R. Since $J \subseteq R_i$, this says $I \cong I_i$. Let K be a left ideal of R such that $K \cong I$. Then $K \cong I_i$ which implies $K \subseteq R_i$.

Claim: $K \subseteq J$.

Proof: As R is semisimple, there exists a left ideal I' such that $I \oplus I' = R$. Then $1 = e + e'$ for $e \in I, e' \in I'$ where $e \neq 0$. Then $e = e^2 + ee'$. Since $I \cap I' = (0)$, we have $e = e^2$ and thus $Ie \neq 0$. As $Ie \subseteq I$ and I is simple, this says $I = Ie$. Let $\phi : I \to K$ be a left $R-$module isomorphism. Then $K = \phi(I) = \phi(Ie) = I\phi(e) \subseteq J\phi(e) \subseteq J$ as J is two sided.

Since K was arbitrary, this says $J \supseteq R_i$ which implies $J = R_i$. □

Corollary 3.107. *Let R be a semisimple ring. TFAE*

1. *R is simple.*
2. *There exists a unique left simple ideal up to isomorphism.*

Example. Let D be a division ring and $n \geq 1$. Then $M_n(D)$ is simple and semisimple.

Proof. Let $R = M_n(D)$ and e_i be the matrix with a 1 in the i,i-spot and zeroes elsewhere. Then $R = Re_1 \oplus \cdots \oplus Re_n$, where Re_i are simple left ideals and $\phi : Re_i \to Re_j$ defined by $re_i \mapsto re_iE_{ij}$ is an isomorphism. Then R has a unique maximal simple left ideal. Thus R is simple. □

Notation. Let R be a ring and E an $R-$module. Let $R'(E) = End_R(E)$. If $a \in R$, define $r_a : E \to E$ by $e \mapsto ea$. Then $r_a \in R'(E)$. Let $R''(E) = End_{R'}(E)$. (Note that if E is an $R'-$module, then for $\phi \in R', e \in E$, we can define $\phi e := \phi(e)$). For $a \in R$, define $\ell_a : E \to E$ by $e \mapsto ae$.

Claim: $\ell_a \in R''(E)$.

Proof: Let $f \in R', e \in E$. Then $f\ell_a(e) = f(ae) = af(e) = \ell_a(f(e))$.

This gives yield to the natural homomorphism $\lambda : R \to R''(E)$ defined by $a \mapsto \ell_a$. Note that λ is injective if and only if $\ell_a \neq 0$ for all $a \in R \setminus \{0\}$ which is if and only if $ann_R(E) = (0)$ (that is, E is a **faithful** $R-$module).

Schur's Lemma: *Let R be a ring and E a simple $R-$module. Then $R'(E)$ is a division ring.*

Proof. Let $\phi \in R'(E) \setminus \{0\}$. It is enough to show ϕ is an isomorphism. Of course, $\ker \phi$ is a submodule of E (which is simple) and since $\phi \neq 0$ we have $\ker \phi \neq E$ and so $\ker \phi = (0)$. Similarly, $im\phi$ is a submodule of E and since $\phi \neq (0)$ we have $im\phi = E$. □

Theorem 3.108. *Let R be a simple ring and $I \neq (0)$ a left ideal. Then $\lambda : R \to R''(I)$ is an isomorphism.*

Proof. (Rieffel) Since $\ker \lambda$ is a two sided ideal and R is simple, $\ker \lambda = 0$ or R. Since $1 \mapsto \ell_1$, which is clearly not zero, we see $\ker \lambda = 0$. Thus λ is injective. Note that $IR \neq (0)$ is a two sided ideal of R. Thus $IR = R$. Then $\{\sum \lambda(i_k)\lambda(r_k) | i_k \in I, r_k \in R\} = \lambda(I)\lambda(R) = \lambda(IR) = \lambda(R)$.

Claim: $\lambda(I)$ is a left ideal of R''.

Proof: Let $f \in R''$, $\ell_a \in \lambda(I)$ where $a \in I$. Let $i \in I$. Then $f\ell_a(i) = f(ai) = f(r_i(a)) = r_i(f(a)) = f(a)i = \ell_{f(a)}(i)$. Thus $f\ell_a = \ell_{f(a)} \in \lambda(I)$ as $f(a) \in I$.

Now, $\underbrace{R'' = R''\lambda(R)}_{\text{since } 1=\ell_1\in\lambda(R)} = R''\lambda(I)\lambda(R) = \lambda(I)\lambda(R) = \lambda(R)$. Thus λ is onto. □

Theorem 3.109 (Artin-Wedderburn)**.** *Let R be a simple ring. TFAE*

1. *R is semisimple.*
2. *R is left Artinian.*
3. *$R \cong M_n(D)$, $n \in \mathbb{N}, D$ a division ring.*

Proof. $3 \Rightarrow 1 \Rightarrow 2$ already done.

$2 \Rightarrow 3$ Since a minimal nonzero left ideal is a simple left ideal and R is left Artinian, we see that there exists a simple left ideal, call it I. By the Theorem, $\lambda : R \to R''(I) = End_{R'}(I)$ is an isomorphism. Since I is simple, $R' = End_R(I)$ is a division ring by Schur's Lemma.

Claim: I is finitely generated as an R' module.
Proof: Suppose not. Then there exists an infinite set $\{e_1, e_2, ...\} \subseteq I$ which is linearly independent over R'. For each $n \in \mathbb{N}$, let $J_n = \{f \in R''(I) | f(e_1) = \cdots = f(e_n) = 0\}$. Note J_n is a left ideal of R'' and $J_n \supsetneq J_{n+1}$ for all n. This says $R'' \cong R$ is not left Artinian, a contradiction.

Thus I is finitely generated as an R'−module. So $I \cong (R')^n$ for some n. Thus $R'' = End_{R'}((R')^n) \cong M_n((R')^{op})$ as $(R')^n$ is a free module. Let $D = (R')^{op}$, a division ring. □

Corollary 3.110. *Let R be a ring. TFAE*

1. *R is semisimple.*
2. *$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_\ell}(D_\ell)$ for $n_i \in \mathbb{N}, D_i$ division rings.*

Proof. $2 \Rightarrow 1$ Done, as products of semisimple rings are semisimple.

$1 \Rightarrow 2$ $R \cong R_1 \times \cdots \times R_\ell$ where R_i are left Artinian simple rings. □

Corollary 3.111. *If R is semisimple, then R is left/right Artinian and left/right Noetherian. Also, left semisimple if and only if right semisimple.*

Proof. Clear as $M_{n_1}(D_1) \times \cdots \times M_{n\ell}(D_\ell)$ are. □

Notation. Let R be a ring, E an $R-$module, $R' = R'(E) = End_R(E)$ and $R'' = R''(E) = End_{R'}(E)$. Let $E^n = \oplus_{i=1}^n E$ and $E_i := 0 \oplus \cdots \oplus 0 \oplus E \oplus 0 \oplus \cdots \oplus 0$. Let $\pi_i : E^n \to E_i$ and $\mu_i : E_i \to E^n$ be the natural maps. Let $\psi \in End_R(E^n)$ and $\psi_{ij} = \pi_i \psi \mu_j : E_j \to E_i$. So $\psi_{ij} \in Hom_R(E_i, E_j) \cong End_R(E) = R'$. Thus we can represent ψ as a matrix $(\psi_{ij})_{n\times n}$ where

$$\psi \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (\psi_{ji}) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n \psi_{ji}(x_j) \\ \vdots \\ \vdots \\ \sum_{j=1}^n \psi_{jn}(x_j) \end{pmatrix} \text{ for } \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in E^n.$$

Thus $End_R(E^n) \cong M_n(End_R(E))$, that is $R'(E^n) \cong M_n(R')$.

Remark. Let $f \in R''(E)$. So $f : E \to E$ and $f(\phi(x)) = \phi((f(x))$ for all $\phi \in R', x \in E$. Thus $f\phi = \phi f$ for all $\phi \in R'$. Define $f^{(n)} : E^n \to E^n$ by $f^{(n)}(x_1, ..., x_n) = (f(x_1), ..., f(x_n))$. As a matrix, this says $f^{(n)} = fI_n$. Let $\psi \in R'(E^n)$. Then $(fI_n)(\psi_{ij}) = \psi_{ij}(fI_n)$ since $f\psi_{ij} = \psi_{ij}f$ for all i, j. Thus $f^{(n)}\psi = \psi f^{(n)}$ for all $\psi \in R'(E^n)$. Thus $f^{(n)} \in R''(E^n)$ (its clearly additive and we just showed we can pull out elements from R'.) Therefore,

$$f \in R''(E) \Rightarrow f^{(n)} \in R''(E^n).$$

Lemma 3.112. *Let R be a ring, E a semisimple $R-$module. Let $f \in R''(E), x \in E$. Then there exists $\alpha \in R$ such that $f(x) = \alpha x$ (note that α depends on x).*

Proof. Fix $x \in E$. Since E is semisimple and Rx is a submodule of E, we have $E = Rx \oplus F$ for some left submodule F. Define $\pi : E \to E$ by $rx + f \mapsto rx$ (the projection onto Rx). So $\pi \in R'$ and since $\pi(x) = x$ we have $f(x) = f(\pi(x)) = \pi(f(x)) \in Rx$. $\square$

Theorem 3.113 (**Jacobson Density Theorem**)**.** *Let R be a ring and E a semisimple left $R-$module. Let $f \in R''(E)$ and $x_1, ..., x_n \in E$. Then there exists $\alpha \in R$ such that $f(x_i) = \ell_\alpha(x_i)$ for all $i \in [n]$.*

Proof. Let $f^{(n)} : E^n \to E^n$ be as above and $x = (x_1, ..., x_n) \in E^n$. By the remark, $f^{(n)} \in R''(E^n)$ and E^n is semisimple. By the lemma, there exists $\alpha \in R$ such that $f^{(n)}(x) = \alpha x$ which implies $f(x_i) = \alpha x_i$ for all $i \in [n]$. $\square$

Corollary 3.114. *If E is finitely generated over R', then $\lambda : R \to R''(E)$ defined by $\alpha \mapsto \ell_\alpha$ is surjective.*

Proof. Let $x_1, ..., x_n$ be generators for E as an $R'-$module. If $f \in R''$ and $f(x_i) = \ell_\alpha(x_i)$ for $i \in [n]$, then $f = \ell_\alpha$. $\square$

Corollary 3.115. *Let R be a semisimple ring and $E = R^n$ a left $R-$module. Then $\lambda : R \to R''(E)$ defined by $\alpha \mapsto \ell_\alpha$ is an isomorphism.*

Proof. As R is semisimple, E is. So $\ker \lambda = Ann_R(E) = (0)$ as R^n is faithful (it's free!). Note that E is generated over R' by $\{e_1\}$ (Let $x \in E$. As $\{e_1\}$ is part of an $R-$basis for E, there exists an endomorphism $\phi e_1 = \phi(e_1) = x$. Thus $R'e_1 = E$). By the previous corollary, x is surjective. $\square$

Corollary 3.116. *Let D be a division ring and E a finitely generated $D-$module. Then $D \cong End_{D'}(E)$, that is, $\lambda : D \to D''(E)$ is an isomorphism.*

Proof. D is semisimple and $E = D^n$ for some n. Done by previous corollary. $\square$

In matrix notation, this says $End_D(D^n) \cong M_n(D^{op}) =: D'$. So D^n is an $M_n(D^{op})-$module. Then $End_{D'}(D^n) = D$.

Corollary 3.117 (Wedderburn)**.** *Let R be a finite dimensional $k-$algebra, where k is a field. Let E be a simple $R-$module. Then $\lambda : R \to R''(E)$ is surjective. If, in addition, we have E is faithful, then λ is an isomorphism.*

Proof. By the first corollary, it is enough to show E is finitely generated as an $R'-$module. Since E is simple, $E = Rx$ for $x \in E$. So $\dim_k E < \infty$. Since $k \subset Z(R)$, we have $k \hookrightarrow R'(E)$ via $\alpha \mapsto \ell_\alpha$ ($\ell_\alpha \in R'(E)$ as k is commutative). So E finitely generated over k implies E is finitely generated over R'. $\square$

Note. E a finitely generated $R-$module does NOT imply E is a finitely generated $R'(E)-$module.

> **Example.** Let A be the ring from Exam 1 #6. A is called the (first) Weyl algebra of F and is denoted $A_1(F)$. An equivalent definition for A is $A_1(F) \cong F\{x,y\}/ < xy - yx - 1 >$ where $F\{x,y\}$ is the free algebra generated by x, y (i.e., x, y do not commute). Let I be a maximal left ideal of A and $E = A/I$. Then E is a simple $A-$module. Thus $A'(E) = End_A(E)$ is a division ring (as E is simple). If E is finitely generated as an $A'-$module, then $E \cong (A')^n$ and by the corollary, $\lambda : A \to A''(E)$ would be surjective, where $A''(E) = End_{A'}((A')^n) = M_n(A')$ is semisimple. Since A is simple, $\ker \lambda = 0$ which implies $A \cong M_n(A')$, a contradiction as A is not Artinian by $M_n(A')$ is. Thus E is *not* a finitely generated $A'-$module.

Remark. Let R be a ring, E an $R-$module. Let $r \in Z(R)$. Then $\ell_r \in R'(E)$. Thus there exists a ring homomorphism $\phi : Z(R) \to R'(E)$ mapping $r \mapsto \ell_r$. Denote $\phi(Z(R))$ by $Z(R) \cdot I_E$ where I_E is the identity map on E. If E is a finitely generated $Z(R)-$module, then E is a finitely generated $Z(R)I_E-$module (the actions on E are the same).

Observation. If E is a finitely generated $R-$module and R a finitely generated $Z(R)-$module, then E is a finitely generated $Z(R)-$module and hence a finitely generated $R'-$module.

Proof. Let $E = Rz_1 + ... + Rz_m, R = Zu_1 + ... + Zu_n$. Then $E = \sum_{i,j} Zu_i x_j$. Now, $Z(R)I_E$ is a subring of R' and thus E finitely generated over $Z(R)$ implies E is finitely generated over R'. $\square$

Proposition 3.118. *Suppose R is finitely generated over $Z(R)$ and E is a finitely generated semisimple $R-$module. Then $\lambda : R \to R''$ is onto.*

Note. Suppose $r \in Z(R)$. Then $\ell_r \in R'(E)$. In fact, $\ell_r \in Z(R')$ as for $f \in R'$, $f\ell_r(x) = f(rx) = rf(x) = (\ell_r f)(x)$ for all $x \in E$. Hence $Z(R)I_E \subset Z(R')$.

