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Class Notes for Math 921/922: Real Analysis, Instructor Mikil Foss

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Topics include: Semicontinuity, equicontinuity, absolute continuity, metric spaces, compact spaces, Ascoli's theorem, Stone Weierstrass theorem, Borel and Lebesque measures, measurable functions, Lebesque integration, convergence theorems, L^p spaces, general measure and integration theory, Radon-Nikodyn theorem, Fubini theorem, Lebesque-Stieltjes integration, Semicontinuity, equicontinuity, absolute continuity, metric spaces, compact spaces, Ascoli's theorem, Stone Weierstrass theorem, Borel and Lebesque measures, measurable functions, Lebesque integration, convergence theorems, Lp spaces, general measure and integration theory, Radon-Nikodyn theorem, Fubini theorem, Lebesque-Stieltjes integration.

Prepared by Laura Lynch, University of Nebraska-Lincoln

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Course Information

Office Hours: 11:30-1 MWF
Optional Meeting Time: Thursdays at 5pm

Assessment

Homework	6	34pts
Midterm		33pts
Final		33pts (Dec 13 : $1 - 3pm$)

1 Chapter 1

Drawbacks to Riemann Integration

1. Not all bounded functions are Riemann Integrable.
2. All Riemann Integrable functions are bounded.
3. To use the following theorem, we must have $f \in \mathcal{R}[a, b]$.

   Theorem 1 (Dominated Convergence Theorem for Riemann Integrals, Arzela)**.** *Let* $\{f_n\}_{n=1}^{\infty} \subset \mathcal{R}[a, b]$ *and* $f \in \mathcal{R}[a, b]$ *be given. Suppose there exists* $g \in \mathcal{R}[a, b]$ *such that* $|f_n(x)| < g(x)$ *for all* $x \in [a, b]$ *and* $\lim_{n\to\infty} f_n(x) = f(x)$ *for all* $x \in [a, b]$*, then* $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.

   Example. Define $f_n(x) = \begin{cases} 1 & \text{if } x = \frac{p}{q} \text{ in lowest terms with } 1 \leq q \leq n \text{ on } [0,1], \\ 0 & \text{otherwise.} \end{cases}$

   Then $f_n(x) \to \chi_{\mathbb{Q}\cap[0,1]} =: f$. Notice here $|f_n(x)| < 2$ for all $x \in [0, 1]$ but f is not Riemann Integrable. Thus we can not use the theorem.
4. The space $\mathcal{R}[a, b]$ is not complete with respect to many useful metrics.

Good Properties of Riemann Integration

1. $\mathcal{R}[a, b]$ is a vector space
2. The functional $f \mapsto \int_a^b f(x)dx$ is linear on $\mathcal{R}[a, b]$.
3. $\int_a^b f(x)dx \geq 0$ when $f(x) \geq 0$ for all $x \in [a, b]$.
4. Theorem 1 holds.

1.1 Measurable and Topological Spaces

Definition (p 113,119)**.** *Let* X *be a nonempty set.*

1. *A collection* $\mathcal{T}$ *of subsets of* X *is called a* ***topology*** *of* X *if it possesses the following properties:*
   (a) $\emptyset \in \mathcal{T}$ *and* $X \in \mathcal{T}$.
   (b) *If* $\{U_j\}_{j=1}^n \subseteq \mathcal{T}$*, then* $\cap_1^n U_j \in \mathcal{T}$.
   (c) *If* $\{U_\alpha\}_{\alpha\in A} \subseteq \mathcal{T}$*, then* $\cup_{\alpha\in A} U_\alpha \in \mathcal{T}$.
2. *If* $\mathcal{T}$ *is a topology on* X*, then* $(X, \mathcal{T})$ *is called a* ***topological space****. If* $\mathcal{T}$ *is understood, we may just call* X *itself a topological space. The members of* $\mathcal{T}$ *are called* ***open sets*** *in* X*. The complements of open sets in* X *are called* ***closed sets****.*

3. *If* X *and* Y *are topological spaces and* $f : \mathrm{X} \to \mathrm{Y}$, *then* f *is called* ***continuous*** *if* $f^{-1}(\mathrm{V})$ *is open in* X *for all* V *that are open in* Y.

Examples.

1. If $\mathrm{X} = \mathbb{R}$, then $\{\emptyset, \mathbb{R}\}$ is a topology on $\mathbb{R}$.
2. If $\mathrm{X} = \mathbb{R}$, then the power set $\{\mathcal{P}(\mathbb{R})\}$ is a topology on $\mathbb{R}$.
3. $\{\emptyset, \cup_{a>0}\{(-a, a)\}, \mathbb{R}\}$ is a topology on $\mathbb{R}$.
4. $\{\emptyset, \cup_{a<b\in\mathbb{R}}\{(a, b)\}, \cup_{a\in\mathbb{R}}\{(-\infty, a), (a, \infty)\}, \overline{\mathbb{R}}\}$ is a topology on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.

Definition (pg21,25,43)**.** *1. A collection* $\mathcal{M}$ *of subsets of* X *is called a* $\sigma-$***algebra*** *if*

  (a) $\emptyset \in \mathcal{M}$ *and* $\mathrm{X} \in \mathcal{M}$.
  (b) If $E \in \mathcal{M}$, *then* $E^C \in \mathcal{M}$.
  (c) If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$, *then* $\cup_{j=1}^{\infty} E_j \in \mathcal{M}$.

2. *If* $\mathcal{M}$ *is a* $\sigma-$*algebra on* X, *then* $(\mathrm{X}, \mathcal{M})$ *is called a* ***measurable space***. *If* $\mathcal{M}$ *is understood, then we may just call* X *itself a measurable space. The members of* $\mathcal{M}$ *are called* ***measurable sets***.

3. *If* $(\mathrm{X}, \mathcal{M})$ *and* $(\mathrm{Y}, \mathcal{N})$ *are measurable spaces, then* $f : \mathrm{X} \to \mathrm{Y}$ *is called* $(\mathcal{M}, \mathcal{N})-$***measurable*** *or* ***measurable*** *if* $f^{-1}(E) \in \mathcal{M}$ *whenever* $E \in \mathcal{N}$.

Examples.

1. $(\mathbb{R}, \{\emptyset, \mathbb{R}\})$ is a $\sigma-$algebra
2. $(\mathbb{R}, \{\emptyset, \{1\}, \mathbb{R} \setminus \{1\}, \mathbb{R}\})$ is a $\sigma-$algebra
3. $(\mathbb{R}, \{E \subseteq \mathbb{R} | E \text{ or } E^C \text{ is countable}\})$ is a $\sigma-$algebra

Lemma 1. *If* $\{E_j\}_{j=1}^{\infty} \subset \mathcal{P}(\mathrm{X})$, *then* $\{F_j\}_{j=1}^{\infty} \subseteq \mathcal{P}(\mathrm{X})$ *defined by* $F_j = E_j \setminus \cup_{k=1}^{j-1} E_k$ *is a sequence of mutually disjoint sets and* $\cup_{j=1}^{\infty} E_j = \cup_{j=1}^{\infty} F_j$.

Note. As a result of Lemma 1, we can actually modify part (c) of our definition of a $\sigma-$algebra to say

(c) If $\{E_j\}_1^{\infty} \subset \mathcal{M}$ is a sequence of mutually disjoint sets, then $\cup_1^{\infty} E_j \in \mathcal{M}$.

Remarks.

1. Property (1a) of our definition for $\sigma-$ algebra could be replaced with "$\emptyset \in \mathcal{M}$ or $\mathrm{X} \in \mathcal{M}$."
2. If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$, then $\{E_j^C\}_{j=1}^{\infty} \subseteq \mathcal{M}$ and $\cap_{j=1}^{\infty} E_j = [\cup_{j=1}^{\infty} E_j^C]^C \subseteq \mathcal{M}$.
3. If $E, F \in \mathcal{M}$, then $E \setminus F = E \cap F^C \in \mathcal{M}$.

Theorem 2. *If* $\mathcal{E}$ *is a collection of subsets of* X, *then there exists a unique smallest* $\sigma-$*algebra* $\mathcal{M}(\mathcal{E})$ *that contains the members of* $\mathcal{E}$. *Note: By smallest, we mean any other* $\sigma-$*algebra will contain all the sets in* $\mathcal{M}(\mathcal{E})$.

Proof. Let Ω be the family of all $\sigma-$algebras containing $\mathcal{E}$. Note $\Omega \neq \emptyset$ as $\mathcal{P}(\mathrm{X}) \in \Omega$. Define $\mathcal{M}(\mathcal{E}) = \cap_{\mathcal{M}\in\Omega} \mathcal{M}$. We want to show $\mathcal{M}(\mathcal{E})$ is a $\sigma-$algebra.

1. $\emptyset, \mathrm{X} \in \mathcal{M}(\mathcal{E})$ since $\emptyset, \mathrm{X} \in \mathcal{M}$ for all $\mathcal{M} \in \Omega$.
2. Let $E \in \mathcal{M}(\mathcal{E})$. Then $E \in \mathcal{M}$ for all $\mathcal{M} \in \Omega$ which implies $E^C \in \mathcal{M}$ for all $\mathcal{M} \in \Omega$ which implies $E^C \in \mathcal{M}(\mathcal{E})$.
3. If $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{E})$, then $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ for all $\mathcal{M} \in \Omega$ which implies $\cup_1^{\infty} E_j \in \mathcal{M}$ for all $\mathcal{M} \in \Omega$ which implies $\cup_1^{\infty} E_j \in \mathcal{M}(\mathcal{E})$. □

Remark. $\mathcal{M}(\mathcal{E})$ is called the $\sigma-$algebra generated by $\mathcal{E}$.

Definition. *Let* $(\mathrm{X}, \mathcal{T})$ *be a topological space. The* $\sigma-$*algebra generated by* $\mathcal{T}$ *is called the* ***Borel*** $\sigma-$***algebra*** *on* X *and is denoted* $\mathcal{B}_{\mathrm{X}}$. *The members of a Borel* $\sigma-$*algebra are called* ***Borel sets****.*

Proposition 1 (p 22)**.** $\mathcal{B}_{\mathbb{R}}$ *is generated by each of the following:*

1. $\mathcal{E}_1 = \{(a, b) : a < b\}$
2. $\mathcal{E}_2 = \{[a, b] : a < b\}$
3. $\mathcal{E}_3 = \{(a, b] : a < b\}$ *or* $\mathcal{E}_4 = \{[a, b) : a < b\}$
4. $\mathcal{E}_5 = \{(a, \infty) : a \in \mathbb{R}\}$ *or* $\mathcal{E}_6 = \{(-\infty, a) : a \in \mathbb{R}\}$
5. $\mathcal{E}_7 = \{[a, \infty) : a \in \mathbb{R}\}$ *or* $\mathcal{E}_8 = \{(-\infty, a] : a \in \mathbb{R}\}$

Proof. In text. □

Remark. The Borel $\sigma-$algebra on $\overline{\mathbb{R}}$ is $\mathcal{B}_{\overline{\mathbb{R}}} = \{E \subseteq \overline{\mathbb{R}} : E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}$. It can be generated by $\mathcal{E} = \{(a, \infty] : a \in \mathbb{R}\}$.

Proposition 2. *If* $(\mathrm{X}, \mathcal{M})$ *is a measurable space and* $f : \mathrm{X} \to \mathbb{R}$, *then TFAE*

1. f *is measurable*
2. $f^{-1}((a, \infty)) \in \mathcal{M}$ *for all* $a \in \mathbb{R}$
3. $f^{-1}([a, \infty)) \in \mathcal{M}$ *for all* $a \in \mathbb{R}$
4. $f^{-1}((-\infty, a)) \in \mathcal{M}$ *for all* $a \in \mathbb{R}$
5. $f^{-1}((-\infty, a]) \in \mathcal{M}$ *for all* $a \in \mathbb{R}$

Proposition 3 (p 43)**.** *Let* $(\mathrm{X}, \mathcal{M})$ *and* $(\mathrm{Y}, \mathcal{N})$ *be measurable spaces. If* $\mathcal{N}$ *is generated by* $\mathcal{E} \subseteq \mathcal{P}(\mathrm{Y})$, *then* $f : \mathrm{X} \to \mathrm{Y}$ *is* $(\mathcal{M}, \mathcal{N})-$*measurable if and only if* $f^{-1}(E) \in \mathcal{M}$ *for all* $E \in \mathcal{E}$.

Proof. (⇒) Since $\mathcal{E} \subseteq \mathcal{N}$, if f is measurable then $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$ by definition.

(⇐) Define $O = \{E \in \mathrm{Y} : f^{-1}(E) \in \mathcal{M}\}$. Want to show O is a $\sigma-$algebra. Then since $\mathcal{E} \subseteq O$ and $\mathcal{N}$ is generated by $\mathcal{E}$, we will be able to conclude $\mathcal{N} \subseteq \mathcal{O}$. Recall (p4 of text) $f^{-1}(E^C) = [f^{-1}(E)]^C$ and $f^{-1}(\cup_1^\infty E_j) = \cup_1^\infty f^{-1}(E_j)$.

Claim: O is a $\sigma-$algebra.

Proof:

1. Since $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(\mathrm{Y}) = f^{-1}(\emptyset^C) = (f^{-1}(\emptyset))^C = (\emptyset)^C = \mathrm{X} \in \mathcal{M}$, we have $\emptyset, \mathrm{Y} \in O$.
2. Suppose $F \in O$. Then $f^{-1}(F) \in \mathcal{M}$ by definition of O and $f^{-1}(F^C) = [f^{-1}(F)]^C \in \mathcal{M}$ since $\mathcal{M}$ is a $\sigma-$algebra.
3. Suppose $\{F_j\}_{j=1}^\infty \in O$. Then $\{f^{-1}(F_j)\}_{j=1}^\infty \in \mathcal{M}$ and $f^{-1}(\cup_{j=1}^\infty F_j) = \cup_{j=1}^\infty f^{-1}(F_j) \in \mathcal{M}$ as $\mathcal{M}$ is a $\sigma-$algebra. Thus $\cup_{j=1}^\infty F_j \in O$.

Hence O is a $\sigma-$algebra on Y, which contains $\mathcal{E}$. Then $\mathcal{N} \in O$ implies f is measurable. □

Definition. *Let* X *be a nonempty set and let* $\{\mathrm{Y}_\alpha, \mathcal{N}_\alpha\}$ *be a family of measurable spaces. If* $f_\alpha : \mathrm{X} \to \mathrm{Y}_\alpha$ *is a map for all* $\alpha \in A$ *(some index set), then the* $\sigma-$***algebra on*** X ***generated by*** $\{f_\alpha\}_{\alpha \in A}$ *is the unique smallest* $\sigma-$*algebra on* X *that makes each* f_α *measurable. It is generated by* $\{f^{-1}(E) : \alpha \in A$ *and* $E \in \mathcal{N}_\alpha\}$.

Proposition 4. *If* $(\mathrm{X}, \mathcal{M})$ *is a measurable space, then* $f : \mathrm{X} \to \overline{\mathbb{R}}$ *is measurable if and only if* $f^{-1}((a, \infty]) \in \mathcal{M}$.

Theorem 3. *Let* $(\mathrm{X}, \mathcal{M})$ *be a measurable space, let* Y, Z *be topological spaces. Let* $\phi : \mathrm{Y} \to \mathrm{Z}$ *be a continuous function. If* $f : \mathrm{X} \to \mathrm{Y}$ *is* $(\mathcal{M}, \mathcal{B}_{\mathrm{Y}})-$*measurable, then* $\phi \circ f$ *is* $(\mathcal{M}, \mathcal{B}_{\mathrm{Z}})-$*measurable.*

Proof. By definition, we need to check $(\phi \circ f)^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{B}_{\mathrm{Z}}$. By Proposition 3, we need only to check $(\phi \circ f)^{-1}(E) \in \mathcal{M}$ for all E open in Z. Let E be open in Z. Then $\phi^{-1}(E)$ is open in Y since ϕ is continuous. Since f is $(\mathcal{M}, \mathcal{B}_{\mathrm{Y}})$−measurable and $\phi^{-1}(E)$ is open in Y, we have $f^{-1}(\phi^{-1}(E)) \in \mathcal{M}$. □

Fact. If V is an open set in $\mathbb{R}^2$, then there exists a family $\{R_j\}_{j=1}^{\infty}$ of open rectangles in $\mathbb{R}^2$ satisfying

1. $R_j \subseteq \mathrm{V}$ for all $j = 1, 2, 3...$
2. $\cup_{j=1}^{\infty} R_j = \mathrm{V}$

Proposition 5. *Let* $(\mathrm{X}, \mathcal{M})$ *be a measurable space. Let* $u, v : \mathrm{X} \to \mathbb{R}$ *be* $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$*−measurable. If* $\phi : \mathbb{R}^2 \to \mathbb{R}$ *is continuous, then* $h : \mathrm{X} \to \mathbb{R}$ *defined by* $h(x) = \phi(u(x), v(x))$ *is* $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$*−measurable.*

Proof. Define $f : \mathrm{X} \to \mathbb{R}^2$ by $f(x) = (u(x), v(x))$. Since $h = \phi \circ f$ and ϕ is continuous, by Theorem 3 it is enough to show that f is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}^2})$−measurable. First we show that if $R \subseteq \mathbb{R}^2$ is an open rectangle, then $f^{-1}(R) \in \mathcal{M}$. Let $(a, b), (c, d) \subseteq R$ be open intervals such that $R = \{(y, z) \in \mathbb{R}^2 | a < y < b, c < z < d\}$. If $(u(x), v(x)) \in R$, then $u(x) \in (a, b)$ and $v(x) \in (c, d)$ implies $x \in u^{-1}((a, b)) \cap v^{-1}((c, d))$. Hence $f^{-1}(R) = u^{-1}((a, b)) \cap v^{-1}((c, d))$. Since u, v are $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$−measurable, we see $f^{-1}(R) \in \mathcal{M}$ as $\mathcal{M}$ is closed under countable intersections. Now let V be an open set in $\mathbb{R}^2$. By the above fact, there is a family $\{R_j\}_{j=1}^{\infty}$ of open rectangles such that $\bigcup_{1}^{\infty} R_j = \mathrm{V}$. So $f^{-1}(\mathrm{V}) = f^{-1}\left(\bigcup_{1}^{\infty} R_j\right) = \bigcup_{1}^{\infty} f^{-1}(R_j) \in \mathcal{M}$. Thus f is $(\mathrm{X}, \mathcal{B}_{\mathbb{R}^2})$−measurable. By Theorem 3, $h = \phi \circ f$ is $(\mathrm{X}, \mathcal{B}_{\mathbb{R}})$−measurable. □

Proposition 6 (p45)**.** *Let* $(\mathrm{X}, \mathcal{M})$ *be a measurable space. If* $c \in \mathbb{R}$ *and* $f, g : \mathrm{X} \to \mathbb{R}$ *are* $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$*−measurable, then*

1. cf *is* $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$*−measurable*
2. $f + g$ *is* $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$*−measurable*
3. fg *is* $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$*−measurable*

Proof.
1. Define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(y) = cy$. Then $\phi \circ f = cf$ and by Theorem 3, cf is measurable.
2. Define $\phi : \mathbb{R}^2 \to \mathbb{R}$ by $\phi(y, z) = y + z$. Then $\phi(f, g) = f + g$ and since ϕ is continuous, by Proposition 5, $f + g$ is measurable.
3. Define $\phi : \mathbb{R}^2 \to \mathbb{R}$ by $\phi(y, z) = yz$. Then $\phi(f, g) = fg$ and since ϕ is continuous, by Prop 5, fg is measurable. □

Note. In the above proposition, points (1) and (2) imply its a vector space and adding on point (3) implies it is an algebra. Also, the proposition is true if we consider $f, g : \mathrm{X} \to \overline{\mathbb{R}}$.

Proposition 7. *If* $\{f_j\}_{j=1}^{\infty}$ *is a sequence of* $\overline{\mathbb{R}}$*−valued measurable functions on* $(\mathrm{X}, \mathcal{M})$ *then*

$$g_1(x) = \sup_{j \geq 1} f_j(x) \qquad g_3(x) = \limsup_{j \geq 1} f_j(x)$$
$$g_2(x) = \inf_{j \geq 1} f_j(x) \qquad g_4(x) = \liminf_{j \geq 1} f_j(x)$$

are all $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$*−measurable. Moreover, if* $f(x) = \lim_{j \to \infty} f_j(x)$ *exists for all* $x \in \mathrm{X}$*, then* f *is measurable.*

Proof. Let $a \in \mathbb{R}$ be given. Then $\{x \in \mathrm{X} : g_1(x) > a\} = \bigcup_{j=1}^{\infty} \{x \in \mathrm{X} : f_j(x) > a\}$ implies $g_1^{-1}((a, \infty]) = \bigcup_{1}^{\infty} f_j^{-1}((a, \infty]) \in \mathcal{M}$ since f_j is measurable. Thus g_1 is measurable. Also $\{x \in \mathrm{X} : g_3(x) > a\} = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \{x \in \mathrm{X} : f_k(x) > a\}$ implies $g_3^{-1}((a, \infty]) = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} f_k^{-1}((a, \infty]) \in \mathcal{M}$. So g_3 is measurable. Since $g_2(x) = \inf_{j \geq 1} f_j(x) = -\sup_{j \geq 1} -f_j(x)$ and $g_4(x) = -\limsup -f_j(x)$, we see g_2 and g_4 are measurable. □

Corollary 1. *If* $f, g : \mathrm{X} \to \overline{\mathbb{R}}$ *are measurable functions, then* $\max\{f, g\}$ *and* $\min\{f, g\}$ *are also measurable.*

Corollary 2. *If $f : \mathrm{X} \to \overline{\mathbb{R}}$ is measurable, then so are $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$.*

Corollary 3. *If $f : \mathrm{X} \to \overline{\mathbb{R}}$ is measurable, then so is $|f| = f^+ + f^-$.*

1.2 Simple Functions (Generalized Step Functions)

Recall that for $E \subseteq \mathrm{X}$, the characteristic function of E is

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

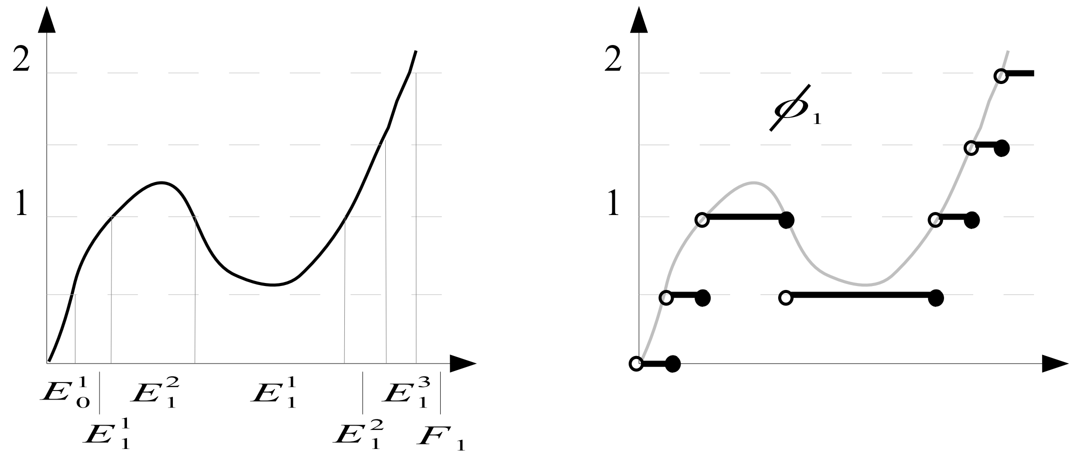
Definition (p46)**.** *A **simple function** on* X *is a measurable function whose range consists of a finite number of values in* $\mathbb{R}$.

If ϕ is a simple function with range $\{a_1, ..., a_n\}$, then for all $j = 1, 2, ..., n$, the set $E_j = \phi^{-1}(a_j)$ is measurable. The standard representation for ϕ is $\phi(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x)$.

Theorem 4 (p47)**.** *Let $(\mathrm{X}, \mathcal{M})$ be a measurable space.*

1. *If $f : \mathrm{X} \to [0, \infty]$ is a measurable function, then there exists a sequence $\{\phi_n\}_{n=0}^{\infty}$ of simple functions such that*
   - $0 \le \phi_0 \le \phi_1 \le ... \le f$
   - $\phi_n(x) \to f(x)$ *for all* $x \in \mathrm{X}$
   - ϕ_n *converges uniformly to f on the sets where f is uniformly bounded.*
2. *If $f : X \to \mathbb{C}$ is measurable, there is a sequence $\{\phi_n\}$ of simple functions such that $0 \le |\phi_1| \le |\phi_2| \le \cdots \le |f|$, $\phi_n \to f$ pointwise and $\phi_n \to f$ uniformly on any set on which f is bounded.*

Proof. (*of 1*) For all $n = 0, 1, 2, ...$ and $k = 0, 1, ..., 2^{2n} - 1$, set $E_n^k = f^{-1}((k2^{-n}, (k+1)2^{-n}])$ and $F_n = f^{-1}((2^n, \infty])$.



Define $\phi_n(x) = \sum_{k=0}^{2^{2n}-1} k2^{-n}\chi_{E_n^k} + 2^n \chi_{F_n}(x)$. We see that

$$E_n^k = f^{-1}\left(\left(k2^{-n}, \left(k + \frac{1}{2}\right)2^{-n}\right]\right) \cup f^{-1}\left(\left(\left(k + \frac{1}{2}\right)2^{-n}, (k+1)\,2^{-n}\right)\right) = E_{n+1}^{2k} \cup E_{n+1}^{2k+1}.$$

On the set E_n^k, we see $\phi_n = k2^{-n}\chi_{E_n^k}$ and $\phi_{n+1} = (2k)2^{-n-1}\chi_{E_{n+1}^{2k}} + (2k+1)2^{-n-1}\chi_{E_{n+1}^{2k+1}} = k2^{-n}\chi_{E_{n+1}^{2k}} + (k+\frac{1}{2})2^{-n}\chi_{E_{n+1}^{2k+1}}$. So $\phi_{n+1} \ge \phi_n$ on each E_n^k. Also, we can see $\phi_{n+1} \ge \phi_n$ on F_n. Therefore $\phi_{n+1} \ge \phi_n$. Since $\phi_n \le f$, on each E_n^k we see $0 \le f - \phi_n \le (k+1)2^{-n} - k2^{-n}$. It follows that $\phi_n \to f$ and on the sets where f is bounded, it converges uniformly (as these sets fall into some E_n^k.) ∎

Definition. *Let $(\mathrm{X}, \mathcal{M})$ be a measurable space.*

1. *A* ***positive measure*** *on $\mathcal{M}$ is a function $\mu : \mathcal{M} \to [0,\infty]$ with the properties $\mu(\emptyset) = 0$ and if $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ is a sequence of mutually disjoint sets, then $\mu\left(\bigcup_{j=1}^{\infty} E_k\right) = \sum_{j=1}^{\infty} \mu(E_j)$. To avoid trivialities, we assume $\mu(E) < \infty$ for some $E \subseteq \mathcal{M}$. Usually, we refer to a positive measure as just a* ***measure****.*

2. *A* ***measure space*** *is a triple $(\mathrm{X}, \mathcal{M}, \mu)$ where μ is a measure on $\mathcal{M}$.*

Theorem 5. *Let $(\mathrm{X}, \mathcal{M}, \mu)$ be a measure space. Then*

1. *$\mu(\emptyset) = 0$*

2. *(monotonicity) If $E, F \in \mathcal{M}$ and $E \subseteq F$, then $\mu(E) \leq \mu(F)$.*

3. *(subadditivity) If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$, then $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu(E_j)$.*

4. *(continuity from above) If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ and $E_1 \subseteq E_2 \subseteq ...$, then $\mu(\cup E_j) = \lim_{j\to\infty} \mu(E_j)$.*

5. *(continuity from below) If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ and $E_1 \supseteq E_2 \supseteq ...$ and $\mu(E_1) < \infty$, then $\mu(\cap E_j) = \lim_{j\to\infty} \mu(E_j)$.*

Proof. 1. Since there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$, we see $\mu(E) = \mu(E \cup \emptyset) = \mu(E) + \mu(\emptyset)$ since E and $\emptyset$ are disjoint. Now, subtracting $\mu(E)$ from both sides, we see $\mu(\emptyset) = 0$.

2. Let $E, F \in \mathcal{M}$ such that $E \subseteq F$. Then $F = E \cup (F \setminus E)$. Since E and $F \setminus E$ are disjoint, we see

$$\mu(F) = \mu(E \cup (F \setminus E)) = \mu(E) + \mu(F \setminus E) \geq \mu(E).$$

3. Use Lemma 1

4. Let $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ satisfying $E_1 \subseteq E_2 \subseteq \cdots$ with $E_0 = \emptyset$. Define $F_j = E_j \setminus \cup_{k=1}^{j-1} E_k = E_j \setminus E_{j-1}$. By Lemma 1, $\cup_{j=1}^{\infty} F_j = \cup_{j=1}^{\infty} E_j$. Thus

$$\mu(\cup E_j) = \mu(\cup F_j) = \sum \mu(F_j) = \sum \mu(E_j \setminus E_{j-1}) = \sum \mu(E_j) - \mu(E_{j-1}) = \lim_{n\to\infty} \mu(E_n)$$

as $\mu(E_0) = 0$.

5. Similar ☐

Definition. 1. *Let $(\mathrm{X}, \mathcal{M}, \mu)$ be a measure space. Then a $\mu-$**null set**, or simply **null set**, is a set in $\mathcal{M}$ that has measure 0.*

2. *If some statement P is true for all points in X except possibly those points in a null set, then we say P holds* ***almost everywhere (a.e.)*** *or we may say P holds for almost every $x \in \mathrm{X}$ or P holds $\mu-$a.e.*

1.3 Integration

Let $(\mathrm{X}, \mathcal{M}, \mu)$ be a measure space. We set $L^+ = \{f : \mathrm{X} \to [0,\infty] : f \text{ is measurable}\}$.

Definition. *Let $\phi \in L^+$ be a simple function. Then there exists $\{a_1, a_2, \ldots, a_n\} \subseteq [0,\infty)$ and $\{E_j\}_{j=1}^{n} \subseteq \mathcal{M}$ such that $\phi = \sum_{j=1}^{n} a_j \chi_{E_j}$. We define the* ***Lebesgue Integral of*** *ϕ with respect to μ by $\int_{\mathrm{X}} \phi d\mu := \sum_{j=1}^{n} a_j \mu(E_j)$. More generally, if $A \in \mathcal{M}$ is measurable, then we define the* ***Lebesgue Integral of*** *ϕ* ***over*** *A with respect to μ as $\int_A \phi d\mu := \int_{\mathrm{X}} \phi \chi_A d\mu = \sum_{j=1}^{n} a_j \mu(E_j \cap A)$.*

Definition. *Let $f \in L^+$ be any function. Then the* ***Lebesgue Integral of*** f *with respect to μ is*

$$\int_X f d\mu = \sup\left\{\int_X \phi d\mu | 0 \leq \phi \leq f, \phi \in L^+, \phi \text{ is simple}\right\}.$$

Also, if $A \in \mathcal{M}$ is measurable, then the ***Lebesgue Integral of*** f ***over*** A *with respect to μ is given by* $\int_A f d\mu = \int_X f \chi_A d\mu$.

Proposition 8. *Let $f, g \in L^+$ and $c \in [0, \infty]$. Then*

1. *If $f \leq g$, then $\int_X f d\mu \leq \int_X g d\mu$.*
2. *If $A, B \in \mathcal{M}$ and $A \subseteq B$ then $\int_A f d\mu \leq \int_B f d\mu$.*
3. *If $A \in \mathcal{M}$, then $\int_A c f d\mu = c \int_A f d\mu$.*
4. *If $f(x) = 0$ for all $x \in A \subseteq \mathcal{M}$, then $\int_A f d\mu = 0$.*
5. *If $A \in \mathcal{M}$ and $\mu(A) = 0$, then $\int_A f d\mu = 0$.*

Proposition 9. *Let $\phi \in L^+$ be a simple function. Define $\lambda : \mathcal{M} \to [0, \infty]$ by $\lambda(E) = \int_E \phi d\mu$. Then λ is a measure on $\mathcal{M}$.*

Proof. Since ϕ is simple, there exists $\{a_1, a_2, \ldots, a_n\} \in [0, \infty)$ and $\{E_j\}_{j=1}^n \subseteq \mathcal{M}$ such that $\phi = \sum_{j=1}^n a_j \chi_{E_j}$. Let $\{A_k\}_1^\infty \subseteq \mathcal{M}$ be mutually disjoint sets. Then

$$\begin{aligned}
\lambda\left(\bigcup_{k=1}^\infty A_k\right) &= \int_{\bigcup_{k=1}^\infty A_k} \phi d\mu \\
&= \int_X \phi \chi_{\bigcup_{k=1}^\infty A_k} d\mu \\
&= \sum_{j=1}^n a_j \mu\left(E_j \cap \left(\bigcup_{k=1}^\infty A_k\right)\right) \\
&= \sum_{j=1}^n a_j \mu\left(\bigcup_{k=1}^\infty (E_j \cap A_k)\right) \\
&= \sum_{j=1}^n a_j \sum_{k=1}^\infty \mu(E_j \cap A_k) \\
&= \sum_{k=1}^\infty \sum_{j=1}^n a_j \mu(E_j \cap A_k) \\
&= \sum_{k=1}^\infty \int_{A_k} \phi d\mu = \sum_{k=1}^\infty \lambda(A_k)
\end{aligned}$$

□

Theorem 6 (**Monotone Convergence Theorem**). *Let $\{f_n\}_{n=1}^\infty \subseteq L^+$ be given. Suppose that*

1. *$f_j \leq f_{j+1}$ for all $j = 1, 2...$*
2. *$f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in X$*

Then $f \in L^+$ and $\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu$.

Proof. Since $f(x) = \sup_{n\geq 1} f_n(x)$, by Prop 7, $f \in L^+$. By Prop 8(1), we see $\{\int_X f_n d\mu\}_{n=1}^\infty \subset [0, \infty]$ is a nondecreasing sequence of real numbers and thus by the MCT for $\mathbb{R}$, there exists $M \in [0, \infty]$ such that $\lim_{n\to\infty} \int_X f_n d\mu = M$. Since $f_n \leq f$ for all n, Prop 8(1) also tells us $\int_X f_n d\mu \leq \int_X f d\mu$. Thus $M \leq \int_X f d\mu$. Thus we just need to show $M \geq \int_X f d\mu$.

Let $\alpha \in (0, 1)$ and $\phi \in L^+$ be a simple function such that $0 \leq \phi \leq f$. Set $E_n := \{x \in X | f_n(x) \geq \alpha\phi(x)\}$. Since $f_j \leq f_{j+1}$, we see $E_1 \subseteq E_2 \subseteq \cdots$. Since $\alpha\phi \leq f$, we also have $\cup_{n=1}^\infty E_n = X$. Thus

$$\int_X f_n d\mu \geq \int_{E_n} f_n d\mu \geq \alpha \int_{E_n} \phi d\mu. \tag{1}$$

By Prop 9 and Thm 5(4), $\lim_{n\to\infty} \int_{E_n} \phi d\mu = \int_X \phi d\mu$. Thus, taking the limit of Equation 1 $M = \lim_{n\to\infty} \int_X f_n d\mu \geq \alpha \int_X \phi d\mu$. By definition of the Lebesgue Integral for f, taking the sup over ϕ and α gives us $M \geq \int_X f d\mu$. □

Proposition 10. *Let $\phi, \psi \in L^+$ be simple functions. Then $\int_X (\phi + \psi) d\mu = \int_X \phi d\mu + \int_X \psi d\mu$.*

Proof. Let $\sum_{j=1}^n a_j \chi_{E_j}$ and $\sum_{k=1}^m b_k \chi_{F_k}$ be the standard representations for ϕ and ψ respectively. Clearly, $E_j = \cup_{k=1}^m (E_j \cap F_k)$ for each j and $F_k = \cup_{j=1}^n (E_j \cap F_k)$ for each k. So

$$\begin{aligned}
\int_X (\phi + \psi) d\mu &= \sum_{j=1}^n \sum_{k=1}^m (a_j + b_k)\mu(E_j \cap F_k) \\
&= \sum_{j=1}^n a_j \sum_{k=1}^m \mu(F_k \cap E_j) + \sum_{k=1}^m b_k \sum_{j=1}^n \mu(F_k \cap E_j) \\
&= \sum_{j=1}^n a_j \mu\left(\bigcup_{k=1}^m F_k \cap E_j\right) + \sum_{k=1}^m b_k \mu\left(\bigcup_{j=1}^n F_k \cap E_j\right) \\
&= \sum_{j=1}^n a_j \mu(E_j) + \sum_{k=1}^m a_k \mu(F_k) \\
&= \int_X \phi d\mu + \int_X \psi d\mu
\end{aligned}$$

□

Theorem 7. *If $\{f_n\}_{n=1}^\infty \subseteq L^+$ and $f(x) = \sum_{n=1}^\infty f_n(x)$ for all $x \in X$, then $\int_X f d\mu = \sum_{n=1}^\infty \int_X f_n d\mu$.*

Proof. First we will show for a sum of two functions, then n functions, then an infinite series of functions. So let $f_1, f_2 \in L^+$, then by Theorem 4, there exists $\{\phi_j\}_{j=1}^\infty, \{\psi_j\}_{j=1}^\infty \subseteq L^+$ such that ϕ_j, ψ_j are simple with

- $0 \leq \phi_1 \leq \phi_2 \leq \ldots \leq f_1$ and $0 \leq \psi_1 \leq \psi_2 \leq \ldots \leq f_2$
- $\lim \phi_j = f_1$ and $\lim \psi_j = f_2$.

From these it follows that

- $0 \leq \phi_1 + \psi_1 \leq \phi_2 + \psi_2 \leq \ldots \leq f_1 + f_2$
- $\lim \phi_j + \psi_j = f_1 + f_2$.

By the Monotone Convergence Theorem and Proposition 10,

$$\int_X (f_1 + f_2) d\mu = \lim_{j\to\infty} \int_X (\phi_j + \psi_j) d\mu = \lim_{j\to\infty} \int_X \phi_j d\mu + \int_X \psi_j d\mu = \int_X f_1 d\mu + \int_X f_2 d\mu$$

Using Induction, we can show for n functions. To show for an infinite series, note that

- $0 \leq \sum_{n=1}^1 f_n \leq \sum_{n=1}^2 f_n \leq \ldots \leq \sum_{n=1}^\infty f_n$
- $\lim_{N\to\infty} \sum_{n=1}^N f_n(x) = f(x)$

Thus, applying the Monotone Convergence Theorem again, we see

$$\int_X f d\mu = \lim_{N\to\infty} \int_X \sum_{n=1}^N f_n d\mu = \lim_{N\to\infty} \sum_{n=1}^N \int_X f_n d\mu = \sum_{n=1}^\infty \int_X f_n d\mu.$$

□

Lemma (Fatou's Lemma- P.52). *If $\{f_n\}_{n=1}^\infty \subseteq L^+$, then $\int_X (\liminf f_n) d\mu \leq \liminf \int_X f_n d\mu$.*

Proof. Define $g_k(x) = \inf_{n\geq k} f_n(x)$ for all k and for all $x \in \mathrm{X}$. By Proposition 7, $g_k \in L^+$ for all k. Also $(g_k)_{k=1}^{\infty}$ is a monotone sequence with $g(x) := \lim_{k\to\infty} g_k(x) = \liminf_{n\to\infty} f_n(x)$. By the Monotone Convergence Theorem,

$$\lim_{k\to\infty} \int_{\mathrm{X}} g_k d\mu = \int_{\mathrm{X}} \lim_{k\to\infty} g_k d\mu = \int_{\mathrm{X}} \liminf_{n\to\infty} f_n d\mu.$$

We also see

$$\lim_{k\to\infty} \int_{\mathrm{X}} g_k d\mu = \liminf_{k\to\infty} \int_{\mathrm{X}} g_k d\mu \leq \liminf_{k\to\infty} \int_{\mathrm{X}} f_k d\mu$$

since $g_k \leq f_k$ for all k. Combining these two equations, we get what we wanted. □

Proposition 11 (P 51)**.** *If $f \in L^+$, then $\int_{\mathrm{X}} f d\mu = 0$ if and only if $f = 0$ a.e.*

Proof. First, we will show for simple functions. Let $\phi \in L^+$ be a simple function and say $\phi = \sum_{j=1}^{n} a_j \chi_{E_j}$. Suppose $\phi = 0$ a.e. Then either $a_j = 0$ or $\mu(E_j) = 0$ for all $j = 1, ..., n$. Thus $\int_{\mathrm{X}} \phi d\mu = \sum_{j=1}^{n} a_j \mu(E_j) = 0$. Now suppose $\int_{\mathrm{X}} \phi d\mu = 0$. Then $a_j \mu(E_j) = 0$ for all $j = 1, ..., n$, which implies either a_j or $\mu(E_j) = 0$ for all j. Thus $\phi = 0$ a.e.

Now let $f \in L^+$. Suppose $f = 0$ a.e. Then for all simple $\phi \in L^+$ such that $0 \leq \phi \leq f$, $\phi = 0$ a.e. Then $\int_{\mathrm{X}} \phi d\mu = 0$ and by the definition of a Lebesgue Integral, $\int_{\mathrm{X}} f d\mu = \sup\{\int_{\mathrm{X}} \phi d\mu : \phi \in L^+, 0 \leq \phi \leq f, \text{ and } \phi \text{ is simple}\} = 0$. Now suppose $f \neq 0$ a.e. Then for sufficiently large n, $\mu(\{x \in \mathrm{X} : f(x) > \frac{1}{n}\}) > 0$. Set $E = \{x \in \mathrm{X} : f(x) > \frac{1}{n}\}$. Then $\mu(E) > 0$. Consider the simple functions $\frac{1}{n}\chi_E$. We see $0 \leq \frac{1}{n}\chi_E \leq f$. By Proposition 8(1),

$$\int_{\mathrm{X}} f d\mu \geq \int_{\mathrm{X}} \frac{1}{n}\chi_E d\mu = \frac{1}{n}\mu(E) > 0.$$

□

Remark. This shows for $f \in L^+$, the Lebesgue Integral does not see values of f on the null sets.

Corollary 4. *If $\{f_n\}_{n=1}^{\infty} \subseteq L^+$ and $\liminf_{n\to\infty} f_n(x) \geq f(x)$ a.e. with $f \in L^+$, then $\int_{\mathrm{X}} f d\mu \leq \liminf_{n\to\infty} \int_{\mathrm{X}} f_n d\mu$.*

Proof. Set $E = \{x \in \mathrm{X} : \liminf f_n(x) < f\}$. By hypothesis, $\mu(E) = 0$. Thus we have

$$\liminf_{n\to\infty} f_n \chi_{E^C} \geq f\chi_{E^C} \text{ for all } x \in \mathrm{X}, \text{ and} \qquad (*)$$

$$f\chi_E = 0 \text{ a.e. implies } \int_{\mathrm{X}} f\chi_E d\mu = 0. \qquad (**)$$

Using these together with Fatou's Lemma, we see

$$\begin{aligned}
\liminf \textstyle\int_{\mathrm{X}} f_n d\mu &\geq \liminf \textstyle\int_{\mathrm{X}} f_n \chi_{E^C} d\mu \text{ by Proposition 8(1)} \\
&\geq \textstyle\int_{\mathrm{X}} \liminf f_n \chi_{E^C} d\mu \text{ by Fatou} \\
&\geq \textstyle\int_{\mathrm{X}} f\chi_{E^C} d\mu \text{ by Proposition 8(1) and } (*) \\
&= \textstyle\int_{\mathrm{X}} f\chi_{E^C} d\mu + \int_{\mathrm{X}} f\chi_E d\mu \text{ by Prop 11} \\
&= \textstyle\int_{\mathrm{X}} f\chi_{E^C} + f\chi_E d\mu \text{ by } (**) \\
&= \textstyle\int_{\mathrm{X}} f d\mu.
\end{aligned}$$

□

Definition. *Let $(\mathrm{X}, \mathcal{M}, \mu)$ be a measure space. Then the measure μ is **complete** if whenever $E \in \mathcal{M}$ is a nullset, we find $F \in \mathcal{M}$ for all $F \subseteq E$.*

Note. If μ is complete, then for $E \in \mathcal{M}$ with $\mu(E) = 0$ and $F \subseteq E$, we must have $\mu(F) = 0$.

Theorem 8. *Suppose $(\mathrm{X}, \mathcal{M}, \mu)$ is a measure space. Set $\mathcal{N} = \{N \in \mathcal{N} | \mu(N) = 0\}$ and $\overline{\mathcal{M}} = \{E \cup F | E \in \mathcal{M}, F \subseteq N$ for some $N \in \mathcal{N}\}$. Then $\overline{\mathcal{M}}$ is a $\sigma-$algebra and there exists a unique extension of μ to a measure $\overline{\mu}$ on $\overline{\mathcal{M}}$. Say $\overline{\mu}$ is the **completion** of μ.*

Proof. 1. Clearly $\emptyset, X \in \overline{\mathcal{M}}$.

2. Let $G \in \overline{\mathcal{M}}$. Want to show $G^C \in \overline{\mathcal{M}}$. Find $E \in \mathcal{M}$ and $F' \subseteq N' \in \mathcal{N}$ such that $G = E \cup F'$. Define $N = N' \setminus E$ and $F = F' \setminus E$. Then $G = E \cup F$ and $E \cap N = E \cap F = \emptyset$. Also $F \subseteq N \in \mathcal{N}$. Then $E \cup F' = E \cup F = (E \cap N^C) \cup F = ((E \cup N) \cap N^C) \cup ((E \cup N) \cap F) = (E \cup N) \cap (N^C \cup F)$. So $G^C = (E \cup F')^C = [(E \cup N) \cap (N^C \cup F)]^C = (E \cup N)^C \cup (N^C \cup F)^C = (E \cup N)^C \cup (N \cap F^C)$. Now $E \cup N \in \mathcal{M}$ which implies $(E \cup N)^C \in \mathcal{M}$. Also $N \cap F^C \subseteq N$. So $G^C \in \overline{\mathcal{M}}$ by definition.

3. If $\{G_j\}_{j=1}^{\infty} \subseteq \overline{\mathcal{M}}$, then there exists $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ and $\{N_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ and $\{F_j\}_{j=1}^{\infty}$ such that $F_j \subseteq N_j$ and $G_j = E_j \cup F_j$. Then $\bigcup_{j=1}^{\infty} G_j = \bigcup_{j=1}^{\infty} E_j \cup F_j = \left(\bigcup_{j=1}^{\infty} E_j\right) \cup \left(\bigcup_{j=1}^{\infty} F_j\right)$. Notice that $\bigcup_{j=1}^{\infty} F_j \subseteq \bigcup_{j=1}^{\infty} N_j$ and $\mu(\cup N_j) \leq \sum \mu(N_j) = 0$. So $\cup F_j \subseteq N \in \mathcal{N}$. So $\bigcup_{j=1}^{\infty} G_j \subseteq \overline{\mathcal{M}}$.

□

Definition. *Define* $\overline{\mu} : \overline{\mathcal{M}} \to [0, \infty]$ *by* $\overline{\mu}(E) = \mu(E)$ *for all* $E \in \mathcal{M}$ *and* $\overline{\mu}(E \cup F) = \mu(E)$ *for all* $E \in \mathcal{M}$ *and* $F \subseteq N \in \mathcal{N}$.

Notes.

1. $\overline{\mu}$ defines a measure. (prove)

2. $\overline{\mu}$ is well-defined and unique.

   - Well-defined: Suppose $E_1 \cup F_1 = E_2 \cup F_2$ with $E_1, E_2 \in \mathcal{M}$ and $F_1 \subseteq N_1 \in \mathcal{N}, F_2 \subseteq N_2 \in \mathcal{N}$. Then $\overline{\mu}(E_1 \cup F_1) = \mu(E_1) \leq \mu(E_2 \cup N_2) = \mu(E_2) = \overline{\mu}(E_2 \cup F_2)$. Similarly, $\geq$. So $\overline{\mu}(E_1 \cup F_1) = \overline{\mu}(E_2 \cup F_2)$.
   - Unique: Suppose $\overline{\nu} : \overline{\mathcal{M}} \to [0, \infty]$ is another completion. Let $E \cup F \in \overline{\mathcal{M}}$. Then $\overline{\nu}(E \cup F) \leq \overline{\nu}(E \cup N) = \mu(E \cup N) = \mu(E) = \overline{\mu}(E \cup F) = \mu(E) = \overline{\nu}(E) \leq \overline{\nu}(E \cup F)$. Thus $\overline{\nu}(E \cup F) = \overline{\mu}(E \cup F)$.

Definition. *Let* $(X, \mathcal{M}, \mu)$ *be a measure space. If* $E = \bigcup_{j=1}^{\infty} E_j$ *with* $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ *and* $\mu(E_j) < \infty$ *for all* j, *then* E *is* $\sigma-$***finite***.

Proposition 12 (p. 52)**.** *If* $f \in L^+$ *and* $\int_X f d\mu < \infty$, *then* $\{x \in X | f(x) = \infty\}$ *is a null set and* $\{x \in X | f(x) > 0\}$ *is a* $\sigma-$*finite set.*

Proof. Set $E = \{x \in X | f(x) = \infty\}$. Then $\infty > \int_X f d\mu \geq \int_E f d\mu = \infty \cdot \mu(E)$, which implies $\mu(E) = 0$. Also for all $j \geq 1$, set $E_j = \{x \in X | f(x) > \frac{1}{j}\}$. Then $\{x \in X | f(x) > 0\} = \cup E_j$ and $\infty > \int_X f d\mu \geq \int_{E_j} f d\mu > \frac{1}{j}\mu(E_j)$, which says $\mu(E_j) < \infty$. □

Definition. *Let* $(X, \mathcal{M}, \mu)$ *be a measure space. Define* $\widetilde{L}'(X, \mathcal{M}, \mu)$ *to be the collection of all measurable functions* $f : X \to \overline{\mathbb{R}}$ *such that* $\int_X |f| d\mu < \infty$.

Note. If f is measurable, so is $|f| \in L^+$ and $|f| = f^+ + f^-$. Thus $\int f^{\pm} d\mu \leq \int |f| d\mu < \infty$ if $f \in \widetilde{L}'$.

Definition. *If* $f \in \widetilde{L}'(X, \mathcal{M}, \mu)$, *then* f *is* ***integrable*** *and define* $\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu$.

Proposition 13 (p. 53)**.** *If* $f \in \widetilde{L}'(X, \mathcal{M}, \mu)$, *then* $|\int_X f d\mu| \leq \int_X |f| d\mu$.

Proof. By Theorem 7,

$$\left|\int_X f d\mu\right| = \left|\int_X f^+ d\mu - \int_X f^- d\mu\right| \leq \left|\int_X f^+ d\mu + \int_X f^- d\mu\right| \leq \left|\int_X |f| d\mu\right| = \int_X |f| d\mu.$$

□

Proposition 14 (p. 54)**.** *If* $f, g \in \widetilde{L}'(X, \mathcal{M}, \mu)$, *then TFAE*

1. $\int_E f d\mu = \int_E g d\mu$ *for all* $E \in \mathcal{M}$.

2. $\int_X |f - g| d\mu = 0$

3. $f = g$ a.e.

