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Class Notes for Math 918: Cohen Macaulay Modules, Instructor Roger Wiegand

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Topics covered are: Cohen Macaulay modules, zero-dimensional rings, one-dimensional rings, hypersurfaces of finite Cohen-Macaulay type, complete and henselian rings, Krull-Remak-Schmidt, Canonical modules and duality, AR sequences and quivers, two-dimensional rings, ascent and descent of finite Cohen Macaulay type, bounded Cohen Macaulay type.

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§1. Depth and Dimension

1.0. Conventions Throughout Chapter 1, R is a commutative Noetherian ring. When we say that $(R, \mathfrak{m}, k)$ is a local ring, we mean R has a unique maximal ideal $\mathfrak{m}$ and that $k = R/\mathfrak{m}$, the residue field. (When the residue field is not an issue, we sometimes just write $(R, \mathfrak{m})$.) In general, a commutative ring with exactly one maximal ideal is said to be *quasi-local.* Thus "local" = "quasi-local + Noetherian".

1.1. Regular sequences Let M be a finitely generated R-module. A sequence $(\underline{x}) = (x_1, \ldots, x_r)$ of elements of R is said to be M-*regular* (or to be an M-*sequence*) provided (a) x_i is a non-zerodivisor on $M/(x_1, \ldots, x_{i-1})M$ for each $i = 1, \ldots, r$ and (b) $M \neq (x_1, \ldots, x_r)M$. The integer r is called the *length* of the regular sequence. The M-regular sequence $(x_1, \ldots, x_r)$ is *maximal* provided there is no element $x \in R$ such that $(x_1, \ldots, x_r, x)$ is M-regular.

Note that we use the same symbol for the regular sequence as we do for the ideal generated by its entries. This never seems to cause much confusion. By the way, since the ideal generated by $\emptyset$ is (0), the case $i = 1$ in the definition says that x_1 is a non-zerodivisor on M. The inductive nature of the definition will be exploited shamelessly: $(x_1, \ldots, x_r)$ is M-regular if and only if x_1 is M-regular and $(x_2, \ldots x_r)$ is M/x_1M-regular.

Check that if $(x_1, \ldots, x_r)$ is M-regular then the chain $(\emptyset) \subset (x_1) \subset (x_1, x_2) \subset \cdots \subset (x_1, \ldots, x_r)$ of ideals is strictly increasing. (The non-degeneracy condition (b) in the defintion is essential here.) Since R is Noetherian by assumption, we see that every M-regular sequence can be extended to a maximal one.

Usually we will restrict our attention to finitely generated modules over local rings. In that context, as long as $M \neq 0$, the degeneracy condition is equivalent to the condition that $(x_1, \ldots, x_r) \subseteq \mathfrak{m}$.

Recall that a prime ideal $p \in \operatorname{Ass} M$ if and only if $p \in SpecR$ and $p = (0 : x)$ for some $x \in M$. Moreover, $\operatorname{Ass} M$ is finite and, as long as $M \neq 0$, $\operatorname{Ass} M \neq 0$. Also, if M is a finitely generated $R-$module, then $ZD(M) = \cup \operatorname{Ass}(M)$. We will use this fact in the proof of the following.

1.2. Lemma *Let $(R, \mathfrak{m}, k)$ M be a finitely generated R-module, and let $(\underline{x}) = (x_1, \ldots, x_r)$ be an M-sequence. Then $(\underline{x})$ is a maximal M-sequence if and only if* $\mathfrak{m} \in \operatorname{Ass}(M/\underline{x}M)$.

Proof. We know that $(\underline{x})$ is a maximal $M-$sequence if and only if $m \subseteq ZD(M/(\underline{x})M) \subseteq \cup \operatorname{Ass}(M/(\underline{x})M)$. By the prime avoidance theorem, this is if and only if $m \in \operatorname{Ass}(M/(\underline{x})M)$. □

1.3. Definition Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module. If $M \neq (0)$, then $\operatorname{depth}(M)$ is the supremum of integers r such that there exists an M-regular sequence of length r. If $M = (0)$, we put $\operatorname{depth}(M) = \infty$.

Verify that $\operatorname{depth}(M) = 0$ if and only if $\mathfrak{m} \in \operatorname{Ass}(M)$. This observation is the key to characterizing depth homologically over a local ring $(R, \mathfrak{m}, k)$. We have $\operatorname{depth}(M) = 0$ if and only if $m \in \operatorname{Ass} M$. Note, however, that if $p \in \operatorname{Spec} R$, then $p \in \operatorname{Ass} M$ if and only if there is an injection $R/p \hookrightarrow M$ (as $p = (0 : x)$ if and only if $Rs \cong R/p$ and $Rx \subseteq M$). Thus, we have $\operatorname{depth}(M) = 0$ if and only if $k \hookrightarrow M$, which is if and only if $\operatorname{Hom}_R(k, M) \neq 0$ as k is simple. Better: $\operatorname{depth}(M) \geq 1$ if and only if $\operatorname{Hom}_R(k, M) = 0$. Actually, k can be replaced by any non-zero module of finite length.

1.3.1 Lemma *Let N be any nonzero $R-$module of finite length. Then* $\operatorname{Hom}_R(k, M) \neq 0$ *if and only if* $\operatorname{Hom}_R(N, M) \neq 0$

Proof. The forward direction is clear by factoring out a maximal submodule (as R is Noetherian) and noting $N \twoheadrightarrow k \hookrightarrow M$. For the backward direction, let $\phi : N \to M$ be nonzero. Then $0 \neq \phi(N)$ has finite length. Thus $\phi(N)$ contains a copy of k, which implies $k \hookrightarrow M$. □

Thus, for any nonzero module N of finite length, we have that $\operatorname{depth} M = 0$ if and only if $\operatorname{Hom}_R(N, M) \neq 0$. Moreover, we can show, by induction:

1.4. Theorem *Let $(R, \mathfrak{m}, k)$ be local, let N be a non-zero R-module of finite length, let M be a finitely generated R-module, and let r be a positive integer. Then* $\operatorname{depth}(M) \geq r$ *if and only if* $\operatorname{Ext}_R^i(N, M) = 0$ *for $i = 0, \ldots, r-1$.*

Proof. We may assume throughout that $M \neq 0$. For $r = 1$, we have $\operatorname{depth} M \geq 1$ if and only if $Ext_R^0(N, M) = Hom_R(N, M) \neq 0$. So assume $r > 1$ and proceed by induction. Suppose $\operatorname{depth}(M) \geq r$, and let $(x_1, \ldots, x_r)$ be M-regular. Put $M_1 := M/x_1M$ (which has depth at least $r-1$ as $(x_2, ..., x_r)$ is an M_1-regular sequence), and look at the short exact sequence

$$0 \to M \xrightarrow{x_1} M \to M_1 \to 0.(\dagger)$$

Given $i < r$, we look at the following piece of the long exact sequence obtained by applying $\operatorname{Hom}_R(N, -)$ to the short exact sequence ($\dagger$):

$$\operatorname{Ext}_R^{i-1}(N, M_1) \to \operatorname{Ext}_R^i(N, M) \xrightarrow{x_1} \operatorname{Ext}_R^i(N, M)$$

The first module is 0 by induction, and this means that multiplication by x_1 is an injection, that is, x_1 is a non-zerodivisor on $\operatorname{Ext}_R^i(N, M)$. Since N has finite length, we have $\mathfrak{m}^t N = 0$ for some $t \geq 1$. Therefore $x_1^t N = 0$. Then x_1^t is a non-zerodivisor on $\operatorname{Ext}_R^i(N, M)$ even though $x_1^t \operatorname{Ext}_R^i(N, M) = 0$. We conclude that $\operatorname{Ext}_R^i(N, M) = 0$.

For the converse, we assume that $\operatorname{Ext}_R^i(N, M) = 0$ for each $i < n$. In particular, $\operatorname{Hom}_R(N, M) = 0$, so $\operatorname{depth}(M) \geq 1$. Choose an M-regular element $x_1 \in \mathfrak{m}$, form the exact sequence ($\dagger$) as before, and apply $\operatorname{Hom}_R(N, -)$. The long exact sequence shows immediately that $\operatorname{Ext}_R^j(N, M_1) = 0$ for each $j < r-1$. By induction, $\operatorname{depth}_R(M_1) \geq r-1$; therefore $\operatorname{depth}(M) \geq r$. ☐

Next we want to show that $\operatorname{depth}(M) < \infty$ for every non-zero R-module M. We know that there exist maximal M-sequences (by ACC), so the following obvservation will do the job:

1.5. Proposition *Let $(R, \mathfrak{m}, k)$ be local, and let M be a non-zero finitely generated module. Let $(\underline{x}) = (x_1, \ldots, x_r)$ be a maximal M-regular sequence. Then* $\operatorname{Ext}_R^r(k, M) \neq 0$.

Proof. We've already done the case $r = 0$, so assume $r > 0$. Put $M_i := M/(x_1, \ldots, x_i)M$, for $i = 0, \ldots, r$. Since $\operatorname{depth}(M_i) \geq r - i$, we have, by (1.1.3),

$$\operatorname{Ext}_R^j(k, M_i) = 0 \text{ for all } j < r - i.(\ddagger)$$

Applying $\operatorname{Hom}_R(k, -)$ to the short exact sequences $0 \to M_i \xrightarrow{x_{i+1}} M_i \to M_{i+1} \to 0$, we get the following exact sequences:

$$\to \operatorname{Ext}_R^{r-(i+1)}(k, M_i) \to \operatorname{Ext}_R^{r-(i+1)}(k, M_{i+1}) \to \operatorname{Ext}_R^{r-i}(k, M_i) \xrightarrow{x_{i+1}} \operatorname{Ext}_R^{r-i}(k, M_i) \to$$

The first module is 0, by ($\ddagger$), and the map x_{i+1} is 0 since x_{i+1} kills k. It follows that $\operatorname{Ext}_R^{r-(i+1)}(k, M_{i+1}) \cong \operatorname{Ext}_R^{r-i}(k, M_i)$ for $i = 0, \ldots, r$. Putting all of these isomorphisms together, we get $\operatorname{Ext}_R^r(k, M) = \operatorname{Ext}^r(k, M_0) \cong \ldots \cong \operatorname{Ext}_R^0(k, M_r) = \operatorname{Hom}_R(k, M/(\underline{x})M)$. Since $(\underline{x})$ is a maximal M-regular sequence, we have $\operatorname{Hom}_R(k, M/(\underline{x})M) \neq 0$. ☐

1.6. Corollary *Let $(R, \mathfrak{m}, k)$ be local, and let M be a non-zero finitely generated R-module.*

(1) $\operatorname{depth}(M) < \infty$.
(2) $\operatorname{depth}(M) = \inf\{r \mid \operatorname{Ext}^r(k, M) \neq 0\}$.
(3) *For any non-zero R-module N of finite length,* $\operatorname{depth}(M) = \inf\{r \mid \operatorname{Ext}_R^r(N, M) \neq 0\}$.
(4) *All maximal M-sequences have length equal to* $\operatorname{depth}(R)$.
(5) *Let $x \in \mathfrak{m}$, and assume that x is a non-zerodivisor on M. Then* $\operatorname{depth}(M/xM) = \operatorname{depth}(M) - 1$. ☐

1.7. Proposition *Let M be a non-zero finitely generated module over a local ring $(R, \mathfrak{m})$. Then* $\operatorname{depth}(M) \leq \dim(M)$.

Proof. We proceed by induction on $d := \operatorname{depth}(M)$. If $d = 0$, then $M \neq 0$, so $\dim(M) \geq 0$. Suppose $d > 0$, and let $(x_1, \ldots, x_d)$ be an M-sequence. Put $M_1 = M/x_1M$. Given any minimal element P of $\operatorname{Supp}(M)$, we see that $x_1 \notin P$, since x_1 is a non-zerodivisor on M and $P \in \operatorname{Ass}(M)$; it follows that $(M_1)_P = 0$. This shows that $\operatorname{Supp}(M_1)$ contains none of the minimal elements of $\mathrm{Supp}(M)$, and since $\mathrm{Supp}(M_1) \subset \mathrm{Supp}(M)$, we see that $\dim(M_1) < \dim(M)$. Since $\mathrm{depth}(M_1) = d - 1$, we have, by induction, $d - 1 \leq \dim(M_1) < \dim(M)$; thus $d \leq \dim(M)$. $\square$

1.8. Definition Let $(R, \mathfrak{m}, k)$ be a local ring. An R-module M is *Cohen-Macaulay* (CM) provided M is finitely generated and $\mathrm{depth}(M) = \dim(M)$. The ring R is CM provided R is CM as an R-module. The module M is *maximal Cohen-Macaulay* (MCM) provided M is CM and $\mathrm{depth}(M) = \dim(R)$.

1.9. Background Here we summarize basic results that may have been deployed somewhat flippantly in the notes. We keep the conventions established in §1.0. In particular, R is always commutative and Noetherian.

1.9.1. Zero-divisors and associated primes An element $x \in R$ is a *zerodivisor on* the module M provided M has a non-zero element m such that $rm = 0$. A *non-zerodivisor on* M is an element $r \in R$ that is not a zerodivisor on M. (Thus 0 is a zerodivisor on M if and only if $M \neq (0)$.) A prime ideal P of R is an *associated prime of* M provided there is some element $m \in M$ such that $P = (0 :_R m) := \{r \in R \mid rm = 0\}$. Equivalently, there is an injection $R/P \hookrightarrow M$. Notation: $\mathrm{Ass}(R) = \{\text{associated primes of } R\}$.

1.9.2. Proposition *Let M be a finitely generated R-module. Then $\mathrm{Ass}(M)$ is a finite set, and it is non-empty if and only if $M \neq (0)$. The set of zerodivisors on M is $\bigcup \mathrm{Ass}(M)$.*

1.9.3. Support of a module Let M be an R-module. The *support* of M is the set $\mathrm{Supp}(M) := \{P \in \mathrm{Spec}(R) \mid M_P \neq (0)\}$. Assume M is finitely generated. Then $\mathrm{Supp}(M) = \mathrm{V}(0 :_R M)$, the closed set consisting of primes that contain the annihilator of M. The *dimension* of M is the Krull dimension of the ring $R/(0 :_R M)$. Thus $\dim(M)$ is the supremum of chains of primes in the support of M. Every minimal element of $\mathrm{Supp}(M)$ is in $\mathrm{Ass}(M)$.

1.9.4. Prime filtrations Every finitely generated R-module M has a *prime filtration*, that is, a sequence $\mathbf{F} = (M = M_0 \supset M_1 \supset \cdots \supset M_t = (0))$ such that, for each $i = 0, \dots, t-1$, $M_i/M_{i+1} \cong R/P_i$ for some prime ideal P_i. We put $\mathrm{Supp}(\mathbf{F}) :== \{P_0, \dots, P_{t-1}\}$. For any prime filtration of M, we have $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(\mathbf{F}) \subseteq \mathrm{Supp}(M)$.

1.9.5. Finite-length modules Let $(R, \mathfrak{m}, k)$ be a local ring, and let M be an R-module. Recall that M *has finite length* provided M has a composition series (in which case the length of M is the length of any composition series, that is, the number of proper inclusions in the series). The composition factors are, of course, $R/\mathfrak{m}$, the only simple module around. These conditions on an R-module are equivalent:

(1) M has finite length.
(2) M has both ACC and DCC on submodules.
(3) M is finitely generated and $\mathfrak{m}^t M = 0$ for some $t \geq 1$.
(4) M is finitely generated and $\mathrm{Supp}(M) \subseteq \{\mathfrak{m}\}$.

It's sometimes useful to look at the "finite-length part" of a finitely generated R-module. This is, by definition, the sum of all of the finite-length submodules of M. The standard notation for the finite-length part of a finitely generated module M is $\mathrm{H}^0_{\mathfrak{m}}(M)$. Since M has ACC, it's clear that $\mathrm{H}^0_{\mathfrak{m}}(M)$ has finite length, and that $\mathrm{H}^0_{\mathfrak{m}}(M) = \bigcup_{t \geq 1}(0 :_M \mathfrak{m}^t)$. We note that $\mathrm{H}^0_{\mathfrak{m}}(M) = 0$ if and only if $\mathrm{Hom}_R(k, M) = 0$.

1.9.6. Torsion Let S be the multiplicative set consisting of non-zerodivisors of R. The *total quotient ring* of R is the ring of fractions $K := S^{-1}R$. The *torsion submodule* M_{tors} of M is the kernel of the natural map $M \to S^{-1}M$ (or $M \to K \otimes_R M$). The module M is *torsion* provided $M = M_{\mathrm{tors}}$ and *torsion-free* provided $M_{\mathrm{tors}} = 0$ (that is, $M \to K \otimes M$ is injective).

1.9.7. Crash course on Ext For details, consult [11]. First of all, for R-modules M and N, $\mathrm{Hom}_R(N, M))$ is an abelian group, and it's an additive functor of each variable, contravariant in the first variable, covariant in the second. Thus $f : N_1 \to N_2$ and $g : M_1 \to M_2$ induce a homomorphism $\mathrm{Hom}(f, g) : \mathrm{Hom}_R(N_2, M_1) \to \mathrm{Hom}(N_1, M_2)$ taking $\varphi : N_2 \to M_1$ to $g\varphi f : N_1 \to M_2$.

$$N_1 \xrightarrow{f} N_2 \xrightarrow{\varphi} M_1 \xrightarrow{g} M_2$$

Additivity just means $\mathrm{Hom}(f_1 + f_2, g) = \mathrm{Hom}(f_1, g) + \mathrm{Hom}(f_2, g)$ and similarly for the other variable. Functoriality means various diagrams commute. All of this works fine even for non-commutative rings, but since R commutes we can do much more: $\mathrm{Hom}_R(N, M)$ is an R-module: For $r \in R$ and $f \in \mathrm{Hom}_R(N, M)$, we can define rf by either $(rf)(n) := f(rn)$ or $(rf)(n) := r(f(n))$. Of course these give the same result. But there's another way to look at this: Consider the multiplication maps $r_N : N \to N$ and $r_M : M \to M$. Then multiplication by r on $\mathrm{Hom}_R(N, M)$ is given by $\mathrm{Hom}(r_N, 1_M)$ and is also given by $\mathrm{Hom}(1_N, r_M)$.

Now, what about $\mathrm{Ext}_R(N, M)$? We have an abelian group $\mathrm{Ext}^i_R(N, M)$ for each $i \geq 0$. (Sometimes, to avoid boundary effects, it is useful to define $\mathrm{Ext}^j_R(N, M) = 0$ for all $j < 0$.) We have $\mathrm{Ext}^0_R(N, M) = \mathrm{Hom}_R(N, M)$; the higher Ext's are cleverly designed to repair the lack of exactness of $\mathrm{Hom}_R(-, -)$. Each $\mathrm{Ext}^i_R(-, -)$ is an additive functor in each variable, contravariant in the first and covariant in the second. Moreover, each has an R-module structure, obtained exactly as for $\mathrm{Hom}_R(N, M)$, via the R-action on either variable. Finally, each short exact sequence $0 \to M' \to M \to M'' \to 0$ gives rise to a long exact sequence

$$\cdots \to \mathrm{Ext}^i_R(N, M') \to \mathrm{Ext}^i_R(N, M) \to \mathrm{Ext}^i_R(N, M'') \xrightarrow{\delta} \mathrm{Ext}^{i+1}_R(N, M') \to \dots,$$

and each short exact sequence $0 \to N' \to N \to N'' \to 0$ yields a long exact sequence

$$\cdots \to \mathrm{Ext}^i_R(N'', M) \to \mathrm{Ext}^i_R(N, M) \to \mathrm{Ext}^i_R(N', M) \xrightarrow{\delta} \mathrm{Ext}^{i+1}_R(N'', M) \to \dots.$$

Moreover, the connecting homomorphisms δ are natural, in the sense that various diagrams commute.

Often it is important to know that $\mathrm{Ext}^i_R(N, M)$ is a finitely generated R-module. If M and N are both finitely generated (and R is Noetherian), then each R-module $\mathrm{Ext}^i_R(N, M)$ is indeed finitely generated as an R-module.

1.10. Exercises Assume throughout that R is a Noetherian ring and M is a finitely generated non-zero module.

1.1. (2 points) If (x, y) is M-regular, then x is a non-zerodivisor on M/yM. However, (y, x) need not be regular.

1.2. (4 points) If R is local, then every permutation of an M-sequence is an M-sequence.

1.3. (3 points) With $R = k[X, Y, Z]$ (k a field), give an example of an R-sequence $(\underline{x})$ and a permutation $(\underline{y})$ of x such that (y) is not an R-sequence. (Don't peek at the example in [10].)

1.4. (15 points) Let R be a commutative ring (containing an infinite field). Let $(x_1, \dots, x_n)$ be an $R-$sequence. Then there are elements $c_{ij} \in R$ such that

$$(x_1, x_2 + c_{21}x_1, x_3 + c_{32}x_2 + c_{31}x_1, \dots, x_n + c_{nn-1}x_{n-1} + ... + c_{n1}x_1)$$

is a permutable $R-$sequence (that is, every permutation of it is also $R-$regular).

1.5. (5 points) Let $(x_1, \dots, x_r)$ be M-regular, and let $t_1, \dots, t_r$ be positive integers. Prove that $(x_1^{t_1}, \dots, x_r^{t_r})$ is M-regular.

1.6. (4 points) Let A be the ring of continuous real-valued functions on the unit interval $[0, 1]$. Either find an A-regular sequence of length 2 or prove that none exists.

1.7. (3 points) Assume R is a local CM ring of dimension 1. Prove that M is MCM if and only if M is torsion-free. Show by example that this is false if the assumption that R is CM is deleted.

1.8. (2 points) Prove that every reduced local ring of dimension one is CM. ("Reduced" means there are no non-zero nilpotent elements.)

1.9. (3 points) Find a two-dimensional local integral domain that is not CM.

1.10. (1 point) Find an example of a $\mathbb{Z}$-module whose support is not closed in $\mathrm{Spec}(\mathbb{Z})$.

1.11. (3 points) Prove or disprove: Let $A \to B$ be a homomorphism of commutative rings, and let M be a finitely generated B-module. Then $\mathrm{Supp}_A(M)$ is closed in $\mathrm{Spec}(A)$.

1.12. (2 points) Find an example of a finitely generated module M over a Noetherian ring R, together with a prime ideal $P \notin \mathrm{Ass}(M)$, such that $P \in \mathrm{Supp}(\mathbf{F})$ for *every* prime filtration F of M.

1.13. (4 points) Suppose R is an integral domain and M is a finitely generated torsion-free R-module. Find, and verify, a simply-stated necessary and sufficient condition for M to have a filtration $\mathbf{F}$ such that $\mathrm{Supp}(\mathbf{F}) = \mathrm{Ass}(M)$.

1.14 (??? points) Given R and M, how does one characterize those prime $P \notin \mathrm{Ass}(M)$ that show up in the support of *every* prime filtration of M? (Cf. Eisenbud, Herzog, Landsburg, Aihua Li, clean filtrations, shellable simplicial complexes, ...)

§2. Cohen Macaulay Modules and Finite CM Type.

2.1 Theorem. *Let (R, m, k) be a local ring and ${}_RM \neq 0$ a finitely generated R-module. Then $depth(M) \leq dim(M)$.*

$$dim(M) := dim\frac{R}{(0 :_R M)} = sup\{n \mid p_0 \subset \ldots \subset p_n, \quad p_i \in Supp(M)\}$$

$$Supp(M) := \{p|M_p \neq 0\} = V(0 :_R M) := \{p|p \supset (0 :_R M)\}$$

Proof. Use induction on $depth(M)$. We can assume that $depth(M) \geq 1$ since there is nothing to prove in the base case i.e. depth is non-negative. Choose $x \in M$, x a NZD on M. Let $M_1 = M/xM$. Let p be a minimal element of $Supp(M)$ then $p \in Ass(M) \implies p \subset ZD(M) \implies x \notin p$. Clearly $Supp(M_1) \subset Supp(M)$. But x avoids all the minimal elements of $Supp(M)$. Since $x \in (0 : M_1)$, so for all $p \in minSupp(M)$ we have $(0 :_R M_1) \nsubseteq p$. Thus no element in $minSupp(M_1)$ can be minimal in $Supp(M)$. So $\dim M_1 < \dim M$. Recall, as x is a non-zerodivisor on M that $depth(M_1) = depth(M) - 1$. By induction, $depth(M_1) \leq \dim(M_1) < \dim(M)$. $\implies depth(M) \leq \dim(M)$. $\square$

2.2 Definition. M is *Cohen Macaulay* (CM) provided $M \neq 0$, and $depth(M) = dim(M)$. M is *maximal Cohen Macaulay* (MCM) provided $depth(M) = dim(R)$. A ring R is CM if it is as an $R-$module, that is, if $depth R = \dim R$.