Proposition 3.119. *Suppose $\lambda : R \to R''(E)$ is an isomorphism. Then $Z(R') = Z(R)I_E = \{\ell_r | r \in Z(R)\}$.*

Proof. Only need to show ($\subset$). Let $f \in Z(R')$. Then for all $\phi \in R'$, $f(\phi x) = \phi f(x)$ which implies $f \in End_{R'}(E) = R''(E)$. So $f = \ell_r$ for some $r \in R$. Want to show $r \in Z(R)$. Let $s \in R$. Then $rsx = \ell_r(sx) = f(sx) = sf(x) = s\ell_r(x) = srx$. Thus $rs(x) = sr(x)$ for all $x \in E$ which says $(rs - sr)E = 0$. Of course, E is faithful which implies $rs = sr$. $\square$

Corollary 3.120. *Let D be a division ring. Then $Z(M_n(D)) = \{xI_n | x \in Z(R)\}$.*

Proof. Let $R = D, E = D^n$. Then $R'(E) = M_n(D^{op})$ and since E is a finitely generated semisimple ring over a division ring, we've seen $\lambda : R \to R''(E)$ is an isomorphism. Thus $Z(R') = Z(R)I_E$. Now, note that $Z(M_n(D)) = Z(M_n(D^{op}))$. $\square$

Proposition 3.121. *Let D_1, D_2 be division rings, V_1, V_2 finitely generated D_1, D_2 vectors spaces. Then $End_{D_1}(V_1) \cong End_{D_2}(V_2)$ if and only if $D_1 \cong D_2$ and $\dim_{D_1} V_1 = \dim_{D_2} V_2$.*

Proof. Let $R = End_{D_1} V_1$ and $\phi : R \to End_{D_2} V_2$. Then V_1 is an $R-$module and V_2 is an $R-$module through ϕ. Note V_1 is a simple $R-$module (let $v \in V \setminus \{0\}$ and $u \in V_1$. Then there exists $\sigma \in End_{D_1} V_1 = R$ such that $\sigma v = u$. Thus $Rv = V_1$). Similarly, V_2 is simple over $End_{D_2}(V_2) \cong R$. Recall R is simple Artinian and thus has a unique simple $R-$module. Thus $V_1 \cong V_2$. So $D_1 \cong D_1''(V_1) \cong End_R(V_1) = End_R(V_2) \cong D_2''(V_2) \cong D_2$. If $V_1 = D_1^{n_1}$, then $dim_{D_1} End_{D_1} V_1 = n_1^2$. So $n_1^2 = dim_{D_1} R = dim_{D_2} R = n_2^2$. Thus $n_1 = n_2$. $\square$

Proposition 3.122. *Suppose $A_1 \times \cdots \times A_k \cong B_1 \times \cdots \times B_\ell$ as a ring isomorphism where A_i's and B_j's are nonzero simple rings. Then $k = \ell$ and $A_i = B_j$ after reordering.*

Proof. Suppose they are isomorphic via ϕ. A_1 is an ideal of $A_1 \times \cdots \times A_k$. Thus $\phi(A_1)$ is an ideal of $B_1 \times \cdots \times B_\ell$. Since ideals of $B_1 \times \cdots \times B_\ell$ are of the form $I_1 \times \cdots \times I_\ell$ where I_i is an ideal of B_i, but $I_i = (0)$ or $I_i = B_i$, we have $\phi(A_1) = B_1 \times \cdots \times B_t \times (0) \times \cdots \times (0)$ (after reordering). If $t > 1$, then $\phi(A_1)$ has nontrivial proper ideals, a contradiction as A_1 simple. So $\phi(A_1) = B_1$. Use induction (mod out and repeat) to get $A_i = B_i$ and $k = \ell$. $\square$

Theorem 3.123. *Let R be a semisimple ring. Then there exist unique division rings $D_1, ..., D_k$ and natural numbers $n_1, ..., n_k$ such that $R \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$. Furthermore, every such R is semisimple.*

Definition 3.124. *An ideal I is **nilpotent** if $I^n = 0$ for some n and I is called **nil** if every element in I is nilpotent.*

Note. I nilpotent implies I nil, but the converse is false.

Example. $R = k[[x_1, ..., x_n]]/(x_1, x_2^2, x_3^3, ...,)$. R is quasilocal and $m = (x_1, ..., x_n)$ is nil, but not nilpotent.

In 1907, Wedderburn proved: *If R is a finite dimensional $k-$algebra (where k is a field), then there exists a largest nilpotent left ideal of R* (that is, it contains all other nilpotent ideals). 20 years later, Artin proved the same result for left Artinian rings. This largest nilpotent ideal is called the **Wedderburn radical**. Wedderburn defined a finite dimensional $k-$algebra to be semisimple if the Wedderburn radical was 0. In 1945, Jacobson extended the definition of the Wedderburn radical:

Definition 3.125. *Let R be a ring. The **Jacobson Radical** of R is $radR = J(R) = \cap m$, where the intersection runs over all maximal left ideals.*

Note. If R has DCC, then $J(R)$ is exactly the Wedderburn radical.

Lemma 3.126. *Let R be a ring and $y \in R$. TFAE*

1. *$y \in J(R)$*
2. *$1 - xy$ is left invertible for all $x \in R$.*
3. *$yM = 0$ for all simple left $R-$modules.*

Proof. $1 \Rightarrow 2$ If $1 - xy$ is not left invertible, then $R(1 - xy) \neq R$, which says $R(1 - xy) \subseteq m$ for some maximal left ideal m. Since $y \in m$, we know $xy \in m$ and thus $1 \in m$, a contradiction.

$2 \Rightarrow 3$ Suppose $yM \neq 0$. Then $yu \neq 0$ for some $u \in M$. Then $Ryu \neq 0$ which implies $Ryu = M$ as M is simple. So $u = xyu$ for some $x \in R$ which says $(1 - xy)u = 0$. By 2, $u = 0$, a contradiction.

$3 \Rightarrow 1$ Let m be a left maximal ideal. Then R/m is simple which implies $y(R/m) = 0$ and thus $y \in m$. Since m was arbitrary, $y \in J(R)$. $\square$

Definition 3.127. *For all $R-$modules M, the **annihilator** of M is defined as $Ann_R(M) = \{r \in R | rM = 0\}$.*

Recall that $Ann_R(M)$ is a two-sided ideal ($Ann_R(M) = \ker(\lambda : M \to End_R M)$).

Corollary 3.128. *$J(R) = \cap Ann_R M$, where the intersection runs over all simple left $R-$modules. In particular, $J(R)$ is an ideal.*

Proposition 3.129. *Let R be a ring and $y \in R$. TFAE*

1. *$y \in J(R)$*
2. *$1 - xyz$ is a unit for $x, z \in R$.*

Proof. **2 $\Rightarrow$1** Let $z = 1$ and use previous lemma.

1 $\Rightarrow$2 By the corollary, $yz \in J(R)$. Thus $1 - xyz$ is left invertible. Let u be its left inverse (so u is right invertible). Then $u(1-xyz) = 1$ which implies $u = 1 + uxyz$. Note $uxyz \in J(R)$ and thus $u = 1 + uxyz$ is left invertible. Thus u is a unit, which implies its left inverse is its right inverse and thus $1 - xyz$ is a unit. □

Corollary 3.130. *Let R be a ring. Then $J(R) = \cap m$, where the intersection runs over all maximal* right *ideals.*

Proof. We can prove the above results for the "right" Jacobson radical and then (2) of the proposition says they must be the same. □

Definition 3.131. *A ring is called **semiprimitive/Jacobson semisimple/J-semisimple** if $J(R) = 0$.*

Remark. Semisimple rings are semiprimitive.

Proof. Let R be semisimple and $y \in J(R)$. Now $R = I_1 \oplus \cdots \oplus I_k$ where I_j are simple. Now $yI_j = 0$ for all j which implies $yR = 0$ and in particular $y \cdot 1 = 0$. □

Examples. $\mathbb{Z}$, $F[x]$ for a field F are semiprimitive, but not semisimple.

Theorem 3.132. *Let R be a ring. TFAE*

1. *R is semisimple*

2. *R is left Artinian and $J(R) = 0$.*

Proof. Note that $1 \Rightarrow 2$ is done by the remark. For the other direction, note that by DCC, every nonzero left ideal of R contains a simple (that is, minimal nonzero) left ideal.

Claim: Every simple left ideal is a direct summand of R.

Proof: Let I be a simple left ideal (so $I \neq 0$). Since $J(R) = 0$, $I \not\subset m$ for some maximal m. Since m is maximal, this says $I + m = R$. Since I is simple, $I \cap m = 0$. Thus $I \oplus m = R$.

Let I_1 be a simple left ideal of R. Then $R = I_1 \oplus J_1$ for some ideal J_1 by the claim. If $J_1 = 0$, done. Otherwise, J_1 contains a simple ideal I_2. By the Claim, $R = I_2 \oplus A_2$ and thus $J_1 = I_2 \oplus A_2 \cap J_1$. Let $J_2 := A_2 \cap J_1$. Then $R = I_1 \oplus I_2 \oplus J_2$. Continuous in this manner. By DCC, the chain must eventually end at a simple J_n. Then R is the direct sum of simple modules and therefore semisimple. □

Proposition 3.133. *Let R be a commutative ring, x an indeterminant. Then $J(R[x]) = Nilrad(R[x]) = (Nilrad(R))[x]$.*

Proof. Note that $Nilrad(R[x]) = \cap_{p \in SpecR[x]} p \subseteq \cap_{m \in SpmR[x]} m = J(R[x])$. Let $f = a_0 + ... + a_n x^n \in J(R[x])$. Then $1 - xf = 1 - a_0 x - a_1 x^2 - ... - a_n x^{n+1}$ is a unit in $R[x]$. By a previous exercise, this implies $a_0, ..., a_n$ are nilpotent. Thus $f \in Nilrad(R[x])$. □

Corollary 3.134. *If R is reduced (that is, $NilradR = 0$), then $R[x]$ is semiprimitive. In fact, $R[x_\alpha | \alpha \in I]$ is semiprimitive.*

Lemma 3.135. *Let $I_1, ..., I_k$ be nilpotent left ideals. Then $I_1 + ... + I_k$ is nilpotent.*

Proof. By induction, it suffices to prove for $k = 2$. Let n be such that $I_1^n = I_2^n = 0$. Then we see $(I_1 + I_2)^{2n-1} = 0$ by showing $(a_1 + b_1)...(a_{2n-1} + b_{2n-1}) = 0$ for $a_i \in I_1, b_i \in I_2$. □

Corollary 3.136. *If R is a left Noetherian ring, then there exists a nilpotent left ideal containing all other nilpotent ideals (and is itself contained in $J(R)$).*

Remark. The set of nilpotents in a noncommutative ring does not necessarily form a left or right ideal.

Lemma 3.137. *If I is a nil left ideal, then $I \subseteq J(R)$.*

Proof. Let $y \in I$. It is enough to show $1 - xy$ is a unit for all $x \in R$. Now $y \in I$ implies $xy \in I$ and therefore xy is nilpotent. In general, we've seen if $a^n = 0$, then $(1-a)^{-1} = 1 + ... + a^{n-1}$. Thus $1 - xy$ is a unit and $y \in J(R)$. □

Theorem 3.138. *Let R be a left Artinian ring. Then $J(R)$ is nilpotent. Hence $J(R)$ is the largest nilpotent left or right ideal and so $J(R)$ is the Wedderburn Radical.*

Proof. Let $J = J(R)$. By DCC, the descending chain $J \supseteq J^2 \supseteq J^3 \supseteq \cdots$ stabilizes. So there exists k such that $J^k = J^{k+1} = \cdots$. Let $I = J^k \subseteq J(R)$.

Claim: $I = 0$.

Proof: Suppose not. Consider $\Lambda = \{J | J \text{ is a left ideal such that } IJ \neq 0\}$. Note $R \in \Lambda$ so $\Lambda \neq \emptyset$. So there exists a minimal element $J \in \Lambda$ by DCC. Choose $y \in J$ such that $Iy \neq 0$. Note $Iy \subseteq J$ is a left ideal and $I(Iy) = I^2y = Iy \neq 0$. Thus $Iy \in \Lambda$ and by minimality, we have $Iy = J$. Now $y \in J$ implies $y = iy$ for some $i \in I$. Thus $(1-i)y = 0$ but $i \in J(R)$ implies $1 - i$ is a unit. Thus $y = 0$, a contradiction. □

Remark. Let R be a semisimple ring and M a left $R-$module. TFAE

1. M is (left) Artinian
2. M is (left) Noetherian
3. M is finitely generated
4. $\lambda_R(M) < \infty$.

Proof. If R is semisimple, then M is. Thus $M = \oplus_{i \in \Lambda} I_i$ for I_i simple. If Λ is finite, we have a composition series. If Λ is infinite, then we can find an ascending/descending chain that does not stabilize (just add on/pluck off components). □

Theorem 3.139. *Let R be a left Artinian ring. Then R is left Noetherian (and hence $\lambda(R) < \infty$ where R is considered a left $R-$module).*

Proof. Let $J = J(R)$. Note that R/J is semisimple (as R is left Artinian, R/J is left Artinian and $J(R/J) = 0$ by the bijection of maximal ideals of R and R/J). For any i, we see J^i/J^{i+1} is an $R/J-$module as $J(J^i/J^{i+1}) = 0$. Since R is left Artinian and $J^i \subset R$, we see J^i is left Artinian and thus J^i/J^{i+1} is left Artinian as an R module and thus as an $R/J-$module. Thus $\lambda_{R/J}(J^i/J^{i+1}) < \infty$ by the remark which says J^i/J^{i+1} satisfies ACC as an $R/J-$module and thus as an $R-$module and so $\lambda_R(J^i/J^{i+1}) < \infty.(*)$

Claim: $\lambda(R/J^i) < \infty$ for all i.

Proof: For $i = 1$, we see $\lambda(R/J) < \infty$ by the $i = 0$ case of $(*)$. For $i > 1$, consider the short exact sequence $0 \to J^{i-1}/J^i \to R/J^i \to R/J^{i-1} \to 0$. Since $\lambda_R(J^{i-1}/J^i) < \infty$ by $(*)$ and $\lambda_R(R/J^{i-1}) < \infty$ by induction, we have $\lambda_R(R/J^i) < \infty$.

By the Theorem, $J^n = 0$ for some n and thus we get $\lambda_R(R) = \lambda_R(R/J^n) < \infty$. □

Proposition 3.140. *Let R be a commutative Artinian ring. Then R has only finitely many prime ideals, each of which is maximal (that is,* $\dim R = 0$*).*

Proof. Recall that $\dim R = \sup\{n | p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n, p_i \in SpecR\}$.

Claim: R has only finitely many maximal ideals

Proof: Suppose not. Let $m_1, m_2, ...,$ be an infinite list of distinct maximal ideals. Then $m_1 \supset m_1 \cap m_2 \supset m_1 \cap m_2 \cap m_3 \supset \cdots$ is a descending chain of ideals. By DCC, there exists k such that $m_1 \cap \cdots \cap m_k = m_1 \cap \cdots \cap m_k \cap m_{k+1}$. Since maximal ideals are prime, $m_{k+1} \supseteq m_i$ for some $i = 1, ..., k$. Since both are maximal, this says $m_{k+1} = m_i$, a contradiction as they are distinct.

Thus $J(R) = m_1 \cap \cdots \cap m_k$. Let $p \in SpecR$. As $J(R)$ is nilpotent, $p \supseteq J(R)$ (as the nilradical is the intersection of all primes). Then $p \supseteq m_1 \cap \cdots \cap m_k$ which implies $p \supseteq m_i$. Since m_i is maximal, $p = m_i$. Thus every prime is maximal. $\square$

Definition 3.141. *Let R be a commutative ring, I an ideal. Say $Min_R R/I = \{p \in SpecR | p \text{ is minimal over } I\}$ (Recall by p minimal over I, we mean there does not exist $q \in SpecR$ such that $p \supsetneq q \supseteq I$.)*

By the bijection between primes p of R/I and primes $I \subseteq p$ in R, these are the minimal primes of R/I. Thus $Min_R R/I \leftrightarrow Min_{R/I} R/I$. Also, note that $Min_R R/(0)$ are just the minimal primes of R.

Remarks.

1. $\sqrt{I} = \cap_{p \in Min_R R/I} p$.
2. $Min_R R/I$ is a finite set if and only if $\sqrt{I}$ is the intersection of finitely many prime ideals.

Proposition 3.142. *Let R be commutative and Noetherian, I an ideal. Then $Min_R(R/I)$ is finite.*

Proof. Let $\Gamma = \{I \subsetneq R | Min_R R/I \text{ is not finite}\}$. By way of contradiction, suppose $\Gamma \neq \emptyset$. Choose I maximal in Γ by ACC. Then, by maximality, $I = \sqrt{I}$ as they have the same minimal primes. Replacing R/I with R, we have a Noetherian ring R such that

1. $Min_R R$ is infinite
2. $Min_R R/J$ is finite for all $j \neq 0$
3. R is reduced (as $I = \sqrt{I}$).