Proof. (2)⇒(3) By Prop 11

(3)⇒(2) If $f = g$ a.e., then $f - g = 0$ a.e. which implies $|f-g| = 0$ a.e. and thus $\int_X |f-g| d\mu = 0$ by Prop 11.

(2)⇒(1) If $\int_X |f-g| d\mu = 0$, then for all $E \in \mathcal{M}$

$$\left| \int_E f d\mu - \int_E g d\mu \right| = \left| \int_X (f-g)\chi_E d\mu \right| \leq \int_X |f-g|\chi_E d\mu \leq \int_X |f-g| d\mu = 0.$$

Thus $\int_E f d\mu = \int_E g d\mu$.

(1)⇒(3) Contrapositive. Then $\mu(\{x \in X | f(x) - g(x) \neq 0\}) > 0$. Define $E_1 = \{x \in X | f(x) - g(x) > 0\}$ and $E_2 = \{x \in X | f(x) - g(x) < 0\}$. Then either $\mu(E_1), \mu(E_2)$ or both are > 0. Suppose $\mu(E_1) > 0$. Then $(f-g)\chi_{E_1} \in L^+$ and so $\int_{E_1} (f-g) d\mu > 0$. This implies $\int_{E_1} f d\mu > \int_{E_1} g d\mu$ and so(1) does not hold. Similarly for $\mu(E_2) > 0$. □

Note. We say f and g are related if $f = g$ $\mu-$a.e. This defines an equivalence relation between functions in $\widetilde{L}'(X, \mathcal{M}, \mu)$.

Definition. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Define $L^1(X, \mu)$ to be the collection of all equivalence classes of integrable functions with respect to the relation just described.*

Notation. If we write $f \in L^1(\mu)$, then we really mean f is a representative for its equivalence class.

Proposition 15 (p.47)**.** *Suppose μ is a complete measure. Then*

1. *If f is measurable and $g = f$ $\mu-$a.e., then g is measurable.*
2. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions and $\lim_{n\to\infty} f_n(x) = f(x)$ a.e., then f is measurable.*

Proposition 16 (p.48)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space and $(X, \overline{\mathcal{M}}, \overline{\mu})$ be its completion. If $f : X \to \overline{\mathbb{R}}$ is $(\overline{\mathcal{M}}, \mathcal{B}_{\overline{\mathbb{R}}})-$measurable, then there exists an $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})-$measurable function g such that $f = g$ $\overline{\mu}-$a.e.*

Note. We identify $L^1(\mu)$ with $L^1(\overline{\mu})$.

Definition. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Define $\rho_1 : L^1(\mu) \times L^1(\mu) \to [0, \infty)$ by $\rho_1(f, g) = \int_X |\widetilde{f} - \widetilde{g}| d\mu$ where $\widetilde{f}, \widetilde{g} \in \widetilde{L}'(X, \mathcal{M}, \mu)$ are representatives for the equivalence classes f and g.*

Proposition 17. *The function ρ_1 is a metric on $L^1(\mu)$.*

Proof. Clearly, $\rho_1(f, g) = \rho_1(g, f)$. Also if $f, g, h \in L^1(\mu)$, then

$$\rho_1(f, g) = \int_X |f-g| d\mu = \int_X |f - h + h - g| d\mu \leq \int_X |f-h| + |h-g| d\mu = \rho_1(f, h) + \rho_1(h, g).$$

Finally, let $f, g \in L^1(\mu)$ and $\widetilde{f}, \widetilde{g} \in \widetilde{L}'(X, \mathcal{M}, \mu)$ be representatives for f and g. Then $\rho_1(f, g) = \int_X |\widetilde{f} - \widetilde{g}| d\mu = 0$ if and only if $\widetilde{f} = \widetilde{g}$ a.e. which happens if and only if f, g are in the same equivalence class. □

Definition. *If $\{f_n\}_{n=1}^{\infty} \subseteq L^1(\mu)$ and $f \in L^1(\mu)$ satisfies $\lim_{n\to\infty} \rho_1(f_n, f) = 0$, then we write $f_n \to f$ in $L^1(\mu)$ and say f_n* ***converges (strongly)*** *to f in $L^1(\mu)$.*

Theorem (Lebesgue's Dominated Convergence Theorem). *Let $\{f_n\}_{n=1}^{\infty} \subseteq L^1(\mu)$ be a sequence such that $\lim\limits_{n\to\infty} f_n(x) = f(x)$ $\mu-$a.e. and there exists $g \in L^1(\mu)$ such that $|f_n(x)| \leq g$ $\mu-$a.e. for all n. Then $f \in L^1(\mu)$ and $\displaystyle\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu$.*

Proof. By Propositions 15 and 16, we may assume f is measurable. By hypothesis, we see $|f(x)| \leq g(x)$ $\mu-$a.e. which implies $\int_X |f|d\mu \leq \int_X gd\mu < \infty$. So $f \in L^1(\mu)$. Since $|f_n(x)| < g(x)$ $\mu-$a.e., we also see that $g+f_n \geq 0$ and $g-f_n \geq 0$ $\mu-$a.e. for all n. Notice that since $\lim_{n\to\infty} f_n = f$ $\mu-$a.e., $\liminf_{n\to\infty}(g+f_n)(x) = g(x)+f(x)$ $\mu-$a.e. and $\liminf_{n\to\infty}(g-f_n)(x) = g(x)-f(x)$ $\mu-$a.e. By the corollary to Fatou's Lemma (Corollary 4), $\int_X (g+f)d\mu \leq \liminf_{n\to\infty} \int_X (g+f_n)d\mu = \int_X gd\mu + \liminf_{n\to\infty} \int_X f_n d\mu$ and $\int_X (g-f)d\mu \leq \liminf_{n\to\infty} \int_X (g-f_n)d\mu = \int_X gd\mu - \limsup_{n\to\infty} \int_X f_n d\mu$. Thus $\limsup_{n\to\infty} \int_X f_n d\mu \leq \int_X f d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu$. Since $\geq$ is obvious, we see they are all $=$ and thus $\int_X f d\mu = \lim_{n\to\infty} f_n d\mu$. □

Corollary 5. *Suppose $\{f_n\}_{n=1}^\infty \subseteq L^1(\mu)$ satisfies the hypotheses of the LDC Theorem. Then $f_n \to f$ in $L^1(\mu)$, that is, $\lim_{n\to\infty} \int_X |f_n - f|d\mu = 0$.*

Proof. Notice

- $\lim_{n\to\infty} |f_n(x) - f(x)| = 0$ $\mu-$a.e.
- $|f_n(x) - f(x)| \leq 2g(x)$ $\mu-$a.e. for all n.

Then by the LDC Theorem, $\int_X |f_n - f|d\mu = \int_X 0 d\mu = 0$. □

Theorem 9 (p 55)**.** *Suppose $\{f_n\}_{n=1}^\infty \subseteq L^1(\mu)$ satisfies $\sum_{n=1}^\infty \int_X |f_n|d\mu < \infty$. Then $\sum_{n=1}^\infty f_n$ converges $\mu-$a.e. to some function $f \subseteq L^1(\mu)$ and $\int_X \sum_{n=1}^\infty f_n d\mu = \sum_{n=1}^\infty \int_X f_n d\mu$.*

Proof. Define $g(x) = \sum_{n=1}^\infty |f_n(x)|$ for all $x \in X$. By Theorem 7 and our hypotheses

$$\int_X g(x)d\mu = \int_X \sum_{n=1}^\infty |f_n|d\mu = \sum_{n=1}^\infty \int_X |f_n|d\mu < \infty.$$

Then $g \in L^1(\mu)$. By Proposition 12, $\sum_{n=1}^\infty |f_n(x)| < \infty$ $\mu-$a.e. Hence $\sum f_n$ convergence absolutely $\mu-$a.e. So we may put $f(x) = \sum_{n=1}^\infty f_n(x)$ for those x where the series converges and $f(x) = 0$ everywhere else (i.e., on a null set). Moreover, $|\sum_{n=1}^\infty f_n(x)| \leq \sum_{n=1}^\infty |f_n(x)| \leq g(x)$ $\mu-$a.e. By the LDC Theorem, $f \in L^1(\mu)$ and $\int_X f d\mu = \int_X \lim_{N\to\infty} \sum_{n=1}^N f_n(x) = \lim_{N\to\infty} \sum_1^N \int_X f_n(x) = \sum_1^\infty \int_X f_n d\mu$. □

Types of Convergence

- $f_n \to f$ pointwise if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$.
- $f_n \to f$ a.e. if $\lim_{n\to\infty} f_n(x) = f(x)$ $\mu-$a.e.
- $f_n \to f$ uniformly if for all $\epsilon > 0$ there exists N_ϵ such that for all $n > N_\epsilon$ we have $|f_n - f| < \epsilon$ for all $x \in X$.
- $f_n \to f$ in L^1 if $\lim_{n\to\infty} \int |f_n - f|d\mu = 0$. (**strong convergence**)
- $f_n \to f$ in measure if for all $\epsilon > 0$ we have $\lim_{n\to\infty} \mu(\{x \in X | |f_n(x) - f(x)| \geq \epsilon\}) = 0$.

Definition. *We say that $\{f_n\}_{n=1}^\infty \subseteq L^1(\mu)$ is **Cauchy in measure** if for all $\epsilon > 0$ we have $\lim_{m,n\to\infty} \mu(\{x \in X | |f_n(x) - f_m(x)| \geq \epsilon\}) = 0$.*

Proposition 18. *Suppose $\{f_n\}_{n=1}^\infty \subseteq L^1(\mu)$ and $f \in L^1(\mu)$. If $f_n \to f$ in L^1, then $f_n \to f$ in measure.*

Proof. Let $\epsilon > 0$ be given. Set $E_n = \{x \in X | |f_n(x) - f(x)| \geq \epsilon\}$. Now

$$0 = \frac{1}{\epsilon} \lim_{n\to\infty} \int_X |f_n - f|d\mu \geq \lim_{n\to\infty} \frac{1}{\epsilon} \int_{E_n} |f_n - f|d\mu \geq \lim_{n\to\infty} \frac{1}{\epsilon} \int_{E_n} \epsilon d\mu = \lim_{n\to\infty} \mu(E_n) \geq 0.$$

Thus $\mu(E_n) = 0$. □

Theorem 10. *Suppose that $\{f_n\}_{n=1}^\infty$ is a sequence of measurable functions that are Cauchy in measure. Then there exists a measurable function f such that $f_n \to f$ in measure.*

Proof. Choose $\{g_j\}_{j=1}^\infty = \{f_{n_j}\}_{j=1}^\infty \subseteq \{f_n\}_{j=1}^\infty$ such that for all j we have

$$\mu(\underbrace{\{x \in \mathrm{X} : |g_j(x) - g_{j+1}(x)| \geq 2^{-j}\}}_{E_j}) \leq 2^{-j}.$$

Set $F_k = \bigcup_{j=k}^\infty E_j$. Then $\mu(F_k) \leq \sum_{j=k}^\infty \mu(E_j) \leq \sum 2^{-j} = 2^{1-k}$. For $x \notin F_k$, we have for all $i \geq j \geq k$ $|g_j(x) - g_i(x)| \leq \sum_{\ell=j}^{i-1} |g_{\ell+1}(x) - g_\ell(x)| \leq \sum_{\ell=j}^{i-1} 2^{-\ell} \leq 2^{1-j}$. It follows that $\{g_j\}_{j=1}^\infty$ is pointwise Cauchy on F_k^C for all k. Then there exists $f : \mathrm{X} \to \overline{\mathbb{R}}$ such that $g_j \to f$ on F_k^C for all k, that is, $g_j \to f$ pointwise on $\mathrm{X} \setminus (\bigcap_1^\infty F_k)$ and $f = 0$ on the null set.
Since $\mu(F_1) = \sum \mu(E_j) \leq \sum 2^{-j} = 1$ and $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$, we find that $0 \leq \mu(\bigcap_1^\infty F_k) = \lim \mu(F_k) \leq \lim 2^{1-k} = 0$. Thus $\mu(\bigcup_1^\infty F_k) = 0$. Thus $g_j \to f$ $\mu-$a.e. and by Proposition 15, f is measurable.
For each $x \notin F_j$ we see

$$|g_j(x) - f(x)| \leq \lim_{i\to\infty} |g_j(x) - g_i(x)| + \lim_{i\to\infty} |g_i(x) - f(x)| \leq \lim_{i\to\infty} \sum_{\ell=j}^{i-1} |g_{\ell+1}(x) - g_\ell(x)| \leq \lim_{i\to\infty} \sum_{l=j}^{i-1} 2^{-\ell} \leq 2^{1-j}.$$

(We know $\lim |g_i(x) - f(x)| = 0$ as $x \notin F_j$ implies $x \notin F_i$.)Thus $g_j \to f$ in measure.
Observe $|f_n(x) - f(x)| \leq |f_n(x) - g_j(x)| + |g_j(x) - f(x)|$. So if $|f_n(x) - f(x)| \geq \epsilon$ then either $|f_n(x) - g_j(x)| \geq \frac{\epsilon}{2}$ or $|g_j(x) - f(x)| \geq \frac{\epsilon}{2}$. Thus

$$\mu\left(\{x \in \mathrm{X} : |f_n(x) - f(x)| \geq \epsilon\}\right) \leq \mu\left(\left\{x \in \mathrm{x} : |f_n(x) - g_j(x)| \geq \frac{\epsilon}{2}\right\}\right) + \mu\left(\left\{x \in \mathrm{X} : |g_j(x) - f(x)| \geq \frac{\epsilon}{2}\right\}\right).$$

So taking the limit of both sides as $n, j \to \infty$, we get

$$\lim_{n\to\infty} \mu(\{x \in \mathrm{X} : |f_n(x) - f(x)| \geq \epsilon\} = 0$$

since $\lim \mu\left(\left\{x \in \mathrm{X} : |f_n(x) - g_j(x)| \geq \frac{\epsilon}{2}\right\}\right) = 0$ for f_n is Cauchy in measure and $\lim \mu\left(\left\{x \in \mathrm{X} : |g_j(x) - f(x)| \geq \frac{\epsilon}{2}\right\}\right) = 0$ for f_n converges in measure. □

Theorem 11. *Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of measurable functions such that $f_n \to f$ in measure with f measurable. Then there exists $\{f_{n_j}\}_{j=1}^\infty \subseteq \{f_n\}_{n=1}^\infty$ such that $f_{n_j} \to f$ $\mu-$a.e.*

Proof. Choose a subsequence $\{f_{n_j}\}_{j=1}^\infty$ such that $\mu(\underbrace{\{x \in \mathrm{C} : |f_{n_j} - f| \geq 2^{-j}\}}_{E_j}) \leq 2^{-j}$. Setting $F_k = \cup_{j=k}^\infty E_j$, $\mu(F_k) \leq 2^{1-k}$.
For $x \notin F_k$ and $j \geq k$ we see that $|f_{n_j}(x) - f(x)| \leq 2^{-j}$. It follows that $f_{n_j} \to f$ pointwise in $\mathrm{X} \setminus \cap_{k=1}^\infty F_k$. Thus $f_{n_j} \to f \mu-$a.e. since $\mu(\cup_{k=1}^\infty F_k) = 0$. □

Theorem 12. *Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of measurable functions and $f_n \to f$ and $f_n \to g$ in measure for some measurable f and g. Then $f = g$ $\mu-$a.e.*

Corollary 6. *If $\{f_n\}_{n=1}^\infty \subseteq L^1(\mu)$ and $f \in L^1(\mu)$ with $f_n \to f$ in L^1, then there exists a subsequence $\{f_{n_j}\}_{j=1}^\infty$ such that $f_{n_j} \to f$ $\mu-$a.e.*

Examples. Take $\mathrm{X} = \mathbb{N}, \mathcal{M} = \mathcal{P}(\mathbb{N}), \mu(E) =$ the number of elements in E. (that is, the counting measure).

- If $f \in L^+(\mu)$, then $\int_\mathrm{X} f d\mu = \sum_1^\infty f(k)$
- If $f \in L^1(\mu)$, then $\sum_{k=1}^\infty |f(k)| = \int_\mathbb{N} |f| d\mu < \infty$. So $\sum_{k=1}^\infty f(k)$ is absolutely convergent.
- Suppose that $f_n(k) = \frac{k}{n}$. Then $f_n(k) \to 0$ pointwise (and thus $\mu-$a.e.), but not uniformly (as for all $\epsilon > 0$, $\frac{k}{n} \geq \epsilon$ when $k \geq n\epsilon$) and not in measure (as $\mu(\{k \in \mathbb{N} : \frac{k}{n} \geq \epsilon\}) = \infty$)
- Suppose that $f_n(k) = \frac{1}{n}$. Then $f_n(k) \to 0$ pointwise, uniformly, in measure, and $\mu-$a.e., but not in L^1.

- Suppose that $f_n(k) = \begin{cases} \frac{k}{n^2} & \text{for } k \le n, \\ 0 & \text{otherwise.} \end{cases}$ Then $f_n(k) \to 0$ pointwise, uniformly, μ–a.e., and in measure, but not in L^1.

Theorem (**Egoroff's Theorem**). *Suppose $\mu(\mathrm{X}) < \infty$ and $f_1, f_2, ..., f$ are complex valued and measurable functions on X such that $f_n \to f$ a.e. Then for all $\epsilon > 0$ there exists $E \subseteq \mathrm{X}$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on $\mathrm{X} \setminus E$.*

Proof. Let N be the set of all points where $f_n(x) \nrightarrow f(x)$. So $\mu(N) = 0$. For each $k, n \in \mathbb{N}$, define $E_{n,k} = \cup_{m=n}^{\infty} \{x \in \mathrm{X} \setminus N : |f_m(x) - f(x)| \ge \frac{1}{k}\}$. Observe for all k that $E_{n+1,k} \subseteq E_{n,k}$ and $\cap_{n=1}^{\infty} E_{n,k} = \emptyset$. Since $\mu(E_{1,k}) \le \mu(\mathrm{X}) < \infty$, we may use Theorem 5 to conclude that $0 = \mu(\cap_{n=1}^{\infty} E_{n,k}) = \lim \mu(E_{n,k})$. So for all k there exists n_k such that $\mu(E_{n_k,k}) < 2^{-k}\epsilon$. Set $E = N \cup (\cup_{k=1}^{\infty} E_{n_k,k})$. Then $\mu(E) \le \mu(N) + \mu(\cup E_{n_k,k}) \le \epsilon \sum_{k=1}^{\infty} 2^{-k} < \epsilon$. If $x \notin E$, then for all $n > n_k$, $|f_n(x) - f(x)| < \frac{1}{k}$, that is, $f_n \to f$ uniformly on $\mathrm{X} \setminus E$. ☐

1.4 L^p Spaces

Definition. *A function $F : (a, b) \to \mathbb{R}$ is **convex** on $(a, b) \subseteq \mathbb{R}$ if*

$$F(\lambda x + (1 - \lambda)y) \le \lambda F(x) + (1 - \lambda)F(y)$$

for all $\lambda \in [0, 1]$ and $x, y \in (a, b)$.

Theorem 13. *If F is convex on $(a, b) \subseteq \mathbb{R}$, then for all $[x, y] \subset (a, b)$ with $x < y$, we find that there exists $M < \infty$ such that $F(s) \ge -M$ for all $s \in [x, y]$.*

Proof. Suppose there does not exist $M < \infty$. Then for all $n \in \mathbb{N}$ there exists $s_n \in [x, y]$ such that $F(s_n) < -n$. Since $[x, y]$ is compact, there exists a subsequence (call it s_n for simplicity) such that $s_n \to s^*$ for some $s^* \in [x, y]$. Let $s \in [x, y] \setminus \{s^*\}$ be given. For each $\lambda \in [0, 1)$, we have

$$F(\lambda s + (1 - \lambda)s_n) \le \lambda F(s) + (1 - \lambda)F(s_n) < \lambda F(s) + (1 - \lambda)(-n) \to -\infty.$$

It follows that $F(\lambda s + (1 - \lambda)s^*) = -\infty$ for all $\lambda \in [0, 1)$. So $F(s) = -\infty$ for all $s \in [x, y] \setminus \{s^*\}$, which contradicts the fact that $F : (a, b) \to \mathbb{R}$. ☐

Theorem 14. *If F is convex on $(a, b) \subseteq \mathbb{R}$, then F is continuous on (a, b).*

Proof. We will first prove a claim.

Claim: For each $x, y, z \in (a, b)$ satisfying $x < y < z$, we have $\frac{F(y)-F(x)}{y-x} < \frac{F(z)-F(y)}{z-y}$.
Proof: Let $y = \lambda x + (1 - \lambda)z$ with $\lambda = \frac{z-y}{z-x}$. Then

$$F(y) \le \frac{z - y}{z - x}F(x) + \frac{y - x}{z - x}F(z)$$

as F is convex. This implies

$$F(x) \ge \frac{z - x}{z - y}F(y) - \frac{y - x}{z - y}F(z)$$

and thus

$$\frac{-1}{y - x}F(x) \le \frac{1}{z - y}F(z) - \frac{z - x}{(y - x)(z - y)}F(y).$$

Thus

$$\frac{F(y)-F(x)}{y-x} \le \frac{1}{y-x}F(y) + \frac{1}{z-y}F(z) - \frac{z-x}{(y-x)(z-y)}F(y) = \frac{1}{y-z}F(y) + \frac{1}{z-y}F(z) = \frac{F(z)-F(y)}{z-y}.$$

Let $[x, y] \subset (a, b)$ with $x < y$ be given. Then F is uniformly bounded from below by Theorem 13. Let $s \in (x, y)$ and $t \in (s, y)$. So $x < s < t < y$. Then

$$\frac{F(s) - F(x)}{s - x} \le \frac{F(t) - F(s)}{t - s} \le \frac{F(y) - F(t)}{y - t}$$

which implies

$$\tfrac{t-s}{s-x}[F(s)-F(x)]+F(s)\leq F(t)\leq \tfrac{t-s}{y-t}[F(y)-F(t)]+F(s).$$

Since F is uniformly bounded, the RHS does not blow up, so as $t\to s$ we see $F(t)\to F(s)$. Similarly for $t\in(x,s)$. Thus $\lim_{t\to s}F(t)=F(s)$. □

Theorem 15 (Jensen's Inequality). *Suppose that* $(\mathrm{X},\mathcal{M},\mu)$ *is a measure space with* $\mu(\mathrm{X})<\infty$*. If* F *is a convex function on* $\mathbb{R}$ *and* $f\in L^1(\mu)$*, then*

$$F\left(\frac{1}{\mu(\mathrm{X})}\int_{\mathrm{X}} f d\mu\right)\leq \frac{1}{\mu(\mathrm{X})}\int_{\mathrm{X}} F\circ f d\mu.$$

Proof. Since $f\in L^1(\mu)$, $\int_{\mathrm{X}}|f|d\mu<\infty$. By Proposition 12, $\{x\in \mathrm{X}:|f|=+\infty\}$ is a nullset. So WLOG we may assume f is $\mathbb{R}-$valued (just redefine it to be 0 on the nullset). Put $t=\frac{1}{\mu(\mathrm{X})}\int_{\mathrm{X}} f d\mu$. For each $s\in(-\infty,t)$ and $u\in(t,\infty)$, the claim above gives us

$$\frac{F(t)-F(s)}{t-s}\leq\frac{F(u)-F(t)}{u-t}.$$

Let $\beta=\sup_s\frac{F(t)-F(s)}{t-s}$. Then $\beta\leq\frac{F(u)-F(t)}{u-t}$ which implies $F(u)\geq F(t)+\beta(u-t)$ for $u\in(t,\infty)$. If $u\in(-\infty,t)$, then $\frac{F(t)-F(u)}{t-u}\leq\beta$ by definition of supremum. Thus $F(t)\leq F(u)+\beta(t-u)$ which implies $F(u)\geq F(t)+\beta(u-t)$. Let $u=f(x)$. Then $F(f(x))$ is measurable and $F(f(x))\geq F(t)+\beta(f(x)-t)$ which implies

$$\int_{\mathrm{X}}F(f(x))d\mu\geq\int_{\mathrm{X}}F(t)d\mu+\beta\left(\int_{\mathrm{X}}f(x)d\mu-\int_{\mathrm{X}}td\mu\right)=F(t)\mu(\mathrm{X})+\beta\left(\int_{\mathrm{X}}f(x)d\mu-t\mu(\mathrm{X})\right).$$

Note that if $F(f(x))$ is not in L^1 then it integrates to ∞, in which case this inequality is still true. Substituting the value for t, we see

$$\int_{\mathrm{X}}F(f(x))d\mu\geq F\left(\frac{1}{\mu(\mathrm{X})}\int f d\mu\right)\mu(\mathrm{X}).$$

□

Let $\mathrm{X}=[n],\mathcal{M}=\mathcal{P}(\mathrm{X}),\mu(k)=a_k$ where $\sum_{k=1}^n a_k=1$ and $a_k>0$. So $\int_{\mathrm{X}}fd\mu=\sum f(k)a_k$. Put $F=e^t$, which is convex on $\mathbb{R}$. Then by Jensen's Inequality, since $\mu(\mathrm{X})=1$, we have

$$\exp\left(\sum f(k)a_k\right)=\exp\left(\int_{\mathrm{X}}fd\mu\right)\leq\int_{\mathrm{X}}e^f d\mu=\sum a_k\exp(f(k)).$$

Put $y_k=e^{f(k)}$, that is, $f(k)=\ln y_k$. Then $\exp(\sum\ln y_k^{a_k})\leq\sum a_k\exp(\ln y_k)$ which implies

$$\prod_{k=1}^n y_k^{a_k}=\sum_{k=1}^n a_k y_k.$$

Theorem (Young's Inequality). *Let* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p,q>1$*. Then* $|ab|\leq\frac{1}{p}|a|^p+\frac{1}{q}|b|^q$*.*

Proof. Use the above with $\alpha_1=\frac{1}{p},\alpha_2=\frac{1}{q},y_1=|a|^p,y_2=|b|^p$. □

Theorem (Hölder's Inequality). *Let* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p,q>1$*. Let* $f,g\in L^+$*. Then*

$$\int fgd\mu\leq\left(\int f^pd\mu\right)^{1/p}\left(\int g^qd\mu\right)^{1/q}.$$

Proof. If $\int f^pd\mu=0$, then $f=0$ a.e. which implies $fg=0$ a.e. and thus $\int fgd\mu=0$. Similarly if $\int g^qd\mu=0$. So assume $\int f^pd\mu,\int g^qd\mu>0$. If $\int f^pd\mu=\infty$ or $\int g^qd\mu=\infty$, the inequality is clear. So assume $0<\int f^pd\mu,\int g^qd\mu<\infty$.Put

$$F=\frac{f}{\left(\int f^pd\mu\right)^{1/p}}\text{ and }G=\frac{g}{\left(\int g^qd\mu\right)^{1/q}}.$$

Observe $\int F^p d\mu = \frac{1}{\int f^p d\mu}\int f^p d\mu = 1$. Similarly, $\int G^q d\mu = 1$. Using Young'is Inequality,

$$\int_X FG d\mu \leq \int \frac{1}{p}F^p d\mu + \frac{1}{q}G^q d\mu = \frac{1}{p}\int F^p d\mu + \frac{1}{q}\int G^q d\mu = \frac{1}{p} + \frac{1}{q} = 1.$$

So $\int_X \frac{fg}{(\int f^p d\mu)^{1/p}(\int g^q d\mu)^{1/q}} d\mu \leq 1$ which implies

$$\int_X fg d\mu \leq \left(\int f^p d\mu\right)^{1/p}\left(\int g^q d\mu\right)^{1/q}.$$

□

Theorem (**Minkowski's Inequality**). *Suppose $p \geq 1$. Let $f, g \in L^+$ be given. Then*

$$\left(\int (f+g)^p d\mu\right)^{1/p} \leq \left(\int f^p d\mu\right)^{1/p} + \left(\int g^p d\mu\right)^{1/p}.$$

Proof. If $p = 1$, then it is trivial. So assume $p > 1$. Then

$$\begin{aligned} \int (f+g)^p d\mu &= \int f(f+g)^{p-1} d\mu + \int g(f+g)^{p-1} d\mu \\ &\leq (\int f^p d\mu)^{1/p}(\int (f+g)^p d\mu)^{p-1/p} + (\int g^p d\mu)^{1/p}(\int (f+g)^p d\mu)^{p-1/p}. \end{aligned}$$

If $\int (f+g)^p d\mu = 0$, clear. If it is ∞ then $(f+g)^p = 2^p(\frac{1}{2}f + \frac{1}{2}g)^p \leq 2^{p-1}f^p + 2^{p-1}g^p = 2^{p-1}(f^p + g^p)$ (since x^p is convex) which implies one of $\int f^p$ and $\int g^p$ is ∞. Thus we can divide by $(\int (f+g)^p d\mu)^{p-1/p}$ to get Minkowski's Inequality. □

Definition. *For each $p \in [1, \infty)$ and each measurable function f, define $||f||_p = (\int_X |f|^p d\mu)^{1/p}$ and $||f||_\infty = ess\ sup_{x \in X}|f(x)| = \int\{a \geq 0 : \mu(\{x \in X : |f(x)| > a\}) = 0\}$ (where $\inf \emptyset = \infty$. This is called the **essential supremum**.*

Definition. *For each $p \in [1, \infty]$ define $L^p(X, \mu) = \{f \in L^1(\mu) : ||f||_p < \infty\}$.*

1.5 Normed Vector Spaces

Let K denote $\mathbb{R}$ or $\mathbb{C}$. Recall that a **vector space** $\mathfrak{X}$ is a set of elements with addition and scalar multiplication. By a **subspace**, we mean a vector subspace of $\mathfrak{X}$. If $x \in \mathfrak{X}$, denote by Kx the subspace $\{kx \in \mathfrak{X} : k \in K\}$. If $\mathcal{M}$ and $\mathcal{N}$ are subspaces of $\mathfrak{X}$, then $\mathcal{M} \oplus \mathcal{N} = \{x + y \in \mathfrak{X} : x \in \mathcal{M}, y \in \mathcal{N}\}$.

Definition. *A **seminorm** on $\mathfrak{X}$ is a function $||\cdot|| : \mathfrak{X} \to [0, \infty)$ such that*

- $||x + y|| \leq ||x|| + ||y||$ *for all $x, y \in \mathfrak{X}$.*
- $||\lambda x|| = |\lambda|\ ||x||$ *for all $x \in \mathfrak{X}$ and $\lambda \in K$.*

If $||\cdot||$ also satisfies

- $||x|| = 0$ *if and only if $x = 0$*

*then $||\cdot||$ is called a **norm** on $\mathfrak{X}$. A pair $(\mathfrak{X}, ||\cdot||)$ is called a **normed vector space**.*

Examples.

- $\mathbb{R}^n$ is a VS and the function $||x||_p = (\sum_{k=1}^n |x_k|^p)^{1/p}$ for $p \in [1, \infty)$ is a norm. So is $||x||_\infty = \max\{|x_1|, |x_2|, ..., |x_n|\}$.
- For each $p \in [1, \infty]$, the space $L^P(\mu)$ is a VS and the function $||\cdot||_p$ is a norm on L^p.

Fact. If $(\mathfrak{X}, ||\cdot||)$ is a NVS, then $\rho_{||\cdot||}(x, y) = ||x - y||$ for $x, y \in \mathfrak{X}$ is a metric on $\mathfrak{X}$. The topology induced by this metric is called the **norm (or strong) topology**.

Definition. *If ρ is a metric on a set $\mathfrak{X}$, the* ***topology induced by*** *ρ is generated by $\mathcal{E} = \{U \in \mathfrak{X} :$ there exists $\epsilon > 0, x \in \mathfrak{X}$ such that $\rho(y,x) < \epsilon$ for all $y \in U\}$. (In Euclidean Space, these are the open balls of radius ϵ.) If $\mathcal{E} \subseteq \mathcal{P}(\mathfrak{X})$, then the smallest topology $\mathcal{T}(\mathcal{E})$ containing $\mathcal{E}$ is called the* ***topology generated by*** *$\mathcal{E}$.*

Note. Each set in $\mathcal{E}$ is open in the topology generated by $\mathcal{E}$ by definition.

Definition. *Two norms $||\cdot||$ and $||\cdot||_1$ are* ***equivalent*** *if there exists constants $0 < c_1, c_2 < \infty$ such that $c_1||x|| \leq ||x||_1 \leq c_2||x||$ for all $x \in \mathfrak{X}$.*

Examples.

- If $\mathfrak{X} = \mathbb{R}^n$, then for all $p, q \in [1,\infty]$, the norms $||\cdot||_p$ and $||\cdot||_q$ are equivalent.
- If $\mathfrak{X} = \mathbb{R}^{\mathbb{N}}$ (that is, the space of infinite sequences of real numbers), then for each $p \neq q \in [1,\infty)$, the norms $||\cdot||_p$ and $||\cdot||_q$ are not equivalent.

Definition. *If $(\mathfrak{X}, ||\cdot||)$ is a NVS that is complete with respect to $\rho_{||\cdot||}$, then we say that $(\mathfrak{X}, ||\cdot||)$, or just $\mathfrak{X}$, is a* ***Banach Space****.*

Definition. *Let $\{x_n\}_{n=1}^\infty \subseteq \mathfrak{X}$ be given. The series $\sum_{n=1}^\infty x_n$* ***converges*** *to $x \in \mathfrak{X}$ if $\lim_{N\to\infty} \sum_{n=1}^N x_n = x$ (i.e., $\lim_{N\to\infty} ||\sum_{n=1}^N x_n - x|| = 0$). The series $\sum_1^\infty x_n$ is* ***absolutely convergent*** *if $\sum_{n=1}^\infty ||x_n|| < \infty$.*

Theorem 16 (p. 152). *A NVS $(\mathfrak{X}, ||\cdot||)$ is complete if and only if every absolutely convergent series is convergent.*

Proof. ($\Rightarrow$) Suppose $(\mathfrak{X}, ||\cdot||)$ is complete. Let $\{x_n\}_{n=1}^\infty \subset \mathfrak{X}$ such that $\sum_{n=1}^\infty ||x_n|| < \infty$. Then for all N we can define $S_N = \sum_{n=1}^N x_n \in \mathfrak{X}$. Want to show S_N is Cauchy. Let $M > N$ be given. Then $||S_M - S_N|| = ||\sum_{N+1}^M x_n|| \leq \sum_{N+1}^M ||x_n|| \to 0$ as $M, N \to \infty$. Thus $\{S_N\}_{N=1}^\infty$ is a Cauchy Sequence in $\mathfrak{X}$ and since $\mathfrak{X}$ is complete there exists $x \in \mathfrak{X}$ such that $\lim_{N\to\infty} ||S_N - x|| = 0$. Thus $\sum_{n=1}^\infty x_n$ converges to $x \in \mathfrak{X}$.

($\Leftarrow$) Suppose every absolutely convergent series converges. Let $\{x_n\}_{n=1}^\infty \subseteq \mathfrak{X}$ be a Cauchy Sequence. Select a subsequence $\{x_{n_j}\}_{j=1}^\infty$ such that for all j and $n, m \geq n_j$, we have $||x_n - x_m|| < 2^{-j}$. Put $y_1 = x_{n_1}$ and $y_j = x_{n_j} - x_{n_{j-1}}$ for all $j > 1$. Then $x_{n_k} = \sum_{j=1}^k y_j$. Also,

$$\sum_{j=1}^\infty ||y_j|| = ||y_1|| + \sum_{j=2}^\infty ||y_j|| = ||x_{n_1}|| + \sum_{j=2}^\infty ||x_{n_j} - x_{n_{j-1}}|| \leq ||x_{n_1}|| + \sum_{j=2}^\infty 2^{-j} \leq ||x_{n_1}|| + 1 < \infty.$$

Then, by hypothesis, there exists $x \in \mathfrak{X}$ such that $\lim_{J\to\infty} \sum_{j=1}^J y_j = x$. Then $\lim_{J\to\infty} x_{n_J} = x$. Of course $||x_n - x|| \leq ||x_n - x_{n_j}|| + ||x_{n_j} - x||$ and for $n \geq n_j$, $||x_n - x_{n_j}|| \leq 2^{-j}$ and $||x_{n_j} - x|| \to 0$. Thus $||x_n - x|| \to 0$. □

Corollary 7. *If $(\mathrm{X}, \mathcal{M}, \mu)$ is a measure space, then $(L^1(\mu), ||\cdot||)$ is a Banach Space.*

Theorem 17 (p. 183). *For $p \in [1,\infty]$, the space $(L^P(\mu), ||\cdot||_p)$ is a Banach Space.*

Proof. Case 1: $p \in [1,\infty)$. Suppose that $\{f_k\}_{k=1}^\infty \subset L^p(\mu)$ satisfying $\sum_{k=1}^\infty ||f_k|| < \infty$. Put $A^p = \sum_1^\infty ||f_k||_p^p$ and $G_n = \sum_1^n |f_k|$, with $G = \sum_1^\infty |f_k|$. Clearly, $G_1^p < G_2^p < \cdots < G^p$ and by Minkowski's inequality

$$||G_n||_p^p = \int_{\mathrm{X}} \left(\sum_1^n |f_k|\right)^p d\mu \leq \sum_1^n \int_{\mathrm{X}} |f_k|^p d\mu = \sum_1^n ||f_k||_p^p \leq A^p.$$

By the Monotone Convergence Theorem,

$$\lim_{n\to\infty} ||G_n||_p^p = \lim_{n\to\infty} \int_{\mathrm{X}} \left(\sum_1^n |f_k|\right)^p d\mu = \int_{\mathrm{X}} \lim \left(\sum_1^n |f_k|\right)^p d\mu = \int_{\mathrm{X}} \left(\sum_1^\infty |f_k|\right)^p d\mu = ||G||_p^p \leq A^p < \infty$$

since $||G_n||_p^p \leq A^p$ for all n. This implies that $(\sum_1^\infty |f_k|)^p < \infty$ a.e. and so $\sum_1^\infty |f_k| < \infty$ a.e. Thus for almost every x there exists $F(x) < \infty$ such that $F(x) = \lim_{N\to\infty} \sum_1^N f_k(x)$. Define $F(x) = 0$ for those x where the sum is infinite. We need to show $F \in L^p(\mu)$ and $\lim ||F - \sum_1^N f_k|| = 0$. We have

$$\int_{\mathrm{X}} |F|^p d\mu = \int \left|\sum_1^\infty f_k(x)\right|^p d\mu \leq \int_{\mathrm{X}} \left(\sum_1^\infty |f_k|\right)^p d\mu = ||G||_p^p \leq A^p < \infty.$$

Thus $F \in L^p(\mu)$. Finally, for all n, $|F - \sum_1^n f_k|^p = |\sum_{n+1}^\infty f_k|^p \leq (\sum_1^\infty |f_k|)^p = G \in L^1$. Also $\lim_{n\to\infty} F(x) - \sum f_k(x) = 0$ a.e. So by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} ||F - \sum_1^n f_k||_p^p = \lim_{n\to\infty} \int |F - \sum_1^n f_k|^p d\mu = 0.$$

Thus by Theorem 16, $L^p(\mu)$ is complete.

Case 2: $p = \infty$. Let $\{f_k\} \subset L^\infty(\mu)$ satisfying $\sum_1^\infty ||f_k||_\infty < \infty$. For each k, set $A_k = \{x \in \mathrm{X} : |f_k(x)| > ||f_k||_\infty\}$. By the definition of $|| \cdot ||_\infty$, each A_k is a null set. Also $A = \bigcup_1^\infty A_k$ is a null set. For each $x \in \mathrm{X} \setminus A$, $\sum_1^\infty |f_k(x)| \leq \sum_1^\infty ||f_k||_\infty < \infty$. So there exists F such that $F(x) = \sum_1^\infty f_k(x) < \infty$ for all $x \in \mathrm{X} \setminus A$. Put $F(x) = 0$ for all $x \in A$. So $F = \lim_{N\to\infty} \sum_1^N f_k(x)$ $\mu-$a.e. Now for $x \in \mathrm{X} \setminus A$, $|F(x)| = |\sum_1^\infty f_k(x)| \leq \sum_1^\infty |f_k(x)| \leq \sum_1^\infty ||f_k||_\infty < \infty$. So $F \in L^\infty$ as $\mu(A) = 0$. Finally

$$\lim_{n\to\infty} ||F - \sum_1^n f_k||_\infty = \lim_{n\to\infty} ||\sum_{n+1}^\infty f_k||_\infty \leq \lim_{n\to\infty} \sum_{n+1}^\infty ||f_k||_\infty = 0$$

since $\sum_1^\infty ||f_k||_\infty < \infty$. □

Proposition 19. *Let $S = \{$simple functions on $\mathrm{X}\}$. For each $p \in [1, \infty]$, the set $S \cap L^P$ is dense in L^P.*

Proof. The case $p = \infty$ is covered by Theorem 4. Suppose $p \in [1, \infty)$ and let $f \in L^p(\mu)$ be given. Want to find a sequence $\{f_n\}_{n=1}^\infty \subseteq S \cap L^p$ such that $\lim_{n\to\infty} ||f_n - f||_p = 0$. By Theorem 4 (applied to f^+ and f^-), there exists a sequence $\{f_n\}_{n=1}^\infty \subseteq S$ such that $|f_n| \leq |f|$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathrm{X}$. Since $f \in L^p$, we find that $\int_{\mathrm{X}} |f_n|^p d\mu \leq \int_{\mathrm{X}} |f|^p d\mu < \infty$ which implies $f_n \in L^p$ for all n. So $\{f_n\}_{n=1}^\infty \subseteq S \cap L^p$. Moreover, $|f_n - f|^p \leq (|f_n| + |f|)^p \leq 2^p |f|^p \in L^1$ and $|f_n - f|^p \to 0$ for all $x \in X$. By the LDC, $\lim_{n\to\infty} \int_{\mathrm{X}} |f_n - f|^p d\mu = 0$ which implies $\lim ||f_n - f||_p = 0$. □

Proposition 20. *If $1 \leq p \leq q \leq r \leq \infty$, then $L^p \cap L^r \subseteq L^q$ and $||f||_q \leq ||f||_p^\lambda ||f||_q^{1-\lambda}$ where $\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}$.*

Proof. If $p = q = r = \infty$, trivial. If $p < \infty$ and $q = r = \infty$ then clearly $L^p \cap L^\infty \subseteq L^\infty$ and if we take $\lambda = 0$, then we see $||f||_q \leq ||f||_r = ||f||_p^0 ||f||_r^1$. So suppose $p, q < \infty$. If $r = \infty$, then

$$\begin{aligned}
\left(\int_{\mathrm{X}} |f|^q d\mu\right)^{1/q} &= \left(\int_{\mathrm{X}} |f|^p |f|^{q-p} d\mu\right)^{1/q} \\
&\leq \left(\int_{\mathrm{X}} |f|^p ||f||_\infty^{q-p} d\mu\right)^{1/q} \\
&= ||f||_\infty^{\frac{q-p}{q}} \left(\int_{\mathrm{X}} |f|^p d\mu\right)^{1/q} \\
&= ||f||_p^{p/q} ||f||_\infty^{1-p/q}.
\end{aligned}$$

Let $\lambda = \frac{p}{q}$. If $f \in L^p \cap L^\infty$, then $||f||_p, ||f||_\infty < \infty$, so $||f_q|| \leq ||f||_p^\lambda ||f||_\infty^{1-\lambda} < \infty$. Thus $f \in L^q$. Now suppose $r < \infty$. Note that $\lambda q < p$.

$$\begin{aligned}\left(\int_X |f|^q d\mu\right)^{1/q} &= \left(\int_X |f|^{\lambda q}|f|^{(1-\lambda)q} d\mu\right)^{1/q} \\ &\leq \left(\int \left(|f|^{\lambda q}\right)^{p/\lambda q} d\mu\right)^{\frac{1}{q}\left(\frac{\lambda q}{p}\right)} \left(\int \left(|f|^{(1-\lambda)q}\right)^{\frac{p}{p-\lambda q}} d\mu\right)^{\frac{1}{q}\left(\frac{p-\lambda q}{p}\right)} \quad \text{by Holder's Inequality} \\ &= \left(\int |f|^p d\mu\right)^{\lambda/p} \left(\int |f|^r d\mu\right)^{\frac{1-\lambda}{r}} \\ &= ||f||_p^{\lambda} ||f||_r^{1-\lambda}.\end{aligned}$$

By the same argument as above, $f \in L^P \cap L^\infty$ implies $f \in L^q$. □

Proposition 21. *If $\mu(X) < \infty$, then for all $1 \leq p \leq q \leq \infty$, we have $L^q \subseteq L^p$ and $||f||_p \leq ||f||_q \mu(X)^{\frac{1}{p}-\frac{1}{q}}$.*

Proof. If $q < \infty$, then

$$\begin{aligned}\left(\int_X |f|^p d\mu\right)^{1/p} &= \left(\int_X 1 \cdot |f|^p d\mu\right)^{1/p} \\ &\leq \left(\int_X \left(|f|^p\right)^{q/p} d\mu\right)^{\frac{p}{q}\left(\frac{1}{p}\right)} \left(\int_X |f|^{\frac{q}{q-p}} d\mu\right)^{\frac{1}{p}\left(\frac{q-p}{q}\right)} \\ &= ||f||_q \mu(X)^{\frac{1}{p}-\frac{1}{q}}.\end{aligned}$$

□

2 Measure Theory

The Lebesgue Measure on $\mathbb{R}^n$. Suppose $(\mathbb{R}^n, \mathcal{M}, m^n)$ is a measure space where the measure $m^n : \mathcal{M} \to [0, \infty]$ with $\mathcal{M} \subseteq \mathcal{P}(\mathbb{R}^n)$ is the unique measure such that

$$m^n\left(\prod_{k=1}^{n}(a_k, b_k)\right) = \prod(b_k - a_k)$$

where $(a_k, b_k) \subseteq \mathbb{R}$ for all k. What can we say about $\mathcal{M}$ if we want to measure all the open boxes $\prod(a_k, b_k)$?

- Since any open set is a countable union of open boxes, all open sets in the usual topology must be in $\mathcal{M}$.
- The smallest $\sigma-$algebra $\mathcal{M}$ must be the Borel $\sigma-$algebra.

So we want to somehow extend m^n from the boxes to all of $\mathcal{B}_{\mathbb{R}^n}$.

Definition. *Let $X \neq \emptyset$. A family of sets $\mathcal{C} \subseteq \mathcal{P}(X)$ is a **semialgebra** if*

1. *$\emptyset, X \in \mathcal{C}$*
2. *If $E_1, E_2 \in \mathcal{C}$, then $E_1 \cap E_2 \in \mathcal{C}$ (and thus all finite intersections are in $\mathcal{C}$).*
3. *If $E \in \mathcal{C}$, then there exists a finite sequence $\{E_i\}_{i=1}^k \in \mathcal{C}$ with $E_i \cap E_j = \emptyset$ for all $i \neq j$ such that $E^C = \bigcup_{i=1}^k E_i$.*

Examples. The following are semialgebras:

- $I = \{$ open, half-open, closed intervals on $\mathbb{R}\}$.
- $I^n = \{$crossproduct of any n elements of $I\}$.

Notation. Denote any interval with endpoints a and b by $I(a, b)$.

Definition. *Let $X \neq \emptyset$. A family of sets $\mathcal{F} \subseteq \mathcal{P}(X)$ is an **algebra** if*

1. *$\emptyset, X \in \mathcal{F}$*
2. *If $E_1, E_2 \in \mathcal{F}$, then $E_1 \cap E_2 \in \mathcal{F}$ (and thus all finite intersections are in $\mathcal{F}$).*

3. If $E \in \mathcal{F}$, then $E^C \in \mathcal{F}$.

Note that by 2 and 3, we are only allowing finite unions to be in $\mathcal{F}$, unlike in a $\sigma-$algebra.

Examples. The following are algebras

- $\mathcal{F}(I) = \{E \subseteq \mathbb{R} | E = \bigcup_{k=1}^{\ell} I_k, I_k \in I, I_j \cap I_k = \emptyset \text{ for } j \neq k\}$.
- $\mathcal{F}(I)^n = \{E \subseteq \mathbb{R}^n | E = \bigcup_{k=1}^{\ell} I_k, I_k \in I^n, I_j \cap I_k = \emptyset \text{ for } j \neq k\}$.

In general, if $\mathcal{C}$ is a semialgebra, then

$$\mathcal{F}(\mathcal{C}) = \left\{ E \subseteq \mathrm{X} | E = \bigcup_{k=1}^{\ell} E_k, E_k \in \mathcal{C}, E_j \cap E_k = \emptyset \text{ for } j \neq k \right\}$$

is an algebra.

Definition. *Let $\mathcal{C} \subseteq \mathcal{P}(\mathrm{X})$. A set function $\mu : \mathcal{C} \to [0, \infty]$ is called*

- ***monotone** if for all $A, B \in \mathcal{C}$ satisfying $A \subseteq B$, we have $\mu(A) \leq \mu(B)$.*
- ***finite additive** if $\{E_k\}_{k=1}^{\ell} \subseteq \mathcal{C}$ such that $E_j \cap E_k = \emptyset$ and $\bigcup_{k=1}^{\ell} E_k \in \mathcal{C}$ implies $\mu(\bigcup_{k=1}^{\ell} E_k) = \sum_{k=1}^{\ell} \mu(E_k)$.*
- ***countably additive** if $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{C}$ such that $E_j \cap E_k = \emptyset$ and $\bigcup_{k=1}^{\infty} E_k \in \mathcal{C}$ implies $\mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k)$.*
- ***countably subadditive** if $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{C}$ such that $\bigcup_{k=1}^{\infty} E_k \in \mathcal{C}$ implies $\mu(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} \mu(E_k)$.*

1st Goal: Given a monotone countably additive set function μ defined on a semialgebra $\mathcal{C}$, we want to extend μ to a monotone countably additive function $\widetilde{\mu}$ defined on an algebra $\mathcal{F}(\mathcal{C})$ generated by $\mathcal{C}$.

Proposition 22. *Let $\mathcal{C} \subseteq \mathcal{P}(\mathrm{X})$. Then there exists a unique algebra $\mathcal{F}(\mathcal{C}) \subseteq \mathcal{P}(\mathrm{X})$ such that $\mathcal{C} \subseteq \mathcal{F}(\mathcal{C})$ and if $\mathcal{A} \subseteq \mathcal{P}(\mathrm{X})$ is an algebra such that $\mathcal{C} \subseteq \mathcal{A}$, then $\mathcal{F}(\mathcal{C}) \subseteq \mathcal{A}$. So $\mathcal{F}(\mathcal{C})$ is the "smallest" algebra containing $\mathcal{C}$.*

Proof. Define $\mathcal{F}(\mathcal{C}) = \bigcap \{A | \mathcal{C} \subseteq A \subseteq \mathcal{P}(\mathrm{X}), \mathcal{A} \text{ is an algebra}\}$. □

Definition. *Given $\mathcal{C} \subseteq \mathcal{P}(\mathrm{X})$, the algebra $\mathcal{F}(\mathcal{C})$ provided by Prop 22 is called the **algebra generated by** $\mathcal{C}$.*

Proposition 23. *If $\mathcal{C}$ is a semialgebra, then the algebra generated by $\mathcal{C}$ is $\mathcal{F}(\mathcal{C}) := \{E \subseteq X : E = \cup_{k=1}^{\ell} E_k, E_j \cap E_k = \emptyset, j \neq k, E_k \in \mathcal{C}\}$.*

Example. Recall I was a semialgebra. What kind of properties does $m : I \to [0, \infty]$ defined by $m(I(a, b)) = b - a$ have? It is monotone, finitely additive, countably additive (2 cases: if the union is an interval which is finite or infinite), countably subadditive (by monotonicity, countable additivity and Lemma 1).