2.3 Examples.

- If $\dim M = 0$, then M is CM.
- The standard simple example of a non-Cohen Macaulay ring is $R = \frac{k[[x,y]]}{(x^2,xy)}$. It has dimension 1 (as $m \supsetneq (x) \supsetneq 0$) and depth 0 (as $x \neq 0$ but $xm = 0$, so $m \in \mathrm{Ass}\, R$).
- $R/(x) \simeq k[[y]]$ is an example of a MCM module for a non-Cohen Macaulay ring. It has $depth = 1$.

2.4 Definition. The local ring (R, m) has *finite CM type* provided there are only finitely many isomorphism classes of indecomposable MCM R-modules.

2.5 Theorem. *Let (R, m) be zero-dimensional (i.e., Artinian). Then R has finite CM type if and only if R is a principle ideal ring (PIR), i.e. the only ideals of R are $m = (x) \supset (x^2) \supset (x^3) \supset \ldots \supset (x^n) = m^n = 0$.*

Proof. For the backward direction, let ${}_RM$ be an indecomposable finitely generated module then M is cyclic (as every finitely generated module is a sum of cyclics), which implies M is isomorphic to one of $R/(x)$, $R/(x^2)$, ..., $R/(x^n)$ or R.

For the forward direction, assume R isn't a PIR then M needs ≥ 2 generators (If $m = (x)$, then the only ideals are (x^i) making R a PIR). We want to show that for all $n \geq 1$ that R has an indecomposable module needing exactly n generators. Then, R will have infinitely many indecomposable modules as the number of generators is invariant among isomorphic modules. Note that if $I \subset R$ and we can do this for R/I then we can do it for R. So, write $m = Rx_1 + \ldots + Rx_t$, $t \geq 2$ and pass to $R/(x_3, \ldots, x_t)$. Thus, we may assume m needs exactly 2 generators. Next pass to $R/(m^2)$, so we may assume $m^2 = 0$ (as the number of generators don't drop by NAK.)

Thus we are in the situation where (R, m, k) is local, $m^2 = 0$, and $m = Rx+Ry$. Note that m is a two-dimensional vector space over k with basis $\{x, y\}$. Fix $n \geq 2$ let

$$\phi = \begin{bmatrix} x & y & 0 & \cdots & 0 & 0 \\ 0 & x & y & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & x & y & 0 \\ 0 & 0 & \cdots & 0 & x & y \\ 0 & 0 & 0 & \cdots & 0 & x \end{bmatrix}$$

be $n \times n$. Let $M = coker(\phi)$. Then we have the exact sequence

$$R^n \xrightarrow{\phi} R^n \longrightarrow M \longrightarrow 0$$

Note that M is minimally generated by exactly n elements as $M = R/Im(\phi)$ and $Im(\phi) \subset mR^n$, by Nakayama's lemma.

Claim. M is indecomposable.

Proof. Recall that a finitely generated $R-$module N_R is indecomposable if and only if $End_R(N)$ has exactly 2 idempotents, 0 and 1. Let $E = End_R(M)$ and suppose $e = e^2 \in End_R(M)$. We know $e(1-e) = 0$. We will show that $e = 0$ or 1.

Suppose e is surjective. Then, since M is Noetherian, e is an automorphism. So $1-e = 0$, which implies $e = 1$. So assume e is not surjective. As R^n is projective and $R^n \to M$ is surjective, we can find α, β lifting e so that the following diagram commutes, that is $\phi\beta = \alpha\phi$.

$$\begin{tikzcd} R^n \arrow[r, "\phi"] \arrow[d, "\exists\beta"] & R^n \arrow[r] \arrow[d, "\exists\alpha"] & M \arrow[r] \arrow[d, "e"] & 0 \\ R^n \arrow[r, "\phi"] & R^n \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

We have $\phi = xI+yJ$, where J equals a matrix with ones on the super-diagonal and zeros elsewhere. Then, by commutativity, we have

$$(xI + yJ)\rho = \alpha(xI + yJ)$$

$$(\beta - \alpha)x + (J\beta - \alpha J)y = 0$$

Let $\overline{\alpha}$ and $\overline{\beta}$ denote the reductions module the maximal ideal, putting our entries in k. Since x and y are linearly independent $\overline{\beta} = \overline{\alpha}$ and $J\overline{\beta} = \overline{\alpha}J$. Therefore $\overline{\alpha}$ commutes with J. Since J is non-derogatory (only linear combinations of J commute with J), we have $\overline{\alpha} \in k[J]$. Therefore,

$$\overline{\alpha} = \begin{bmatrix} a & b & c & \cdots & \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & a & b & c \\ 0 & \cdots & 0 & a & b \\ 0 & \cdots & 0 & 0 & a \end{bmatrix}$$

If $a \neq 0$, then $\overline{\alpha}$ is invertible. This says $\overline{\alpha}$ is surjective, which implies α and hence e is surjective, a contradiction. Thus $a = 0$ and $\overline{\alpha}^n = 0$. This implies $Im(\alpha^n) \subset mR^n$ and so $Im(e^n) \subset mM$. But $e = e^n$ and so $Im(e) \subset mM$. Thus $(1-e)$ is surjective (NAK) and hence invertible. That is, $e = 0$. □

Now let (R, m, k) be one dimensional and CM (so m contains a non-zerodivisor).

2.6 Lemma. (RW, 1994) *If R has finite CM type then $\overline{R}$, the integral closure of R in the total quotient ring $K := \{NZD\}^{-1}R$ is finitely generated as an R-module. Equivalently, $\overline{R}$ is finitely generated as an R-algebra.*

Proof. Assume $\overline{R}$ is not finitely generated. Then we get elements $x_n \in K$ such that

$$R \subset R(x_1) \subset R(x_1, x_2) \subset \dots$$

Let $S_i := R(x_1, \dots, x_i)$. Note that S_i is a subring of $K = \{NZD\}^{-1}R$ and each is finitely generated as an R-module. Moreover, each is torsion-free as an $R-$module (as we can find a non-zerodivisor in the maximal ideal) and therefore MCM. Note further that each S_i is indecomposable. By Exercise 2.2 below, S_i is not isomorphic to S_j as R-modules if $i \neq j$. Thus R does not have finite CM type. □

2.7 Corollary. *R is reduced, i.e. it has no nilpotent elements other than 0.*

Proof. Let $t \in m$ be a non-zerodivisor. Suppose R is not reduced, and let x be a nonzero nilpotent element. We will show that the integral closure $\overline{R}$ of R is not finitely generated. (Then R has infinite CM type by the earlier result.) Let K be the total quotient ring, and look at the elements x/t^n, $n \geq 1$, in K. These are all in $\overline{R}$. (If x satisfies $x^r = 0$, then x/t^n is a root of the monic polynomial X^r.) Check easily that $R(x/t^n)$ is properly contained in $R(x/t^{n+1})$ (as t a non-zerodivisor implies $t \in m$ and hence $1 - t$ is a unit). Thus $\overline{R}$ is not Noetherian and therefore not finitely generated. □

2.8 Exercises.

2.1. (2 points) Show that every finitely generated CM module has a decomposition into indecomposable submodules not necessarily unique. [Note that this representation may not be unique]

2.2. (4 points) Let R be any commutative ring with total quotient ring K let A and B be subrings of K with $R \subset A \subsetneq B \subset K$ show that A is not isomorphic to B as an R-module.

- Note that we have $R = R_0 \subset R_1 \subset R_2 \subset \dots \subset K$. Each R_i is finitely generated as an R-Module. $\forall i \quad \exists c_i \in R$ s.t. c_i is a NZD and $c_i R_i \subset R$. In any Noetherian ring $\exists G \in \mathbb{N}$ such that $\nexists I_1 \oplus \dots \oplus I_{G+1}$ inside R with $I_j \neq 0$. The least G is called the Goldie Dimension.

2.3 (2 points) Let (R, m) be one-dimensional local with total quotient ring K. Then $R = K$ if and only if R is not CM.

§3. Artinian Pairs

3.0. Conventions. Throughout §3, (R, m) is a one-dimensional, local, reduced ring with finitely generated integral closure $\overline{R}$. Also, M, N are finitely generated torsion-free $R-$modules. [Recall this means M, N are MCM in the case that R is CM].

3.1. The Conductor Square. We define the conductor to be $\mathfrak{f} := \{r \in R | r\overline{R} \subseteq R\} = (R :_R \overline{R}) = (R :_{\overline{R}} \overline{R})$. Note that $\mathfrak{f}$ contains a NZD of R. Therefore $\mathfrak{f} \not\subseteq \bigcup Ass(R) = \bigcup minSpec(R)$. So $dim\ R/\mathfrak{f} = 0$, that is, $R/\mathfrak{f}$ is Artinian, which implies $\overline{R}/\mathfrak{f}$ is Artinian. The "Conductor Square" is as follows

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, two heads] & \overline{R} \arrow[d, two heads] \\ R/\mathfrak{f} \arrow[r, hook] & \overline{R}/\mathfrak{f} \end{tikzcd}$$

The bottom line of the conductor square is an example of:

3.2. Definition. An *Artinian Pair* is a module-finite extension $A \hookrightarrow B$ of commutative Artinian rings.

3.3. Definition. An *$(A \hookrightarrow B)$-module* is a pair $W \hookrightarrow V$, where

(1) V is a finitely generated projective $B-$module.

(2) W is a finitely generated $A-$submodule of V.

(3) $BW = V$.

Given ${}_RM$, let $\overline{R}M$ denote the $\overline{R}$-submodule of $K \otimes_R M$ generated by the image of M in the map $M \hookrightarrow K \otimes_R M$ (injective as M is torsion free).

3.4. Fact. $\overline{R}M \cong \frac{\overline{R} \otimes_R M}{torsion}$

Since $\overline{R}M$ is torsion free, $\overline{R}M$ is $\overline{R}$-projective. So $\overline{R}M = D_1^{(n_1)} \times \cdots \times D_t^{(n_t)}$. Therefore, $\overline{R}M/\mathfrak{f}M$ is $\overline{R}/\mathfrak{f}$-projective. This gives us the following pullback diagram, where the pullback $M = \pi^{-1}(\frac{M}{\mathfrak{f}M})$.

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d] & \overline{R}M \arrow[d, "\pi"] \\ M/\mathfrak{f}M \arrow[r, hook] & \overline{R}M/\mathfrak{f}M \end{tikzcd}$$

Note that the bottom line is an $(R/\mathfrak{f} \hookrightarrow \overline{R}/\mathfrak{f})$-module.

3.5. Examples of pullback diagrams.

(1) Let $R = k[[t^2, t^3]]$. Then $\mathfrak{f} = t^2k[[t]] = (t^2, t^3)R$. The conductor square is:

$$\begin{tikzcd} k[[t^2, t^3]] \arrow[r, hook] \arrow[d] & k[[t]] \arrow[d] \\ k \arrow[r, hook] & k[\,\overline{t}\,] = k \oplus k\overline{t} \end{tikzcd}$$

where $\overline{t}^2 = 0$.

(2) Let $R = k[[t^4, t^5, t^6]]$. Then $\overline{R} = k[[t]]$ and $\mathfrak{f} = t^8k[[t]] = (t^8, t^9, t^{10}, t^{11})R$. [Note: We use t^8 because 8 is the smallest exponent such that $t^n \in R$ for all $n \geq 8$.] The corresponding conductor square is

$$\begin{tikzcd} k[[t^4, t^5, t^6]] \arrow[r, hook] \arrow[d] & k[[t]] \arrow[d] \\ k[[\,\overline{t}^4, \overline{t}^5, \overline{t}^6\,]] \arrow[r, hook] & k[\,\overline{t}\,] = k \oplus k\overline{t}^4 \oplus k\overline{t}^5 \oplus k\overline{t}^6 \end{tikzcd}$$

with $\overline{t}^8 = 0$.

(3) Let $R = \frac{k[[x,y]]}{(xy)}$. Note that we have the idempotent $e := (\frac{y}{x+y})^2 = \frac{y^2}{x^2+y^2} = \frac{y}{x+y}$, since $y^2(x+y) = (x^2+y^2)y$. So, $1 - e = (\frac{x}{x+y})^2 = \frac{x}{x+y}$. Therefore, $\overline{R} = Re \times R(1-e)$ as $Re \cong R/(x) \cong k[[y]]$, $R(1-e) \cong k[[x]]$. Since, $\mathfrak{f} = \mathfrak{m}$ the conductor square is

$$\begin{tikzcd} \dfrac{k[[x,y]]}{(xy)} \arrow[r, hook] \arrow[d] & k[[y]] \times k[[x]] \arrow[d] \\ k \arrow[r, hook, "t \mapsto (t,t)"] & k \times k \end{tikzcd}$$

Given ${}_RM$ f.g. torsion-free, let $M_{art} := M/\mathfrak{f}M \hookrightarrow \overline{R}M/\mathfrak{f}M$. Note, $\mathfrak{f} = \mathfrak{f}\overline{R}$.

3.6. Definition. A *homomorphism of Artinian pairs* $(V_1, W_1) \to (V_2, W_2)$ is a map $W_1 \xrightarrow{\varphi} W_2$ such that $\varphi(V_1) \subseteq V_2$ and such that the following diagram commutes.

$$\begin{tikzcd} V_1 \arrow[r, hook] \arrow[d, "\psi"] & W_1 \arrow[d, "\varphi"] \\ V_2 \arrow[r, hook] & W_2 \end{tikzcd}$$

3.7. Theorem. *Assume $R \neq \overline{R}$. Then $M_{art} \cong N_{art} \Leftrightarrow M \cong N$.*

Proof. We prove only the forward direction. Consider the following diagram.

$$
\begin{tikzcd}
N \arrow[rrr, hook] \arrow[ddd, two heads] & & & \overline{R}N \arrow[ddd, two heads] \\
& M \arrow[ul, "\exists!\varphi"'] \arrow[r, hook] \arrow[d, two heads] & \overline{R}M \arrow[ur, "\exists\dagger"] \arrow[d, two heads] & \\
& M/\mathfrak{f}M \arrow[r, hook] \arrow[dl, "\cong"'] & \overline{R}M/\mathfrak{f}M \arrow[dr, "\cong"] & \\
M/\mathfrak{f}N \arrow[rrr, hook] & & & \overline{R}N/\mathfrak{f}N
\end{tikzcd}
$$

As $\overline{R}M$ is projective, there is a map $(\dagger)$ making the right-hand trapezoid commute. The universal property of pullbacks then gives a unique homomorphism ϕ making the top and left trapezoids commute. NAK implies φ is surjective as $\mathfrak{f} \subseteq m$. Thus we get maps $M \twoheadrightarrow N$ and by symmetry $N \twoheadrightarrow M$. So we have a surjective endomorphism of finitely generated modules $M \overset{\varphi}{\twoheadrightarrow} N \overset{\psi}{\twoheadrightarrow} M$, which says $\psi\varphi$ is as isomorphism, since M is a Noetherian module. Thus, φ is 1-1 and so φ is an isomorphism. $\square$

3.8. Exercises

3.1. (5 points) Prove that $\overline{R} \otimes_R M$ is torsion-free as an $\overline{R}-$module if and only if M is a free $R-$module.

3.2. (5 points) $\overline{R}$ is a direct-product $D_1 \times \ldots \times D_t$, where each D_i is a semilocal PID.

3.3. (2 points) Let $R = \dfrac{k[x,y]_{(x,y)}}{(y^2 - x^3 - x^2)}$. Show $R \cong k[t^2-1, t^3-t]_{(t^2-1, t^3-t)}$ and thus the conductor square for R is:

$$
\begin{tikzcd}
R \arrow[r, hook] \arrow[d, two heads] & \overline{R} = k[t]_{(t+1)\cup(t-1)} \arrow[d, two heads] \\
k \arrow[r, hook] & k \times k
\end{tikzcd}
$$

3.4. (5 points) Let $c_1 < \ldots < c_n \in \mathbb{R}$. Let $S = \{f \in \mathbb{R}[t] | f^{(k)}(c_i) = f^{(k)}(c_j), \forall i,j,k = 0,\ldots,3\}$, where $f^{(k)}(c_i)$ is the k^{th} derivative of f at c_i. Let $S' = \mathbb{R}[t]_{(t-c_1)\cup\ldots\cup(t-c_n)}$. Let $\mathfrak{m} = \{f \in S | f(c_1) = 0\}$ and $R := S_{\mathfrak{m}}$. Show that $\mathfrak{m}$ is a maximal ideal of S and that the pullback for R is

$$
\begin{tikzcd}
R \arrow[r, hook] \arrow[d, two heads] & S' \arrow[d, two heads] \\
k \arrow[r, hook] & k[t_1] \times \ldots \times k[t_n]
\end{tikzcd}
$$

with $t_i^4 = 0$.

§4. More on Artinian Pairs

4.0 Conventions. As before, we take (R, m) to be a one-dimensional, local, reduced ring with finitely generated integral closure $\overline{R}$. Note, by previous exercise, this says R is Cohen Macaulay.

4.1 General form of $\overline{R}$. Let $P_1, ..., P_t$ be the minimal primes in R. Then, for all $i = 1, ..., t$, $D_i = R/P_i$ is a one dimensional, local domain. Further, its integral closure is finitely generated, as $\overline{R} = \overline{D_1} \times \cdots \times \overline{D_t}$; thus $\overline{D_i}$ is finitely generated over R and therefore over D_i). Each $\overline{D_i}$ is a semilocal Dedekind domain, and thus a PID. This yields the following pullback diagram:

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, two heads] & \overline{R} = \overline{D}_1 \times \cdots \times \overline{D}_t \arrow[d, two heads] \\ R/\mathfrak{f} \arrow[r, hook] & \overline{R}/\mathfrak{f} = B_1 \times \cdots \times B_t \end{tikzcd}$$

where each B_i is an Artinian principal ideal ring. Furthermore, each B_i has the same number of maximal ideals as $\overline{D}_i$. So, we can decompose each $B_i = B_{i,1} \times \cdots \times B_{i,s_i}$, where each $B_{i,j}$ is local (as any Artinian ring can be decomposed into Artinian local rings). Thus $\overline{R}/\mathfrak{f} = \prod_{i,j} B_{i,j}$.

A projective $\overline{R}-$module looks like $L = \overline{D}_1^{(e_1)} \times \cdots \times \overline{D}_t^{(e_t)}$, as L must be projective over each $\overline{D}_i$ and a projective over a PID is free. Similarly, as projectives over a local ring are free, a projective $\overline{R}/\mathfrak{f}-$module looks like $W = \prod B_{i_j}^{(e_{i_j})}$. So W "comes from" a projective $\overline{R}-$module if and only if $e_{i,j} = e_{i,k}$ for all i, j, k. (By "comes from," we mean $W \cong L/\mathfrak{f}L$).

4.2 Theorem. *Assume $R \neq \overline{R}$.*

(1) *Let M, N be MCM $R-$modules. Then $M \cong N$ if and only if $M_{art} \cong N_{art}$.*
(2) *Let (V, W) be any $(R/\mathfrak{f} \hookrightarrow \overline{R}/\mathfrak{f})-$module. Then there exists a MCM $R-$module M such that $M_{art} \cong (V, W)$ if and only if there exists a finitely generated projective $\overline{R}-$module $L \neq 0$ such that $L/\mathfrak{f}L \cong W$.*
(3) *Let (V, W) be any $(R/\mathfrak{f} \hookrightarrow \overline{R}/\mathfrak{f})-$module. Then there exists another $(R/\mathfrak{f} \hookrightarrow \overline{R}/\mathfrak{f})-$module (V', W') and a MCM $R-$module M such that $M_{art} \cong (V, W) \oplus (V', W')$.*
(4) *Let M, N be MCM $R-$modules. Suppose $M_{art} \oplus (V, W) \cong N_{art}$ for some $(R/\mathfrak{f} \hookrightarrow \overline{R}/\mathfrak{f})-$module (V, W). Then there exists a MCM $R-$module U such that $U_{art} \cong (V, W)$ and $M \oplus U \cong N$.*

Proof. (1) See Section 3 of the notes.

(2) For the forward direction, note that $\overline{R}M$ is a finitely generated torsion-free module over $\overline{R}$. Since $\overline{R}$ is a direct product of finitely many principal ideal domains, $L := \overline{R}M$ is a projective $\overline{R}$-module. Therefore $L/\mathfrak{f}L$ is a projective $\overline{R}/\mathfrak{f}$-module. But by definition $M_{art} = (M/\mathfrak{f}M, \overline{R}M/\mathfrak{f}M)$. Therefore $W \cong L/\mathfrak{f}L$.

For the backward direction, choose an isomorphism $h : W \to L/\mathfrak{f}L$. Let $\pi : L \to L/\mathfrak{f}L$ be the natural map, and define M by the following pullback diagram:

$$\begin{tikzcd} M \arrow[rr, hook] \arrow[d, two heads, "\pi'"] & & L \arrow[d, two heads, "\pi"] \\ V \arrow[r, hook] & W \arrow[r, tail, two heads, "h"] & L/\mathfrak{f}L \end{tikzcd}$$

(That is, $M = \pi^{-1}(h(W))$.)

Then M is a MCM R-module. Moreover, since $\overline{R}V = W$, it follows that $\overline{R}M = L$. As the pullback of a surjection is always surective, the map π' is surjective. Moreover, $\ker \pi' = \mathfrak{f}M$. (To see this, note that $\ker \pi = \mathfrak{f}L = \mathfrak{f}\overline{R}M = \mathfrak{f}M \subseteq M$.) We conclude that $M_{art} \cong (V, W)$. □

4.3 Proposition. *Let (V, W) be an indecomposable $(A \hookrightarrow B)-$module, where $A \hookrightarrow B$ is an Artinian pair. Then $E := End(V, W)$ is a "local" ring.*

Proof. First note that as E is not necessarily commutative, by "local" we mean $E/J(E)$ is a division ring. Recall $E = \{\phi \in Hom_B(W, W) | \phi(V) \subseteq V\}$. Since B is Noetherian and W finitely generated, E is a finitely generated $B-$module. As B is Artinian, E is also left and right Artinian. Thus $J = J(E)$ is nilpotent. Since idempotents lift modulo nil ideals and E has no nontrivial idempotents (as (V, W) is indecomposable), we see E/J has no nontrivial idempotents. Thus it is semisimple and Wedderburn's Theorem applies. Write $E/J = M_{n_1}(D_1) \times \cdots \times M_{n_t}(D_t)$, where each D_i is a division ring. If $t > 1$, then we would have nontrivial idempotents. Thus $E/J \cong M_{n_1}(D_1)$. Of course, if $n_1 > 1$, we again have nontrivial idempotents. Thus $E/J \cong D_1$. $\square$

Since indecomposables have "local" endomorphism rings, the category of $(A \hookrightarrow B)-$modules has the Krull-Remak-Schmidt property, that is, everything is uniquely written as a direct sum of indecomposables up to permutations. We shall see later that this is not true in the category of $R-$modules.

4.4 Definition. An Artinian pair $(A \hookrightarrow B)$ has *finite representation type* (FRT) if and only if there are only finitely many indecomposable $(A \hookrightarrow B)-$modules up to isomorphism.

4.5 Theorem. *Assume $R \neq \overline{R}$. Then R has finite CM type if and only if $R_{art} := (R/\mathfrak{f} \hookrightarrow \overline{R}/\mathfrak{f})$ has finite representation type.*

Proof. For the forward direction, let $X_1, ..., X_t$ represent all MCM $R-$modules up to isomorphism. Let $(X_i)_{art} = Y_{i,1} \oplus \cdots \oplus Y_{i,n_i}$, where $Y_{i,j}$ are indecomposable $R_{art}-$modules.

Claim. Every indecomposable $R_{art}-$module is isomorphic to some $Y_{i,j}$.

Proof. Let Y be an indecomposable $R_{art}-$module. By (3) of the previous theorem, there exists an $R_{art}-$module Z such that $Y \oplus Z \cong M_{art}$ for some MCM $R-$module M. Write $M = X_1^{(r_1)} \oplus \cdots \oplus X_t^{(r_t)}$, so that $Y \oplus Z = (X_1)_{art}^{(r_1)} \oplus \cdots \oplus (X_t)_{art}^{(r_t)} = \oplus_{i,j} Y_{i,j}^{(s_{i,j})}$. Thus $Y \cong Y_{i,j}$ for some i, j by KRS.

For the backward direction, let $Y_1, ..., Y_n$ represent all indecomposable $R_{art}-$modules. Given a MCM $R-$module M, write $M_{art} \cong Y_1^{(r_1)} \oplus \cdots \oplus Y_n^{(r_n)}$. By KRS, the r_i's are uniquely determined by M. Define a function ϕ : {isomorphism classes of MCM $R-$modules} $\to \mathbb{N}^{(n)}$ (where $\mathbb{N} = \{0, 1, 2, ...\}$) by $[M] \mapsto (r_1, ..., r_n)$. By (1) of the previous theorem, ϕ is injective. Give each copy of $\mathbb{N}$ its natural order and give $\mathbb{N}^{(n)}$ the product poset structure: $(r_1, ..., r_n) \leq (s_1, ..., s_n)$ if and only if $r_i \leq s_i$ for all i.

