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On the $\eta - \kappa$ Distribution

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Comments and Corrections

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***Index Terms*—Fast fading distribution, $\eta-\kappa$ distribution.**

The recent paper by Yacoub *et al.* [1] introduces what is referred to as the $\eta-\kappa$ *distribution* to describe the statistical variation of the envelope in a fast fading environment. The paper discusses several properties of the distribution. Two of the properties discussed are the nth moment, $E(P^n)$, and the cumulative probability function (cpf), $F_P(\cdot)$, where P is a random variable representing the normalized envelope. The expression given for $E(P^n)$ (see equation (10) in Yacoub *et al.* [1]) is a doubly infinite sum of the Gauss hypergeometric function (which, itself, is an infinite sum). That given for $F_P(\cdot)$ (see equation (11) in Yacoub *et al.* [1]) is a triple sum of the incomplete gamma function.

We feel that the expressions in equations (10) and (11) of Yacoub *et al.* [1] are too complicated for practical purposes. In the following, we show how one can derive much simpler forms for $E(P^n)$ and $F_P(\cdot)$. Using equations (5)–(8) in Yacoub *et al.* [1], the probability density function (pdf) of P can be expressed as

$$f_P(p)\frac{\sqrt{h}(1+\kappa)}{\pi\exp(\kappa)}\int_0^{2\pi} p\exp(Ap-Bp^2)d\theta, \tag{1}$$

where $A=2\sqrt{h\kappa(1+\kappa)}\cos\theta$ and $B=(1+\kappa)h+H(1+\kappa)\cos(2(\theta+\phi))$. Thus, the nth moment, $E(P^n)$, can be expressed as

$$\begin{aligned}E(P^n) &= \frac{\sqrt{h}(1+\kappa)}{\pi\exp(\kappa)}\int_0^{\infty}\int_0^{2\pi} p^{n+1}\exp(Ap-Bp^2)d\theta dp\\ &= \frac{\sqrt{h}(1+\kappa)}{\pi\exp(\kappa)}\int_0^{2\pi}\int_0^{\infty} p^{n+1}\exp(Ap-Bp^2)dpd\theta\\ &= \frac{\sqrt{h}(1+\kappa)}{\pi\exp(\kappa)}\int_0^{2\pi} I(\theta)d\theta. \end{aligned}\tag{2}$$

By equation (2.3.15.3) in Prudnikov *et al.* [2], $I(\theta)$ can be calculated as

$$I(\theta)=\Gamma(n+2)(2B)^{-(n/2+1)}\exp\left(\frac{A^2}{8B}\right)D_{-n-2}\left(-\frac{A}{\sqrt{2B}}\right), \tag{3}$$

where $D_p(\cdot)$ denotes the parabolic cylinder function defined by

$$D_p(x)=\frac{\exp(-x^2/4)}{\Gamma(-p)}\int_0^{\infty}\frac{\exp\left\{-(tx+t^2/2)\right\}}{t^{p+1}}dt.$$

Combining (2) and (3) yields the formula

$$E(P^n)=\frac{\Gamma(n+2)\sqrt{h}(1+\kappa)}{2^{n/2+1}\pi\exp(\kappa)}\int_0^{2\pi}B^{-(n/2+1)} \times\exp\left(\frac{A^2}{8B}\right)D_{-n-2}\left(-\frac{A}{\sqrt{2B}}\right)d\theta. \tag{4}$$

This formula applies for any real number $n>-2$. If n is a positive integer then, using equation (2.3.15.7) in Prudnikov *et al.* [2], $I(\theta)$ can be calculated as

$$I(\theta)=\frac{(-1)^{n+1}\sqrt{\pi}}{2\sqrt{B}}\frac{\partial^{n+1}}{\partial q^{n+1}}\times\left[\exp\left(\frac{q^2}{4B}\right)\mathrm{erfc}\left(\frac{q}{2\sqrt{B}}\right)\right]\Big|_{q=-A}, \tag{5}$$

where $\mathrm{erfc}(\cdot)$ denotes the complementary error function defined by

$$\mathrm{erfc}(x)=-\frac{2}{\sqrt{\pi}}\int_0^{x}\exp(-t^2)dt.$$

Combining (2) and (5) yields the simpler formula

$$E(P^n)=\frac{(-1)^{n+1}\sqrt{h}(1+\kappa)}{2\sqrt{\pi}\exp(\kappa)}\int_0^{2\pi}B^{-1/2}\frac{\partial^{n+1}}{\partial q^{n+1}} \times\left[\exp\left(\frac{q^2}{4B}\right)\mathrm{erfc}\left(\frac{q}{2\sqrt{B}}\right)\right]\Bigg|_{q=-A}d\theta. \tag{6}$$