Theorem 18. *Suppose μ is a finitely additive and countable subadditive set function on a semialgebra $\mathcal{C}$ such that $\mu(\emptyset) = 0$. Then there exists a unique countably additive set function $\widetilde{\mu}$ on $\mathcal{F}(\mathcal{C})$ such that $\widetilde{\mu}(E) = \mu(E)$ for all $E \in \mathcal{C}$.*

Proof. For all $E \in \mathcal{F}(\mathcal{C})$, by Prop 23, there exists $\{E_k\}_{k=1}^n \subseteq \mathcal{C}$ such that $E = \cup_{k=1}^n E_k$ and $E_j \cap E_k = \emptyset$ if $j \neq k$. Define $\widetilde{\mu}(E) = \sum_{k=1}^n \mu(E_k)$.

Claim 1: $\widetilde{\mu}$ is well-defined.

Proof: Let $E \in \mathcal{F}(\mathcal{C})$ and suppose there exists $\{E_k\}_{k=1}^n$ and $\{F_\ell\}_{\ell=1}^m \subseteq \mathcal{C}$ such that $E_j \cap E_k = \emptyset$ for $j \neq k$ and $F_j \cap F_\ell = \emptyset$ for $j \neq \ell$ and $\cup_{k=1}^n E_k = E = \cup_{\ell=1}^m F_\ell$. Then for all $\ell = 1, 2, ..., m$, $F_\ell = F_\ell \cap E = F_\ell \cap (\cup_{k=1}^n E_k) = \cup_{k=1}^n (F_\ell \cap E_k)$ and for all $k = 1, 2, ..., n$, $E_k = E_k \cap E = E_k \cap (\cup F_\ell) = \cup_{\ell=1}^m (E_k \cap F_\ell)$. So

$$\begin{aligned}
\widetilde{\mu}(E) = \textstyle\sum_{k=1}^n \mu(E_\ell) &= \textstyle\sum_{k=1}^n \mu(\cup_{\ell=1}^m E_k \cap F_\ell) \\
&= \textstyle\sum_{k=1}^n \sum_{\ell=1}^m \mu(E_k \cap F_\ell) \\
&= \textstyle\sum_{\ell=1}^m \sum_{k=1}^n \mu(E_k \cap F_\ell) \\
&= \textstyle\sum_{\ell=1}^m \mu(\cup_{k=1}^n E_k \cap F_\ell) \\
&= \textstyle\sum_{\ell=1}^m \mu(F_\ell).
\end{aligned}$$

Claim 2: $\widetilde{\mu}$ is finitely additive on $\mathcal{F}(\mathcal{C})$.

Proof: Suppose $\{E_k\}_{k=1}^n \subseteq \mathcal{F}(\mathcal{C})$ with $E_k \cap E_j = \emptyset$ for $j \neq k$, then $\cup_{k=1}^n E_k \subseteq \mathcal{F}(\mathcal{C})$ since $\mathcal{F}(\mathcal{C})$ is an algebra. By Prop 23, there exists $\{G_r\}_{r=1}^s \subseteq \mathcal{C}$ such that $\cup_{r=1}^s G_r = \cup_{k=1}^n E_k$. Also, for all $k = 1, 2, ..., n$, there exist mutually disjoint $\{F_{k,\ell}\}_{\ell=1}^{m_k} \subseteq \mathcal{C}$ such that $E_k = \cup_{\ell=1}^{m_k} F_{k,\ell}$. Then for all $k = 1, ..., n$

$$E_k = E_k \cap \left(\bigcup_{k=1}^n E_k\right) = E_k \cap \left(\bigcup_{r=1}^s G_r\right) = \left(\bigcup_{\ell=1}^{m_k} F_{k,\ell}\right) \cap \left(\bigcup_{r=1}^s G_r\right) = \bigcup_{\ell=1}^{m_k}\bigcup_{r=1}^s (F_{k,\ell} \cap G_r).$$

Also for all $r = 1, .., s$, $G_r = G_r \cap \left(\bigcup_{k=1}^n E_k\right) = G_r \cap \left(\bigcup_{k=1}^n \bigcup_{\ell=1}^{m_k} F_{k,\ell}\right) = \bigcup_{k=1}^n \bigcup_{\ell=1}^{m_k} G_r \cap F_{k,l}$. Now

$$\begin{aligned}\widetilde{\mu}\left(\bigcup_{k=1}^n E_k\right) &= \widetilde{\mu}\left(\bigcup_{r=1}^s G_r\right) = \sum_{r=1}^s \mu(G_r) = \sum_{r=1}^s \mu\left(\bigcup_{k=1}^n \bigcup_{\ell=1}^{m_k} G_r \cap F_{k,\ell}\right) \\ &=^* \sum_{r=1}^s \sum_{k=1}^n \sum_{\ell=1}^{m_k} \mu(G_r \cap F_{k,\ell}) = \sum_{k=1}^n \widetilde{\mu}\left(\bigcup_{r=1}^s \bigcup_{\ell=1}^{m_k} G_r \cap F_{k,\ell}\right) = \sum_{k=1}^n \widetilde{\mu}(E_k).\end{aligned}$$

Claim 3: $\widetilde{\mu}$ is countably subadditive.

Proof: Same as above, except replace n with ∞ and change the $=^*$ to $\leq$.

Note that the countable additivity of $\widetilde{\mu}$ follows from the next theorem (Theorem 19) □

Theorem 19. *Let $\mathcal{F}$ be an algebra of sets on* X *and $\widetilde{\mu} : \mathcal{F} \to [0, \infty]$ be a set function such that $\widetilde{\mu}(\emptyset) = 0$. Then $\widetilde{\mu}$ is countably additive if and only if it is both finitely additive and countably subadditive.*

Proof. First note that if $\widetilde{\mu}$ is finitely additive, then (since $\mathcal{F}$ is an algebra) for $A, B \in \mathcal{F}$ with $A \subseteq B$, we see $\widetilde{\mu}(B) = \widetilde{\mu}(A \cup B \setminus A) = \widetilde{\mu}(A) + \widetilde{\mu}(B \setminus A) \geq \widetilde{\mu}(A)$. Thus $\widetilde{\mu}$ is monotone.

(⇒:) Suppose $\widetilde{\mu}$ is countably additive. Clearly $\widetilde{\mu}$ is finitely additive as $\widetilde{\mu}(\emptyset) = 0$. To show subadditive, let $\{E_k\}_{k=1}^\infty \subseteq \mathcal{F}$ such that $\cup_{k=1}^\infty E_k \in \mathcal{F}$. By Lemma 1, there exists a sequence $\{F_k\}_{k=1}^\infty$ of mutually disjoint sets such that $\cup_1^\infty F_k = \cup_1^\infty E_k$. Using countable additivity and monotonicity, we see $\widetilde{\mu}(\cup E_k) = \widetilde{\mu}(\cup F_k) = \sum \widetilde{\mu}(F_k) \leq \sum \widetilde{\mu}(E_k)$.

(⇐) Suppose $\widetilde{\mu}$ is finitely additive and countably subadditive. Let $\{E_k\}_1^\infty \subseteq \mathcal{F}$ be mutually disjoint sets such that $\cup_1^\infty E_k \in \mathcal{F}$. Since $\widetilde{\mu}$ is countably subadditive, $\widetilde{\mu}(\cup_1^\infty E_k) \leq \sum_1^\infty \widetilde{\mu}(E_k)$. To show the opposite inequality, we use finite additivity and monotonicity to conclude $\widetilde{\mu}(\cup_1^\infty E_k) \geq \widetilde{\mu}(\cup_1^n E_k) = \sum_1^n \widetilde{\mu}(E_k)$ for all n. Taking the limit as $n \to \infty$, we get $\widetilde{\mu}(\cup_1^\infty E_k) \geq \sum_1^\infty \widetilde{\mu}(E_k)$. Thus $\widetilde{\mu}(\cup_1^\infty E_k) = \sum_1^\infty \widetilde{\mu}(E_k)$. □

Definition. *Suppose $\mathcal{A} \subseteq \mathcal{P}(\mathrm{X})$ is an algebra. A function $\widetilde{\mu} : \mathcal{A} \to [0, \infty]$ is called a* ***premeasure*** *if $\widetilde{\mu}(\emptyset) = 0$ and $\widetilde{\mu}$ is countably additive.*

Theorem 18 shows how to construct a premeasure on an algebra, generated from a semialgebra, from a finitely additive countably subadditive function on that semialgebra.

Notation. Define $\widetilde{I} := \{(a, b] : a < b \in \mathbb{R}\} \cup \{(-\infty, b] : b \in \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\} \cup \{(-\infty, \infty)\} \cup \{\emptyset\}$. Note the $\sigma-$algebra generated by $\widetilde{I}$ is $\mathcal{B}_\mathbb{R}$. Also $\widetilde{I}$ is a semialgebra.

Proposition 24. *Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function. Define $\mu_F : \widetilde{I} \to [0, \infty]$ by $\mu_F((a, b]) = F(b) - F(a)$, $\mu_F((-\infty, b]) = F(b) - \lim_{x\to-\infty} F(x)$, $\mu_F((a, \infty)) = \lim_{x\to\infty} F(x) - F(a)$, $\mu_F((-\infty, \infty)) = \lim_{x\to\infty} F(x) - \lim_{x\to-\infty} F(x)$, $\mu_F(\emptyset) = 0$. Then μ_F is well-defined, finitely additive and monotone. Moreover, if F is right continuous, then μ_F is countably subadditive.*

Proof. It is clear that μ_F is well-defined. Suppose $\{I_k\}_{k=1}^n \subseteq \widetilde{I}$ are disjoint. First suppose each I_k is of the form $(a_k, b_k]$ and $\cup_{k=1}^n (a_k, b_k] = (a, b) \in \widetilde{I}$. Then WLOG, assume $a = a_1 < b_1 = a_2 < b_2 = ... = a_n < b_n = b$. So

$$\sum_1^n \mu_F(I_k) = \sum_1^n \mu_F((a_k, b_k]) = \sum_1^n F(b_k) - F(a_k) = F(b) - F(a) = \mu_F((a, b]).$$

Now suppose $\cup_{k=1}^n I_k = (-\infty, b] \in \widetilde{I}$. WLOG, assume $I_1 = (-\infty, b_1]$ and $I_k = (a_k, b_k]$ with $b_1 = a_2 < b_2 = ... = a_n < b_n = b$. So

$$\begin{aligned} \textstyle\sum_1^n \mu_F(I_k) = \mu_F(I_1) + \sum_2^n \mu_F((a_k, b_k]) &= F(b_1) - \lim_{x\to -\infty} F(x) + F(b_n) - F(a_2) \\ &= F(b_n) - \lim_{x\to-\infty} F(x) \\ &= \mu_F((-\infty, b]). \end{aligned}$$

Similarly, the other cases hold. Thus μ_F is finitely additive. Monotonicity follows. Thus we need only to show countable subadditivity in the case that F is right continuous. Suppose $I = (a, b] \subseteq \cup_{k=1}^\infty I_k$ with $\{I_k\}_{k=1}^\infty \subseteq \widetilde{I}$. Let $\epsilon \in (0, b-a)$. For each k, define

$$I_k' = \begin{cases} (a_k, b_k + \delta_k) & \text{if } I_k = (a_k, b_k], \\ (-\infty, b_k + \delta_k) & \text{if } I_k = (-\infty, b_k], \\ I_k & \text{otherwise}, \end{cases}$$

where δ_k satisfies $F(b_k + \delta_k) - F(b_k) < \epsilon 2^{-k}$. Now $\{I_k'\}_{k=1}^\infty$ is an open cover for $[a+\epsilon, b]$. Since compact, Heine Borel says there exists a finite subcover, call it $\{I_k'\}_{k=1}^n$ for simplicity. WLOG, assume $a + \epsilon \in I_1'$. If $b \notin I_1'$, then $b_1 + \delta_1 < b$ and thus $[b_1 + \delta_1, b] \subseteq \cup_{k=2}^n I_k'$. WLOG, assume $b_1 + \delta_1 \in I_2'$. Then $a_2 < b_1 + \delta_1$. If $b \notin I_2'$, then $b_2 + \delta_2 < b$ and so $[b_2 + \delta_2, b] \subseteq \cup_{k=3}^n I_k'$. Continue to find $m < n$ such that $a < a + \epsilon < b_1 + \delta_1 < b_2 + \delta_2 < \cdots < b < b_m + \delta_m$, that is, $b \in I_m'$. Note that this also says $a_{i+1} < b_i + \delta_i$. Now

$$\begin{aligned} F(b) - F(a) &= F(b) - F(a+\epsilon) + F(a+\epsilon) - F(a) \\ &\le F(b_m + \delta_m) - F(a_1) + F(a+\epsilon) - F(a) \\ &= \textstyle\sum_{k=1}^{m-1}((F(b_{k+1} + \delta_{k+1}) - F(b_k - \delta_k)) + F(b_1 + \delta_1) - F(a_1) + F(a+\epsilon) - F(a) \\ &\le \textstyle\sum_{k=1}^{m-1}((F(b_{k+1} + \delta_{k+1}) - F(a_{k+1})) + F(b_1 + \delta_1) - F(a_1) + F(a+\epsilon) - F(a) \\ &= \textstyle\sum_{k=1}^{m}((F(b_k + \delta_k) - F(a_k)) + F(a+\epsilon) - F(a) \\ &= \textstyle\sum_{k=1}^{m}((F(b_k + \delta_k) - F(b_k) + F(b_k) - F(a_k)) + F(a+\epsilon) - F(a) \\ &\le \textstyle\sum_{k=1}^{m} \epsilon 2^{-k} + \sum_{k=1}^m \mu_F(I_k) + F(a+\epsilon) - F(a) \\ &\le \epsilon + F(a+\epsilon) - F(a) + \textstyle\sum_{k=1}^m \mu_F(I_k) \end{aligned}$$

Letting $\epsilon \to 0^+$, the right continuity of F yields

$$\mu_F(I) = F(b) - F(a) \le \sum_1^\infty \mu_F(I_k).$$

Now suppose I is an infinite interval. If $I = (-\infty, b]$, then for each $M > -\infty$, the same argument shows that $\mu((M, b]) = F(b) - F(M) \le \sum_{k=1}^\infty \mu_F(I_k)$. Now letting $M \to -\infty$, we see $\mu_F((-\infty, b]) = F(b) - \lim_{M\to-\infty} F(M) \le \sum_{k=1}^\infty \mu_F(I_k)$. Similarly for the other cases. □

Note. It is also the case that μ_F is countably additive, but we don't prove that here. For reference, Folland refers to this as μ_0. This is similar to Prop 1.15 in Folland.

Proposition 25. *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous. Define $\mu_F : \widetilde{I} \to [0, \infty]$ as in Proposition 24. Then $\widetilde{\mu}_F : \mathcal{F}(\widetilde{I}) \to [0, \infty]$ defined by $\widetilde{\mu}_F(\cup_1^n I_k) = \sum_1^n \mu_F(I_k)$ whenever $\{I_j\}_{k=1}^n \subseteq \widetilde{I}$ satisfies $I_j \cap I_k = \emptyset$ for $j \neq k$ is a premeasure on $\mathcal{F}(\widetilde{I})$.*

Proof. Follows from Prop 23, Thm 18, and Prop 24. □

Remark. If $F = x$, then μ_F is the usual length of an interval.

Proposition 26. *Suppose $\mu : \widetilde{I} \to [0, \infty]$ is finitely additive and $\mu((a, b]) < \infty$ for each $a, b \in \mathbb{R}$. Then there exists an increasing function $F : \mathbb{R} \to \mathbb{R}$ such that $\mu((a, b]) = F(b) - F(a)$ for all $(a, b] \subset \mathbb{R}$. If μ is also countably additive on $\widetilde{I}$, then F is right continuous and $\mu_F = \mu$.*

Proof. For all $x \in \mathbb{R}$, define $F(x) = \begin{cases} \mu((0,x]) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\mu((x,0]) & \text{if } x < 0 \end{cases}$. We want to verify that $\mu((a,b]) = F(b) - F(a)$ for all $(a,b] \in \mathbb{R}$. Since μ is finitely additive, if $0 < a < b$

$$\mu((a,b]) = \mu((0,b] \setminus (0,a]) = \mu((0,b]) - \mu((0,a]) = F(b) - F(a).$$

Similarly for $a \leq 0 < b$ and $a < b \leq 0$. To show F is increasing, note that for $0 < a < b$, $F(b) - F(a) = \mu((a,b]) \geq 0$. Similarly for the other two cases. Now, suppose μ is countably additive. We want to show F is right continuous. Let $x \in \mathbb{R}$ and $\{x_k\}_{k=1}^{\infty} \subset (x, \infty)$ such that $x_k \to x$ as $k \to \infty$ and $\{x_k\}_{k=1}^{\infty}$ is decreasing. Notice $\{F(x_k)\}_{k=1}^{\infty}$ is decreasing and bounded below by $F(x)$, so it converges.

Case 1: Let $x > 0$. Then

$$\begin{aligned} F(x_1) = \mu((0,x_1]) &= \mu((0,x]) + \mu(\cup_{k=1}^{\infty}(x_{k+1}, x_k]) \\ &= F(x) + \lim_{N\to\infty} \textstyle\sum_1^N \mu((x_{k+1}, x_k]) \\ &= F(x) + \lim_{N\to\infty} \textstyle\sum_1^N F(x_k) - F(x_{k+1}) \\ &= F(x) + \lim_{N\to\infty} F(x_1) - F(x_{N+1}). \end{aligned}$$

Thus $F(x) = \lim_{N\to\infty} F(x_{N+1})$. Similarly if $x = 0$.

Case 2: Let $x < 0$. Then for some $m \in \mathbb{N}$ we find $x_m < 0$. Then

$$\begin{aligned} F(x) = -\mu((x,0]) &= -(\mu((x_m,0]) + \mu((x,x_m])) \\ &= F(x_m) - \mu(\textstyle\bigcup_{k=n}^{\infty}(x_{k+1}, x_k]) \\ &= F(x_m) - \lim_{n\to\infty} \textstyle\sum_{k=n}^{\infty} F(x_k) - F(x_{k+1}) \\ &= F(x_m) - F(x_m) + \lim F(x_{n+1}). \end{aligned}$$

Thus $F(x) = \lim_{n\to\infty} F(x_{n+1})$.

To show $\mu = \mu_F$, we need only to compare $\mu(I)$ and $\mu_F(I)$ on infinite intervals. Suppose $I = (-\infty, \infty)$. Then $I = \cup_{k=1}^{\infty}((-k, -k+1] \cup (k-1, k])$. Since $\mu(I) \geq 0$ we have

$$\begin{aligned} \mu(I) &= \lim_{n\to\infty} \sum_1^n \mu((-k,-k+1]) + \mu((k-1,k]) \\ &= \lim_{n\to\infty} \sum_1^n F(-k+1) - F(-k) + \lim_{n\to\infty} \sum_1^n F(k) - F(k-1) \\ &= \lim_{n\to\infty} F(-n) + \lim_{n\to\infty} F(n) = \mu_F(I). \end{aligned}$$

□

<u>2nd Goal</u>: Given a premeasure on an algebra A, we want to extend a measure on the $\sigma-$algebra generated by A.

<u>Intermediate Goal</u>: Approximate "measure" of any subset of a nonempty X using the premeasure on an algebra A.

<u>Idea</u>: Recall $m : I \to [0, \infty]$ was given by $m(I(a,b)) = b - a$ (where $I(a,b)$ is any interval with endpoints a and b) for $a, b \in \overline{\mathbb{R}}$. The algebra generated by I is $\mathcal{F}(I)$, which is the collection of all finite unions of disjoint intervals in I. The extension of m to $\widetilde{m}$ on $\mathcal{F}(I)$ is $\widetilde{m}(E) = \sum_{k=1}^{n}(b_k - a_k)$ for $E = \cup_{k=1}^{n} I(a_k, b_k)$, with $\{I(a_k,b_k)\}_{k=1}^{n}$ mutually disjoint. Now, we want to extend $\widetilde{m}$ to a set function that "measures" any subset of $\mathbb{R}$. Suppose $E \subseteq \mathbb{R}$. Then we can find at least one countable family $\{I_k\}_{k=1}^{\infty} \subseteq \mathcal{F}(I)$ such that $E \subseteq \cup_{k=1}^{\infty} I_k$ (take $I_k = \mathbb{R}$ for all k). Since $E \subseteq \cup_{k=1}^{\infty} I_k$, we expect the "measure" of E to be $\leq \widetilde{m}(\cup_{k=1}^{\infty} I_k) \leq \sum_1^{\infty} \widetilde{m}(I_k)$. So, in general, we want

$$\text{"measure of" } E \leq \sum_{k=1}^{\infty} \widetilde{m}(I_k).$$

So we should define it as

$$\text{“measure” of } E = \inf\left\{\sum \widetilde{m}(I_k) : \{I_k\} \subseteq \mathcal{F}(I), E \subseteq \cup I_k\right\}.$$

Note that we do not get anything new by trying to approximate the “measure” from the inside, since if $E \subseteq \mathbb{R} \in \mathcal{F}(I)$, then the “inner measure of” E is $\widetilde{m}(\mathbb{R})-$ “the outer measure of” $R \setminus E$.

Definition. *Let $\mathcal{E} \subseteq \mathcal{P}(\mathrm{X})$ be such that $\emptyset, \mathrm{X} \in \mathcal{E}$ and $\rho : \mathcal{E} \to [0, \infty]$ satisfy $\rho(\emptyset) = 0$. For each $E \in \mathrm{X}$, define $\mu^*(E) = \inf\{\sum \rho(E_k) : \{E_k\}_{k=1}^{\infty} \subseteq \mathcal{E}, E \subseteq \cup E_k\}$ to be the **outermeasure of E induced by** ρ.*

In general,

Definition. *If $\mu^* : \mathcal{P}(\mathrm{X}) \to [0, \infty]$ is monotone, countably subadditive, and satisfies $\mu^*(\emptyset) = 0$, call μ^* an **outermeasure** on X.*

Proposition 27. *The set function in the former definition is an outer measure in the sense of the latter definition.*

Proof. Clearly, $\mu^*(\emptyset) = 0$. Suppose $A, B \subseteq \mathrm{X}$ with $A \subseteq B$. Observe that there is at least one collection $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{E}$ such that $B \subseteq \cup_{k=1}^{\infty} E_k$. Then $A \subseteq \cup_{k=1}^{\infty} E_k$. This is true for all covers of B. So $\mu^*(A) \leq \mu^*(B)$. To show subadditivity, let $\epsilon > 0$ and $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{P}(\mathrm{X})$. Then for all j we can find a sequence $\{E_{k,j}\}_{j=1}^{\infty} \subseteq \mathcal{E}$ such that $A_k \subseteq \cup E_{k,j}$. Then $\mu^*(A_k) \geq \sum_{j=1}^{\infty} \rho(E_{k,j}) - \epsilon 2^{-k}$. Now $\cup_{k=1}^{\infty} A_k \subseteq \cup_{k=1}^{\infty} \cup_{j=1}^{\infty} E_{k,j}$. By the definition of μ^*,

$$\mu^*(\cup A_k) \leq \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \rho(E_{k,j}) \leq \sum_{k=1}^{\infty}(\mu^*(A_k) + \epsilon 2^{-k}) \leq \sum_{k=1}^{\infty} \mu^*(A_k) + \epsilon.$$

Since true for all ϵ, we get $\mu^*(\cup A_k) \leq \sum \mu^*(A_k)$. □

Roughly speaking, if μ^* were a measure and $A \subset E$, then $\mu^*(A) + \mu^*(E \setminus A) = \mu^*(E)$ if A, E are measurable. Then $\mu^*(E \cap A) + \mu^*(E \cap A^C) = \mu^*(E)$.

Definition. *If μ^* is an outer measure on X, then a set $A \subseteq \mathrm{X}$ is called μ^***-measurable** if and only if $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C)$ for all $E \subseteq \mathrm{X}$.*

Remark. By countable subadditivity, we already have the $\leq$ direction as $E = (E \cap A) \cup (E \cap A^C)$.

Theorem (Carathéodory's Theorem). *If μ^* is an outermeasure on X, then the collection of all μ^*-measurable sets, call it $\mathcal{M}$ is a $\sigma-$algebra. Moreover, μ^* is a complete positive measure on $\mathcal{M}$.*

Proof. We prove that $\mathcal{M}$ is a $\sigma-$algebra:

1. Let $E \subseteq \mathrm{X}$ be given. Then
$$\mu^*(E) = \mu^*(\emptyset \cap E) + \mu^*(\mathrm{X} \cap E).$$
Thus $\emptyset, \mathrm{X}$ are μ^* measurable sets.

2. Suppose A is μ^* measurable and $E \subseteq \mathrm{X}$. Then
$$\mu^*(E) = \mu^*(A \cap E) + \mu^*(A^C \cap E) = \mu^*(A^C \cap E) + \mu^*((A^C)^C \cap E).$$
Thus A^C is μ^*-measurable.

3. First, we will show μ^* is finitely additive on $\mathcal{M}$. Let $A, B \in \mathcal{M}$ with $A \cap B = \emptyset$ and let $E \subseteq \mathrm{X}$. Notice
$$A \cup B = (A \cap \mathrm{X}) \cup (A^C \cap B) = (A \cup (B \cap B^C)) \cup (A^C \cap B) = (A \cap B) \cup (A \cap B^C) \cup (A^C \cap B).$$

Now

$$\begin{aligned}\mu^*(E) &= \mu^*(E\cap A)+\mu^*(E\cap A^C)\\ &= \mu^*(E\cap A\cap B)+\mu^*(E\cap A\cap B^C)+\mu^*(E\cap A^C\cap B)+\mu^*(E\cap A^C\cap B^C)\\ &\geq \mu^*(E\cap (A\cap B)\cup(A\cap B^C)\cup(A^C\cap B))+\mu^*(E\cap A^C\cap B^C)\\ &= \mu^*(E\cap(A\cup B))+\mu^*(E\cap(A\cup B)^C).\end{aligned}$$

By the remark, we know $\leq$ is true, thus we have equality and $A\cup B$ is μ^* measurable. Now, let $E=A\cup B$. Then, as A is μ^* measurable, we see

$$\mu^*(A\cup B)=\mu^*(A\cap(A\cup B))+\mu^*(A^C\cap(A\cup B))=\mu^*(A)+\mu^*(B).$$

Thus μ^* is finitely additive. Now we want to show that $\mathcal{M}$ is closed under countable unions. Let $\{A_k\}_{k=1}^\infty\subset\mathcal{M}$ be mutually disjoint. For all n, set $B_n=\cup_{k=1}^n A_k$ and set $B=\cup_{k=1}^\infty A_k$. Let $E\subseteq X$. Notice that $B_n\cap A_n=A_n$ and $B_n\cap A_n^C=B_{n-1}$. Thus

$$\mu^*(E\cap B_n)=\mu^*(E\cap B_n\cap A_n)+\mu^*(E\cap B_n\cap A_n^C)=\mu^*(E\cap A_n)+\mu^*(E\cap B_{n-1})=\sum_{k=1}^n\mu^*(E\cap A_k)$$

by iterative applications. Since $B_n^C\supset B^C$, we see

$$\mu^*(E)=\mu^*(E\cap B_n)+\mu^*(E\cap B_n^C)\geq\sum_{k=1}^n\mu^*(E\cap A_k)+\mu^*(E\cap B^C).$$

Since this is true for all n, we get

$$\mu^*(E)\geq\sum_{k=1}^\infty\mu^*(E\cap A_k)+\mu^*(E\cap B^C)\geq\mu^*(E\cap B)+\mu^*(E\cap B^C)$$

by countable subadditivity. Thus $B=\cup_{k=1}^\infty A_k\subseteq\mathcal{M}$.

Thus $\mathcal{M}$ is a $\sigma-$algebra. By Thm 19, we see that μ^* is countable on $\mathcal{M}$ (as $\sigma-$algebra implies algebra).

To show μ^* is complete, let $N\in\mathcal{M}$ with $\mu^*(N)=0$. Let $A\subseteq N$. Notice $0\leq\mu^*(A)\leq\mu^*(N)=0$. So $\mu^*(A)=0$. Let $E\subseteq \mathrm{X}$. Then

$$\mu^*(E\cap A)+\mu^*(E\cap A^C)=\mu^*(E\cap A^C)\leq\mu^*(E)$$

as $\mu^*(E\cap A)\leq\mu^*(N)=0$. Thus $A\in\mathcal{M}$. □

Example. Let $\mathcal{E}=\{\emptyset,\{x\},\mathrm{X}\}$ with $x\in\mathrm{X}$ and $\mathrm{X}\setminus\{x\}\neq\emptyset$. Consider $\rho:\mathcal{E}\to[0,\infty]$ defined by $\rho(\emptyset)=0,\rho(\mathrm{X})=1,\rho(\{x\})=2$. Then, by definition $\mu^*(\emptyset)=0$, $\mu^*(\mathrm{X})=\inf\{\sum\rho(A_j):\{A_j\}\subseteq\mathcal{E}\text{ and }\mathrm{X}\subseteq\cap A_j\}=1$. Let $A\subseteq\mathrm{X}$ with $A\neq\emptyset$. Then $\mu^*(A)=1$ (as X covers A). What sets are μ^*-measurable?

- Clearly $\emptyset,\mathrm{X}$ are.
- Let $A\subsetneq\mathrm{X}$ such that $A\neq\emptyset$. Note that $\mu^*(\mathrm{X}\cap A)+\mu^*(\mathrm{X}\cap A^C)=1+1\neq1=\mu^*(\mathrm{X})$. Thus A is not μ^*-measurable.

This example shows that $\mathcal{M}$ is *not* generated by $\mathcal{E}$ as $\mathcal{E}\subsetneq\mathcal{M}$. As we shall see, if $\mathcal{E}$ is an algebra, then $\mathcal{M}$ *is* the $\sigma-$algebra generated by $\mathcal{E}$.

Proposition 28. *If $\widetilde{\mu}$ is a premeasure on an algebra $\mathcal{A}\subseteq\mathcal{P}(\mathrm{X})$ and μ^* is the outer measure induced by $\widetilde{\mu}$, then*

1. $\mu^*|_A=\widetilde{\mu}$
2. *Every set in $\mathcal{A}$ is μ^*-measurable.*

Proof. 1. Let $E \in \mathcal{A}$. We will show $\mu^*(E) = \widetilde{\mu}(E)$. Since $\mu^*(E) = \inf\{\sum_1^\infty \widetilde{\mu}(A_j) : \{A_j\} \subseteq \mathcal{A}, E \subseteq \cup A_j\}$, we see $\mu^*(E) \le \widetilde{\mu}(E)$ (take $A_1 = E$ and $A_j = \emptyset$ for $j > 1$). Now, let $\{A_j\}_1^\infty \subseteq \mathcal{A}$ such that $E \subseteq \cup A_j$. By Lemma 1, the sequence $\{B_j\} \subseteq \mathcal{A}$ defined by $B_j = A_j \setminus \cup_1^{j-1} A_k$ is such that B_j's are mutually disjoint and $\cup B_j = \cup A_j$. We see

$$\bigcup_{j=1}^\infty (B_j \cap E) = \left(\bigcup_{j=1}^\infty B_j\right) \cap E = \left(\bigcup_{j=1}^\infty A_j\right) \cap E = E.$$

Since $\widetilde{\mu}$ is a premeasure and $B_j \cap E \subseteq \mathcal{A}$ for all j,

$$\widetilde{\mu}(E) = \widetilde{\mu}\left(\bigcup_{j=1}^\infty (B_j \cap E)\right) = \sum_{j=1}^\infty \widetilde{\mu}(B_j \cap E) \le \sum_{j=1}^\infty \widetilde{\mu}(A_j).$$

Now, since this holds for all $\{A_j\}$, taking the infimum gives us $\widetilde{\mu}(E) \le \mu^*(E)$.

2. Let $A \in \mathcal{A}$ and $E \subseteq \mathrm{X}$. By definition of μ^*, for all $\epsilon > 0$ there exists a sequence of A_j such that $E \subseteq \cup A_j$ and $\mu^*(E) \ge \sum_{j=1}^\infty \widetilde{\mu}(A_j) - \epsilon$. Since $\widetilde{\mu}$ is additive and $A \cap A_j, A^C \cap A_j \in \mathcal{A}$, we see

$$\begin{aligned}
\mu^*(E) &\ge \sum \widetilde{\mu}(A_j) - \epsilon \\
&= \sum \widetilde{\mu}((A \cap A_j) \cup (A^C \cap A_j)) - \epsilon \\
&= \sum \widetilde{\mu}((A \cap A_j)) + \widetilde{\mu}((A^C \cap A_j)) - \epsilon \\
&= \sum \widetilde{\mu}((A \cap A_j)) + \sum \widetilde{\mu}((A^C \cap A_j)) - \epsilon \\
&= \sum \mu^*((A \cap A_j)) + \sum \mu^*((A^C \cap A_j)) - \epsilon \\
&\ge \mu^*((A \cap (\cup A_j))) + \mu^*((A^C \cap (\cup A_j))) - \epsilon \\
&\ge \mu^*((A \cap E)) + \mu^*((A^C \cap E)) - \epsilon
\end{aligned}$$

Since this is true for all ϵ, we see $\mu^*(E) \ge \mu^*((A \cap E)) + \mu^*((A^C \cap E))$. Thus $\mathcal{A} \in \mathcal{M}$. □

Definition. *Let $\mathcal{A} \subseteq \mathcal{P}(\mathrm{X})$ be an algebra and $\widetilde{\mu}$ a premeasure on $\mathcal{A}$. Then $\widetilde{\mu}$ is called*

- ***finite*** *if $\widetilde{\mu}(\mathrm{X}) < \infty$.*
- *$\sigma-$**finite** if there exists $\{A_j\}_{j=1}^\infty \subseteq \mathcal{A}$ such that $\mathrm{X} = \cup_1^\infty A_j$ and $\widetilde{\mu}(A_j) < \infty$.*
- ***semifinite*** *if for all $E \in \mathcal{A}$ with $\widetilde{\mu}(E) = \infty$, then there exists $A \subseteq E$ such that $0 < \widetilde{\mu}(A) < \infty$.*

Theorem 20 (p 31)**.** *Let $\mathcal{A} \in \mathcal{P}(\mathrm{X})$ be an algebra. Let $\widetilde{\mu}$ be a premeasure on $\mathcal{A}$ and $\mathcal{M}$ the $\sigma-$algebra generated by $\mathcal{A}$.*

1. *Then there exists a measure μ on $\mathcal{M}$ such that $\mu|_{\mathcal{A}} = \widetilde{\mu}$.*
2. *If there exists another measure ν such that $\nu|_{\mathcal{A}} = \widetilde{\mu}$ then $\nu(E) \le \mu(E)$ for all $E \in \mathcal{M}$. If $\mu(E) < \infty$, then $\nu(E) = \mu(E)$.*
3. *If $\widetilde{\mu}$ is $\sigma-$finite, then μ is the unique extension of $\widetilde{\mu}$ to $\mathcal{M}$.*

Proof. 1. Follows from Caratheodory's Theorem and Prop 28 if we take $\mu = \mu^*$ (the outer measure induced by $\widetilde{\mu}$.)

2. Suppose ν is another measure on $\mathcal{M}$ which extends $\widetilde{\mu}$. If $\mu(E) = \infty$, then $\nu(E) \le \mu(E)$. So assume $\mu(E) < \infty$. Then for all $\epsilon > 0$, there exists $\{A_j\}_{j=1}^\infty \subseteq \mathcal{A}$ such that $\mu(E) \ge \sum_{j=1}^\infty \widetilde{\mu}(A_j) - \epsilon$. Since ν is a measure, $\nu(E) \le \nu(\cup A_j) \le \sum \nu(A_j) = \sum \mu(A_j) \le \mu(E) + \epsilon$. Since ϵ was arbitrary, $\nu(E) \le \mu(E)$. To show $\nu(E) = \mu(E)$ if $\mu(E) < \infty$, let $\epsilon > 0$ and take $\{A_j\}$ as above. By continuity of measures from above (Theorem 5),

$$\nu(\cup_1^\infty A_j) = \lim_{n\to\infty} \nu(\cup_1^n A_j) = \lim_{n\to\infty} \mu(\cup_1^n A_j) = \mu(\cup_1^\infty A_j).$$

Now $\mu(\cup_1^\infty A_j \setminus E) = \mu(\cup_1^\infty A_j) - \mu(E)$ and $\mu(\cup_1^\infty A_j) \le \sum \mu(A_j) = \sum \widetilde{\mu}(A_j) < \mu(E) + \epsilon$. Thus $\mu(\cup A_j \setminus E) < \epsilon$. So $\mu(E) \le \mu(\cup_1^\infty A_j) = \nu(\cup_1^\infty A_j) = \nu(E) + \nu(\cup A_j \setminus E) \le \nu(E) + \mu(\cup A_j \setminus E) < \nu(E) + \epsilon$. Since ϵ was arbitrary, we see $\nu(E) > \mu(E)$. Thus $\nu(E) = \mu(E)$.

3. If $\widetilde{\mu}$ is $\sigma-$finite, then $X = \cup_1^\infty A_j$ with $\widetilde{\mu}(A_j) = \mu(A_j) = \nu(A_j) < \infty$. Let $E \in \mathcal{M}$. Consider $E \cap \cup_1^n A_j$. By (2), we see $\nu(E \cap \cup_1^n A_j) = \mu(E \cap \cup_1^n A_j)$. Thus

$$\nu(E) = \nu(E \cap \cup_1^\infty A_j) = \lim \nu(E \cap \cup_1^n A_j) = \lim \mu(E \cap \cup_1^n A_j) = \mu(E \cap \cup_1^\infty A_j) = \mu(E).$$

□

Recall the premeasure μ_F obtained in Prop 25 where $\widetilde{\mu}_F|_{\widetilde{I}} = \mu_F$. By Caratheodory's Theorem, the μ_F^*- measurable sets form a $\sigma-$algebra where $\mu_F^*(E) = \inf\{\sum \mu_F((a_j, b_j]) : E \subseteq \cup(a_j, b_j]\}$. We denote this $\sigma-$algebra by $\mathcal{M}_{\mu_F}$ and $\mu_F^*|_{\mathcal{M}_{\mu_F}}$ by μ_F. This is the extension of μ_F on $\widetilde{I}$ to all of $\mathcal{M}_{\mu_F}$. It follows that $\mathcal{B}_\mathbb{R} \subseteq \mathcal{M}_{\mu_F}$ (note that in general this is a strict containment).

Notes.

1. F is called the distribution function for μ_F.
2. μ_F is a complete measure on $\mathcal{M}_{\mu_F}$ (by Caratheodory's Theorem).
3. The measure μ_F is called the **Lebesgue-Stieltjes Measure** associated with F. If $F = x$, then μ_F is called the **Lebesgue Measure** and $\mathcal{M}_{\mu_F}$ is called the Lebesgue Measurable sets.

Note that for any $E \subseteq \mathcal{M}_{\mu_F}$, we define $\mu_F(E)$ to be $\mu_F^*(E)$ as we defined it above.

Fix an F which is right continuous and increasing.

Lemma 2. *For any* $E \in \mathcal{M}_{\mu_F}$, $\mu_F(E) = \inf\{\sum_1^\infty \mu_F((a_j, b_j)) : E \subseteq \cup(a_j, b_j)\}$.

Proof. Define $\nu(E) = \inf\{\sum \mu_F((a_j, b_j)) : E \subseteq \cup(a_j, b_j)\}$. We want to show $\nu(E) \geq^1 \mu_F(E) \geq^2 \nu(E)$.

1. Let $\{(a_j, b_j)\}_1^\infty$ be such that $E \subseteq \cup(a_j, b_j)$. For all j, let $\{c_{k,j}\}_{k=1}^\infty$ be a sequence in (a_j, b_j) such that $c_{k,j}$ increases up to b_j. Then $(a_j, b_j) = (a_j, c_{i,j}] \cup \cup_1^\infty (c_{k,j}, c_{k,j+1}]$. So $\mu_F((a_j, b_j)) = \mu_F((a_j, c_{1,j}]) + \sum_1^\infty \mu_F((c_{k,j}, c_{k,j+1}])$. It follows that $\sum_1^\infty \mu_F((a_j, b_j)) \geq \sum_1^\infty \mu_F((a_j, c_{1,j}]) + \sum_{j,k} \mu_F((c_{k,j}, c_{k,j+1}]) \geq \mu_F(E)$. Since this holds for all intervals $\{(a_j, b_j)\}$ such that $E \subseteq \cup(a_j, b_j)$, we see $\nu(E) \geq \mu_F(E)$.
2. Let $\epsilon > 0$. By definition of μ_F, there exists $\{a_j, b_j\}$ such that $\mu_F(E) \geq \sum \mu_F((a_j, b_j]) - \epsilon$. Since F is right continuous, for all j there exists δ_j such that $F(b_j + \delta_j) - F(b_j) \leq \epsilon 2^{-j}$. Since $E \subseteq \cup_1^\infty (a_j, b_j + \delta_j)$, we see

$$\begin{aligned} \nu(E) &\leq \sum \mu_F((a_j, b_j + \delta_j)) \\ &= \sum \mu_F((a_j, b_j]) + \mu_F((b_j, b_j + \delta_j)) \\ &\leq \mu_F(E) + \epsilon + \sum F(b_j + \delta_j) - F(b_j) \\ &\leq \mu_F(E) + 2\epsilon. \end{aligned}$$

Of course, ϵ is arbitrary. Thus $\nu(E) \leq \mu_F(E)$. □

Theorem 21. *If* $E \subseteq \mathcal{M}_{\mu_F}$, *then*

1. $\mu_F(E) = \inf\{\mu_F(U) : U$ *is open* $, E \subseteq U\}$ *(that is,* μ_F *is outer-regular)*
2. $\mu_F(E) = \sup\{\mu_F(K) : K$ *is compact* $, E \supseteq K\}$ *(that is,* μ_F *is inner-regular)*

Proof. 1. Let $E \in \mathcal{M}_{\mu_F}$. By Lemma 2, for all ϵ there exists $\{(a_j, b_j)\}_1^\infty \subset \mathcal{P}(\mathbb{R})$ such that $E \subset \cup(a_j, b_j)$ and $\mu_F(E) \geq \sum \mu_F(a_j, b_j) - \epsilon$. Since μ_F is subadditive, $\mu_F(E) \geq \mu_F(\cup(a_j, b_j)) - \epsilon$. Let $U = \cup(a_j, b_j)$, an open set. Then $\mu_F(U) \leq \mu_F(E) + \epsilon$. Now ϵ is arbitrary and since all open sets are the union of open intervals, we see $\inf\{\mu_F(U)\} \leq \mu_F(E)$. Of course, $\geq$ is true by monotonicity, so they are equal.

2. If E is compact, clearly $\mu_F(E) = \sup\{\mu_F(K) : K \subseteq U, K$ is compact$\}$. If E is bounded, then the closure of E, $\overline{E}$, is compact. Note that it is also measurable as it is a Borel Set. Thus by (1) for all $\epsilon > 0$ we can find an open U such

that $\overline{E} \setminus E \subseteq U$ and $\mu_F(\overline{E} \setminus E) \geq \mu_F(U) - \epsilon$. Note that $\overline{E} \setminus U$ is compact and $\overline{E} \setminus U \subseteq E$. Since $\overline{E} \setminus U = E \setminus (E \cap U)$, we have

$$\begin{aligned}
\mu_F(\overline{E} \setminus U) &= \mu_F(E \setminus E \cap U) \\
&= \mu_F(E) - \mu_F(E \cap U) \text{ since } E \text{ is bounded, } \mu_F(E) < \infty \\
&= \mu_F(E) - \mu_F(U \setminus (U \setminus E)) \\
&= \mu_F(E) - \mu_F(U) + \mu_F(U \setminus E) \text{ since } \mu_F(U) \leq \mu_F(\overline{E} \setminus E) + \epsilon < \infty \\
&\geq \mu_F(E) - \mu_F(\overline{E} \setminus E) - \epsilon + \mu_F(U \setminus E) \\
&\geq \mu_F(E) - \mu_F(\overline{E} \setminus E) - \epsilon + \mu_F(\overline{E} \setminus E) = \mu_F(E) - \epsilon
\end{aligned}$$

So for all compact sets $\overline{E} \setminus U$ we have $\mu_F(E) \leq \mu_F(\overline{E} \setminus U) + \epsilon \leq \mu_F(E) + \epsilon$. Since this is true for all ϵ, we see $\mu_F(E) = \sup\{\mu_F(K) : K \subseteq E, K \text{ is compact}\}$. If E is not necessarily bounded or closed, consider $E_j = E \cap (j, j+1]$. Clearly $\cup_{j=-\infty}^{\infty} E_j = E$ and E_j is bounded for all j. Let $\epsilon > 0$. By previous argument, for all j there exists K_j such that K_j is compact, $K_j \subseteq E_j$, and $\mu_F(K_j) \leq \mu_F(E_j) \leq \mu_F(K_j) + \epsilon 2^{-|j|}$. Put $H_n = \cup_{j=-n}^{n} K_j$. Then H_n is compact and $H_n \subseteq E$. So $\mu_F(H_n) \leq \mu_F(\cup_{j=-n}^{n} E_j) = \sum_{j=-n}^{n} \mu_F(E_j) \leq \sum_{j=-n}^{n} \mu_F(K_j) + 3\epsilon = \mu_F(H_n) + 3\epsilon$. If $\mu_F(E) = +\infty$, then $\lim_{n\to\infty} \mu_F(\cup_{-n}^{n} E_j) = \infty$ which implies $\mu_F(H_n) \to \infty$. Then $\sup\{\mu_F(K) : K \subseteq E, K \text{ is compact}\} = \infty$. If $\mu_F(E) < \infty$, then $\lim_{n\to\infty} \mu_F(\cup_{-n}^{n} E_j) = \mu_F(E)$. Then there exists $N \in \mathbb{N}$ such that $|\mu_F(E) - \mu_F(\cup_{-N}^{N} E_j)| < \epsilon$. Hence $\mu_F(H_N) \leq \mu_F(E) \leq \mu_F(H_N) + 4\epsilon$. It follows, since ϵ was arbitrary, that $\mu_F(H_N) \to \mu_F(E)$ and $\mu_F(E) = \sup\{\mu_F(K) : K \subseteq E, K \text{ is compact}\}$. □

Theorem 22. *If $A \subseteq \mathbb{R}$, then there exists $E \in \mathcal{B}_{\mathbb{R}}$ such that $\mu_F(E) = \mu_F^*(A)$ and $A \subseteq E$.*

Proof. If $\mu_F(A) = \infty$, let $E = \mathbb{R}$. Otherwise, assume $\mu_F(A) < \infty$. For all j, we may select $\{(a_{j,k}, b_{j,k}]\}_{k=1}^{\infty}$ such that $A \subseteq \cup_1^{\infty}(a_{j,k}, b_{j,k}]$ and $\mu^*(A) \geq \sum_{k=1}^{\infty} \mu_F((a_{j,k}, b_{j,k}]) - \frac{1}{j}$. Put $B_j = (a_{j,k}, b_{j,k}]$. Then $B_j \in \mathcal{B}_{\mathbb{R}}$ and we may assume $\mu_F(B_j) < \infty$. Then $\mu_F^*(A) \geq \mu_F(B_j) - \frac{1}{j}$ and $\mu_F^*(A) \leq \mu_F(B_j)$ as $A \subseteq B_j$. Let $B = \cap_{j=1}^{\infty} B_j \in \mathcal{B}_{\mathbb{R}}$. Then $\mu_F(B) = \lim_{\ell\to\infty} \mu_F(\cap_1^{\ell} B_j) \leq \lim_{\ell\to\infty} \mu_F(B_\ell) \leq \lim_{\ell\to\infty} \mu_F^*(A) + \frac{1}{\ell} = \mu_F^*(A)$. Since $A \subseteq B_j$ for all j, we know $A \subseteq B$ and so $\mu_F^*(A) \leq \mu_F(B)$. Combining these two equations, we get equality. □

Definition. *Suppose g is an $(\mathcal{M}_{\mu_F}, \mathcal{B}_{\mathbb{R}})-$measurable function, with μ_F a Lebesgue-Stieltjes measure. Then $\int_{\mathbb{R}} g d\mu_F$ is called the* ***Lebesgue-Stieltjes Integral****.*

Theorem 23. *Suppose F is increasing and differentiable on $\mathbb{R}$. Then*

$$\int_{\mathbb{R}} g\chi_{(a,b]} d\mu_F = \int_a^b gF' dx.$$

Note. If $F = x$, then $\mu_F = m$ is the Lebesgue measure and Thm 23 reduces to $\int_{\mathbb{R}} g\chi_{(a,b]} d\mu_F = \int_a^b g dx$.

Theorem 24. *If $E \subseteq \mathbb{R}$, then TFAE*

1. *$E \in \mathcal{M}_{\mu_F}$*
2. *$E = V \setminus N_1$, where $V \in G_\delta = \{\cap_1^{\infty} U_j | U_j \text{ is open}\}$ and $\mu_F(N_1) = 0$.*
3. *$E = H \cup N_2$, where $H \in F_\sigma = \{\cup_1^{\infty} K_j | K_j \text{ is closed}\}$ and $\mu_F(N_2) = 0$.*

Proof. Since $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}_{\mu_F}$, we see that $(b) \Rightarrow (a)$ and $(c) \Rightarrow (a)$. So suppose $E \in \mathcal{M}_{\mu_F}$. First, suppose $\mu_F(E) < \infty$. Then by Theorem 21, there exists $\{U_j\}_{j=1}^{\infty} \subseteq \mathcal{P}(\mathrm{X})$ of open sets and $\{K_j\}_{j=1}^{\infty} \subseteq \mathcal{P}(\mathrm{X})$ of compact sets such that $E \subseteq U_j$ and $K_j \subseteq E$ and $\mu(U_j) - 2^{-j} \leq \mu_F(E) \leq \mu_F(K_j) + 2^{-j}$. Put $V = \cap_1^{\infty} U_j$ and $H = \cup_1^{\infty} K_j$. So $H \subseteq E \subseteq V$. Then $\mu_F(H) \leq \mu_F(E) \leq \mu_F(V)$ and

$$\mu_F(V) = \lim_{\ell\to\infty} \mu_F(\cap_1^{\ell} U_j) \leq \mu_F(E) \leq \lim_{\ell\to\infty} \mu_F(\cup_1^{\ell} K_j) = \mu_F(H).$$

Thus $\mu_F(H) = \mu_F(E) = \mu_F(V)$. It follows that $\mu_F(V \setminus E) = 0$ and $\mu_F(E \setminus H) = 0$. Let $N_1 = V \setminus E$ and $N_2 = E \setminus H$. Note that $V \in G_\delta$ and $H \in F_\sigma$. Now suppose $\mu_F(E) = \infty$. Consider $E_j = E \cap (j, j+1]$ for all $j \in \mathbb{Z}$. Follow the argument in Theorem 21. □

The Lesbesgue Measure is the most commonly used measure on $\mathbb{R}^n$. Let $\mathcal{L}$ denote the set of Lebesgue Measurable sets. Note that $\mathcal{L}$ is complete and is Borel, that is $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{L}$.