Claim. M is indecomposable if and only if $\phi([M])$ is a minimal element of $\mathrm{im}(\phi)$.

Proof. The backward direction is clear as $\phi([U \oplus V]) = \phi([U]) + \phi([V])$. For the forward direction, suppose $\phi([M])$ is not minimal. Then there exists a MCM $R-$module N such that $\phi([N]) < \phi([M])$. Say $\phi([N]) = (r_1, ..., r_n)$ and $\phi([M]) = (s_1, ..., s_n)$. Let $(V, W) = Y_1^{(s_1 - r_1)} \oplus \cdots \oplus Y_n^{(s_n - r_n)} \neq 0$. Thus $N_{art} \oplus (V, W) \cong M_{art}$. By (4) of the previous theorem, there exists a MCM $R-$module U such that $N \oplus U \cong M$.

Now, we want to show $\mathrm{im}(\phi)$ has only finitely many minimal elements. Of course, the set of minimal elements of $\mathrm{im}(\phi)$ is a clutter (i.e., a totally disorded set) and, as the Exercise 4.1 below will show, it must therefore be finite! As ϕ is injective, this says there are only finitely many isomorphism classes of indecomposable MCM $R-$modules. $\square$

Note in the above theorem that if $R = \overline{R}$, then there is exactly one indecomposable module, namely R.

4.6 Theorem. *Let (R, m) be a one dimensional, local, reduced ring with $\overline{R}$ finitely generated. Then R has finite CM type if and only if*

(1) *$\overline{R}$ is generated by ≤ 3 elements as an $R-$module.*
(2) *$m\overline{R}/m$ is cyclic as an $R-$module.*

4.7 Examples. Let $R = k[[t^{a_1}, ..., t^{a_m}]]$, where $a_1 < \cdots < a_m$ and $(a_1, ..., a_m) = 1$. Then $\overline{R} = k[[t]]$ and is minimally generated by $1, t, ..., t^{a_1 - 1}$. Thus (1) of the theorem is satisfied if and only if $a_1 \leq 3$.

- This shows that $k[[t^4, t^5]]$ has infinite CM type.
- It can be seen that $k[[t^3, t^5]], k[[t^3, t^4]], k[[t^3, t^4, t^5]]$, and $k[[t^3, t^5, t^7]]$ all have finite CM type.
- Although it satisfies condition (1) of the theorem, $k[[t^3, t^7]]$ has infinite CM type.

4.8 Exercises.

4.1. (5 points) Let A and B be posets satisfying (a) DCC and (b) the finite clutter property (i.e., each clutter is finite). Then $A \times B$ also satisfies (a) and (b). From this, one can deduce that every clutter of $\mathbb{N}^{(n)}$ is finite.

§5. Towards Classifying Rings with Finite CM Type

5.0 Observations. Assume throughout (R, m, k) is a local, reduced, one-dimensional ring with ${}_R\bar{R}$ finitely generated, $A := R/\mathfrak{f} \hookrightarrow \bar{R}/\mathfrak{f} =: B$ is an Artinian pair. The minimal number of generators of an R-module M will be denoted $\mu_R(M)$.

5.1 Proposition. [5, Thm 2.1] *Since R is one-dimensional, $e(R)$, the multiplicity of R, is the number of generators required for $\mathfrak{m}^n$, for $n >> 0$. Then every ideal of R is generated by at most e elements. Moreover, e is exactly the number of generators required for $\bar{R}$ as an R-module.*

5.2 Lemma. $\mu_R(\frac{R+m\bar{R}}{R}) < \mu_R(\bar{R})$.

Proof. Notice that $1 \notin m(R+m\bar{R})$, so 1 is part of a minimal generating set for $R+m\bar{R}$. Therefore, as R is generated by 1 and we are modding out by R,

$$\mu_R(\tfrac{R+m\bar{R}}{R}) = \mu_R(R+m\bar{R}) - 1.$$

But $R+m\bar{R}$ is isomorphic to an ideal of R (this can be done by clearing denominators as R is finitely generated and multiplication by non-zero-divisors is an isomorphism on torsion-free modules). Therefore

$$\mu_R(R+m\bar{R}) < e_R(R) = \mu_R(\bar{R})$$

(see Proposition 5.1). Therefore

$$\mu_R(\tfrac{R+m\bar{R}}{R}) < \mu_R(R+m\bar{R}) \leq \mu_R(\bar{R})$$

□

The goal of this section is to develop the tools needed to prove the following theorem.

5.3 Theorem. *Assume $R \neq \overline{R}$ and either $\mu_R(\bar{R}) \geq 4$ or $\mu(\frac{m\bar{R}}{m}) \geq 2$. Then for all $n \geq 1$, there exists an indecomposable MCM ${}_RM$ such that $\bar{R}M \cong \bar{R}^n$.*

In particular, if R satisfies the condition of this theorem, R has infinite CM type. It is in fact the case that if $\mu(\overline{R}) \leq 3$ and $\mu_R(m\overline{R}/m) = 1$, then R has finite CM type.

5.4 Reductions for the Proof of Theorem 5.3.

(1) If there exists an indecomposable $(A \hookrightarrow B)$-module (V, W) with $W \cong B^n$ then we're done. For, take M to be the pullback

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d] & \bar{R}^n \arrow[d, "\pi"] \\ V \arrow[r, hook] & B^n \cong W \end{tikzcd}$$

where $M = \pi^{-1}(V)$. As M is torsion-free, M is MCM, and $V = M/\mathfrak{f}M$. If M is decomposable, say $M = P \oplus Q$, then we have a decomposition of the quotient as by NAK $P, Q \neq 0$ as $\mathfrak{f} \subseteq m$.

- Examining $\frac{m\bar{R}}{m}$; we have the following useful isomorphisms

$$\frac{m\bar{R}}{m} \cong \frac{m\bar{R}}{(m\bar{R} \cap R)} \cong \frac{R+m\bar{R}}{R} \cong \frac{\frac{R+m\bar{R}}{\mathfrak{f}}}{\frac{R}{f}} \cong \frac{A+mB}{A}$$

where the first isomorphism comes from the fact that $\overline{R}$ is finitely generated and so $m\overline{R} \neq \overline{R}$ by NAK (i.e., $m\overline{R}$ does not contain 1) and the second isomorphism comes from the Isomorphism Theorems.

- Based on Lemma 5.2, we have that $\mu_R(\frac{m\bar{R}}{m}) < \mu_R(\bar{R})$. Consequently, if $\mu_R(\bar{R}) \leq 2$, there is nothing to prove in regards to Theorem 5.3 as $\mu_R(m\overline{R}/m) \leq 1$. Also, if $\mu_R(\bar{R}) = 3$, then $\mu_R(\frac{m\bar{R}}{R}) \leq 2$.

(2) Suppose we can find a ring C with $A \subseteq C \subseteq B$ and an indecomposable $(A \hookrightarrow C)$-module (V, C^n). Then (V, B^n) is an indecomposable $(A \hookrightarrow B)$-module, and we are done.

(3) If we can build an indecomposable $(k \hookrightarrow C/mC)$-module $(V, (C/mC)^n)$, then we get an indecomposable $(A \hookrightarrow C)$-module, namely, the pullback

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & C \arrow[d] \\ A/m = k \arrow[r, hook] & C/mC \end{tikzcd} \qquad \begin{tikzcd} \pi^{-1}(V) =: V' \arrow[r, hook] \arrow[d] & C \arrow[d, "\pi"] \\ V \arrow[r, hook] & (C/mC)^n \end{tikzcd}$$

Check that this pullback works by checking idempotents as in the notes from the first day of class.

5.5 The Plan for Proving Theorem 5.3.

(a) If $\mu_R(\overline{R}) \geq 4$, then $\mu_A(B) \geq 4$ by NAK, so $\dim_k(B/mB) \geq 4$. We will work with $k \hookrightarrow D := B/mB$ by Reduction 3.

(b) If $\mu_R(\bar{R}) = 3$ and $\frac{A+mB}{A} \cong \frac{m\bar{R}}{m}$ is not cyclic (i.e., $\mu_R\left(\frac{A+mB}{A}\right) = 2$), let $C = A + mB$. Choose $x, y \in mB$ such that $A + mB = A \cdot 1 + Ax + Ay$, minimally. As $C = A + mB$, $C/mC = \frac{A+mB}{m+m^2B.}$ Reduced mod m, $C/mC = k \oplus k\bar{x} \oplus k\bar{y}$ where $\bar{x}^2 = \bar{y}^2 = \bar{x}\bar{y} = 0$. Nakayama's lemma gives us that x, y are minimal generators.

5.6 Basic Pathological Construction. Let k be a field and D a finite dimensional k-algebra (for example B/mB or C/mC). Suppose there are $a, b \in D$ such that $\{1, a, b\}$ is k-linearly independent (i.e., it is sufficient to assume that $\dim_k D \geq 3$.) Let

$$\phi = \begin{bmatrix} a & b & & & \\ & a & b & & \\ & & \ddots & \ddots & \\ & & & a & b \\ & & & & a \end{bmatrix}_{n \times n}$$

be a matrix over D. Take $W = D^n$ and let $V = \{x + \phi y | x, y \in k^n\}$. Note that $k^n \subseteq V$ by setting $y = 0$, so $DV = W$.

5.7 Theorem. *Assume that one of the following conditions hold.*

(i) $\{1, a, a^2, b\}$ *is k-linearly independent.*

(ii) $\{1, a, b\}$ *is k-linearly independent and* $a^2 = ab = b^2 = 0$.

(iii) $\{1, a, b, ab\}$ *is k-linearly independent and* $a^2 = b^2 = 0$.

(iv) (D, n, ℓ) *is local,* $\operatorname{char}(k) = 2$, *and* $\bar{a}^2, \bar{b}^2 \in k$ *(where* $\bar{a} = a + n$, $\bar{b} = b + n$*), and* $\{1, \bar{a}, \bar{b}, \bar{a}\bar{b}\}$ *is k-linearly independent.*

Then (V, W) *is indecomposable.*

Proof. Case (i): Assume that $\alpha \in \operatorname{End}(V, W)$. So α is an $n \times n$ matrix over W and $\alpha V \subseteq V$. Note that for all $x \in k^n$, $x \in V$, so $\alpha x \in V$ and $\alpha x = x' + \phi y'$ for $x', y' \in k^n$. Using the linear independence of $\{1, a, b\}$, one shows that x', y' are unique.

Note that $\phi = aI + bJ$ where I is the $n \times n$ identity matrix and

$$J = \begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & 0 & \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ & \ddots & \ddots & \ddots & 1 \\ & & & & 0 \end{bmatrix}_{n\times n}.$$

Also, $\alpha x = x' + ay' + bJy'(*)$ so $x' = \beta x$ and $y' = \gamma x$ for some $n \times n$ matrices β, γ over k. So $\alpha x = \beta x + a\gamma x + bJ\gamma x \Rightarrow \alpha = \beta + \alpha\gamma + bJ\gamma(**)$.

Given any $y \in k^n$, we know that $ay + bJy = \phi y \in V$. Therefore $\alpha(ay + bJy) \in V$. By $(*)$,

$$\alpha(ay + bJy) = \delta y + a\epsilon y + bJ\epsilon y,$$

where δ, ϵ are $n \times n$ matrices over k. Thus $\alpha(aI + bJ) = \delta + a\epsilon + bJ\epsilon$, and after simplification from $(**)$ we have,

$$(\dagger)\ 1 \cdot \delta = a(\beta - \epsilon) + a^2\gamma + ab(J\gamma + \gamma J) + b(\beta J - J\epsilon) + b^2 J\gamma J.$$

Our goal is to show that $\gamma = 0$. To do this we need the following claim.

Claim. $J^i\gamma J^j = 0$ for all $i, j = 1, \ldots, n$.

Proof. The proof will use "backwards double induction". Since $J^n = 0$, the claim holds if $i = n$ or $j = n$. Assume that $J^{i+1}\gamma J^j = 0$ and $J^i\gamma J^{j+1} = 0$; we will show that $J^i\gamma J^j = 0$. Multiplying by J^i, J^j to get $J^i \cdot (\dagger) \cdot J^j$ yields:

$$1 \cdot J^i\delta J^j = 1 \cdot aJ^i(\beta - \epsilon)J^j + a^2 J^i\gamma J^j + ab(\boldsymbol{J^{j+1}\gamma J^j + J^i\gamma J^{j+1}}) + b(J^i\beta J^{j+1} - J^{i+1}\epsilon J^j) + b^2(\boldsymbol{J^{i+1}\gamma J^{j+1}})$$

The terms in bold are zero by induction; by examining the entries of the "a^2" term we see by linear independence that $J^i\gamma J^j = 0$, as desired. This completes the proof

Now set $j = i = 0$. Thus $\gamma = 0$, as desired. So $(\dagger)$ simplifies to $1 \cdot \delta = a(\beta - \epsilon) + b(\beta J - J\epsilon)$. Therefore $\beta = \epsilon$ and $\beta J = J\epsilon$ which implies $\beta J = J\beta$. Further, $(**)$ shows $\alpha = \beta$.

So $\alpha \in k[J]$, and

$$\beta = \begin{bmatrix} a_1 & a_2 & \multicolumn{4}{c}{\dotfill} \\ 0 & a_1 & a_2 & \multicolumn{3}{c}{\dotfill} \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \ldots & 0 & a_1 & a_2 \\ 0 & \multicolumn{4}{c}{\dotfill} & a_1 \end{bmatrix}.$$

If $\alpha^2 = \alpha$, then $\alpha = 1$ or 0. Hence (V, W) is indecomposable.

Cases (ii) and (iii): In each case, we look at the "1", "a", and "b" terms of equation $(\dagger)$ and deduce that $\delta = 0$ (irrelevant), $\beta = \epsilon$, and $\beta J = J\epsilon$. Thus β commutes with J and is therefore in $k[J]$. (In case (iii) we look also at the "ab" term, and get the irrelevant information that $J\gamma + \gamma J = 0$.) Thus we have $\alpha = \beta + a\gamma + bJ\gamma$, and we know that $\beta \in k[J]$.

Now assume α is idempotent, and expand the equation $\alpha = \alpha^2$, getting

$$\beta + a\gamma + bJ\gamma = \beta^2 + a(\beta\gamma + \gamma\beta) + b(\beta J\gamma + J\gamma\beta),$$

plus an irrelevant "ab" term in case (iii).

Comparing the "1" terms, we get $\beta = \beta^2$. Since, as we noted before, $k[J]$ has no idempotents other than 0 or 1, we see that $\beta = 0$ or $\beta = 1$ (i.e., β is either the 0-matrix or the identity matrix).

Now compare the "a" terms in the equation above: We see that $\gamma = \beta\gamma + \gamma\beta$. Obviously either possibility for β (0 or 1) forces $\gamma = 0$. Thus $\alpha = \beta = 0$ or 1, and we're done. $\square$

5.8 Exercises.

(5.1) (6 points) Prove Case (iv) of Theorem 5.7. (This exercise comes highly recommended.)[**Hint:** Note that the parenthetical "irrelevant" remarks that appear in the proof of case (ii) and (iii) *are* relevant for this case.]

(5.2) (6 points) Let k be a field and D a finite dimensional k-algebra with $\dim_k D \geq 4$. Then either

  (a) there exist $a, b \in D$ such that $\{1, a, b\}$ is k-linearly independent and one of (i), (ii) (iii), (iv) of Theorem 5.7 hold.

  (b) $|k| = 2$ and $D = \Pi_{i \geq 4} k$.

(5.3) (5 points) Suppose k is a field and $D = k^{(s)}$, where $s \geq 4$. Then for all n there exists an indecomposable $(k \hookrightarrow D)$-module $(V, D^{(n)})$. Prove this. To do so, let $V = \{(x, y, x + y, x + Jy, x, \dots, x) \mid x, y \in k^{(n)}\}$. Then, show that $(V, D^{(n)})$ is an indecomposable $(k \hookrightarrow D)$-module, that is, $VD = D^{(n)}$. [Hint: Take an idempotent endomorphism of $D^{(n)}$ and show it is 0 or 1. D has s orthogonal idempotents, say $e_1, \dots, e_s$ with sum 1: $e_1 = (1, 0, \dots, 0)$, etc. A D-linear map $\alpha : D^{(n)} \to D^{(n)}$ has to satisfy $\alpha(e_i z) = e_i \alpha(z)$. Thus $\alpha = (\alpha_1, \dots, \alpha_s)$, where $\alpha_i : k^{(n)} \to k^{(n)}$.]

§6. Matrix Factorizations

Let $(S, \mathfrak{N}, k)$ be a regular local ring of dimension $d \geq 1$. Here are some basic facts about regular local rings:

(1) Each finitely generated S-module has finite projective dimension. I.e., there exists an exact sequence of the form
$$0 \to F_s \to \cdots \to F_0 \to M \to 0,$$
where each F_i is a finitely generated free S-module. The least such s is called the projective dimension. In fact, we have
$$\mathrm{pd}_S M = d - \mathrm{depth(M)},$$
for each finitely generated S-module (this is the Auslander-Buchsbaum Formula). Note that if $\mathrm{pd}_S M = 0$, then M is isomorphic to a free module.

(2) S is Cohen-Macaulay.

(3) S is an integral domain. (Note CM is key to proving this. The proof is by induction.)

(4) Let $0 \neq f \in \mathfrak{N}$, and put $R := S/(f)$. We almost always will assume that $f \in \mathfrak{N}^2$. [Note that $(R, \mathfrak{M}, k)$ is a regular local ring if and only if $f \notin \mathfrak{N}^2$. Here we really do have the same residue field as $R/\mathfrak{M} = (S/f)/(\mathfrak{N}/f) = S/\mathfrak{N} = k$.] With R defined in this way, Krull's Principal Ideal Theorem says $\dim R \geq d - 1$. Of course, $f \neq 0$ and $f \in \mathfrak{N}$ implies $\dim R = d - 1$, as we have exactly one less prime in R.

(5) R is Cohen-Macaulay.

(6) If M is a finitely generated R-module, then $\mathrm{depth}_S M = \mathrm{depth}_R M$. [See exercise 6.1]

Now, suppose ${}_R M$ is MCM. Then
$$\mathrm{depth}_S M = \mathrm{depth}_R M = \dim R = d - 1,$$
and therefore $\mathrm{pd}_S M = 1$ by the Auslander-Buchsbaum Formula. Let $\mu_S M = n = \mu_R M$ (note that n is unique since it is taken to be minimal). Then we get a resolution
$$0 \to S^{(r)} \to S^{(n)} \to M \to 0.$$
Note that $fM = 0$ and $f \neq 0$, so M is a torsion S-module. Therefore $K \otimes_S M = 0$ ($K = S_{(0)}$ and $M_{(0)} = 0$). So tensoring the above exact sequence gives the exact sequence
$$0 \to K^{(r)} \to K^{(n)} \to 0,$$

whence $r = n$. Therefore we have the exact sequence

$$(*) \qquad 0 \to S^{(n)} \xrightarrow{\phi} S^{(n)} \to M \to 0.$$

Notice $M = S^{(n)}/\operatorname{im}\phi$. If $x \in S^{(n)}$, then $fx \subseteq \operatorname{im}\phi$, since $fM = 0$. Therefore there exists a unique $x' \in S^{(n)}$ such that $\phi(x') = fx$ (unique as ϕ is 1-1). Since we have uniqueness, we can define a map $\psi : S^{(n)} \to S^{(n)}$ by $\psi(x) = x'$. Then $\phi(\psi(x)) = fx$. Notice also that

$$\phi(\psi(x_1 + x_2)) = f(x_1 + x_2) = fx_1 + fx_2 = \phi(\psi(x_1)) + \phi(\psi(x_2)) = \phi(\psi(x_1) + \psi(x_2)),$$

and as ϕ is 1-1, we have $\psi(x_1 + x_2) = \psi(x_1) + \psi(x_2)$. Similarly, we can show that ψ respects scalar multiplication, so that $\psi : S^{(n)} \to S^{(n)}$ is S-linear. Now, thinking of maps as $n \times n$ matrices, we have $\phi\psi = fI_n$ (where I_n is the $n \times n$ identity). Therefore

$$\phi\psi\phi = (fI_n)\phi = \phi f I_n,$$

implying that $\psi\phi = fI_n$ because ϕ is 1-1. It can also be shown that ψ is 1-1, so we have two maps ψ, ϕ such that

$$\phi = \begin{bmatrix} f & & \\ & \ddots & \\ & & f \end{bmatrix} \quad \text{and} \quad \psi = \begin{bmatrix} f & & \\ & \ddots & \\ & & f \end{bmatrix},$$

which give rise to the next definition.

6.1 Definition. A *matrix factorization* (of f with rings given, etc.) is a pair (ϕ, ψ) of $n \times n$ matrices such that

$$\phi\psi = fI_n = \psi\phi.$$

We have seen that every MCM R-module gives rise to a matrix factorization. Suppose now that ${}_RM_1 \cong {}_RM_2$, both MCM. We can draw the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & S^{(n)} \arrow[r, "\phi_1"] \arrow[d, dotted, "\exists!\beta"] & S^{(n)} \arrow[r] \arrow[d, dotted, "\exists\alpha"] & M_1 \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & S^{(n)} \arrow[r, "\phi_2"] & S^{(n)} \arrow[r] & M_2 \arrow[r] & 0
\end{tikzcd}$$

As $S^{(n)}$ is projective and $M_1 \xrightarrow{\cong} M_2$, we can lift to $\alpha : S^{(n)} \to S^{(n)}$. Further, α is surjective by Nakayama's Lemma, since the entries of ϕ_1 are in $\mathfrak{N}$ (n is minimal if and only if the entries are in $\mathfrak{N}$). Therefore, (by Exercise 6.2) α is 1-1. Similarly, there exists a unique $\beta : S^{(n)} \to S^{(n)}$ making the diagram commute. By the Five Lemma, β is an isomorphism.

We've just shown that if $M_1 \cong M_2$ and $(\phi_1, \psi_1), (\phi_2, \psi_2)$ are the corresponding matrix factorizations, then there exists a commutative diagram (see exercise 6.3)

$$(\dagger) \qquad \begin{tikzcd}
S^{(n)} \arrow[r, "\psi_1"] \arrow[d, "\alpha", "\cong"'] & S^{(n)} \arrow[r, "\phi_1"] \arrow[d, "\beta", "\cong"'] & S^{(n)} \arrow[d, "\alpha", "\cong"'] \\
S^{(n)} \arrow[r, "\psi_2"] & S^{(n)} \arrow[r, "\phi_2"] & S^{(n)}.
\end{tikzcd}$$

The above shows that isomorphic MCM modules give equivalent matrix factorizations. In fact, the converse is true (see exercise 6.4). A matrix factorization gives rise to a MCM module and equivalent matrix factorizations give isomorphism MCM modules.

6.2 Example. If ${}_RM$ is MCM with R, S as above and $0 \to S \xrightarrow{f} S \to R \to 0$, the corresponding matrix factorization is $(f, 1)$.

6.3 Definition. A matrix factorization (ϕ, ψ) is *reduced* provided the entries of both ϕ and ψ are in $\mathfrak{N}$.

6.4 Fact. (in Yoshino) Let (ϕ, ψ) correspond to ${}_RM$, MCM. Then (ϕ, ψ) is reduced iff M has no direct summand isomorphic to R. This is sometimes said as, "M has no free summand."

6.5 Example. Let $S = k[[x, y]]$, and let $f = -x^4 + y^5$. Find a matrix factorization of f. To do this, we cheat and resolve k over R first:

$$0 \leftarrow k \leftarrow R \xleftarrow{[x,y]} R^{(2)} \xleftarrow{\phi} R^{(2)} \xleftarrow{\psi} R^{(2)} \leftarrow \cdots,$$

where $\phi = \begin{bmatrix} y & -x^3 \\ -x & y^4 \end{bmatrix}$ and $\psi = \begin{bmatrix} y^4 & x^3 \\ x & y \end{bmatrix}$. Now forget about R and think of ϕ, ψ as matrices over S. Check that $\phi\psi = \begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix}, \psi\phi = \begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix}$.

6.6 Theorem. Let (ϕ, ψ) be a matrix factorization of f $(R = S/(f))$. Let ${}_RM = \operatorname{coker} \phi$, a MCM by exercise. Then the sequence

$$\cdots \xrightarrow{\overline{\phi}} R^{(n)} \xrightarrow{\overline{\psi}} R^{(n)} \xrightarrow{\overline{\phi}} R^{(n)} \to M \to 0$$

is exact, where $\overline{\phi}$ and $\overline{\psi}$ are reductions modulo f (and $n \times n$ with entries in R).