Note also that R is not a domain as otherwise $Min_R R = (0)$. Choose $a \in R \setminus \{0\}$ such that a is a zero divisor. Consider $ann_R a \subseteq ann_R a^2 \subseteq \cdots$. By ACC, there exists n such that $ann_R a^n = ann_R a^{n+1}$. Let $b = a^n$. Then $ann_R b = ann_R b^2$.

Claim: $(b) \cap ann_R b = (0)$.

Proof: First note that since R is reduced, $b \neq 0$ and since a is a zero divisor, $ann_R b \neq 0$. Now, let $x \in (b) \cap ann_R b$. So $x = rb \in ann_R b$ which implies $xb = rb^2 = 0$. Thus $r \in ann_r b^2 = ann_r b$. So $x = rb = 0$.

Thus $(0) = \sqrt{(0)} = \sqrt{(b) \cap ann_R b} = \sqrt{(b)} \cap \sqrt{ann_R b} = (P_1 \cap \cdots \cap P_\ell) \cap (Q_1 \cap \cdots \cap Q_k)$ (since $Min_R(R/J) < \infty$, for an ideal J we have $\sqrt{J}$ is the intersection of finitely many primes). Thus 0 is the intersection of finitely many primes which implies $Min_R R$ is finite, a contradiction. $\square$

Theorem 3.143. *Let R be a commutative, Noetherian ring. Then every ideal has only finitely many minimal primes.*

Proof. Let $\Lambda = \{I : I \text{ has infinitely many min'l primes}\}$. Let $I \in \Lambda$ be maximal. Clearly, I is not prime. Choose $a, b \in R$ such that $a, b \notin I$ but $ab \in I$. Let $J_1 = (I, a) = I + aR$ and $J_2 = (I, b) = I + bR$. Then $J_i \supsetneq I$ and $J_1 J_2 \subseteq I$. Note that $Min_R R/I \subseteq Min_R R/J_1 \cup Min_R R/J_2$, which are both finite (as $J_1, J_2 \notin \Lambda$). Thus $Min_R R/I$ is finite, a contradiction. $\square$

Theorem 3.144. *If V is a vector space over a division ring, then TFAE*

1. *V is Noetherian.*
2. *V is Artinian.*
3. *$\lambda(V) < \infty$.*
4. *$\dim V < \infty$.*
5. *V is finitely generated.*

Theorem 3.145. *Let M be a semisimple left $R-$module. TFAE*

1. *M is left Noetherian.*

2. *M is left Artinian.*

3. *$\lambda_R(M) < \infty$.*

4. *M is finitely generated.*

Proof. To show any of 1,2,3 implies 4, use contrapositive. To show 4 implies any of 1,2, or 3, note that $M \cong \oplus_{i=1}^n Re_i$. Thus submodules are of the form $\oplus_{j\in J} Re_j$ which says there are finitely many submodules. $\square$

Theorem 3.146. *Let R be a commutative ring. TFAE*

1. *R is Artinian.*

2. *$\lambda(R) < \infty$.*

3. *R is Noetherian and $\dim R = 0$.*

Proof. Recall that R Artinian implies all prime ideals are maximal and so $\dim R = 0$. Thus,the only thing needed to prove is $3 \Rightarrow 2$. Let $J = J(R)$. Since R is Noetherian and every prime ideal is maximal (as $\dim R = 0$), $SpecR = \{m_1, ..., m_r\}$. So $J(R) = \cap_{i=1}^r m_i$. So $R/J = R/(m_1 \cap \cdots \cap m_r)$. Now $m_i + m_j = R$ for all $i \neq j$, thus by the Chinese Remainder Theorem, we have $R/J \cong R/m_1 \times \cdots \times R/m_r$. So R/J is semisimple. Now, since J is nilpotent as $J = m_1 \cap \cdots \cap m_r = \sqrt{(0)}$ and J is finitely generated, there exists n such that $J^n = 0$. Consider $R = J^0 \supseteq J \supseteq \cdots \supseteq J^n = (0)$. Note that J^i/J^{i+1} is a finitely generated R/J module for all i which implies it is semisimple R/J module as R/J is. Thus it is a semisimple $R-$module. (Recall an $R-$module M is simple if and only if M is a simple R/J-module). Now, R Noetherian implies J^i is finitely generated and thus $\lambda_R(J^i/J^{i+1}) < \infty$ for all i. But $\lambda(R) = \sum_{i=0}^{n-1} \lambda_r(J^i/J^{i+1}) < \infty$. $\square$

Example. $R = k[x,y,z]/(x^3, xy, y^2, xz, z^6)$ where k is a field. Note that $SpecR = \{(x,y,z)R\}$ which implies $\dim 0$. Now R is Noetherian as k is. Consider $k[x,y,z]/(x^3, xy, xz, z^6)$. Here, $(x,y,z) \subsetneq (x,z)$ which implies it has $\dim > 0$ and is thus not Artinian.

Definition 3.147. *Let R be a ring. R is called* ***von Neumann regular*** *if for all $a \in R$, there exists $x \in R$ such that $axa = a$.*

Examples.

1. Division rings are von Neumann regular

2. Products of von Neumann regular rings are von Neumann regular.

3. Example of a commutative von Neumann regular ring which is not a product of fields: Let F be a finite field and $S = \prod_{i=1}^\infty F$. Consider S as an $F-$algebra via $F \to S$ defined by $1 \mapsto (1, 1, ...)$. Let $R = F1_S + \oplus_{i=1}^\infty F = \{(a_i) \in S :$ there exists $c \in F$ such that $a_i = c$ for all but finitely many $i\}$. R is easily seen to be von Neumann regular (take $x_i = a_i^{-1}$).

   The idempotents of R fall into disjoint sets $A = \{(e_i) : e_i = 1$ for all but finitely many $i\}$ and $B = \{1 - e : e \in A\}$. Observe $e \in A$ if and only if $1 - e \in B$. If $e \in B$, then $|Re| < \infty$. Thus there do not exist idempotents $e \in R$ such that $|Re| = \infty$ and $|R(1-e)| = \infty$. But any infinite product of fields has such idempotents: $e = (1, 0, 1, 0, ...)$ and $1 - e = (0, 1, 0, 1, ...)$.

Proposition 3.148. *Let R be a ring. TFAE*

1. *R is von Neumann regular*

2. *Every finitely generated left ideal is generated by an idempotent.*

3. *Every finitely generated left ideal is a direct summand of R.*

Proof. $1 \Rightarrow 2$ Let $I = Ra_1 + ... + Ra_n$. If $n = 1$, then there exists $x \in R$ such that $a = axa$. Let $e = xa \in Ra$. Then $e^2 = xaxa = xa = e$. Clearly, $Re \subseteq Ra$. But $a = ae \in Re$. So $Ra = Re$. For $n > 1$, note that it is enough to show the $n = 2$ case. Let $I = Ra_1 + Ra_2$. By the $n = 1$ case, we have $I = Re_1 + Re_2$ where $e_1^2 = e_1$ and $e_2^2 = e_2$. Note that $I = Re_1 + Re_2(1-e_1)$ as $re_1 + se_2(1-e_1) = re_1 + se_2 - se_2e_1$. Let f be an idempotent such that $Rf = Re_2(1-e_1)$. Then $fe_1 \in Re_2(1-e_1)e_1 = 0$. So $f(f+e_1) = f$.

Claim: $I = R(f + e_1)$.
Proof: We've shown $f \in R(f + e_1)$. Thus $e_1 \in R(f + e_1)$. So $Rf + Re_1 \subseteq R(f + e_1)$. Of course, $I = Re_2(1 - e_1) + Re_1 = Rf + Re_1 \subseteq R(f + e_1)$ and since $I \supseteq R(f + e_1)$, we see $I = R(f + e_1)$.

By the $n = 1$ case, $R(f + e_1)$ is generated by an idempotent.

$2 \Rightarrow 3$ Let I be a finitely generated ideal. Then $I = Re, e^2 = e$. Then $R = Re \oplus R(1-e) = I \oplus R(1-e)$.

$3 \Rightarrow 1$ Let $a \in R$. Then $R = Ra \oplus J$. So $1 = ra + j$ such that $j \in J$. This implies $a = ara + aj$. Now, $aj = a - ara = (1 - ar)a \in Ra$ and $aj \in J$. Thus $aj = 0$ which implies $a = ara$. □

Corollary 3.149. *Let R be a ring. TFAE*

1. *R is semisimple.*
2. *R is von Neumann regular and left Noetherian.*

Example. $\prod_{i=1}^{\infty} F$ is von Neumann regular but not semisimple for a field F.

Proposition 3.150. *von Neumann regular rings are semiprimitive.*

Proof. Let $a \in J(R)$. Then there exists $x \in R$ such that $a = axa$. Then $a(1 - xa) = 0$. As $a \in J(R)$, $1 - xa$ is a unit which implies $a = 0$. □

Example. Let F be a field, V an infinite dimensional $F-$vector space. Then $End_F V$ is not Artinian and hence not semisimple. It is also not Noetherian.

Proof. Let $\{e_1, e_2, ...\}$ be part of an $F-$basis for V. Let $I_n = \{f \in End_F V | f(e_1) = ... = f(e_n) = 0\}$. These are left ideals of $End_F V$ and $I_1 \supsetneq I_2 \supsetneq I_3 \supsetneq \cdots$. Thus it is not left Artinian. □

Proposition 3.151. *Let M be a semisimple left $R-$module. Then $End_R M$ is von Neumann regular.*

Proof. Let $f \in End_R M$. Want to find $g \in End_R M$ such that $fgf = f$. Let $K = \ker f$. Then there exists N such that $M = K \oplus N$ since M is semisimple. Also, there exists K' such that $M = K' \oplus f(N)$. Note $f|_N : N \to f(N)$ is an isomorphism as $N \cap K = 0$. Define $g : M \to M$ by $g|_{K'} = 0$ and $g|_{f(N)} = (f|_N)^{-1}$. Then $g \in End_R M$ and $fgf = f$. □

Theorem 3.152 (Wedderburn 1905). *Every finite division ring is a field.*

Proof. Let D be a finite division ring. Let $F = Z(D)$, a subfield of D. Say $F = \mathbb{F}_q$ (that is, $|F| = q = p^m$, $charF = p$). Let $n = \dim_F D$ (so that $|D| = q^n$) as D is an $F-$vector space. For each $a \in D$, let $C(a) = \{d \in D | da = ad\}$, the centralizer. It is easily seen that $F \subseteq C(A)$ is a division subring of D (If d commutes with a so does d^{-1}). Let $r_a = \dim_F C(a), m_a = \dim_{C(a)} D$. Just as in the proof for fields, we can show $m_a r_a = n$. In particular, $r_a | n$. By the class equation, $|D^*| = |Z(D^*)| + \sum \frac{|D^*|}{|C(a)^*|}$, where the sum runs over the distinct conjugacy classes. Since $|Z(D^*)| = |F^*|$, we see $(*) |D^*| = q - 1 + \sum_a \frac{q^n - 1}{q^{r_a} - 1}$ where $r_a < n$ as $a \notin F$. Suppose, by way of contradiction, that $n > 1$. Recall $x^n - 1 = \prod_{d|n} \phi_d(x)$. Then for all $a \notin F$, we see $r_a | n$ and $r_a < n$. This says $x^n - 1 = (x^{r_a} - 1)\phi_n h_a(x)$ for some $h_a(x) \in \mathbb{Z}[x]$. Letting $x = q$ we see $\phi_n(q) | \frac{q^n - 1}{q^{r_a} - 1}$ in $\mathbb{Z}$ for all $a \notin F$. By $(*)$, we have $\phi_n(q) | q - 1$. Of course, $\phi_n(q) = \prod(q - w)$ where w are the primitive n^{th} roots of unity. So $|q - 1| = |q - w_1| \cdots |q - w_t||z|$. By the triangle inequality and the fact that $w \notin \mathbb{R}^+$, we see $|q - w| > |q| - |w| = q - 1$, a contradiction. □

Corollary 3.153. *Any finite subring of a division ring is a field.*

Proof. Any finite subring of a division ring is a division ring. □

Corollary 3.154. *Let D be a division ring with char$D > 0$. Then any finite subgroup of D^* is cyclic.*

Proof. Note that $\mathbb{F}_p \subseteq Z(D)$. Let $G = \{g_1, ..., g_n\}$ be a finite subgroup of D^*. Let $R = \{\sum \alpha_i g_i | \alpha_i \in \mathbb{Z}_p, g_i \in G\}$. Then R is a finite subgroup of D which implies R is a field. Now, G is a finite subgroup of R^* which implies G is cyclic. □

Example. The division ring of quaternions $D = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$. Now, Q_8 is a finite subgroup of D^* which is not cyclic.

4 Representation Theory

Exercise. Let M be a finitely generated semisimple left $R-$module. Then $M \cong n_1V_1 \oplus ... \oplus n_kV_k$ where n_i are positive integers, V_i are simple left $R-$modules with $V_i \neq V_j$ for all $i \neq j$, and $n_iV_i = \underbrace{V_i \oplus \cdots \oplus V_i}_{n_i\ times}$. Furthermore, if $M = m_1W_1 \oplus ... \oplus m_\ell W_\ell$, then $k = \ell$ and, after reordering, $n_i = m_i$ and $V_i \cong W_i$ for all i.

Proof. The first statement is the additive version of $M \cong \prod M_i^{e_i}$, which is proven in HW5#2. For uniqueness, note that these yield composition series which are unique by Jordan Hölder. □

Definition 4.1. *The n_i's in the above exercise are called the* ***multiplicity*** *of V_i in M.*

Recall. Let R be a semisimple ring, $I_1, ..., I_t$ the distinct simple left ideals of R. Then $R \cong n_1I_1 \oplus \cdots \oplus n_tI_t \cong B(I_1) \times \cdots \times B(I_1)$ as rings where $B(I_j) = \sum_{J \cong I_j} J$(see Exam 1#1). Note that $B(I_j)$ are two sided ideals of R. They are not subrings of R (as they have different identities), but $B(I_j)$ are simple Artinian rings (where I_j is the unique simple left ideal of $B(I_j)$). Furthermore, $End_{B(I_j)}I_j = End_R I_j$ (Write $r = b_1 + ... + b_t$. Then $rI_j = b_jI_j$), which is a division ring, say D_j. By Artin Wedderburn, $B(I_j) = End_{D_j}I_j \cong M_{n_j}(D_j^{op})$ where $n_j = \dim_{D_j} I_j$.

Theorem 4.2. *Let R be a semisimple finite dimensional k algebra for $k = \overline{k}$ a field. Let $R \cong n_1I_1 \oplus \cdots \oplus n_tI_t$ where I_i are simple left ideals and $I_i \ncong I_j$ for all $i \neq j$. Then*

1. $n_j = \dim_k I_j$ *for all* $j = 1, ..., t$.
2. $\dim_k R = \sum_{j=1}^t n_j^2$.

Proof. Clearly $1 \Rightarrow 2$. So its only left to prove 1. Let $m_j = \dim_k I_j$. Since $\dim_k I_j \leq \dim_k R < \infty$, we see $m_j < \infty$. Let $D_j = End_R I_j$. Note that $\dim_k D_j \leq \dim_k End_k I_j = \dim_k M_{m_j}(k) = m_j^2 < \infty$. Now, $k \subseteq Z(R)$. Hence, multiplication by elements of k are in $End_R I_j$. So $k \hookrightarrow End_R I_j$. In fact, $k \subseteq Z(D_j)$ (Let $f \in D_j$ and μ_a multiplication by a. Then $(f\mu_a)(i) = f(ai) = af(i) = (\mu_a f)(i)$). Now, $k = \overline{k}$ and $k \subseteq Z(D_j)$ which implies $k = D_j$ for all j (Choose $\alpha \in D_j$. Then $k(\alpha)/k$ is algebraic, but $k = \overline{k}$ so $k(\alpha) = k$). Now $n_jI_j \cong B(I_j) \cong End_{D_j}I_j = End_k I_j \cong M_{m_j}(k)$. Thus $n_jm_j = \dim_k n_jI_j = \dim_k M_{m_j}(k) = m_j^2$. Thus $n_j = m_j$. □

Theorem 4.3 (**Maschke's Theorem**). *Let G be a finite group and F a field. If char $F \nmid |G|$, then $F[G]$ is semisimple.*

Proof. We proved this shortly after the definition of semisimple. □

Note. The converse is true!