Theorem 25. *If $E \in \mathcal{L}$, then $E + s = \{x + s \in \mathbb{R} | x \in E\}, rE = \{rx \in \mathbb{R} | x \in E\} \in \mathcal{L}$ for all $r, s \in \mathbb{R}$. Moreover, $m(E + s) = m(E)$ and $m(rE) = |r|m(E)$.*

Proof. If E is open, then so is $E+s$ and rE (for $r \neq 0$). It follows if $E \in \mathcal{B}_{\mathbb{R}}$, then so is $E+s$ and rE for all $r, s \in \mathbb{R}$. Denote $m(E + s)$ by $m_s(E)$ and $m(rE)$ by $m^r(E)$ for all $E \in \mathcal{L}$. Clearly

$$m_s(I(a, b)) = m(I(a + s, b + s)) = (b + s) - (a + s) = b - a = m(I(a, b))$$

and

$$m^r(I(a, b)) = m(I(|r|a, |r|b)) = |r|b - |r|a = |r|m(I(a, b))$$

for all intervals $I(a, b)$. Since $\mathbb{R}$ is $\sigma-$finite with respect to m_s, m^r, m, our earlier propositions imply that the extension of m_s, m^r, m from the left open, right closed intervals to $\mathcal{B}_{\mathbb{R}}$ is unique. Thus we see $m_s(E) = m(E)$ and $m^r(E) = |r|m(E)$ for all $E \in \mathcal{B}_{\mathbb{R}}$. Suppose $E \in \mathcal{L}$ and $m(E) = 0$. By Theorem 24, there exists $V \in \mathcal{B}_{\mathbb{R}}$ and a null set N such that $E = V \setminus N$. Now $0 = m(E) = m(V \setminus N) = m(V) - m(N) = m(V)$. So V is a Lebesgue null set. Since $V \in \mathcal{B}_{\mathbb{R}}$, we have $m_s(V) = m(V) = 0$ and $m^r(V) = |r|m(V) = 0$. By monotonicity and completeness, as $E \subseteq V$, we see $m_s(E) = 0 = m^r(E)$. In general, if $E \in \mathcal{L}$, then use Theorem 24(3) to conclude $E + s, rE \in \mathcal{L}$ and $m_s(E) = m(E)$ and $m^r(E) = |r|m(E)$. □

Example. Let $E = \mathbb{Q} \cap [0, 1]$. Then there exists an enumeration $\{r_j\}_1^\infty$ of E (since $\mathbb{Q}$ is countable). Observe that $m(\{r_j\}) = 0$. So $m(E) = m(\cap\{r_j\}) = \sum m(\{r_j\}) = 0$. So this is measure-theoretically small. On the other hand, $\overline{E} = [0, 1]$, so this is topologically large.

Now set $U_j = (r_j - \epsilon 2^{-j}, r_j + \epsilon 2^{-j}) \cap [0, 1]$ for all $j \in \mathbb{N}$. So $m(\cup U_j) \leq \sum m(U_j) = \sum \epsilon 2^{-j+1} = 2\epsilon$. So again, this is set is measure-theoretically very small. However, we still see $\overline{\cup_1^\infty U_j} = [0, 1]$, and so the set is topologically large.

Define $F = [0, 1] \setminus (\cup_1^\infty U_j \cap [0, 1])$. Notice F is compact (and thus $F = \overline{F}$). Clearly, $F \neq [0, 1]$, in fact F is nowhere dense (that is, there does not exist an open interval contained in F). So this set is topologically small. However, $m(F) = m([0, 1] \setminus (\cup U_j \cap [0, 1])) = m([0, 1]) - m(\cup U_j \cap [0, 1]) \geq 1 - 2\epsilon$. Thus this is measure-theoretically large.

Proposition 29. *Let C be the Cantor Set.*

1. *C is compact, nowhere dense, and totally disconnected. However, C has no isolated points.*
2. *$m(C) = 0$.*
3. *C is uncountable.*

Proof. 1. That C is compact follows from the fact that it is a countable intersection of closed sets (and therefore closed) and clearly bounded (as $C \subseteq [0, 1]$). For the other properties, we will use decimal expansions base 3, that is for $x \in [0, 1]$, we will find $a_j \in \mathbb{Z}_3$ such that $x = \sum_1^\infty \frac{a_j}{3^j}$ and write $x = .a_1a_2a_3 \ldots_3$. Note that $\sum_{j=2}^\infty \frac{2}{3^j} = 2(\frac{1}{1-\frac{1}{3}} - 1 - \frac{1}{3}) = \frac{1}{3}$, so $.0\overline{2}_3 = .1_3$. Now we will apply this to our Cantor Set:

$$\begin{aligned} S_0 &= [0_3, 1_3] \\ S_1 &= [0_3, .1_3] \cup [.2_3, 1_3] = [0_3, .0\overline{2}_3] \cup [.2_3, .2\overline{2}_3] \\ S_2 &= [0_3, .00\overline{2}_3] \cup [.02_3, .02\overline{2}_3] \cup [.2_3, .20\overline{2}_3] \cup [.22_3, .22\overline{2}_3] \end{aligned}$$

It follows that C contains all the points that have only 0's and 2's in its base 3 expansion. It seems all be in C, however this is not the case:

$$\frac{1}{4} = \sum \frac{2}{9^j} = .\overline{02}_3 \text{ which implies } \frac{1}{4} \in C$$

yet it is clear that $\frac{1}{4}$ is not an endpoint. Now C has no open intervals as we can always choose a number with a 1 in its base 3 expansion inside any open interval of $[0, 1]$. This says C is totally disconnected. To show it has no isolated points, we will use a particular example (as all other points will follow from there). Consider $\frac{1}{4} = .02\overline{02}_3$ and $x_1 = .00\overline{02}_3$.

These are in C and differ by $.02_3 = \frac{2}{9}$. Now consider $x_2 = .0200\overline{02}_3$. This is also in C and $\frac{1}{4} - x_2 = .0002 = \frac{2}{9^2}$. We can continually do this, finding a sequence $\{x_i\} \subseteq C$ such that $\frac{1}{4} - x_i = \frac{2}{9^i}$. Then $x_i \to \frac{1}{4}$, which says it is not an isolated point.

2. Now $m(C) = m(\cap S_n) = \lim m(S_n)$ (we can do that as $m(S_0) = 1 < \infty$). Note that $m(S_0) = 1, m(S_1) = 1 - \frac{1}{3} = \frac{2}{3}, m(S_2) = \frac{2}{3} - \frac{2}{9} = \frac{4}{9}, m(S_3) = \frac{4}{9} - \frac{4}{27} = \frac{8}{27}$, etc. Thus $m(S_n) = \frac{2^n}{3^n} \to 0$. Thus $m(C) = 0$.

3. To show C is uncountable, we will show there exists a surjective map $C \to [0,1]$, as we know $[0,1]$ is uncountable. For all $x = \sum \frac{a_j}{3^j}$ with $a_j = 0, 2$ (i.e., for all $x \in C$), define $f(x) = \sum \frac{a_j}{2 \cdot 2^j}$, a binary expansion. Now let $y \in [0,1]$. Then $y = \sum \frac{b_j}{2_j}$ for some $b_j \in \mathbb{Z}_2$. Let $x = \sum \frac{2b_j}{3^j}$. Then $x \mapsto y$. □

Generalized Cantor Set

Let I be a bounded interval and call J the open α^{th} middle of I. If J is open, then $m(J) = \alpha m(I)$ and the midpoint of J is the same as I. (Here, we take $\alpha \in [0,1]$.) Inductively define $S_0 = [0,1]$ and S_n to be S_{n-1} with the α_n^{th} middle of each interval in S_{n-1} removed. The generalized Cantor set is then $C = \cap S_n$. If $\{\alpha_n\}_{n=1}^{\infty} \subseteq (0,1)$, then C is compact, totally disconnected, and uncountable. For n, we see $m(S_n) = (1 - \alpha_n) m(S_{n-1}) = \prod_1^n (1 - \alpha_i)$. So $m(C) = \prod_1^{\infty} (1 - \alpha_i)$. If $\{a_n\}$ are bounded away from 0 uniformly, then $m(C) = 0$. If $\{a_n\}$ go to 0 slowly enough, then $m(C) = 0$. If $\{a_n\}$ go to 0 too fast, then $m(C) > 0$.

Vitali Function, AKA Cantor-Lebesgue Function

Definition (1)**.** *The complement of the Cantor Set C in $[0,1]$ is*

$$O = [0,1] \setminus C = [0,1] \setminus \cap_{n=0}^{\infty} S_n = \cup_{n=1}^{\infty} [0,1] \setminus S_n = \cup_{j=1}^{\infty} \cup_{\alpha \in \{0,2\}^j} O_\alpha$$

where $\{0,2\}^0 = \emptyset, \{0,2\}^1 = \{(0),(2)\}, \{0,2\}^2 = \{(0,0),(0,2),(2,0),(2,2)\}$, *etc and* $O_\emptyset = (\frac{1}{3}, \frac{2}{3}), O_{(0)} = (\frac{1}{9}, \frac{2}{9}), O_{(2)} = (\frac{7}{9}, \frac{8}{9}), O_{(0,0)} = (\frac{1}{27}, \frac{2}{27})$, *and for* $\alpha = (a_1, ..., a_n)$, *we see* $O_\alpha = (\sum_{i=1}^n a_i 3^i + 3^{-(n+1)}, \sum_{i=1}^n a_i 3^i + 2 \cdot 3^{-(n+1)})$.

Now, for $x \in O_\emptyset$, *define* $f(x) = \frac{1}{2}$ *and for* $x \in O_\alpha$ *for* $\alpha \neq \emptyset$, *define* $f(x) = \sum_{i=1} 6n \frac{a_i}{2} 2^{-i} + 2^{-(n+1)}$. *Note that f is uniformly continuous on* 0. *Thus let F be the unique extension of f to $[0,1]$ that is continuous. Note* $F(0) = 0, F(1) = 1$, *and F is non-decreasing.*

Definition (2)**.** *Define* $F(x) = \begin{cases} \frac{1}{2}F(3x) & \text{if } 0 \le x \le \frac{1}{3} \\ \frac{1}{2} & \text{if } \frac{1}{3} < x < \frac{2}{3} \\ \frac{1}{2} + \frac{1}{2}F(3x-2) & \text{if } \frac{2}{3} \le x \le 1 \end{cases}$. *Then, for example,* $F(\frac{1}{4}) = \frac{1}{2}F(\frac{3}{4}) = \frac{1}{2}(\frac{1}{2} + \frac{1}{2}F(\frac{1}{4}))$, *and solving we get* $F(\frac{1}{4}) = \frac{1}{3}$.

Fact. F is piecewise constant on O, so F is constant a.e. Also, F is differentiable a.e. and $F'(x) = 0$ a.e. In fact, $F'(x) = 0$ for all $x \in O$. However, F is not constant as $F(0) = 0$ and $F(1) = 1$.

Another Amazing Function

Define $g : [0,1] \to [0,2]$ by $g(x) = F(x) + x$. Then g is continous and strictly increasing. Also $g(0) = 0, g(1) = 2$. So $m(g([0,1])) = 2$. What is $m(g(O))$? Since $O = \cup_{j=1}^{\infty} \cup_{\alpha \in \{0,2\}^j} O_\alpha$ with O_α mutually disjoint and g is strictly increasing, $g(O) = \cup \cup g(O_\alpha)$. Thus $m(g(O)) = \sum m(g(O_\alpha)) = \sum m(O_\alpha) = 1$ (since $O_\alpha, g(O_\alpha) = O_\alpha + c$ and m is translation invariant). So $m(g(C)) = m(g([0,1] \setminus O)) = m(g([0,1])) - m(g(O)) = 1$. So g maps the Cantor Set to a set of measure 1 (in a continuous way!).

2.1 Hausdorff Measure (p 350)

For all $\delta > 0$, define $\mathcal{E}_\delta = \{E \subseteq \mathbb{R}^n : diam(E) < \delta\}$, where $diam(E) = \sup\{||x - y|| : x, y \in E\}$. Then, for all $p \ge 0$, define $H_{p,\delta}(A) = \inf\{\sum_{j=1}^{\infty} (diam(B_j))^p : \{B_j\}_{j=1}^{\infty} \subseteq \mathcal{E}_\delta \text{ and } A \subseteq \cup_1^{\infty} B_j\}$. Note that this is an outer measure. Define the **Hausdorff Outer-Measure** H_p by $H_p(A) = \lim_{\delta \to 0^+} H_{p,\delta}(A)$.

Notes.

1. $\mathcal{E}_{\delta_1} \subset \mathcal{E}_{\delta_2}$ whenever $\delta_1 < \delta_2$. So $H_{p,\delta_1}(A) \geq H_{p,\delta_2}(A)$. Thus $\{H_{p,\delta}\}$ is increasing and thus the limit exists.

2. You may restrict $\mathcal{E}_\delta$ (and still get the same results) to $E \subseteq \mathbb{R}^n$ where E is open or closed (see p 350).

Proposition 30. *H_p is an outermeasure on $\mathbb{R}^n$.*

Proof. We see that H_p is nonnegative and $H_p(\emptyset) = 0$. If $A_1 \subseteq A_2$, then since $H_{p,\delta}$ is an outermeasure, $H_{p,\delta}(A_1) \leq H_{p,\delta}(A_2) \leq H_p(A_2)$. Since this is true for all δ, take the limit to get $H_p(A_1) \leq H_p(A_2)$. To show subadditivity, let $\{A_j\}_{j=1}^\infty \subset \mathbb{R}^n$. As $H_{p,\delta}$ are outermeasures,

$$H_{p,\delta}(\cup A_j) \leq \sum H_{p,\delta}(A_j) \leq \sum H_p(A_j).$$

Since true for all δ, take the limit to get $H_p(\cup A_j) \leq \sum H_p(A_j)$. □

Let (X, ρ) be a metric space. We define $\rho(A, B) = \inf\{\rho(x, y) : x \in A, y \in B\}$. If $\rho(A, B) > 0$, then $A \cap B = \emptyset$. If $\rho(A, B) = 0$, anything may happen.

Definition. *Suppose μ^* is an outermeasure on X. We say that μ^* is a metric outer measure if $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ whenever $\rho(A, B) > 0$.*

Proposition 31. *If μ^* is a metric outer measure, then every Borel Set is μ^* measurable.*

Proof. Recall that the Borel Sets are generated by the closed sets. Thus it suffices to show all closed sets are μ^* measurable. Suppose $F \subseteq \mathrm{X}$ is closed. We need to show $\mu^*(A) = \mu^*(A \cap F) + \mu^*(A \setminus F)$ for all $A \subseteq \mathrm{X}$. Recall that "$\leq$" follows from subadditivity. If $\mu^*(A) = \infty$, clear. So suppose $\mu^*(A) < \infty$. If $\rho(A \cap F, A \setminus F) > 0$, done by the definition. Define $B_n = \{x \in A \setminus F | \rho(x, F) \geq \frac{1}{n}\}$. Now $\rho(B_n, F) \geq \frac{1}{n}$. Since μ^* is a metric outermeasure, $\mu^*(A) = \mu^*((A \cap F) \cup (A \setminus F)) \geq \mu^*((A \cap F) \cup B_n) = \mu^*(A \cap F) + \mu^*(B_n)$. So we need only show $\lim_{n \to \infty} \mu^*(B_n) = \mu^*(A \setminus F)$.

Claim: Let $C_{n+1} = B_{n+1} \setminus B_n$. Then $\rho(C_{n+1}, B_n) \geq \frac{1}{n(n+1)}$.

Proof: Note that the distance between points in B_n and F is $\geq \frac{1}{n}$ and the distance between points in C_{n+1} and F is $\leq \frac{1}{n+1}$. So the distance between C_{n+1} and B_n is $\geq \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$.

Now $B_{2k+1} = C_{2k} \cup B_{2k} \supseteq C_{2k} \cup B_{2k-1}$. So

$$\begin{aligned} \mu^*(B_{2k+1}) &\geq \mu^*(C_{2k} \cup B_{2k-1}) \\ &= \mu^*(C_{2k}) + \mu^*(B_{2k-1}) \\ &\geq \mu^*(C_{2k}) + \mu^*(C_{2k-2} \cup B_{2k-3}) \\ &\geq \textstyle\sum_{j=1}^k \mu^*(C_{2j}) + \mu^*(B_1) \\ &\geq \textstyle\sum_{j=1}^k \mu^*(C_{2j}). \end{aligned}$$

Thus we have $\mu^*(B_{2k+1}) \geq \sum_{j=1}^k \mu^*(C_{2j})$ and it follows that $\mu^*(B_{2k}) \geq \sum_{j=1}^k \mu^*(C_{2j-1})$. Since $\mu^*(A) < \infty, \mu^*(B_n) < \mu^*(A) < \infty$. Thus $\sum_1^\infty \mu^*(C_{2j}) < \infty$ and $\sum_{j=1}^k \mu^*(C_{2j-1}) < \infty$. Since these sums both converge absolutely, $\sum_{j=n}^\infty \mu^*(C_j) \to 0$. By subadditivity of μ^*, $\mu^*(A \setminus F) = \mu^*(B_n \cup (\cup_{j=n}^\infty C_j)) \leq \mu^*(B_n) + \sum_{j=n}^\infty \mu^*(C_j)$. Taking this limit as $n \to \infty$, we see

$$\mu^*(A \setminus F) \leq \liminf \mu^*(B_n) \leq \limsup \mu^*(B_n) \leq \mu^*(A \setminus F).$$

Thus $\mu^*(A \setminus F) = \lim \mu^*(B_n)$. □

Note. For the above proof, we assumed F was closed in order to deduce that $\cup_1^\infty B_n = A \setminus F$ as if $x \in A \setminus F$, then as F is closed, we know $\rho(x, F) \geq \epsilon > 0$ which implies $x \in B_n$ for $n \geq \frac{1}{\epsilon}$.

Proposition 32. *H_p is a metric outer measure on $\mathbb{R}^n$.*

Proof. By Proposition 30, we know that H_p is an outermeasure. Thus we need only show $H_p(A \cup B) = H_p(A) + H_p(B)$ whenever $\rho(A, B) > 0$. Let $A, B \subseteq \mathbb{R}^n$ with $\rho(A, B) > 0$. Since H_p is an outermeasure, we already have "$\leq$". To show "$\geq$", we select $\delta \in (0, \rho(A, B))$ and $\{E_j\}_{j=1}^\infty \subseteq \mathcal{E}_\delta$ such that $A \cup B \subseteq \cup E_j$ and $H_{p,\delta}(A \cup B) \geq \sum_{j=1}^\infty (diam(E_j))^p - \epsilon$ for a given ϵ.

Since $\delta < \rho(A,B)$, no E_j can intersect both A and B. So we split our covering for $A \cup B$ into 2 families: $\{C_j\}_{j=1}^{\infty}$ (which are the E_j such that $E_j \cap A \neq \emptyset$) and $\{D_j\}_{j=1}^{\infty}$ (which are all the other sets). Then $\{C_j\}$ is a cover for A and $\{D_j\}$ is a cover for B. Now,

$$H_{p,\delta}(A) + H_{p,\delta}(B) \leq \sum_{j=1}^{\infty}(diam(C_j))^p + \sum_{j=1}^{\infty}(diam(D_j))^P = \sum_{j=1}^{\infty}(diam(E_j))^p \leq H_{p,\delta}(A \cup B) + \epsilon.$$

Now, since ϵ is arbitrary, we have $H_{p,\delta}(A) + H_{p,\delta}(B) \leq H_{p,\delta}(A \cup B) \leq H_p(A \cup B)$. Again, this is true for all δ, so letting $\delta \to 0$, we see $H_p(A) + H_p(B) \leq H_p(A \cup B)$. ☐

Corollary 8. *All the Borel Sets of $\mathbb{R}^n$ are H_p-measurable.*

Example. The Cantor Set. Define it as $S_0 = C_0, S_1 = C_{1,1} \cup C_{1,2}, S_2 = C_{2,1} \cup C_{2,2} \cup C_{2,3} \cup C_{2,4}$ and in general $S_k = \cup_1^{2^k} C_{k,j}$ where $diam(C_{j,k}) = (\frac{1}{3})^k$. Since $C = \cap S_k$, each S_k covers C. It follows that $C \subseteq \cup_1^{2^j} C_{j,k}$. So if $\delta = (\frac{1}{3})^k$, we see that

$$H_p(C) = \lim_{\delta \to 0^+} H_{p,\delta}(C) \leq \lim_{k \to 0^+} \left(\frac{2}{3^p}\right)^k = \begin{cases} 0 & \text{if } p > \frac{\ln 2}{\ln 3} \\ 1 & \text{if } p = \frac{\ln 2}{\ln 3} \\ \infty & \text{if } p < \frac{\ln 2}{\ln 3} \end{cases}$$

Interestingly, it can be show the inequality is actually equality.

Proposition 33 (p 351). *If $H_p(A) < \infty$, then $H_q(A) = 0$ for all $q > p$. If $H_p(A) > 0$, then $H_q(A) = \infty$ for all $q < p$. It follows that*

$$\inf\{p | H_p(A) = 0\} = \sup\{p \geq 0 | H_p(A) = \infty\}.$$

*The **Hausdorff Dimension** of A is the above number.*

2.2 Product Measures

Goal: Given measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$, we want to define a measure on $X \times Y$ (the cartesian product) such that the measure of $E \times F$, with $E \in \mathcal{M}$ and $F \in \mathcal{N}$ is $\mu(E)\nu(F)$.

Definition. *Let $\{(X_\alpha, \mathcal{M}_\alpha)\}_{\alpha \in A}$ be measurable spaces. The **product $\sigma-$algebra** on $X = \prod_{\alpha \in A} X_\alpha$ is the $\sigma-$algebra generated by $\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\}$ where $\pi_\alpha : X \to X_\alpha$ is the α^{th} coordinate map.*

Notation. The product $\sigma-$algebra is denoted by $\otimes_{\alpha \in A} \mathcal{M}_\alpha$.

Proposition 34. *If A is countable, then $\otimes_{\alpha \in A} \mathcal{M}_\alpha$ is the $\sigma-$algebra generated by $\{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha\}$.*

Proof. See page 22-23. ☐

Proposition 35. *Let $X_1, ..., X_n$ be metric spaces and let $X = \prod_1^n X_j$ be equipped with the product measure (p 13). Then $\otimes_1^n \mathcal{B}_{X_j} \subseteq \mathcal{B}_X$. If each X_j is separable, then $\otimes_1^n \mathcal{B}_{X_j} = \mathcal{B}_X$.*

Corollary 9. $\mathcal{B}_{\mathbb{R}^n} = \otimes_1^n \mathcal{B}_{\mathbb{R}}$.

Definition. *If $E \in \mathcal{M}$ and $F \in \mathcal{N}$, we call $E \times F$ a **measurable rectangle**. We denote the set of all measurable rectangles by $\mathcal{R}$.*

By HW3 #3, since $\mathcal{M}$ and $\mathcal{N}$ and semialgebras, we see $\mathcal{R} = \mathcal{M} \times \mathcal{N}$ is a semialgebra. So we can use the Carathéodory construction to extend it to an algebra:

Theorem 26. *Let $\Pi : \mathcal{R} \to [0, \infty]$ be defined by $\Pi(A \times B) = \mu(A)\nu(B)$. Then Π is well-defined, countably additive, and* $\Pi(\emptyset) = 0$.

Proof. Clearly $\Pi(\emptyset) = 0$ and Π is well-defined. To show countably additive, suppose we have $\{A_n\}_1^\infty \subseteq \mathcal{M}$ and $\{B_n\}_1^\infty \subseteq \mathcal{N}$ which satisfy

- $\cup_1^\infty (A_n \times B_n) = A \times B$ with $A \in \mathcal{M}$ and $B \in \mathcal{N}$.
- $(A_m \times B_m) \cap (A_n \times B_n) = \emptyset$ if $n \neq m$.

Then we need to show $\Pi(A \times B) = \sum \Pi(A_n \times B_n)$. Note that

$$\chi_A(x)\chi_B(y) = \chi_{A\times B}((x,y)) = \sum_1^\infty \chi_{A_j\times B_j}((x,y)) = \sum_1^\infty \chi_{A_j}(x)\chi_{B_j}(y).$$

By Theorem 7, we see

$$\begin{aligned} \mu(A)\chi_B(y) &= \int_X \chi_A(x)\chi_B(y)d\mu(x) \\ &= \int_X \sum_1^\infty \chi_{A_j}(x)\chi_{B_j}(y)d\mu(x) \\ &= \sum_1^\infty \int_X \chi_{A_j}(x)\chi_{B_j}(y)d\mu(x) \\ &= \sum_1^\infty \mu(A_j)\chi_{B_j}(y). \end{aligned}$$

Now, using Theorem 7 again, we have

$$\begin{aligned} \mu(A)\nu(B) &= \int_Y \mu(A)\chi_B(y)d\nu(y) \\ &= \int_Y \sum_1^\infty \mu(A_j)\chi_{B_j}(y)d\nu(y) \\ &= \sum_1^\infty \int_Y \mu(A_j)\chi_{B_j}(y)d\nu(y) \\ &= \sum_1^\infty \mu(A_j)\nu(B_j). \end{aligned}$$

Thus, by our definition of Π, we see Π is countably additive. □

Theorem 27. *With Π defined as in Theorem 26, there exists a unique extension of Π to a premeasure $\widetilde{\Pi}$ on $\mathcal{F}(\mathcal{R})$, the algebra generated by $\mathcal{R}$.*

Proof. Done, by Theorems 19 and 18. □

Theorem 28. *The premeasure $\widetilde{\Pi}$ generates an outer measure Π^* on $X \times Y$ whose restriction to $\mathcal{M} \otimes \mathcal{N}$ is a measure extending Π. Moreover, if μ and ν are $\sigma-$finite, then so is $\Pi^*|_{\mathcal{M}\otimes\mathcal{N}}$ and Π^* is unique.*

Proof. Done, by Proposition 27, Carathéodory's Theorem, and Theorem 20. □

Notation. We denote $\Pi^*|_{\mathcal{M}\otimes\mathcal{N}}$ by $\mu \times \nu$.

Note that by iterative applications of the above we can define a product measure for any finite number of measure spaces. In that case, we denote the product measure on $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$ by $\prod_1^n \mu_j$.

How do we find $\mu \times \nu(E)$ for $E \in \mathcal{M} \otimes \mathcal{N}$?

Simple Case: Let $E \subseteq \mathbb{R}^2$ with $E = \{(x,y) \in \mathbb{R}^2 | a \leq x \leq b, f(x) \leq y \leq g(x)\}$. Then the measure of E is $\int_a^b g - f(x)dx$. Now $g - f(x) = meas(E_x)$ where $E_x = \{y \in \mathbb{R} | (x,y) \in E\}$. So we see

$$meas(E) = \int_{\mathbb{R}} m(E_x)dm.$$

Questions

1. Given $E \in \mathcal{M} \otimes \mathcal{N}, x \in X$, is $E_x \in \mathcal{N}$?
2. Is the function $x \mapsto \nu(E_x)$ $\mu-$measurable?
3. Is $\mu \times \nu(E) = \int_X \nu(E_x)d\mu$?
4. Can we interchange μ and ν?

Definition. *If $E \subseteq X \times Y$, then for all $x \in X$ define the $x-$**section** $E_x = \{y \in Y|(x,y) \in E\}$ and for all $y \in Y$ define the $y-$**section** $E^y = \{x \in X|(x,y) \in E\}$. If $f : X \times Y \to \overline{\mathbb{R}}$, we define the $x-$section f_x and the $y-$section f^y as $f_x(y) = f^y(x) = f(x,y)$.*

Example. If $E = A \times B$ with $A \in \mathcal{M}$ and $B \in \mathcal{N}$ then $E_x = \emptyset$ if $x \notin A$ and $E_x = B$ if $x \in A$.

Proposition 36 (p 65)**.** *1. If $E \in \mathcal{M} \otimes \mathcal{N}$, then $E_x \in \mathcal{N}$ for all $x \in X$ and $E^y \in \mathcal{M}$ for all $y \in Y$.*

2. If f is a $(\mathcal{M} \otimes \mathcal{N})-$measurable function, then f_x is $\mathcal{N}-$measurable and f^y is $\mathcal{M}-$measurable.

Proof. 1. Let $\mathcal{O}$ be the collection of all $E \subseteq X \times Y$ satisfying $E_x \in \mathcal{N}$ for all $x \in X$ and $E^y \in \mathcal{M}$ for all $y \in Y$. We show $\mathcal{O}$ is a $\sigma-$algebra:

- Clearly, $\emptyset, X \times Y \in \mathcal{O}$.
- Let $E \in \mathcal{O}$. Then $E_x \in \mathcal{N}$ which implies $(E^C)_x = E_x^C \in \mathcal{N}$ for all $x \in X$ as $\mathcal{N}$ is a $\sigma-$algebra and $E^y \in \mathcal{M}$ which implies $(E^C)^y = (E^y)^C \in \mathcal{M}$ for all $y \in Y$ as $\mathcal{M}$ is a $\sigma-$algebra. Thus $E^C \in \mathcal{O}$.
- Suppose $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{O}$. Then $(\cup E_j)_x = \cup (E_j)_x \in \mathcal{N}$ for all $x \in X$ and similarly $(\cup E_j)^y \in \mathcal{M}$ for all $y \in Y$. This $\cup E_j \in \mathcal{O}$.

Now we observe that all measurable rectangles are in $\mathcal{O}$ and since $\mathcal{M} \otimes \mathcal{N}$ is defined to be the smallest $\sigma-$algebra which contains the measurable rectangles, we see $\mathcal{M} \otimes \mathcal{N} \subseteq \mathcal{O}$.

2. Suppose $E \subseteq \mathcal{B}_{\overline{\mathbb{R}}}$. Then $f^{-1}(E) \in \mathcal{M} \otimes \mathcal{N}$. By part 1, $(f^{-1}(E))_x \in \mathcal{N}$ and $(f^{-1}(E))^y \in \mathcal{M}$. Notice $(f^{-1}(E))_x = \{y \in Y : f(x,y) \in E\} = \{y \in Y : f_x(y) \in E\} = f_x^{-1}(E)$. Thus $f_x^{-1}(E) \in \mathcal{N}$ which implies f_x is measurable. Similarly, f^y is measurable. □

Definition. *A subset $\mathcal{C} \subseteq \mathcal{P}(X)$ is a* ***monotone class*** *if it possesses the following properties:*

- *If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{C}$ and $E_1 \subseteq E_2 \subseteq \cdots$, then $\cup_1^{\infty} E_j \in \mathcal{C}$.*
- *If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{C}$ and $E_1 \supseteq E_2 \supseteq \cdots$, then $\cap_1^{\infty} E_j \in \mathcal{C}$.*

Note. A $\sigma-$algebra is a monotone class.

Given a subset $\mathcal{E} \subseteq \mathcal{P}(X)$, there exists a smallest monotone class $\mathcal{C}(\mathcal{E})$ containing $\mathcal{E}$. We say $\mathcal{C}(\mathcal{E})$ is the **monotone class generated by** $\mathcal{E}$.

Theorem 29. *$\mathcal{M} \subset \mathcal{P}(X)$ is a $\sigma-$algebra if and only if $\mathcal{M}$ is a monotone class and an algebra.*

Proof. **(⇒:)** Clear

(⇐:) Suppose $\mathcal{M}$ is an algebra and a monotone class. Then

1. $\emptyset, X \in \mathcal{M}$ as $\mathcal{M}$ is an algebra.
2. $\mathcal{M}$ is closed under complements as $\mathcal{M}$ is an algebra.
3. As $\mathcal{M}$ is an algebra, it is closed under finite unions. Let $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$. Define $A_k := \cup_{j=1}^{k} E_j$. Then $A_k \in \mathcal{M}$ for all k and $A_1 \subseteq A_2 \subseteq \cdots$. Since $\mathcal{M}$ is a monotone class, we see $\cup_{j=1}^{\infty} E_j = \cup_{k=1}^{\infty} A_k \in \mathcal{M}$. □

Lemma (**Monotone Class Lemma** (p66))**.** *If $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra, then the monotone class $\mathcal{C}(\mathcal{A})$ generated by $\mathcal{A}$ and the $\sigma-$algebra $\mathcal{M}(\mathcal{A})$ generated by $\mathcal{A}$ are equal.*

Proof. Since $\mathcal{M}(\mathcal{A})$ is a monotone class, we see $\mathcal{C}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$. So it is enough to show $\mathcal{C}(\mathcal{A})$ is a $\sigma-$algebra. By Theorem 29, it is enough to show $\mathcal{C}(\mathcal{A})$ is an algebra.

1. Since $\mathcal{A}$ is an algebra, $\emptyset, X \in \mathcal{A} \subseteq \mathcal{C}(\mathcal{A})$.

2. Define $\mathcal{E} := \{E \subseteq X | E^C \in \mathcal{C}(\mathcal{A})\}$. We show $\mathcal{E}$ is a monotone class.

   - If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{E}$ such that $E_1 \subseteq E_2 \subseteq \cdots$, then $\{E_j^C\}_{j=1}^{\infty} \in \mathcal{C}(\mathcal{A})$ and $E_1^C \supseteq E_2^C \supseteq \cdots$ which implies $(\cup E_j)^C = \cap E_j^C \in \mathcal{C}(\mathcal{A})$ which implies $\cup E_j \in \mathcal{E}$.
   - Similar

   So $\mathcal{C}(\mathcal{A}) \subseteq \mathcal{E}$ which implies $\mathcal{C}(\mathcal{A})$ is closed under complements.

3. We want to show $\mathcal{C}(\mathcal{A})$ is closed under finite unions. Define $\mathcal{E}(F) := \{E \subseteq X | E \cup F \in \mathcal{C}(\mathcal{A})\}$ for all $F \in \mathcal{C}(\mathcal{A})$. Now, suppose $F \in \mathcal{A}$. Then $\mathcal{A} \subseteq \mathcal{E}(F)$ as $\mathcal{A}$ is an algebra. Continuing under the assumption that $F \in \mathcal{A}$, we want to show $\mathcal{E}(F)$ is a monotone class (as then $\mathcal{C}(\mathcal{A}) \subseteq \mathcal{E}(F)$.). So

   - Let $\{A_j\}_{j=1}^{\infty} \subseteq \mathcal{E}(F)$ with $A_1 \subseteq A_2 \subseteq \cdots$. Then $\{A_j \cup F\}_{j=1}^{\infty} \subseteq \mathcal{C}(\mathcal{A})$ and of course $A_1 \cup F \subseteq A_2 \cup F \subseteq \cdots$ which implies $(\cup A_i) \cup F = \cup(A_i \cup F) \in \mathcal{C}(\mathcal{A})$. Thus $\cup A_i \in \mathcal{E}(F)$.
   - Similar

   Now, suppose $E \in \mathcal{C}(\mathcal{A})$. Then $E \in \mathcal{E}(F)$ if and only if $E \cup F \in \mathcal{C}(\mathcal{A})$ if and only $F \in \mathcal{E}(E)$. Thus for all $E \in \mathcal{C}(\mathcal{A})$, we have $\mathcal{A} \in \mathcal{E}(E)$. By the above, $\mathcal{E}(E)$ is a monotone class. So $\mathcal{C}(\mathcal{A}) \subseteq \mathcal{E}(E)$ for all $E \in \mathcal{C}(\mathcal{A})$. Thus $\mathcal{C}(\mathcal{A})$ is closed under finite unions.

Thus $\mathcal{C}(\mathcal{A})$ is an algebra, which implies it is a $\sigma-$algebra and thus $\mathcal{C}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$. □

Theorem 30 (p 66)**.** *Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are $\sigma-$finite measure spaces. Let $E \in \mathcal{M} \otimes \mathcal{N}$. Then*

1. *$x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are measurable in X and Y, respectively.*
2. *$\mu \times \nu(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$.*

Proof. Let $\mathcal{E} := \{E \in \mathcal{M} \otimes \mathcal{N} : (1), (2) \text{ hold}\}$. We want to show $\mathcal{E} = \mathcal{M} \otimes \mathcal{N}$, that is $\mathcal{E} \supseteq \mathcal{M} \otimes \mathcal{N}$. For this, we show $\mathcal{E}$ is a monotone class that contains the algebra $\mathcal{F}(\mathcal{R})$. Since $\mathcal{M} \otimes \mathcal{N}$ is the $\sigma-$algebra generated by $\mathcal{R}$, it is also generated by $\mathcal{F}(\mathcal{R})$. So it is the monotone class generated by $\mathcal{F}(\mathcal{R})$ by the Monotone Class lemma. Thus, if we show $\mathcal{F}(\mathcal{R}) \subseteq \mathcal{E}$ and $\mathcal{E}$ is a monotone class, then $\mathcal{M} \otimes \mathcal{N} \subseteq \mathcal{E}$.

First, we assume μ, ν are finite measures. Now, if $A \times B \in \mathcal{R}$, then $x \mapsto \nu((A \times B)_x) = \nu(B)\chi_A(x)$ and $y \mapsto \mu((A \times B)^y) = \mu(A)\chi_B(y)$, which are measurable. Thus property (1) holds for $\mathcal{R}$. For property (2),

$$\int_X \nu((A \times B)_x) d\mu(x) = \int_X \nu(B)\chi_A(x) d\mu(x) = \nu(B)\mu(A) = \mu \times \nu(A \times B).$$

and similarly $\int_Y \mu((A \times B)^y) d\nu(y) = \mu \times \nu(A \times B)$. Thus $\mathcal{R} \in \mathcal{E}$. Since $\mathcal{R}$ is a semialgebra, it is enough to show $\mathcal{E}$ is closed under finite disjoint unions (by Proposition 23) to show $\mathcal{F}(\mathcal{R}) \subseteq \mathcal{E}$. Let $E_1, E_2 \in \mathcal{E}$ with $E_1 \cap E_2 = \emptyset$. Observe $(E_1 \cup E_2)_x = (E_1)_x \cup (E_2)_x$ with $(E_1)_x \cap (E_2)_x = \emptyset$. Similarly for the $y-$sections. Thus

1. $x \mapsto \nu((E_1 \cup E_2)_x) = \nu((E_1)_x \cup (E_2)_x) = \nu((E_1)_x) + \nu((E_2)_x)$ and similarly $y \mapsto \mu((E_1)^y) + \mu((E_2)^y)$, which are measurable.
2. Since $\mu \times \nu$ is a measure, we see $\mu \times \nu(E_1 \cup E_2) = \mu \times \nu(E_1) + \mu \times \nu(E_2) = \int \nu((E_1)_x) d\mu + \int \nu((E_2)_x) d\mu = \int \nu((E_1)_x) + \nu((E_2)_x) d\mu = \int \nu((E_1 \cup E_2)_x) d\mu$ and similarly $\mu \times \nu(E_1 \cup E_2) = \int \mu((E_1 \cup E_2)^y) d\nu$.

Thus $\mathcal{E}$ is closed under disjoint unions of two sets and (by induction) thus is closed under finite disjoint unions. Thus $\mathcal{F}(\mathcal{R}) \subseteq \mathcal{E}$. Now, we need to show $\mathcal{E}$ is a monotone class.

- Suppose $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{E}$ with $E_1 \subseteq E_2 \subseteq \cdots$. Now, $(E_1)_x \subseteq (E_2)_x \subseteq \cdots$ and $(E_1)^y \subseteq (E_2)^y \subseteq \cdots$. So $\{x \mapsto \nu((E_n)_x)\}_n$ and $\{y \mapsto \mu((E_n)^y)\}_n$ are increasing sequences of measurable functions. By Theorem 5,

$$\lim_{n\to\infty} \nu((E_n)_x) = \nu\left(\bigcup_{n=1}^{\infty} (E_n)_x\right) = \nu\left(\left(\bigcup_{n=1}^{\infty} E_n\right)_x\right) \text{ and } \lim_{n\to\infty} \mu((E_n)^y) = \mu\left(\left(\bigcup_{n=1}^{\infty} E_n\right)^y\right).$$

So $x \mapsto \nu((\cup E_n)_x)$ and $y \mapsto \mu((\cup E_n)^y)$ are measurable functions (as the supremum of measurable functions is measurable). For property 2, by the MCT and Theorem 5,

$$\int_y \mu((\cup_{n=1}^{\infty} E_n)^y) d\nu(y) = \lim_{n\to\infty} \int \mu((E_n)^y) d\nu = \lim_{n\to\infty} \mu \times \nu(E_n) = \mu \times \nu(\cup_{n=1}^{\infty} E_n)$$

and similarly for the $x-$sections. Thus $\cup_{n=1}^{\infty} E_n \subseteq \mathcal{E}$.

- Similar

Thus, if μ and ν are finite, we see $\mathcal{E}$ is a monotone class containing $\mathcal{F}(\mathcal{R})$ which implies $\mathcal{M} \otimes \mathcal{N} \subseteq \mathcal{E}$. If μ and ν are $\sigma-$finite, then X and Y are the unions of finite increasing sets, in which case we can use the above with the MCT to get the limit. □

Theorem (**Fubini-Tonelli Theorem** p.67)**.** *Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are $\sigma-$finite measure spaces. Then*

1. *(Tonelli) If $f \in L^+(X \times Y)$, then the functions $g(x) = \int_Y f_x d\nu$ and $h(y) = \int_X f^y d\mu$ are in $L^+(X), L^+(Y)$, respectively and $(*)$ $\int_{X\times Y} f(x,y) d(\mu \times \nu) = \int_X \left[\int_Y f(x,y) d\nu\right] d\mu = \int_Y \left[\int_X f(x,y) d\mu\right] d\nu$.*
2. *(Fubini) If $f \in L^1(\mu \times \nu)$, then $g \in L^1(X)$ and $h \in L^1(Y)$ and $(*)$ holds almost everywhere.*

Proof.

1. If $f = \chi_E$ for $E \in \mathcal{M} \otimes \mathcal{N}$, then $f_x = \chi_{E_x}$. So $g(x) = \int_Y f_x d\nu = \int_Y \chi_{E_x} d\nu = \nu(E_x)$, which is measurable. Thus $g \in L^+$ by Theorem 30. Similarly, $h(y) = \mu(E^y)$ and $h \in L^+$. Also by Theorem 30,

$$\int f d(\mu \times \nu) = \int \chi_E d(\mu \times \nu) = \mu \times \nu(E) = \int_X \nu(E_x) d\mu = \int_X \int_Y f_x(y) d\nu d\mu$$

and similarly $\int f d(\mu \times \nu) = \int_Y \int_X f^y(x) d\mu d\nu$. Since $f_x(y) = f^y(x) = f(x,y)$, the theorem holds for all characteristic functions of measurable sets. If f is a simple function in L^+, then it is a finite linear combination of characteristic functions of measurable sets. Thus, the theorem holds for all simple functions.

If $f \in L^+$, not necessarily simple, then we may select $\{\phi_n\}_{n=1}^{\infty} \subseteq L^+$ such that ϕ_n is simple, $0 \le \phi_1 \le \phi_2 \le \cdots \le f$, and $\phi_n \to f$ pointwise everywhere. Clearly, $(\phi_n)_x \to f_x, (\phi_n)^y \to f^y$, and $(\phi_n)_x, (\phi_n)^y$ are increasing sequences. Define $g_n(x) = \int_Y (\phi_n)_x d\nu$ and $h_n(x) = \int_X (\phi_n)^y d\mu$. By the MCT,

$$\lim_{n\to\infty} g_n(x) = \int_Y \lim_{n\to\infty} (\phi_n)_x d\nu = \int_Y (f)_x d\nu = g(x)$$

and similarly $\lim_{n\to\infty} h_n(x) = h(x)$. We also see that $0 \le g_1 \le g_2 \le \cdots \le g$ and $0 \le h_1 \le h_2 \le \cdots \le h$. Thus again by the MCT

$$\int_X \int_Y f(x,y) d\nu d\mu = \int_X g d\mu = \lim_{n\to\infty} \int_X g_n d\mu = \lim_{n\to\infty} \int_X \int_Y (\phi_n)_x d\nu d\mu = \lim_{n\to\infty} \int_{X\times Y} \phi_n d(\mu \times \nu) = \int_{X\times Y} f d(\mu \times \nu).$$

Similarly $\int_Y \int_X f(x,y) d\mu d\nu = \int_{X\times Y} f d(\mu \times \nu)$.

2. Follows from applying part (a) to f^+ and f^- separately (as if $f \in L^1$, then $\int f^+, \int f^-$ are finite). □

Note. A common way to use this theorem is to use part a in order to use part b. That is, if f is measurable, then $|f| \in L^+$. Then we have $\int_{X\times Y} |f(x,y)| d(\mu \times \nu) = \int_X \int_Y |f(x,y)| d\nu d\mu = \int_Y \int_X |f(x,y)| d\mu d\nu$ and we can show that one of those integrals is finite to conclude that $f \in L^1$. Then, we can use part b.

Definition. *The $n-$dimensional Lebesgue measure m^n is the completion of $(\mathbb{R}^n, \mathcal{L} \otimes \cdots \otimes \mathcal{L}, m \times \cdots \times m)$. The domain of m^n is $\mathcal{L}^n$, the class of $n-$dimensional Lebesgue measurable sets.*

Remarks.

1. Often, the superscript n is dropped. For example, just write $(\mathbb{R}, \mathcal{L}, m)$ for $(\mathcal{R}^n, \mathcal{L}^n, m^n)$. Integrals with respect to the Lebesgue measure are usually written as $\int_{\mathbb{R}^n} f dx$ instead of $\int_{\mathbb{R}^n} f dm$.

2. By Theorem 8 (1.9 in Folland), if $\mathcal{N} = \{N \in \mathcal{B}_{\mathbb{R}^n} : m(N) = 0\}$, then $\mathcal{L}^n = \{E \cup F : E \in \mathcal{B}_{\mathbb{R}^n}, F \subseteq N \text{ for some } N \in \mathcal{N}\}$.

3. If $\{E_j\}_{j=1}^n \subseteq \mathcal{L} \subseteq P(\mathbb{R})$, then $m^n(\prod_{j=1}^n E_j) = \prod_{j=1}^n m(E_j)$.

If $E = \prod_{j=1}^n E_j$, then we will refer to each E_j as a **side/edge** of E. Recall that $E \triangle F = (E \setminus F) \cup (F \setminus E)$. Let $\mathcal{R} = \{\prod_{k=1}^n E_k : \{E_k\}_{k=1}^n \subseteq \mathcal{L} \subseteq \mathcal{P}(\mathbb{R})\}$.

Theorem 31 (2.41 in Folland)**.** *Suppose $E \in \mathcal{L}^n$. Then*

1. $m(E) = \inf\{m(U) : E \subseteq U$ and U is open$\}$.

2. $m(E) = \sup\{m(K) : K \subseteq E$ and K is compact$\}$.

3. $E = V \setminus N_1$, where V is a G_δ set and $m(N_1) = 0$, where $G_\delta = \{\cap_{j=1}^\infty U_j : U_j$ is open$\}$.

4. $E = H \cup N_2$, where H is an F_σ set and $m(N_2) = 0$, where $F_\sigma = \{\cup_{j=1}^\infty D_j : D_j$ is closed$\}$.

Note that this is just the $n-$dimensional version of Theorem 21 (1.18 in Folland) and Theorem 24 (1.19 in Folland).

Proof. 1. Recall that m is the restriction to $\mathcal{L}^n$ of the outer measure m^* which is induced by $\prod_{k=1}^n A_k \mapsto \prod_{k=1}^n m(A_k)$. Thus for a given $E \in \mathcal{L}^n$, we have $m(E) = m^*(E) = \inf\{\sum_{j=1}^\infty m(E_j) : \{E_j\}_{j=1}^\infty \subseteq \mathcal{R}, E \subseteq \cup_{j=1}^\infty E_j\}$. Let $\epsilon \in (0,1)$. Then there exists $\{T_j\}_{j=1} \subseteq \mathcal{R}$ such that $E \subseteq \cup_{j=1}^\infty T_j$, and $\sum_{j=1}^\infty m(T_j) \le m(E) + \frac{1}{2}\epsilon$. Set $\mathcal{Q}_0 = \{\prod_{k=1}^n [a_k, a_k+1) : a_k \in \mathbb{Z}\}$. Notice that this is a countable collection of mutually disjoint sets such that $\cup_{Q \in \mathcal{Q}_0} Q = \mathbb{R}^n$. Let $\{Q_r\}_{r=1}^\infty$ be an enumeration of $\mathcal{Q}_0$. Let $j \in \mathcal{N}$ be given. We have that $T_j = \cup_{r=1}^\infty Q_r \cap T_j = \cup_{r=1}^\infty Q_r \cap \prod E_{j,k}$ where $\{E_{j,k}\}_{k=1}^n \subseteq \mathcal{L} \subseteq \mathcal{P}(\mathbb{R})$ (that is, they are the one dimensional edges of T_j). Let $r \in \mathbb{N}$ be given. Since $Q_r = \prod_{k=1}^n [a_k, a_k+1)$ for some $\{a_k\}_{k=1}^n \subseteq \mathbb{Z}$, we conclude $Q_r \cap T_j = \prod_{k=1}^n [a_k, a_k+1) \cap E_{j,k}$. Now $[a_k, a_k+1) \cap E_{j,k} \in \mathcal{L} \subseteq \mathcal{P}(\mathbb{R})$, so by Theorem 21, for all $k = 1, ..., n$, there exists $F_{r,j,k} \subseteq \mathbb{R}$ that is open and satisfies $F_{r,j,k} \supseteq [a_k, a_k+1) \cap E_{j,k}$ and $m(F_{r,j,k}) \le m([a_k, a_k+1) \cap E_{j,k}) + \frac{1}{2^n n}\epsilon 2^{-r-j}$. It follows that $Q_r \cap T_j \subseteq \prod_{k=1}^n F_{r,j,k}$, which is open and

$$\begin{aligned}
m\left(\prod_{k=1}^n F_{r,j,k}\right) &= \prod_{k=1}^n m(F_{r,j,k}) \le \prod_{k=1}^n \left(m([a_k, a_k+1) \cap E_{j,k}) + \frac{1}{2^n n}\epsilon 2^{-r-j}\right) \\
&= \prod_{k=1}^1 \left(m([a_k, a_k+1) \cap E_{j,k}) + \frac{1}{2^n n}\epsilon 2^{-r-j}\right) \prod_{k=2}^n \left(m([a_k, a_k+1) \cap E_{j,k}) + \frac{1}{2^n n}\epsilon 2^{-r-j}\right) \\
&= \prod_{k=1}^1 \{m([a_k, a_k+1) \cap E_{j,k})\} \prod_{k=2}^n \left\{m([a_k, a_k+1) \cap E_{j,k}) + \frac{1}{2^n n}\epsilon 2^{-r-j}\right\} \\
&\quad + \frac{1}{2^n n}\epsilon 2^{-r-j} \prod_{k=2}^n \left\{m([a_k, a_k+1) \cap E_{j,k}) + \frac{1}{2^n n}\epsilon 2^{-r-j}\right\} \\
&\le \prod_{k=1}^1 \{m([a_k, a_k+1) \cap E_{j,k})\} \prod_{k=2}^n \left\{m([a_k, a_k+1) \cap E_{j,k}) + \frac{1}{2^n n}\epsilon 2^{-r-j}\right\} \\
&\quad + \frac{1}{2^n n}\epsilon 2^{-r-j} \left(1 + \frac{1}{2^n n}\epsilon 2^{-r-j}\right)^{n-1} \\
&\le \prod_{k=1}^2 \{m([a_k, a_k+1) \cap E_{j,k})\} \prod_{k=3}^n \left\{m([a_k, a_k+1) \cap E_{j,k}) + \frac{1}{2^n n}\epsilon 2^{-r-j}\right\} \\
&\quad + \frac{1}{2^n n}\epsilon 2^{-r-j} m([a_1, a_1+1) \cap E_{j,1}) \prod_{k=3}^n \left\{m([a_k, a_k+1) \cap E_{j,k}) + \frac{1}{2^n n}\epsilon 2^{-r-j}\right\} \\
&\quad + \frac{1}{n 2^n}\epsilon 2^{-r-j} \left(1 + \frac{1}{2^n n}\epsilon 2^{-r-j}\right)^{n-1}
\end{aligned}$$

$$\begin{aligned}
&\leq \prod_{k=1}^{2}\{m([a_k,a_k+1)\cap E_{j,k})\}\prod_{k=3}^{n}\left\{m([a_k,a_k+1)\cap E_{j,k})+\frac{1}{2^n n}\epsilon 2^{-r-j}\right\}\\
&\quad +2\frac{1}{2^n n}\epsilon 2^{-r-j}\left(1+\frac{1}{2^n n}\epsilon 2^{-r-j}\right)^{n-1}\\
&\leq \prod_{k=1}^{n} m([a_k,a_k+1)\cap E_{j,k})+n\left(\frac{1}{2^n n}\epsilon 2^{-r-j}\right)\left(1+\frac{1}{2^n n}\epsilon 2^{-r-j}\right)^{n-1}\\
&\leq \prod_{k=1}^{n} m([a_k,a_k+1)\cap E_{j,k})+\frac{1}{2}\epsilon 2^{-r-j} \text{ as } \left(1+\frac{1}{2^n n}\epsilon 2^{-r-j}\right)^{n-1}\leq 2^{n-1}\\
&= m(Q_r\cap T_j)+\tfrac{1}{2}\epsilon 2^{-r-j}
\end{aligned}$$

Thus for all $r,j\in\mathbb{N}$, there exists an open set of the form $\prod F_{r,j,k}$ such that $Q\cap T_j\subseteq\prod F_{r,j,k}$ and $m(\prod F_{r,j,k})\leq m(Q_r\cap T_j)+\frac{1}{2}\epsilon 2^{-r-j}$. Now, for all j set $U_j=\cup_{r=1}^{\infty}\prod_{k=1}^{\infty}F_{r,j,k}$. So $T_j\subseteq U_j$, with U_j open and

$$\begin{aligned}
m(U_j) &\leq \textstyle\sum_{r=1}^{\infty} m(\prod_{k=1}^{n}F_{r,j,k})\\
&\leq \textstyle\sum_{r=1}^{\infty}(m(Q_r\cap T_j)+\frac{1}{2}\epsilon 2^{-r-j})\\
&\leq m(\cup_{r=1}^{\infty}Q_r\cap T_j)+\tfrac{1}{2}\epsilon 2^{-j}\textstyle\sum_{r=1}^{\infty}2^{-r}\\
&= m(T_j)+\tfrac{1}{2}\epsilon 2^{-j}
\end{aligned}$$

Set $U=\cup_{j=1}^{\infty}U_j$, so U is open and $E\subseteq\cup_{j=1}^{\infty}T_j\subseteq U$ and $m(U)\leq\sum_{j=1}^{\infty}m(U_j)\leq\sum_{j=1}^{\infty}(m(T_j)+\frac{1}{2}\epsilon 2^{-j}=\sum_{j=1}^{\infty}m(T_j)+\frac{1}{2}\epsilon\leq m(E)+\epsilon$. Since $\epsilon\in(0,1)$ was arbitrary, we're done.