Simple Singularities

Let S be a regular local ring, $(S, \mathfrak{N}, k)$, and let $f \in \mathfrak{N}^2 \setminus \{0\}$. Define $R = S/(f)$, $(R, \mathfrak{M}, k)$. We say R is a *simple singularity* (relative to the presentation $S \twoheadrightarrow R$) provided there exist only finitely many ideals $I \subseteq S$ such that $I \subseteq \mathfrak{N}$ and $f \in I^2$.

6.7 Theorem. If R has finite CM-type, then R is a simple singularity.

Proof. Let M be a maximal CM R-module with no free summand. Let (ϕ, ψ) be a corresponding reduced matrix factorization. "Define" $I(M)$ to be the ideal of S generated by the entries of ϕ and ψ. Note that $I(M)$ depends only on the isomorphism class of M and $f \in I(M)^2$. To complete the proof, see exercise 6.5. □

6.8 Exercises.

(6.1) (4 points) If M is a finitely generated R-module, then $\operatorname{depth}_S M = \operatorname{depth}_R M$. (Proof uses stuff from day 1)

(6.2) (3 points) Let X be a Noetherian left A-module. Then every surjective A-endomorphism of X is injective.

(6.3) (2 points) In the commutative diagram (†), using the fact that $\alpha\phi_1 = \phi_2\beta$, show that $\psi_2\alpha = \beta\psi_1$ (I.e., the matrix factorizations are equivalent).

(6.4) (4 points) Let (ϕ, ψ) be a matrix factorization, and let $M := \operatorname{coker} \phi$. Show that $fM = 0$. Therefore, M is an R-module. Show M is MCM (as an R-module).

(6.5) (6 points)

(a) Show that $I(M \oplus N) = I(M) + I(N)$.

(b) Show that if $I \subseteq S$ is an ideal, $f \in I^2$, and $I \subseteq \mathfrak{N}$, then there exists a reduced matrix factorization of f such that $I(M) = I$, for the corresponding MCM ${}_RM$.

Hint: Let $I = (x_0, \ldots, x_r)$. Write $f = x_0y_0 + \cdots + x_ry_r$, where $y_i \in I$. Let $f_s = x_0y_0 + \cdots + x_sy_s$ for $0 \leq s \leq r$. Show for each $s = 0, \ldots, r$, there exist $2^s \times 2^s$ matrices ϕ_s, ψ_s with entries in I such that $\phi_s\psi_s = f_sI_{2^s \times 2^s}$. Then

$$\phi_{s+1} = \begin{bmatrix} x_{s+1} & \phi_s \\ \psi_s & -y_{s+1} \end{bmatrix} \quad \text{and} \quad \psi_{s+1} = \begin{bmatrix} -y_{s+1} & \phi_s \\ \psi_s & x_{s+1} \end{bmatrix}.$$

(c) Complete the proof.

7.0 Convention. Let $(R,\mathfrak{m},k)$ be a CM local ring containing a field.

7.1 The Goal. We wish to classify those R that have finite CM-type. To do so, we will assume, at first, that R is complete and k is algebraically closed with $char\ k=0$ (in fact, we only need that $char\ k\neq 2,3,5$). Later we'll relax these assumptions.

Step 1. List the 1-dimensional ADE singularities.
Step 2. If $\dim(R)=1$ and R satisfies (DR): $\mu_R\left(\bar{R}\right)\leq 3$ and $\mu_R\frac{\mathfrak{m}\bar{R}}{\mathfrak{m}}\leq 1$, then R dominates an ADE singularity. (We've shown the DR conditions are necessary for finite CM type. Now we will show the converse).
Step 3. Prove that 2-dimensional ADE singularities have finite CM-type. (This will be a bit sketchy.)
Step 4. Deduce that 1-dimensional ADE singularities have finite CM-type (and therefore are simple).
Step 5. If $R=\dfrac{k\left[\left[x_0,x_1,...,x_n\right]\right]}{(f)}$ where $0\neq f\in\left(x_0,...,x_n\right)^2$, k is algebraically closed and char$k=0$, then R has finite CM-type if and only if R is ADE if and only if R is simple.

7.2 (Step 1) ADE singularities in dimension one.

Let $k=\bar{k}$, char$k=0$. The ADE singularities are all of the form $\dfrac{k\left[\left[x,y\right]\right]}{(f)}$ where f is one of the following:

$$\begin{array}{lll} (A_n) & x^2+y^{n+1} & (n\geq 1)\\ (D_n) & x^2y+y^{n-1} & (n\geq 4)\\ (E_6) & x^3+y^4 & \\ (E_7) & x^3+xy^3 & \\ (E_8) & x^3+y^5 & \end{array}$$

We stipulate $n\geq 1$ for (A_n) as $(A_0):\dfrac{k\left[\left[x,y\right]\right]}{(x^2+y)}\cong k\left[\left[x\right]\right]$, a discrete valuation ring. Here, x^2+y is not a singularity, but rather is smooth!

The requirement that $n\geq 4$ for (D_n) is a bit more complicated. Consider $(D_3):\dfrac{k\left[\left[x,y\right]\right]}{(x^2y+y^2)}=\dfrac{k\left[\left[x,y\right]\right]}{(y\left(x^2+y\right))}$. Notice $y\left(x^2+y\right)$ is the graph of two smooth curves tangent to each other. This is in fact isomorphic to (A_3) as for (A_3) we have $f=x^2+y^4=\left(x+iy^2\right)\left(x-iy^2\right)$, which is also two smooth curves tangent to each other. The change of coordinates $(A_3)\to(D_3)$ given by $x\mapsto\sqrt{-2i}y$ and $y\mapsto x+iy^2$ shows that $(A_3)\cong(D_3)$.

7.3 Fact. Let $(R,\mathfrak{m})$ be local, CM, with $\mu_R\bar{R}\leq 2$. Then R is Gorenstein. Consequently, every MCM R-module is reflexive (ie, $M\to M^{**}$ is an isomorphism where $M^*=Hom_R(M,R)$). This will be useful in proving the following result.

7.4 Proposition. (A_n), *n even, has finite CM-type.*

Proof. Note that $R=\dfrac{k\left[\left[x,y\right]\right]}{(x^2+y^{n+1})}\cong k\left[\left[t^2,t^{n+1}\right]\right]$ via the map $x\mapsto t^{n+1}$ and $y\mapsto -t^2$. So $\overline{R}=k\left[\left[t\right]\right]=R\cdot 1+R\cdot t$. Let $R_i=k[[t^2,t^{n+1-2i}]]$. Then, we have the chain

$$\underbrace{R=R_1}_{(A_n)}\subset\underbrace{R_2}_{(A_{n-2})}\subset\cdots\subset R_{\frac{n}{2}-1}\subset\underbrace{R_{\frac{n}{2}}}_{k[[t]]=\overline{R}}.$$

Let ${}_RM$ be MCM, with no free summand (that is, R is not a summand). We'll show M is an R_2-module. Let $M^*=Hom_R(M,R)$. Note if there were a surjective homomorphism $M\twoheadrightarrow R$ then $M=R\oplus N$; therefore, $Hom_R(M,R)=Hom_R(M,\mathfrak{m})$. $Hom_R(M,\mathfrak{m})$ is a module over $S:=Hom_R(\mathfrak{m},\mathfrak{m})$. Therefore, so is M^{**} and so, by the fact, M is an S-module. Now, $Hom_R(\mathfrak{m},\mathfrak{m})\cong(\mathfrak{m}:_K\mathfrak{m})=\{\alpha\in K:\alpha\mathfrak{m}\subseteq\mathfrak{m}\}$ where K is the quotient field. Furthermore, as $\mathfrak{m}=\left(t^2,t^{n+1}\right)=\left\{\text{power series of the form } at^2+bt^4+\cdots+ct^n+dt^{n+1}+et^{n+2}+\cdots\right\}$, we see $t^i\mathfrak{m} \subseteq \mathfrak{m}$ if and only if i is even or $i \geq n-1$. Therefore, $Hom_R(\mathfrak{m},\mathfrak{m}) \cong (\mathfrak{m} :_K \mathfrak{m}) = k\left[\left[t^2, t^{n-1}\right]\right] = R_2\ (A_{n-2})$. Thus M is an R_2-module.

Now look at ${}_{R_2}M$. Write $M = R_2^{(r)} \oplus N$ where ${}_{R_2}N$ has no free summand. If $N \neq 0$, repeat! (Note we can repeat since $\bar{R}$ is the integral closure of R_2 and $\bar{R} = R_2 \cdot 1 + R_2 \cdot t$.) If, by the time we get to $(A_2) = k\left[\left[t^2, t^3\right]\right]$, we still have $N' \neq 0$, then N' is an $k[[t]] = \bar{R}$ - module. Since $\bar{R}$ is a DVR, $N' \cong \bar{R}^{(s)}$ (finitely generated torsion free modules over a DVR are free). Thus, every MCM module is a direct sum of copies of $R_1, R_2, ..., R_{\frac{n}{2}} = \bar{R}$. Therefore, R has exactly $\frac{n}{2}$ non-isomorphic indecomposable MCM modules. $\square$

7.5 Proposition. (E_6) *is simple.* (Note that (E_6) being simple means that $k[[x,y]]$ has only finitely many ideals I such that $x^3 + y^4 \in I^2$.)

Proof. Suppose that $x^3 + y^4 \in I^2$. We'll show that I is either $\mathfrak{n} = (x,y)$ or $\left(x, y^2\right)$. Note if $f, g \in I$, then $\frac{\partial}{\partial x}(fg) = \left(\frac{\partial}{\partial x}f\right)g + f\left(\frac{\partial}{\partial x}g\right) \in I$. Therefore, $\frac{\partial}{\partial x}\left(x^3+y^4\right) \in I$ and $\frac{\partial}{\partial y}\left(x^3+y^4\right) \in I$. So $x^2 \in I$ and $y^3 \in I$ (assuming characteristic not equal to 2 or 3). Thus, $\left(x^2, y^3\right) \subseteq I$. If $I \subseteq \mathfrak{n}^2$, then $x^3 + y^4 \in \mathfrak{n}^4$, which is false. $\therefore I \nsubseteq \mathfrak{n}^2$.

Choose $g \in I - \mathfrak{n}^2$, say $g = ax + by +$ (higher degree terms), with $a \neq 0$ or $b \neq 0$. Then $g \notin \mathfrak{n}^2$, so $\mathfrak{n}/(g)$ is principal. Therefore $k[[x,y]]/(g)$ is a DVR, and it follows that I has to be $\mathfrak{m}^j + (g)$ for some $j \geq 1$ (since the non-zero ideals of $k[[x,y]]/(g)$ are the ideals $(\mathfrak{m}^j + (g))/(g)$).

Case 1. $a = 0$. Then $b \neq 0$, so wlog, $b = 1$. We have $g = y + cxy + dy^2 + exy^2 + \underbrace{\left(\text{elt of } \left(x^2, y^3\right)\right)}_{\in I} \in I$.

Therefore, $y + cxy + dy^2 + exy^2 = y\underbrace{(1 + cx + y + exy)}_{\text{a unit}} \in I$, which implies $y \in I$. So $\left(x^2, y\right) \subseteq I \subseteq (x,y)$.

But $\dfrac{k[[x,y]]}{(y)} \cong k[[x]]$ is a DVR (whose ideals are only the powers of x), so $\dfrac{(x,y)}{(x^2,y)}$ is simple. Therefore, $I = \left(x^2, y\right)$ or $I = (x,y)$. But $x^3 + y^4 \notin \left(x^2, y\right)^2$. So $I = (x,y)$.

Case 2. $a \neq 0$, wlog, $a = 1$. Then $g = x + by + cxy + dy^2 + exy^2 + \left(\text{elt of } \left(x^2, y^3\right)\right) \in I$. $\therefore h := x + by + cxy + dy^2 + exy^2 \in I$.

Suppose $b \neq 0$. Then $h = x + y(b + cx + dy + exy) \in I$, and so I contains

$$h\left(x - y\left(b + cx + dy + exy\right)\right) = \underbrace{x^2}_{\in I} - y^2\left(b + cx + dy + exy\right)^2.$$

Therefore, $y^2\underbrace{(b + cx + dy + cxy)}_{\text{a unit}} \in I \Rightarrow y^2 \in I$. Thus, $x + by + cxy = h - y^2(d + ex) \in I$. Multiply by y to get $xy \in I$. Therefore, looking back at g we see $x + by \in I$. We now have $(x + by) + \mathfrak{n}^2 \subseteq I \subseteq \mathfrak{n}$, and since $k[[x,y]]/(x+by)$ is a DVR we conclude that I is either (x,y) or $(x+by)+\mathfrak{n}^2$. If $I = (x+by)+\mathfrak{n}^2$, then $x^3 + y^4 \in I^2 = (x+by)^2 + (x+by)\mathfrak{n}^2 + \mathfrak{n}^4 \Rightarrow x^3 \in (x+by)^2 + (x+by)\mathfrak{n}^2 + \mathfrak{n}^4$, which is a contradiction. (Set $x = -by$. Then this would say $-b^3y^3 \in \mathfrak{n}^4$, which is false.) So $I = (x,y)$ in this case.

Finally, suppose $b = 0$. Then we have $x + cxy + dy^2 + exy^2 \in I$. Multiply by y to get $xy + cxy^2 \in I$. Then $xy\underbrace{(1 + cy)}_{\text{a unit}} \in I \Rightarrow xy \in I$. Now, $x + dy^2 \in I$. Now $(x + dy^2) + \mathfrak{n}^3 \subseteq I$. Since $k[[x,y]]/(x+dy^2)$ is a DVR, I has to be one of the following: $(x+dy^2)+\mathfrak{n}^3$, $(x+dy^2)+\mathfrak{n}^2$, or $\mathfrak{n}$. The second possibility forces $I = (x, y^2)$, so we just have to rule out the first possibility. We assume that $I = (x+dy^2)+\mathfrak{n}^3$ and seek a contradiction. We have $x^3 + y^4 \in I^2 = (x+dy^2)^2 + (x+dy^2)\mathfrak{n}^3 + \mathfrak{n}^6$. Substituting $x = -dy^2$, we get $-d^3y^6 + y^4 \in \mathfrak{n}^6$, contradiction. $\square$

7.6 Exercises.

7.1 (3 points) Show (A_n) is a simple singularity for any n.

§8. Dominating an ADE Singularity (Step 2)

8.0 Convention. Let $(R, \mathfrak{m}, k)$ be a one-dimensional reduced local ring, with finitely generated integral closure. Assume further that R is complete, contains a field, and char $k \neq 2, 3, 5$. Also, assume that R satisfies the Drozd-Roĭter conditions: $e(R) \leq 3$ and $\mu_R(\mathfrak{m}\bar{R}/\mathfrak{m}) \leq 1$.

8.1 Remarks.

(1) Recall that Cohen's Structure Theorem implies that R contains a coefficient field; that is, a subring that maps isomorphically onto k via the natural map $R \to R/\mathfrak{m}$. So, we assume that $k \subseteq R$.
(2) Note that the first Drozd-Roĭter condition is equivalent to saying that $\mu_R(\bar{R}) \leq 3$, since $e(R) = \mu_R(\bar{R})$ in this situation.
(3) Also note that $\mu_R(\mathfrak{m}\bar{R}/\mathfrak{m}) = \dim_k \frac{\mathfrak{m}\bar{R}+R}{\mathfrak{m}^2\bar{R}+R}$.

8.2 Theorem. *The ring R above birationally dominates an ADE singularity. That is, there exists an ADE singularity A such that $A \subseteq R \subseteq \bar{A}$.*

We should discuss what this means. We say that two domains are *birationally equivalent* if they have the same quotient field. A ring R *dominates* enough ring A if the maximal ideal of R contracts to the maximal ideal of A. To show R birationally dominates A, then really does mean that $A \subseteq R \subseteq \overline{A}$. The point of this theorem is that if A has finite Cohen Macaulay type, then R has finite Cohen Macaulay type. Thus it is enough to show that the ADE singularities have finite Cohen Macaulay type to show that R does. In order to prove this theorem, we must first take a brief excursion to Henselian rings.

8.3 Definition. A local ring (S, n, l) is *Henselian* provided it satisfies one of the following equivalent conditions:

(1) If $f \in S[t]$ is a monic polynomial, and $\alpha \in l$ is a simple root (that is, not a multiple root) of $\bar{f} \in l[t]$, then there exists a $\beta \in S$ so that $f(\beta) = 0$ and $\bar{\beta} = \alpha$, where we are reducing modulo n.
(2) Every commutative, module-finite extension ring of S is a direct product of finitely many local rings. In other words, $S \hookrightarrow T$ with T finitely generated over S implies that $T \cong \Lambda_1 \times \cdots \times \Lambda_n$, where each Λ_i is local.

8.4 Hensel's Lemma. *Complete rings are Henselian.*

Note that Hensel verified condition 1.

Complete rings are often nicer to work with because of their qualities. In essence, while local rings capture local behaviour of a singularity, the completion of a local ring at its maximal ideal captures the *extremely* local behavior of the singularity. The following example illustrates this point.

8.5 Example. Let $R = k[x, y]/(y^2 - x^3 - x^2)$, localized at the maximal ideal (x, y). This is the local ring at the singularity in the nodal cubic. Before completing, R is a domain. However, $\hat{R}$ is *not*. Indeed, $\hat{R} \cong [\![x, y]\!]/(y^2 - x^3 - x^2) \cong k[\![t]\!] \times k[\![t]\!]$.

- To prove this, we first show $y^2 - x^3 - x^2 = y^2 - x^2(x+1)$ factors as a difference of squares. By the binomial expansion, we see $g := (1+x)^{\frac{1}{2}} \in R$:

$$g = 1 + \frac{1}{2}x + \binom{\frac{1}{2}}{2}x^2 + \cdots \in k[\![x, y]\!] \text{ where } \binom{\frac{1}{2}}{n} = \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!}.$$

Note here that when we simplify the coefficients, we see $g = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots$, where all the denominators are powers of 2. We rely on the fact that the characteristic is not 2 for this to make sense.

So, $\hat{R} \cong k[\![x, y]\!]/(y + xg)(y - xg)$, and hence

$$\bar{\hat{R}} \cong k[\![x, y]\!]/(y + xg) \times k[\![x, y]\!]/(y - xg) \cong k[\![t]\!] \times k[\![t]\!].$$

Thus the ring R gives us the singularity of the nodal cubic whereas $\hat{R}$ gives us the extremely local view of this similarity, that is, we get essentially the curves $1+x, 1-x$.

Along these lines, we first prove a preliminary lemma on Henselian local rings, which says we can extract roots of units (like $1+x$ above).

8.6 Lemma. *Let (S, n, l) be a Henselian local ring, let $u \in S$ be a unit, and let n be a positive integer. Suppose that $\bar{u} \in l$ has an n^{th} root and char $l \nmid n$. Then u has an n^{th} root in S.*

Proof. Consider the monic polynomial $h = X^n - u \in S[X]$. Then $\bar{h}$ has a simple root, as char $l \nmid n$. So, we can choose $v \in S$ such that $h(v) = 0$, i.e., $v^n = u$. □

We now begin the proof of Theorem 8.2.

Proof of Theorem 8.2. If $P_1, \ldots, P_s$ are the minimal primes of R, then

$$\overline{R} \cong \overline{(R/P_1)} \times \cdots \times \overline{(R/P_s)},$$

which needs at least s generators as an R-module. Therefore, $s \leq 3$ by the Drozd-Roĭter conditions.

Case $s = 1$: Since R is reduced with a unique minimal prime, R is a domain. Also, $\bar{R}$ is a domain that is a direct product of DVRs by Hensel's Lemma (part 2), and so $\bar{R} \cong k[\![t]\!]$. We have $k \subseteq R \subseteq \bar{R} = k[\![t]\!]$. Our goal is to find copies of t^2, t^3, etc in R. However, we cannot do this without a little work.

Since $\dim_k \bar{R}/\mathfrak{m}\bar{R} = e$, $\mathfrak{m}\bar{R} = t^r\bar{R}$ for some r, and $\dim_k \bar{R}/t^r\bar{R} = r$, we see that $r = e$. Thus, $\mathfrak{m}\bar{R} = t^e\bar{R}$. But, $\mathfrak{m}\bar{R} = b_1\bar{R} + \cdots + b_p\bar{R}$ for suitable $b_i \in \mathfrak{m}$. Since $b_i\bar{R} = t^{m_i}\bar{R}$ for some m_i, we must have $\mathfrak{m}\bar{R} = b\bar{R}$ for $b = b_j$ satisfying $m_j \leq m_i$ for $1 \leq i \leq p$. Therefore, we must have $b\bar{R} = t^e\bar{R}$. Hence, there is a unit u of $\bar{R}$ such that $ut^e = b$.

Since char $k \neq 2, 3$, we can extract an e^{th} root v such that $v^e = u$. Now $(vt)^e = b$. After the change of variable $t \mapsto vt$, we have $t^e = b \in \mathfrak{m}$. We break the remainder of this case into the following subcases:

- $e = 1$: In this situation, $R = \bar{R} = k[\![t]\!]$, and so take $A = k[\![t^2, t^3]\!]$ to get $A \subseteq R \subseteq \overline{A}$. Note that this is essentially a trivial case.
- $e = 2$: We have that $t^2 \in R$. Let $\mathfrak{f} = (R :_R \bar{R})$ be the conductor, say $\mathfrak{f}\bar{R} = t^n\bar{R}$, for some n. It is an exercise to prove n is even. Then, $t^{n+1} = t^n t \in \mathfrak{f} \subseteq R$, so $t^{n+1} \in R$. Therefore R birationally dominates the (A_n) singularity $k[\![t^2, t^{n+1}]\!]$.
- $e = 3$: Exercises 8.2-8.5 lead us to find an element $a \in k$ such that either $t^4 + at^5 \in R$ (if $v(\gamma) = 4$) or $t^5 + at^7 \in R$ (if $v(\gamma) = 5$) Then either $k[\![t^3, t^4 + at^5]\!] \subseteq R$ or $k[\![t^3, t^5 + at^7]\!] \subseteq R$. In the following section, we look at these rings to show they are in fact isomorphic to ADE singularities.

□

8.6. Exercises

8.1. (2 Points) Prove that n is even (when $e = 2$).

8.2. (3 Points) Show that there are elements α and β in $\mathfrak{m}\bar{R}$ with $v(\alpha) = 4$ and $v(\beta) = 5$, such that the images of t^3, α, and β form a k-basis of $\frac{\mathfrak{m}\bar{R}}{\mathfrak{m}^2\bar{R}}$.

8.3. (3 Points) Conclude that there exists $\gamma \in R$ such that $v(\gamma) = 4$ or $v(\gamma) = 5$.

8.4. (5 points) Assuming that $t^3 \in R$ and R contains an element γ with $v(\gamma) = 4$, prove the following:
   (1) For each positive number n, R contains an element δ_n such that $v(t^7 - \delta_n) \geq n$ (and therefore $t^7 \in R$).
   (2) For each positive integer n, R contains an element ϵ_n such that $v(t^8 - \epsilon_n) \geq n$ (and therefore $t^8 \in R$).

8.5. (3 Points) Suppose $v(\gamma) = 4$. Then there exists $a \in k$ such that $t^4 + at^5 \in R$.

§9. Completing the proof of Theorem 8.2

We now complete the proof of Theorem 8.2. We will finish the case $s = 1$ and $e = 3$ and then examine the case $s = 2$. We leave the proofs for $s = 2, e = 3$ and $s = 3$ for the next section.

Proof. (Continued) (*Case* $s = 1, e = 3$) First, suppose $k[\![t^3, t^4 + at^5]\!] \subseteq R$. Notice that $t^6, t^7, t^8, \ldots \in k[\![t^3, t^4 + at^5]\!]$. Define an automorphism ϕ of $k[\![t]\!]$ (which will take $k[\![t^3, t^4]\!]$ to $k[\![t^3, t^4 + at^5]\!]$) as follows:

$$t \mapsto t + \frac{1}{4}at^2 + \frac{3}{16}a^2t^3 = t + \frac{1}{4}at^2\left(1 + \frac{3}{4}at\right)$$

(note that we use characteristic not 2 here).

Claim. $\phi(k[\![t^3, t^4]\!]) = k[\![t^3, t^4 + at^5]\!]$, so the rings are isomorphic.