Proof. Let $|G| = n$ and $e = \sum_{g \in G} e_g \in F[G]$. Observe $e_ge = e = ee_g$ for all $g \in G$. Thus Fe is a two sided ideal. Furthermore, $e^2 = ee_{g_1} + ... + ee_{g_n} = ng$ as $ee_g = e$. Thus, if char $F|n$, then $(1 - xey)$ is a unit for all $x, y \in F[G]$ as $(1 - xey)(1 + xey) = 1 - (x^2)e^2(y^2) = 1 - (x^2)ne(y^2) = 1$. Thus $e \in J(F[G])$ and since e is not zero (the e_g are linearly independent), we see $F[G]$ is not semisimple. □

Proposition 4.4. *Let G be a finite group, F a field. Let $C_1, ..., C_r$ be the distinct conjugacy classes of G. Let $z_i = \sum_{g \in C_i} g \in F[G]$. Then $\{z_1, ..., z_r\}$ is an $F-$basis for $Z(F[G])$.*

Proof. For all i and for all $g \in G$, $gC_ig^{-1} = C_i$. Thus $gz_ig^{-1} = z_i$. Of course, z_i commutes with elements in F and so $z_i \in Z(F[G])$ for all i. As $C_1, ..., C_r$ are disjoint, $\{z_1, ..., z_r\}$ is linearly independent over F. Let $c \in Z(F[G])$. Say $c = \sum_{g \in G} \gamma_g g$, where $\gamma_g \in F$. For $h \in G$, we see $c = hch^{-1} = \sum_{g \in G} \gamma_g hgh^{-1} = \sum_{g \in G} \gamma_{h^{-1}gh} g$. As the g's form a basis for $F[G]$, we see $\gamma_g = \gamma_{h^{-1}gh}$ for all $h \in G$. Hence, if g_1, g_2 are in the same conjugacy class, then $\gamma_{g_1} = \gamma_{g_2}$. Thus c is a linear combination of $z_1, ..., z_r$. ☐

Theorem 4.5. *Let G be a finite group, F an algebraically closed field, char $F \nmid |G|$. Then the number of distinct simple $F[G]-$modules is equal to the number of conjugacy classes of G.*

Proof. By Maschke's Theorem, $F[G]$ is semisimple. By Artin-Wedderburn, $F[G] \cong n_1I_1 \oplus \cdots \oplus n_tI_t$ and thus $F[G] \cong M_{n_1}(D_1) \times \cdots \times M_{n_t}(D_t)$, where $D_j = End_{F[G]}(I_j)$. Moreover, t is the number of distinct simple $F[G]-$modules.

Claim: $D_i = F$.

Proof: By definition of the group ring, $F \subseteq Z(F[G])$. Thus multiplication by any element of F induces an $F[G]-$endomorphism of I_j. Thus $F \subseteq D_j$. Furthermore, $F \subseteq Z(D_j)$ as multiplication by F commutes with elements of $End_{F[G]}(I_j)$. As $F[G]$ is left Noetherian, we see I_j is a finitely generated ideal. Further, as $F[G]$ is a finitely generated $F-$vector space, we can conclude I_j is a finitely generated $F-$vector space. Since $D_j = End_{F[G]}(I_j) \subseteq End_F(I_j)$ and $End_F(I_j)$ is a finite dimensional $F-$vector space, we see D_j is a finite dimensional $F-$vector space. Thus we have $F \subseteq Z(D_j)$ where D_j is a finite dimensional $F-$vector space. Now, for $u \in D_j$ we have $F[u]$ is a domain (it is contained in D_j), is a finite dimensional $F-$vector space, and is also commutative. Thus $F[u]$ is a field. Of course, $F = \overline{F}$ and so $F = F[u]$. Since $u \in D_j$ was arbitrary, we see $F = D_j$.

Therefore, $Z(F[G]) \cong Z(M_n(F)) \times \cdots \times Z(M_{n_t}(F))$. Recall $Z(M_n(F)) = \{\lambda I_n | \lambda \in F\} \cong F$. Hence $Z(F[G]) \cong \underbrace{F \times \cdots \times F}_{t \text{ times}}$. Recall the number of conjugacy classes of G is $\dim_F Z(F[G]) = \dim_F F^t = t$. ☐

For simplicity, we will refer to the assumptions "G a finite group, $F = \overline{F}$ a field, char $F \nmid |G|$" as the **Standard Hypothesis**. Summarizing, under the standard hypothesis, let $I_1, ..., I_t$ be the distinct simple left ideals of $F[G]$. Let $n_i = \dim_F I_i$. Then

1. $\sum_{i=1}^t n_i^2 = |G|$.
2. t is the number of conjugacy classes of G.
3. n_i is the number of times I_i appears in a decomposition into simple submodules of $F[G]$ (the decomposition is called the "regular representation" of G).

Corollary 4.6. *Under the standard hypothesis, G is abelian if and only if* $\dim_F V = 1$ *for all simple $F[G]-$modules V.*

Proof. Now G is abelian if and only if t (the number of conjugacy classes) is $|G|$ which is if and only if $n_i = 1$ for all i by property (1) above. ☐

Remark. Let M be an $F[G]-$module. Then M is an $F-$vector space. In general, we want M to be finitely generated. So then $M = F^n$. So an $F[G]-$ module structure is determined by how g acts on F^n for all $g \in G$. Now $\widetilde{g} : M \to M$ defined by $m \mapsto gm$ is an $F-$endomorphism of M which implies $\widetilde{g}$ can be represented by an invertible matrix.

Example. Let $G = C_n$. Let M be a simple $F[G]-$module. By the corollary, $M = Fu$. Let $C_n =< a >$. Then $\widetilde{a} : M \to M$ defined by $u \mapsto au = \lambda u$ for some $\lambda \in F$. Of course, $a^n = 1$ and so $u = \widetilde{1}u = \widetilde{a^n}u = \widetilde{a}^n u = \lambda^n u$ which implies $\lambda^n = 1$. So λ is an n^{th} root of unity (not necessarily primitive). Thus each n^{th} root of unity determines an $F[G]-$module structure on F via $a^iu = \lambda^iu$. Since $F[G]$ has n distinct simple $F[G]-$modules, all of these simple modules given by the roots of unity are non-isomorphic.

Example. Let $G = V_4 = \{1, a, b, ab\}$ and $M = Fu$. Since a, b are order 2 elements, $\widetilde{a} : M \to M$ and $\widetilde{b} : M \to M$ are defined by $u \mapsto \pm u$. This yields 4 $F[G]-$module structures. Since G is abelian, there must be exactly 4 simple $F[G]-$modules which says these maps are distinct and determine all of the simple $F[G]-$modules.

Example. Let $G = S_3$. Then S_3 has 3 conjugacy classes which means there are 3 simple $F[G]-$modules, call them V_1, V_2, V_3 where $n_i = \dim_F V_i$. Recall that $n_1^2 + n_2^2 + n_3^2 = 6$. So WLOG, $n_1 = n_2 = 1$ and $n_3 = 2$. Then

- $V_1 = F$ with G acting trivially on F (there is always the trivial representation, which means we may always assume $n_1 = 1$)
- $V_2 = Fu$. So $1u = u, (12)u = \lambda u, (123)u = \omega u$ where $\lambda = \pm 1$ and $\omega^3 = 1$. Now, $(23)u = (13)(12)(13)u$. Say $(13)u = \delta u$ (so $\delta = \pm 1$). Then $(23)u = \delta^2 \lambda u = \lambda u$. Thus everything in the same conjugacy class of (12) maps u to the same scalar multiple of u. Also, $u = (123)(132)u = \omega^2 u$. So $\omega^2 = 1 = \omega^3$ which implies $\omega = 1$. We can similar show all 3-cycles act trivially. So V_2 is given by $(1)u = u, (12)u = -u, (123)u = u$ (where everything in the same conjugacy class act the same on u).

Definition 4.7. *Let F be a field, V an $F-$vector space. Let $GL_F(V) := End_F(V)^*$. Let G be a group. A **(linear) $F-$representation** of G is a group homomorphism $\rho : G \to GL_F(V)$ for some $F-$vector space V. The **degree** of ρ is $dim_F V$.*

Remarks.

1. Let $\rho : G \to GL_F(V)$ be a representation of G. Define a left $F[G]-$module V_ρ by $V_\rho = V$ as an $F-$vector space. For $g \in G$ and $v \in V$, define $gv := \rho(g)v$. One can check that V_ρ is an $F[G]-$module.

   Composition: $g_1(g_2 v) := \rho(g_1)(\rho(g_2)(v)) = (\rho(g_1)\rho(g_2))(v) = \rho(g_1 g_2)(v) = (g_1 g_2)v.$

2. Conversely, let M be a left $F[G]-$module. For each $g \in G$, define $\widetilde{g} : M \to M$ by $m \mapsto gm$. Then $\widetilde{g} \in End_F(M)$ (as $F \in Z(G)$ and thus F commutes with everything). Since $(\widetilde{g})^{-1} = \widetilde{g^{-1}}$, we see $\widetilde{g} \in GL_F(M)$. Define $\rho : G \to GL_F(M)$ by $g \mapsto \widetilde{g}$. It is easily checked that ρ is a group homomorphism.

This gives us a correspondence between $F-$representations of G and $F[G]-$modules.

Definition 4.8. *Let $\rho_i : G \to GL_F(V_i)$ for $i = 1, 2$ be two $F-$representations of G. We say ρ_1 is **isomorphic** (or **similar** or **equivalent**) to ρ_2 if $(V_1)_{\rho_1} \cong (V_2)_{\rho_2}$ as $F[G]-$modules. An $F-$representation $\rho : G \to GL_F(V)$ is called **irreducible** if V_ρ is a simple $F[G]-$module. A **subrepresentation** of ρ is a representation $\phi : G \to GL_F(W)$ where W is a subspace of V and $\phi(G) = \rho(G)|_W$ for all $g \in G$. Equivalently, W_ϕ is an $F[G]-$submodule of V_ρ.*

In particular, if ρ_1 is isomorphic to ρ_2 then $V_1 \cong V_2$ as $F-$vector spaces and thus have the same dimension.

Notes.

- The zero representation of G is $\rho : G \to \{1\} = End_F(0)$.
- Any degree 1 representation is irreducible (as $\deg 1 \leftrightarrow \dim V = 1$ which has no subrepresentations).

Examples.

1. The trivial representation: $\rho : G \to GL_F(F)$ where $\rho(g) = 1$ for all g. This is a degree 1 representation and F_ρ is the $F[G]-$module F where $gf = f$ for all $g \in G$.
2. The sign representation: Let $G = S_n$ and define $\rho : G \to GL_F(F) = End_F(F) = F^*$ by $\sigma \mapsto (-1)^{sgn(\sigma)}$ where $sgn(\sigma)$ is 1 if its an even permutation and -1 if its odd. This is a degree 1 representation and note ρ is nontrivial if and only if $n > 1$ and char $F \neq 2$.

3. Let $G = C_n$ and suppose $w \in F$ where w is a primitive n^{th} root of unity. Define $\rho_i : C_n \to GL_F(F) = F^*$ by $a \mapsto \omega^i$. Now $\deg \rho_i = 1$ and thus the representation is irreducible. As we saw earlier, if char $F \nmid n$, then $\rho_i \not\cong \rho_j$ for all $0 \le i \ne j \le n-1$.

4. $G = S_3$. Recall there were 2 degree 1 representations and 1 degree 2 representation. We've seen ρ_1 is the trivial representation and ρ_2 is the sign representation where $\rho_1 \nmid \rho_2$ as long as char $F \ne 2$. Now let us figure out ρ_3. Let V be a 3-dimensional $F-$vector space with basis $\{e_1, e_2, e_3\}$. Define $\rho : S_3 \to GL_F(V)$ by $\sigma \mapsto \widetilde{\sigma}$ where $\widetilde{\sigma}(e_i) = e_{\sigma(i)}$. So ρ is a degree 3 representation of S_3. Since we've seen the only irreducible representations have degree 1 or 2, this is not irreducible. So there exists a subrepresentation. Let $W = F(e_1 + e_2 + e_3) \subseteq V$. Note $\widetilde{\sigma}$ fixes W for all $\sigma \in S_3$. So W is an $F[S_3]-$ submodule of V_ρ. Consider the $F[S_3]-$module $U = V/W \cong Fe_1 \oplus Fe_2 \oplus Fe_3/F(e_1 + e_2 + e_3)$. To show this is an irreducible representation, we can show it has no proper submodules. Note that $\dim V = 2$.

   Claim: U is a simple $F[S_3]-$module if and only if char $F \ne 3$.
   Proof: Suppose char $F \ne 3$. Note that $U = F\overline{e}_1 \oplus F\overline{e}_2$ where $\overline{e}_3 = -\overline{e}_1 - \overline{e}_2$. Let $u = r\overline{e}_1 + s\overline{e}_2 \ne 0$ in U.

   Case 1: $r \ne -s$. Then $(13)u + (123)u = r\overline{e}_3 + s\overline{e}_2 + r\overline{e}_2 + s\overline{e}_3 = -(r+s)\overline{e}_1$. If $r \ne -s$, then $\overline{e}_1 \in F[S_3]u$ which implies $\overline{e}_2 = (12)e_1 \in F[S_3]u$. So $F[S_3]u = U$.

   Case 2: $r = -s \ne 0$. Then, as we can divide by r, it is enough to show for $u = \overline{e}_1 - \overline{e}_2$. Note $(23)u + (123)u = \overline{e}_1 - \overline{e}_3 + \overline{e}_2 - \overline{e}_3 = 3(\overline{e}_1 + \overline{e}_2)$. Since char $F \ne 3$, this says $\overline{e}_1 + \overline{e}_2 \in F[S_3](\overline{e}_1 - \overline{e}_2)$. If char $F \ne 2$, this says $\overline{e}_1, \overline{e}_2 \in F[S_3](\overline{e}_1 - \overline{e}_2)$. Now, suppose char $F = 2$. Then $\overline{e}_1 - \overline{e}_2 = \overline{e}_1 + \overline{e}_2$ and $(13)(\overline{e}_1 + \overline{e}_2) = \overline{e}_1 + 2\overline{e}_2 = \overline{e}_1$. Thus $\overline{e}_1, \overline{e}_2 \in F[S_3](\overline{e}_1 - \overline{e}_2)$ and therefore $F[S_3]u = U$.

   We have just shown that $F[S_3]u = U$ for all $u \in U$. Thus U is simple. The char $F = 3$ case is left as an exercise.

   Thus $\rho_3 : G \to GL_F(F^2) = GL_F(F)$ defined by $(12) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $(123) \mapsto \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ is the last representation.

Definition 4.9. *Let $\rho_1, \rho_2 : G \to GL_F(V_i)$ for $i = 1, 2$ be two $F-$representations of G. The* ***direct sum*** *$\rho_1 \oplus \rho_2$ is $\rho_1 \oplus \rho_2 : F \to GL_F(V_1 \oplus V_2)$ defined by $g \mapsto \rho_1(g) \oplus \rho_2(g)$.*

Note. $(V_1 \oplus V_2)_{\rho_1 \oplus \rho_2} \cong (V_1)_{\rho_1} \oplus (V_2)_{\rho_2}$ as $F[G]-$modules.