2. Follows exactly from Theorem 21b (1.18 in Folland)

3. Follows exactly from Theorem 24 (1.19 in Folland)

4. Follows exactly from Theorem 24. □

For each $k\in\mathbb{Z}$, define $\mathcal{Q}_k^n=\{\prod_{j=1}^{n}[a_j2^{-k},(a_j+1)2^{-k}]:a_j\in\mathbb{Z}\}$, the set of n dimensional dyadic cubes.

Remarks.

- For each $k\in\mathbb{Z}$, $\mathbb{R}^n=\cup_{Q\in\mathcal{Q}_k^n}Q$.
- If $Q_1\in\mathcal{Q}_k^n$ and $Q_2\in\mathcal{Q}_\ell^n$ with $k<\ell$, then either $Q_2\subset Q_1$ or $Q_2\cap Q_1=\emptyset$.
- If $Q\in\mathcal{Q}_k^n$, then $m(Q)=2^{-kn}$.
- If $Q\in\mathcal{Q}_\ell^n$, then there are exactly $2^{(k-\ell)n}$ elements of $\mathcal{Q}_k^n$ contained in Q.

Lemma 3. *Let $U\subseteq\mathbb{R}^n$ be an open set, then there exists a countable collection of disjoint dyadic cubes $\{Q_r\}_{r=1}^{\infty}\subseteq\cup_{k=0}^{\infty}\mathcal{Q}_k$ such that $U=\cup_{r=1}^{\infty}Q_r$.*

Proof. See Rudin. □

Theorem 32 (2.40c in Folland)**.** *Suppose $E\in\mathcal{L}^n$ and $m(E)<\infty$. Then for all $\epsilon>0$, there exists a finite collection $\{Q_r\}_{r=1}^{N}$ of disjoint dyadic cubes such that $m(E\triangle\cup_{r=1}^{N}Q_r)<\epsilon$.*

Proof. By Theorem 31a (2.40a), there exists an open set $U\subseteq\mathbb{R}^n$ such that $m(U)<m(E)+\frac{1}{2}\epsilon$. By Lemma 3, there exists a collection $\{Q_r\}_{r=1}^{\infty}$ of disjoint dyadic cubes such that $U=\cup_{r=1}^{\infty}Q_r$. Then $\sum_{r=1}^{\infty}m(Q_r)=m(U)<m(E)+\frac{1}{2}\epsilon<\infty$. Since $\sum m(Q_r)$ is absolutely convergent, there exists an $N\in\mathbb{N}$ such that $\sum_{r=N+1}^{\infty}m(Q_r)<\frac{1}{2}\epsilon$. Thus

$$\begin{aligned}
m(E\triangle\cup_{r=1}^{N}Q_r) &= m((E\setminus\cup_{r=1}^{N}Q_r)\cup(\cup_{r=1}^{N}Q_r\setminus E))\\
&= m(E\setminus\cup_{r=1}^{N}Q_r)+m(\cup_{r=1}^{N}Q_r\setminus E)\\
&\leq m(U\setminus\cup_{r=1}^{N}Q_r)+m(U\setminus E)\\
&= m(U)-\textstyle\sum_{r=1}^{N}m(Q_r)+m(U)-m(E)\\
&< m(E)+\tfrac{1}{2}\epsilon-\textstyle\sum_{r=1}^{\infty}m(Q_r)+\sum_{r=N+1}^{\infty}m(Q_r)+m(U)-m(E)=\epsilon.
\end{aligned}$$

□

Theorem 33 (2.42 in Folland)**.** *The $n-$dimensional Lebesgue measure is translation invariant. To be more precise, for all $a \in \mathbb{R}^n$, define $\tau_a : \mathbb{R}^n \to \mathbb{R}^n$ by $\tau_a(x) = x + a$. Then*

1. *If $E \in \mathcal{L}^n$, then $\tau_a(E) \in \mathcal{L}^n$ and $m(\tau_a(E)) = m(E)$.*

2. *If $f : \mathbb{R}^n \to \mathbb{C}$ is Lebesgue measurable, then so is $f \circ \tau_a$. Moreover, if either $f \geq 0$ is real valued or $f \in L^1(m)$, then $\int_{\mathbb{R}^n} f \circ \tau_a dm = \int_{\mathbb{R}^n} f dm$.*

Proof. **Key Observation:** Suppose λ is a Borel measure on $\mathbb{R}^n$ and there exists a constant c such that $\lambda(Q) = cm(Q)$ for all dyadic cubes. Then, by Lemma 3, $\lambda(U) = \sum_{r=1}^{\infty} \lambda(Q_r) = \sum_{r=1}^{\infty} cm(Q_r) = cm(U)$ for all open sets $U \subseteq \mathbb{R}^n$. Thus $\lambda(E) = cm(E)$ for all $E \in \mathcal{B}_{\mathbb{R}^n}$.

We will prove 1. Fix $a \in \mathbb{R}^n$ and define $\lambda : \mathcal{B}_{\mathbb{R}^n} \to [0, \infty]$ by $\lambda(E) = m(\tau_a(E))$. It is easy to verify λ is a Borel measure. Let Q be a dyadic cube. Then $\tau_a(Q)$ is still a dyadic cube and has the same volume. Thus $\lambda(Q) = m(Q)$. By the Key Observation, $(*)$ $m(\tau_a(E)) = \lambda(E) = m(E)$ for all $E \in \mathcal{B}_{\mathbb{R}^n}$. Of course, we want to show this for a general $E \in \mathcal{L}^n$. If $N \in \mathcal{L}^n$ and $m(N) = 0$, then by Theorem 31(3) (2.40b), there exists $V \in \mathcal{B}_{\mathbb{R}^n}$ such that $N \subseteq V$ and $m(V) = 0$ by $(*)$. Since m is complete, it follows that $\tau_a(N) \in \mathcal{L}^n$ and $m(\tau_a(N)) = 0$. In general, if $E \in \mathcal{L}^n$, then by Theorem 31(4), there is an $H \in \mathcal{B}_{\mathbb{R}^n}$ and a null set $N \in \mathcal{L}^n$ such that $E = H \cup N$, so $\tau_a(E) = \tau_a(H) \cup \tau_a(N) \in \mathcal{L}^n$. Thus the translation of a Lebesgue measurable set is still Lebesgue measurable. Furthermore,

$$\begin{aligned} m(E) &= \inf\{m(U) | U \text{ is open and } E \subseteq U\} \\ &= \inf\{m(\tau_a(U)) | U \text{ is open and } E \subseteq U\} \\ &= \inf\{m(U) | U \text{ is open and } \tau_a(E) \subseteq U\} \\ &= m(\tau_a(E)) \end{aligned}$$

Thus, we conclude $m(\tau_a(E)) = m(E)$ for all $E \in \mathcal{L}^n$. □

Theorem 34. *Suppose μ is a Borel measure satisfying $\mu(\tau_a(E)) = \mu(E)$ for all $E \in \mathcal{B}_{\mathbb{R}^n}$ and $a \in \mathbb{R}^n$. Suppose also $\mu(Q_0) < \infty$ for some unit dyadic cube. Then $\mu(E) = \mu(Q_0)m(E)$ for all $E \in \mathcal{B}_{\mathbb{R}^n}$.*

Proof. Since $\mu(\tau_a(Q)) = \mu(Q)$ for all $a \in \mathbb{R}^n$ and dyadic cubes Q, we may assume that $Q_0 \in \mathcal{Q}_0^n$. Let $Q \in \mathcal{Q}_k^n$ for some $k \in \mathbb{N}$. Now $Q_0 = \cup_{r=1}^{2^{nk}} Q_r$ for some family $\{Q_r\}_{r=1}^{2^{nk}} \subseteq Q_k^n$, where $\mu(Q_r) = \mu(Q_s)$ for all $r, s = 1, ..., 2^{nk}$. Thus $\mu(Q_0) = \mu(\cup_{r=1}^{2^{nk}} Q_r) = \sum_{r=1}^{2^{nk}} \mu(Q) = 2^{nk}\mu(Q)$. Thus $\mu(Q) = 2^{-nk}\mu(Q_0) = m(Q)\mu(Q_0)$. By the Key Observation, $\mu(E) = \mu(Q_0)m(E)$ for all $E \in \mathcal{B}_{\mathbb{R}^n}$. □

Corollary 10 (11.20 in Folland)**.** *If H_p is the $p-$dimensional Hausdorff measure on $\mathbb{R}^n$, then there is a constant $j_{p,n} \geq 0$ such that $H_p(E) = j_{p,n}m(E)$ for each $E \in \mathcal{B}_{\mathbb{R}^n}$ (we assume $p \geq n$).*

- *If $p = n$, then $j_{p,n} = H_n(Q_0) = \frac{1}{m(B)}$ where B is a ball of radius 1.*

- *If $p > n$, then $j_{p,n} = 0$.*

Theorem 35. *Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation (that is, $T(ax + by) = aT(x) + bT(y)$ for all $x, y \in \mathbb{R}^n, a, b \in \mathbb{R}$). Then there exists a number $\delta < \infty$ such that $m(T(E)) = \delta m(E)$ for all $E \in \mathcal{L}^n$.*

Proof. If the dimension of the range of T is less than n then $m(T(\mathbb{R}^n)) = 0$ which implies $m(T(E)) = 0$ for all $E \in \mathcal{L}^n$, so we have $\delta = 0$. If the dimension of the range of T is n, then T can be represented by an invertible matrix. In particular, T^{-1} exists and is also linear (and thus continuous). It follows that T^{-1} is a Borel measurable mapping. Thus $T(E) \in \mathcal{B}_{\mathbb{R}^n}$ whenever $E \in \mathcal{B}_{\mathbb{R}^n}$. Define $\mu : \mathcal{B}_{\mathbb{R}^n} \to [0, \infty]$ by $\mu(E) = m(T(E))$. Since T is linear, it is easy to verify that μ is a measure. Let $a \in \mathbb{R}^n$ by given. Then $\mu(\tau_a(E)) = \mu(a+E) = m(T(a+E)) = m(T(a)+T(E)) = m(T(E)) = \mu(E)$ (as m is translation invariant) for all $E \in \mathcal{B}_{\mathbb{R}^n}$. By Theorem 34, $\mu(E) = \mu(Q_0)m(E)$ for all $E \in \mathcal{B}_{\mathbb{R}^n}$ with Q_0 a unit cube.

For the general case where $E \in \mathcal{L}^n$, use essentially the same argument used at the end of the proof for Thm 33. □

2.3 Signed Measures and Differentiation

Major Goal: Develop a theory of differentiation for measures.

Suppose that $g \in C^1(\mathbb{R})$ with $g(0) = 0$. Then by the Fundamental Theorem of Calculus, there exists $f \in C(\mathbb{R})$ such that $g(x) = \int_0^x f(s)ds$. (Here, of course, $f = g'$.) We want to do something similar for measures:

- Suppose that μ, ν are measures on a $\sigma-$algebra $\mathcal{M}$. When is it true that there is a $\mathcal{M}-$measurable function such that for each $A \in \mathcal{M}$, $\nu(A) = \int_A f d\mu$? In some sense, f is the derivative of ν with respect to μ.

To develop an answer to this question, we extend our notion of measures to signed measures.

Definition. *Let $(X, \mathcal{M})$ be a measurable space. A **signed measure** on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that*

- $\nu(\emptyset) = 0$.
- *ν assumes at most one of the values $\pm\infty$, that is, if there exists $A \in \mathcal{M}$ such that $\nu(A) = \infty$, then there does not exist $B \in \mathcal{M}$ such that $\nu(B) = -\infty$.*
- *if $\{E_j\}_{j=1}^\infty \subseteq \mathcal{M}$ are mutually disjoint sets then $\nu(\cup_{j=1}^\infty E_j) = \sum_{j=1}^\infty \nu(E_j)$ and $\sum_{j=1}^\infty |\nu(E_j)| < \infty$ whenever $|\sum_{j=1}^\infty \nu(E_j)| < \infty$.*

Remark. Since countable unions are invariant under rearrangement, if $|\sum_{j=1}^\infty \nu(E_j)| < \infty$, then one can show $\sum_{j=1}^\infty |\nu(E_j)| < \infty$.

Examples.

1. Suppose $\alpha, \beta \in \mathbb{R}$ and μ_1, μ_2 are positive measures on $\mathcal{M}$ such that either $\mu_1(X) < \infty$ or $\mu_2(X) < \infty$. Then $\alpha\mu_1 + \beta\mu_2$ is a signed measure. [The condition that one must be finite is to prevent $\alpha\mu_1 + \beta\mu_2$ from taking on values of both $\pm\infty$.]
2. If $f \in L^1(\mu)$ where μ is a positive measure on $\mathcal{M}$, then the function $\nu : \mathcal{M} \to (-\infty, \infty)$ defined by $\nu(A) = \int_A f d\mu$ is a signed measure.

Proposition 37 (3.1)**.** *Let ν be a signed measure on $(X, \mathcal{M})$.*

1. *If $\{E_j\}_{j=1}^\infty \subseteq \mathcal{M}$ and $E_1 \subseteq E_2 \subseteq \cdots$, then $\nu(\cup_{j=1}^\infty E_j) = \lim_{j\to\infty} \nu(E_j)$.*
2. *If $\{E_j\}_{j=1}^\infty \subseteq \mathcal{M}$ and $E_1 \supseteq E_2 \supseteq \cdots$ and $|\nu(E_1)| < \infty$, then $\nu(\cap_{j=1}^\infty E_j) = \lim_{j\to\infty} \nu(E_j)$.*

Proof. This is similar to the proof for Theorem 5 (1.8 in Folland). Thus we will prove only (1). If there exists $N \in \mathbb{N}$ such that $|\nu(E_N)| = \infty$, then for all $j \geq N$ $\nu(E_j) = \nu(E_j \setminus E_N) + \nu(E_N) = \nu(E_N) = \pm\infty$ by property 3 of the definition of signed measures. Thus $|\nu(E_j)| = \infty$ and $\lim_{j\to\infty} \nu(E_j) = \pm\infty$. Also, $\nu(\cup_{j=1}^\infty E_j) = \nu(\cup_{j=1}^\infty E_j \setminus E_N) + \nu(E_N) = \infty = \nu(E_N) = \lim \cup_{j=1}^\infty \nu(E_j)$. So we may assume $|\nu(E_j)| < \infty$ for all $j \in \mathbb{N}$. Define $\{F_j\}_{j=1}^\infty \subseteq \mathcal{M}$ by $F_1 := E_1$ and $F_j := E_j \setminus \cup_{k=1}^{j-1} E_k$ for all $j \geq 2$. Then by Lemma 1, $\nu(\cup_{j=1}^\infty E_j) = \nu(\cup_{j=1}^\infty F_j) = \sum_{j=1}^\infty \nu(F_j)$. For each $j \geq 2$, we see $\nu(E_j) = \nu(F_j) + \nu(\cup_{k=1}^{j-1} E_k) = \nu(F_j) + \nu(E_{j-1})$. Since $|\nu(E_k)| < \infty$, this says $\nu(F_j) = \nu(E_j) - \nu(E_{j-1})$. Thus we have

$$\nu(\cup_{j=1}^\infty E_j) = \sum_{j=1}^\infty \nu(F_j) = \nu(E_1) + \sum_{j=1}^\infty \nu(E_j) - \nu(E_{j-1}) = \lim_{j\to\infty} \nu(E_j).$$

□

Definition. *Suppose ν is a signed measure. A set $E \in \mathcal{M}$ is called*

1. ***positive** if $\nu(F) \geq 0$ for all $F \subseteq E$ such that $F \in \mathcal{M}$,*
2. ***negative** if $\nu(F) \leq 0$ for all $F \subseteq E$ such that $F \in \mathcal{M}$,*
3. ***null** if $\nu(F) = 0$ for all $F \subseteq E$ such that $F \in \mathcal{M}$,*

Lemma 4. *Suppose $\{P_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ are positive sets with respect to ν. Then $\cup_{j=1}^{\infty} P_j$ is also positive.*

Theorem (**Hahn Decomposition Theorem-** p.86)**.** *Suppose ν is a signed measure. Then there exists a positive set P and a negative set N such that $X = P \cup N$ and $P \cap N = \emptyset$. Moreover, if P', N' are another such pair, then $P \triangle P' = N \triangle N'$ are null sets.*

Proof. WLOG, assume $\nu(A) < \infty$ for all $A \in \mathcal{M}$ (if not, work with $-\nu$). Put $M = \sup\{\nu(P) : P \text{ is positive}\}$. Then there exists $\{P_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ such that each P_j is a positive set and $\lim_{j\to\infty} \nu(P_j) = M$. WLOG, assume $P_1 \subseteq P_2 \subseteq \cdots$ as otherwise we can just use $s_j = \cup_{k=1}^{j} P_k$ where still $\nu(s_j) \to M$. Set $P = \cup_{j=1}^{\infty} P_j$. Then by Proposition 37, $\nu(P) = \lim_{j\to\infty} \nu(P_j) = M < \infty$. Set $N = X \setminus P$.

Claim: N is a negative set.

Proof: Suppose not. Then there exists $A \in \mathcal{M}$ such that $A \subseteq N$ and $\nu(A) > 0$.

Subclaim: There exists a positive set E such that $E \subseteq A$ and $\nu(E) > 0$.

Proof: If A is a positive set, done. Otherwise, there exists $C \in \mathcal{M}$ such that $C \subseteq A$ and $\nu(C) < 0$. Put $L_1 = \inf\{\nu(C) : C \in \mathcal{M}, C \subseteq A\} < 0$. Let $n_1 \in \mathbb{N}$ be the smallest such integer such that $L_1 < -\frac{1}{n_1}$. Then there exists $C_1 \in \mathcal{M}$ such that $C_1 \subseteq A$ and $\nu(C_1) < -\frac{1}{n_1}$. Set $A_1 = A \setminus C_1$. If A_1 is a positive set, done. Otherwise, there exists $C \in \mathcal{M}$ such that $C \subseteq A$ and $\nu(C) < 0$. Put $L_2 = \inf\{\nu(C) : C \in \mathcal{M}, C \subseteq A_1\} < 0$. Let $n_2 \in \mathbb{N}$ be the least such integer such that $L_2 < -\frac{1}{n_2}$. Then there exists $C_2 \in \mathcal{M}$ such that $C_2 \subseteq A_1$ and $\nu(C_2) < -\frac{1}{n_2}$. Set $A_2 = A_1 \setminus C_2$ and continue inductively to get sequences of sets $\{A_j\}_{j=1}^{\infty}, \{C_j\}_{j=1}^{\infty}$, and positive integers $\{n_j\}_{j=1}^{\infty}$ such that for $j \geq 2$ we have $C_j \subseteq A_{j-1}$ and for all $j \in \mathbb{N}$, $\nu(C_j) < -\frac{1}{n_j}$. Notice $\nu(A_j) > \nu(A) + \sum_{k=1}^{j} \frac{1}{n_k}$. Put $E = \cap_{j=1}^{\infty} A_j$. Since $A_1 \supseteq A_2 \supseteq \cdots$ and $\nu(A_1) < \infty$, by Proposition 37, we have $\nu(E) = \lim \nu(A_k) > \nu(A) + \sum_{k=1}^{\infty} \frac{1}{n_k} > 0$. Since $\nu(E) < \infty$, we have $\sum \frac{1}{n_k} < \infty$ and thus $n_k \to \infty$ as $k \to \infty$. Now, suppose E is not a positive set. Then there exists $C \in \mathcal{M}$ such that $C \subseteq E$ and $\nu(C) < 0$. Since $n_k \to \infty$, there exists k_0 such that $\nu(C) < -\frac{1}{n_{k_0-1}}$. Since $C \subseteq E \subseteq \cap_{j=1}^{n_{k_0-1}} A_j \subseteq A_{k_0-1}$ and $L_{k_0} = \inf\{\nu(\widetilde{C}) : \widetilde{C} \in \mathcal{M}, \widetilde{C} \subseteq A_{k_0-1}\}$, we see $L_{k_0} < -\frac{1}{n_{k_0-1}}$. But $n_{k_0} - 1 < n_{k_0}$ and n_{k_0} was chosen to be the smallest integer, a contradiction. Thus E is a positive set and $\nu(E) > \nu(A) > 0$.

By the subclaim, if N was not a negative set, then there exists a positive set E such that $\nu(E) > 0$. But this contradicts the fact that $\nu(P) = \sup\{\nu(\widetilde{P}) : \widetilde{P} \text{ is a positive set}\}$ as $P \cup E$ is a positive set with $\nu(P \cup E) > \nu(P)$. Thus N is a negative set.

If P', N' is another such decomposition, then $P \setminus P' \subseteq P$ and $P \setminus P' \subseteq N'$. Thus $\nu(P \setminus P') = 0$. Similarly for $P' \setminus P$ and thus $P \triangle P'$ is a null set. ☐

Definition (p 87)**.** *Any decomposition of X into a positive set P and a negative set N (that is, $P \cup N = X$ and $P \cap N = \emptyset$) is called a **Hahn Decomposition**.*

Definition (p 87)**.** *Suppose that μ and ν are signed measures on $(X, \mathcal{M})$. We say μ and ν are **mutually singular**, denoted $\mu \perp \nu$, if there exists a set $E \in \mathcal{M}$ such that E is a null set for μ and $X \setminus E$ is a null set for ν. We also say μ is **singular with respect to** ν and vice versa.*

Example. Suppose m is the Lebesgue measure and ν any discrete signed measure, that is, there exists a countable set $K \subset \mathbb{R}^n$ such that $\mathbb{R}^n \setminus K$ is a $\nu-$null set (for example, the counting measure on $\mathbb{Z}$). Then $m \perp \nu$. (since $m(k) = 0$)

Example. Put $D = \{(x, y) \in \mathbb{R}^2 | x = y\}$. Define $\nu : \mathcal{B}_{\mathbb{R}^2} \to [0, \infty]$ by $\nu(E) = m(\{x \in \mathbb{R} : (x, x) \in E\})$. Then $m|_{\mathcal{B}_{\mathbb{R}^2}}(D) = 0$ and $\nu(D^C) = 0$.

Theorem (**Jordan Decomposition Theorem** p.87)**.** *If ν is a signed measure on $(X, \mathcal{M})$, then there exist unique positive measures ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.*

Proof. Let P, N be a Hahn Decomposition for ν. Define $\nu^+, \nu^- : \mathcal{M} \to [0, \infty]$ by $\nu^+(E) = \nu(E \cap P)$ and $\nu^- = -\nu(E \cap N)$. Its easy to check ν^+, ν^- are positive measures. Also $\nu^+ \perp \nu^-$ as $P \cap N = \emptyset$. Finally,

$$\nu(E) = \nu((E \cap P) \cup (E \cap N)) = \nu(E \cap P) + \nu(E \cap N) = \nu^+(E) - \nu^-(E).$$

To show uniqueness, suppose there exist mutually singular positive μ^+, μ^- such that $\nu = \mu^+ - \mu^-$. Since $\mu^+ \perp \mu^-$, there exists $E \in \mathcal{M}$ such that $\mu^-(E) = \mu^+(E^C) = 0$. Now, for any $A \in \mathcal{M}$ such that $A \subseteq X \setminus E$, $\nu(A) = \mu^+(A) - \mu^-(A) = -\mu^-(A)$ by monotonicity. So $X \setminus E$ is a negative set and similarly E is a positive set. Thus $E, X \setminus E$ is another Hahn Decomposition of ν which implies $\nu(E \triangle P) = \nu(E^C \triangle N) = 0$. Let $A \in \mathcal{M}$ be given. Then

$$\begin{aligned} \mu^+(A) = \mu^+(A \cap E) = \nu(A \cap E) &= \nu(A \cap ((E \setminus P) \cup (P \cap E))) \\ &= \nu(A \cap E \setminus P) + \nu(A \cap (P \cap E)) \\ &= \nu((A \cap E) \cap P) = \nu^+(A \cap E). \end{aligned}$$

Also

$$\begin{aligned} \nu^+(A) = \nu^+((A \cap E) \cup (A \setminus E)) &= \nu^+(A \cap E) + \nu^+(A \setminus E) \\ &= \nu^+(A \cap E) + \nu((A \setminus E) \cap P) \\ &= \nu^+(A \cap E) \text{ as } (A \setminus E) \cap P \subseteq E \triangle P. \end{aligned}$$

□

Definition. *The decomposition of a signed measure ν into a difference of two positive mutually singular measures ν^+, ν^- is called a **Jordan Decomposition**. The positive measure ν^+ is called the **positive variation** of ν and ν^- is called the **negative variation** of ν. The **total variation** of ν is defined by $|\nu|(E) = \nu^+(E) + \nu^-(E)$ for $E \in \mathcal{M}$.*

Note. This is a generalization of bounded variation.

Remarks.

1. $|\nu|$ is a positive measure on $\mathcal{M}$.
2. $A \in \mathcal{M}$ is a null set for ν if and only if it is for $|\nu|$.
3. If ν is a signed measure on $\mathcal{M}$, then $\nu \perp \mu$ if and only if $|\nu| \perp \mu$ if and only if $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.
4. If P, N is a Hahn Decomposition for ν, then $\nu(A) = \nu^+(A) - \nu^-(A) = |\nu|(A \cap P) - |\nu|(A \cap N) = \int_{A \cap P} 1 d|\nu| - \int_{A \cap N} 1 d|\nu| = \int_A |\chi_P - \chi_N| d|\nu|$.

Definition. *Suppose ν is a signed measure on $(X, \mathcal{M})$. We set $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$ and define $\int f d\nu = \int f d\nu^+ - \int f d\nu^-$ for $f \in L^1(\nu)$.*

Example. Define $f \in C^\infty(\mathbb{R})$ by $f(x) = x^2 + 2$. Define $\delta_0, \delta_1 : \mathcal{B}_\mathbb{R} \to [0, \infty]$ by $\delta_i(A) = 0$ if $i \notin A$ and 1 if $i \in A$. Put $\nu = m - \delta_0 - \delta_1$. This is a signed measure on $\mathcal{B}_\mathbb{R}$ as $\delta_0 + \delta_1$ is finite. Notice

$$\int_{[0,1)} f d\nu = \int_{[0,1)} f dm - \int_{[0,1)} f d\delta_0 - \int_{[0,1)} f d\delta_1 = \frac{1}{3}x^3 + 2|_0^1 - f(0) - 0 = \frac{1}{3}$$

(as $1 \notin [0, 1)$) but

$$\int_{[0,1]} f d\nu = \frac{1}{3} - f(0) - f(1) = -\frac{8}{3}.$$

2.4 The Lebesgue-Radon-Nikodym Theorem

Definition. *Suppose ν is a signed measure and μ a positive measure on $(X, \mathcal{M})$. We say ν is **absolutely continuous** with respect to μ, denoted $\nu << \mu$, if $\nu(E) = 0$ whenever $\mu(E) = 0$.*

Remarks.

- If $\nu << \mu$, then each null set for μ is a null set for ν.
- $\nu << \mu$ if and only if $|\nu| << \mu$ if and only if $\nu^+ << \mu$ and $\nu^- << \mu$.
- If $\nu << \mu$ and $\nu \perp \mu$, then $\nu = 0$.

Examples.

1. Suppose $f \in L^1(\mu)$ and define $\nu : \mathcal{M} \to (-\infty, \infty)$ by $\nu(E) = \int_E f d\mu$ for all $E \in \mathcal{M}$. Then $\nu << \mu$.
2. Recall the dirac measures δ_0, δ_1. Then $\delta_0 << \delta_0 + \delta_1$ and $\delta_1 << \delta_0 + \delta_1$, but $\delta_0 + \delta_1$ is not absolutely continuous with respect to δ_0, δ_1.

Theorem 36 (3.5)**.** *Let ν be a finite signed measure and let μ be a positive measure. Then $\nu << \mu$ if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|\nu(E)| < \epsilon$ whenever $\mu(E) < \delta$.*

Proof. If the $\epsilon - \delta$ condition holds and $\mu(E) = 0$, then for all $\epsilon > 0$, we have $|\nu(E)| < \epsilon$. Thus $\nu(E) = 0$. Now suppose $\nu << \mu$, but there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists $E \in \mathcal{M}$ such that $|\nu(E)| \geq \epsilon$ and $\mu(E) < \delta$. Then for all $n \in \mathbb{N}$, find $E_n \in \mathcal{M}$ such that $\mu(E_n) < \frac{1}{2^n}$ yet $|\nu(E_n)| \geq \epsilon$. Set $F = \liminf_{j\to\infty} E_j = \cap_{j=1}^{\infty} \cup_{k=j}^{\infty} E_k$. So for $j \in \mathbb{N}$, we have $0 \leq \mu(F) \leq \mu(\cup_{k=j}^{\infty} E_k) \leq \sum_{k=j}^{\infty} \frac{1}{2^k} = \frac{1}{2^{j-1}}$. Since this holds for all $j \in \mathbb{N}$, we have $\mu(F) = 0$. Since $\nu << \mu$, $|\nu| << \mu$ and thus $|\nu|(F) = 0$. Observe $|\nu|(\cup_{k=1}^{\infty} E_k) < \infty$ and $\cup_{k=j}^{\infty} E_k \supseteq \cup_{k=j+1}^{\infty} E_k$. Since $|\nu|$ is a positive measure, Theorem 5 gives $0 = |\nu|(F) = \lim_{j\to\infty} |\nu|(\cup_{k=j}^{\infty} E_k) \geq \lim_{j\to\infty} |\nu|(E_j) \geq \lim_{j\to\infty} |\nu(E_j)| \geq \epsilon$, a contradiction. ☐

Corollary 11. *If $f \in L^1(\mu)$, then for all $\epsilon > 0$ there exists $\delta > 0$ such that $|\int_A f d\mu| < \epsilon$ whenever $\mu(A) < \delta$.*

Notation. If $\nu(E) = \int_E f d\mu$ for $E \in \mathcal{M}$ we write $\frac{d\nu}{d\mu}$ for f. Also, write $d\nu$ for $f d\mu$.

Lemma 5 (3.7)**.** *Suppose ν, μ are finite positive measures. Either $\nu \perp \mu$ or there exists $\epsilon > 0$ and $E \in \mathcal{M}$ such that $\mu(E) > 0$ and E is a positive set for $\nu - \epsilon\mu$.*

Proof. For all $n \in \mathbb{N}$, let λ_n be the signed measure $\nu - \frac{1}{n}\mu$ and P_n, N_n be a Hahn Decomposition for λ_n. Set $P = \cup_{n=1}^{\infty} P_n, N = \cap_{n=1}^{\infty} N_n$. Note $N = X \setminus P$. We see N is a negative set for all λ_n. Thus $0 \geq \lambda_n(N) = (\nu - \frac{1}{n}\mu)(N)$ which implies $\frac{1}{n}\mu(N) \geq \nu(N)$. Taking the limit as $n \to \infty$, since ν is a positive measure, $\nu(N) = 0$. If P is a null set for μ, then $\nu \perp \mu$. So suppose P is not a null set for μ, that is, $\mu(P) > 0$ (since μ is a positive measure). Then there exists $n_0 \in \mathbb{N}$ such that $\mu(P_{n_0}) > 0$ and since P_{n_0} is a positive set for λ_{n_0}, the lemma is proved (that is, take $E = P_{n_0}$). ☐

Theorem (**Lebesgue Radon Nikodym Theorem-** p.90)**.** *Let ν be a $\sigma-$finite signed measure and μ be a $\sigma-$finite positive measure. There are unique $\sigma-$finite signed measures λ, ρ on $(X, \mathcal{M})$ such that $\lambda \perp \mu, \rho << \mu$ and $\lambda + \rho = \nu$ (this is called the **Lebesgue Decomposition** of ν and μ). Moreover, there exists an $\overline{\mathbb{R}}-$valued $\mu-$integrable function f such that $d\rho = f d\mu$. Any other such function is equal to f $\mu-a.e.$*

Note. By $\mu-$integrable, we mean either $\int f^+ d\mu$ or $\int f^- d\mu$ is finite.

Convention: If ν is a signed measure, we may refer to $d\nu$ as a signed measure, but we are actually referring to $E \mapsto \int_E d\nu$.

Proof. Step 1: First, we will assume μ, ν are finite positive measures. Set $\mathcal{F} = \{f \in L^1(\mu) : \int_E f d\mu \leq \nu(E) \text{ for all } E \in \mathcal{M}\}$. Note that $0 \in \mathcal{F}$, thus it is non-empty. Also, if $f, g \in \mathcal{F}$, so is the function $x \mapsto max\{f(x), g(x)\}$ as if $A = \{x \in X | f(x) > g(x)\}$, then for $E \in \mathcal{M}$ $\int_E max\{f(x), g(x)\} d\mu = \int_{E\cap A} f d\mu + \int_{E\setminus A} g d\mu \leq \nu(E \cap A) + \nu(E \setminus A) = \nu(E)$. Put $a := \sup\{\int_X f d\mu : f \in \mathcal{F}\}$ so that $a \leq \nu(X) < \infty$. We may select $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ such that $\lim_{n\to\infty} \int f_n d\mu = a$. For each $n \in \mathbb{N}$, define $g_n = max\{f_1, ..., f_n\}$. Then $\{g_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ is an increasing sequence. Define $f : X \to \overline{\mathbb{R}}$ by $f(x) := \sup_{n\in\mathbb{N}} f_n(x)$.

Claim: $f \in \mathcal{F}$.

Proof: Observe $g_n \to f$ pointwise and $g_1 \leq g_2 \leq \cdots \leq f$. By the Monotone Convergence Theorem, for all $E \in \mathcal{M}$, $\int_E f d\mu = \lim_{n\to\infty} \int_E g_n d\mu \leq \nu(E)$. Then $f \in \mathcal{F}$.

Note that $\int_X f d\mu = a$.

Claim: The measure $d\lambda = d\nu - f d\mu$ is singular with respect to μ.

Proof: Note $d\lambda$ is a positive measure as $\nu(E) - \int f d\mu \geq 0$ for all $E \in \mathcal{M}$. Suppose λ was not singular with respect to μ. By Lemma 5(3.7), there exists $\epsilon > 0$ and $E \in \mathcal{M}$ such that $\mu(E) > 0$ and $\lambda(A) - \epsilon\mu(A) \geq 0$ for all $A \in \mathcal{M}$ with $A \subseteq E$. This implies for all $A \in \mathcal{M}$ that $\epsilon\mu(A \cap E) \leq \lambda(A \cap E) = \nu(A \cap E) - \int_{A\cap E} f d\mu$. Thus $\int_A \{f + \epsilon\chi_E\} d\mu = \int_A f d\mu + \epsilon\mu(A \cap E) \leq \int_A f d\mu + \nu(A \cap E) - \int_{A\cap E} f d\mu = \int_{A\setminus E} f d\mu + \nu(A \cap E) \leq \nu(A)$. Therefore, $f + \epsilon\chi_E \in \mathcal{F}$ but $\int_X f + \epsilon\chi_E d\mu = \int f d\mu + \epsilon\mu(E) > \int f d\mu$, a contradiction. Thus $\lambda \perp \mu$.

For uniqueness, suppose there exists λ', ρ', f' satisfying the conclusion of the theorem. Then $d\nu = d\lambda + fd\mu = d\lambda' + f'd\mu$ which implies $d\lambda - d\lambda' = (f' - f)d\mu$. Since $\lambda \perp \mu$ and $\lambda' \perp \mu$, we see $\lambda - \lambda' \perp \mu$. Also, since $f'd\mu << d\mu$ and $fd\mu << d\mu$ we have $(f' - f)d\mu << d\mu$. This implies that $\lambda - \lambda'$ is singular and absolutely continuous with respect to μ which says $\lambda - \lambda' = 0$. Now, $\int_X |f - f'|d\mu = 0$ and thus by Proposition 14 (2.23), $f' = f$ $\mu-$a.e. Thus, the theorem is proved when μ, ν are finite positive measures.

Step 2: Now, assume μ, ν are positive $\sigma-$finite measures. We may find a sequence $\{A_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ of mutually disjoint sets such that $\cup_{j=1}^{\infty} A_j = X$ and $\nu(A_j), \mu(A_j) < \infty$. For each $j \in \mathbb{N}$, define the positive finite measures ν_j and μ_j by $\mu_j(E) = \mu(E \cap A_j)$ and $\nu_j(E) = \nu(E \cap A_j)$ for all $E \in \mathcal{M}$. Apply Step 1 to each pair (μ_j, ν_j) to obtain $\{\lambda_j\}_{j=1}^{\infty}$ of signed measures and $\{f_j\}_{j=1}^{\infty}$ of μ_j-integrable functions (in fact, in $L^1(\mu_j)$, since $\mu_j(X) < \infty$) such that for all $j \in \mathbb{N}$, $\lambda_j \perp \mu_j$ and $d\nu_j = d\lambda_j + f_j d\mu_j$. Since $\mu_j(X \setminus A_j) = 0$, WLOG, assume $f_j = 0$ on $X \setminus A_j$. Also, observe for all $E \in \mathcal{M}$ such that $E \subseteq X \setminus A_j$ we have $\lambda_j(E) = \nu_j(E) - \int_E f_j d\mu_j = 0$. So $X \setminus A_j$ is a null set for λ_j. Put $\lambda := \sum_{j=1}^{\infty} \lambda_j, f := \sum_{j=1}^{\infty} f_j$. Then it can be shown $\lambda \perp \nu$ and $d\nu = d\lambda + fd\mu$. Also, λ and $fd\mu$ are $\sigma-$finite.

Step 3: If ν is a $\sigma-$finite signed measure, then $\nu = \nu^+ - \nu^-$ where ν^+, ν^- are $\sigma-$finite positive measures. Apply Step 2 to ν^+ and ν^- separately and take the difference of the results. □

Definition. *The function f in the Lebesgue-Radon-Nikodym Theorem is called the* ***Radon-Nikodym derivative*** *of ν with respect to μ. It is traditionally denoted by $\frac{d\nu}{d\mu}$ and if $\nu << \mu$ then $d\nu = \frac{d\nu}{d\mu}d\mu$.*

Example. Let ν be a $\sigma-$finite signed measure on $(X, \mathcal{M})$. We see $\nu << |\nu|$. Also, we observed $\nu(E) = \int_E [\chi_P - \chi_N]d|\nu|$ where P, N is a Hahn Decomposition of ν. Thus $\chi_P - \chi_N$ is the Radon-Nikodym derivative of ν with respect to $|\nu|$.

Examples. Let $F : \mathbb{R} \to \mathbb{R}$ be given by $F(x) = \begin{cases} 0 & \text{if } x < 0, \\ 3 - e^{-x} & \text{if } 0 \le x < 1 \\ 4 - e^{-x} & \text{if } 1 \le x < \infty. \end{cases}$ Thus F is nondecreasing, right continuous. So there exists a Lebesgue-Stieltjes measure μ_F with F as its distribution. What is the Lebesgue Decomposition of μ_F with respect to m? Note that $m(\{0\}) = 0$ but $\mu_F(\{0\}) = \mu_F((-\infty, 0] \setminus (-\infty, 0)) = 2$. Also, $m(\{1\}) = 0$ but $\mu_F(\{1\}) = 1$. Then there exists a singular measure on $\{0, 1\}$. Define $G : \mathbb{R} \to \mathbb{R}$ by $G(x) = \begin{cases} 0 & \text{if } x < 0, \\ e^{-x} & \text{if } 0 \le x < \infty. \end{cases}$ Notice $\int_0^x G(s)ds$ is 0 if $x < 0$ and $-e^{-x}+1$ if $x \ge 0$. So define $\rho : \mathbb{B}_{\mathbb{R}} \to [0, \infty]$ by $\rho(E) = \int_E G(x)dx$. We see that $\mu_F = \rho + 2\delta_0 + \delta_1$ and $\frac{d\rho}{dm} = G(x)$.

Expansion of discussion on p 106

Definition. *Let $(X, \mathcal{M})$ be a measurable space such that $\{x\} \in \mathcal{M}$ for all $x \in X$. Let ν be a signed measure on $(X, \mathcal{M})$. We say $x \in X$ is an* ***atom*** *of ν if $\nu(\{x\}) \neq 0$.*

Definition. *Let $(X, \mathcal{M})$ be a measurable space such that $\{x\} \in \mathcal{M}$ for all $x \in X$. Let ν be a signed measure. Then,*

- *We say ν is* ***continuous*** *if $\nu(\{x\}) = 0$ for all $x \in X$.*
- *We say ν is* ***discrete*** *if there exists a countable set $k \subseteq \mathcal{M}$ such that $|\nu|(k^C) = 0$.*

Definition. *Let $(X, \mathcal{M})$ be a measure space such that $\{x\} \in \mathcal{M}$ for all $x \in X$. For each $x \in X$ define the* ***dirac measure*** *concentrated at x by $\delta_x = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E \end{cases}$ for all $E \in \mathcal{M}$.*

Examples.

- The Lebesgue measure, the 0 measure, and all Lebesgue Stieltjes measures with continuous distribution functions are continuous.
- The 0 measure and the dirac measures are discrete.
- There exist measures which are neither continuous nor discrete. For example $m + \delta_0$.

Proposition 38. *Let $(X, \mathcal{M})$ be a measurable space such that $\{x\} \in \mathcal{M}$ for all $x \in X$. Let ν be a $\sigma-$finite positive measure on $(X, \mathcal{M})$. Then there exist $\sigma-$finite positive measures ν_c and ν_d such that $\nu_c \perp \nu_d, \nu = \nu_c + \nu_d$, ν_c is continuous, and ν_d is discrete.*

Proof. Step 1: Assume ν is finite.

Claim: ν has only a countable number of atoms.

Proof: Let $F \in \mathcal{M}$ be a set consisting of a countable number of atoms. Then $\nu(F) = \sum_{x \in F} \nu(\{x\}) \leq \nu(X) < \infty$. Put $\alpha := \sup\{\sum_{x \in F} \nu(\{x\}) : F \in \mathcal{M}$ and F is countable$\}$. Then $\alpha < \infty$ and there exists a sequence $\{F_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ such that each F_n is countable and $\alpha = \lim_{n \to \infty} \sum_{x \in F_n} \nu(\{x\})$. Set $F = \cup_{n=1}^{\infty} F_n$. Then F is countable and $\alpha = \sum_{x \in F} \nu(\{x\})$. If there were an uncountable number of atoms for ν, then there would exist $x_0 \in X$ such that x_0 is an atom but $x_0 \notin F$. But then $F \cup \{x_0\}$ would be a countable set where $\sum_{x \in F \cup \{x_0\}} \nu(\{x\}) > \alpha$, a contradiction. Thus there exists a set $k = \{\text{atoms}\}$ which is countable.

Define $\nu_d : \mathcal{M} \to [0, \infty)$ by $\nu_d(E) = \sum_{x \in k} \nu(\{x\}) \delta_x(E)$ for all $E \in \mathcal{M}$. Put $\nu_c = \nu - \nu_d$. Clearly, ν_c is countably additive as ν and ν_d are and $\nu_c(\emptyset) = \nu(\emptyset) - \nu_d(\emptyset) = 0$. To show ν_c is non-negative, let $E \in \mathcal{M}$. Then $\nu_c(E) = \nu(E) - \nu_d(E) = \nu(E \cap k^C) + \nu(E \cap k) - \nu_d(E \cap k^C) - \nu_d(E \cap k) = \nu(E \cap k^C) + \sum_{x \in E \cap k} \nu(\{x\}) - \sum_{x \in E \cap k} \nu(\{x\}) = \nu(E \cap k^C) \geq 0$ (since $\delta_x = 1$ for all $x \in E \cap k$). Thus ν_c is a positive finite measure. Need to show ν_c is continuous. Let $x \in X$ be given. Then $\nu_c(\{x\}) = \nu(\{x\} \cap k^C) = 0$. Since $\nu_c = \nu(E \cap k^C)$ and $\nu_d = \nu(E \cap k)$, clearly $\nu_c \perp \nu_d$. We leave uniqueness as an exercise.
Step 2: Extend to $\sigma-$finite measures. (Again, an exercise). ☐

Theorem 37. *Let $(X, \mathcal{M})$ be a measurable space such that $\{x\} \in \mathcal{M}$ for all $x \in X$. Let μ be a $\sigma-$finite positive measure and ν a $\sigma-$finite signed measure. Then there exist unique $\sigma-$finite signed measures ν_{ac}, ν_{sc}, and ν_d such that*

1. $\nu_{ac} << \mu, \nu_{sc} \perp \mu$ *and* $\nu_d \perp \mu$.
2. ν_{sc} *is continuous,* ν_d *is discrete and* $\nu_{sc} \perp \nu_d$.
3. $\nu = \nu_{ac} + \nu_{sc} + \nu_d$.

Proof. Use Lebesgue Decomposition for part (1), then use Jordan Decomposition and Proposition 38 for the rest. ☐

Example. Let m be the Lebesgue measure on $\mathbb{R}$. Define $\nu : \mathcal{L} \to [0, \infty]$ by $\nu(E) = m(E) + \delta_0(E) + \delta_1(E)$. Define $\mu : \mathcal{L} \to [0, \infty]$ by $\mu(E) = \sum_{x \in E \cap \mathbb{N}} 1$. Then $\nu_{ac} = \delta_1, \nu_{sc} = m, \nu_d = \delta_0$ and $\frac{d\nu_{ac}}{d\mu} = \frac{d\delta_1}{d\mu} = \chi_{\{1\}}$.

2.5 Differentiation on Euclidean Space

We consider the setting where $(X, \mathcal{M}) = (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$. In this setting, we will look at computing $\frac{d\nu}{dm}$ more explicitly.

Consider a positive Borel measure μ on $\mathbb{R}$ such that $\mu << m$. By the Radon-Nikodym Theorem there exists $f \in L^+(\mathbb{R})$ such that $\mu(E) = \int_E f dx$ for all $E \in \mathcal{B}_{\mathbb{R}}$. Can we find a formula for f in terms of μ?

Special Case: Suppose f is continuous. Then for all $x_0 \in \mathbb{R}$ and $h > 0$, we have $\mu((x_0 - h, x_0 + h)) = \int_{x_0-h}^{x_0+h} f dx$ and

$$\frac{\mu((x_0 - h, x_0 + h))}{m((x_0 - h, x_0 + h))} = \frac{1}{2h} \int_{x_0-h}^{x_0+h} f dx \to f(x_0)$$

as $h \to 0^+$ by the Fundamental Theorem of Calculus. Thus $\frac{d\mu}{dm} = \lim_{h \to 0^+} \frac{\mu((x_0-h, x_0+h))}{m((x_0-h, x_0+h))} = f(x_0)$.

Let $B(r, x) \subset \mathbb{R}^n$ be the open ball of radius r centered at $x \in \mathbb{R}^n$. Want to show that if ν is a signed measure on $\mathcal{B}_{\mathbb{R}^n}$ such that $\nu << m$, then $d\nu = f dm$ for some $m-$integrable function and for $m-$a.e. $x \in \mathbb{R}^n$ we have $\frac{d\nu}{dm} = f(x) = \lim_{\nu \to 0^+} \frac{\nu(B(r,x))}{m(B(r,x))}$.