Proof. First we consider the images of t^3 and t^4:

$$\begin{aligned} t^3 \mapsto \quad & t^3 + \tfrac{3}{4}at^4\left(1 + \tfrac{3}{4}at\right) + \tfrac{3}{16}a^2t^5\left(1 + \tfrac{3}{4}at\right)^2 + (\text{stuff}) \\ & = t^3 + \tfrac{3}{4}at^4 + \tfrac{9}{16}a^2t^5 + \tfrac{3}{16}a^2t^5 + (\text{stuff}) \\ & = t^3 + \tfrac{3}{4}at^4 + \tfrac{3}{4}a^2t^5 + (\text{stuff}) \\ & = t^3 + \tfrac{3}{4}a(t^4 + at^5) + (\text{stuff}) \end{aligned}$$

$$\begin{aligned} t^4 \mapsto \quad & t^4 + at^5(1 + \tfrac{3}{4}at) + (\text{stuff}) \\ & = t^4 + at^5 + (\text{stuff}) \end{aligned}$$

where the higher order terms lumped into (stuff) are contained in $t^6k[\![t]\!] \subset k[\![t^3, t^4 + at^5]\!]$. Therefore, $\phi(k[\![t^3, t^4]\!]) \subseteq k[\![t^3, t^4 + at^5]\!]$.

It remains to show that $k[\![t^3, t^4 + at^5]\!] \subseteq \phi(k[\![t^3, t^4]\!])$. Notice that the degree 5 terms in $\phi(t^3)$ and $\phi(\frac{3}{4}at^4)$ are equal. So $t^3 = \phi(t^3 - \frac{3}{4}at^4) - q$ where $q \in t^6k[\![t]\!] \in k[\![t^3, t^4]\!]$. Now we can continue to refine by removing higher powers of t using that the smallest degree term in $\phi(xt^n)$ is xt^n. Similarly, $t^4 + at^5 = \phi(t^4) - q$ for some $q \in k[\![t]\!]$ and so we can continue to refine before we apply ϕ to remove higher degree terms.

Now suppose $k[\![t^3, t^5 + at^7]\!] \subset R$. Define an automorphism $\phi : k[\![t]\!] \to k[\![t]\!]$ by

$$t \mapsto t + \frac{1}{5}at^3 + \frac{4}{25}a^2t^5$$

(note that we use characteristic not 5 here). Similar arguments show that $\phi(k[\![t^3, t^5]\!]) = k[\![t^3, t^5 + at^7]\!]$.

Thus, up to isomorphism, $k[\![t^3, t^4 + at^5]\!]$ is an E_6-singularity and $k[\![t^3, t^5 + at^7]\!]$ is an E_8-singularity. (Note that the result is also true in characteristic 2, 3, and 5, but the proof is drastically different).

Case $s = 2$: Then R has two minimal primes, say P and Q. Since R is reduced, $P \cap Q = (0)$. So $\overline{R} = \overline{R/P} \times \overline{R/Q}$. Notice that $e(R) = \mu_R(\overline{R}) = \mu_R(\overline{R/P}) + \mu_R(\overline{R/Q}) = \mu_{R/P}(\overline{R/P}) + \mu_{R/Q}(\overline{R/Q})$. Since $e(R) \leq 3$, one of the terms must be 1. Without loss of generality, assume $\mu_{R/P}(\overline{R/P}) = 1$. That is, $\overline{R/P} = R/P$.

Recall, since R has two minimal primes, $\overline{R} = k[\![t]\!] \times k[\![t]\!]$. So $e > 1$. Suppose $e = 2$ (we omit the proof of $e = 3$). Then $\overline{R/Q} = R/Q$, so $R/P \simeq k[\![t]\!]$ and $R/Q \simeq k[\![t]\!]$. Define $v : k(\!(t)\!) \times k(\!(t)\!) \to (\mathbb{Z} \cup \{\infty\}) \times (\mathbb{Z} \cup \{\infty\})$ by $v(\alpha, \beta) = (v_1(\alpha), v_2(\beta))$ where $v_i(\gamma) = m$ if $\gamma k[\![t]\!] = t^m k[\![t]\!]$ and $v_i(0) : = \infty$. By exercise 9.1 there is some $c \in R$ such that $c\overline{R} = tk[\![t]\!] \times tk[\![t]\!] = (t,t)\overline{R}$. Then, there is a unit $u \in \overline{R}$ such that $u(t,t) = c$ (since they generate the same ideal). Write $u = (v, w)$ where v, w are units in $k[\![t]\!]$. Define an automorphism of $k[\![t]\!] \times k[\![t]\!]$ by $(t_1, t_2) \mapsto (t_1v, t_2w) = c$. So, after a change of variables, we get $(t, t) \in R$.

By exercise 9.2, $\mathfrak{f} = P + Q$ and so $\sqrt{P + Q} = \mathfrak{m}$. So there is some n such that $(t,t)^n \in \mathfrak{f}$. Therefore, $(it^n, -it^n) = (i, -i)(t,t)^n \in \mathfrak{f}$ where $i = \sqrt{-1} \in \overline{R}$. (Again, we are using that the characteristic is not 2.) So $k[\![(t,t), (it^n, -it^n)]\!] \simeq \frac{k[\![x,y]\!]}{(y^2 + x^{2n})}$ via the map $x \mapsto (t,t), y \mapsto (it^n, -it^n)$, which is an A_{2n-1}-singularity. $\square$

9.1. Exercises.

9.1 (5 points) Relax the assumption that $e = 2$, but still assume $s = 2$, that is, $\overline{R} = k[\![t]\!] \times k[\![t]\!]$. Let $v(\mathfrak{m}) = \{v(x) \mid x \in \mathfrak{m}\} \subset (\mathbb{N} \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\})$.

(1) Show that $v(\mathfrak{m})$ has a unique minimal element (a, b). (Show any two incomparable elements have something below them.)

(2) Show $a > 0, b > 0$.

(3) Show that ${}_R\overline{R}$ is minimally generated by $(1, 0), (t, 0), \ldots, (t^{a-1}, 0), (0, 1), (0, t), \ldots, (0, t^{b-1})$. (Hint: Use NAK and show these generate modulo the maximal ideal)

(4) Show that $e(R) = a + b$.

9.2 (2 points) $\overline{R} = (R/P) \times (R/Q)$. Show that $\mathfrak{f} = P + Q$.

9.3 (5 points; 3 points if you find it in the literature) $k[\![t^3, t^7]\!] \simeq k[\![t^3, t^7 + t^8]\!]$ if and only if $\mathrm{char}(k) = 3$. That is, you can't get rid of the higher term. (Note: This satisfies DR1, but not DR2, so does not have finite representation type.

End of Step 2 and Start of Step 3

We now complete the proof of Theorem 8.2

Proof. (Continued) Suppose R has two minimal primes and $e(R) = 3$. Then $\overline{R} = k[\![t]\!] \times k[\![t]\!]$. Use the second Drozd-Roĭter condition to get an element $(0, \alpha) \in \overline{R}$ such that

$$v_2(\alpha) = 2 \quad \text{or} \quad v_2(\alpha) = 3\ ,$$

where v_2 is the valuation on the second component of $\overline{R} = k[\![t]\!] \times k[\![t]\!]$. By definition (see notes from last time),

$$\begin{aligned} v_2(\alpha) = 2 \quad &\Longrightarrow \quad \alpha k[\![t]\!] = t^2 k[\![t]\!]\ , \\ v_2(\alpha) = 3 \quad &\Longrightarrow \quad \alpha k[\![t]\!] = t^3 k[\![t]\!]\ . \end{aligned}$$

(1) If $v_2(\alpha) = 2$, then after a change of coordinates $t \to vt$ (where v is a unit) we can get an element $(0, t^2) \in R$. Eventually, we can get $(t, t^m) \in R$ for some $m \geq 2$, m odd. Then

$$k[\![(0, t^2), (t, t^m)]\!] \cong \frac{k[\![x, y]\!]}{y(x^2 + y^m)},\ \text{the } (D_{m+2}) \text{ singularity},$$

via the isomorphism

$$\begin{aligned} x \quad &\longmapsto \quad i(t, t^m) \\ y \quad &\longmapsto \quad (0, t^2)\ . \end{aligned}$$

Thus

$$k[\![(0, t^2), (t, t^m)]\!] \subseteq R \subseteq \overline{k[\![(0, t^2), (t, t^m)]\!]}\ ,$$

and R birationally dominates the (D_{m+2}) singularity $k[\![(0, t^2), (t, t^m)]\!]$.

(Note that if m were even, then $y(x^2 + y^m)$ would factor and R would have three minimal primes instead of two.)

(2) If $v_2(\alpha) = 3$, then we can get $(0, t^3 + at^4) \in R$. Note we also showed that we could find γ, δ such that $v_1(\gamma) = 1, v_2(\delta) = 2$ and, after a change of variables, $(t, t^2) \in R$.Use a Tschirnhausen transformation to eliminate a. Then

$$k[\![(t, t^2), (0, t^3)]\!] \cong \frac{k[\![x, y]\!]}{x^3 + xy^3} \ \text{the } (E_7) \text{ singularity},$$

via the isomorphism

$$\begin{aligned} x \quad &\longmapsto \quad (0, t^3) \\ y \quad &\longmapsto \quad -(t, t^2)\ . \end{aligned}$$

Thus

$$k[\![(t,t^2),(0,t^3)]\!] \subseteq R \subseteq \overline{k[\![(t,t^2),(0,t^3)]\!]}\ ,$$

and R birationally dominates the (E_7) singularity $k[\![(t,t^2),(0,t^3)]\!]$.

If R has three minimal primes, tt can be shown that R birationally dominates a (D_n) singularity, where n is even. Thus, in all cases, we see R birationally dominates an ADE singularity. $\square$

The following Proposition shows that if R birationally dominates a ring of finite CM type, then R has finite CM type. (Later we will show that simple singularities have finite CM type and we can conclude that any R as in the Theorem has finite CM type.)

10.1. Propsition. *Let (R,m,k) be a one-dimensional, reduced ring with finitely generated integral closure $\overline{R}$. Let (S,n,l) be a local ring with $R \subseteq S \subseteq \overline{R}$. If R has finite CM type, then so does S.*

Proof. Let M_1, M_2 be MCM S-modules (i.e. $M_1, M_2 \neq 0$ are torsion free).

Claim. $\operatorname{Hom}_R(M_1,M_2) = \operatorname{Hom}_S(M_1,M_2)$.

Proof. The containment $(\supseteq)$ is clear. To show $(\subseteq)$, let $\varphi \in \operatorname{Hom}_R(M_1,M_2)$, $m \in M_1$ and $s \in S$. Write $s = \dfrac{r}{t}$ for some $r,t \in R$ and t a non-zerodivisor on R. Then

$$t\varphi(sm) = \varphi(tsm) = \varphi(rm) = r\varphi(m) = ts\varphi(m) \quad \Longrightarrow \quad t(\varphi(sm) - s\varphi(m)) = 0\ .$$

Since M_2 is torsion free, $\varphi(sm) = s\varphi(m)$.

Now suppose $M_1, M_2, \ldots$ is an infinite list of pairwise non-isomorphic indecomposable S-modules.

Suppose $M_i \cong M_j$, $i \neq j$, as R-modules. Then choose a bijection $\varphi \in \operatorname{Hom}_R(M_i,M_j)$. Since $\operatorname{Hom}_R(M_i,M_j) \cong \operatorname{Hom}_S(M_i,M_j)$ by the Claim, $\varphi \in \operatorname{Hom}_S(M_i,M_j)$ is an S-module isomorphism between M_i and M_j, which gives a contradiction. Thus $M_1, M_2, \ldots$ is an infinite list of pairwise non-isomorphic R-modules.

Suppose the R-module M_i decomposes. Then $\operatorname{Hom}_R(M_i,M_i)$ has an idempotent different from $0,1$. By the Claim, $\operatorname{Hom}_S(M_i,M_i)$ has an idempotent different from $0,1$ and so the S-module M_i decomposes, which is a contradiction. So M_i is an indecomposable R-module.

Thus $M_1, M_2, \ldots$ is an infinite list of pairwise non-isomorphic indecomposable R-modules, contradicting the fact that R has finite CM type. Therefore S has finite CM type. $\square$

In the above proposition, since $\overline{R}$ is finitely generated and the rings are Noetherian, the R-module S is finitely generated. By NAK, $mS \neq S$, so $mS \subseteq n$ and $n \cap R = m$. Thus S birationally dominates R.

10.2. Example. (If S has finite CM type, then R need not have finite CM type.)
We know $R = k[\![t^3,t^7]\!]$ does not have finite CM type (see exercise 10.1). But we can find a ring S between $R = k[\![t^3,t^7]\!]$ and its integral closure $\overline{R} = k[\![t]\!]$ that has finite CM type. In particular,

$$k[\![t^3,t^7]\!] \hookrightarrow k[\![t^3,t^4]\!] \hookrightarrow k[\![t]\!]\ ,$$

where $S = k[\![t^3,t^4]\!]$ is a (E_6) singularity of finite CM type (simple singularities have finite CM type, as we will show later).

Working toward Step 3 of the Grand Plan

10.3. Defining $R^\#$. Let (S,n,l) be a regular local ring, let $f \in n^2$, $f \neq 0$ and let $R := S/(f)$ denote the corresponding hypersurface defined by f in S. Assume S is complete and char $l \neq 2$. Suppose $\dim R = d$, so $\dim S = d+1$. Let z be an indeterminate and let

$$R^\# := S[\![z]\!]/(f+z^2)\ .$$

10.4. MCM over $R^{\#}$. We want to study the MCM modules over $R^{\#}$ compared to those over R. Note that $\dim S[\![z]\!] = d+2$ and so $\dim R^{\#} = d+1$. Also, $S[\![z]\!]$ is a regular local ring and so $R^{\#}$ is a hypersurface. Also, we have a surjective ring homomorphism

$$\begin{array}{rcl} R^{\#} & \twoheadrightarrow & R = R^{\#}/(\bar{z}) \\ \bar{z} & \mapsto & 0\,. \end{array}$$

Suppose N is a MCM $R^{\#}$-module (so that $\operatorname{depth} N = d+1$). Then $\overline{N} := N/\bar{z}N$ is a MCM R-module. Similarly, suppose M is a MCM R-module and let $M^{\#}$ be the first syzygy as an $R^{\#}$-module of M,

$$M^{\#} := \operatorname{syz}^1_{R^{\#}}(M)\,.$$

(*Warning:* we are using a different notation from Yoshino.) We want to show that $M^{\#}$ is a MCM $R^{\#}-$module. To do so, we first need some basic facts on minimal free resolutions and depth.

Minimal free resolutions over local rings. Let A be a local ring and V a finitely generated A-module.
Let $n = \mu_A(V)$ be the number of generators of V. Then

$$\begin{tikzcd} A^{(m)} \arrow[rr, "\varphi"] \arrow[dr] & & A^{(n)} \arrow[r, "\pi"] & V \arrow[r] & 0 \\ & \ker\pi \arrow[ur] \arrow[dr] & & & \\ 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

where n, m are chosen as small as possible so that φ is a matrix with entries in $\mathfrak{m}$. Then $\ker\pi = \operatorname{syz}^1_A(V)$ is the first syzygy of V, which is unique up to isomorphism.

Depth Lemma. Let $0 \to U \to V \to W \to 0$ be a short exact sequence of finitely generated modules over a local ring A.

(1) If $\operatorname{depth} W < \operatorname{depth} V$, then $\operatorname{depth} U = 1 + \operatorname{depth} W$.
(2) $\operatorname{depth} U \geq \min\{\operatorname{depth} V, \operatorname{depth} W\}$.

[These results follow from the long exact sequence of Ext obtained by applying $\operatorname{Hom}_A(A/\mathfrak{m}_A, -)$ to the short exact sequence. Recall that the vanishing of Ext gives the depth.]

Our Situation Suppose A is CM and $\dim A = d$. Then

$$\operatorname{depth} \operatorname{syz}^1_A(V) = \begin{cases} \operatorname{depth}_A(V) + 1 & \text{if } \operatorname{depth}_A(V) < d \\ d & \text{if } \operatorname{depth}_A(V) = d\,. \end{cases}$$

To see this, apply the Depth Lemma to the short exact sequence

$$0 \longrightarrow \operatorname{syz}^1_A(V) \longrightarrow A^n \longrightarrow V \longrightarrow 0\ ,$$

where $\operatorname{depth} A^n = d$ since A is CM.

Thus, $\operatorname{depth} M^{\#} = d+1$ and so $M^{\#}$ is a MCM $R^{\#}-$module.

10.5. Theorem.

(1) Assume M is a MCM R-module with non free summand. Then

$$\overline{M^{\#}} \cong M \oplus \operatorname{syz}^1_R(M)\,.$$

(2) Let N be a MCM $R^{\#}$-module. Then

$$\overline{N}^{\#} \cong N \oplus \operatorname{syz}^1_{R^{\#}}(N)\,.$$

Note. Since syzygies preserve direct sums, we have:

$$\begin{aligned}\overline{(\overline{M^{\#}})^{\#}} &\cong \overline{M^{\#}} \oplus \operatorname{syz}^1_R \overline{M^{\#}}\\ &\cong M \oplus \operatorname{syz}^1_R(M) \oplus \operatorname{syz}^1_R(M \oplus \operatorname{syz}^1_R(M))\\ &\cong M \oplus \operatorname{syz}^1_R(M) \oplus \operatorname{syz}^1_R(M) \oplus M\ .\end{aligned}$$

10.6. Exercises.

10.1. Show that $k[\![t^3, t^7]\!]$ does not have finite CM type. (*Hint:* the second Drozd-Roĭter condition fails.)

§11. Comparing the CM type of R and $R^{\#}$

11.1. Lemma. *Let $(S, \mathfrak{n}, k)$ be a regular local ring, char $(k) \neq 2$ and $R := S/(f)$ for $0 \neq f \in \mathfrak{n}^2$. Let (φ, ψ) be a reduced matrix factorization of f and let $M := \operatorname{coker} \varphi$. Then*

(i) $\left(\begin{bmatrix}\psi & -z\\ z & \varphi\end{bmatrix}, \begin{bmatrix}\varphi & z\\ -z & \psi\end{bmatrix}\right)$ *is a reduced factorization of $f + z^2$ over $S[\![z]\!]^{(2n)}$.*

(ii) $\operatorname{syz}^1_{R^{\#}}(M) \cong \operatorname{coker} \begin{bmatrix}\psi & -z\\ z & \varphi\end{bmatrix}$

(iii) *There exists an exact sequence*

$$R^{\#^{(2n)}} \xrightarrow{\begin{bmatrix}\psi & -z\\ z & \varphi\end{bmatrix}} R^{\#^{(2n)}} \xrightarrow{\begin{bmatrix}\varphi & zI_n\end{bmatrix}} R^{\#^{(n)}} \xrightarrow{\pi} M \longrightarrow 0$$

Proof. We first prove (iii). Note that M is MCM as (φ, ψ) is reduced. So M has no free summands. We have the following exact sequences

$$\cdots \longrightarrow R^{(n)} \xrightarrow{\overline{\varphi}} R^{(n)} \xrightarrow{\overline{\psi}} R^{(n)} \xrightarrow{\overline{\varphi}} M \longrightarrow 0$$

and

$$R^{\#^{(n)}} \xrightarrow{z} R^{\#^{(n)}} \twoheadrightarrow R^{(n)} \longrightarrow 0$$

since $R = \frac{R^{\#}}{zR^{\#}}$. Combining these, we get the following commutative diagram with exact columns

$$\begin{tikzcd}
R^{\#^{(n)}} \arrow[r, "\psi"] \arrow[d, "z"] & R^{\#^{(n)}} \arrow[r, "\varphi"] \arrow[d, "z"] & R^{\#^{(n)}} \arrow[d, "z"] & & \\
R^{\#^{(n)}} \arrow[r, "\psi"] \arrow[d] & R^{\#^{(n)}} \arrow[r, "\varphi"] \arrow[d] & R^{\#^{(n)}} \arrow[d] \arrow[dr, dashed, "\pi"] & & \\
R^{(n)} \arrow[r, "\overline{\psi}"] \arrow[d] & R^{(n)} \arrow[r, "\overline{\varphi}"] \arrow[d] & R^{(n)} \arrow[r] \arrow[d] & M \arrow[r] & 0 \quad \text{exact} \\
0 & 0 & 0 & &
\end{tikzcd}$$

Note that the first two rows in the diagram are not exact as $\varphi\psi = -z^2 \neq 0$ in $R^{\#}$.

By diagram chasing we see that $\ker \pi = \operatorname{im} \varphi + (z)$ and $\ker \begin{bmatrix}\varphi & z\end{bmatrix} = \operatorname{im} \begin{bmatrix}\psi & -z\\ z & \psi\end{bmatrix}$ so (iii) holds.

Note that (i) follows from the matrix multiplication:

$$\begin{bmatrix}\psi & -z\\ z & \varphi\end{bmatrix} \cdot \begin{bmatrix}\varphi & z\\ -z & \psi\end{bmatrix} = \begin{bmatrix}\psi\varphi + z^2 & 0\\ 0 & z^2 + \varphi\psi\end{bmatrix} = \begin{bmatrix}f + z^2 & 0\\ 0 & z^2 + f\end{bmatrix} = (f + z^2) I_n.$$

For (ii), note that the entries of ψ, φ and z are in $\overline{z}\frac{S[\![z]\!]}{(f+z^2)}$ (the maximal ideal of $R^{\#}$). Thus our exact sequence from (iii) is the beginning of a minimal resolution of M over $R^{\#}$. Hence $\operatorname{syz}^1_{R^{\#}}(M) \cong \ker \pi \cong \operatorname{coker} \begin{bmatrix} \psi & -z \\ z & \psi \end{bmatrix}$ by diagram chasing. □

11.2. Theorem. *Assume the hypothesis of the previous lemma. Assume further that S is complete and k is closed square roots.*

(1) *Let ${}_RM$ be MCM with no non-zero free direct summands . Then $\overline{M^{\#}} \cong M \oplus \operatorname{syz}^1_R(M)$.*
(2) *Let ${}_{R^{\#}}N$ be MCM. Then $\overline{N}^{\#} \cong N \oplus \operatorname{syz}^1_{R^{\#}}(N)$.*

Proof. (2) Let $d = \dim R$. Recall that $\dim R^{\#} = d + 1 = \dim S$

$$\begin{tikzcd} & S[\![z]\!] \arrow[dl, shift left, two heads] \arrow[dr, two heads] & & \\ S \arrow[ur, hook, shift left] \arrow[dr, two heads] & & R^{\#} \arrow[dl, two heads] & = S[\![z]\!]/(f+z^2) \\ & R = \frac{S}{(f)} & & \end{tikzcd}$$

As exercise 11.1 shows $R^{\#}$ is finitely generated over S,we can use exercise 6.1 to say $\operatorname{depth}_S N = \operatorname{depth}_{R^{\#}} N = \dim R^{\#} = \dim S$. Thus ${}_SN$ is MCM. Now, S is a regular local ring and so $\operatorname{pd}_S(N) < \infty$. Thus, we can use the Auslander-Buchsbaum formula to get $\operatorname{pd}_S(N) = 0$, that is, ${}_SN$ is free, say of rank n.

Let $\varphi : \ S^{(n)} \longrightarrow S^{(n)}$ represent multiplication by $\overline{z}$ on N. This yields the following commutative diagram

$$\begin{tikzcd} S^{(n)} \arrow[r, "\varphi"] \arrow[d, "\cong"] & S^{(n)} \arrow[r] \arrow[d, "\cong"] & \operatorname{coker}\varphi \arrow[r] \arrow[d, dashed, "\exists!\cong"] & 0 \\ {}_SN \arrow[r, "\overline{z}"] & {}_SN \arrow[r, two heads] & \overline{N} \arrow[r] & 0 \end{tikzcd}$$

so $\overline{N} \cong \operatorname{coker}\varphi$. Note that $(\varphi, -\varphi)$ is a matrix factorization of f, as $\overline{-z^2} = \overline{f}$ in $R^{\#}$. So $\overline{N}^{\#} = \operatorname{syz}^1_{R^{\#}}(\overline{N})$ is a MCM $R^{\#}-$module.