Remark. If $|G| < \infty$ and char $F \nmid |G|$, then every $F-$representation of G is a direct sum of irreducible representations.

Example. The regular representation. Let G be a group, F a field, and V an $F-$vector space of dim $|G|$. Let $\{e_g | g \in G\}$ be a basis for V. For $h \in G$, define the $F-$linear map $\widetilde{h} : V \to V$ by $e_g \mapsto e_{hg}$. Clearly $\widetilde{h_1 h_2} = \widetilde{h_1}\widetilde{h_2}$ and $\widetilde{h^{-1}} = \widetilde{h}^{-1}$. So $\widetilde{h} \in GL_F(V)$ and $\rho : G \to GL_F(V)$ defined by $h \mapsto \widetilde{h}$ is an $F-$representation of G, called the **regular representation** of G. Note that $V_\rho \cong F[G]$. If $F[G]$ is semisimple, then every $F[G]-$module appears in any decompositions of $F[G]$ into simple left $F[G]$ modules. Thus every irreducible F representation of G appears in any decomposition of the regular representation into irreducible representations.

Recall. If $F = \overline{F}$ and char $F \nmid |G|$, then $F[G] \cong n_1 I_2 \oplus \cdots \oplus n_t I_t$ where $I_1, ..., I_t$ are the distinct simple left ideals (up to isomorphism) and $n_i = \dim_F I_i$. Let ρ be the regular representation and $\rho_1, ..., \rho_t$ the distinct irreducible $F-$representations of G corresponding to I_i. Then $\rho = n_1\rho_1 \oplus \cdots \oplus n_t\rho_t$ where $n_i = \deg \rho_i$.

4.1 Characters

Let k be a field and R a finite dimensional $k-$algebra. Let M be a finitely generated left $R-$module. So $\dim_k M < \infty$. Let $r \in R$ and define $\widetilde{r}_M : M \to M$ by $m \mapsto rm$. Since $k \subseteq Z(R)$, we see $\widetilde{r}_M \in End_k(M)$. So $tr(\widetilde{r}_M) \in F$ is defined. Define the **character** χ_M associated with M by $\chi_M : R \to k$ where $r \mapsto \mathrm{tr}(\widetilde{r}_M)$.

Remarks.

1. Let $B = \{u_1, ..., u_n\}$ be a $k-$basis for R. Let $r \in R$. Then $r = \sum a_i u_i$ for $a_i \in k$. It is easy to see $\widetilde{r} = \sum a_i \widetilde{u}_{i,M}$ which implies $\operatorname{tr}(\widetilde{r}_M) = \sum a_i \operatorname{tr}(\widetilde{u}_{i,M})$. So $\chi_M(r) = \sum_{i=1}^n a_i \chi_M(u_i)$. So χ_M is determined by $\chi_M|_B$.

2. If $R = F[G]$ and M is a left $R-$module, since G is an $F-$basis for R we often consider χ_M to be a function from $G \to F$ as opposed to $R \to F$.

   Note. If $\rho : G \to GL_F(V)$ is an $F-$representation of G, we define the character χ_ρ associated to ρ by $\chi_\rho := \chi_{V_\rho} : G \to F$. Explicitly, $\chi_\rho(g) = \operatorname{tr}(\rho(g))$.

3. If char $k = 0$, then $\chi(1) = \dim_k M$. If $\rho : G \to GL_F(V)$, then $\chi_\rho(1) = \dim_F V = \deg \rho$.

Proposition 4.10. *Let R be a finite dimensional $k-$algebra. Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence of finitely generated left $R-$modules. Then $\chi_M = \chi_L + \chi_N$.*

Proof. Let $r \in R$ and consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "\widetilde{r}_L"'] & M \arrow[r, "g"] \arrow[d, "\widetilde{r}_M"'] & N \arrow[r] \arrow[d, "\widetilde{r}_N"'] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0
\end{tikzcd}$$

Claim: This is a diagram of $k-$linear maps.

Proof: Let $\ell \in L$. Then $f\widetilde{r}_L(\ell) = f(r\ell) = rf(\ell) = \widetilde{r}_M(f(\ell))$. Similarly for the other square.

Since the rows split as $k-$vector spaces, we see $M \cong L \oplus N$. So we have

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\widetilde{r}_M"'] & L \oplus N \arrow[d, "\widetilde{r}_L \oplus \widetilde{r}_N"'] \\
M \arrow[r, "f"] & L \oplus N
\end{tikzcd}$$

and $\widetilde{r}_L \oplus \widetilde{r}_N$ corresponds to $\begin{bmatrix} \widetilde{r}_L & \\ & \widetilde{r}_N \end{bmatrix}$. This says $\operatorname{tr}(\widetilde{r}_M) = \operatorname{tr}(\widetilde{r}_L) + \operatorname{tr}(\widetilde{r}_N)$ and thus $\chi_M(r) = \chi_L(r) + \chi_N(r)$. ☐

Corollary 4.11. *1. If $N \subseteq M$ are finitely generated $R-$modules, then $\chi_M = \chi_N + \chi_{M/N}$.*

2. $\chi_{M \oplus N} = \chi_M + \chi_N$.

3. If $M \cong N$ as $R-$modules, then $\chi_M = \chi_N$.

Examples. The converse of 3 is not true in general.

1. Let k be a field, $R = k[x]/(x^2) \cong k \oplus kx$ as $k-$vector spaces. Let $M = R/(x) \oplus R/(x) \cong k \oplus k$ as $k-$vector spaces. Then $M \not\cong R$ since $xM = 0$ and $xR = kx \neq 0$.

   Claim: $\chi_M = \chi_R$.

   Proof: It is enough to show they agree on the basis $\{1, x\}$. Of course, $\chi_M(1) = \dim M = 2 = \chi_R(1)$. Also, $\chi_M(x) = 0$ as multiplication by x is the 0 map and since $\widetilde{x}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ we see $\chi_R(x) = \operatorname{tr}(\widetilde{x}_R) = 0$.

2. Let $R = \mathbb{F}_2, M = \mathbb{F}_2 \oplus \mathbb{F}_2$. Then $\chi_M(1) = 2 = 0$ but obviously $M \not\cong 0$.

Exercise. If R is semisimple and finitely generated over k and M is simple, then $\chi_M \neq 0$.

Proof. Note that M is isomorphic to a simple left ideal of R, say I_i, where $R = n_1 I_1 + ... + n_t I_t$. Then $\chi_M = \chi_i$. Of course, $\chi_i(1) = \dim_k I_i \neq 0$. ☐

Theorem 4.12. *Let R be a finite dimensional $k-$algebra (for a field k), char $k = 0$. Let M, N be finitely generated semisimple left $R-$modules. Then $\chi_M = \chi_N$ if and only if $M \cong N$ as $R-$modules.*

Proof. We've already shown $\Leftarrow$, thus we need only show $\Rightarrow$. Let $J = J(R)$. Since M, N are sums of simple modules, $JM = JN = 0$. Thus M, N are left $R/J-$modules. Since R/J is semisimple (R is Artinian and $J(R/J) = 0$), we know $R/J \cong R_1 \times \cdots \times R_t$, where R_i is Artinian, simple with left simple modules I_i. Let $I_1, ..., I_t$ be the distinct simple $R/J-$modules. Then $I_1, ..., I_t$ are the distinct simple left $R-$modules. (Any simple R module is a simple R/J module and vice versa). Thus

$$M \cong m_1 I_1 \oplus \cdots \oplus m_t I_t \text{ and } N \cong n_1 I_1 \oplus \cdots \oplus n_t I_t$$

for $m_i, n_i \geq 0$. Thus it is enough to show $m_i = n_i$ for all $i = 1, ..., t$. Let $e_i \in R$ be such that $e_i + J$ is the identity of R_i. Then $e_i I_j = 0$ for all $i \neq j$ and $e_i|_{I_i} = 1|_{I_i}$. Consider $(\widetilde{e}_i)_M : M \to M$ defined by $m \mapsto e_i m$, a $k-$endomorphism. Choose a basis for M by taking the union of bases for $m_j I_j$ for $j = 1, ..., t$. Then $(\widetilde{e}_i)_M = \begin{bmatrix} 0 &&&&&& \\ & \ddots &&&&& \\ && 0 &&&& \\ &&& 1_{m_i \dim I_i} &&& \\ &&&& 0 && \\ &&&&& \ddots & \\ &&&&&& 0 \end{bmatrix}$, where $1_{m_i \dim I_i}$ is the identity matrix of size $m_i \dim I_i$. So $\chi_M(e_i) = \mathrm{tr}((\widetilde{e}_i)_M) = m_i \dim I_i$. Similarly, $\chi_N(e_i) = \mathrm{tr}((\widetilde{e}_i)_N) = n_i \dim I_i$. As char $k = 0$, we have $m_i = n_i$. $\square$

Corollary 4.13. *Let R be a semisimple finite dimensional $k-$algebra with char $k = 0$. Let $I_1, ..., I_t$ be the distinct left simple ideals. Let $\chi_i = \chi_{I_i}$ for $i = 1, ..., t$. Then $\chi_1, ..., \chi_t$ are distinct irreducible $k-$characters of R. Given any finitely generated left $R-$module M, there exist unique $n_1, ..., n_t \in \mathbb{Z}$ such that $\chi_M = n_1\chi_1 + ... + n_t\chi_t$ (since characters are additive). If $n_i > 0$, say χ_i is an* ***irreducible constituent*** *of χ_M.*

Example. Let R be as above and $k = \overline{k}$. For $R = n_1 I_1 \oplus \cdots \oplus n_t I_t$, we know $n_i = \dim_k I_i = \chi_i(1)$. Thus $\chi_R = \chi_1(1)\chi_1 + \cdots + \chi_t(1)\chi_t$.

Proposition 4.14. *Let G be a group, F a field. Let χ be an $F-$character of G. Then for all $g, x \in G$, we have $\chi(g) = \chi(xgx^{-1})$, that is, χ is constant on conjugacy classes.*

Proof. Let $\rho : G \to GL_F(V)$ be an $F-$representation of G with character χ. Then $\chi(xgx^{-1}) = \mathrm{tr}(\rho(xgx^{-1})) = tr(\rho(x)\rho(g)\rho(x)^{-1}) = \mathrm{tr}(\rho(g)) = \chi(g)$. $\square$

Examples. Let $k = \overline{k}$ and char $k = 0$.

1. $G = C_n =< a >$. Since G is abelian, all representations have deg 1. Then irreducible $k-$representations are $\rho_i = C_n \to k^*$ defined by $a \mapsto \omega^i$ for $i = 0, ..., n-1$ where ω is a fixed primitive n^{th} root of unity. The character χ_i associated to ρ_i is $\chi_i(a^j) = \omega^{ij}$. Thus we can construct the **character table**:

	1	a	a^2
χ_0	1	1	1
χ_1	1	ω	ω^2
χ_2	1	ω^2	ω

where the top row consists of representations for each conjugacy class and the first column consists of the irreducible characters.

2. $G = V_4 = \{1, a, b, ab\}$. Recall the representations are $\rho_{ij} : G \to k^*$ defined by $a^i \mapsto (-1)^i$ and $b^j \mapsto (-1)^j$ for $i, j \in \{0, 1\}$.

		1	a	b	ab
$(\rho_{00} \leftrightarrow\ )$	χ_0	1	1	1	1
$(\rho_{01} \leftrightarrow\ )$	χ_1	1	-1	1	-1
$(\rho_{10} \leftrightarrow\ )$	χ_2	1	1	-1	-1
$(\rho_{11} \leftrightarrow\ )$	χ_3	1	-1	-1	1

3. $G = S_3$. Recall that there were two degree 1 representations: the trivial representation ρ_0 and the signed representation ρ_1 and one degree 2 representation: $\rho_2 : S_3 \to GL_2(k)$ defined by $(12) \mapsto \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$ and $\begin{bmatrix} & -1 \\ 1 & -1 \end{bmatrix}$. Thus the character table is given by:

	(1)	(12)	(123)	
χ_0	1	1	1	
χ_1	1	-1	1	
χ_2	2	0	-1	$\leftarrow$ Remember we just want the trace of the matrices

$\leftarrow$ as there only 3 conjugacy classes

Note that the first column is always just the degree of the representation.

4. $G = Q_8 =< a, b | a^4 = b^4 = 1, a^2 = b^2, ab = b^3 a >$. Note that any normal subgroup of Q_8 induces the homomorphism $G \to G/H \to GL_F(V)$. Let $Z = Z(G) = \{\pm 1\}$. Then $G/Z \cong V_4$. Recall that this has 4 degree 1 representations, namely $\rho_{ij} : Q_8 \to k^*$ by $a \mapsto (-1)^i$ and $b \mapsto (-1)^j$ where $i, j \in \{0, 1\}$. As G is not abelian, we know there must exist a representation of degree ≥ 2. Furthermore, as $\sum_{i=1}^t n_i^2 = |G| = 8$, we see there can exist only one more representation, which must have degree 2. Note that $\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$ and $\begin{bmatrix} & \omega \\ \omega & \end{bmatrix}$ (where ω is a primitive 4^{th} root of unity (for $k = \mathbb{C}, \omega = \pm i$)) satisfy the relations of G. So define $\rho_4 : Q_8 \to GL_2(k)$ by $a \mapsto \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$ and $b \mapsto \begin{bmatrix} & \omega \\ \omega & \end{bmatrix}$ Note that then $c \mapsto \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} & \omega \\ \omega & \end{bmatrix} = \begin{bmatrix} \omega & \\ & -\omega \end{bmatrix}$. Thus our character table is given by

	1	-1	a	b	c
χ_0	1	1	1	1	1
χ_1	1	1	1	-1	-1
χ_2	1	1	-1	1	-1
χ_3	1	1	-1	-1	1
χ_4	2	-2	0	0	0

For -1, just note that since $-1 \in Z(G)$, -1 is $1 \in G/Z(G)$. So $\chi_i(-1) = \chi_i(1)$ for $i = 0, ..., 3$. For χ_4, we know $-1 \mapsto -I$, which has trace -2.

Let G be a finite group, $k = \bar{k}$, with char $k \nmid |G|$. Recall $k[G] \cong B_1 \times \cdots \times B_t$ with B_i simple and Artinian. Let $e_i \in B_i$ be the identity element. Then $\{e_1, ..., e_t\}$ are uniquely determined by $k[G]$. Recall $Z(k[G]) = Z(B_1) \times \cdots \times Z(B_t)$ where $Z(B_i) = Z(M_n(k)) = \{\lambda I_n | \lambda \in k\} = ke_i$. Thus $Z(k[G]) = ke_1 \times \cdots \times ke_t$. On the other hand, we know $Z(k[G]) = kz_1 \oplus \cdots \oplus kz_t$ where $z_i = \sum_{g \in C_i} g$ where $C_1, ..., C_t$ are the distinct conjugacy classes of G.

Let $\chi_1, ..., \chi_t$ be the irreducible characters of G associated to the simple left ideals $I_1, ..., I_t$, respectively and where $B_i \cong n_i I_i$ (as $k-$vector spaces). Recall $\dim_k I_i = n_i$. Let $m_i = |C_i|$ for $i = 1, ..., t$.

Theorem 4.15. *With the above notation,*

1. $e_i = \frac{n_i}{|G|} \sum_{g \in G} \chi_i(g^{-1}) g$ *for* $i = 1, ..., t$.

2. $z_i = m_i \sum_{j=1}^{t} \frac{\chi_i(g)e_j}{n_j}$ *for* $g \in C_i$.

In particular, (1) says char $k \nmid n_i$.

Proof. Let ϕ be the character associated to the regular representation of G. Recall

(a) $\phi = n_1\chi_1 + ... + n_t\chi_t$.

(b) $\phi(1) = |G|$.