Definition. *A measurable function $f : \mathbb{R}^n \to \mathbb{R}$ is* ***locally integrable*** *with respect to the Lebesgue measure if $\int_K |f| dx < \infty$ for all bounded measurable sets $K \in \mathcal{L}$. We denote the space of locally integrable functions by $L^1_{loc}(\mathbb{R}^n)$ or just by L^1_{loc}.*

Definition. *Let $f \in L^1_{loc}$. Then for all bounded measurable sets $K \in \mathcal{L}$ with $m(K) > 0$, we define the* ***mean (average) value*** *of f over K by $\frac{1}{m(K)} \int_K f dx$. We denote the mean value of f over K by $⨍_K f dx$. (Note that this notation is different, but more standard, from Folland's notation).*

Definition. *Let $f \in L^1_{loc}$. The* ***Hardy-Littlewood Maximal Function*** *$Hf : \mathbb{R}^n \to \mathbb{R}$ (also notated as Mf) is given by $Hf := \sup_{r>0} ⨍_{B(r,x)} |f| dx$.*

Recall that a function $h : \mathbb{R}^n \to \mathbb{R}$ is **lower semi-continuous** if the set $\{x \in \mathbb{R}^n : h(x) > a\}$ is open for all $a \in \mathbb{R}$. Also, we say h is lower semicontinuous if $\liminf_{y \to x} h(y) \geq h(x)$.

Proposition 39. *Let $f, g \in L^1_{loc}$ be given. Then*

1. *$0 \leq Hf \leq +\infty$.*
2. *$H(f + g) \leq Hf + Hg$*
3. *$H(cf) = |c| Hf$ for all $c \in \mathbb{R}$.*
4. *Hf is a lower semicontinuous function in $\mathbb{R}^n$.*
5. *Hf is a Borel measurable function.*

Proof. (1-3) obvious

(4) Let $a \in \mathbb{R}$. Consider the set $U_a := \{x \in \mathbb{R}^n : Hf > a\}$. We want to show U_a is open. Let $x_0 \in U_a$. Then there exists a sequence $\{r_n\}_{n=1}^\infty \subseteq (0, \infty)$ such that $\lim_{n\to\infty} ⨍_{B(r_n,x_0)} |f| dx > a$. Thus there exists $r_0, \epsilon > 0$ such that $⨍_{B(r_0,x_0)} |f| dx \geq a + \epsilon$. Now, the measure $E \mapsto \int_E |f| dx$ is absolutely continuous with respect to m. So by Theorem 36, there exists $\delta > 0$ such that whenever $m(E) < \delta$ we find $\int_E |f| dx < \frac{1}{2} \epsilon m(B(r_0, x_0))$. Consider the set $D = \{x \in \mathbb{R}^n : m(B(x_0, r_0) \triangle B(x, r_0)) < \delta\}$. This is an open set containing x_0. Let $x \in D$. Then by Theorem 33

$$\begin{aligned}
⨍_{B(r_0,x_0)} |f| dy &= \frac{1}{m(B(r_0,x_0))} \int_{B(r_0,x)} |f| dy \\
&= \frac{1}{m(B(r_0,x_0))} \left[\int_{B(r_0,x)} |f| dy - \int_{B(r_0,x_0)} |f| dy \right] + ⨍_{B(r_0,x_0)} |f| dy \\
&\geq \frac{1}{m(B(r_0,x_0))} \left[- \int_{B(r_0,x) \triangle B(r_0,x_0)} |f| dy \right] + ⨍_{B(r_0,x_0)} |f| dy \\
&\geq \frac{1}{m(B(r_0,x_0))} \left[-\frac{1}{2} \epsilon m(B(r_0,x_0)) \right] + a + \epsilon \\
&= a + \frac{1}{2}\epsilon
\end{aligned}$$

(5) Since for all $a \in \mathbb{R}$ the set $(Hf)^{-1}((a, \infty]) \in \mathcal{B}_{\mathbb{R}^n}$ by (4), we see Hf is Borel measurable. ☐

Example. Consider the function $\chi_{[0,1)} \in L^1_{loc}$. We see $H\chi_{[0,1)} = \sup_{r>0} \frac{1}{2r} m(B(r, x) \cap [0, 1)) = \begin{cases} 1 & \text{if } x \in (0,1), \\ \frac{1}{2x} & \text{if } x \geq 1 \\ \frac{1}{2(1-x)} & \text{if } x \leq 0. \end{cases}$ Note that even though $\chi_{[0,1)} \in L^1 \cap L^\infty$, the maximal function $H\chi_{[0,1)} \notin L^1$. In general, $Hf \notin L^1(\mathbb{R}^n)$ unless $f = 0$ a.e.

Theorem (Chebyshev's Inequality- p.193**).** *Let $(X, \mathcal{M}, \mu)$ be a measure space. If $f \in L^p(\mu)$ for some $p \in [1, \infty)$, then for each $\alpha > 0$ we have $\mu(\{x \in X : |f(x)| > \alpha\}) \leq \frac{1}{\alpha^p} ||f||^p_{L^p}$.*

Proof. Set $E_\alpha := \{x \in X : |f(x)| > \alpha\}$. Then $\int_X |f|^p d\mu \geq \int_{E_\alpha} |f|^p d\mu \geq \alpha^p \int_{E_\alpha} 1 d\mu = \alpha^p \mu(E_\alpha)$. ☐

Definition. *Let $(X, \mathcal{M}, \mu)$ be a measure space. For each measurable function $f : X \to \overline{\mathbb{R}}$, define $[f]_p := \sup_{\alpha>0} [\alpha^p \mu(\{x \in X : |f(x)| > \alpha\})]$. We say f is in* ***weak-L^p*** *if and only if $[f]_p < \infty$.*

Remarks.

- $[f]_p$ is not a norm (it does not satisfy the triangle inequality).
- By Chebyshev's Inequality, for all $p \in [1, \infty)$ we find $L^p(\mu) \subsetneq weak - L^p(\mu)$. (This is strict as $\frac{1}{x^{1/p}} \notin L^p$, but is in $weak - L^p$.)

Lemma 6 (Simple Vitali Covering Lemma 3.15**).** *Let $\mathcal{C}$ be a collection of open balls in $\mathbb{R}^n$ and set $U = \cup_{B \in \mathcal{C}} B$. If $c < m(U)$, then there exists disjoint balls $\{B_j\}_{j=1}^k \subseteq \mathcal{C}$ such that $\sum_{j=1}^k m(B_j) > \frac{c}{3^n}$.*

Proof. Let $c < m(U)$. By Theorem 31(b), there exists a compact set $K \subseteq U$ such that $c < m(K) < m(U)$. Since K is compact, there exists a finite subcover $\{A_j\}_{j=1}^m \subseteq \mathcal{C}$ of K. WLOG, assume $m(A_1) \geq m(A_2) \geq \cdots \geq m(A_m)$. Set $B_1 = A_1$. Then, pick B_2 to be the next ball in the list $A_2, ..., A_m$ such that $B_2 \cap B_1 = \emptyset$. Then pick B_3 to be the next ball after B_2 in the collection of $\{A_j\}$ such that $B_3 \cap (\cup_{j=1}^2 B_j) = \emptyset$. Continue until the list is exhausted. So we end up with a disjoint collection of $\{B_j\}_{j=1}^k$ such that $A_i \cap (\cup_{j=1}^k) B_j \neq \emptyset$ for all $i = 1, ..., m$. Let $1 \leq i \leq m$ be given. Then there exists at least one j such that $A_i \cap B_j \neq \emptyset$. Pick the smallest such j. Then $m(B_j) \geq m(A_i)$. Hence the radius of B_j is greater than the radius of A_i. Let B_j^* be the ball that is concentric to B_j but has 3 times the radius. Then $A_i \subset B_j^*$. It follows that $\cup_{j=1}^k B_j^* \supseteq \cup_{j=1}^m A_j \supseteq K$. Thus $c < m(K) \leq m(\cup_{j=1}^k B_j^*) \leq \sum_{j=1}^k m(B_j^*) = 3^n \sum_{j=1}^k m(B_j)$. ☐

Theorem (The Maximal Theorem, AKA The Hardy-Littlewood Theorem). *There exists a constant $c > 0$ such that for all $f \in L^1$ and all $\alpha > 0$ we have $m(\{x \in \mathbb{R}^n : Hf(x) > \alpha\}) \leq \frac{c}{\alpha} ||f||_{L^1}$, that is, $Hf \in weak - L^1$.*

Proof. Set $E_\alpha = \{x \in \mathbb{R}^n : Hf(x) > \alpha\}$. Then for all $x \in E_\alpha$, we must have $\sup_{r>0} ⨍_{B(r,x)} |f|dy > \alpha$ which implies $⨍_{B(r_x,x)} |f|dy > \alpha$ for some $r_x > 0$. Define $\mathcal{C} = \{B(r_x, x) : x \in E_\alpha\}$. Then $E_\alpha \subseteq \cup_{B \in \mathcal{C}} B$. For each $c < m(E_\alpha)$, we may select (by Lemma 6) a finite number of points $\{x_j\}_{j=1}^k \subseteq E_\alpha$ such that $\{B(r_{x_j}, x_j)\}_{j=1}^k$ are disjoint and $\sum_{j=1}^k m(B(r_{x_j}, x_j)) > \frac{c}{3^n}$. So

$$\begin{aligned} c < 3^n \textstyle\sum_{j=1}^k m(B(r_{x_j}, x_j)) &= \tfrac{3^n}{\alpha}(\textstyle\sum_{j=1}^k m(B(r_{x_j}, x_j))\alpha) \\ &\leq \tfrac{3^n}{\alpha}(\textstyle\sum_{j=1}^k m(B(r_{x_j}, x_j)) ⨍_{B(r_{x_j},x_j)} |f|dy) \\ &= \tfrac{3^n}{\alpha}(\textstyle\sum_{j=1}^k \int_{B(r_{x_j},x_j)} |f|dy) \\ &= \tfrac{3^n}{\alpha}(\textstyle\int_{\cup_{j=1}^k B(r_{x_j},x_j)} |f|dy) \\ &\leq \tfrac{3^n}{\alpha} \textstyle\int_{\mathbb{R}^n} |f|dy. \end{aligned}$$

Since true for all $c < m(E_\alpha)$, it follows that $m(E_\alpha) \leq \frac{3^n}{\alpha} \int_{\mathbb{R}^n} |f|dx$. ☐

Theorem 38 (2.41)**.** *If $f \in L^1(m)$ and $\epsilon > 0$, then there exists a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ such that $\int_{\mathbb{R}^n} |f - g|dx < \epsilon$.*

Recall that $\lim_{r \to R} \phi(r) = c$ if and only if $\limsup_{r \to R} |\phi(r) - c| = 0$.

Theorem 39 (3.18)**.** *Let $f \in L^1_{loc}$. Then for almost every $x \in \mathbb{R}^n$ we have $\lim_{r \to 0} ⨍_{B(r,x)} f(y)dy = f(x)$.*

Proof. Let $N \in \mathbb{N}$. Since $f \in L^1_{loc}$, we find $f_N := f\chi_{B(N,0)} \in L^1$. Let $\epsilon > 0$. By Theorem 38, there exists a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ such that $\int_{\mathbb{R}^n} |f_N - g|dx < \epsilon$.

Claim: $\lim_{r \to 0^+} ⨍_{B(r,x)} gdy = g(x)$ for all $x \in \mathbb{R}^n$.

Proof: Let $x \in \mathbb{R}^n$ and $\delta > 0$. Since g is continuous, there exists $r > 0$ such that $|g(y) - g(x)| < \delta$ for all $y \in B(r, x)$. Thus $⨍_{B(r,x)} |g(y) - g(x)|dy < \delta$. Since $\delta > 0$ was arbitrary, we see

$$\begin{aligned} 0 = \lim_{r \to 0} ⨍_{B(r,x)} |g(y) - g(x)|dy &\geq \lim_{r \to 0} |⨍_{B(r,x)} g(y) - g(x)dy| \\ &= \lim_{r \to 0} |⨍_{B(r,x)} g(y)dy - g(x)| \geq 0. \end{aligned}$$

Hence, $\lim_{r \to 0^+} ⨍_{B(r,x)} g(y)dy = g(x)$.

Now, estimate $\limsup_{r\to 0^+} |\fint_{B(r,x)} f_N(y)dy - f_N(x)|$ by comparing f_N and g. We have

$$\begin{aligned}\limsup_{r\to 0^+} |\fint_{B(r,x)} f_N(y)dy - f_N(x)| &= \limsup_{r\to 0^+} |\fint_{B(r,x)} f_N(y) - g(y)dy + \fint_{B(r,x)} g(y) - g(x)dy + g(x) - f_N(x)| \\ &\le \limsup_{r\to 0^+} [\fint_{B(r,x)} |f_N(y) - g(y)|dy] + |g(x) - f_N(x)| \\ &\le H(f_N - g)(x) + |f_N(x) - g(x)|.\end{aligned}$$

It follows that for all $\alpha > 0$, if $\limsup_{r\to 0^+} |\fint_{B(r,x)} f_N(y)dy - f_N(x)| > \alpha$, then either $H(f_N - g)(x) > \frac{\alpha}{2}$ or $|f_N(x) - g(x)| > \frac{\alpha}{2}$. Set $E_\alpha = \{x \in \mathbb{R}^n : \limsup_{r\to 0^+} |\fint_{B(r,x)} f_N(y)dy - f_N(x)| > \alpha\}$, $F_\alpha = \{x \in \mathbb{R}^n : H(f_N - g)(x) > \alpha\}$, and $G_\alpha = \{x \in \mathbb{R}^n : |f_N(x) - g(x)| > \alpha\}$. Note that $E_\alpha \subseteq F_{\alpha/2} \cup G_{\alpha/2}$. By the Maximal theorem and Chebyshev's inequality,

$$m(E_\alpha) \le m(F_{\alpha/2}) + m(G_{\alpha/2}) \le \frac{2c}{\alpha}\int_{\mathbb{R}^n} |f_N - g|dx + \frac{2}{\alpha}\int_{\mathbb{R}^n} |f_N - g|dx \le \epsilon\left(\frac{2c}{\alpha} + \frac{2}{\alpha}\right).$$

Since $\epsilon > 0$ was arbitrary, we deduce $m(E_\alpha) = 0$ for all $\alpha < 0$. Set $E = \cup_{k=1}^\infty E_{1/k}$. Then $m(E) = 0$ and for all $x \in E^C$ we see $0 \le \limsup_{r\to 0^+} |\fint_{B(r,x)} f_N(y)dy - f_N(x)| = 0$. Thus $\lim_{r\to 0^+} \fint_{B(r,x)} f_N(y)dy = f_N(x)$. Since $N \in \mathbb{N}$ was arbitrary and $f_N = f$ on $B(N,0)$, we conclude $\lim_{r\to 0^+} \fint_{B(r,x)} f(y)dy = f(x)$ for almost every $x \in \mathbb{R}^n$. ☐

Definition. *For each $f \in L^1_{loc}$, set $L_f := \{x \in \mathbb{R}^n : \lim_{r\to 0^+} \fint_{B(r,x)} |f(y) - f(x)|dy = 0\}$. The set L_f is called the **Lebesgue set** for f. The points in L_f are called the **Lebesgue points** for f.*

Theorem 40 (3.20)**.** *If $f \in L^1_{loc}$, then $m(\mathbb{R}^n \setminus L_f) = 0$.*

Proof. For each $\alpha \in \mathbb{R}$, define $g_\alpha := |f - \alpha|$. Since $f \in L^1_{loc}$ and $\alpha < \infty$, $g_\alpha \in L^1_{loc}$. Then, by Theorem 39, $E_\alpha = \{x \in \mathbb{R}^n | \lim_{r\to 0^+} \fint_{B(r,x)} |f - \alpha|dy$ DNE or $\neq f(x) - \alpha\}$ is a null set for all $\alpha \in \mathbb{R}$. Put $E := \cup_{\alpha\in\mathbb{Q}} E_\alpha$ so $m(E) = 0$. Let $x \notin E$ and $\epsilon > 0$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, select $\alpha \in \mathbb{Q}$ such that $|f(x) - \alpha| < \epsilon$. Since $x \notin E$,

$$0 \le \limsup_{r\to 0} \fint_{B(r,x)} |f(y) - f(x)|dy \le \underbrace{\limsup_{r\to 0} \fint_{B(r,x)} |f(y) - \alpha|dy + |\alpha - f(x)| = 2|f(x) - \alpha|}_{\text{by Theorem 39}} \le 2\epsilon.$$

Since ϵ was arbitrary, $\lim_{r\to 0} \fint_{B(r,x)} |f(y) - f(x)|dy = 0$. ☐

Definition. *A family of sets $\{E_r\}_{r>0} \subset \mathcal{B}_{\mathbb{R}^n}$ is said to **shrink nicely** to x if*

- *$E_r \subseteq B(r, x)$ for all $r > 0$.*
- *There exists a constant $\alpha > 0$ such that for all $r > 0$ we see $m(E_r) > \alpha m(B(r, x))$.*

Theorem (**Lebesgue's Differentiation Theorem**)**.** *Suppose $f \in L^1_{loc}$. For each $x \in L_f$ we have $\lim_{r\to 0^+} \fint_{E_r} |f(y) - f(x)|dy = 0$ and $\lim_{r\to 0^+} \fint_{E_r} f(y)dy = f(x)$ for all families $\{E_r\}_{r>0} \subseteq \mathcal{B}_{\mathbb{R}^n}$ that shrink nicely to x.*

Proof. By definition, there exists $\alpha > 0$ such that $m(E_r) > \alpha m(B(r, x))$ for all $r > 0$. Thus

$$\fint_{E_r} |f(y) - f(x)|dy \le \frac{1}{m(E_r)}\int_{B(r,x)} |f(y) - f(x)|dy \le \frac{1}{\alpha m(B(r,x))}\int_{B(r,x)} |f(y) - f(x)|dy = \frac{1}{\alpha}\fint_{B(r,x)} |f(y) - f(x)|dy.$$

If $x \in L_f$, then it follows that $\lim_{r\to 0^+} \fint_{E_r} |f(y) - f(x)|dy = 0$. Also

$$0 \le \lim_{r\to 0^+} \left|\fint_{E_r} f(y)dy - f(x)\right| \le \lim_{r\to 0^+} \fint_{E_r} |f(y) - f(x)|dy = 0.$$

☐

Remark. Recall that the members of L^1 (and L^1_{loc}) are actually equivalence classes. By Theorem 3p, for all $f \in L^1_{loc}$ we have $\lim_{r\to 0^+} \fint_{B(r,x)} f(y)dy$ exists for almost every $x \in \mathbb{R}^n$. The function $f^*(x) = \begin{cases} \fint_{B(r,x)} f(y)dy & \text{if it exists} \\ 0 & \text{otherwise} \end{cases}$ is called the **precise representative** for the equivalence class $f \in L^1_{loc}$. Note that if $f, g \in L^1_{loc}$ and $f = g$ a.e., then $f^* = g^*$ for all $x \in \mathbb{R}^n$.

Definition. *Let ν be a signed measure on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$. Then we say ν is **regular** if*

1. *$|\nu(K)| < \infty$ for all compact $K \subseteq \mathcal{R}^n$.*

2. *For all $E \in \mathcal{B}_{\mathbb{R}^n}$, we find $|\nu|(E) = \inf\{|\nu|(U) : U$ is open, $E \subseteq U\}$.*

Remarks.

1. If ν is regular, then ν is $\sigma-$finite (by 1).

2. Property 1 implies Property 2.

3. If $d\nu = fdm$, then ν is regular if and only if $f \in L^1_{loc}$.

   Proof. By the Jordan Decomposition, $d\nu^+ = f^+dm$ and $d\nu^- = f^-dm$. So $d|\nu| = |f|dm$. If ν is regular, then (1) implies $f \in L^1_{loc}$. So assume $f \in L^1_{loc}$. Then clearly (1) is satisfied. For (2), let $E \in \mathcal{B}_{\mathbb{R}^n}$. Put $E_1 := E \cap B(1,0)$ and for $j = 2, 3, ...$ define $E_j := E \cap (B(j,0) \setminus B(j-1,0))$. Note $f\chi_{\overline{B(j+1,0)}} \in L^1$ for all $j \in \mathbb{N}$. Let $\epsilon > 0$. By Corollary 10 (3.6), for all $j \in \mathbb{N}$ there exists δ_j such that $\int_F |f\chi_{B(j+1,0)}|dx < \epsilon 2^{-j}$ whenever $m(F) < \delta_j$. By Theorem 31a(2.40), for all $j \in \mathbb{N}$ there exists $U_j \subseteq B(j+1,0)$ such that U_j is open, $E_j \subseteq U_j$ and $m(U_j) < m(E_j) + \delta_j$. This implies $m(U_j \setminus E_j) = m(U_j) - m(E_j) < \delta_j$. Put $U = \cup_{j=1}^{\infty} U_j$. Then

$$\begin{aligned} |\nu|(U) = \int_U |f|dx &\leq \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} |f\chi_{U_j}|dx \\ &= \sum_{j=1}^{\infty} \int_{U_j} |f|dx \\ &= \sum_{j=1}^{\infty} \int_{E_j} |f|dx + \sum_{j=1}^{\infty} \int_{U_j \setminus E_j} |f|dx \\ &= \int_E |f|dx + \sum_{j=1}^{\infty} \epsilon 2^{-j} \\ &= |\nu|(E) + \epsilon \leq |\nu|(U) + \epsilon \end{aligned}$$

   Since $\epsilon > 0$ is arbitrary, $\nu(E) = \inf\{|\nu|(U) : U$ is open, $E \subseteq U\}$. ☐

Theorem 41 (3.22)**.** *Let ν be a regular signed measure on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$. Let $d\nu = d\lambda + fdm$ be the Lebesgue-Radon-Nikodym decomposition of ν with respect to m. Then for almost every $x \in \mathbb{R}^n$, we have $\lim_{r \to 0^+} \frac{\nu(E_r)}{m(E_r)} = f(x)$ for every family $\{E_r\}_{r>0}$ that shrinks nicely to x.*

Proof. As in the proof of the remark, the Jordan Decomposition implies $d|\nu| = d|\lambda| + |f|dm$.

Claim: λ and fdm are regular.

Proof: If K is compact, then $|\lambda|(K) + \int_K |f|dm = |\nu|(K) < \infty$ by property 1 of regularity. Thus $|\nu(K)|, \int_K |f|dm < \infty$. Thus $f \in L^1_{loc}$ and by the previous remarks, $|f|dm$ satisfies property 2 of regularity. For $|\lambda|$, note that $d|\lambda| = d|\nu| - |f|dm$ where ν and fdm are regular. So we can use the same exact argument to show if $f \in L^1_{loc}$ then fdm satisfies property 2.

Since $f \in L^1_{loc}$, the Lebesgue Differentiation Theorem implies $\lim_{r \to 0^+} ⨍_{E_r} fdy = f(x)$ whenever x is a Lebesgue point for f. It only remains to show $\lim_{r \to 0^+} \frac{\lambda(E_r)}{m(E_r)} = 0$ for $m-$almost every $x \in \mathbb{R}^n$ and all $\{E_r\}_{r>0}$ which shrink nicely to x (as then $\lim_{r \to 0^+} \frac{\nu(E_r)}{m(E_r)} = \lim \frac{\int_{E_r} fdm}{m(E_r)} = f(x)$). Since $\lambda \perp m$, there exists $A \in \mathcal{B}_{\mathbb{R}^n}$ such that $\lambda(A) = 0 = m(A^C)$. Since A^C is an $m-$null set, we need only consider $x \in A$. For each $k \in \mathbb{N}$, define $F_k := \{x \in A : \limsup_{r \to 0^+} \frac{|\lambda|(B(r,x))}{m(B(r,x))} > \frac{1}{k}\}$. We claim $m(F_k) = 0$ for all $k \in \mathbb{N}$. Let $\epsilon > 0$. Then there exists $U \in \mathbb{R}^n$ that is open such that $A \subseteq U$ and $|\lambda|(U) < \lambda(A) + \epsilon = \epsilon$ by property 2 of regularity. Let $x \in F_k$. Then there exists $r_x < 0$ such that $B(r_x, x) \subseteq U$ and $\frac{1}{k} \leq \frac{|\lambda|(B(r_x,x))}{m(B(r_x,x))}$. Set $V = \cup_{x \in F_k} B(r_x, x) \subseteq U$. For each $c < m(V)$, by Lemma 6 there exists a finite collection of disjoint balls $\{B(r_{x_j}, x_j)\}_{j=1}^{\ell}$ such that $c < 3^n \sum_{j=1}^{\ell} m(B(r_{x_j}, x_j)) \leq 3^n k \sum_{j=1}^{\ell} |\lambda|(B(r_{x_j}, x_j)) \leq 3^n k |\lambda|(V) \leq 3^n k |\lambda|(U) \leq 3^n k \epsilon$. Since $c < m(V)$ and $\epsilon > 0$ were arbitrary, we conclude $m(F_k) = 0$. Setting $F = \cup_{k=1}^{\infty} F_k$, we see $m(F) = 0$. Suppose $x \in A$ and there exists $\{E_r\}_{r>0} \subset \mathcal{B}_{\mathbb{R}^n}$

that shrink nicely to x such that $\limsup_{r\to 0^+} |\frac{\nu(E_r)}{m(E_r)}| = \beta > 0$. Then there exists $\alpha > 0$ such that $m(E_r) > \alpha m(B(r,x))$ for all $r > 0$ and $K > \frac{1}{\alpha\beta}$, we have $\limsup_{r\to 0^+} \frac{|\lambda|(B(r,x))}{m(B(r,x))} \geq \limsup_{r\to 0^+} \frac{|\lambda|(E_r)}{m(B(r,x))} \geq \limsup_{r\to 0} \frac{\alpha|\lambda|(E_r)}{m(E_r)} \geq \alpha\beta > \frac{1}{k}$. So $x \in F$. Thus $\limsup_{r\to 0^+} |\frac{\lambda(E_r)}{m(E_r)}| = 0$ for almost every $x \in \mathbb{R}^n$. □

Example. Consider the earlier example $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = \begin{cases} 0 & x < 0, \\ 3 - e^{-x} & 0 \leq x < 1. \\ 4 - e^{-x} & 1 \leq x. \end{cases}$ Let μ_F be the Lebesgue-Stieltjes measure with distribution function F. So $\mu_F((-\infty, x]) = F(x)$. Previously, we found $d\mu_F = d\lambda + f dm$ where $\lambda = 2\delta_0 + \delta_1$ and $f(x) = \begin{cases} 0 & x < 0, \\ e^{-x} & x \geq 0. \end{cases}$ By Theorem 41, for $m-$almost every x, we see

$$f(x) = \lim_{r\to 0} \frac{\mu_F((x-r, x+r))}{m((x-r, x+r))} = \lim_{r\to 0} \frac{F(x+r) - F(x-r)}{2r} = F'(x)$$

if F is differentiable. Note that since $F'(x) = \begin{cases} 0 & x < 0, \\ e^{-x} & x > 0, \end{cases}$ we have a formula for $f(x)$ almost everywhere. Put $d\rho = f dm$. So then if $x \leq 0$, we see $\rho((-\infty, x]) = 0$ and if $x > 0$, we see $\rho((-\infty, x]) = \int_{(-\infty,x]} f ds = \int_{[0,x]} e^{-s} ds = 1 - e^{-x}$. Now, $\lambda = \mu_F - \rho$. In particular, $\lambda(\{0\}) = \mu_F(\{0\}) - \rho(\{0\}) = \mu_F((-\infty, 0]) - \mu_F((-\infty, 0)) - 0 = 2$ and $\lambda(\{1\}) = \mu_F(\{1\}) = 1$. Thus $\lambda = 2\delta_0 + \delta_1$.

Theorem (Lusin's Theorem). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable function. Suppose there exists $A \in \mathcal{B}_{\mathbb{R}^n}$ such that $m(A) < \infty$ and $f(x) = 0$ for all $x \in A^C$. Then for all $\epsilon > 0$ there exists a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ such that*

- *$g(x) = 0$ for all $x \in \mathbb{R}^n \setminus B(R, 0)$ for some $R > 0$.*
- *$\sup_{x\in\mathbb{R}^n} |g(x)| \leq \sup_{x\in\mathbb{R}^n} |f(x)|$.*
- *$m(\{x \in \mathbb{R}^n | f(x) \neq g(x)\}) < \epsilon$.*

Proof. (Of Theorem 38) WLOG, assume $f \geq 0$. By Proposition 19 (6.7), the set of simple functions in L^1 is dense in L^1. Thus we may select a simple function $\phi \in L^1$ such that $\int_{\mathbb{R}^n} |f - \phi| dx < \frac{\epsilon}{2}$. Since ϕ is simple and in L^1, there is an $A \in \mathcal{B}_{\mathbb{R}^n}$ such that $m(A) < \infty$ and $\phi(x) = 0$ for all $x \in A^C$. By Lusin's Theorem, there exists a continuous $g : \mathbb{R}^n \to \mathbb{R}$ such that $\sup_{x\in\mathbb{R}^n} |g(x)| < \infty$ and $m(\{x \in \mathbb{R}^n | g(x) \neq \phi(x)\}) < \frac{\epsilon}{4\sup_{x\in\mathbb{R}^n} |\phi(x)|+1}$. Thus

$$\int_{\mathbb{R}^n} |\phi - g| dx \leq \int_{\{x\in\mathbb{R}^n | g(x)\neq\phi(x)\}} |\phi - g| dx \leq \sup_{x\in\mathbb{R}^n} |\phi - g| m(\{x \in \mathbb{R}^n : \phi(x) \neq g(x)\}) \leq \frac{(2\sup_{x\in\mathbb{R}^n} |\phi|)\epsilon}{4\sup_{x\in\mathbb{R}^n} |\phi(x)| + 1} < \frac{\epsilon}{2}$$

by Holder's Inequality. Hence $\int_{\mathbb{R}^n} |f - g| dx \leq \int_{\mathbb{R}^n} |f - \phi| + |\phi - g| dx < \epsilon$. □

Theorem 42. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable function. Let $A \in \mathcal{B}_{\mathbb{R}^n}$ be such that $m(A) < \infty$. For all $\epsilon > 0$, there exists a compact set $K \subseteq \mathbb{R}^n$ such that $m(A \setminus K) < \epsilon$ and $f|_K$ is continuous.*

Proof. Recall that $\mathcal{Q}_k^1$ is the collection of dyadic intervals in $\mathbb{R}$ with length 2^{-k}. Fix $k \in \mathbb{N}$. Let $\{Q_i\}_{i=1}^\infty$ be an enumeration of $\mathcal{Q}_k^1$. For each $i \in \mathbb{N}$, set $A_i := A \cap f^{-1}(Q_i)$. Since $A \in \mathcal{B}_{\mathbb{R}^n}$ and f is Borel measurable, each $A_i \in \mathcal{B}_{\mathbb{R}^n}$. Also, the $A_i's$ are mutually disjoint and $A = \cup_{i=1}^\infty A_i$. By Theorem 31a, for each $i \in \mathbb{N}$ there exists a compact set $K_i \subset \mathbb{R}^n$ such that $K_i \subseteq A_i$ and $m(A_i \setminus K_i) < \frac{\epsilon}{2^{i+k}}$. Now $\{K_i\}_1^\infty$ are disjoint and so $m(A \setminus \cup K_i) = m(\cup A_i \setminus \cup K_i) = m(\cup(A_i \setminus K_i)) = \sum m(A_i \setminus K_i) < \frac{\epsilon}{2^k}$. Thus there exists N_k such that $m(A \setminus \cup_1^{N_k} K_i) < \frac{\epsilon}{2^k}$. Set $D_k = \cup_1^{N_k} K_i$. Then D_k is compact and $D_k \subseteq A$. For each $i = 1, ..., N_k$, select $b_i \in Q_i$ (note $Q_i \supseteq f(K_i)$) and define $g_k : D_k \to \mathbb{R}$ by $g_k(x) = \sum_{i=1}^{N_k} b_i \chi_{K_i}(x)$. Now $\{K_i\}_{i=1}^{N_k}$ are compact disjoint sets, so there exists a strictly positive distance between K_i and K_j whenever $i \neq j$. It follows that g_k is continuous on D_k. Moreover $|f(x) - g_k(x)| \leq 2^{-k}$ for all $x \in D_k$. Put $K = \cap_{k=1}^\infty D_k$, so K is compact and $m(A \setminus K) \leq \sum_{k=1}^\infty m(A \setminus D_k) < \epsilon$. Since $|f(x) - g_k(x)| \leq 2^{-k}$ for all $x \in D_k$, we see $g_k \xrightarrow{unif} f$ on K which implies f is continuous on K. □

Remark. Sometimes Theorem 42 is called Lusin's Theorem.

Theorem 43. *Suppose $K \subseteq \mathbb{R}^n$ is a compact set and $f : K \to \mathbb{R}$ is continuous. Then there exists a continuous function $\overline{f} : \mathbb{R}^n \to \mathbb{R}$ such that $\sup_{x\in\mathbb{R}^n} |\overline{f}(x)| \leq \sup_{x\in K} |f(x)|$ and $\overline{f}(x) = f(x)$ for all $x \in K$.*

Proof. Note that this follows from Tietze's Extension Theorem, but we will prove it without. First, we construct a candidate for the extension of f to $\mathbb{R}^n$: Put $U = \mathbb{R}^n \setminus K$, so that U is open. For each $s \in K$, put $v_s(x) = \max\{2 - \frac{||x-s||}{dist(x,K)}, 0\}$. Notice that $0 \leq v_s(x) \leq 1$ for all $x \in U$. Note that $x \mapsto v_s(x)$ is continuous (as $2, ||x - s||, dist(x, K)$ are continuous and $dist(x, K) > 0$). Let $\{s_j\}_{j=1}^{\infty}$ be a countably dense subset of K. Define $\sigma : U \to [0, 1]$ by $\sigma(x) = \sum_1^{\infty} v_{s_j}(x)\frac{1}{2^j}$. Since $\{s_j\}$ are dense, we see $\sigma(x) > 0$. Now, we define $w_k : U \to [0, 1]$ by $x \mapsto \frac{\frac{1}{2^k} v_{s_k}(x)}{\sigma(x)}$. Observe that $\{w_k\}_{k=1}^{\infty}$ forms a partition of unity in U :

- $x \mapsto w_k(x)$ is continuous on U
- $0 \leq w_k \leq 1$
- $\sum w_k(x) = 1$ for all $x \in U$.

Now, our candidate for $\overline{f}$ is $\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in K \\ \sum_{k=1}^{\infty} w_k(x) f_s(x) & \text{if } x \in U. \end{cases}$ Observe that $\sup_{x\in\mathbb{R}^n} |\overline{f}(x)| \leq \sup_{x\in K} |f(x)|$. To show $\overline{f}$ is continuous on U, let $E \subseteq U$ be compact. Then $\min_{x\in E} \sigma(x) > 0$ which implies $\max_{x\in E} |w_k(x)f_s(x)| \leq \frac{\sup_{x\in K} |f(x)|}{2k \min_{x\in E} \sigma(x)} =: M_k$. Now $\sum_{k=1}^{\infty} M_k = \frac{\sup |f(x)|}{\min \sigma(x)} \sum \frac{1}{2^k} < \infty$. By the M-test and the fact that if $\sum u_k(X)$ is a uniform convergent series of continuous functions on E then the function $x \mapsto \sum u_k(x)$ is continuous on E, we conclude $\sum_{k=1}^{\infty} w_k(x)f(s_k)$ is continuous on E. Hence $\overline{f}$ is continuous on U.

We now show $\overline{f}$ is an extension of f. We need only show that for all $x \in K$ we have $\lim_{x\to a, x\in U} \overline{f}(x) = f(a)$. Fix $\alpha > 0$. Since f is continuous, there exists $\delta > 0$ such that $|f(s) - f(a)| < \alpha$ for all $s \in K$ satisfying $||s - a|| < \delta$. Suppose $x \in U$ and $||x - a|| < \frac{\delta}{4}$. Notice that whenever $||a - s_k|| \geq \delta$, we have

$$\delta \leq ||a - s_k|| < ||a - x|| + ||x - s_k|| < \frac{\delta}{4} + ||x - s_k||.$$

This says $||x - s_k|| > \frac{3\delta}{4} > 3||x - a|| > 3dist(x, K)$. Thus $2 - \frac{||x-s_k||}{dist(x,K)} < -1$ which implies $v_{s_k}(x) = 0$ and so $w_k(x) = 0$. Since $\sum w_k(x) = 1$ for all $x \in U$ and $w_k(x) = 0$ whenever $||x - a|| < \frac{\delta}{4}, ||a - s_k|| \geq \delta$, and $|f(a) - f(s_k)| < \alpha$ when $||s_k - a|| < \delta$, we see that

$$|\overline{f}(x) - f(a)| = |\sum_{k=1}^{\infty} w_k(x)[f(s_k) - f(a)]| \leq \sum_{k=1}^{\infty} w_k(x)|f(s_k) - f(a)| \leq \sum_{k\in\mathbb{N}, |s_k - a|<\delta} w_k(x)|f(s_k) - f(a)| < \sum_{k\in\mathbb{N}} w_k(x)\alpha = \alpha.$$

Since α was arbitrary, we see $\lim_{x\to a, x\in U} \overline{f}(x) = f(a)$. □

Lemma 7. *Let $K \subseteq \mathbb{R}^n$ be compact and $U \subseteq \mathbb{R}^n$ be an open set such that $K \subseteq U$. Then there exists a continuous function $\Psi : \mathbb{R}^n \to \mathbb{R}$ such that $\Psi(x) = 1$ if $x \in K$, $\Psi(x) = 0$ if $x \notin U$, and $0 \leq \Psi(x) \leq 1$ for $x \in \mathbb{R}^n$.*

Proof. Let $d := min\{dist(x, \mathbb{R}^n \setminus U) : x \in K\}$. Since $K, \mathbb{R}^n \setminus U$ are closed, $d > 0$. Set $\widetilde{K} = \cup_{x\in K} B(\frac{d}{2}, x)$ and note $dist(x, \mathbb{R}^n \setminus U) > \frac{d}{2}$ for all $x \in \widetilde{K}$. Define $\Psi : \mathbb{R}^n \to \mathbb{R}$ by $\Psi(x) = \fint_{B(\frac{d}{2},x)} \chi_{\widetilde{K}}(y)dy$. By HW3 #4, the function $\Psi(x)$ is continuous. Noting $B(\frac{d}{2}, x) \subseteq \widetilde{K}$ whenever $x \in K$ and $B(\frac{d}{2}, x) \cap \widetilde{K} = \emptyset$ if $x \in \mathbb{R}^n \setminus U$, we see all the properties for Ψ as stated in the lemma are verified. □

Proof. (Of Lusin's Theorem) Let $\epsilon > 0$. By Theorem 42, there exists a compact set $K \subseteq \mathbb{R}^n$ such that $m(A \setminus K) < \frac{\epsilon}{2}$ and $f|_K$ is continuous. By Theorem 43, there exists an $\overline{f} : \mathbb{R}^n \to \mathbb{R}$ such that $\overline{f}$ is continuous and $\overline{f}(x) = f(x)$ for all $x \in K$ and $\sup_{x\in\mathbb{R}^n} |\overline{f}(x)| \leq \sup_{x\in K} |f(x)| \leq \sup_{x\in\mathbb{R}^n} |f(x)|$. By Theorem 31a and since A is compact, we may find an open set U and $R > 0$ such that $K \subseteq U \subseteq B(R, 0)$ and $m(U) \leq m(K) + \frac{\epsilon}{2}$. Then $m(U \setminus K) \leq \frac{\epsilon}{2}$. Let ψ be continuous such that $\psi = 1$ on K, $\psi = 0$ on $\mathbb{R}^n \setminus U$ and $0 \leq \psi \leq 1$ by Lemma 7. Put $g = \overline{f}\psi$. Then

- $g(x) = 0$ for all $x \in \mathbb{R}^n \setminus B(R, 0)$
- $\sup_{x\in\mathbb{R}^n} |g(x)| \leq \sup_{x\in\mathbb{R}^n} |\overline{f}(x)| \leq \sup_{x\in\mathbb{R}^n} |f(x)|$
- If $x \in K$, then $f(x) = g(x)$. If $x \in A^C \cap U^C$, then $f(x) = 0 = g(x)$. So

$$\begin{aligned} \{x \in \mathbb{R}^n | f(x) \neq g(x)\} &\subseteq (\mathbb{R}^n \setminus K) \cap (U^C \cap A^C)^C \\ &= (\mathbb{R}^n \setminus K) \cap (U \cup A) \\ &= ((\mathbb{R}^n \setminus K) \cap U) \cup (\mathbb{R}^n \setminus K) \cap A) \\ &= (U \setminus K) \cup (A \setminus K) \end{aligned}$$

Hence $m(\{x \in \mathbb{R}^n | f(x) \neq g(x)\}) \leq m((U \setminus K) \cup (A \setminus K)) \leq m(U \setminus K) + m(A \setminus K) < \epsilon$. □

2.6 Functions of Bounded Variation

Recall. There exists a correspondence between regular Borel measures and increasing right continuous functions. We just established a nice differentiation theorem for regular Borel measures in $\mathbb{R}^n$. We can use this to establish differentiation theorems for the distribution function F. Recall that if μ is a regular Borel measure, then $\lim_{r\to 0^+} \frac{\mu_F(E_r)}{m(E_r)} = F(x)$ for almost every $x \in \mathbb{R}^n$. Now, in $\mathbb{R}^1$, this implies $\lim_{r\to 0^+} \frac{\mu_F((x,x+r))}{r} = \frac{F(x+r)-F(x)}{r}$, the derivative.

Theorem 44 (3.23)**.** *Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function and define $G : \mathbb{R} \to \mathbb{R}$ by $G(x) := F(x+) = \lim_{a\mapsto x^+} F(a)$. Then*

1. *The set of points of discontinuity for F is countable (and thus has measure 0)*
2. *F and G are differentiable and $F' = G'$ a.e.*

Proof. 1. Define intervals $I_x := (F(x-), F(x+)) \subseteq \mathbb{R}$ for all $x \in \mathbb{R}$. Then $I_x \neq \emptyset$ if and only if x is a point of discontinuity for F. Since F is increasing, $\{I_x\}_{x\in\mathbb{R}}$ is a disjoint family of intervals. For each $N \in \mathbb{N}$, we have

$$\begin{aligned} \textstyle\sum_{x\in(-N,N)} m(I_x) &= \sup\{\textstyle\sum_{x\in E} m(I_x) : E \subseteq (-N, N) \text{ is finite}\} \\ &= \sup\{m(\cup_{x\in E} I_x) | E \subset (-N, N) \text{ is finite}\} \\ &\leq m(F((-N, N))) \\ &\leq m((F(-N-), F(N+))) = F(N+) - F(-N-) < \infty \end{aligned}$$

Thus there exists countable many $x \in (-N, N)$ such that $m(I_x) > 0$. It follows that $F(x-) = F(x+)$ except for at most a countable number of $x \in (-N, N)$.

2. By definition, G is increasing and right continuous. Also, $G = F$ at all points of continuity for F. In particular, $G = F$ a.e. For all $h \neq 0$, we have $G(x + h) - G(x) = \begin{cases} \mu_G((x, x + h]) & \text{if } h > 0 \\ -\mu_G((x + h, x]) & \text{if } h < 0. \end{cases}$ Observe $\{(x - r, x]\}_{r>0}, \{(x, x + r]\}_{r>0}$ shrink nicely to x as $r \to 0^+$. Also, by Theorem 21a(1.18), μ_G is a regular Borel measure on $\mathbb{R}$. Thus by Theorem 41 (3.22), we have

$$\lim_{r\to 0^+} \frac{\mu_G((x - r, x])}{m((x - r, x])} = \lim_{r\to 0^+} \frac{G(x) - G(x - r)}{r} = G' \text{ and } \lim_{r\to 0^+} \frac{\mu_G((x, x + r])}{m((x, x + r])} = \lim_{r\to 0^+} \frac{G(x + r) - G(x)}{r} = G'$$

for almost every $x \in \mathbb{R}$. Put $H := G - F$. If we show $H' = 0$ a.e., then $F' = G'$ a.e. By part 1, we know $H = 0$ a.e. Let $\{x_j\}_{j=1}^{\infty}$ be an enumeration of those points for which $H \neq 0$. (We assume x_j are distinct and note that j may be over a finite index set). Since $G \geq F$, we see $H \geq 0$ and for all $j \in \mathbb{N}$, we have $H(x_j) = G(x_j) - F(x_j) = \lim_{x\to x_j} F(x) - F(x_j) \leq \lim_{x\to x_j^+} F(x) - \lim_{x\to x_j^-} F(x) = F(x_j+) - F(x_j-)$. As in part 1, we see $0 \leq \sum_{x_j\in(-N,N)} H(x_j) < \infty$ for all $N \in \mathbb{N}$. Put $\nu := \sum_{j=1}^{\infty} H(x_j)\delta_{x_j}$. We see that if K is compact, then $K \subseteq (-N, N)$ for some $N \in \mathbb{N}$. Then

$$\nu(K) = \sum_{x_j\in K} \nu(\{x_j\}) \leq \sum_{x\in(-N,N)} H(x_j) < \infty.$$

By Proposition 2.6 (1.16) and Theorem 21a, ν is regular. We can use the Lebesgue Differentiation Theorem. Moreover, $\nu \perp m$ and so by Theorem 41, $\lim_{r\to 0^+} \frac{\nu(E_r)}{m(E_r)} = 0$ for almost every $x \in \mathbb{R}$. (Here $\{E_r\}$ shrink nicely to x). Thus $\lim_{h\to 0} |\frac{H(x+h)-H(x)}{h}| \leq \lim_{h\to 0} \frac{H(x+h)-H(x)}{|h|} \leq \lim_{h\to 0} \frac{2r((x-2|h|,x+2|h|))}{|h|} = 0$ a.e. by above as $\{(x-2|h|, x+2|h|)\}$ shrinks nicely to x. Thus $H'(x) = 0$ a.e. □

Definition. *Let $F : \mathbb{R} \to \mathbb{R}$. Define $T_F : \mathbb{R} \to \mathbb{R}$ by $T_F(x) = \sup\{\sum_{i=1}^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < x_1 < \cdots < x_n = x\}$. We call T_F the* ***total variation function*** *of F. If $a < b$, then we call $T_F(b) - T_F(a)$ the total variation of F over $[a, b]$.*

Remarks.

- T_F is nondecreasing.
- It can be shown that $T_F(b) - T_F(a) = \sup\{\sum_{i=1}^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, a = x_0 < x_1 < \cdots < x_n = b\}$.

Definition. *Let $F : \mathbb{R} \to \mathbb{R}$ be given. If $\lim_{x\to\infty} T_F(x) < \infty$, we say F is of* ***bounded variation*** *on $\mathbb{R}$. We set $BV(\mathbb{R}) = \{F : \mathbb{R} \to \mathbb{R} | \lim_{x\to\infty} T_F(x) < \infty\}$. Let $F : [a, b] \to \mathbb{R}$. If $T_F(b) - T_F(a) < \infty$, then F is of bounded variation on $[a, b]$ and we set $BV([a, b]) = \{F : [a, b] \to \mathbb{R} | T_F(b) - T_F(a) < \infty\}$.*

Remarks.

- $BV(\mathbb{R}), BV([a, b])$ are vector spaces.
- If $F \in BV([a, b])$, then $\overline{F} : \mathbb{R} \to \mathbb{R}$ defined by $\overline{F}(x) = \begin{cases} F(x) & x \in [a, b] \\ F(a) & x < a \\ F(b) & x > b \end{cases}$ is in $BV(\mathbb{R})$. Thus any $F \in BV([a, b])$ can be extended to $F \in BV(\mathbb{R})$.
- If $F \in BV(\mathbb{R})$, then $F \in BV([a, b])$.

Lemma 8. *If $F \in BV(\mathbb{R})$, then $T_F + F, T_F - F$ are increasing.*

Proof. Let $x, y \in \mathbb{R}$ with $x < y$. We want to show $T_F(y) \pm F(y) \geq T_F(x) \pm F(x)$. Let $\epsilon > 0$. We may find $\{x_j\}_{j=0}^n \subset (-\infty, x]$ such that $-\infty < x_0 < ... < x_n = x$ and $T_F(x) - \epsilon \leq \sum_{j=1}^n |F(x_j) - F(x_{j-1})|$. Then

$$T_F(y) \geq \sum_{j=1}^n |F(x_j) - F(x_{j-1})| + |F(y) - F(x)| \geq T_F(x) - \epsilon + |F(y) - F(x)|.$$

Now, $T_F(y) \geq T_F(x) - \epsilon + F(x) - F(y)$ and $T_F(y) \geq T_F(x) - \epsilon + F(y) - F(x)$. Thus $T_F(y) \pm F(y) \geq T_F(x) \pm F(x) - \epsilon$. Since ϵ was arbitrary, done. □

Theorem 45 (3.27b)**.** *Suppose $F : \mathbb{R} \to \mathbb{R}$. Then $F \in BV(\mathbb{R})$ if and only if F can be written as the difference of two bounded increasing functions.*

Proof. Since $BV(\mathbb{R})$ is a vector space and any bounded increasing function is in $BV(\mathbb{R})$, the backward direction is done. Let $F \in BV(\mathbb{R})$. Then $\frac{1}{2}(T_F \pm F)$ are increasing and their difference is F. Just need to show they are bounded. Of course, since $F \in BV(\mathbb{R})$, we have T_F is bounded by definition. Also, $|F(x)| \leq |F(x) - F(0)| + |F(0)| \leq T_F(x) + |F(0)| < \infty$. So F is (uniformly) bounded. Thus $\frac{1}{2}(T_F \pm F)$ is bounded. □

Theorem 46 (3.27)**.** *Let $F \in BV(\mathbb{R})$. Then*

1. *$F(x+), F(x-)$ exist for all $x \in \mathbb{R}$ and $\lim_{x\to\infty} F(x), \lim_{x\to-\infty} F(x)$ exists.*
2. *The set of points of discontinuity of F is countable.*
3. *If $F : \mathbb{R} \to \mathbb{R}$ is defined by $G(x) = F(x+)$, then F', G' exist a.e. and $F' = G'$ a.e.*

Proof. Note that (1) follows from Theorem 45. To show $F(x+)$ exists, just note there exists increasing bounded functions f_1, f_2 such that $F = f_1 - f_2$. Then $\lim_{y\to x+} f_1(y), \lim_{y\to x+} f_2(y)$ exist and thus $\lim_{y\to x+} f(y)$ exists. For the limits, use the fact that F is bounded. Now parts (2) and (3) follow from Theoerem 44 and 45. □

Definition. *If $F \in BV(\mathbb{R})$, then the representation $F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$ is called the* ***Jordan representation*** *of F. We say $\frac{1}{2}(T_F + F)$ is the* ***positive variation*** *and $\frac{1}{2}(T_F - F)$ is the* ***negative variation*** *for F.*

Remark. If $-\infty < x_0 < ... < x_n = x$, then $F(x) = F(x_0) + \sum_{j=1}^n F(x_j) - F(x_{j-1})$. Thus

$$\begin{aligned}\tfrac{1}{2}(T_f + F) &= \tfrac{1}{2}\sup\{\textstyle\sum |F(x_j) - F(x_{j-1})| + F(x)\} \\ &= \tfrac{1}{2}\sup\{\textstyle\sum |F(x_j) - F(x_{j-1})| + F(x_0) + \sum_{j=0}^n F(x_j) - F(x_{j-1})\} \\ &= \sup\{\textstyle\sum_{j=1}^n [F(x_j) - F(x_{j-1})]^+ + \frac{1}{2}F(x_0)\} \\ &= \sup\{\textstyle\sum_{j=1}^n [F(x_j) - F(x_{j-1})]^+\} + \frac{1}{2}\lim_{x\to-\infty} F(x).\end{aligned}$$

Define $NBV(\mathbb{R}) = \{F \in BV(\mathbb{R}) : \lim_{x\to-\infty} F(x) = 0 \text{ and } F \text{ is right cont}\}$.