Apply (iii) of the previous lemma to the module $\overline{N} = \operatorname{coker}\varphi$ and the matrix factorization $(\varphi, -\varphi)$ to get the exact sequence

$$R^{\#(2n)} \xrightarrow{\begin{bmatrix} -\varphi & -z \\ z & \varphi \end{bmatrix}} R^{\#(2n)} \xrightarrow{\begin{bmatrix} \varphi & z \end{bmatrix}} R^{\#(n)} \xrightarrow{\pi} \overline{N} \longrightarrow 0 \ .$$

By (ii) of the previous lemma $\qquad \overline{N}^{\#} = \operatorname{syz}^1_{R^{\#}}(\overline{N}) \cong \operatorname{coker} \begin{bmatrix} -\varphi & -z \\ z & \varphi \end{bmatrix}.$

Proceed to a base change to get $\overline{N}^{\#}$ into a diagonal matrix

$$\underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{=A} \begin{bmatrix} -\varphi & -z \\ z & \varphi \end{bmatrix} \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{A^{-1}} = \begin{bmatrix} 0 & z-\varphi \\ -(z+\varphi) & 0 \end{bmatrix}$$

and so

$$\overline{N}^{\#} \cong \operatorname{coker} \begin{bmatrix} -\varphi & -z \\ z & \varphi \end{bmatrix} \cong \operatorname{coker} \begin{bmatrix} 0 & z-\varphi \\ -(z+\varphi) & 0 \end{bmatrix} \cong \operatorname{coker} \begin{bmatrix} z-\varphi & 0 \\ 0 & z+\varphi \end{bmatrix} = \operatorname{coker}(z-\varphi) \oplus \operatorname{coker}(z+\varphi).$$

Note that $(z-\varphi, z+\varphi)$ is a matrix factorization of $f+z^2$ over $S[\![z]\!]$. Now

$$\cdots R^{\#(n)} \xrightarrow{\overline{z}+\overline{\varphi}} R^{\#(n)} \xrightarrow{\overline{z}-\overline{\varphi}} R^{\#(n)} \xrightarrow{\overline{z}+\overline{\varphi}} R^{\#(n)} \xrightarrow{\overline{z}-\overline{\varphi}} N \longrightarrow 0$$

is an $R^{\#}-$resolution of N, so $\operatorname{coker}(\overline{z}+\overline{\varphi}) = \operatorname{syz}^1_{R^{\#}}(N)$. From exercise 11.2, $N \cong \operatorname{coker}(z-\varphi)$ so

$$\overline{N}^{\#} \cong \operatorname{coker}(z-\varphi) \oplus \operatorname{coker}(z+\varphi) \cong N \oplus \operatorname{syz}^1_{R^{\#}}(N).$$

(1) ${}_RM$ is MCM with no free summands, so it admits a reduced matrix factorization (φ, ψ) of f over R with $\operatorname{coker}\varphi = M$. By the lemma, $M^{\#} = \operatorname{coker}\begin{bmatrix} \psi & -z \\ z & \varphi \end{bmatrix}$. Thus

$$\overline{M^{\#}} = \operatorname{coker}\begin{bmatrix} \overline{\psi} & 0 \\ 0 & \overline{\varphi} \end{bmatrix} = \operatorname{coker}(\overline{\psi}) \oplus \operatorname{coker}(\overline{\varphi}).$$

As $R^{(n)} \xrightarrow{\overline{\varphi}} R^{(n)} \xrightarrow{\overline{\psi}} R^{(n)} \longrightarrow M \longrightarrow 0$ is a minimal exact sequence, $\operatorname{coker}\overline{\psi} \cong \operatorname{syz}^1_R M$ and $\operatorname{coker}\overline{\varphi} \cong M$, so $\overline{M^{\#}} = \operatorname{syz}^1_R(M) \oplus M$. □

11.3. Corollary (Knörrer). *Adopting the notation and assumptions of the theorem*

R has finite CM type if and only if $R^{\#}$ has finite CM type.

Note that as $\dim R < \dim R^{\#}$ we can increase dimension without losing finite CM type.

Proof. We will prove the forward direction. The other direction follows similarly. Let $M_1, \cdots, M_t$ be a representative list of the indecomposable non-free MCM $R-$modules. Consider an indecomposable, non-free MCM $R^{\#}-$module ${}_{R^{\#}}N$. If $\overline{N}$ has a free summand, write $\overline{N} = X \oplus R$ for some $R-$module X. Then $\overline{N}^{\#} \cong N \oplus \operatorname{syz}^1_{R^{\#}}(N) \cong X^{\#} \oplus R^{\#}$. By Krull-Remak-Schmidt, this says that either N or $\operatorname{syz}^1_{R^{\#}}(N)$ has a free direct summand, deriving a contradiction. Hence $\overline{N}$ doesn't have a free $R-$summand. Say $\overline{N} = M_1^{(e_1)} \oplus \cdots \oplus M_t^{(e_t)}$, then $N \oplus \operatorname{syz}^1_{R^{\#}}(N) \cong \overline{N}^{\#} \cong M_1^{\#(e_1)} \oplus \cdots \oplus M_t^{\#(e_t)}$. As N is indecomposable over a complete ring, use the Krull-Remak-Schmidt theorem to conclude that N is a direct summand of some $M_i^{\#}$. Each $M_i^{\#}$ has finitely many indecomposable direct summands. Therefore so does $M_i^{\#(e_i)}$ and $M_1^{\#(e_1)} \oplus \cdots \oplus M_t^{\#(e_t)}$. So N has finite CM type. As a consequence of exercise 11.3, this shows if R has n MCM indecomposable modules, $R^{\#}$ has at most $2(n-1)+1$ indecomposable MCM modules. □

11.4. Example. Assume k is a field with char $k \neq 2$ and let $R = \frac{k[\![x,y]\!]}{(x^2+y^{n+1})}$ (R is an (A_n) ADE singularity hence has finite CM type). Then $R^{\#} = \frac{k[\![x,y]\!]}{(x^2+y^{n+1}+z^2)}$ has finite CM type. In fact, this arguement also shows $\frac{k[\![x_0,\ldots,x_n]\!]}{(x_0^2+\ldots+x_n^2)}$ has finite CM type.

11.5. Exercises.

11.1 (3 points) Under the assumptions of the theorem, prove that $R^{\#}$ is a finitely generated $S-$module. (We are assuming S is complete.)

11.2 (5 points) Show that the sequence $S[\![z]\!]^{(n)} \xrightarrow{z-\varphi} S[\![z]\!]^{(n)} \longrightarrow N \longrightarrow 0$ is exact. (Hint: Show that $\{1, z\}$ is a basis for $R^{\#}$ over S).

11.3 (3 points) If ${}_RM$ is an indecomposable MCM module, then $M^{\#}$ is either indecomposable or the direct summand of two indecomposables.

11.4 (3 points) Under the hypothesis of the theorem, show that if $R^{\#}$ is a simple singularity then so is R.

§12. The Weierstrass Preparation Theorem

12.1. Lemma. *Let $R = S/(f)$, where $(S, \mathfrak{n}, k)$ is a complete regular local ring and $0 \neq f \in \mathfrak{n}^2$. We also assume that k is infinite, $k \subseteq S$, and therefore $S = k[[x_0, x_1, \ldots, x_n]]$ by the Cohen Structure Theorem. Suppose further that R is a simple singularity, and $dim(R) \geq 1$. Then:*

(1) *R is reduced*
(2) *$e(R) \leq 3$*
(3) *If $dim(R) \geq 2$, then $e(R) = 2$.*

Proof. (1) Since S is a RLR, S is a UFD. Suppose R is not reduced. Then f is not squarefree, and hence has a repeated prime factor. Put $f = gh^2$, where $g, h \in S$, and $h \in \mathfrak{n}$. Therefore, $dim(S/(h)) = dim(S) - 1 = dim(R) \geq 1$. Therefore $S/(h)$ has infinitely many ideals. Note that for any ideal I with $(h) \subseteq I \subseteq \mathfrak{n}$, we get that $f \in I^2$. Of course, there exist infinitely many such ideals. This contradicts R being a simple singularity.

(2) Suppose $e(R) \geq 4$. Then $f \in \mathfrak{n}^4$. If J is any ideal such that $\mathfrak{n}^2 \subsetneq J \subsetneq \mathfrak{n}$, then $f \in J^2$. However, $\mathfrak{n}/\mathfrak{n}^2$ is a $k-$vector space, and $\mu_S(\mathfrak{n}) = dim_k(\mathfrak{n}/\mathfrak{n}^2) = dim(S) = dim(R) + 1 \geq 2$. Hence, such J's are 1-dimensional subspaces of $\mathfrak{n}/\mathfrak{n}^2$. Thus, there are infintely many such J's, again contradicting R being a simple singularity. Therefore $e(R) \leq 3$.

(3) Set $d = dim(R) \geq 2$. Since $f \in \mathfrak{n}^2$, $3 \geq e(R) \geq 2$. Suppose $e(R) = 3$. We will show R is not simple. Put $f = f_3 + f_4 + \ldots$, where $f_i \in k[x_0, x_1, \ldots, x_d]$ is homogeneous of degree i (since $e(R) = 3$, we may assume $f_3 \neq 0$ and that there are nonzero terms of degree ≤ 2.) Set $V = \left\{p \in \mathbb{P}^d_k : f_3(p) = 0\right\}$. Then $dim(V) = d - 1 \geq 1$. Thus V is infinite. Given $\lambda = (a_0 : a_1 : \cdots : a_d) \in V$, with at least one $a_i \neq 0$, set $I_\lambda = (\{a_i x_j - a_j x_i : 0 \leq i, j \leq d\}) + \mathfrak{n}^2$.

Claim. $f \in I_\lambda^2$.

Proof. By making a projective change of coordinates, we may assume that $\lambda = (1 : 0 : \cdots : 0)$. Then $I_\lambda = (x_1, x_2, \ldots, x_d) + \mathfrak{n}^2 = (x_0^2, x_1, \ldots, x_d)S$. Hence, I_λ^2 is generated by x_0^4, $x_0^2 x_i$ for $1 \leq i \leq d$, and $x_i x_j$ for $1 \leq i, j \leq d$. Therefore, f_3 may be written as: $f_3 = ax_0^3 +$ elements of I_λ^2, for some $a \in k$. However, $a = f_3(1 : 0 : \cdots : 0) = 0$, as $(1 : 0 : \cdots : 0) \in V$. Hence, $f_3 \in I_\lambda^2$ and so $f \in I_\lambda^2$.

Claim. $I_\lambda \neq I_{\lambda'}$ for $\lambda \neq \lambda'$.

Proof. Set $P = k[x_0, x_1, \ldots, x_d]$. Then $P \cong gr_{\mathfrak{n}} P = k \oplus \mathfrak{n}/\mathfrak{n}^2 \oplus \ldots$ Note that $I_\lambda/\mathfrak{n}^2$ is a subspace of $\mathfrak{n}/\mathfrak{n}^2$ corresponding to $L_\lambda := k\langle a_i x_j - a_j x_i\rangle \subseteq P$ under this isomorphism. But since the $a_i x_j - x_i a_j$ are the defining equation of λ, $V(L_\lambda) = \{\lambda\}$. Now suppose that $I_\lambda = I_{\lambda'}$. Then $\frac{I_\lambda}{\mathfrak{n}^2} = \frac{I_{\lambda'}}{\mathfrak{n}^2}$ which implies $L_\lambda = L_{\lambda'}$. So $V(L_\lambda) = V(L_{\lambda'})$, and thus $\lambda = \lambda'$.

We now know that for every $\lambda \in V$, we get a unique I_λ whose square contains f. Since V is infinite, we contradict R being a simple singularity. Hence $e(R) = 2$ as claimed. □

12.2. The Weierstrass Preparation Theorem. *Let $(D, \mathfrak{m})$ be a complete local ring, with $f \in D[[x]]$. Suppose $f = a_0 + a_1 x + \cdots + a_t x^t$ + higher degree terms, with $\{a_0, a_1, \ldots, a_{t-1}\} \subset \mathfrak{m}$ and $a_t \in D - \mathfrak{m}$. Then there exists $\{b_0, b_1, \ldots, b_{t-1}\} \subset \mathfrak{m}$ and u a unit in $D[[x]]$ such that $f = (x^t + b_{t-1}x^{t-1} + \cdots + b_0)u$.*

Proof. See: Lang, 93/99, Chapter IV, Theorem 9.2. □

12.3. Lemma. (Zariski and Samuel, Vol II, Page 147) *Let k be a field, and $0 \neq f \in k[\![x_0, x_1, \ldots, x_n]\!]$. Then there exists a change of variables (which is linear when k is infinite) such that f is as in the Weierstrass Preparation Theorem with $t = ord(f) = \sup\{r : f \in (x_0, x_1, \ldots, x_n)^r\}$.*

Proof. We will sketch the proof, assuming k is infinite. Let $f = f_t +$ higher degree terms. Then $x_n f_t \neq 0$, and, since k is infinite, there exists $c = (c_0, c_1, \ldots, c_n) \in k^n$ such that $(x_n f_t)(c) \neq 0$. Thus, $c_n \neq 0$. Since f_t is homogeneous, we can scale and assume that $c_n = 1$. Now we perform the change of coordinates:

$$\phi : X_i \mapsto \begin{cases} X_i + c_i X_n & \text{if } i < n \\ X_n & \text{if } i = n \end{cases}$$

Now, $\phi(f) = \phi(f_t)$ + higher degree terms. Now $\phi(f_t)$ contains the term $f_t(c_0, c_1, \ldots, c_{n-1}, 1)x_n^t = cx_n^t$, where $c \neq 0$, as desired. $\square$

Now, let $S = k[[x_0, x_1, \ldots, x_n]]$, and suppose that $\operatorname{char}(k) \neq 2$. Let $0 \neq f \in \mathfrak{n}^2$, where $\mathfrak{n} = (x_0, x_1, \ldots, x_n)$. Set $R = S/(f)$, and assume R is a simple singularity with $e(R) = 2$. We remark that if u is a unit of S, $(f) = (u^{-1}f)$. So, by the WPT, as well as the lemma, we can assume that $f = x_n^2 + b_1 x_n + b_0$, where $b_i \in (x_0, x_1, \ldots, x_{n-1})k[[x_0, x_1, \ldots, x_{n-1}]]$. Since $\operatorname{char}(k) \neq 2$, we can complete the square by $x_n \mapsto x_n - \frac{b_1}{2}$. We can therefore put $f = x_n^2 + b$, with b a non-unit of $k[[x_0, x_1, \ldots, x_{n-1}]]$. Finally, this implies that $R = B^\#$, where $B = \frac{k[[x_0, x_1, \ldots, x_{n-1}]]}{(b)}$.

§13. One Dimensional ADE Singularities have finite CM type

13.1. Theorem *Let $(R, \mathfrak{m}, k)$ be a one-dimensional ADE-singularity. Assume $k \subseteq R$, R is complete, $\operatorname{char}(k) = 0$, $k = \overline{k}$. Then R has finite CM-type.*

We will reduce the proof to a simpler case. First note, it is enough to show that $R^\#$ has finite CM type. Possibilities for $R^\#$:

$$(A_n) \quad \frac{k[[x, y, z]]}{(x^2 + y^{n+1} + z^2)} \quad (n \geq 1),$$

$$(D_n) \quad \frac{k[[x, y, z]]}{(x^2 y + y^{n-1} + z^2)} \quad (n \geq 4),$$

$$(E_6) \quad \frac{k[[x, y, z]]}{(x^3 + y^4 + z^2)}$$

$$(E_7) \quad \frac{k[[x, y, z]]}{(x^3 + xy^3 + z^2)}$$

$$(E_8) \quad \frac{k[[x, y, z]]}{(x^3 + y^5 + z^2)}$$

13.2 Fact. Note $SL_2(k)$ acts on $P := k[[u, v]]$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} au + bv \\ cu + dv \end{pmatrix}$$

that is, $u \longmapsto au + bv$ and $v \longmapsto cu + dv$. If G is a finite subgroup of $SL_2(k)$, we let $P^G = \{f \in P : \sigma f = f, \forall \sigma \in G\}$, the **ring of invariants of** G. Klein showed each of the above possibilities for $R^\#$ is the ring of invariants of a finite subgroup of $SL_2(k)$ acting on $k[[u, v]]$ linearly. Furthermore, he showed every finite subgroup of $SL_2(k)$ is conjugate to one of these:

$$(A_n) \quad C_n = \left\langle \begin{pmatrix} \zeta_{n+1} & 0 \\ 0 & (\zeta_{n+1})^{-1} \end{pmatrix} \right\rangle, \text{ where } \zeta_{n+1} = e^{\frac{2\pi i}{n+1}} \text{ and the order of } C_n \text{ is } n+1.$$

$$(D_n) \quad D_n = \left\langle \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} \zeta_{2n-4} & 0 \\ 0 & (\zeta_{2n-4})^{-1} \end{pmatrix} \right\rangle, \text{ where the order of } D_n \text{ is } 4(n-2),\ n \geq 4.$$

(E_6) $\quad T =$ tetrahedral group, and the order of T is 24.

(E_7) $\quad O =$ octahedral group, and the order of O is 48.

(E_8) $\quad I =$ icosahedral group, and the order of I is 120.

Thus each of the possibilities of $R^{\#}$ is isomorphic to the ring of invariants for one of these 5 subgroups of $SL_2(k)$.

13.3 Example. For $n \geq 1$,

$$P^{C_n} \cong \frac{k[[x,y,z]]}{(x^2+y^{n+1}+z^2)}.$$

To see this, first we need to find what is fixed by C_n. So map $u \longmapsto \zeta u$ and $v \longmapsto \zeta^{-1}v$. Note that u^{n+1}, v^{n+1}, uv. are fixed. In fact, $P^{C_n} = k[[u^{n+1}, v^{n+1}, uv]]$.

One can show we have an epimorphism $k[[x,y,z]] \twoheadrightarrow P^{C_n}$ via $x \mapsto u^{n+1}$, $z \mapsto v^{n+1}$, $y \mapsto uv$ where the kernel of this map is $(y^{n+1} - xz)$. [To see this, note that it is irreducible and hence a height 1 prime. Modding out gives us a two-dimensional domain mapping onto a two-dimensional domain.]

Using another change of variables, we let $x = x' + iz'$, $z = -(x' - iz')$ and $y = y'$. Then $y^{n+1} - xz = (y')^{n+1} - (-1)((x')^2 + (z')^2) = (y')^{n+1} + (x')^2 + (z')^2$, proving our assertion.

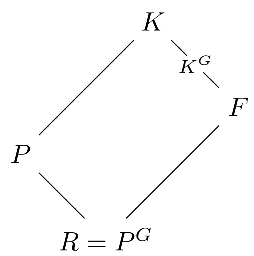
We leave P^{D_n} as an exercise. Now, to prove Theorem 13.1, it will suffice to prove the following theorem.

13.4 Theorem. *Let P be a complete RLR of dimension 2. Let G be a finite group of automorphisms of (P, n, k). Assume $R := P^G$ is a complete local ring and that char(k) does not divide $|G|$. Then, R has finite CM type.*

Proof. Note G acts on K:=quotient field of P in the obvious way, i.e., if $\sigma \in G$, then

$$\sigma\left(\frac{f}{g}\right) = \frac{\sigma(f)}{\sigma(g)} \quad \text{for} \quad f, g \in P \text{ and } g \neq 0.$$

Let F be the quotient field of R. Obviously, $F \subseteq K^G$. We know K/K^G is a finite Galois extension with Galois group G.



Claim. $F = K^G$ (See Lang's Algebra book, pages 263-264)

Proof. Given $\alpha \in P$, let Ω be the orbit of α under G. Let $\Omega = \{\sigma_1\alpha, ..., \sigma_r\alpha\}$, where $\sigma_i \in G$ and $|\Omega| = r$. Let $h(T) := \prod_{i=1}^{r}(T - \sigma_i\alpha)$. Then $h(T) \in R[T]$ and $h(\alpha) = 0$. This shows that P is an integral extension of R. Note also that $h \in F[T]$. Therefore each element of P is separable algebraic of degree $\leq n := |G|$ over F. Since $K = F(P)$, K/F is separable algebraic.

Choose $\alpha \in P$ so as to maximize $[F(\alpha) : F]$ (as these are all bounded by n, we can choose the maximum). Then, we must have $K = F(\alpha)$. If not, then there exists $\beta \in P$ such that $\beta \notin F(\alpha)$. Let $L_t := F(\alpha + t\beta)$ for each $t \in F$. Since P is uncountable and K/F is algebraic, F is infinite (otherwise K would be countable). Since $F(\alpha, \beta)/F$ is a finite separable extension, there exist only finitely many intermediate fields. So, there exist t and u such that $t \neq u$ and $L_t = L_u$. Then

$\alpha, \beta \in L_t$ so that $L_t \supsetneqq F(\alpha)$ and $[L_t : F] > [F(\alpha) : F]$, a contradiction.

$$\begin{tikzcd}
K = F(\alpha) \arrow[dd, dash, "\leq n"'] \arrow[dr, dash, "=n"] & \\
& K^G \arrow[dl, dash] \\
F &
\end{tikzcd}$$

Therefore, we see by the diagram that $F = K^G$.

Now, we know K/F is a finite algebraic extension.

$$\begin{tikzcd}
& K \arrow[dl, dash] \arrow[dr, dash, "<\infty"] & \\
P \arrow[dr, dash, "integral"'] & & F = K^G \arrow[dl, dash] \\
& R = P^G &
\end{tikzcd}$$

Note P is the integral closure of R in K (P is a UFD so integrally closed and P/R is integral). Since complete rings are excellent, P is a finitely generated R-module.

> **Fact.** Let R be an integrally closed local domain of dimension 2 and let $M \neq 0$ be an R-module. Then, M is MCM $\Leftrightarrow$ M is reflexive, i.e the natural map $M \to M^{**}$ is an isomorphism where $()^* = \operatorname{Hom}_R(-, R)$. [In particular, R is CM.]

Note that R is integrally closed (this follows from the above diagram) and so the fact applies. Define $\rho : P \longrightarrow R$ by $\rho(x) = \dfrac{1}{|G|} \displaystyle\sum_{\sigma \in G} \sigma(x)$. Then, $\begin{tikzcd} R \arrow[r, hook] \arrow[rr, bend right, "1_R"'] & P \arrow[r, two heads, "\rho"] & R \end{tikzcd}$ and ρ is an R-homomorphism. Therefore,

$$\begin{aligned}
P &\cong R \oplus X \text{ as } R\text{-modules, where } X \text{ is finitely generated} \\
&\cong R \oplus X_1 \oplus ... \oplus X_t \text{ where } {}_R X_i \text{ are indecomposable modules}
\end{aligned}$$

Note that there are finitely many X_i as X is finitely generated.

It is now left to show that $R, X_1, ..., X_t$ are the only indecomposable MCM R-modules. Let ${}_R M$ be an indecomposable MCM module. Then,

$$P \otimes_R M \cong \underbrace{(R \otimes_R M)}_{\cong M} \oplus (X_1 \otimes_R M) \oplus \cdots \oplus (X_t \otimes_R M)$$

Therefore,

$$\begin{aligned}
(P \otimes_R M)^{**} &\cong M^{**} \oplus (X_1 \otimes_R M)^{**} \oplus ... \oplus (X_t \otimes_R M)^{**} \\
&\cong M \oplus (X_1 \otimes_R M)^{**} \oplus ... \oplus (X_t \otimes_R M)^{**} \text{ since } M \text{ is reflexive.}
\end{aligned}$$

Note that $(P \otimes_R M)^{**}$ is a P-module and a MCM R-module. So it is reflexive and thus a MCM P-module. This implies it is free as a P-module (as P is a RLR) and so ${}_R M$ is a direct summand of $P^{(m)} = R^{(m)} \oplus X_1^{(m)} \oplus ... \oplus X_t^{(m)}$. Therefore, $M \cong R$ or $M \cong X_i$ for some i. $\square$

13.5 Exercises.

13.1 (10 points) Prove that for $n \geq 4$,

$$P^{D_n} \cong \frac{k[[x,y,z]]}{(x^2y + y^{n-1} + z^2)}.$$

(5 points for $n = 4$ case). To do so, consider the maps ϕ defined by $u \mapsto iv$, $v \mapsto iu$ and ψ defined by (in the $n = 4$ case) $u \mapsto iu$ and $v \mapsto -iv$. We see $u^2v^2, u^2 - v^2$ are fixed by ϕ and u^4, v^4, uv are fixed by ψ. Invariants for our ring, must be fixed for both. Its easy to see u^2v^2 and $u^4 + v^4$ are fixed, but what is another?

§14. Hypersurfaces of Finite CM Type.

14.1 Theorem. *Let $(R, \mathfrak{m})$ be a local integrally closed domain of dimension 2. Let ${}_RM$ be a nonzero finitely generated module. Then,*

$$M \text{ is MCM} \iff M \text{ is reflexive.}$$

Proof. First we must show R is CM. Choose any $x \in R - \{0\}$. This is a non-zerodivisor as R is a domain. If R is not CM, then $\mathfrak{m} \in Ass(R/xR)$. Then, there exists $y \in R - xR$ such that $\mathfrak{m}y \subseteq Rx$. As $y \in R - xR$, $\frac{y}{x} \notin R$ and $\frac{y}{x}\mathfrak{m} \subseteq R$. If $\frac{y}{x}\mathfrak{m} = R$, then $\mathfrak{m} = R\frac{x}{y}$, contradicting $\mu_R(\mathfrak{m}) \geq 2$. Therefore $\frac{y}{x}\mathfrak{m} \subsetneqq R$. So, $\frac{y}{x}\mathfrak{m} \subseteq \mathfrak{m}$. Then $\mathfrak{m}$ is a faithful $R\left[\frac{y}{x}\right]$-module which is finitely generated as an R-module. Thus $\frac{y}{x}$ is integral over R. Since $\frac{y}{x} \notin R$, this says R is not integrally closed. Thus R is CM.