(c) $\phi(g) = 0$ for all $g \neq 1$ (as for $V = \{e_h | e_h \in G\}$, $\rho : G \to GL_k(V)$ defined by $g \cdot e_h = e_{gh} \neq e_h$ if $g \neq 1$. Thus $\text{tr}(\rho(g)) = 0$ if $g \neq 1$.)

1. Let $e_i = \sum_{g\in G} a_{ig} g$ for $a_{ig} \in k$. Want to show $a_{ig} = \frac{n_i\chi_i(g^{-1})}{|G|}$. Let $h \in G$ and consider $\phi(e_i h^{-1}) = \sum_{g\in G} a_{ig}\phi(gh^{-1}) = a_{ih}|G|$ by (b) and (c). By (a), $\phi(e_i h^{-1}) = \sum_{j=1}^{t} n_j\chi_j(e_i h^{-1})$, where $\chi_j(e_i h^{-1}) = \text{tr}(\widetilde{e_i h^{-1}}_{I_j}) = \text{tr}(\delta_{ij}\widetilde{h^{-1}}_{I_j}) = \delta_{ij}\chi_j(h^{-1})$ as $\widetilde{e_i}$ annihilates I_j but is the identity on I_i. Thus $a_{ih}|G| = \phi(e_i h^{-1}) = n_i\chi_i(h^{-1})$. Thus $a_{ih} = \frac{n_i\chi_i(h^{-1})}{|G|}$.

2. Let $g \in C_i, z_i = \sum_{j=1}^{t} b_{gj} e_j$. Then $\chi_j(z_i) = m_i\chi_j(g)$ as $z_i = \sum_{h\in C_i} h$ and $\chi_j(\sum_{\ell=1}^{t} b_{g\ell}e_\ell) = \sum_{\ell=1}^{t} b_{g\ell}\chi_j(e_\ell) = b_{gj}\chi_j(e_j) = b_{gj}\text{tr}(id_{I_j}) = b_{gj}n_j$. Thus $b_{gj} = \frac{m_i\chi_j(g)}{n_j}$ which implies $z_i = m_i\sum_{j=1}^{t}\frac{\chi_j(g)e_j}{n_j}$. [It should be noted here that we mean $\overline{n_j} \in k$, however, we will just say n_j for simplicity] ☐

Corollary 4.16. *With the above notation ($|G| < \infty$, char $k \nmid |G|, k = \overline{k}$), let $\chi_1, ..., \chi_t$ be the irreducible characters of G. Then*

1. *For i, j we have $\sum_{g\in G} \chi_i(g)\chi_j(g^{-1}) = \delta_{ij}|G|$.*

2. *For all $g, h \in G$, we have $\sum_{i=1}^{t} \chi_i(g)\chi_i(h^{-1}) = \delta|C_G(g)|$, where $C_G(g) := \{x \in G | xg = gx\}$ and $\delta = 1$ if g, h are in the same conjugacy class and $\delta = 0$ otherwise.*

3. *If $g \neq 1$, then $\sum_{i=1}^{t} \chi_i(1)\chi_i(g) = 0$.*

Proof.

1. By the Theorem, $e_i = \frac{n_i}{|G|}\sum \chi_i(g^{-1})g$. Apply χ_j to both sides. Then $\chi_j(e_i) = \delta_{ij}n_i$. So $\delta_{ij}n_i = \frac{n_i}{|G|}\sum\chi_i(g^{-1})\chi_j(g)$. Thus $\delta_{ij}|G| = \sum\chi_i(g^{-1})\chi_j(g)$.

2. Plug 1 of the theorem into 2 of the theorem to get for $g \in C_i$, $z_i = \frac{m_i}{|G|}\sum_{h\in G}(\sum_{j=1}^{t}\chi_j(g)\chi_j(h^{-1}))h$. Comparing coefficients, $\frac{m_i}{|G|}\sum_{j=1}^{t}\chi_j(g)\chi_j(h^{-1}) = 1$ if and only if $h \in C_i$ (and 0 otherwise). Now, $m_i = |C_i| = \frac{|G|}{|C_G(g)|}$.

3. Follows from 2 be letting $h = 1$. ☐

Definition 4.17. *A k-**class function** on G is a function $\phi : G \to k$ which is constant on conjugacy classes, that is, $\phi(g) = \phi(xgx^{-1})$ for all $x, g \in G$. Let $F_k(G)$ be the set of $k-$class functions of G.*

Remark. $F_k(G)$ is a $k-$vector space in a natural way

$$(\phi + \psi)(g) = \phi(g) + \psi(g) \text{ and } (a\phi)(g) = a\phi(g) \text{ for all } g \in G, a \in k.$$

The $\dim_k F_k(G)$ is the number of conjugacy classes. We can define an inner product (which is bilinear) on $F_k(G)$ via

$$< \phi, \psi >= \frac{1}{|G|}\sum_{g\in G}\phi(g^{-1})\psi(g).$$

Proposition 4.18. *With the above notation, the set of irreducible characters on G, $\{\chi_1, ..., \chi_t\}$, is an orthonormal basis for $F_k(G)$.*

Proof. We've shown $< \chi_i, \chi_k >= \delta_{i,j}$. Since $\dim_k F_k(G) = t$, we see that it is a basis. ☐

Examples.

1. $G = A_4$ (where char $k \neq 2, 3$). First, we need to find the conjugacy classes. Let $H = \{(1), (12)(34), (13)(24), (14)(23)\}$. Then $H \lhd A_4$ and the conjugacy classes are $\{1\}, H \setminus \{1\}, (123)H, (132)H$. Thus there are 4 irreducible characters. Note that $G/H \cong C_3$, which gives us 3 degree 1 representations. Since $\sum n_i^2 = |G|$, we see there is only one other, which has degree 3. Now, we can fill out the character table:

	(1)	(12)(34)	(123)	(132)	
χ_1	1	1	1	1	
χ_2	1	1	ω	ω^2	
χ_3	1	1	ω^2	ω	
χ_4	3	-1	0	0	$\leftarrow$ for this row, recall $\chi_4(1) = \deg \rho_4$ and $0 = \sum_{i=1}^{t} \chi_i(g)\chi_i(1)$.

   Since $(12)(34) \in H$, it acts like (1) on χ_1, χ_2, χ_3.

   What is a representation with character χ_4? Let $V = ke_1 \oplus ke_2 \oplus ke_3 \oplus ke_4 / k(e_1 + e_2 + e_3 + e_4) \cong k\overline{e}_1 \oplus k\overline{e}_2 \oplus k\overline{e}_3$, where $\overline{e}_4 = -\overline{e}_1 - \overline{e}_2 - \overline{e}_3$. Now, make V into a $k[A_4]-$module by defining $\sigma\overline{e}_i = \overline{e}_{\sigma(i)}$ for all $\sigma \in A_4, i = 1, 2, 3$. This is well-defined as $\sigma e_i = e_{\sigma(i)}$ is well-defined and σ fixes $e_1 + e_2 + e_3 + e_4$. Thus V gives rise to a degree 3 representation of A_4. Let χ be the associated character.

   Claim: $\chi = \chi_4$ (that is, χ is irreducible)

   Proof: If $\chi \neq \chi_4$, then it is reducible. Thus it is a sum of irreducible characters, which implies $\chi = \chi_1 + \chi_2 + \chi_3$. Then, $\chi((12)(34)) = \chi_1 + \chi_2 + \chi_3 = 3$. However, $\chi((12)(34)) = \mathrm{tr}(\rho((12)(34))) = \mathrm{tr} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} = -1$.

   This shows that, since χ is irreducible, V is simple.

2. $G = S_4$. Here, the conjugacy classes are $(1), (12), (12)(34), (123), (1234)$. Note that H above is still normal in S_4. Here, $|S_4/H| = 6$. Since every element of S_4 has order ≤ 4, we see $S_4/H \cong S_3$.

	(1)	(12)	(12)(34)	(123)	(1234)	
χ_1	1	1	1	1	1	
χ_2	1	-1	1	1	-1	
χ_3	2	0	2	-1	0	
χ_4	3	1	-1	0	-1	$\leftarrow V$, the $k[A_4]-$module above is also a simple $k[S_4]-$module
χ_5	3	-1	-1	0	1	$\leftarrow$ Use the fact that $\chi_{i=1}^{t}\chi_i(1)\chi_i(g) = 0$.

   For χ_1, χ_2, χ_3, note that $(12)(34)$ maps to 1 in S_4/H and (1234) maps to a transposition in S_4/H.

By HW6# 6, if $k = \mathbb{C}$, we see $\frac{1}{|G|}\sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = \delta_{ij}$.

Corollary 4.19. *Suppose $k = \mathbb{C}$. With the above notation, $\frac{1}{|G|}\sum_{g \in G} |\chi_i(g)|^2 = 1$ and $\sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = 0$ for $i \neq j$.*

Let $g_i \in C_i$ for $i \in [t]$ and $m_i = |C_i|$. Then $\sum_{i=1}^{t} m_i \chi_j(g_i)\overline{\chi_\ell(g_i)} = \delta_{j\ell}|G|$.

Facts. Let G be a finite group, $\rho : G \to GL_{\mathbb{C}}V$ a finite dimensional representation with associated character χ. Say $\deg \rho = n$. Then $GL_{\mathbb{C}}V = GL_n(\mathbb{C})$.

1. For all $g \in G, |\chi(g)| \leq \chi(1)$.

2. $\chi(g) = \chi(1)$ if and only if $g \in \ker \rho$.

Proof. 1. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of $\rho(g)$. Then $\lambda_1, ..., \lambda_n$ are roots of unity. So $|\chi(g)| = |\lambda_1 + ... + \lambda_n| \leq |\lambda_1| + ... + |\lambda_n| = n \cdot 1 = \chi(1)$.

2. The backward direction is clear. So assume $\chi(g) = \chi(1)$. Then, by (1), $\lambda_1 + ... + \lambda_n = n$. By Cauchy Schwarz, $|\lambda_1 + ... + \lambda_n| = |\lambda_1| + ... + |\lambda_n|$ if and only if $\lambda_i = \lambda_j$ for all i, j. Then $n\lambda_1 = n$ which implies $\lambda_1 = 1$. Thus $\lambda_i = 1$ for all i. Since the minimal polynomial divides $x^n - 1$, $\rho(g)$ is diagonalizable and since it is similar to the identity matrix, it is in fact the identity matrix. □

Examples.

1. Recall $(12)(34) \in H \lhd A_4$ and $\chi_i((12)(34)) = \chi_i(1)$ for $i = 1, 2, 3$.

2. Let $F = \mathbb{R}, G = C_4 =< g >$. Then

$$\mathbb{R}[G] = \mathbb{R} \cdot 1 \oplus \mathbb{R}g \oplus \mathbb{R}g^2 \oplus \mathbb{R}g^3 = \mathbb{R}[x]/(x^4 - 1) \cong \mathbb{R}[x]/(x - 1) \oplus \mathbb{R}[x]/(x + 1) \oplus \mathbb{R}[x]/(x^2 + 1).$$

Let $\rho_1 : G \to \mathbb{R}^*$ be defined by $g \mapsto 1$, $\rho_2 : G \to \mathbb{R}^*$ be defined by $g \mapsto -1$, and $\rho_3 : G \to \mathbb{R}^*$ be defined by $g \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then ρ_i is a representation of the i^{th} summand on the right hand side of the above equation.

Now, let $\phi : G \to GL_4(\mathbb{R})$ be defined by $g \mapsto \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{bmatrix}$. Then $\phi = \rho_1 \oplus \rho_2 \oplus \rho_3$. Also, if we define $\rho : C_4 \to GL_4(\mathbb{R})$ by $g \mapsto \begin{bmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{bmatrix}$. Then $\rho \cong \phi$.

4.2 Integral Extensions

Our goal is to work towards proving Burnside's Theorem, which says *Every group of order $p^a q^b$ (for p, q primes) is solvable.*

Definition 4.20. *Let $R \subset S$ be commutative rings, $u \in S$. Then u is **integral** over R if $f(u) = 0$ for some monic polynomial $f(x) \in R[x]$. We say S is integral over R if every element of S is integral over R.*

Remark. If E/F is a field extension, then $\alpha \in E$ is integral over F if and only if α is algebraic over F.

Proposition 4.21. *Let $R \subset S$ be commutative rings, $u \in S$. TFAE*

1. *u is integral over R.*

2. *$R[u]$ is a finitely generated $R-$module*

3. *There exists a faithful $R[u]-$submodule M of S which is finitely generated as an $R-$module. (Recall faithful means $Ann_{R[u]}M = 0$)*

Note. The above are also equivalent to "There exists a finitely generated $R-$submodule M of S such that $1 \in M$ and $uM \subseteq M$."

Proof. (1) ⇒ (2) There exists an equation of the form $u^n + r_1 u^{n-1} + ... + r_n = 0$, $r_i \in R$. Then $u^{n+k} \in R \cdot 1 + ... + Ru^{n-1}$ for all $k \geq 0$. So $R[u] = R \cdot 1 + ... + Ru^{n-1}$, a finitely generated $R-$module.

(2) ⇒ (3) Let $M = R[u]$. M is faithful as $1 \in M$. Of course $uR[u] \subseteq R[u]$.

(3) $\Rightarrow$ (1) "determinant trick." Recall: *Let R be a commutative ring, $A \in M_n(R)$. Define the adjoint of A by $adjA = (b_{ij})_{n\times n}$ where $b_{ij} = (-1)^{i+j}\det(A_{ji})$ where A_{ji} is the $(n-1)\times(n-1)$ matrix obtained by deleting the j^{th} row and i^{th} column.* Also, $A\cdot(adjA) = (detA)I_n = (adjA)\cdot A$ (p 511). Let $M = Rx_1 + ... + Rx_n \subseteq S, Ann_R M = 0, uM \subseteq M$. For $j, i = 1, ..., n$ there exists $r_{ij} \in R$ such that $ux_i = \sum r_{ij}x_j$, that is, $uI_n \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ for $A \in M_n(R)$. Then $(uI_n - A)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0$. Say $B := uI_n - A$. Multiply both sides by $adjB$. Then $0 = adj(B)B\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (detB)I\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ which implies $(detB)M = 0$. But M is faithful and $\det B \in R[u]$. Thus $detB = 0$. One can show $detB$ has the form $u^n + t_1u^{n-1} + ... + t_n$ for $t_i \in R$. Thus $u^n + t_1u^{n-1} + ... + t_n = 0$ which implies u is integral over R. □

Corollary 4.22. *S/R as above, $u \in S$. Then TFAE*

1. *u is integral over R.*
2. *$R[u]$ is a finitely generated $R-$module.*
3. *$R[u]$ is integral over R.*

Proof. Let $\beta \in R[u]$ and $M = R[u]$ from theorem. Then $\beta M \subseteq M$ (that is, M is an $R[\beta]-$module), $1 \in M$. By (2), M is a finitely generated $R-$module. By (3) of the theorem, β is integral. □

Exercise. S/R as above and $u_1, ..., u_n \in S$. Then TFAE

1. $u_1, ..., u_n$ are integral over R.
2. $R[u_1, ..., u_n]$ is a finitely generated $R-$module.
3. $R[u_1, ..., u_n]$ is integral over R.

Corollary 4.23. *$R \subseteq S$ as above.*

1. *If S is finitely generated as an $R-$module, then S is integral over R.*
2. *If S is integral over R, then S is finitely generated over R as an algebra if and only if S is finitely generated over R as a module.*

Examples.

1. Let K be a field, $R = k[t^2, t^3], S = k[t]$. Then t is integral over R (it is a root of $x^2 - t^2 \in R[t]$) and so S is integral over R. Note that S is contained in the field of fractions. Also, S is integral over R and finitely generated as an $R-$algebra. Thus S is finitely generated as as $R-$module ($S = R + Rt$).
2. Let $S = \mathbb{Z}[\frac{3+\sqrt{5}}{2}]$ and $R = \mathbb{Z}[\sqrt{5}]$. Note $S \subseteq Q(R) = \mathbb{Q}(\sqrt{5})$. Note $\frac{3+\sqrt{5}}{2}$ is integral over R as it is a root of $x^2 - 3x + 1 \in \mathbb{Z}[x]$. Thus S is integral over R and finitely generated as an $R-$module.