Lemma 9 (3.28)**.** *If $F \in BV(\mathbb{R})$, then $\lim_{x\to-\infty} T_F(x) = 0$. If F is right continuous, then T_F is also right continuous.*

Proof. Let $\epsilon > 0$. With $x \in \mathbb{R}$, select $\{x_j\}_{j=1}^n \subset \mathbb{R}$ such that $-\infty < x_0 < ... < x_n = x$ and $T_F(x) - \epsilon \le \sum_{j=1}^\infty |F(x_j) - F(x_{j-1})|$. Thus $T_F(x) - T_F(x_0) = \sup\{\sum_{j=1}^m |F(y_j) - F(y_{j-1})| : m \in \mathbb{N}, x_0 = y_0 < ... < y_n = x\} \ge \sum_{j=1}^m |F(x_j) - F(x_{j-1})| \ge T_F(x) - \epsilon$. Thus $T_f(x_0) \le \epsilon$ which implies $T_F(y) \le \epsilon$ for all $y \le x_0$. Hence $\lim_{x\to-\infty} T_F(x) = 0$. Now, suppose F is right continuous. We want to show T_F is right continuous. Put $\alpha = \lim_{y\to x^+} T_F(y) - T_F(x) \ge 0$. Let $\epsilon > 0$. Choose $\delta > 0$ such that $|F(y) - F(x)| < \epsilon$ whenever $x \le y \le x + \delta$ and (by definition) when $T_F(y) - \lim_{z\to x^+} T_F(z) < \epsilon$ whenever $x \le y \le x + \delta$. (Note that δ may initially be different, but then choose the smaller one). Let $y \in (x, x + \delta)$ be given. We may choose $\{x_j\}_{j=1}^n \subset \mathbb{R}$ such that $x = x_0 < ... < x_n = y$ and

$$\sum_{j=1}^n |F(x_j) - F(x_{j-1})| \ge [T_F(y) - T_F(x)] - \frac{1}{4}[T_F(y) - T_F(x)] \ge \frac{3}{4}[\lim_{z\to x^+} T_F(z) - T_F(x)] = \frac{3}{4}\alpha.$$

Since $x_1 \in (x, x + \delta)$, we have $|F(x_1) - F(x_0)| < \epsilon$. So $\sum_{j=2}^n |F(x_j) - F(x_{j-1})| \ge \frac{3}{4}\alpha - \epsilon$. Now, we may find $\{t_j\}_{j=1}^m \subset \mathbb{R}$ such that $x = t_0 < ... < t_m = x_1$ and $\sum |F(t_j) - F(t_{j-1})| \ge \frac{3}{4}\alpha$. Now, $x = t_0 < ... < t_m = x_1 < ... < x_n = y$ is a partition. So $T_F(y) - T_F(x) \ge \sum_{j=1}^m |F(t_j) - F(t_{j-1})| + \sum_{j=2}^n |F(x_j) - F(x_{j-1})| \ge \frac{3}{2}\alpha - \epsilon$. Now, $\alpha + \epsilon \ge T_F(y) - T_F(x) \ge \frac{3}{2}\alpha - \epsilon$, which implies $\alpha \le \epsilon$. Since $\epsilon > 0$ was arbitrary, $\alpha = 0$. □

Theorem 47 (3.29)**.** *If μ is a finite signed Borel measure on $\mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ is defined by $F(x) = \mu((-\infty, x])$, then $F \in NBV(\mathbb{R})$. If $F \in NBV(\mathbb{R})$, then there exists a unique finite signed regular measure μ_F on $\mathbb{R}$ such that $\mu_F((-\infty, x]) = F(x)$ for all $x \in \mathbb{R}$. Moreover, $|\mu_F| = \mu_{T_F}$, that is, $\mu_{T_F}((-\infty, x]) = T_F(x)$ for all $x \in \mathbb{R}$.*

Proof. Suppose μ is a finite signed Borel measure. By the Jordan Decomposition Theorem, we may find positive Borel measures μ^+, μ^- such that $\mu = \mu^+ - \mu^-$. By Proposition 26, the functions $F^+, F^- : \mathbb{R} \to \mathbb{R}$ given by $F^\pm(x) = \mu^\pm((-\infty, x])$ are right continuous and increasing. Moreover, $\lim_{x\to-\infty} F^\pm(x) = 0$ by Theorem 5e (1.8d) and $\lim_{x\to\infty} F^\pm(x) = \mu^\pm(\mathbb{R})$. So $F^\pm \in NBV(\mathbb{R})$. Since $F = F^+ - F^-$ and $NBV(\mathbb{R})$ is a vector space, we see $F \in NBV(\mathbb{R})$. For the other direction, put $F^\pm = \frac{1}{2}(T_F \pm F)$. So $F^\pm$ are bounded increasing and right continuous. Let $\mu_{F^\pm}$ be the associated Lebesgue Stieltjes measures (note they are regular) and put $\mu_F := \mu_{F^+} - \mu_{F^-}$. Thus μ_F is a signed regular Borel measure and $\mu_F((-\infty, x]) = \mu_{F^+}((-\infty, x]) - \mu_{F^-}((-\infty, x]) = F^+(x) - F^-(x) = F(x)$ as they are in $NBV(\mathbb{R})$. To show $|\mu_F| = \mu_{T_F}$, observe that for $[a, b] \in \mathbb{R}$, $|\mu_F|((a, b]) = \mu_{F^+}((a, b]) + \mu_{F^-}((a, b]) = F^+(b) - F^+(a) + F^-(b) - F^-(a) = T_F(b) - T_F(a) = \mu_{T_F}((a, b])$. Since they are equal on the semialgebra of left open-right closed intervals, they are equal on $\mathcal{B}_\mathbb{R}$. Uniqueness is an exercise. □

Remarks. Folland proves everything for $\mathbb{C}-$valued functions. Also, compare Propositions 25 and 26 to Theorem 1.16 in Folland.

Proposition 40 (3.30)**.** *Let $F \in NBV(\mathbb{R})$. Then F' exists a.e. and there exists $f \in L^1$ such that $F' = f$ a.e. Moreover, $\mu_F \perp m$ if and only if $F' = 0$ a.e. and $\mu_F << m$ if and only if $F(x) = \int_{(-\infty, x]} f(t)dt$.*

Proof. If $F \in NBV(\mathbb{R})$, then μ_F is a signed regular Borel measure by Theorem 47 (3.29) and F' exists a.e. by Theorem 46. Let f be the Radon-Nikodym derivative for μ_F so that $f \in L^1(m)$ by the LRN Theorem and $f = F'$ a.e. by Theorem 41 (3.22). The rest follows from the LRN Theorem. □

Definition. *We say $F : \mathbb{R} \to \mathbb{R}$ is* ***absolutely continuous*** *if for all $\epsilon > 0$ there exists $\delta > 0$ such that for any family $\{(a_j, b_j)\}_{j=1}^n$ of disjoint intervals, we have $\sum_{j=1}^N |F(b_j) - F(a_j)| < \epsilon$ whenever $\sum_{j=1}^N (b_j - a_j) < \delta$.*

Proposition 41 (3.32)**.** *If $F \in NBV(\mathbb{R})$, then F is absolutely continuous if and only if $\mu_F << m$.*

Proof. See Folland. □

Corollary 12 (3.33)**.** *If $f \in L^1(m)$, then $F(x) := \int_{(-\infty,x]} f(t)dt$ is in $NBV(\mathbb{R})$, is absolutely continuous, and $F'(x) = f(x)$ a.e. If $F \in NBV(\mathbb{R})$ is absolutely continuous, then $F' \in L^1(m)$ (a.e.) and $F(x) = \int_{(-\infty,x]} F'(t)dt$.*

Proof. The second part follows immediately from Proposition 40 and 41. For the first part, if $f \in L^1(m)$, then $f^+, f^- \in L^1(m)$ and $F(x) = \int_{(-\infty,x]} f^+(t)dt - \int_{(-\infty,x]} f^-(t)dt$, the difference of two increasing bounded functions. Thus $F \in BV(\mathbb{R})$. Of course, F is clearly continuous and $\lim_{x\to-\infty} F(x) = 0$. So $F \in NBV(\mathbb{R})$. We see F is absolutely continuous by Proposition 41 and $F'(x) = f(x)$ a.e. follows from Proposition 40. □

Theorem (**Fundamental Theorem of Calculus**)**.** *Let $[a, b] \subset \mathbb{R}$ and $F : [a, b] \to \mathbb{R}$ be given. TFAE*

1. *F is absolutely continuous on $[a, b]$.*
2. *$F(x) = F(a) + \int_{(a,x]} f(t)dt$ for some $f \in L^1([a, b], m)$.*
3. *F' exists a.e. in $[a, b]$ and there exists $f \in L^1([a, b], m)$ such that $f = F'$ a.e. and $F(x) = F(a) + \int_{(a,x]} f(t)dt$.*

Theorem (**Integration By Parts**)**.** *Suppose $F, G \in NBV(\mathbb{R})$ and either F or G is continuous. Then for all $[a, b] \subset \mathbb{R}$, we have $\int_{(a,b]} F d\mu_G = F(b)G(b) - F(a)G(a) - \int_{(a,b]} G d\mu_F$.*

Proof. WLOG, assume G is continuous. By considering $H(x) := G(-x)$, we see T_G is continuous. Thus $G^\pm := \frac{1}{2}(T_G \pm G)$ are continous. Also $F^\pm := \frac{1}{2}(T_F \pm F)$ are right continuous as $F \in NBV(\mathbb{R})$. Set $\Omega = \{(x, y) : a < x \le y \le b\} \subseteq \mathbb{R}^2$. Now, $\mu_{F^\pm}, \mu_{G^\pm}$ are all positive finite Borel measures. Thus by Fubini's Theorem, we have

$$\begin{aligned}
(\mu_{F^+} \times \mu_{G^+})(\Omega) &= \int_\Omega d(\mu_{F^+} \times \mu_{G^+}) \\
&= \int_{(a,b]} \int_{[x,b]} d\mu_{G^+}(y) d\mu_{F^+}(x) \\
&= \int_{(a,b]} \mu_{G^+}(b) - \mu_{G^+}(x) d\mu_{F^+}(x) \\
&= \int_{(a,b]} G^+(b) - G^+(x) d\mu_{F^+}(x) \\
&= G^+(b)[F^+(b) - F^+(a)] - \int_{(a,b]} G^+(x) d\mu_{F^+}(x).
\end{aligned}$$

and similarly

$$\begin{aligned}
(\mu_{F^+} \times \mu_{G^+})(\Omega) &= \int_{(a,b]} \int_{(a,y]} d\mu_{F^+}(x) d\mu_{G^+}(y) \\
&= \int_{(a,b]} F^+(y) - F^+(a) d\mu_{G^+}(y) \\
&= \int_{(a,b]} F^+(y) d\mu_{G^+}(y) - F^+(a)(G^+(b) - G^+(a)).
\end{aligned}$$

Combining these two equations, we get

$$\int_{(a,b]} F^+ d\mu_{G^+} = F^+(b)G^+(b) - F^+(a)G^+(a) - \int_{(a,b]} G^+ d\mu_{F^+}.(*)$$

Of course, we could easily show $(*)$ holds for F^-, G^+. Then, subtracting these we get that $(*)$ holds for F, G^+. Repeat with G to get that $(*)$ holds for F, G. □

2.7 Measurable Transformations

Recall. Change of Variable Formula: If $g : (a, b] \to (c, d]$ is continuously differentiable and monotone and f is continuous on $(c, d]$, then $\int_a^b f(g(x))|g'(x)|dx = \int_c^d f(y)dy$. We want to generalize this idea to Lebesgue integrals.

Definition. *Let $(X, \mathcal{M}), (Y, \mathcal{N})$ be measurable spaces. A mapping $T : X \to Y$ is called a $(\mathcal{M}, \mathcal{N})-$**measurable transformation** if $T^{-1}(F) \in \mathcal{M}$ whenever $F \in \mathcal{N}$.*

Remarks.

- This is the same definition as a measurable function. The point is to note we are not restricting ourselves to $\mathbb{R}$.
- If $(\mathbb{R}, \mathcal{O})$ is a measurable space and $f : Y \to \mathbb{R}$ is a $(\mathcal{N}, \mathcal{O})-$measurable function, then $f \circ T : X \to \mathbb{R}$ is a $(\mathcal{M}, \mathcal{O})-$measurable function.

Proposition 42. *Let T be a measurable transformation from $(X, \mathcal{M}) \to (Y, \mathcal{N})$. Let μ be a positive measure on $\mathcal{M}$ and define $\mu \circ T^{-1} : \mathcal{N} \to [0, \infty]$ by $\mu \circ T^{-1}(F) = \mu(T^{-1}(F))$. Then $\mu \circ T^{-1}$ is a measure on $\mathcal{N}$.*

Definition. *The measure $\mu \circ T^{-1}$ above is called the **measure induced** by μ and T, or the **pushforward** of μ through T.*

Example. Let $T : [0, 2\pi) \to \mathbb{R}^2$ be given by $T(t) := (\cos t, \sin t)$. So T is a bijection between $[0, 2\pi)$ and the unit circle S'. Let $m = m|_{\mathcal{L}([0,2\pi))}$. Let $\mathcal{N}$ be the $\sigma-$algebra on S' generated by T. Then T is a measurable function. By definition, T is an $(\mathcal{M}, \mathcal{N})-$measurable transformation. The Lebesgue measure on S' is the pushforward of m through T, that is, $m_{S'} = m \circ T^{-1}$.

<u>**General Change of Variable Formula.**</u> Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $(Y, \mathcal{N})$ be a measurable space. Suppose T is an $(\mathcal{M}, \mathcal{N})-$measurable transformation. Then for all $\mathcal{N}-$measurable functions $f : Y \to \mathbb{R}$, we have

$$\int_X f(T(x))d\mu = \int_Y f(y)d(\mu \circ T^{-1})$$

in the sense that if one exists, they both do and are equal.

Proof. Exercise using simple function technique. □

Corollary 13. *Under the same hypotheses, for all $F \in \mathcal{N}$, we have*

$$\int_{T^{-1}(F)} f(T(x))d\mu = \int_F f(y)d(\mu \circ T^{-1})$$

provided one of the integrals exist.

Corollary 14. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $(Y, \mathcal{N}, \nu)$ be a $\sigma-$finite measure space. Suppose T is a $(\mathcal{M}, \mathcal{N})-$measurable transformation such that $\mu \circ T^{-1} << \nu$. Then for all $\mathcal{N}-$measurable functions $f : Y \to \mathbb{R}$, we have*

$$\int_X f(T(x))d\mu = \int_Y f(y)\frac{d(\mu \circ T^{-1})}{d\nu}d\nu.$$

Theorem 48. *Suppose $T : [a, b] \to [c, d]$ is an increasing bijection that is absolutely continuous on $[a, b]$. Let $f \in L^1([a, b], m)$ be given. Then $\int_{[a,b]} f(T(x))T'(x)dx = \int_{[c,d]} f(y)dy$.*

Proof. Want to use Corollary 14. Define $\mu : \mathcal{B}_{[a,b]} \to [0, \infty]$ by $\mu(E) = m(T(E))$. Observe that for $(x, y] \subseteq [a, b]$, we have $\mu((x, y]) = m(T((x, y])) = m((T(x), T(y)]) = T(y) - T(x)$ as T is monotone and continuous. Upon appropriately extending T, we get a Lebesgue-Stieltjes measure $\mu_{\widetilde{T}}$ where $\widetilde{T} : \mathbb{R} \to [0, d - c]$ is defined by

$$\widetilde{T}(x) := \begin{cases} T(x) - c & \text{if } x \in [a, b] \\ 0 & \text{if } x < a \\ d - c & \text{if } x < b \end{cases}.$$

Then $\widetilde{T}$ is absolutely continuous and is in $NBV(\mathbb{R})$. So $\mu_{\widetilde{T}} << m$. Also $\mu_{\widetilde{T}}|_{\mathcal{B}_{[a,b]}} = \mu$. Moreover, $\mu_{\widetilde{T}} \circ T^{-1}|_{\mathcal{B}_{[c,d]}} = m|_{\mathcal{B}_{[c,d]}}$ since for $(x, y) \subseteq [c, d]$ we have

$$\mu_{\widetilde{T}} \circ T^{-1}((x,y]) = \mu_{\widetilde{T}}(T^{-1}((x,y])) = \mu_{\widetilde{T}}((T^{-1}(x), T^{-1}(y)]) = T(T^{-1}(y)) - T(T^{-1}(x)) = y - x = m((x,y]).$$

Now, by Corollary 14, $\int_{[a,b]} f(T(x))d\mu_{\widetilde{T}} = \int_{[c,d]} f(y)\frac{d((\mu_{\widetilde{T}})\circ T^{-1})}{dm}dy$ which implies, by the LRN Theorem that $\int_{[a,b]} f(T(x))\frac{d\mu_{\widetilde{T}}}{dm}dx = \int_{[c,d]} f(y)dy$. By the Fundamental Theorem of Calculus and the LRN Theorem, we find that $\widetilde{T}'(x)$ exists for almost every $x \in \mathbb{R}$ and $\frac{d\mu_{\widetilde{T}}}{dm} = \widetilde{T}'$ almost everywhere. Since $\widetilde{T}'(x) = T'(x)$ for almost every $x \in [a, b]$, $\int_{[a,b]} f(T(x))T'(x)dx = \int_{[c,d]} f(y)dy$. □

Note. This holds if T is decreasing, but then we want $-T'$.

We use a similar idea in higher dimensions for the case where $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation. We needed transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mu = m \circ T$ is a measure and $\mu << m$. So, in particular, we need $m(T(E)) = 0$ whenever $m(E) = 0$. Recall Theorem 35: If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, then there is a number $\delta < \infty$ such that $m(T(E)) = \delta m(E)$ for all $E \in \mathcal{L}^n$. If $E = Q_0$, the unit cube, then $m(T(Q_0)) = \delta m(Q_0) = \delta$. Thus $\delta = |\det T|$.

Theorem 49 (Linear Change of Variables Formula - 2.44)**.** *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective linear transformation. If $f \in L^1(m)$, then $f \circ T \in L^1(m)$ and $\int_{\mathbb{R}^n} f(y)dy = |\det T| \int_{\mathbb{R}^n} f(T(x))dx$.*

Proof. Define $\mu : \mathcal{B}_{\mathbb{R}^n} \to [0,\infty]$ by $\mu(E) = m(T(E))$ for all $E \in \mathcal{B}_{\mathbb{R}^n}$. Since T is a bijection and T^{-1} is continuous, we find that μ is a measure on $\mathcal{B}_{\mathbb{R}^n}$. (check!) Also, if $m(E) = 0$, then $\mu(E) = m(T(E)) = |\det T|m(E) = 0$. Thus $\mu << m$. Also, $\mu \circ T^{-1} = m$. Since $\frac{\mu(E)}{m(E)} = |\det T|$ for all $E \in \mathcal{B}_{\mathbb{R}^n}$ with $m(E) \neq 0$, it follows that $\frac{d\mu}{dm} = |\det T|$ (by the theorem on nicely shrinking sets). Now T is a measurable transformation, so by Corollary 14, $\int_{\mathbb{R}^n} f(T(x))d\mu = \int_{\mathbb{R}^n} f(y)\frac{d(\mu\circ T^{-1})}{dm}dy$, which implies $\int_{\mathbb{R}^n} f(T(x))\frac{d\mu}{dm}dx = \int_{\mathbb{R}^n} f(y)dy$ and thus $|\det T| \int_{\mathbb{R}^n} f(T(x))dx = \int_{\mathbb{R}^n} f(y)dy$. □

Corollary 15. *If $R : \mathbb{R}^n \to \mathbb{R}^n$ is a rotation or a reflection across the $(n-1)-$dimensional plane, then $m(R(E)) = m(E)$ for all $E \in \mathcal{L}^n$.*

Notation. If $G : \Omega \to \mathbb{R}^n$, for $\Omega \subseteq \mathbb{R}^n$ open, has continuously differentiable components, that is $G = (G_1, ..., G_n)$ with $G_j \in \mathcal{C}'$, then $D_xG : \Omega \to \mathbb{R}^{n\times m}$ is given by $[D_xG(x)]_{ij} = \frac{\partial G_j}{\partial x_i}(x)$.

Definition. *We say $G : \Omega \to G(\Omega) \subseteq \mathbb{R}^n$ is a $\mathcal{C}'-$**diffeomorphism** if G is bijective and both G and G^{-1} are continuously differentiable.*

Theorem (Change of Variables Formula - 2.47)**.** *Suppose $\Omega \subseteq \mathbb{R}^n$ is open and $G : \Omega \to G(\Omega)$ is a $\mathcal{C}'-$diffeomorphism. Then*

1. *If $f \in L^1(G(\Omega), m)$, then $\int_{G(\Omega)} f(x)dx = \int_\Omega f(G(x))|\det D_xG(x)|dx$.*
2. *If $E \subseteq \Omega$ and $E \in \mathcal{L}^n$, then $G(E) \in \mathcal{L}^n$ and $m(G(E)) = \int_E |\det D_xG(x)|dx$.*

2.7.1 Integration in Polar Coordinates

2-dimensions: Set $\Omega := \{(r, \theta) \in \mathbb{R}^2 : r > 0, \theta \in (0, 2\pi)\}$ and $W := \mathbb{R}^2 \setminus \{(r, 0) : r > 0\}$. So Ω and W are open. Now, define $G : \Omega \to W$ by $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$. Then G is a $\mathcal{C}'-$diffeomorphism. Indeed, $G^{-1} : (x, y) \mapsto (\sqrt{x^2 + y^2}, \theta(x, y))$ where
$$\theta(x,y) = \begin{cases} \tan^{-1}(y/x) & x > 0, y > 0 \\ \pi + \tan^{-1}(y/x) & x \leq 0 \\ 2\pi + \tan^{-1}(y/x) & x > 0, y < 0. \end{cases}$$
Also, $|\det D_{(r,\theta)}G(r, \theta)| = r$ for all $(r, \theta) \in \Omega$. Suppose $f \in L^1(\mathbb{R}^2, m)$. Then the change of variables formula yields

$$\int_W f(x,y)d(x,y) = \int_\Omega f(G(r,\theta))|\det D_{(r,\theta)}G(r,\theta)|d(r,\theta) = \int_\Omega f(r\cos\theta, r\sin\theta)rd(r,\theta).$$

Noting that $W = \mathbb{R}^2 \setminus$ (a null set) and using Fubini's Theorem, we see

$$\int_{\mathbb{R}^2} f(x,y)d(x,y) = \int_{(0,\infty)} \int_{(0,2\pi)} f(rcos\theta, r\sin\theta) r d\theta dr.$$

Higher dimensions: The same formula can be derived in higher dimensions. Set $S^{n-1} := \{x \in \mathbb{R}^n : ||x|| = 1\}$. For $x \in \mathbb{R}^n \setminus \{0\}$, define $r(x) := ||x||$ and $\theta(x) = \frac{x}{||x||} \in S^{n-1}$. It can be verified that the map $G : (0,\infty) \times S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ defined by $(r,\theta) \mapsto r\theta$ is a $\mathcal{C}'-$diffeomorphism. (Note that $G^{-1}(x) = (r(x), \theta(x))$ and both components are differentiable). Since G is a bijection and has a continuous inverse, we may define the measure m_* on $\mathcal{B}_{(0,\infty)\times S^{n-1}}$ by $m_*(E) = m(G(E))$. Now, define the measure ρ_n on $\mathcal{B}_{(0,\infty)}$ by $\rho_n(E) = \int_E r^{n-1} dr$. We want to define a measure σ_{n-1} on S^{n-1} such that $m_* = \rho_n \times \sigma_{n-1}$.

Theorem 50 (2.49)**.** *There exists a unique Borel measure σ_{n-1} on $\mathcal{B}_{S^{n-1}}$ such that $m_* = \rho_n \times \sigma_{n-1}$.*

Proof. Let $E \in \mathcal{B}_{S^{n-1}}$. For each $\alpha > 0$, set $E_\alpha := G((0,\alpha] \times E)$. So $E_\alpha = \{r\theta : 0 < r \leq a, \theta \in E\}$. Define $\sigma_{n-1}(E) := nm(E_1)$. Consider the map $\lambda : \mathcal{P}(S^{n-1}) \to \mathcal{P}(B(1,0)) \subset \mathcal{P}(\mathbb{R}^n)$. Since λ commutes with unions, intersections, and complements, λ maps Borel sets to Borel sets and thus σ_{n-1} is a Borel measure on S^{n-1}. Also, given $\alpha \in (0,\infty), E_\alpha = T(E_1)$ where $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation such that $T(x) = \alpha x$. Thus $m(E_\alpha) = |\det T| m(E_1) = \alpha^n m(E_1)$. For any $(a,b] \subset (0,\infty)$ and $E \in \mathcal{B}_{S^{n-1}}$, we see

$$\begin{aligned} m_*((a,b] \times E) = m(E_b \setminus E_a) &= m(E_b) - m(E_a) \\ &= (b^n - a^n) m(E_1) \\ &= \tfrac{b^n - a^n}{n} \sigma_{n-1}(E) \\ &= \left(\int_{(a,b]} r^{n-1} dr \right) \sigma_{n-1}(E) \\ &= (\rho_n \times \sigma_{n-1})((a,b] \times E). \end{aligned}$$

Also, $m_*((a,\infty) \times E)$ is 0 when $\sigma_{n-1}(E) = 0$ and ∞ otherwise. This agrees with $\rho_n \times \sigma_{n-1}$ which implies the same formula works. Fix $E \in \mathcal{B}_{S^{n-1}}$. Set $\mathcal{C}_E := \{(a,b] \times E : 0 \leq a \leq b\} \cup \{(a,\infty) : 0 \leq a\}$. Then $\mathcal{C}_E$ is a semialgebra on $(0,\infty) \times E$. Since $m_* = \rho_n \times \sigma_{n-1}$ on $\mathcal{C}_E$, Caratheodory's Extension process and uniqueness (as m_* is $\sigma-$finite on $(0,\infty) \times E$)) imply that $m_* = \rho_n \times \sigma_{n-1}$ on the $\sigma-$algebra $\mathcal{M}_E = \{A \times E : A \in \mathcal{B}_{(0,\infty)}\}$. By Proposition 35 (1.5), $\mathcal{B}_{(0,\infty)\times S^{n-1}} = \mathcal{B}_{(0,\infty)} \otimes \mathcal{B}_{S^{n-1}}$ and by Proposition 34 (1.7), $\mathcal{B}_{(0,\infty)} \otimes \mathcal{B}_{S^{n-1}}$ is generated by $\{\mathcal{M}_E\}_{E \in \mathcal{B}_{S^{n-1}}}$. Thus Caratheodory's Extension process and uniqueness imply that $m_* = \rho_n \times \sigma_{n-1}$ on $\mathcal{B}_{(0,\infty)\times S^{n-1}}$. ∎

Thus by Theorem 50 and the simple function technique, if $f \in L^1(\mathbb{R}^n, m)$, then

$$\int_{\mathbb{R}^n} f(x)dx = \int_{(0,\infty)} \int_{S^{n-1}} f(r\theta) r^{n-1} d\sigma_{n-1}(\theta) dr$$

(as $d\rho = r^{n-1}dr$).

Remarks.

- $\sigma(S') = 2\pi$, the circumference of the unit circle.
- For $E \in S^{n-1}$, one can show $\sigma_{n-1}(E) = \mathcal{H}_{n-1}(E)$, the $n-1$ dimensional Hausdorff measure on $\mathbb{R}^n$. Thus

$$\int_{\mathbb{R}^n} f(x)dx = \int_{(0,\infty)} \int_{\partial B(1,0)} f(r\theta) r^{n-1} d\mathcal{H}_{n-1}(\theta) dr.$$

- One can also show $r^{n-1} d\mathcal{H}_{n-1}$ is the $n-1$ dimensional measure on $\partial B(r,0)$. So

$$\int_{\mathbb{R}^n} f(x)dx = \int_{(0,\infty)} \int_{\partial B(r,0)} f(y) d\mathcal{H}_{n-1}(y) dr.$$

- Notice the function $F(s) := \int_{(0,s)} \int_{\partial B(r,0)} f(y) d\mathcal{H}_{n-1}(y) dr$ is absolutely continuous on $[0,\infty)$. So $F'(s)$ exists almost everywhere and

$$F'(s) = \frac{d}{ds}\left[\int_{(0,s)}\int_{\partial B(r,0)} f(y)d\mathcal{H}_{n-1}dr\right] = \int_{\partial B(s,0)} f(y)d\mathcal{H}_{n-1}(y).$$

3 More about L^p Spaces

Let $(X,\mathcal{M},\mu)$ be a measure space. For $f: X \to \mathbb{R}$ such that $|f| \in L^+$, we define for $p \in [1,\infty)$ $||f||_p := \left(\int_X |f|^p d\mu\right)^{1/p}$ and $||f||_\infty := \inf\{a \in \mathbb{R} : \mu(\{x \in X : |f(x)| \geq a\}) = 0\}$. Also, for $p \in [1,\infty]$, we see $L^p(\mu) := \{f : X \to \mathbb{R} : ||f||_p < \infty\}$.
Properties of L^p Spaces:

- Banach Space
- L^2 is a Hilbert Space
- Simple functions are dense
- If $\mu = m$, then continuous functions are dense.
- Hölder's Inequality: Suppose $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for measurable functions $f, g : X \to \mathbb{R}$, we have $||fg||_1 \leq ||f||_p ||g||_q$. (In particular, if $f \in L^p, g \in L^q$, then $fg \in L^1$.)

3.1 Dual Spaces

Definition. *Let $(\mathfrak{X}, ||\cdot||_{\mathfrak{X}})$ and $(\mathfrak{Y}, ||\cdot||_{\mathfrak{Y}})$ be normed vector spaces. A linear map is* ***bounded*** *if there exists $C \in [0,\infty)$ such that $||Tx||_{\mathfrak{Y}} \leq C||x||_{\mathfrak{X}}$ for all $x \in \mathfrak{X}$.*

Proposition 43 (5.2)**.** *If $(\mathfrak{X}, ||\cdot||_{\mathfrak{X}})$ and $(\mathfrak{Y}, ||\cdot||_{\mathfrak{Y}})$ are normed vector spaces and $T : \mathfrak{X} \to \mathfrak{Y}$ is a linear map, then TFAE*

- *T is continuous on $\mathfrak{X}$.*
- *T is continuous at a single point (generally use 0).*
- *T is bounded.*

Proof. $(1) \Rightarrow (2):$ By definition.

$(2) \Rightarrow (3):$ There is a $\delta > 0$ such that $||Tx||_{\mathfrak{Y}} < 1$ whenever $||x||_{\mathfrak{X}} \leq \delta$. If $x \in \mathfrak{X} \setminus \{0\}$, then $\left|\left|\frac{\delta x}{||x||_{\mathfrak{X}}}\right|\right|_{\mathfrak{X}} = \delta$. Thus $||Tx||_{\mathfrak{Y}} = \frac{||x||_{\mathfrak{X}}}{\delta}\left|\left|T\frac{\delta x}{||x||_{\mathfrak{X}}}\right|\right|_{\mathfrak{Y}} \leq \frac{1}{\delta}||x||_{\mathfrak{X}}$.

$(3) \Rightarrow (1):$ There exists c such that $||Tx||_{\mathfrak{Y}} \leq c||x||_{\mathfrak{X}}$ for all $x \in \mathfrak{X}$. Let $\epsilon > 0$. If $||x_1 - x_2|| < \frac{\epsilon}{c}$, then $||Tx_1 - Tx_2||_{\mathfrak{Y}} = ||T(x_1 - x_2)||_{\mathfrak{Y}} \leq c||x_1 - x_2||_{\mathfrak{X}} < \epsilon$. □

Definition. *If $(\mathfrak{X}, ||\cdot||_{\mathfrak{X}})$ and $(\mathfrak{Y}, ||\cdot||_{\mathfrak{Y}})$ are normed vector spaces, then the space of all bounded linear maps from $\mathfrak{X}$ to $\mathfrak{Y}$ is denoted by $L(\mathfrak{X},\mathfrak{Y})$. The function $|||\cdot||| : L(\mathfrak{X},\mathfrak{Y}) \to [0,\infty)$ defined by $|||T||| = \sup\{||Tx||_{\mathfrak{Y}} : ||x||_{\mathfrak{X}} = 1\}$ is called the* ***operator norm****.*

Remarks.

- $(L(\mathfrak{X},\mathfrak{Y}), |||\cdot|||)$ is a normed vector space.
- $T(|||x|||) = \sup\left\{\frac{||Tx||_{\mathfrak{Y}}}{||x||_{\mathfrak{X}}} : x \neq 0\right\} = \inf\{c \in \mathbb{R} : ||Tx||_{\mathfrak{Y}} \leq c||x||_{\mathfrak{X}} \text{ for all } x \in \mathfrak{X}\}$.

Example. There do exist discontinuous linear maps (other than the obvious $T : \mathfrak{X} \to +\infty$). Let $\mathfrak{X} = \{\{x_k\}_{k=1}^\infty : x_k \in \mathbb{R}, \sum_{k=1}^\infty kx_k < \infty\}$. Define $||\cdot||_1 : \mathfrak{X} \to [0,\infty)$ by $||x||_1 = \sum_{k=1}^\infty |x_k|$ and $||\cdot||_2 : \mathfrak{X} \to [0,\infty)$ by $||x||_2 = \sum k|x_k|$. These are both norms. Define $T : (\mathfrak{X}, ||\cdot||_1) \to (\mathfrak{X}, ||\cdot||_2)$ by $Tx = x$. However, $|||T||| = \sup\{||Tx||_2 : ||x||_1 = 1\}$. Define $x^{(n)} \in \mathfrak{X}$ by $x^{(n)} = (0, ..., 0, 1, 0, ...)$ where 1 appears in the n^{th} spot. Then $||x^{(n)}||_1 = 1$ but $||x^{(n)}||_2 = n$. Thus $|||T||| \geq ||x^{(n)}||_2 = n$ for all n which implies $|||T||| = \infty$. Thus T is unbounded and discontinuous.

Proposition 44 (5.4)**.** *If $(\mathfrak{Y}, ||\cdot||_{\mathfrak{Y}})$ is complete, then so is $(L(\mathfrak{X},\mathfrak{Y}), |||\cdot|||)$.*

Definition. *If $(\mathfrak{X}, ||\cdot||_{\mathfrak{X}})$ is a normed vector space, $(L(\mathfrak{X},\mathbb{R}), |||\cdot|||)$ is called the **(continuous) dual space** of $\mathfrak{X}$. It is denoted by $\mathfrak{X}^*$ or $(\mathfrak{X}^*, ||\cdot||_{\mathfrak{X}^*})$. The members of $L(\mathfrak{X},\mathbb{R})$ are called **linear functionals**.*

Remark. By Proposition 44, $\mathfrak{X}^*$ is complete.

Definition. *If $\mathfrak{X}$ is a vector space, we see that $p : \mathfrak{X} \to \mathbb{R}$ is a **sublinear functional** if $p(x+y) \le p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ for $\lambda \ge 0$ and for all $x, y \in \mathfrak{X}$.*

Theorem (**Hahn Banach Theorem**)**.** *Let $\mathfrak{X}$ be a vector space over $\mathbb{R}$, p a sublinear functional on $\mathfrak{X}$ and $\mathcal{M}$ a linear subspace of $\mathfrak{X}$. If $f : \mathcal{M} \to \mathbb{R}$ is a linear functional such that $f(x) \le p(x)$ for all $x \in \mathcal{M}$. Then there exists a linear functional $\overline{f} : \mathfrak{X} \to \mathbb{R}$ such that $\overline{f}(x) \le p(x)$ for all $x \in \mathfrak{X}$ and $\overline{f}|_{\mathcal{M}} = f$.*

Proof. WLOG, assume $\mathcal{M} \subseteq \mathfrak{X}$. Let $x \in \mathcal{M}^C$. Want to extend f to $\mathcal{M}_x = \{y + \alpha x : y \in \mathcal{M}, \alpha \in \mathbb{R}\}$. To do this, we want to define $f_x : \mathcal{M}_x \to \mathbb{R}$ such that f is linear on $\mathcal{M}_x$, $f_x(y) \le p(y)$ for all $y \in \mathcal{M}_x$ and $f_x|_{\mathcal{M}} = f$. Suppose there exists $\beta \in \mathbb{R}$ such that $\alpha\beta \le p(\alpha x + y) - f(y)$ for all $\alpha \in \mathbb{R}, y \in \mathcal{M}$.$(*)$. Then we could define $f_x(\alpha x + y) = \alpha\beta + f(y)$ for all $\alpha \in \mathbb{R}, y \in \mathcal{M}$ and f_x would be the desired extension to $\mathcal{M}_x$. For each $y_1, y_2 \in \mathcal{M}$, we have

$$\begin{aligned} f(y_1) + f(y_2) = f(y_1 + y_2) &\le p(y_1 + y_2 - x + x) \le p(y_1 - x) + p(y_2 + x) \\ &\Rightarrow f(y_1) - p(y_1 - x) \le p(y_2 + x) - f(y_2) \end{aligned}$$

Since $y_1, y_2 \in \mathcal{M}$ were arbitrary, we have

$$\sup\{f(y) - p(y-x) : y \in \mathcal{M}\} \le \inf\{p(y+x) - f(y) : y \in \mathcal{M}\}.$$

So select β such that $\sup\{f(y) - p(y-x) : y \in \mathcal{M}\} \le \beta \le \inf\{p(y+x) - f(y) : y \in \mathcal{M}\}$. If $\alpha = 0$, then β satisfies $(*)$. If $\alpha > 0$, then

$$\begin{aligned} \alpha\beta &\le \alpha \inf\{p(y+x) - f(y) : y \in \mathcal{M}\} \\ &= \inf\{\alpha p(y+x) - \alpha f(y) : y \in \mathcal{M}\} \\ &= \inf\{p(\alpha y + \alpha x) - f(\alpha y) : y \in \mathcal{M}\} \\ &= \inf\{p(\alpha y + \alpha x) - f(\alpha y) : \alpha y \in \mathcal{M}\} \\ &= \inf\{p(y + \alpha x) - f(y) : y \in \mathcal{M}\} \end{aligned}$$

and lastly, if $\alpha < 0$, then

$$\begin{aligned} \alpha\beta &\le \alpha \sup\{f(y) - p(y-x) : y \in \mathcal{M}\} \\ &= (-\alpha) \inf\{p(y-x) - f(y) : y \in \mathcal{M}\} \\ &= \inf\{p(-\alpha y + \alpha x) - f(-\alpha y) : y \in \mathcal{M}\} \\ &= \inf\{p(y + \alpha x) - f(y) : y \in \mathcal{M}\}. \end{aligned}$$

Thus β satisfies $(*)$ and we have thus extended f to $\mathcal{M}_x$. Now, consider $\mathcal{F}$, the collection of all pairs $(f|_V, V)$ where V is a linear subspace of $\mathfrak{X}$ containing $\mathcal{M}$ and f_V is a linear functional on V such that $f_V|_{\mathcal{M}} = f$ and $f_V(y) \le p(y)$ for all $y \in V$. Define a partial order $\le$ on $\mathcal{F}$ by $(f_{V_1}, V_1) \le (f_{V_2}, V_2)$ if $V_1 \subset V_2$ and $f_{V_2}|_{V_1} = f_{V_1}$. Note that $(f_x, \mathcal{M}_x) \in \mathcal{F}$ and thus it is non empty. Let $\mathcal{G} \subseteq \mathcal{F}$ be a totally ordered subset of $\mathcal{F}$. Since $W = \cup_{(f_V, V) \in \mathcal{G}} V$ is a linear subspace of $\mathfrak{X}$, we see $(f_W, W) \in \mathcal{F}$ and is an upper bound for the chain. Thus, by Zorn's Lemma, there exists a maximal element, call it $(\overline{f}, V)$. If $V \ne \mathfrak{X}$, then there exists $x \in \mathfrak{X} \setminus V$ and $\overline{f}_x$ extends $\overline{f}$, a contradiction to maximality. □

Example. (Generalized/Banach limits). Set $\ell^\infty = \{\{x_k\}_1^\infty \subset \mathbb{R} : \sup |x_k| < \infty\}$ and $\mathcal{M} = \{x \in \ell^\infty : \lim_{k\to\infty} x_k \text{ exists}\}$. So $\mathcal{M}$ is a linear subspace. Consider the linear function $L_0 : \mathcal{M} \to \mathbb{R}$ defined by $x \mapsto \lim_{k\to\infty} x_k$. Define $p : \ell^\infty \to \mathbb{R}$ by $x \mapsto \limsup_{k\to\infty} \frac{1}{k} \sum_1^k x_j$. So p is a sublinear function on ℓ^∞. We can verify $L_0(x) = p(x)$ for $x \in \mathcal{M}$. So by the Hahn Banach Theorem, there exists a linear function $\overline{L} : \ell^\infty \to \mathbb{R}$ such that $\overline{L}|_{\mathcal{M}} = L_0$ and $\overline{L}(x) \le p(x)$ for $x \in \ell^\infty$. Since $p(x) \le \limsup_{k\to\infty} x_k$, we have

$$\liminf x_k = -\limsup(-x_k) \le -p(-x) \le -\overline{L}(-x) = \overline{L}(x) \le p(x) \le \limsup x_k.$$

Also, if we define, for each $x \in \ell^\infty$, the sequence $x^{(n)} \in \ell^\infty$ by $x_k^{(n)} = x_{k+n}$, then you can verify $\overline{L}(x^{(n)}) = \overline{L}(x)$.

Theorem 51 (5.8)**.** *Let $(\mathfrak{X}, ||\cdot||_\mathfrak{X})$ be a normed vector space.*

1. *If $\mathcal{M}$ is a closed subspace of $\mathfrak{X}$ and $x \in \mathfrak{X}\setminus\mathcal{M}$, then there exists $f \in \mathfrak{X}^*$ such that $f(x) \neq 0$ and $f|_\mathcal{M} = 0$. Moreover, $||f||_{\mathfrak{X}^*} = 1$ and $f(x) = \inf_{y\in\mathcal{M}} ||x-y||_\mathfrak{X}$.*
2. *If $x \in \mathfrak{X}\setminus\{0\}$, then there exists $f \in \mathfrak{X}^*$ such that $||f||_{\mathfrak{X}^*} = 1$ and $f(x) = ||x||_\mathfrak{X}$.*
3. *Bounded linear functionals in $\mathfrak{X}^*$ separate points in $\mathfrak{X}$, that is, if $x_1, x_2 \in \mathfrak{X}$ and $x_1 \neq x_2$, then there exists $f \in \mathfrak{X}^*$ such that $f(x_1) \neq f(x_2)$.*
4. *For each $x \in \mathfrak{X}$, define $\hat{x} : \mathfrak{X}^* \to \mathbb{R}$ by $\hat{x}(f) = f(x)$. The map $x \mapsto \hat{x}$ is a linear isometry from $\mathfrak{X}$ into $\mathfrak{X}^{**}$, that is, $||x||_\mathfrak{X} = ||\hat{x}||_{\mathfrak{X}^{**}} = \sup\{|\hat{x}(f)| : ||f||_{\mathfrak{X}^*} = 1\} = \sup\{|f(x)| : ||f||_{\mathfrak{X}^*} = 1\}$.*

Proof. 1. Let $\mathcal{M}_x = \{y \in \mathfrak{X} : y = z + \lambda x \text{ for some } z \in \mathcal{M}, \lambda \in \mathbb{R}\}$ and define $f : \mathcal{M}_x \to \mathbb{R}$ by $f(z+\lambda x) = \lambda \inf_{y\in\mathcal{M}} ||x - y||_\mathfrak{X}$. So $f(x) = \inf_{y\in\mathcal{M}} ||x-y||_\mathfrak{X}, f|_\mathcal{M} = 0$ and f is linear on $\mathcal{M}_x$:

Let $a, b \in \mathbb{R}, y_1, y_2 \in \mathcal{M}_x$. Say $y_i = z_i + \lambda_i x$. Then $f(ay_1 + by_2) = f(az_1 + bz_2 + a\lambda_1 x + b\lambda_2 x) = (a\lambda_1 + b\lambda_2)\inf_{y\in\mathcal{M}} ||x-y|| = af(y_1) + bf(y_2)$.

For all $\lambda \neq 0$, we see $|f(z+\lambda x)| = |\lambda| \inf_{y\in\mathcal{M}} ||x-y|| \leq |\lambda|\ ||x + \lambda^{-1}z|| = ||\lambda x + z||$ (take $y = -\lambda^{-1}z$). So f is linear on $\mathcal{M}$ and $f(y) \leq ||y||$ for all $y \in \mathcal{M}_x$. Thus by the Hahn Banach Theorem, there exists an extension $\overline{f} \in \mathfrak{X}^*$ that is linear and satisfies

$$\overline{f}|_{\mathcal{M}_x} = f, \overline{f}(x) = \inf ||x-y||_\mathfrak{X} \text{ and } ||f||_{\mathfrak{X}^*} \leq 1.$$

since $\mathcal{M}$ is closed, $f(x) > 0$. Let $\epsilon > 0$. Then there exists $y^* \in \mathcal{M}$ such that $||x - y^*|| \leq \inf_{y\in\mathcal{M}} ||x-y|| + \epsilon$. Since $x - y^* \in \mathcal{M}_x$, we see

$$\overline{f}(x-y^*) = f(x-y^*) = \inf_{y\in\mathcal{M}} ||x-y|| \geq ||x-y^*|| - \epsilon.$$

Thus

$$\overline{f}\left(\frac{x-y^*}{||x-y^*||_\mathfrak{X}}\right) \geq \frac{||x-y^*||_\mathfrak{X} - \epsilon}{||x-y^*||_\mathfrak{X}} \geq 1 - \frac{\epsilon}{\inf_{y\in\mathcal{M}} ||x-y||_\mathfrak{X}}.$$

Thus $||f||_{\mathfrak{X}^*} = \sup\{|f(x)| : ||x||_\mathfrak{X} = 1\} \geq 1$. Hence $||f||_{\mathfrak{X}^*} = 1$.

2. Take $\mathcal{M} = \{0\}$ and apply 1.
3. Since $x_1 \neq x_2$, we have $x_1 - x_2 \in \mathfrak{X}\setminus\{0\}$. Now apply 2.
4. If $f, g \in \mathfrak{X}^*$ and $a, b \in \mathbb{R}$, then $\hat{x}(af + bg) = (af+bg)(x) = af(x) + bg(x) = a\hat{x}(f) + b\hat{x}(g)$. So $\hat{x}$ is a linear functional on $\mathfrak{X}^*$. Moreover, if $x_1, x_2 \in \mathfrak{X}$ and $a, b \in \mathbb{R}$, then $\widehat{ax_1 + bx_2}(f) = f(ax_1 + bx_2) = af(x_1) + bf(x_2) = a\hat{x}_1(f) + b\hat{x}_2(f)$. Thus $x \mapsto \hat{x}$ is a linear map from $\mathfrak{X} \to \mathfrak{X}^{**}$. Now for all $f \in \mathfrak{X}^*$, we have $|\hat{x}(f)| = |f(x)| \leq ||f||_{\mathfrak{X}^*}||x||_\mathfrak{X} = ||x||_\mathfrak{X}$ if $||f||_{\mathfrak{X}^*} = 1$. Thus $||\hat{x}||_{\mathfrak{X}^{**}} \leq ||x||_\mathfrak{X}$. By 2, there exists $f \in \mathfrak{X}^*$ such that $||f||_{\mathfrak{X}^*} = 1$ and $f(x) = ||x||_\mathfrak{X}$. So, for this particular f, we see $|\hat{x}(f)| = |f(x)| = ||x||_\mathfrak{X}$. Then $||\hat{x}||_{\mathfrak{X}^{**}} \geq ||x||_\mathfrak{X}$ and so $||\hat{x}||_{\mathfrak{X}^{**}} = ||x||_\mathfrak{X}$. □

Remarks.

1. $\mathfrak{X}^{**}$ is complete.
2. Define $\hat{\mathfrak{X}} := \{\hat{x} \in \mathfrak{X}^{**} | x \in \mathfrak{X}\}$. Then $\hat{\mathfrak{X}}$ is a subspace of $\mathfrak{X}^{**}$. Since we've shown $x \mapsto \hat{x}$ is a linear isometry, we can identify $\hat{\mathfrak{X}}$ with $\mathfrak{X}$ so that $\mathfrak{X} \hookrightarrow \mathfrak{X}^{**}$. By definition, $\mathfrak{X}$ is a dense subspace of $\overline{\hat{\mathfrak{X}}}$ (the closure of $\mathfrak{X}$). Thus $\overline{\hat{\mathfrak{X}}}$ has to be a subset of $\mathfrak{X}^{**}$ as $\mathfrak{X}^{**}$ is complete. Call $\overline{\hat{\mathfrak{X}}}$ the completion of $\mathfrak{X}$.
3. $\mathfrak{X}$ is called **reflexive** if $\mathfrak{X} \to \mathfrak{X}^{**}$ defined by $x \mapsto \hat{x}$ is surjective as well. It is standard to identify $\hat{x}$ with x itself.

Definition. *(p 125) A* ***directed set*** *is a nonempty set A with a relation $\lesssim$ such that*

- $\alpha \lesssim \alpha$ *for all* $\alpha \in A$.
- $\alpha \lesssim \beta$ *and* $\beta \lesssim \gamma$ *implies* $\alpha \lesssim \gamma$.
- *If* $\alpha, \beta \in A$*, then there exists* $\gamma \in A$ *such that* $\alpha \lesssim \gamma$ *and* $\beta \lesssim \gamma$.

An element of A *is called an* ***index****.*

Examples.

- Any nonempty subset of $\mathbb{R}$ with the usual order relation is a directed set. In particular, $\mathbb{N}$ is a directed set.
- Let $\mathcal{B}$ be a neighborhood basis for a topology $\mathcal{T}$ on X, that is $\mathcal{B} \subseteq \mathcal{T}$ and for all $x \in X$ there exists $\mathcal{N} \in \mathcal{B}$ such that $x \in V$ for all $V \in \mathcal{N}$ and if $U \in \mathcal{T}$ and $x \in U$, then there exists $V \in \mathcal{N}$ such that $V \subseteq U$. Set $\mathcal{N}_x : \{U \in \mathcal{B} : x \in U\}$ with $x \in X$ fixed. If we say for $U, V \in \mathcal{N}_x$ that $U \lesssim V$ when $U \supseteq V$, then $\mathcal{N}_x$ is a directed set. (Note: If $U, V \in \mathcal{N}_x$, then $x \in U \cap V \in \mathcal{T}$ which implies there exists $W \in \mathcal{B}$ with $x \in W$ such that $W \subseteq U \cap V$. So $W \gtrsim U$ and $W \gtrsim V$.)