($\Leftarrow$) Resolve M^*

$$R^{(a)} \xrightarrow{\varphi} R^{(b)} \longrightarrow M^* \longrightarrow 0$$

Applying $\mathrm{Hom}_R(-, R)$, we get the exact sequence

$$0 \longrightarrow M^{**} \longrightarrow R^{(b)} \xrightarrow{\varphi^t} R^{(a)} \longrightarrow C \longrightarrow 0$$

where $C = \text{Coker } \varphi^t$ and $M^{**} \cong M$. Thus M is a second syzygy of C. So depth $M \geq 2$ by the depth lemma, and hence M is MCM.

($\Rightarrow$) Build an exact sequence

$$0 \longrightarrow K \longrightarrow M \xrightarrow{i} M^{**} \longrightarrow C \longrightarrow 0$$

where K and C are the kernel and cokernel of the natural map i, respectively. Let $Q = (0)$. Since $(-)^*$ commutes with localization (for finitely generated modules over Noetherian rings), we have an exact sequence

$$0 \longrightarrow K_Q \longrightarrow M_Q \longrightarrow (M_Q)^{**} \longrightarrow C_Q \longrightarrow 0$$

Since R_Q is a field, $M_Q \cong M_Q{**}$ and so $K_Q = 0$. But K is torsion-free, since $K \subseteq M$ and M MCM is torsion free. Thus $K \hookrightarrow K_Q$, which implies $K = 0$. Therefore we have

$$0 \longrightarrow M \xrightarrow{i} M^{**} \longrightarrow C \longrightarrow 0$$

Let p be a height-one prime ideal in R. Then R_p is a one-dimensional integrally closed local domain. So R_p is a DVR. Now M_p is free because it is torsion-free. So $M_p \xrightarrow{i_p} M_p^{**}$ is an isomorphism. Therefore, $C_p = 0$. This shows that $\text{Supp}(C) = \{\mathfrak{m}\}$ if $C \neq 0$, which implies $\text{depth}(C) = 0$. But M^{**} is reflexive and hence MCM. This implies $\text{depth}(M) = 1$ from the exact sequence $0 \longrightarrow M \longrightarrow M^{**} \longrightarrow C \longrightarrow 0$ and the depth lemma. This is a contradiction, as M is MCM. So $C = 0$. $\square$

14.2 Theorem. *Let $(R, \mathfrak{m}, k)$ be a complete hypersurface with $k \subseteq R$, $k = \overline{k}$, $\operatorname{char}(k) = 0$. Let $d = \dim(R) \geq 1$. Then R has finite CM type $\iff R \cong \dfrac{k[[x, y, z_0, ..., z_{d-2}]]}{(g + z_0^2 + ... + z_{d-2}^2)}$ where $g \in k[[x, y]]$ is an ADE singularity.*

If $\dim R = 1$, we have an ADE singularity which we have shown has finite CM type. The key issues of the proof are the following:

(1) A hypersurface A has finite CM type $\iff A^{\#}$ has finite CM type. Thus by sharping the one dimensional ADE singularity, we get all of these.

(2) If $d \geq 2$ and A has finite CM type, then $e(A) \leq 2$. We use Weierstrass Preparation Theorem and completing the square to show if $\dim A \geq 2$ and A has finite CM type, then $A = R^{\#}$.

14.3 Definition. Let $(R, \mathfrak{m}, k)$ be a local CM ring of dimension d. A *canonical module* for R is a finitely generated module ω_R such that

$$\operatorname{Ext}^i_R(k, \omega) = \begin{cases} k\,, & i = d \\ 0\,, & i \neq d \end{cases}$$

Note that ω_R is MCM. Also, not every CM local ring has a canonical module (though complete rings do).

14.4 Proposition. *If M and N are canonical modules for R, then $M \cong N$.*

Sketch of Proof. It is enough to show the following.

(1) If $\dim R = 0$, show that $\omega_R \cong E_R(k)$, the injective hull, as R-module, of the residue field.

(2) Let $n \geq 1$. Let $(\underline{x}) := (x_1, ..., x_d)$ be a regular sequence in $\mathfrak{m}$. Then $(\underline{x})$ is both $R-$ and $M-$regular (it generates $\mathfrak{m}$ up to radical), which implies $(\underline{x}^n) = (x_1^n, ..., x_d^n)$ is. Check that $\omega_R/(\underline{x}^n)\omega_R$ is a canonical module for $R/(\underline{x}^n)R$. Thus, if M and N were two canonical modules for R, then we would have

$$M/(\underline{x}^n)M \cong N/(\underline{x}^n)N \quad \text{for all} \quad n \geq 1 \quad \text{by (1)}$$

Therefore

$$M/\mathfrak{m}^n M \cong N/\mathfrak{m}^n N \quad \text{for all} \quad n \geq 1$$

Then, by the following theorem, the result is proved. □

14.5 Theorem. (Guralnick): *Let M and N be finitely generated modules over a local ring $(R, \mathfrak{m})$.*

(1) *If $M/\mathfrak{m}^n M \cong N/\mathfrak{m}^n N$ for all $n \geq 1$, then $M \cong N$.*

(2) *If $M/\mathfrak{m}^n M \mid N/\mathfrak{m}^n N$ for all $n \geq 1$, then $M \mid N$. Note: $M \mid N$ means there exists X such that $M \oplus X \cong N$.*

14.6 Definition. $(R, \mathfrak{m}, k)$ is *Gorenstein* provided R is CM and $R \cong \omega_R$, i.e,

$$\operatorname{Ext}^i_R(k, R) = \begin{cases} k\,, & i = d \\ 0\,, & i \neq d \end{cases}$$

14.6 Facts.

(1) Suppose (S, η, k) is Gorenstein and there is a surjection $S \twoheadrightarrow R$ where R is a local CM ring. Then, R has a canonical module, namely, $\omega_R = \operatorname{Ext}^{n-d}_S(R, S)$, where $n = \dim S$ and $d = \dim R$.

(2) RLR $\Rightarrow$ Gorenstein.

(3) Cohen-Structure Theorem $\Rightarrow$ Every complete local ring is a homomorphic image of a complete RLR. Therefore, each complete CM local ring has a canonical module.

Let $(R, \mathfrak{m}, k)$ be a CM ring. Let $\mathfrak{C}_R(i) = \{{}_R M : M$ is CM of *dimension* i $\}$ Recall $\dim(M) = \dim(R/(0 : M)) = \dim(\operatorname{Supp}(M))$ and M is CM $\Leftrightarrow$ $\operatorname{depth} M = \dim M$.

14.6 Duality Theorem. *For $M \in \mathfrak{C}_R(i)$, define $M\check{} := \operatorname{Ext}_R^{d-i}(M,\omega)$. (Let $d = \dim(R)$ and assume ω_R exists.) Then*

(1) $M\check{} \in \mathfrak{C}_R(i)$
(2) *The natural map $M \to M\check{}\check{}$ is an isomorphism.*
(3) $Ext_R^n(M,\omega) = 0$ *if* $n \neq d - i$.

(For CM modules, $d = i$ and so $M\check{} = \operatorname{Hom}_R(M,\omega)$).

14.7 Exercises.

14.1 (3 points). Let $(R,\mathfrak{m},k)$ be a local ring and M a MCM R-module. Let $(\underline{x})$ be an R-regular sequence. Prove that $(\underline{x})$ is M-regular.

14.2 (3 points). Let $(R,\mathfrak{m},k)$ be a local CM ring, and let M be a finitely generated R-module. Let $x \in \mathfrak{m}$ be both R-regular and M-regular. Prove that M is a canonical module for R if and only if M/xM is a canonical module for $R/(x)$. (Use Lemma 3 on page 140 of [H. Matsumura, *Commutative Ring Theory*, Cambridge studies in advanced mathematics 8, 1989].)

14.3 (3 points). Let $(R,\mathfrak{m},k)$ be a local ring, let M be a finitely generated R-module, and let N be a proper submodule of M. Suppose $(\underline{x})$ is an (M/N)-regular sequence in $\mathfrak{m}$. Prove that $N \cap (\underline{x})M = (\underline{x})N$.

§15. Gorenstein Local Rings of finite CM type have Hypersurface Completions.

Suppose that $(R,\mathfrak{m},k)$ is a local ring. Let

$$\cdots \longrightarrow R^{\beta_n} \longrightarrow R^{\beta_{n-1}} \longrightarrow \cdots \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow k \longrightarrow 0$$

be a minimal free resolution of k, i.e. take the smallest β_i at each step. So

$$\begin{aligned} \beta_0 &= 1 \\ \beta_1 &= \mu_R(\mathfrak{m}) \\ &\vdots \\ \beta_n &= \mu_R(\operatorname{syz}_R^n(k)). \end{aligned}$$

15.1 Theorem. (Tate and Gulliksen) *The $\beta_n(k)$ are bounded if and only if $\widehat{R}$ is a hypersurface, i.e. $\widehat{R} = S/(f)$ where S is a regular local ring and f is a non-unit of S.*
Note $\beta_i = 0$ for all $i > \dim R$ if R is a regular local ring.

"Proof". We already know that hypersurfaces have $\beta_n(k)$ bounded: Assume $R = \widehat{R}$, with $\dim R = d$. Then $\operatorname{syz}_R^i(k)$ is MCM for all $i \geq d$ (depth $k = 0$ and the depth increases by 1 for each syzygy). Therefore, using matrix factorizations, we see that $\beta_i = \beta_{i+1}$ for $i \geq d$. □

15.2 Lemma. (Direct Sum Cancellation) *Let $(R,\mathfrak{m},k)$ be local, and A,B,C finitely generated R-modules. If $A \oplus C \cong B \oplus C$ then $A \cong B$.*

Proof. Let $\widehat{A} = \widehat{R} \otimes_R A$, etc. We have

$$\widehat{A} \oplus \widehat{C} \cong \widehat{B} \oplus \widehat{C}$$

because the tensor product distributes over direct sums. By Krull-Remak-Schmidt for complete local rings, $\widehat{A} \cong \widehat{B}$.

Recall a result of Guralnick: If $A/\mathfrak{m}^t A \cong B/\mathfrak{m}^t B$ for all t then $A \cong B$. We have

$$A/\mathfrak{m}^t A \cong \widehat{A}/\mathfrak{m}^t \widehat{A} \cong \widehat{B}/\mathfrak{m}^t \widehat{B} \cong B/\mathfrak{m}^t B,$$

since finite length modules are unchanged upon taking the completion. So $A \cong B$. □

Note that we could similarly prove the result for semilocal rings.

15.3 Lemma. *Let $(R, \mathfrak{m}, k)$ be a local CM ring. Let M be a MCM R-module and let*

$$0 \longrightarrow U \xrightarrow{\subset} F \longrightarrow M \longrightarrow 0,$$

be exact with F free and $\mu_R(F) = \mu_R(M)$ (a minimal presentation). Assume M has no free direct summand. Then U has no free direct summand.

Proof. Let $d = \dim R$ and let $(x_1, \ldots, x_d)$ be R-regular. Therefore it's an M-regular sequence by Problem 14.1. We have $U \subset \mathfrak{m}F$ using some flavor of Nakayama's Lemma. By Problem 14.3, we have $U \cap (\underline{x})F = (\underline{x})U$, where $(\underline{x})$ is the ideal generated by $x_1, \ldots, x_d$. Therefore we have a one-to-one map $U/(\underline{x})U \to \mathfrak{m}F/(\underline{x})F$ since something from U is in $(\underline{x})F$ precisely when it's in $U \cap (\underline{x})F = (\underline{x})U$.

Because $U/(\underline{x})U$ sits as a submodule we have $\big(0 :_{R/(\underline{x})} U/(\underline{x})U\big) \supset \big(0 :_{R/(\underline{x})} \mathfrak{m}F/(\underline{x})F\big)$.

But $\mathfrak{m}F/(\underline{x})F = (\mathfrak{m}/(\underline{x}))^{(n)}$ where $F = R^{(n)}$. Since $(\underline{x})$ is a maximal R-regular sequence, $\mathfrak{m} \in \operatorname{Ass}_R(R/(\underline{x}))$. So $\big(0 :_{R/(\underline{x})} \mathfrak{m}/(\underline{x})\big) \neq 0$. Therefore $\big(0 :_{R/(\underline{x})} U/(\underline{x})U)\big) \neq 0$.

Suppose that $U = R \oplus Z$ for the purpose of a contradiction. Then $U/(\underline{x})U \cong R/(\underline{x}) \oplus Z/(\underline{x})Z$. Since $\big(0 :_{R/(\underline{x})} R/(\underline{x}\big) = 0$, we have $\big(0 :_{R/(\underline{x})} U/(\underline{x})U\big) = 0$, a contradiction. $\square$

15.4 Lemma. (Herzog) *Let $(R, \mathfrak{m}, k)$ be Gorenstein. Let M be a MCM R-module that is indecomposable and with $M \ncong R$. Let*

$$0 \longrightarrow U \xrightarrow{\subset} F \longrightarrow M \longrightarrow 0,$$

be a minimal presentation, i.e. F free and $\mu_R(F) = \mu_R(M)$. Then U is indecomposable. (We also know that U is MCM since the depth lemma forces $\operatorname{depth} U \geq \operatorname{depth} M$*.)*

Proof. Apply $\operatorname{Hom}_R(-, R) = (\ )^*$ to the minimal presentation:

$$0 \longrightarrow M^* \longrightarrow F^* \longrightarrow U^* \longrightarrow \operatorname{Ext}^1_R(M, R) \longrightarrow \cdots$$

Recall that Gorenstein implies that $R = \omega_R$. Since M is MCM, we have $\operatorname{Ext}^i_R(M, R) = 0$ for all $i \neq d - \operatorname{depth} M = 0$. So $\operatorname{Ext}^1_R(M, R) = 0$. Thus we have an exact sequence

$$0 \longrightarrow M^* \longrightarrow F^* \longrightarrow U^* \longrightarrow 0.$$

Suppose that $U = U_1 \oplus U_2$, with $U_i \neq 0$ for all i. Then $U^* = U_1^* \oplus U_2^*$, and we claim that $U_i^* \neq 0$. If $U_i^* = 0$ then $U_i^{**} = 0$. But U_i is MCM so $U_i^{**} \cong U_i$, since $X \to X^{**}$ is an isomorphism for any X that is MCM.

Let

$$0 \longrightarrow V_i \longrightarrow G_i \longrightarrow U_i^* \longrightarrow 0,$$

be a minimal presentation of U_i^*. Then

$$0 \longrightarrow V_1 \oplus V_2 \longrightarrow G_1 \oplus G_2 \longrightarrow U_1^* \oplus U_2^* \longrightarrow 0,$$

is a minimal presentation of $U_1^* \oplus U_2^*$. Since $U_1^* \oplus U_2^* = U^*$ we can use our previous sequence for U^* to get $M^* \oplus R^{(s)} \cong V_1 \oplus V_2 \oplus R^{(t)}$, for some s and t.

If $t \geq s$, use direct sum cancellation to get $M^* \cong V_1 \oplus V_2 \oplus R^{(t-s)}$. Therefore M^{**} decomposes. But $M^{**} \cong M$ and M is indecomposable. Contradiction.

If $s > t$, cancel to get $V_1 \oplus V_2 \cong R \oplus X$. Therefore either V_1 surjects onto R or V_2 does. Say V_1 surjects onto R. This map splits, so V_1 has a free summand.

By Lemma 15.3, U_1^* has a free summand. Therefore U_1^{**} has a free summand. Since $U_1^{**} \cong U_1$ we conclude that U has a free summand. By Lemma 15.3 again, M must have a free summand. But M is indecomposable, so $M \cong R$ which implies $U = 0$, contradiction. $\square$

15.5 Theorem. (Herzog) *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring with finite CM type. Then $\widehat{R}$ is a hypersurface.*

Proof. By Tate-Gulliksen we just have to show that $\beta_i := \beta_i(k)$ are bounded.

If $j \geq d := \dim R$ then $\operatorname{syz}_R^j(k)$ is MCM. Let $M = \operatorname{syz}_R^d(k)$. Write $M = M_1 \oplus \cdots \oplus M_t$ with M_i indecomposable and M_i free if and only if $i \geq s$.

Then $\operatorname{syz}_R^1(M) = \operatorname{syz}_R^{d+1}(k)$ is a direct sum of at most s indecomposable MCM modules by Lemma 15.4. Also by Lemma 15.4, the syzygy of an indecomposable module is indecomposable. Continue in this manner: $\operatorname{syz}_R^j(k)$ is a direct sum of at most s indecomposable modules for all $j > d$. Therefore $\beta_j(k) \leq Bs$ for all $j \geq d$ where

$$B = \max\{\mu_R(N) \mid {}_RN \text{ is indecomposable MCM}\}.$$

Taking into account the maximum of the β_j with $j \leq d$, we see that the Betti numbers β_j are all bounded. $\square$

Recall that $(R, \mathfrak{m}, k)$ is *Henselian* provided every commutative module-finite R-algebra is a direct product of local rings.

15.6 Theorem. *Let $(R, \mathfrak{m}, k)$ be Henselian, and let E be a module-finite R-algebra, not necessarily commutative. Then idempotents of E lift modulo the Jacobson radical $J(E)$. That is, if $x \in E$ and $x^2 - x \in J(E)$ then there exists $e = e^2 \in E$ such that $e - x \in J(E)$.*

Proof. Let $A = R[x] \subset E$. This is a commutative subring. Consider the diagram,

$$\begin{tikzcd}
 & & E \arrow[dr] & \\
 & A \arrow[ur, hook] \arrow[dr] & & E/J(E) \\
R \arrow[ur, hook] \arrow[dr] & & A/(A \cap J(E)) \arrow[ur, hook] & \\
 & k \arrow[ur] & &
\end{tikzcd}$$

The rest follows from exercises 15.2 and 15.2. $\square$

15.7 Exercises.

15.1 (4 points) In Theorem 15.6, use the fact that A is the direct product of finitely many commutative local ringsto prove that every idempotent of $A/(A \cap J(E))$ is the image of an idempotent of A.

15.2 (3 points) Finish the proof of Theorem 15.6.

15.3 (4 points) Prove that the presentation $0 \longrightarrow M^* \longrightarrow F^* \longrightarrow U^* \longrightarrow 0$ is minimal (that is, $\mu_R(F^*) = \mu_R(U^*)$. Then indicate briefly how the proof of Lemma 15.4 can be completed without invoking Lemma 15.2 and Lemma 15.3.

§16. Uniqueness of Direct Sum Decomposition

For modules M, N, " $M|N$ " means that there exists a module X such that $X \oplus M \cong N$. Recall a ring E is *"local"* if $E/J(E)$ is a division ring. Also E is "local" if and only if the non-units form an ideal, namely $J(E)$.

16.1 Theorem. *Let $(R, \mathfrak{m}, k)$ be a Henselian local ring and let M be a finitely generated indecomposable R-module. Then $E = \operatorname{End}_R(M)$ is "local."*

Proof. Map a free module onto M:

$$R^{(n)} \longrightarrow M \longrightarrow 0.$$

Apply $\mathrm{Hom}_R(-, M)$ to get:

$$0 \longrightarrow E \longrightarrow M^{(n)}.$$

Since $M^{(n)}$ is Noetherian and $E \subset M^{(n)}$, E is a module-finite R-algebra. Now note that $\mathfrak{m}E \subset J(E)$:

> Indeed it is enough to show that for $f \in \mathfrak{m}E$ and $g \in E$, we have $E(1 + fg) = E$. But by NAK this is equivalent to $E(1 + fg) + \mathfrak{m}E = E$, which is clear as $E(1 + gf) + \mathfrak{m}E = E + \mathfrak{m}E$.

Thus $E/J(E)$ is a module over $R/\mathfrak{m} = k$. Now since E is module-finite over R, $E/J(E)$ is finitely generated and thus a finite dimensional k-algebra. Hence it is Artinian with jacobson radical 0, i.e. semi-simple Artinian. Since M is indecomposable E has no nontrivial idempotents. Since idempotents lift modulo the Jacobson radical by Theorem 15.6, $E/J(E)$ also has no nontrivial idempotents. By the Wedderburn structure theorem $E/J(E)$ is a division ring (as otherwise, we'd have idempotents). □

16.2 Theorem. *Let R be a ring and let*

$$M_1 \oplus \cdots \oplus M_s \cong N_1 \oplus \cdots \oplus N_t$$

where the M_i and N_j are R-modules with "local" endomorphism rings. Then $s = t$ and $M_i \cong N_i$ up to permutation.

Proof. Let $\psi : M_1 \oplus \cdots \oplus M_s \longrightarrow N_1 \oplus \cdots \oplus N_t$ and $\varphi : N_1 \oplus \cdots \oplus N_t \longrightarrow M_1 \oplus \cdots \oplus M_s$ be reciprocal isomorphisms. By the universal properties of direct sums and products we can write $\psi = [\psi_{ji}]$ and $\varphi = [\varphi_{ij}]$ where $\psi_{ji} : M_i \longrightarrow N_j$, $\varphi_{ij} : N_j \longrightarrow M_i$. Since $\varphi \circ \psi = 1_M$ we have that $1 = \sum_{j=1}^{t} \varphi_{1j}\psi_{j1}$ with $\varphi_{1j}\psi_{j1} \in E := \mathrm{End}_R(M_1)$. Since a sum of non-units is not a unit and $1 \notin J(E)$ there exists a j such that $\varphi_{1j}\psi_{j1} \notin J(E)$. Relabel so that $j = 1$. So we now have that $\varphi_{11}\psi_{11}$ is a unit of E:

$$M_1 \xrightarrow{\psi_{11}} N_1 \xrightarrow{\varphi_{11}} M_1 \xrightarrow{(\varphi_{11}\psi_{11})^{-1}} M_1$$

Let $\beta = (\varphi_{11}\psi_{11})^{-1} \circ \varphi_{11} : N_1 \longrightarrow M_1$. We then have the splitting

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r, bend left, "\psi_{11}"] & N_1 \arrow[l, bend left, "\beta"] \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

and thus $M_1 | N_1$. Since N_1 is indecomposable $\psi_{11}(M_1) = N_1$. Thus ψ_{11} is bijective, and so φ_{11} is. Hence $M_1 \cong N_1$.

Now to use induction we must first reduce $\varphi = [\varphi_{ii}]$ to a simpler matrix. To do so, we will use φ_{11} to wipe out all other entries on the top row and left column:

Let

$$\alpha = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\varphi_{21}\varphi_{11}^{-1} & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

and note that $\alpha\varphi$ is the matrix φ but with φ_{21} replaced with 0. As α is invertible, $\alpha\varphi$ is still an isomorphism. Continuous with elementary row and column operations to reduce the matrix φ to an invertible matrix of the form

$$\left[\begin{array}{c|ccc} \varphi_{11} & 0 & \ldots & 0 \\ \hline 0 & & & \\ \vdots & & * & \\ 0 & & & \end{array}\right]$$

One can easily check that $* : N_2 \oplus \cdots \oplus N_t \to M_2 \oplus \cdots \oplus M_s$ is an isomorphism and so we may proceed by induction. □

16.3 Corollary. *Let $(R, \mathfrak{m}, k)$ be a Henselian local ring. Then the Krull-Remak-Schmidt Theorem is valid for all finitely generated R-modules.*

Proof. $\mathrm{End}_R(M)$ is local for all finitely generated indecomposable R-modules. $\square$

16.4 Corollary. *The same conclusion holds for complete local rings.*

Proof. Complete local rings are Henselian. $\square$

The Krull-Remak-Schmidt Theorem fails for general local rings. An idea for a "proof" would be the following. Assume we have

$$M_1 \oplus \cdots \oplus M_s \cong N_1 \oplus \cdots \oplus N_t.$$

Completing both sides gives

$$\widehat{M_1} \oplus \cdots \oplus \widehat{M_s} \cong \widehat{N_1} \oplus \cdots \oplus \widehat{N_t}$$

and since $\widehat{R}$ is complete we may use K-R-S. However $\widehat{M_i}, \widehat{N_j}$ may not be indecomposable.

Let $(R, \mathfrak{m}, k)$ be a local ring, and let M be a finitely generated R-module. Set

$$+(M) = \{\text{isomorphism classes of } R\text{-modules } X \text{ such that } X|M^{(n)} \text{ for some } n \geq 1\}.$$

With this definition $+(M)$ becomes an additive monoid (a monoid is a semigroup with a neutral element, i.e., identity element (0)) under the operation $[M] + [N] = [M \oplus N]$.