Various simple expressions can be obtained from (6) by setting specific values for n. For instance, if $n=1$, $n=3$ and $n=5$ then (6) can be reduced to the simple forms

$$\begin{aligned}E(P) &= \frac{\sqrt{h}(1+\kappa)}{8\pi\exp(\kappa)}\int_0^{2\pi}B^{-3}\exp\left(\frac{A^2}{4B}\right)\\ &\quad\times\left[2B^{3/2}\sqrt{\pi}+A^2\sqrt{B}\sqrt{\pi}+2B^{3/2}\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right)\right.\\ &\quad+A^2\sqrt{B}\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right)\\ &\quad\left.+2AB\exp\left(-\frac{A^2}{4B}\right)\right]d\theta, \end{aligned}\tag{7}$$

$$\begin{aligned}E(P^3) &= \frac{\sqrt{h}(1+\kappa)}{32\pi\exp(\kappa)}\int_0^{2\pi}B^{-5}\exp\left(\frac{A^2}{4B}\right)\\ &\quad\times\left[12B^{5/2}\sqrt{\pi}+12A^2B^{3/2}\sqrt{\pi}+A^4\sqrt{B}\sqrt{\pi}\right.\\ &\quad+12B^{5/2}\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right)+12A^2B^{3/2}\\ &\quad\times\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right)+20AB^2\exp\left(-\frac{A^2}{4B}\right)\\ &\quad+A^4\sqrt{B}\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right)\\ &\quad\left.+2A^3B\exp\left(-\frac{A^2}{4B}\right)\right]d\theta \end{aligned}\tag{8}$$

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and

$$
\begin{aligned}
E(P^5) = &\frac{\sqrt{h}(1+\kappa)}{128\pi \exp(\kappa)} \int_0^{2\pi} B^{-7} \exp\left(\frac{A^2}{4B}\right) \\
&\times \Big[120 B^{7/2}\sqrt{\pi} + 180 A^2 B^{5/2}\sqrt{\pi} + 30 A^4 B^{3/2}\sqrt{\pi} \\
&+ A^6 \sqrt{B}\sqrt{\pi} + 120 B^{7/2}\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right) \\
&+ 180 A^2 B^{5/2}\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right) \\
&+ 264 A B^3 \exp\left(-\frac{A^2}{4B}\right) + 30 A^4 B^{3/2}\sqrt{\pi}\left(\frac{A}{2\sqrt{B}}\right) \\
&+ 56 A^3 B^2 \exp\left(-\frac{A^2}{4B}\right) + A^6 \sqrt{B}\sqrt{\pi}\mathrm{erf}\left(\frac{A}{2\sqrt{B}}\right) \\
&+ 2 A^5 B \exp\left(-\frac{A^2}{4B}\right)\Big] d\theta, \qquad (9)
\end{aligned}
$$

respectively, where $\mathrm{erf}(\cdot)$ denotes the error function defined by

$$
\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt.
$$

If n is an even number then one does not need to use (6) since $E(P^n) = E((X^2+Y^2)^{n/2})/\{E(X^2+Y^2)\}^{n/2}$, where X and Y are independent Gaussian random variables (see equation (15) in Yacoub *et al.* [1]).

The cpf of P, $F_P(\cdot)$, can be expressed as

$$
\begin{aligned}
F_P(p) &= 1 - \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \int_p^{\infty}\int_0^{2\pi} x \exp(Ax - Bx^2) d\theta dx \\
&= 1 - \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \int_0^{2\pi}\int_p^{\infty} x \exp(Ax - Bx^2) dx d\theta \\
&= 1 - \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \\
&\quad \times \int_0^{2\pi} \left[\int_p^{\infty} (x-p) \exp(Ax - Bx^2) dx \right. \\
&\quad \left. + p \int_p^{\infty} \exp(Ax - Bx^2) dx \right] d\theta \\
&= 1 - \frac{\sqrt{h}(1+\kappa)}{\pi \exp(\kappa)} \int_0^{2\pi} \left[I_1(\theta) + p I_2(\theta)\right] d\theta. \qquad (10)
\end{aligned}
$$

By equation (2.3.15.1) in Prudnikov *et al.* [2], $I_1(\theta)$ and $I_2(\theta)$ can be calculated as

$$
I_1(\theta) = (2B)^{-1} \exp\left\{\frac{A^2}{8B} + \frac{p(A-pB)}{2}\right\} D_{-2}\left(\frac{2pB - A}{\sqrt{2B}}\right) \qquad (11)
$$

and

$$
I_2(\theta) = (2B)^{-1/2} \exp\left\{\frac{A^2}{8B} + \frac{p(A-pB)}{2}\right\} D_{-1}\left(\frac{2pB - A}{\sqrt{2B}}\right). \qquad (12)
$$

Combining (10), (11) and (12) yields a formula for the cpf $F_P(\cdot)$.

Note that all of the formulas in (4), (6), (7), (8), (9) and (10) involve just one integral (with respect to θ) and are much simpler than those in equations (10) and (11) of Yacoub *et al.* [1]. We feel that the formulas given can help the readers and authors of this journal with respect to modeling the statistical variation of the envelope in a fast fading environment.

References

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