Corollary 4.24. *Let $R \subseteq S$ as above. Let $T = \{\alpha \in S | \alpha$ is integral over $R\}$. Then T is a subring of S which is integral over R. T is called the* ***integral closure*** *of R in S. If $T = R$, then T is said to be* ***integrally closed*** *in S.*

Proof. Follows from above exercise as $\alpha\beta, \alpha \pm \beta \in R[\alpha, \beta]$ which is integral over R when α, β are integral. □

Example. Let $A = \{\alpha \in \mathbb{C} | \alpha \text{ is integral over } \mathbb{Z}\}$. Then A is a ring. The elements of A are called **algebraic integers**. Note A is integral over $\mathbb{Z}$, but not finitely generated over $\mathbb{Z}$ (as either a module or algebra, by the corollary).

Definition 4.25. *Let R be a commutative domain. Let Q be its field of fractions. The* ***absolute integral closure*** *of R, denoted R^+, is $R^+ = \{\alpha \in \overline{Q} : \alpha \text{ is integral over } R\}$ where $\overline{Q}$ is some algebraic closure of Q.*

Theorem 4.26 (Hochster-Heneke, 1993)**.** *If char $R = p$, then R^+ is a Cohen-Macauly $R-$algebra.*

Definition 4.27. *Let R be a domain. Say R is* ***integrally closed*** *(or* ***normal****) if R is integrally closed in its field of fractions.*

Proposition 4.28. *Let R be a UFD. Then R is integrally closed.*

Proof. Let $\frac{a}{b} \in Q(R)$ be integral over R. WLOG, assume $\gcd(a, b) = 1$. So $\left(\frac{a}{b}\right)^n + r_1 \left(\frac{a}{b}\right)^{n-1} + ... + r_{n-1}\left(\frac{a}{b}\right) + r_n = 0$ where $r_i \in R$. Multiply by b^n to get $a^n + \underbrace{r_1 b a^{n-1} + ... + r_{n-1} b^{n-1} a + r_n b^n}_{b \text{ divides these}} = 0$. Thus $b | a^n$. But $\gcd(a^n, b) = 1$. So b is a unit of R which implies $\frac{a}{b} \in R$. □

Note. This says that PIDs are integrally closed.

Corollary 4.29. *The only rational algebraic integers are integers.*

Remark. Let $R \subseteq S$ be commutative rings, I an ideal of S. Then $\phi : R/(I \cap R) \to S/I$ defined by $r + I \cap R \mapsto r + I$ is an injective ring homomorphism. So we can consider $R/(I \cap R)$ as a subring of S/I, where multiplication is defined by $\overline{r} \cdot \overline{s} = \overline{rs}$ (that is, $(r + I \cap R)(s + I) = rs + I$ is well-defined).

Lemma 4.30. *If S is integral over R and I is an ideal of S, then S/I is integral over $R/I \cap R$.*

Proof. Let $s \in S$. Then $s^n + r_1 s^{n-1} + ... + r_n = 0$ for $r_i \in R$. By the remark, modding out by I gives $\overline{s}^n + \overline{r}_1 \overline{s}^{n-1} + ... + \overline{r}_n = 0$ where $\overline{r}_i \in R/(I \cap R)$. □

Proposition 4.31. *Let S be integral over R. Let $p \in Spec\ S$. Then p is maximal in S if and only if $p \cap R$ is maximal in R.*

Proof. By lemma, S/p is integral over $R/p \cap R$. Also, S/p and $R/p \cap R$ are domains (as $p, p \cap R$ are prime). Thus it is enough to prove:

Claim: If S is integral over R and both are domains, then S is a field if and only if R is a field.

Proof:

- $\Leftarrow$ Suppose R is a field. Let $u \in S \setminus \{0\}$. Then u is integral over R which implies u is algebraic over R. Since $R[u] \subseteq S$ is a domain and is a finite dimensional $R-$vector space, $R[u]$ is a field. Thus $u^{-1} \in R[u] \subseteq S$.
- $\Rightarrow$ Suppose S is a field. Let $u \in R \setminus \{0\}$. Then $u^{-1} \in S$ is integral over R. Then $(u^{-1})^n + r_1(u^{-1})^{n-1} + ... + r_n = 0$ for $r_i \in R$ and multiplication by u^{n-1} gives $u^{-1} + r_1 + ... + r_n u^{n-1} = 0$. Thus $u \in R$.

□

Suppose $R \supseteq S$ are commutative rings, Q a multiplicatively closed subset of R. Since localization is exact, $R_W \subseteq S_W$ (as rings).

Proposition 4.32. *If S/R is integral, W is a multiplicatively closed subset of R, then S_W is integral over R_W.*

Proof. Let $\frac{s}{w} \in S_W$. Since S/R is integral, there exists an equation of the form $s^n + r_1 s^{n-1} + ... + r_1 s + r_n = 0$, for $r_i \in R$. Divide by w^n to get $\left(\frac{s}{w}\right)^n + \frac{r_1}{w}\left(\frac{s}{w}\right)^{n-1} + ... + \frac{r_{n-1}}{w^{n-1}}\left(\frac{s}{w}\right) + \frac{r_n}{w^n} = 0$. Thus $\frac{s}{w}$ is integral over R_W. □

Remark. Let N_1, N_2 be $R-$submodules of M and W a multiplicatively closed subset. Then $(N_1 \cap N_2)_W = (N_1)_W \cap (N_2)_W$.

Lying Over (LO) Theorem. (Cohen - Seidenberg) *Let S/R be an integral extension. Given $p \in SpecR$, there exists $P \in SpecS$ such that $P \cap R = p$.*

Proof. Let $W = R - p$, a multiplicatively closed subset of R. Then p_W is the unique maximal ideal of R_W. As noted, S_W is integral over R_W. Let $P \in SpecS$ be such that P_W is maximal in S_W (as maximal ideals of S_W correspond to maximal ideals of S). By a previous proposition, $P_W \cap R_W$ is maximal in R_W. Since p_W is unique, $p_W = P_W \cap R_W = (P \cap R)_W$. Note $P \cap R \in SpecR$. By the one-to-one correspondence between primes of R which do not intersect W and $SpecR_W$, we have $P \cap R = p$. $\square$

Incomparable (INC) Theorem. *Let S/R be integral and $P_1, P_2 \in SpecS$ such that $P_1 \cap R = P_2 \cap R$. Then P_1, P_2 are incomparable (that is, $P_1 \not\subset P_2$ and $P_2 \not\subset P_1$).*

Proof. Let $p \in P_1 \cap R = P_2 \cap R \in SpecR$. Localize at $W = R - p$. Then $(P_1)_W, (P_2)_W \in SpecS_W$ and are $\neq S$. Also $(P_1)_W \cap R_W = p_W = (P_2)_W \cap R_W$. Therefore, it is enough to show in the case that $P_1 \cap R = P_2 \cap R$ is maximal in R. Then P_1, P_2 are maximal in S. Hence $P_1 \not\subset P_2$ and $P_2 \not\subset P_1$. $\square$

Going Up (GU) Theorem. *Let S/R be integral and $p \subset q$ primes of R. Let $P \in SpecS$ such that $P \cap R = p$. Then there exists $Q \in SpecS$ such that $P \subset Q$ and $Q \cap R = q$.*

Proof. By localizing at $Q = R - q$, we can reduce to the case that q is maximal. Thus it is enough to prove in the case that (R, q) is quasilocal. Let Q be any maximal ideal of S containing P. Then $Q \cap R$ is maximal in R which says $Q \cap R = q$. $\square$

Theorem 4.33. *Let S/R be an integral extension. Then $\dim S = \dim R$.*

Proof. Let $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n$ be a chain of primes of S. Intersect with R to get $Q_0 \cap R \subset Q_1 \cap R \subset \cdots \subset Q_n \cap R$, a chain of primes in R. By the INC Theorem, these are still proper containments. Thus $\dim R \geq \dim S$. Let $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n$ be a chain of primes of R. By the LO Theorem, there exists $Q_0 \in SpecS$ such that $Q_0 \cap R = p_0$. Now use the GU Theorem n times to get $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n$ where $Q_i \cap R = p_i$. Then $\dim S \geq \dim R$. $\square$

Setup: Let G be a finite group, $k = \overline{k}$ a field, char $k \nmid |G|$. Then $k[G]$ is semisimple and thus $k[G] = B_1 \times \cdots \times B_t$ where B_i are Artinian simple rings. Let e_i be the identity of B_i. Let $C_1, ..., C_t$ be the conjugacy classes of G and $z_i = \sum_{g \in C_i} g$. We've proved $Z(k[G]) = ke_1 \times \cdots \times ke_t$ as rings and $Z(k[G]) = kz_1 \oplus \cdots \oplus kz_t$ as $k-$modules. If R is a commutative ring, then $R[G] = \oplus_{g \in G} Rg$ and one can show that $Z(R[G]) = Rz_1 \oplus \cdots \oplus Rz_t$. Now, assume char $k = 0$. Then $\mathbb{Z} \subseteq k$ and as k is a field, this says $\mathbb{Q} \subseteq k$.

Remark. If char $k = 0, k = \overline{k}$, then $Z(\mathbb{Z}[g]) = \mathbb{Z}z_1 \oplus \cdots \oplus \mathbb{Z}z_t \subseteq kz_1 \oplus \cdots \oplus kz_t = Z(k[G])$.

Theorem 4.34. *Let char $k = 0$ and $\chi_1, ..., \chi_t$ be the irreducible characters of G where χ_i correspond to B_i. Let $m_i = |C_i|$. Then for all $i, j \in [t], g \in C_j$ we have $\frac{m_j \chi_i(g)}{\chi_i(1)} \in k$ is integral over $\mathbb{Z}$. Thus $z_i \in Ae_1 + ... + Ae_t$, where A is the integral closure of $\mathbb{Z}$ in k.*

Proof. Recall that $z_j = m_j \sum_{i=1}^t \frac{\chi_i(g) e_i}{\chi_i(1)}$. Now $z_j \in Z(\mathbb{Z}[G]) = \mathbb{Z}z_1 + ... + \mathbb{Z}z_t$, which is a ring and a finitely generated $\mathbb{Z}-$module. Thus z_j is integral over $\mathbb{Z}$. Also, $z_j \in \mathbb{Z}(k[G]) = ke_1 + ... + ke_t$. Say $z_j = \sum_{i=1}^t \alpha_i e_i$ for $\alpha_i \in k$. Let $f(x) \in \mathbb{Z}[x]$ be monic such that $f(z_i) = 0$. Then

$$0 = f(z_i) = f(\alpha_1 e_1 + ... + \alpha_t e_t) = f(\alpha_1)e_1 + ... + f(\alpha_t)e_t$$

as $e_i e_j = \delta_{ij} e_i$. as $e_1, ..., e_t$ are linearly independent over k, we must have that $f(\alpha_i) = 0$ for all i. Thus $\alpha_i \in A$ for all i. Thus $z_i \in Ae_1 + ... Ae_t$. $\square$

Lemma 4.35. *Let A be the integral closure of $\mathbb{Z}$ in k and χ be any character of G. Then $\chi(g) \in A$ for all $g \in G$.*

Proof. Note that $\chi(g) = \sum \lambda_i$, where λ_i are the eigenvalues of $\rho(g)$ for $\rho : G \to GL_k(V)$ a representation associated to χ. Recall λ_i is a root of unity. Thus $\lambda_i \in A$ for all i. Since A is a ring, $\chi(g) \in A$. $\square$

Theorem 4.36. *With the above notation, $n_i \| G|$ for all $i = 1, ..., t$.*

Proof. Recall that $e_i = \frac{n_i}{|G|} \sum_{i=1}^{t} m_j \chi_i(g_j^{-1}) z_j$ where $g_i \in C_i$. Thus $\frac{|G|}{n_i} e_i = \sum_{j=1}^{t} m_j \chi_i(g_j^{-1}) z_j$, were $m_j \chi_i(g_j^{-1}) \in A$. Thus $\frac{|G|}{n_i} e_i \in Az_1 + ... + Az_t \subseteq Ae_1 + ... + Ae_t$. Since the $e_i's$ are linearly independent, we must have $\frac{|G|}{n_i} \in A \cap \mathbb{Q} \subseteq \mathbb{Z}$ as $\mathbb{Z}$ is integrally closed. Thus $n_i \| G|$. $\square$

4.3 Representations of Products of Groups

Let $\rho_i : G_i \to GL_k(V_i)$, for $i = 1, 2$, be $k-$representations of G_i. Define the **tensor product** $\rho_1 \otimes \rho_2$ by $\rho_1 \otimes \rho_2 : G_1 \times G_2 \to GL_k(V_1 \otimes V_2)$ by $(g_1, g_2) \mapsto \rho(g_1) \otimes \rho(g_2)$. This is easily seen to be a representation of $G_1 \times G_2$ of degree $(\deg \rho_1)(\deg \rho_2)$. Now $\chi_{\rho_1 \otimes \rho_2}(g_1, g_2) = \mathrm{tr}_k(\rho_1(g_1) \otimes \rho_2(g_2)) = \mathrm{tr}_k(\rho_1(g_1))\mathrm{tr}_k(\rho_2(g_2)) = \chi_{\rho_1}(g_1)\chi_{\rho_2}(g_2)$ (Exercise). Generally, we will write $\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1}\chi_{\rho_2}$. Let ρ_i, ρ_i' be representations of G_i for $i = 1, 2$. Then

$$< \chi_{\rho_1 \otimes \rho_2}, \chi_{\rho_1' \otimes \rho_2'} >=< \chi_{\rho_1}, \chi_{\rho_1'} >_{G_1} < \chi_{\rho_2}, \chi_{\rho_2'} >_{G_2}$$

(Exercise).

Conclusion. $\rho_1 \otimes \rho_2$ is irreducible if and only if ρ_1, ρ_2 are irreducible. Moreover, if $\{\chi_1, ..., \chi_s\}$ is the set of irreducible characters of G_1 and $\{\phi_1, ..., \phi_t\}$ is the set of irreducible characters for G_2, then $\{\chi_i \phi_j\}$ is the set of irreducible characters of $G_1 \times G_2$ (Use the fact that $\sum n_i^2 = |G|$ to show that this must be all of them).

Another Version of...