Definition. *Let* V *be a set. A* ***net*** *in* X *is a function from a directed set* A *into* X*. We denote the mapping* $\alpha \mapsto x_\alpha$ *by* $< x_\alpha >_{\alpha \in A}$ *. The set* A *is called the* ***index set****.*

Definition. *Let* $(X, \mathcal{T})$ *be a topological space and* $E \subseteq X$*. Let* $< x_\alpha >_{\alpha \in A}$ *be a net. Then*

- $< x_\alpha >_{\alpha \in A}$ *is* ***eventually in*** E *if there exists* α_0 *such that* $x_\alpha \in E$ *for all* $\alpha \gtrsim \alpha_0$.
- $< x_\alpha >_{\alpha \in A}$ *is* ***frequently in*** E *if for every* $\alpha \in A$*, there exists* $\beta \in A$ *such that* $\beta \gtrsim \alpha$ *and* $x_\beta \in E$.
- $< x_\alpha >_{\alpha \in A}$ *converges to a point* $x \in X$ *(that is,* $x_\alpha \to x$*) if for all neighborhoods* U *of* x*,* $< x_\alpha >_{\alpha \in A}$ *is eventually in* U.
- *A point* $x \in X$ *is a* ***cluster point*** *of* $< x_\alpha >_{\alpha \in A}$ *if for all neighborhoods* U *of* x $< x_\alpha >_{\alpha \in A}$ *is frequently in* U.

Examples.

- Let $\mathcal{B}$ be a neighborhood basis for $\mathcal{T}$ and set $\mathcal{N}_x = \{U \in \mathcal{B} : x \in U\}$. Suppose $< x_U >_{U \in \mathcal{N}_x} \subseteq X$ satisfies $x_U \in U$ for all $U \in \mathcal{N}_x$. Then $x_U \to x$.
- Let $f \in L^1([a,b]) \cap L^+$. Let S be the set of all nonnegative simple functions on $[a,b]$ that are dominated by f. Order S with the usual ordering of functions (that is, $\phi_1 \lesssim \phi_2$ if $\phi_1(x) \leq \phi_2(x)$ for all $x \in [a,b]$). So S is a directed set. For $\phi \in S$, put $y_\phi = \int_{[a,b]} \phi dx$. Then $< y_\phi >_{\phi \in S}$ is a net and $y_\phi \mapsto \int_{[a,b]} f dx$.

Definition. *A* ***subnet*** *of a net* $< x_\alpha >_{\alpha \in A}$ *is a net* $< y_\beta >_{\beta \in B}$ *together with a map* $\beta \mapsto \alpha_\beta$ *from* B *into* A *such that*

- *For all* $\alpha_0 \in A$ *there exists* $\beta_0 \in B$ *such that* $\alpha_\beta \gtrsim \alpha_0$ *whenever* $\beta \gtrsim \beta_0$.
- $y_\beta = x_{\alpha_\beta}$.

Note. The map $\beta \mapsto \alpha_\beta$ need not be injective.

Proposition 45 (4.18)**.** *If* $(X, \mathcal{T})$ *is a topological space and* $E \subseteq X$*, then* $x \in X$ *is an* ***accumulation point*** *of* E *if and only if there exists a net* $< x_\alpha >_{\alpha \in A} \subseteq E \setminus \{x\}$ *that converges to* x *and* $x \in \overline{E}$ *if and only there exists* $< x_\alpha >_{\alpha \in A} \subseteq E$ *that converges to* x*.*

Proposition 46 (4.19)**.** *If* $(X, \mathcal{T})$ *and* $(Y, \mathcal{S})$ *are topological spaces and* $f : X \to Y$*, then* f *is continuous at* $x \in X$ *if and only if for all nets* $< x_\alpha >_{\alpha \in A}$ *converging to* x*, we have* $< f(x_\alpha) >_{\alpha \in A} \to f(x)$*.*

Proof. (Sketch)

Claim: Let $\mathcal{N} := \{\cap_{f \in \mathcal{D}} f^{-1}(W_f) : W_f$ is open in $Y_f, \mathcal{D}$ is a finite subset of $\mathcal{F}\}$. Then the topology induced by $\mathcal{N}$ is the weak topology and $\mathcal{N}$ is a neighborhood basis.

Proof: Suppose $< f(x_\alpha) >$ converges to $f(x)$. Let $\mathcal{O} \in \mathcal{T}$ be an open set such that $x \in \mathcal{O}$. There exists a finite collection $f_n \in \mathcal{F}$ and $W_{f_j} \in \mathcal{T}_f$ and $f(x) \in W_f$ such that $\cap_{j=1}^n f_j^{-1}(W_{f_j}) \subseteq \mathcal{O}$. Thus $x \in \cap_{j=1}^n f_j^{-1}(W_{f_j})$. Since $f_j(x_\alpha) \to f(x)$ for all $j = 1, ..., n$ there exists α_j such that $\alpha \gtrsim \alpha_j$ and $f_j(x_\alpha) \in W_{f_j}$. Note the α's are in a directed set. So there exists $\alpha_0 \gtrsim \alpha_j$ for all $j = 1, ..., n$. Thus for all $\alpha \gtrsim \alpha_0$, we have $f_j(x_\alpha) \in W_{f_j}$ which implies $x_\alpha \in f_j^{-1}(W_{f_j})$. Thus $x_\alpha \in \cap_{j=1}^n f_j^{-1}(W_{f_j}) \subseteq \mathcal{O}$. Since $\mathcal{O}$ was arbitrary, done. □

Definition. *Let X be a set and $\{(Y_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in A}$ a family of topological spaces. Given a family $\{f_\alpha : X \to Y_\alpha\}_{\alpha\in A}$, there exists a unique smallest topology that makes each f_α continuous. Call this the **weak topology** generated by $\{f_\alpha\}_{\alpha\in A}$.*

Proposition 47. *Let $(X, \mathcal{T})$ be a topological space and $< x_\alpha >_{\alpha\in A} \subseteq X$ a net.*

1. *If $\mathcal{T}$ is induced by a metric ρ on X, then $< x_\alpha > \to x \in X$ if and only if $< \rho(x_\alpha, x) > \to 0 \in \mathbb{R}$.*
2. *If the topological space is the weak topology generated by a family of functions $\mathcal{F} \subseteq \{f : X \to (Y, \rho)\}$, then $< x_\alpha > \to x \in X$ if and only if $< f(x_\alpha) > \to f(x) \in Y$ for all $f \in \mathcal{F}$.*

Definition. *Let $(\mathfrak{X}, ||\cdot||)$ be a normed vector space. The **weak topology** on $\mathfrak{X}$ is the weak topology generated by $\mathfrak{X}^*$. Convergence in this topology is called **weak convergence**.*

Remark. If $< x_\alpha >_{\alpha\in A} \subset \mathfrak{X}$ is a net, then we say $x_\alpha \to x$ strongly if and only $||x_\alpha - x|| \to 0$ and we say $x_\alpha \to x$ weakly (denoted $x_\alpha \rightharpoonup x$) if and only if $f(x_\alpha) \to f(x)$ for all $f \in \mathfrak{X}^*$.

Recall. $\mathfrak{X} \subseteq \mathfrak{X}^{**}$, so each $x \in \mathfrak{X}$ is a linear functional on $\mathfrak{X}^*$.

Definition. *Let $(\mathfrak{X}, ||\cdot||)$ be a normed vector space. The **weak* topology** on $\mathfrak{X}^*$ is the weak topology generated by $\mathfrak{X}$. Convergence in this topology is called **weak* convergence**.*

Remark. If $< f_\alpha >_{\alpha\in A} \subseteq \mathfrak{X}^*$ is a net, then we say $f_\alpha \to f$ strongly if and only if $||f_\alpha - f||_{\mathfrak{X}^*} \to 0$ and we say $f_\alpha \to f$ weakly (denoted $f_\alpha \to^* f$) if and only if $f_\alpha(x) \to f(x)$ for all $x \in \mathfrak{X}$ (note that this is just like pointwise convergence).

Theorem (Alaoglu's Theorem). *If $(\mathfrak{X}, ||\cdot||)$ is a normed vector space, then the closed unit ball $\overline{B}^* := \{f \in \mathfrak{X}^* : ||f|| \le 1\} \subset \mathfrak{X}^*$ is compact in the weak* topology.*

Definition. *Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in A}$ be a family of topological spaces. The **product topology** on $X = \prod_{\alpha\in A} X_\alpha$ is the weak topology generated by the coordinate maps $\{\pi_\alpha : X \to X_\alpha\}$.*

Definition. *A topological space $(X, \mathcal{T})$ is **compact** if whenever $\{U_\alpha\}_{\alpha\in A}$ is an open cover of X there exists a finite subset $B \subseteq A$ such that $X = \cup_{\alpha\in B} U_\alpha$. A subset $K \subseteq X$ is called compact if it is compact with respect to the relative topology on K.*

Theorem (Tychonoff's Theorem. 4.42). *If $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in A}$ is a family of compact topological spaces, then $X = \prod_{\alpha\in A} X_\alpha$ is compact with respect to the product topology.*

Corollary 16. *Suppose X has the weak topology generated by a family of functions $\mathcal{F}$ and that the following hold:*

1. *$\overline{f(X)}$ is compact for all $f \in \mathcal{F}$*
2. *If $x \neq y$, then there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.*
3. *If $< f(x_\alpha) >_{\alpha\in A}$ is a convergent net for all $f \in \mathcal{F}$, then there exists $x \in X$ such that $f(x_\alpha) \to f(x)$ for all $f \in \mathcal{F}$.*

Then X is compact.

Proof. Each $f \in \mathcal{F}$ is a map into some topological space $(Y_f, \mathcal{T}_f)$. Condition 1 states that $\overline{f(X)}$ is a compact subset of Y_f for each $f \in \mathcal{F}$. Tychonoff's Theorem shows that $Z = \prod_{f\in\mathcal{F}} \overline{f(X)}$ is compact with respect to the product topology. Suppose there exists a map $h : X \to Z$ such that

- $h(X)$ is closed

- h is a continuous bijection from X to $h(X)$.

- h^{-1} is continuous on $h(X)$.

(So h is a homeomorphism from X to $h(X)$). Then if $\{U_\alpha\}_{\alpha\in A}$ is an open cover of X, then we have $\{h(U_\alpha)\}_{\alpha\in A}$ is an open cover of $h(X)$. Since $h(X)$ is a closed subset of the compact set Z, it is compact. So there exists a finite subcover $\{h(U_\alpha)\}_{\alpha\in B}$ for $h(X)$. Thus $\{U_\alpha\}_{\alpha\in B}$ is a finite subcover of X as $X = h^{-1}(h(X)) \subseteq h^{-1}(\cup_{\alpha\in B} h(U_\alpha)) = \cup_{\alpha\in B} U_\alpha$. Thus X is compact.

Define $h : X \to Z$ by $h_f(x) = f(x)$ (the f component of h) for all $f \in \mathcal{F}, x \in X$. To show $h(X)$ is closed, let $< y_\alpha >_{\alpha\in A} \subseteq h(X)$ be a convergent net. Then there exists a net $< x_\alpha >_{\alpha\in A} \subseteq X$ such that $h(x_\alpha) = y_\alpha$ for all $\alpha \in A$. Now $< f(x_\alpha) >=< h_f(x_\alpha) >$ is a convergent net (otherwise $< y_\alpha >$ does not converge). By condition 3, there exists $x \in X$ such that $f(x_\alpha) \to f(x)$ for all $f \in \mathcal{F}$. But this implies $h(x_\alpha) \to h(x)$ and thus $y_\alpha \to h(x) \in h(X)$. By Proposition 45 (4.18), $h(X)$ is closed. We see that h is continuous, since each f is continuous and each coordinate map $\pi_f : Z \to \overline{f(X)}$ is continuous. Now $h : X \to h(X)$ is surjective. So we need only show it is 1-1. By the contrapositive of condition 2, this is clear. Thus h^{-1} exists. To show it is continuous on $h(X)$, we will show for a convergent net $< y_\alpha >_{\alpha\in A}$, that $h^{-1}(y_\alpha) \to h^{-1}(y)$, where $y_\alpha \to y$. Let $x_\alpha = h^{-1}(y_\alpha)$. So $< h(x_\alpha) >=< y_\alpha >$ is convergent. By condition 3, there exists $x \in X$ such that $h(x_\alpha) \to h(x)$. Then $h(x) = y$ since $h(x_\alpha) = y_\alpha \to y$. Thus $h^{-1}(y) = x$. By Proposition 47b, $< x_\alpha >_{\alpha\in A}$ is convergent in X if and only if $< h(x_\alpha) >_{\alpha\in A}$ is convergent in $h(X)$. Since $h(x_\alpha) \to h(x)$, we see $x_\alpha \to x$ by the bijectivity of h. Thus $h^{-1}(y_\alpha) = x_\alpha \to x = h^{-1}(y)$. Thus h^{-1} is continuous. □

Proof. (Of Alaoglu's Theorem) We want to use Corollary 16 with $X = \overline{B}^*$ and the relative weak* topology. Also, $\mathfrak{X}$ is the family $\mathcal{F}$ in the corollary (so we replace the x's with $f \in \mathfrak{X}^*$ and the f's with $x \in \mathfrak{X}$). Now, we just need to verify the three conditions of the corollary hold.

Condition 1: Observe for $f \in \overline{B}^*$ and $x \in \mathfrak{X}$ that $|x(f)| = |f(x)| \le ||f||_{\mathfrak{X}^*}||x||_{\mathfrak{X}} \le ||x||_{\mathfrak{X}}$. Thus $x(\overline{B}^*) \subseteq [-||x||_{\mathfrak{X}}, ||x||_{\mathfrak{X}}]$ which implies $\overline{x(\overline{B}^*)}$ is compact.

Condition 2: Clearly, if $x(f) = x(g)$ for all $x \in \mathfrak{X}$, then $f = g$.

Condition 3: Let $< f_\alpha >_{\alpha\in A} \subseteq \overline{B}^*$ be given and suppose $< x(f_\alpha) >$ is convergent for all $x \in \mathfrak{X}$. Then for all $x \in \mathfrak{X}$, there exists $\ell(x)$ such that $f_\alpha(x) \to \ell(x)$. We need to show $\ell \in \overline{B}^*$. Let $\beta, \gamma \in \mathbb{R}, x, y \in \mathfrak{X}$. Then $f_\alpha(\beta x + \gamma y) = \beta g_\alpha(x) + \gamma f_\alpha(y) \to \beta\ell(x) + \gamma\ell(y)$ and $f_\alpha(\beta x + \gamma y) \to \ell(\beta x + \gamma y)$. Thus $\beta\ell(x) + \gamma\ell(y) = \ell(\beta x + \gamma y)$. Thus ℓ is linear. Now, we show $||\ell||_{\mathfrak{X}^*} \le 1$. We see $|f_\alpha(x)| \le ||f_\alpha||_{\mathfrak{X}^*}||x||_{\mathfrak{X}} \le ||x||_{\mathfrak{X}}$. Thus $|f_\alpha(x)| \to |\ell(x)| \le ||x||_{\mathfrak{X}}$ for all $x \in \mathfrak{X}$ and so $||\ell||_{\mathfrak{X}^*} \le 1$. Therefore $\ell \in \overline{B}^*$. Hence $x(f_\alpha) = f_\alpha(x) \to \ell(x) = x(\ell)$ for all $x \in X$.

Thus, by Corollary 16, $\overline{B}^*$ is compact. □

Corollary 17. *Suppose $< f_\alpha > \subseteq \mathfrak{X}^*$ is a net such that $\sup ||f||_{\mathfrak{X}^*} \le M$ for some $M < \infty$. Then there exists a subnet $< g_\beta >$ of $< f_\alpha >$ that is weak* convergent.*

Proof. The net $< \frac{1}{M} f_\alpha > \subseteq \overline{B}^*$, which is compact by Alaoglu's Theorem. It follows that there exists a subnet $< \frac{1}{M} g_\beta >$ of $< \frac{1}{M} f_\alpha >$ converging weak* in $\overline{B}^*$. □

Application (Sect 6.2) Since $L^p(\mu)$ is a Banach Space, Alaoglu's Theorem implies the closed unit ball in $[L^p]^*$ is weak* compact. In particular, bounded nets in $[L^p]^*$ have weak* convergent subnets. What is $[L^p]^*$?

- Suppose $p \ge 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ (if $p = 1$, take $q = \infty$). For each $g \in L^q$, define $\phi_g : L^p(\mu) \to \mathbb{R}$ by $\phi_g(f) = \int_X gf\, d\mu$. This is a linear functional. To show $\phi_g \in [L^p]^*$, use Hölder's Inequality:

$$\begin{aligned} ||\phi_g||_{[L_p]^*} &= \sup\{|\textstyle\int_X fg\, d\mu| : f \in L^p, ||f||_p = 1\} \\ &\le \sup\{\textstyle\int_x |fg| d\mu : f \in L^p, ||f||_p = 1\} \\ &\le \sup\{||g||_q ||f||_p : f \in L^p, ||f||_p = 1\} \\ &= ||g||_q < \infty. \end{aligned}$$

Proposition 48. *Suppose $\frac{1}{p}+\frac{1}{q}=1$ and $1<p\leq\infty$. If $g\in L^q$, then $||g||_{L^q}=||\phi_g||_{[L^p]^*}=\sup\{|\int_X fgd\mu|:||f||_p=1\}$. If μ is semifinite, then we can allow for $p=1$.*

Recall. μ is semifinite if for all E with $\mu(E)=\infty$, there exists $F\subseteq E$ such that $0<\mu(F)<\infty$.

Proof. If $g=0$, done. By Hölder's Inequality, $||\phi_g||_{[L^p]^*}\leq||g||_{L^q}$. Define $sgn(g(x))=\begin{cases}\frac{g(x)}{|g(x)|} & \text{if } g(x)\neq 0.\\ 0 & \text{if } g(x)=0.\end{cases}$ If $p>1$, define $f:X\to\mathbb{R}$ by $f(x)=\frac{|g(x)|^{q-1}sgn(g(x))}{||g||_q^{q-1}}$. So $||f||_p^p=\frac{1}{||g||_q^{p(q-1)}}\int_X|g(x)|^{p(q-1)}d\mu=\frac{1}{||g||_q^q}\int_X|g(x)|^qd\mu=1$. Thus $||\phi_g||_{[L^p]^*}\geq|\int_X fgd\mu|=\left|\int_X f(x)|g(x)|sgn(g(x))d\mu\right|=\left|\int_X\frac{|g(x)|^q}{||g||_q^{q-1}}d\mu\right|=||g||_{L^q}$. Thus $||\phi_g||_{[L^p]^*}=||g||_{L^q}$. If $p=1$, we assume μ is semifinite. Let $\epsilon>0$ and set $E:=\{x\in X:|g(x)|\geq||g||_\infty-\epsilon\}$. Since μ is semifinite, there exists $F\subseteq E$ such that $0<m(F)<\infty$ (note $\mu(E)>0$). Define $f:X\to\mathbb{R}$ by $f(x)=\frac{1}{\mu(F)}\chi_F(x)sgn(g(x))$. Thus $||f||_1=\frac{1}{\mu(F)}\int_X\chi_Fd\mu=1$ and $||\phi_g||_{[L^1]^*}\geq|\int_X fgd\mu|=\frac{1}{\mu(F)}\int_F|g|d\mu\geq\frac{1}{\mu(F)}(||g||_\infty-\epsilon)\mu(F)=||g||_\infty-\epsilon$. Since $\epsilon>0$ was arbitrary, $||\phi_g||_{[L^1]^*}\geq||g||_\infty$. So $||\phi_g||_{[L^p]^*}=||g||_{L^q}$. □

Theorem 52 (6.14)**.** *Suppose $\frac{1}{p}+\frac{1}{q}=1$ and $g:X\to\mathbb{R}$ is measurable. If*

1. *$fg\in L^1(\mu)$ for all $f\in\Sigma:=\{$simple functions in $L^1(\mu)$ which are 0 outside a set of finite measure$\}$.*
2. *$M_q(g)=\sup\{|\int_X fgd\mu|:f\in\Sigma,||f||_p\leq1\}<\infty$.*
3. *Either $S_g=\{x\in X:g(x)\neq0\}$ is $\sigma-$finite or μ is semifinite.*

Then $g\in L^q$ and $||g||_{L^q}=M_q(g)$.

Proof. Set $\overline{\Sigma}:=\{f:X\to\mathbb{R}:f$ is bounded, measurable, $f=0$ outside a set of finite measure$\}$.

Claim: $\sup\{|\int_X fgd\mu|:f\in\overline{\Sigma},||f||_p\leq1\}\leq M_q(g)$.

Proof: Let $f\in\overline{\Sigma}$ such that $||f||_{L^p}\leq1$. Select $\{\phi_n\}_{n=1}^\infty$ to be simple functions such that $0\leq|\phi_1|\leq|\phi_2|\leq\cdots\leq|f|$ and $\phi_n\to f$ a.e. Observe each $\phi_n\in\Sigma$ and $||\phi_n||_p\leq||f||_p\leq1$. Suppose $f=0$ for $x\in X\setminus E$ with $\mu(E)<\infty$. Then $|\phi_ng|\leq|fg|=||f||_\infty|g\chi_E|\in L^1$ by (1) since χ_E is a simple function. Also $\phi_ng\to fg$ a.e. So by the LDC Theorem,

$$\Big|\int_X fgd\mu\Big|=\Big|\int\lim\phi_ngd\mu\Big|=\lim\Big|\int\phi_ngd\mu\Big|\leq M_q(g).$$

Now suppose $1\leq q\leq\infty$. Note that S_g is $\sigma-$finite (exercise). Let $\{E_n\}_{n=1}^\infty\subseteq\mathcal{M}$ satisfy $E_1\subset E_2\subset\cdots$ with $S_g=\cup_{n=1}^\infty E_n$ and $\mu(E_n)<\infty$ for all $n\in\mathbb{N}$. Let $\{\phi_n\}_{n=1}^\infty$ be a sequence of simple functions such that $0\leq|\phi_1|\leq|\phi_2|\leq\cdots\leq|g|$ and $\phi_n\to g$ a.e. Put $g_n=\phi_n\chi_{E_n}$ so $0\leq|g_1|\leq|g_2|\leq\cdots\leq|g|$ and $g_n\to g$ with $g_n=0$ for all $x\in X\setminus E_n$. Define $f_n(x):=\frac{|g_n|^{q-1}sgn(g(x))}{||g_n||_q^{q-1}}$. So $||f_n||_p=1$ as in the previous proof and $f_n\in\overline{\Sigma}$. By Fatou's Lemma and the first claim,

$$\begin{aligned}||g||_q\leq\liminf||g_n||_q &= \liminf\tfrac{1}{||g_n||_q^{q-1}}\int_X|g_n|^qd\mu\\ &= \liminf\int_X|f_n(x)||g_n(x)|d\mu\\ &\leq \liminf\int_X|f_n||g|d\mu\\ &= \liminf\int_X f_ngd\mu\leq M_q(g)<\infty.\end{aligned}$$

By Hölder's Inequality, $||g||_q\geq M_q(g)$. Thus $||g||_q=M_q(g)$.

Now, suppose $q=\infty$ and let $\epsilon>0$. Set $A=\{x\in X:|g(x)|\geq M_\infty(g)+\infty\}$. If $\mu(A)>0$, select $B\subseteq A$ such that $0<\mu(B)<\infty$ (by 3) and put $f=\frac{1}{\mu(B)}\chi_Bsgn(g(x))$. So $f\in\Sigma$ and $||f||_{L^1}=\int_X|f|d\mu=1$. Thus $M_\infty(g)\geq|\int fgd\mu|=\frac{1}{\mu(B)}\int_B|g(x)|d\mu\geq\frac{1}{\mu(B)}(M_\infty(g)+\epsilon)\mu(B)=M_\infty(g)+\epsilon$, a contradiction. So $\mu(A)=0$. Hence $||g||_\infty\leq M_\infty(g)<\infty$. So $f\in L^\infty$. Also $M_\infty(g)=\sup\{|\int fgd\mu|\}\leq||g||_\infty\sup\{|\int fd\mu|\}\leq||g||_\infty$. Thus $||g||_\infty=M_\infty(g)$. □

Theorem (**Riesz Representation Theorem for L^p Spaces** (6.15))**.** *Suppose $\frac{1}{p}+\frac{1}{q}=1$ with $p\in(1,\infty)$. Then for all $\phi\in[L^p]^*$, there exists $g\in L^q$ such that $\phi(f)=\int_X fgd\mu$ for all $f\in L^p$. If μ is $\sigma-$finite, then we allow for $p=1$.*

Proof. First assume $\mu(X) < \infty$. So all simple functions belong to L^1. Let $\phi \in [L^p]^*$. Define $\nu : \mathcal{M} \to (-\infty, \infty)$ by $\nu(E) = \phi(\chi_E)$. Want to show ν is a finite signed measure. Note that $|\nu(E)| \leq |\phi(\chi_E)| \leq ||\phi||_{[L^p]^*}||\chi_E||_p \leq ||\phi||_{[L^p]^*}\mu(E)^{1/p} \leq ||\phi||_{[L^p]^*}\mu(X)^{1/p} < \infty$. Thus ν is uniformly bounded. To show it is countably additive, let $\{E_j\}_{j=1}^\infty \subseteq \mathcal{M}$ be mutually disjoint. So if $E = \cup_{j=1}^\infty E_j$, then $\chi_E = \sum \chi_{E_j}$. Also, $\sum_{j=1}^\infty \mu(E_j) \leq \mu(X) < \infty$ which implies $\sum_{j=1}^\infty \mu(E_j)$ is absolutely convergent. Since $p < \infty$ $||\chi_E - \sum_{j=1}^N \chi_{E_j}||_p = ||\sum_{j=N+1}^\infty \chi_{E_j}||_p = (\mu(\cup_{j=N+1}^\infty E_j))^{1/p} = (\sum_{j=N+1}^\infty \mu(E_j))^{1/p} \to 0$ as $N \to \infty$ as $\mu(E_j$ is absolutely convergent. Since ϕ is continuous and linear in L^p and $\sum_{j=1}^N \chi_{E_j} \to \chi_E$ in L^p, we have $\nu(E) = \phi(\chi_E) = \phi(\sum_1^\infty \chi_{E_j}) = \lim_{N\to\infty} \phi(\sum_1^N \chi_{E_j}) = \lim_{N\to\infty} \sum_1^N \phi(\chi_{E_j}) = \sum_{j=1}^\infty \nu(E_j)$. So ν is countably additive. Moreover, if $\mu(E) = 0$, then for $F \in \mathcal{M}$ such that $F \subseteq E$, we have $||\chi_F||_p = 0$. So $\phi(\chi_F) = 0$ as ϕ is continuous which implies $\phi(0) = 0$. Thus $\nu(F) = 0$ and thus $\nu << \mu$. By the Radon Nikodym Theorem, there exists $g \in L^1(\mu)$ such that $\phi(\chi_E) = \nu(E) = \int_E g d\mu$. If f is a simple function, by linearity of the integral and ϕ we have $\phi(f) = \int fgd\mu$. Since $|\int_X fgd\mu| = |\phi(f)| \leq ||\phi||_{[L_p]^*}||f||_p$. By Theorem 51, we see $g \in L^q$. Need to show $\phi(f) = \int fgd\mu$ for all $f \in L^p$. By Proposition 19, there exists a sequence of simple functions $\{f_n\}_{n=1}^\infty \subseteq L^p$ such that $f_n \to f$ in L^p, that is $||f_n - f||_p \to 0$. By continuity of ϕ, we see

$$\phi(f) = \lim_{n\to\infty} \phi(f_n) = \lim_{n\to\infty} \int f_n g d\mu = \lim_{n\to\infty} \int (f_n - f) g d\mu + \int fgd\mu \leq \lim_{n\to\infty} ||f_n - f||_p ||g||_q + \int fgd\mu = \int fgd\mu.$$

(Note here that g is in fact unique a.e. by the Radon Nikodym Theorem).

Now we assume μ is a $\sigma-$finite measure. Select $\{E_n\}_{n=1}^\infty \subseteq \mathcal{M}$ such that $E_1 \subseteq E_2 \subseteq \cdots, \cup E_n = X, \mu(E_n) < \infty$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, find $g_n \in L^q(\mu, E_n)$ such that for all $f \in L^p(\mu, E_n)$ we have $\phi(f) = \int_{E_n} fgd\mu$ and $||g_n||_q = ||\phi||_{[L^p(E_n)]^*} \leq ||\phi||_{[L^p(X)]^*}$. We may assume for $m \leq n$ that $g_n|_{E_m} = g_m$ a.e. in E_m (by the uniqueness of the finite case.) Define g a.e. in X by $g(x) = g_n(x)$ if $x \in E_n$. Thus $g|_{E_n} = g_n$ a.e. in E_n. By the Monotone Convergence Theorem, $||g||_q^q = \int_X |g|^q d\mu = \int_X \lim_{n\to\infty} |g\chi_{E_n}|^q d\mu = \lim_{n\to\infty} \int_{E_n} |g_n|^q d\mu \leq ||\phi||_{[L^p(X)]^*}^q < \infty$. Thus $g \in L^q$. For a given $f \in L^p(X)$, we have

- $gf\chi_{E_n} \to gf$ a.e. in X as $n \to \infty$.
- $|gf\chi_{E_n}(x)| \leq \underbrace{|g(x)||f(x)|}_{L^1(x)}$ for a.e. $x \in X$.
- $f\chi_{E_n} - f \to 0$ for a.e. $x \in X$.
- $|f\chi_{E_n} - f|^p \leq 2|f(X)|^p$ for a.e. $x \in X$ (as $|f\chi_{E_n}(x)|^p \leq |f(x)|^p$ for a.e. $x \in X$).

Thus by the Lebesgue Dominated Convergence Theorem and the last two observations, $||f\chi_{E_n} - f||_{L^p} \to 0$ as $n \to \infty$, that is, $f\chi_{E_n} \to f$ in L^p. By the Lebesgue Dominated Convergence Theorem and the continuity of ϕ, we have $\phi(f) = \phi(\lim f\chi_{E_n}) = \lim \phi(f\chi_{E_n}) = \lim \int gf\chi_{E_n} d\mu = \int \lim gf\chi_{E_n} d\mu = \int gfd\mu$. Thus the Representation Theorem holds when μ is $\sigma-$finite and $p \in [1, \infty)$.

Finally, assume $p \in (1, \infty)$ and μ is an arbitrary measure. For any $\sigma-$finite set $E \subseteq X$, there exists an a.e. unique $g_E \in L^q(E)$ such that $\phi(f) = \int_E fgd\mu$ for all $f \in L^p(E)$. If $E \subseteq F$ and F is $\sigma-$finite, then $g_F|_E = g_E$ for a.e. $x \in E$ and $||g_E||_q \leq ||g_F||_q \leq ||\phi||_{[L^p(x)]^*}$. Put $M = \sup\{||g_E||_q : E \text{ is } \sigma - \text{finite}, \phi(f) = \int_E fg_E d\mu \text{ for all } f \in L^p(E)\}$. So $M \leq ||\phi||_{[L^p(x)]^*} < \infty$. Let $\{E_n\}_{n=1}^\infty \subseteq M$ be $\sigma-$finite such that $||g_{E_n}||_q \to M$. Then $F = \cup_{n=1}^\infty E_n$ is $\sigma-$finite (as it is a countable union of $\sigma-$finite sets) and $||g_{E_n}||_q \leq ||g_F||_q$ for all $n \in \mathbb{N}$. It follows that $M = ||g_F||_q$. For any $\sigma-$finite set A such that $F \subseteq A$, we have $\int_F |g_F|^q + \int_{A\setminus F} |g_{A\setminus F}|^q = \int_F |g_A|^q + \int_{A\setminus F} |g_A|^q = \int_A |g_A|^q d\mu \leq M_q = \int_F |g_F|^1$. Thus $g_{A\setminus F} = 0$ a.e. Notice if $f \in L^p(X)$, then $\mu(\{x \in X : |f(x)| > \frac{1}{j}\}) = j^p(\frac{1}{j})^p \int_{\{|f(x)| > \frac{1}{j}\}} d\mu \leq j^p \int_X |f(x)|^p d\mu < \infty$. Thus $\{x \in X : f(x) \neq 0\} = \cup_{j=1}^\infty \{x \in X : f(x) > \frac{1}{j}\}$ is $\sigma-$finite and so is $A = F \cup \{x \in X : f(x) \neq 0\}$. Thus $g_F|_{A\setminus F} = 0$ a.e. and $f|_{X\setminus A} = 0$. We have $\phi(f) = \phi(f\chi_A + f\chi_{X\setminus A}) = \phi(f\chi_A) + \phi(f\chi_{X\setminus A})$. Now $f\chi_{X\setminus A} = 0$ implies $f\chi_{X\setminus A} = 0$ in L^p and thus $\phi(f\chi_{X\setminus A}) = 0$. Thus $\phi(f) = \phi(f\chi_A) = \int_A fg_A d\mu = \int_F fg_F d\mu$ as $g_A = 0$ on $A \setminus F$. Take $g(x) = \begin{cases} g_F(x) & \text{if } x \in F \\ 0 & \text{otherwise.} \end{cases}$ □

Remark. By Proposition 47 (6.13), we see that L^q is isometrically isomorphic to $[L^p]^*$. Functionals in $[L^p]^*$ are usually just identified with functions in L^q (for $1 \leq p < \infty$).

Corollary 18. *If $p \in (1,\infty)$, then L^p is reflexive.*

Corollary 19. *If μ is $\sigma-$finite and $< f_\alpha >_{\alpha \in A}$ is a bounded net in L^∞, then there exists a subnet $< g_\beta >_{\beta \in B}$ and a function $g \in L^\infty$ such that $g_\beta \to^* g$ in L^∞.*

Corollary 20. *If $p \in (1,\infty)$ and $< f_\alpha >_{\alpha \in A}$ is a bounded net in L^p, then there exists a subnet $< g_\beta >_{\beta \in B}$ and a function $g \in L^p$ such that $g_\beta \rightharpoonup g$ in L^p.*

Example. (Fourier Series) Let $1 < p < \infty$. Consider the space $L^P([0,2\pi])$. For each $k \in \mathbb{Z}$, put $e_k(x) = \frac{1}{\sqrt{2\pi}} e^{-ikx}$. Then $e_k \in L^\infty$ for all $k \in \mathbb{Q}$ which implies $e_k \in L^q$ for all $q \in [1,\infty]$. For each $f \in L^p$ and $k \in \mathbb{Z}$, put $\hat{f}(x) := \frac{1}{\sqrt{2\pi}} \int_{[0,2\pi]} f(x) e_k(x) dx = \frac{1}{2\pi} \int_{[0,2\pi]} f(x) e^{-ikx} dx$. Here, $\hat{f}(k)$ is the k^{th} fourier coefficient for f. The series $\sum_{k=-n}^{n} \hat{f}(k) e^{ikx}$ is the n^{th} partial sum of the Fourier Series for f. For each $x \in [0,2\pi]$, define $\phi_x : L^p \to \mathbb{C}$ by $\phi_x(f) = \sum_{k=-m}^{n} \hat{f}(k) e^{ikx}$. From the definition of $\hat{f}(x)$, we see ϕ_x is linear and

$$\begin{aligned} |\phi_x(f)| &\leq \textstyle\sum_{k=-n}^{n} |\hat{f}(k)||e^{ikx}| \text{ note } |e^{ikx}| = 1 \\ &= \textstyle\sum_{k=-n}^{n} |\hat{f}(k)| = \sum_{k=-n}^{n} \frac{1}{2\pi} |\int_{[0,2\pi]} f(x) e^{-kx} dx| \\ &\leq \textstyle\sum_{k=-n}^{n} \frac{1}{2\pi} ||f||_{L^P} \left(\int_{[0,2\pi]} |e^{-ikx}|^{p/p-1} \right)^{p-1/p} \text{ By Holder} \\ &= \textstyle\sum_{k=-n}^{n} \frac{||f||_{L^P}}{(2\pi)^{1/p}} = \frac{2n+1}{(2\pi)^{1/p}} ||f||_p. \end{aligned}$$

Hence $||\phi_x||_{[L^P]^*} < \infty$ and so $\phi_x \in [L^P]^*$. By the Riesz Representation Theorem, there exists $g_x \in L^{p/p-1}([0,2\pi])$ such that $\phi_x(f) = \int_{[0,2\pi]} f(y) g_x(y) dy$. What is g_x? Note

$$\phi_x(f) = \sum_{k=-n}^{n} \frac{1}{2\pi} \int f(y) e^{-iky} e^{ikx} dy = \int_{[0,2\pi]} f(y) \underbrace{\sum_{k=-n}^{n} \frac{1}{2\pi} e^{ik(x-y)}}_{:= g_x(y)} dy.$$

Using trig,

$$g_x(y) = k_n(x-y) = \begin{cases} \frac{\sin(\frac{n+1}{2}(x-y))}{2\pi \sin(\frac{x-y}{2})} & \text{if } x \neq y \\ \frac{2n+1}{2\pi} & \text{if } x = y. \end{cases}$$

Thus $\phi_x(f) = \int_{[0,2\pi]} f(y) k_n(x-y) dy$.

3.2 Dual Spaces for Spaces of Continuous Functions (Ch 7)

Definition. *If $(X, \mathcal{T})$ is a topological space and $f : X \to \mathbb{R}$, then the **support** of f is $supp(f) := \overline{\{x \in X : f(x) \neq 0\}}$.*

Notation. Let $\mathcal{C}(X)$ denote the vector space of all continuous functions from $(X, \mathcal{T}) \to \mathbb{R}$. Set $\mathcal{C}_c(X) := \{f \in \mathcal{C}(X) : supp(f)$ is compact$\}$. Define $|| \cdot ||_{\mathcal{U}} : \mathcal{C}_c(X) \to (0,\infty)$ by $||f||_{\mathcal{U}} := \sup\{|f(x)| : x \in X\}$ (the **uniform norm**). We will assume $\mathcal{C}_c(X)$ is endowed with $|| \cdot ||_{\mathcal{U}}$, making it a normed vector space. Note that, in general, it is not complete.

Definition. *If $I : \mathcal{C}_c(X) \to \mathbb{R}$, then I is a **positive linear function** if I is linear and $I(f) \geq 0$ whenever $f \geq 0$.*

Definition. *If μ is a Borel measure on X and $E \in \mathcal{B}_X$, then*

- *μ is **outerregular** on E if $\mu(E) = \inf\{\mu(U) : U$ is open, $E \subseteq U\}$.*
- *μ is **innerregular** on E if $\mu(E) = \sup\{\mu(K) : K$ is compact, $K \subseteq E\}$.*
- *μ is **regular** if it is both outer and inner regular on all Borel sets.*

Definition. *A Borel measure μ is called a **Radon measure** if*

- *$\mu(K) < \infty$ for all compact $K \subseteq X$.*

- *μ is outer regular on all Borel sets.*
- *μ is inner regular on all open sets.*

Remark. If $(X, \mathcal{T})$ is $\sigma-$compact and $\sigma-$finite, then μ is a Radon measure if and only if μ is regular.

Notation. If U is open and $f \in \mathcal{C}_c(X)$, then we write $f \prec U$ if $0 \leq f \leq \chi_U$ and $supp(f) \subseteq U$ and say f is **subordinate** to U.

Theorem (Riesz Representation Theorem for $\mathcal{C}_c(X)$). *If I is a positive linear function on $\mathcal{C}_c(X)$, then there exists a unique Radon measure μ on X such that $I(f) = \int_X f d\mu$ for all $f \in \mathcal{C}_c(X)$. Also*

- *For all open sets U, $\mu(U) = \sup\{I(f) : f \in \mathcal{C}_c(X), f \prec U\}$.*
- *For all compact K, $\mu(K) = \inf\{I(f) : f \in \mathcal{C}_c(X), f \geq \chi_K\}$.*

Facts. Suppose μ is a Radon measure on X.

- If $1 \leq p < \infty$, then $\mathcal{C}_c(X)$ is dense in $L^p(\mu)$.
- (Lusin's Theorem for Radon measures) If $f : X \to \mathbb{R}$ is measurable and 0 outside a set of finite measure, then for all $\epsilon > 0$, there exists $g \in \mathcal{C}_c(X)$ such that $\sup_{x\in X} |g(x)| \leq \sup_{x\in X} |f(x)|$ and $\mu(\{x \in X : f(x) \neq g(x)\}) < \epsilon$.

Definition. *A topological space $(X, \mathcal{T})$ is called a* ***locally compact Hausdorff space*** *if*

1. *it is* ***locally compact****, that is, for each $x \in X$, there exists $A \subseteq X$ such that x is in the interior of A and A is compact.*
2. *it is* ***Hausdorff****, that is, whenever $x, y \in X$ and $x \neq y$, we may find open $U, V \in \mathcal{T}$ such that $U \cap V = \emptyset$ and $x \in U, y \in V$.*

Let $(X, \mathcal{T})$ be a locally compact Hausdorff space. Let $\mathcal{C}_0(X)$ be the closure of $\mathcal{C}_c(X)$ with respect to the uniform metric. It can be shown that $f \in \mathcal{C}_0(X)$ if and only if $\{x \in X : |f(x)| > \epsilon\}$ is compact for all $\epsilon > 0$ (so f is "vanishing at ∞").

Definition. *We say that a signed Borel measure μ on $\mathcal{B}_X$ is a* ***signed Radon measure*** *if $|\mu|$ is a Radon measure.*

Notation. Let $M(X)$ be the vector space of all finite signed Radon measures. Define $||\cdot||_M : M(X) \to [0, \infty)$ by $||\mu||_M = |\mu|(X)$.

Riesz Representation Theorem for $\mathcal{C}_0(X)$. *Let $(X, \mathcal{T})$ be a locally compact Hausdorff space. For each $\mu \in M(X)$ and $f \in \mathcal{C}_0(X)$, define $I_\mu(f) = \int_X f d\mu$. Then the map $\mu \mapsto I_\mu$ is an isometric isometry from $\mu(X) \to \mathcal{C}_0(X)^*$.*

Corollary 21. *If $< \mu_\alpha >_{\alpha\in A} \subseteq M(X)$ is a bounded net, then there exists a subnet $< \nu_\beta >_{\beta\in B}$ of $< \mu_\alpha >$ and a Radon measure $\nu \in M(X)$ such that $\nu_\beta \rightharpoonup^* \nu$ in $M(X)$, that is, for all $f \in \mathcal{C}_0(X)$, we have $\int_X f d\nu_\beta \to \int_X f d\nu$.*

Corollary 22. *Let μ be a positive Radon measure and $< g_\alpha > \subseteq L^1(\mu)$ satisfies $\sup_{\alpha\in A} ||g_\alpha||_{L^1} < \infty$. Then there exists a subnet $< h_\beta >_{\beta\in B}$ of $< g_\alpha >$ and a Radon measure $\nu \in M(X)$ such that $h_\beta d\mu \rightharpoonup^* d\nu$ in $M(X)$.*

Proof. First, define $\mu_\alpha \in M(X)$ by $\mu_\alpha(E) = \int_E g_\alpha d\mu$. By the previous corollary, there exists a subnet $< \nu_\beta >$ such that $\nu_\beta \rightharpoonup^* \nu$. By definition of μ_α, we have $h_\beta d\mu \rightharpoonup^* d\nu$. □

Note. Although each $g_\alpha d\mu$ is absolutely continuous with respect to μ, this may not be true for $d\nu$.

Example. $X = [-1, 1]$ with the usual topology. Let μ be the Lebesgue measure on $[-1, 1]$ restricted to the Borel sets. Then μ is a Radon measure. Consider the sequence $g_n(x) = \begin{cases} 0 & \text{if } x \notin [-\frac{1}{n}, \frac{1}{n}] \\ \frac{n}{2} & \text{if } x \in (-\frac{1}{n}, \frac{1}{n}). \end{cases}$ Notice that $||g_n||_{L^1} = 1 < \infty$.

Claim: $g_n dx \rightharpoonup^* d\delta_0$ in $M(X)$.

Proof: Consider $\int_{[-1,1]} g_n(x)f(x)dx$ for $f \in \mathcal{C}_0([-1,1])$. We have

$$\frac{n}{2}\int_{(-\frac{1}{n},\frac{1}{n})} f(x)dx = \frac{n}{2}\int_{(-\frac{1}{n},\frac{1}{n})} [f(x)-f(0)]dx + f(0) \leq ||f(x)-f(0)||_{L^\infty(-\frac{1}{n},\frac{1}{n})} + f(0) \to f(0)$$

by continuity of f.

This implies $g_n dx \rightharpoonup^* d\delta_0$, which is not absolutely continuous with respect to m.

How do we fix this? If $\sup ||f_\alpha|| < \infty$ and for all $\epsilon > 0$ there exists $\delta > 0$ for all α such that $\int |f_\alpha| d\mu < \epsilon$ whenever $\mu(E) < \delta$, then there exists a subnet $< g_\beta > \rightharpoonup g$ in L^1. So if $h_\beta d\mu$ is "**uniformly absolutely continuous**", then we are good.

Note. The above example also works to show that weak convergence does not imply strong convergence (consider the sequence $g_n^{1/p}$).

3.3 Baire Category Theorem

Definition. *Let $(X,\mathcal{T})$ be a topological space. A set $E \subseteq X$ is of* ***first category*** *(meager) if E is the countable union of nowhere dense sets (in particular, it does not contain any open sets) and E is of* ***second category*** *if it is not meager.*

Example. $\mathbb{Q}$ is meager as it is a countable union of points.

Baire Category Theorem [p 161]. *Let X be a complete metric space.*

1. *If $\{U_n\}_{n=1}^\infty$ is a sequence of open dense subsets of X, then $\cap_{n=1}^\infty U_n$ is dense in X.*
2. *X is not the countable union of nowhere dense subsets of X.*

Proof. Since X is a metric space, if a set E is not dense in X, then there exists $x \in X$ such that $x \notin \overline{E}$, which implies there exists an open set $A \subseteq X$ such that $x \in A$ and $A \cap E = \emptyset$. To prove (1), it suffices to show that for any open set $W \subseteq X$ we have $W \cap (\cap_{j=1}^\infty U_j) \neq \emptyset$. Since each U_j is dense, we must have $A \cap U_j \neq \emptyset$ for all open sets A. In particular, $W \cap U_1 \neq \emptyset$. Thus there exists a ball $B(r_1,x_1) \subseteq W \cap U_1$. Now, $B(r_1,x_1)$ is open, so we may find $B(r_2,x_2) \subseteq B(\frac{1}{2}r_1,x_1) \cap U_2$ and note $\overline{B(r_2,x_2)} \subseteq B(r_1,x_1) \cap U_2$. Continuing inductively, we obtain $\{B(r_j,x_j)\}_{j=1}^\infty$ such that $B(r_j,x_j) \subseteq B(\frac{1}{2}r_{j-1},x_{j-1}) \cap U_j$ and $\overline{B(r_j,x_j)} \subseteq B(r_{j-1},x_{j-1}) \cap U_j$. Note $r_1 \geq 2r_2 \geq 2^2 r_3 \geq \cdots \geq 2^{j-1}r_j \geq \cdots$. Hence $r_j \to 0$. Also, if $m,n \geq N \in \mathbb{N}$, then $B(r_n,x_m) \leq 2^{1-N}r_1$. Thus $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence. As X is complete, there exists $x \in X$ such that $\rho(x_n,x) \to 0$. In particular, $x \in \overline{B}(r_N,x_N) \subseteq U_N \cap B(r_1,x_1) \subseteq U_N \cap W$ for all $N \in \mathbb{N}$. Thus $x \in (\cap_{j=1}^\infty U_j) \cap W$. Hence, $\cap_{j=1}^\infty U_j$ is dense in X. For (2), if $\{E_j\}_{j=1}^\infty$ are each nowhere dense sets, then $\{X \setminus \overline{E_j}\}_{j=1}^\infty$ would be a sequence of open dense sets. By (1), $\cap_{j=1}^\infty (X \setminus E_j)$ is dense and thus $\neq \emptyset$. Of course

$$\emptyset \neq \cap_{j=1}^\infty (X \setminus \overline{E_j}) = X \setminus (\cup_{j=1}^\infty \overline{E_j}) \subseteq X \setminus \cup_{j=1}^\infty E_j.$$

So $\cup_{j=1}^\infty E_j \neq X$. □

Consequences of the Baire Category Theorem

Open Mapping Theorem. *Let $\mathfrak{X},\mathfrak{Y}$ by Banach Spaces. If $T \in L(\mathfrak{X},\mathfrak{Y})$ is surjective, then T is an open mapping, that is $T(U)$ is open whenever U is open in $\mathfrak{X}$.*

Corollary 23. *If T is a bijection between $\mathfrak{X}$ and $\mathfrak{Y}$, then T and T^{-1} are continuous and so T is an isomorphism.*

Closed Graph Theorem. *If $T : \mathfrak{X} \to \mathfrak{Y}$ is a closed linear map of Banach Spaces, then T is bounded (and so continuous).*

Definition. *A linear map T is* ***closed*** *if the* ***graph of T***, *$\{(x,y) \in \mathfrak{X} \times \mathfrak{Y} : Tx = y\}$, is a closed subspace in $\mathfrak{X} \times \mathfrak{Y}$.*

Remark. To show $Tx_n \to Tx$ whenever $x_n \to x$, it is sufficient to show Tx_n converges to some y in the range of T.

Example. There exists a nowhere differentiable function on $[0,1]$.

Proof: Endow $\mathcal{C}([0,1])$ with the uniform norm. Let $D = \{f \in \mathcal{C}([0,1]) : f'(x_0) \text{ exists for some } x_0 \in [0,1]\}$. We will show that D is a countable union of nowhere dense sets in $\mathcal{C}([0,1])$ and thus $D \neq \mathcal{C}([0,1])$ by the Baire Category Theorem. If $f'(x_0)$ exists then $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists. Furthermore, if $f'(x_0)$ exists, then for some $m, n \in \mathbb{N}$, we must have $\left|\frac{f(x)-f(x_0)}{x-x_0}\right| \leq n$ for all x with $0 < |x - x_0| \leq \frac{1}{m}$. Define $E_{n,m} = \{f \in \mathcal{C}([0,1]) : \text{ for some } x_0 \in [0,1], |f(x) - f(x_0)| \leq n|x - x_0| \text{ for all } x \in [0,1] \text{ with } 0 < |x - x_0| \leq \frac{1}{m}\}$. So $D \subseteq \cup_{n=1}^{\infty} \cup_{m=1}^{\infty} E_{n,m}$.

Claim: $E_{n,m}$ are closed.

Proof: Let $\{f_j\}_{j=1}^{\infty} \subseteq E_{n,m}$ be such that $||f - f_j||_{\mathcal{U}} \to 0$ for some $f \in \mathcal{C}([0,1])$. For each j, we have $|f_j(x) - f_j(x_j)| \leq n|x - x_j|$ for all x such that $0 < |x - x_k| \leq \frac{1}{m}$. Extract a subsequence (unrelabeled, for simplicity) so that $x_n \to x_0 \in [0,1]$. Then

$$\begin{aligned} |f(x) - f(x_0)| &\leq \underbrace{|f(x) - f_j(x)|}_{\to 0} + |f_j(x) - f_j(x_j)| + \underbrace{|f_j(x_j) - f_j(x_0)|}_{\to 0} + \underbrace{|f_j(x_0) - f(x_0)|}_{\to 0} \\ &\leq \tfrac{1}{m}|x - x_j| \\ &\to \tfrac{1}{m}|x - x_0|. \end{aligned}$$

To show that $E_{n,m}$ are nowhere dense, note that it is enough to show that for all $\epsilon > 0$ and $f \in E_{n,m}$ that there exists $g \in \mathcal{C}([0,1])$ such that $||g - f||_{\mathcal{U}} < \epsilon$, but $g \notin E_{n,m}$. Of course, we can do this- just take g to be a piecewise linear continuous function with slope $> n$ everywhere.