Lemma 4.37 (Schur's Lemma). *Let $|G|$ be a finite group, char $k \nmid |G|, k = \overline{k}$. Let $\rho : G \to GL_k(V)$ be an irreducible representation of G and χ its associated character. Then for all $g \in Z(G)$, we have*

1. *$\rho(g) = \lambda I$ for some $\lambda \in k^*$.*
2. *$|\chi(g)| = \chi(1)$ if $k = \mathbb{C}$.*

Proof. Write $k[G] = B_1 \times \cdots \times B_t$ where B_i are simple, Artinian, and $e_i \in B_i$ is the identity. Then $Z(k[G]) = ke_1 \times \cdots \times ke_t$. If $g \in Z(G)$, then $g \in Z(k[G])$. Write $\alpha_1 e_1 + ... + \alpha_t e_t = g$, $\alpha_i \in k$. Now V is an irreducible $k[G]-$module. WLOG, say V is a simple B_1-module (if not, reindex the B_i's). Then $e_1 v = v$ for all $v \in V$ and $e_j v = 0$ for all $j > 1$. Then $gv = \alpha_1 v$ for all $v \in V$ and thus $\rho(g) = \alpha_1 I_V$. $\square$

Theorem 4.38. *Under the "standard notation" above, $n_i \Big| [G : Z(G)]$ for all i.*

Proof. (Tate) Let $n = n_1, \chi = \chi_1$ with $\rho : G \to GL_k(V)$, a representation associated to χ. Let m be a positive integer and consider $\rho_m := \rho \otimes \cdots \otimes \rho : \underbrace{G \times \cdots \times G}_{:=G_m} \to GL_k(V \otimes \cdots \otimes V)$. As ρ is irreducible, so is ρ_m. Define a map $\gamma : Z(G) \to k^*$ by $g \mapsto \alpha$ where $\rho(g) = \alpha I$. It is easily seen that γ is a group homomorphism. Let $D = \{(g_1, ..., g_m) \in Z(G_m) | \lambda(g_1 \cdots g_n) = 1\}$. Let $H = \ker \gamma$ and $g_1, ..., g_{m-1} \in Z(G)$. Note $(g_1, ..., g_m) \in D$ if and only if $g_1, ..., g_m \in H$ which is if and only if $g_m \in g_1^{-1} \cdots g_{m-1}^{-1} H$. Thus $|D| = |Z(G)|^{m-1}|H|$ (as there are $|Z(G)|^{m-1}$ choices for $g_1, ..., g_{m-1}$ and $|H|$ choices for g_m. Now $D \lhd G_m$ as $D \subseteq Z(G_m)$ and $D \subseteq \ker \rho_m$ (To see this, let $(g_1, ..., g_m) \in D$. Then $\rho(g_1, ..., g_m) = \rho(g_1) \otimes \cdots \otimes \rho(g_m) = \alpha_{g_1} I_V \otimes \cdots \otimes \alpha_{g_m} I_V = (\alpha_{g_1} \cdots \alpha_{g_m}) I_{V \otimes \cdots \otimes V} = \gamma(g_1) \cdots \gamma(g_m) I_{V \otimes \cdots \otimes V} = \gamma(g_1 \cdots g_m) I_{V \otimes \cdots \otimes V} = I_{V \otimes \cdots \otimes V}$.) Thus $\overline{\rho}_m : G_m/D \to GL_k(V \otimes \cdots \otimes V)$ defined by $(\overline{g}_1, ..., \overline{g}_m) \mapsto \rho(g_1) \otimes \cdots \otimes \rho(g_m)$ is a well defined irreducible representation of G_m/D. By the previous theorem, $\deg \overline{\rho}_m \Big| |G_m/D|$ which implies $n^m \Big| |G|^m/(|Z(G)|^{m-1} \cdot |H|)$. So $\frac{|G|^m}{n^m |Z(G)|^{m-1}|H|} \in \mathbb{Z}$.

Then $\underbrace{\frac{|Z(G)|}{|H|}}_{\in\mathbb{Z}}\underbrace{\left(\frac{|G|}{n|Z(G)|}\right)^m}_{\in\mathbb{Q}} \in \mathbb{Z}$ for all m. By HW7#5, we see $\frac{|G|}{n|Z(G)|}$ is integral over $\mathbb{Z}$ which says $n\Big|[G : Z(G)]$ as $\mathbb{Z}$ is integrally closed in $\mathbb{Q}$. $\square$

Lemma 4.39. *Let G be a finite group, $\rho : G \to GL_n(\mathbb{C})$ an irreducible representation, and χ its associated character. Let C be a conjugacy class of G such that $\gcd(|C|, n) = 1$. Then for all $g \in C$, either $\chi(g) = 0$ or $|\chi(g)| = 1$.*

Proof. Let $m = |C|$. Then there exists $r, s \in \mathbb{Z}$ such that $rm + sn = 1$. Then for all $g \in C$, we have $\frac{rm\chi(g)}{n} + s\chi(g) = \frac{\chi(g)}{n}$. Let A be the integral closure of $\mathbb{Z}$ in $\mathbb{C}$. We've see $\chi(g) \in A$ for all $g \in G$. By a previous proposition, we have also shown $\frac{m\chi(g)}{n} \in A$ for $g \in C$. Thus $\frac{\chi(g)}{n} \in A$ for all $g \in C$. Let $\chi(g) = \lambda_1 + ... + \lambda_n$ where λ_i are k^{th} roots of unity. Let ω be a primitive k^{th} root of unity and $L = \mathbb{Q}(\omega)$. Then $\lambda_i \in L$ for all i. Let $H = Gal(L/\mathbb{Q})$ and $\sigma \in H$. Note $\sigma(A \cap L) \subseteq A \cap L$. Also, $\sigma(\lambda_i) = \lambda_j$. Let $\alpha = \frac{\chi(g)}{n} = \frac{\lambda_1+...+\lambda_n}{n}$. Then $|\alpha| \leq 1$. Note $|\sigma(\alpha)| = |\frac{\sigma(\lambda_1)+...+\sigma(\lambda_n)}{n} \leq 1$ for all $\sigma \in H$ and $\sigma(\alpha) \in A$. Consider $N = N^L_{\mathbb{Q}} : L \to \mathbb{Q}$ where $N(\beta) = \Pi\sigma(\beta) \in \mathbb{Q}$. So $N(\alpha) = \Pi_{\sigma\in H}\sigma(\alpha) \in \mathbb{Q} \cap A = \mathbb{Z}$. So $|N(\alpha)| = \Pi|\sigma(\alpha)| \leq 1$. Thus $N(\alpha) = 0$ or 1. Now $N(\alpha) = 0$ implies $\frac{\chi(g)}{n} = \alpha = 0$. Thus $\chi(g) = 0$. If $N(\alpha) = 1$, then $|\alpha| = 1$ which says $\lambda_1 = ... = \lambda_n$ so that $\chi(g) = \lambda n$ and $|\chi(g)| = n = \chi(1)$. $\square$

Theorem 4.40. *Let G be a finite simple group, C a conjugacy class of G. Then $|C| \neq p^a$ for p prime and $a > 0$.*

Proof. Assume G is not abelian (as otherwise $|C| = 1$). Suppose there exists C such that $C = p^a$ for $a > 0$. Let $\chi_1, ..., \chi_t$ be the irreducible $\mathbb{C}-$characters of G and $\rho_i : G \to GL_{n_i}(\mathbb{C})$ the irreducible representations associated with χ_i. Let ρ_1 be the trivial representation.

Claim 1: If $p \nmid n_i$ for $i > 1$, then $\chi_i(g) = 0$ for all $g \in C$.

Proof: Let $G_i = \{g \in G|\rho_i(g) = \lambda I, some\ \lambda \in \mathbb{C}\}$. It is easy to see $G_i \lhd G$. But G is simple, so $G_i = \{1\}$ or $G_i = G$. Suppose $G_i = G$. Note $\ker\rho_i \lhd G$ and $\rho_i \neq 1$. Thus $\ker\rho_i = \{1\}$. So $G \cong \rho_i(G) = \{\lambda_g I|g \in G\}$ as $G_i = G$, but this is abelian, a contradiction. Thus $G_i = \{1\}$ and $\rho_i(g) \neq \lambda I$ for all $\lambda \in \mathbb{C}$ and $g \neq 1$. Thus $|\chi_i(g)| < \chi_i(1)$ by HW7. By the lemma, $\chi_i(g) = 0$ for all $g \in C$.

Claim 2: $p|n_i$ for some $i > 1$.

Proof: By an orthogonality relation, for $g \in C$, we have $\sum_{i=1}^t \chi_i(1)\chi_i(g) = 0$. So $0 = 1 + \sum_{i=2}^t \chi_i(1)\chi_i(g)$. Since $0 \neq 1$, there exists $j \geq 2$ such that $\chi_j(g) \neq 0$. Thus $p|n_j$.

Reorder the characters such that $p|n_i$ for $i = 2, ..., s$ and $p \nmid n_i$ for $i = s+1, ..., t$. Thus by Claim 1, $1+\sum_{j=2}^s \chi_j(1)\chi_j(g) = 0$. Since $p|n_j$, we have $\frac{1}{p} = -\sum_{j=2}^s(\frac{n_j}{p})\chi_j(g) \in A \cap \mathbb{Q} = \mathbb{Z}$, a contradiction. $\square$

Corollary 4.41 (Burnside)**.** *Let G be a group of order p^aq^b. Then G is solvable.*

Proof. We will show that G is not simple. We've seen the case where $b = 0$. So assume $a, b \geq 1$. Let P be a Sylow-p subgroup. Let $z \in Z(P) \setminus \{1\}$. Then $C_G(z) \supseteq P$ which implies $[G : C_G(z)] = q^c$, for some c. Of course, $[G : C_G(z)] = |C|$, where C is the conjugacy class of z. By the theorem, if G is simple, then $c = 0$ which implies $z \in Z(G) \setminus \{1\}$ and so G has a nontrivial subgroup. So G is not simple. Let $H \lhd G$. By induction, H and G/H are solvable, which implies G is solvable. $\square$

4.4 Injective Modules

Definition 4.42. *An $R-$module E is injective if given*

$$\begin{tikzcd} & E & & \\ 0 \arrow[r] & M \arrow[r] \arrow[u] & N \arrow[ul, dashed] & \textit{exact} \end{tikzcd}$$

there exists a map $N \to M$ such that the diagram commutes.

Theorem 4.43 (**Baer's Criterion**). *Let E be a left $R-$module. Then E is injective if and only if given a diagram*

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & I \arrow[u] \arrow[r] & R \arrow[ul, dashed, "\exists h"'] \end{tikzcd} \qquad \textit{exact}$$

where I is a left ideal, there exists $h : R \to E$ making the diagram commute.

Proof. The forward direction is clear from the definition. So suppose we are given a diagram

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & M \arrow[u, "f"] \arrow[r, "i"] & N \end{tikzcd} \qquad \text{exact}$$

where WLOG we may assume $M \subset N$ and so i is just the inclusion map. Let $\Lambda = \{(K, f_K)|M \subseteq K \subseteq N, K \text{ a left } R\text{−module}, f_K : K \to E, f_K|_M = f\}$. Partially order in the obvious way. Then $\Lambda \neq \emptyset$ and $(M, f) \in \Lambda$. By Zorn's Lemma, there exists (K, f_K) maximal in Λ.

Claim: $K = N$.

Proof: Suppose not. Choose $x \in N \setminus K$. Let $I = (K :_R x) = \{r \in R|rx \in K\}$. Then I is a left ideal of R. Define $\phi : I \to E$ such that $i \mapsto f_K(ix)$. This is $R-$linear. By hypothesis, there exists $\widetilde{\phi} : R \to E$ such that $\widetilde{\phi}|_I = \phi$. Define $g : K + Rx \to E$ by $k + rx \mapsto f_K(k) + \widetilde{\phi}(r)$. To show g is well-defined, suppose $k + rx = 0$. Then $r \in I$. So $\widetilde{\phi}(r) = \phi(r) = f_K(rx)$. Then $g(k + rx) = f_K(k) + f_K(rx) = f_K(k + rx) = f_K(0) = 0$. Thus $(K + Rx, g) \in \Lambda$, a contradiction to the maximality of (K, f_K).

□

Definition 4.44. *Let R be commutative, M an $R-$module. Say M is* ***divisible*** *if for all $m \in M$ and for all non-zero-divisors $r \in R$, there exists $m' \in M$ such that $rm' = m$.*

Examples.

1. Every vector space over a field is divisible.
2. If R is a domain, then Q, the field of fractions of R, is divisible.
3. Sums, products, quotients of divisible modules are divisible.
4. Submodules of divisible modules are *not* always divisible. For example, $\mathbb{Q}$ is a divisible $\mathbb{Z}-$module, but $\mathbb{Z}$ is not.
5. In particular, $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}$ are divisible $\mathbb{Z}-$modules.

Proposition 4.45. *Let R be commutative. Every injective module is divisible. If R is a PID, then the converse holds.*

Proof. Let E be injective, $e \in E$, and $r \in R$ a non-zero-divisor. Consider the diagram

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & R \arrow[u, "f"] \arrow[r, "r"] & R \arrow[ul, dashed] \end{tikzcd} \qquad \text{exact}$$

where $r : R \to R$ is multiplication by r and $f(1) = e$. As E is injective, we have a map from $R \to E$, say its defined by $1 \mapsto e'$. Then, by commutivity, $re' = e$. Now, suppose R is a PID and E is a divisible module. Let $I = (a)$ be an ideal of R and consider the diagram

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & (a) \arrow[u, "f"] \arrow[r, "r"] & R \arrow[ul, dashed] \end{tikzcd} \qquad \text{exact}$$

If $a = 0$, done. Otherwise, let $e = f(a)$. As a is a non-zero-divisor (R is a domain), there exists e' such that $ae' = e$. Define $\widetilde{f} : R \to E$ by $1 \mapsto e'$. Then $\widetilde{f}(ra) = ra\widetilde{f}(1) = rae' = re = rf(a) = f(ra)$. So $\widetilde{f}|_{(a)} = f$. By Baer's Criterion, E is injective. $\square$

Corollary 4.46. *Any $\mathbb{Z}-$module M can be embedded into an injective $\mathbb{Z}-$module.*

Proof. Consider $0 \to K \to \oplus_{\alpha \in I}\mathbb{Z} \to M \to 0$, which is exact (let $|I|$ be the number of generators of M as a $\mathbb{Z}-$module). So $M \cong \oplus\mathbb{Z}/K \subseteq \oplus\mathbb{Q}/K$. By the above, $\oplus\mathbb{Q}/K$ is a divisible $\mathbb{Z}-$module and so it is injective. Thus M embeds into an injective module. $\square$

Proposition 4.47. *Let $\phi : R \to S$ be a ring homomorphism. Let E be an injective left $R-$module. Then $Hom_R(S, E)$ is an injective left $S-$module.*

Proof. Recall $Hom_R(S, E)$ is a left $S-$module via $(sf) : S \to E$ where $s' \mapsto f(s's)$ for $s \in S, f \in Hom_R(S, E)$. Note sf is $R-$linear. So it is enough to show that if $0 \to M \to N$ is an exact sequence of $S-$modules, $Hom_S(N, Hom_R(S, E)) \to Hom_S(M, Hom_R(S, E))$ is surjective. By Hom-$\otimes$ adjointness and the fact that $S \otimes_S M = M$, we have the following diagram

$$\begin{tikzcd} Hom_S(N, Hom_R(S,E)) \arrow[r, "\sigma"] \arrow[d, "\cong"] & Hom_S(M, Hom_R(S,E)) \arrow[d, "\cong"] & \\ Hom_R(S \otimes_S N, E) \arrow[r] \arrow[d, "\cong"] & Hom_R(S \otimes_S M, E) \arrow[d, "\cong"] & \\ Hom_R(N, E) \arrow[r] & Hom_R(M, E) \arrow[r] & 0 \text{ exact} \end{tikzcd}$$

Note that both squares commute by the "naturality" of the isomorphisms. The bottom row is exact as E is an injective $R-$module. So, we have σ is surjective. $\square$

Theorem 4.48. *Let R be a ring, M a left $R-$module. Then there exists an injective $R-$module E and an injective $R-$module homomorphism $M \to E$.*

Proof. Of course, there exists a ring homomorphism $\phi : \mathbb{Z} \to R$. As M is a $\mathbb{Z}-$module, there exists an injective $\mathbb{Z}-$module I with $M \subseteq I$. By the above proposition, $Hom_{\mathbb{Z}}(R, I)$ is an injective left $R-$module. Define $g : M \to Hom_{\mathbb{Z}}(R, I)$ by $m \mapsto f_m$ where $f_m : R \to I$ is defined by $r \mapsto rm \in M \subseteq I$. We need to show g is $R-$linear. It is enough to show $rf_m = f_{rm}$. For $r' \in R$, we have $(rf_m)(r') = f_m(r'r) = r'rm = f_{rm}(r')$. Also, g is injective as $m = 0$ if and only if $f_m = 0$. $\square$