Let $\widehat{M} = V_1^{(r_1)} \oplus \cdots \oplus V_t^{(r_t)}$ with $r_j \neq 0$, V_j indecomposable $\widehat{R}$-modules, and $V_i \ncong V_j$ for $i \neq j$. If $X \in +(M)$, say $X \oplus Y \cong M^{(n)}$, then

$$\widehat{X} \oplus \widehat{Y} \cong V_1^{(nr_1)} \oplus \cdots \oplus V_t^{(nr_t)}$$

and hence $\widehat{X} \cong V_1^{(s_1)} \oplus \cdots \oplus V_t^{(s_t)}$ with $s_i \leq nr_i$. Now we define $\Phi : +(M) \longrightarrow +(\widehat{M})$ by $\Phi([X]) = [\widehat{X}]$. By a result of Guralnick $\Phi(X)|\Phi(Y)$ if and only if $X|Y$, i.e. Φ has the divisor property and is thus one-to-one.

What is $+(\widehat{M})$? We have an isomorphism $+(\widehat{M}) \longrightarrow \mathbb{N}$ where $[Z] = [V_1^{(u_1)} \oplus \cdots \oplus V_t^{(u_t)}] \mapsto (u_1, \ldots, u_t)$. So we may consider $\Phi : +(M) \hookrightarrow \mathbb{N}^{(t)}$ and hence regard $+(M)$ as a submonoid of $\mathbb{N}^{(t)}$ with the extra property that $+(M) = H \cap \mathbb{N}^{(t)}$ for some $H \leq \mathbb{Z}^t$ (by the divisor property).

16.5 Exercises.

16.1 (3 points) Let R be an associative ring (not necessarily commutative) and M an R-module. Suppose $E := \mathrm{End}_R(M)$ is "local." If $M|U \oplus V$ then $M|U$ or $M|V$.

§17. Relating MCM Modules and Submonoids of $\mathbb{N}^t$.

Let M be a finitely generated R-module and $\widehat{R} \otimes_R M = V_1^{(n_1)} \oplus \cdots \oplus V_t^{(n_t)}$ where V_j is an indecomposable $\widehat{R}$-module for all j, $V_i \ncong V_j$ for $i \neq j$, and $n_j > 0$.

If $[N] \in +(M)$, write $\widehat{R} \otimes_R N \cong V_1^{(m_1)} \oplus \cdots \oplus V_t^{(m_t)}$. Define $\Phi\colon +(M) \hookrightarrow \mathbb{N}^{(t)}$ by $[N] \mapsto (m_1, \ldots, m_t)$. This is injective by Guralnick's result. Thus $+(M)$ is isomorphic to a submonoid of $\mathbb{N}^{(t)}$, an additive monoid.

17.1 Lemma. *Let $[N], [N'] \in +(M)$. The following are equivalent:*

(1) $N'|N$
(2) $\widehat{R} \otimes_R N'|\widehat{R} \otimes N$
(3) $\Phi(N') \leq \Phi(N)$

Proof. (1) $\Leftrightarrow$ (2) follows from a result of Guralnick. We leave (2) $\Leftrightarrow$ (3) as an exercise. $\square$

Note that Bruns & Herzog use the term " semigroup," but all of their semigroups have an identity element. Thus they really are referring to monoids.

17.2 Definition. A *positive affine monoid* is a finitely generated monoid that is isomorphic to a submonoid of $\mathbb{N}^{(t)}$ for some t. (Positive means: $x+y=0 \Rightarrow x=y=0$, that is, the identity is the only unit.)

17.3 Definition. A submonid H of $\mathbb{N}^{(t)}$ is a *full submonoid* provided: If $h_1, h_2 \in H$ and $h_1|h_2$ in $\mathbb{N}^{(t)}$, then $h_1|h_2$ in H. That is, if $h_1 + x = h_2$ with $x \in \mathbb{N}^{(t)}$, then $x \in H$.

In $\mathbb{N}^{(t)}$, $x|y \Leftrightarrow x \leq y$ (in the product partial ordering). In H, $h_1|h_2 \Leftrightarrow h_1 \leq h_2$ in $\mathbb{N}^{(t)}$. Indecomposable elements of H (atoms of H) are the minimal non-zero elements of H. Thus, by exercise 4.1, $\{$atoms of $H\}$ is finite.

17.4 Definition. A *positive normal affine monoid* is isomorphic to a positive affine monoid H that can be embedded in $\mathbb{N}^{(t)}$ in such a way that $h \in \mathbb{Z}H$ and $n \in \mathbb{N} - \{0\}$ and $nh \in \mathbb{N} \Rightarrow h \in H$. (By $\mathbb{Z}H$ we need the quotient monoid or the group generated by the monoid. So it is just the set of $a-b$ where $a, b \in H$).

17.5 Definition. A submonoid $H \leq \mathbb{N}^{(t)}$ is an *expanded submonoid* of $\mathbb{N}^{(t)}$ provided $H = L \cap \mathbb{N}^{(t)}$ for some $\mathbb{Q}$-subspace $L \leq \mathbb{Q}^{(t)}$. (Clearly, expanded $\Rightarrow$ full as $L \cap \mathbb{N}^{(t)} = (L \cap \mathbb{Z}^{(t)}) \cap \mathbb{N}^{(t)}$.)

17.6 Examples.

(1) $2\mathbb{N} = \{0, 2, 4, \dots\}$ is a full submonoid of $\mathbb{N}$, but it is not expanded (if $2 \in L$, then $1 \in L$ as you can multiply by $\frac{1}{2}$). However $2\mathbb{N} \cong \mathbb{N}$, which is an expanded submonoid of $\mathbb{N}$.

(2) $H = \{(x,y) \in \mathbb{N}^{(2)} \mid 3|(x-y)\}$. This a full submonoid of $\mathbb{N}^{(2)}$ as $H = \{(x,y) \in \mathbb{Z}^{(2)} \mid 3|(x-y)\} \cap \mathbb{N}^{(2)}$.

But H is not expanded: If $H = L \cap \mathbb{N}^{(2)}$, we have $(0,3) \in L \cap \mathbb{N}^{(2)}$ thus, $(0,1) \in L \cap \mathbb{N}^{(2)}$. Contradiction! But $H \cong H_1 = \{(x,y,z) \in \mathbb{N}^{(3)} \mid x + 2y = 3z\}$ where H_1 is expanded as it is defined by linear equations and thus is a linear subspace. The isomorphism is given by $(x,y) \mapsto (x, y, \frac{x+2y}{3})$.

Note that $H \leq \mathbb{N}^{(t)}$ is expanded $\Leftrightarrow$ $H = \mathbb{N}^{(t)} \cap \operatorname{Ker}\psi$, where ψ is an $m \times t$ matrix over $\mathbb{Z}$. (Reason: $L \leq \mathbb{Q}^{(t)}$, so $L = \operatorname{Ker}\psi$, where ψ is an $m \times t$ matrix over $\mathbb{Z}$.)

As exercise 17.5 shows, we can have elements α, β, γ in a positive normal affine monoid so that $3\alpha = \beta + \gamma$. Can we have atoms a, b in a positive normal affine monoid H such that $3a = 4b$? No! Embed $H \leq \mathbb{N}^{(t)}$. We have $a > b$. Thus $b \mid a$. So, $a = b + c$ where $b \neq 0, c \neq 0$. Contradiction!

17.7 Theorem. [R. Wiegand, Journal of Algebra, "Direct Sum Decompositions over Local Rings", 2001.] *Let H be an expanded submonoid of $\mathbb{N}^{(t)}$, say $H = \operatorname{Ker}(\psi) \cap \mathbb{N}^{(t)}$ where ψ is an $m \times t$ matrix over $\mathbb{Z}$. Assume $\exists \alpha \in H$ such that $\alpha = (a_1, \dots, a_t)$ with $a_i > 0$ for all i. Let $c_1 < \cdots < c_n \in \mathbb{Q}$ with $n = m+1$. Let R be the ring of Exercise 3.4. Then there exists a finitely generated torsion-free (MCM) R-module M and indecomposable finitely generated torsion-free $\widehat{R}$-modules $V_1, \dots, V_t$ such that $\widehat{R} \otimes_R M \cong V_1^{(a_1)}, \dots, V_t^{(a_t)}$ and $\Phi(+(M)) = H$.*

There are two main ingredients in the proof:

Theorem I. *With R as above, let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the minimal prime ideals of $\widehat{R}$ and let $(r_1, \dots, r_n) \in \mathbb{N}^{(n)} - \{(0, \dots, 0)\}$. Then there is an indecomposable torsion-free $\widehat{R}$-module X such that $X_{\mathfrak{p}_i} \cong R_{\mathfrak{p}_i}^{(r_i)}$ for all i.*

Theorem II. *With R and $\mathfrak{p}_i$ as above, let Y be a finitely generated $\widehat{R}$-module. Let $Y_{\mathfrak{p}_i} \cong R_{\mathfrak{p}_i}^{(r_i)}$. Then there is an R-module Z such that $Y \cong \widehat{R} \otimes_R Z$ if and only if $r_1 = \cdots = r_n$.*

17.8 Exercises.

17.1 (2 points) Prove (2) $\Leftrightarrow$ (3).

17.2 (3 points) Let H be a full submonoid of $\mathbb{N}^{(t)}$. Then each non-zero element of H is a sum of atoms of H. Therefore H is finitely generated.

17.3 (5 points) Let $H \leq \mathbb{N}^{(t)}$ be a positive affine monoid. Show that these are equivalent:

(1) H is a full submonoid of $\mathbb{N}^{(t)}$.

(2) $H = G \cap \mathbb{N}^{(t)}$

(3) $H = \mathbb{Z}H \cap \mathbb{N}^{(t)}$

These conditions imply that H is normal. Conversely, if H_1 is a positive normal affine monoid, then H_1 is isomorphic to a monoid H satisfying (1) – (3).

(Hint: See (6.1.5) in Bruns and Herzog. We have $H \hookrightarrow \mathbb{Z}H \cong \mathbb{Z}^{(t)}$ for some t. Let $\langle\ ,\rangle$ be the scalar product on $\mathbb{Z}H$ corresponding (via the isomorphism) to the ordinary dot product on $\mathbb{A}^{(t)}$. Fact: $\exists\ a_1, \ldots, a_t \in \mathbb{Z}H$ such that if $g \in \mathbb{Z}H$ then $g \in H \Leftrightarrow\ \langle a_i, g\rangle$ for all i. You may use this without proof.)

17.4 (8 points) Let H be a full submonoid of $\mathbb{N}^{(t)}$. Then $H \cong H_1 \leq \mathbb{N}^{(u)}$ for some u, where H_1 is expanded.

(Hint: See Exercise 6.4.16 (c) in Bruns and Herzog. There is an outline of the proof there.)

17.5 (2 points) Let $H = \{(x, y, z) \in \mathbb{N}^{(3)} |\ x + 2y = 3z\}$. Prove that H has 3 atoms, namely $\alpha = (1, 1, 1), \beta = (3, 0, 1), \gamma = (0, 3, 2)$. (Note that $3\alpha = \beta + \gamma$ so, unique factorization fails and so $H \not\cong \mathbb{N}^{(3)}$.)

17.6 (3 points) If there is no such α as described in Theorem 17.7, show that there exists a full embedding $H \leq \mathbb{N}^{(u)}$, where $u < t$ and "$\exists\alpha$".

§18. More on Submonoids of $\mathbb{N}^t$.

We assume Theorems I and II from the previous section.

18.1 Theorem. *Let H be an expanded submonoid of $\mathbb{N}^{(t)}$, say $H = (ker\psi) \cap \mathbb{N}^{(t)}$, where ψ is some $n \times t$ matrix over $\mathbb{Z}$ ($\mathbb{N}^{(t)} \subseteq \mathbb{Z}^{(t)} \xrightarrow{\psi} \mathbb{Z}^{(m)}$). Assume there exists $\alpha = (a_1, \ldots, a_t) \in H$ with $a_i > 0, \forall i$. Let $n = m + 1$ and let $c_1 < \ldots < c_n$ be distinct elements of a field k. Let R be the funny ring of exercise 3.4. Then there exists a finitely generated torsion-free R-module M and f.g. t-f $\widehat{R}$ modules $V_1, \ldots, V_t$, all indecomposable and pairwise non-isomorphic, such that $\widehat{R} \otimes_R M = V_1^{(a_1)} \oplus \ldots \oplus V_t^{(a_t)}$ and $\Phi(+(M)) = H$. In particular, $+(M) \cong H$ via $[M] \leftrightarrow \alpha$.*

Proof. Write $\psi = [q_{ij}]$ ($\mathbb{Z}$ matrix). Choose $h \in \mathbb{N} - \{0\}$ such that $q_{ij} + h \geq 0, \forall ij$. Let $P_1, \ldots, P_n$ be the minimal prime ideals of $\widehat{R}$. By Theorem I, $\exists$ f.g. indecomposable t-f $\widehat{R}$-modules $V_1, \ldots, V_t$ such that

$$\dim_{R_{P_i}} (V_j)_{P_i} = \begin{cases} q_{ij} + h, & \text{if } i \leq m \\ h, & \text{if } i = m + 1 = n \end{cases}$$

Given a $\beta = (b_1, \ldots, b_t) \in \mathbb{N}^{(t)}$, let $N(\beta) := V_1^{(b_1)} \oplus \ldots \oplus V_t^{(b_t)}$. Theorem II says an $\widehat{R}$-module N "comes from" an R-module (i.e., $\exists X$ such that $\widehat{R} \otimes_R X \cong N$) $\Leftrightarrow \dim_{R_{P_i}} N_{P_i} = \dim_{R_{P_j}} N_{P_j}, \forall i, j$. As

$$\dim_{R_{P_i}} (N(\beta)_{P_i}) = \begin{cases} \sum_{j=1}^{t} (q_{ij} + h) b_j, & \text{if } i \leq m \\ \sum_{j=1}^{t} h b_j, & \text{if } i = m + 1 = n \end{cases}$$

$N(\beta)$ comes from an R-module $\Leftrightarrow \sum_{j=1}^{t} q_{ij} b_j\ = 0, \forall i$, that is, $\Leftrightarrow \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in (ker\psi) \cap \mathbb{N}^{(t)} = H$. Since $\alpha \in H$, we see that $N(\alpha) \cong \widehat{R} \otimes_R M$, for some R-module M, necessarily finitely generated and torsion-free. Thus $\widehat{R} \otimes_R M = V_1^{(a_1)} \oplus \ldots \oplus V_t^{(a_t)}$. We just have to show $\Phi(+(M)) = H$.

$\subseteq$: Given $X \in +(M)$, we can write $\widehat{R} \otimes_R X = V_1^{(b_1)} \oplus \ldots \oplus V_t^{(b_t)} = N(\beta)$, where $\beta = (b_1, \ldots, b_t) = \Phi([X])$. Then $N(\beta)$ comes from an R-module, so $\beta \in H$.

$\supseteq$: If $\beta \in H$, $\exists$ an R-module Z such that $\widehat{R}\otimes_R Z \cong N(\beta) = V_1^{(b_1)} \oplus \ldots \oplus V_t^{(b_t)}$. But $\widehat{R}\otimes_R M = V_1^{(a_1)} \oplus \ldots \oplus V_t^{(a_t)}$ since $a_i > 0, \forall i$, choose $u \in \mathbb{N}$ such that $ua_i \geq b_i, \forall i$. Then $\widehat{R} \otimes_R Z \mid (\widehat{R} \otimes_R M)^{(u)} = \widehat{R} \otimes_R (M^{(u)})$. By Guralnick's result, we see $Z \mid M^{(u)}$. Therefore $[Z] \in +(M)$. $\square$

The pullbacks for the funny ring R and for its completion $\widehat{R}$ are as follows.

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d] & k[t]_{(t-c_1)\cup\ldots\cup(t-c_n)} \arrow[d] \\ k \arrow[r, hook] & \dfrac{k[t]}{(t-c_1)^4\cdots(t-c_n)^4} = \dfrac{k[t]}{t^4} \times \cdots \times \dfrac{k[t]}{t^4} \end{tikzcd} \qquad \begin{tikzcd} \widehat{R} \arrow[r, hook] \arrow[d] & k[[t]] \times \cdots \times k[[t]] \quad (n\ factors) \arrow[d] \\ k \arrow[r, hook] & \dfrac{k[t]}{t^4} \times \cdots \times \dfrac{k[t]}{t^4} := D = D_1 \times \cdots \times D_n \end{tikzcd}$$

[Using the facts that $\frac{k[t]}{(t-c_1)^4\cdots(t-c_n)^4} = \frac{k[t]}{(t-c_1)^4} \times \cdots \times \frac{k[t]}{(t-c_n)^4}$ and $k[t]_{(t-c)} \cong k[t]_{(t)}$.]

Now Theorem I says given $(r_1, \ldots, r_n)$, with $r_i \geq 0$, not all $r_i = 0$, $\exists$ indecomposable t-f $\widehat{R}$-module N such that $N_{P_i} \cong (\widehat{R}_{P_i})^{(r_i)}, \forall i$. So it is enough to build an indecomposable (k, D)-module (V, W) such that $W \cong D_1^{(r_1)} \times \cdots \times D_n^{(r_n)}$. (Reason: There exists projective $\overline{\widehat{R}}$-module L such that $L/\mathfrak{f}L \cong W$, namely $k[t]^{(r_1)} \times \cdots \times k[t]^{(r_n)}$) By symmetry we can assume $r_1 \geq r_i$ for all i. Thus, $r_1 > 0$.

To make things simple, we prove the following more general result.

18.2 Proposition. (Lemma 2.2 of [R. Wiegand, 2001, Journal of Algebra]) *Let $\mathbb{F}$ be a field, and let $D_1, \ldots, D_n$ be finite-dimensional $\mathbb{F}$-algebras. Assume there exists $a_1, b_1 \in D$ such that $\{1, a_1, a_1^2, b_1\}$ is linearly independent over k. Let $r_1, \ldots, r_n \in \mathbb{N}^{(n)}$ with $0 < r_1 \geq r_i, \forall i$. Then there exists indecomposable $(\mathbb{F}, D)$-module where $D = D_1 \times \cdots \times D_n$, namely, (V, W), where $W = D_1^{(r_1)} \times \cdots \times D_n^{(r_n)}$.*

Proof. Let $W_i = D_i^{(r_i)}$, so $W = W_1 \times \cdots \times W_n$. Define $C := \mathbb{F}^{(r_i)}$ (columns). Define $\partial : C \to W$ by

$$\begin{bmatrix} c_1 \\ \vdots \\ c_{r_1} \end{bmatrix} \mapsto \left(\begin{bmatrix} c_1 \\ \vdots \\ c_{r_2} \end{bmatrix}, \begin{bmatrix} c_1 \\ \vdots \\ c_{r_2} \end{bmatrix}, \ldots, \begin{bmatrix} c_1 \\ \vdots \\ c_{r_n} \end{bmatrix} \right).$$

Let $a = (a_1, 0, \ldots, 0), b = (b_1, 0, \ldots, 0) \in D$. Fix $i \leq n, k \leq r_i$. Let $e = (0, \ldots, 1, 0, \ldots, 0) \in D$ (1 in the ith spot) and $u = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}_{r_1 \times 1} \in C$ (with 1 is in the kth spot) so that $e\partial(u) = \left(0, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, 0 \right)$.

Let $J_i = \begin{bmatrix} 0 & 1 & & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}_{(r_i \times r_i)}$ and let $V = \mathbb{F}$-subspace of W consisting of elements $\partial(w) + a\partial(v) + b\partial(J_1 v)$, where w, v range over C. Note, $\partial(u) \in V$, so $e\partial(u) \in DV$. Everything in W is a D-linear combination of elements of the form $e\partial(u)$. $\therefore W = DV$. The proof that (V, W) is indecomposable is similar to (but ickier than) the proof of Theorem 5.5 Case (i). $\square$

§19. Comparing $\hat{R}-$ and $R-$modules.

Let (R, m, k) be a one-dimensional local ring with completion $\hat{R}$. We are interested in determining which finitely generated $\hat{R}-$modules arise from $R-$modules. To do so, we first consider some general results.

19.1 Proposition. *Let R, S be commutative rings and $R \to S$ a flat ring homomorphism (i.e., ${}_RS$ is flat). Let M, N be $R-$modules with M finitely presented. Then*

$$\phi_M : S \otimes_R \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_S(S \otimes_R M, S \otimes_R N)$$

is an isomorphism.

Proof. If $M = R^{(n)}$, then one can check directly that ϕ_M is an isomorphism by choosing a basis. In general, M finitely presented gives us an exact sequence

$$R^{(n)} \to R^{(m)} \to M \to 0.$$

By applying $\operatorname{Hom}_R(-, N)$ and then $S \otimes_R -$, we get the exact sequence

$$S \otimes_R \operatorname{Hom}_R(R^n, N) \leftarrow S \otimes_R \operatorname{Hom}_R(R^{(m)}, N) \leftarrow S \otimes_R \operatorname{Hom}_R(M, N) \leftarrow 0$$

as $\operatorname{Hom}_R(-, N)$ is left exact and S is flat. Similarly, we get an exact sequence by applying $S \otimes_R -$ and then $\operatorname{Hom}_S(-, S \otimes_R N)$. Thus we have the following diagram with exact rows:

$$\begin{tikzcd}
S \otimes_R \operatorname{Hom}_R(R^{(n)}, N) \arrow[d, "\cong"] & S \otimes_R \operatorname{Hom}_R(R^{(m)}, N) \arrow[l] \arrow[d, "\cong"] & S \otimes_R \operatorname{Hom}_R(M, N) \arrow[l] \arrow[d, "\phi_M"] & 0 \arrow[l] \arrow[d, "\cong"] \\
\operatorname{Hom}_S(S^{(n)}, S \otimes_R N) & \operatorname{Hom}_S(S^{(m)}, S \otimes_R N) \arrow[l] & \operatorname{Hom}_S(S \otimes_R M, S \otimes_R N) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

One needs only show the diagram is in fact commutative and then the Five Lemma yields the desired result. □

Similarly, if there exists $\cdots \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$ with each F_i finitely generated and free, then for all i

$$S \otimes_R \operatorname{Ext}^i_R(M, N) \cong \operatorname{Ext}^i_S(S \otimes_R M, S \otimes_R N).$$

This holds, for example, when R is coherent and ${}_RM$ finitely presented or R Noetherian and ${}_RM$ finitely generated. In particular, as $R \to S^{-1}R$ is flat for a multiplicatively closed set S, if R is Noetherian and ${}_RM$ finitely generated we have

$$\operatorname{Ext}^i_R(M, N)_p = \operatorname{Ext}^i_{R_p}(M_p, N_p).$$

Returning to our situation with (R, m, k) local, let (S, n, ℓ) be a local ring and $\phi : R \to S$ a local homomorphism satisfying

$(*)$ $\quad mS = n$ and the induced map on residue fields is an isomorphism

(i.e., $R + n = S$). The completion or Henselization are examples for the ring S where this holds.

19.2 Lemma. *Let $R \to S$ be a flat local homomorphism satisfying $(*)$. If ${}_RM$ has finite length, then $M \to S \otimes_R M$ is an isomorphism.*

Proof. Induct on $\lambda_R(M)$. If $\lambda_R(M) = 1$, then $M = k$. We know $S \otimes_R k = S \otimes_R R/m = S/mS = S/n = k$. If $\lambda_R(M) > 1$, we get a short exact sequence $0 \to M' \to M \to M'' \to 0$ where $\lambda(M) > \lambda(M'), \lambda(M'')$ by the additivity of length. Apply $S \otimes_R -$ to get an exact sequence and the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\cong"] & M' \arrow[r] \arrow[d, "\cong"] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d, "\cong"] & 0 \arrow[d, "\cong"] \\
0 \arrow[r] & S \otimes_R M' \arrow[r] & S \otimes_R M \arrow[r] & S \otimes_R M'' \arrow[r] & 0
\end{tikzcd}$$

By the Five Lemma, done. □

In particular, this shows that finite length modules do not change when taking the completion. Similarly, we have the following result.

19.3 Lemma. *Let $R \to S$ be flat, satisfying $(*)$. Let ${}_SM$ have finite length n. Then $\lambda_R({}_RN) = n$.*

Given a commutative Noetherian ring R, let $\alpha(R) = R_{p_1 \cup \cdots \cup p_t}$ where p_i are the minimal primes of R. If R is Cohen Macaulay, then $\alpha(R)$ is just the total quotient ring. Notice that $R \to \alpha(R)$ is not injective if R is a one-dimensional local ring that is not Cohen-Macaulay.

19.4 Theorem. *Let (R, m, k) be local with $\dim R = 1$. Let $R \to S$ be a flat local homomorphism satisfying $(*)$. Let ${}_SN$ be finitely generated. Then N comes from an $R-$module if and only if $\alpha(S) \otimes_S N$ comes from a $\alpha(R)-$module.*

Proof. The forward direction is clear by associativity of tensor products and commutativity of the diagram in Exercise 19.1: If ${}_SN$ comes from ${}_RM$ (i.e., $N \cong S \otimes_R M$), then $\alpha(S) \otimes_S N$ comes from $\alpha(R) \otimes_R M$.

For the backward direction, assume $\alpha(S) \otimes_S N \cong \alpha(S) \otimes_{\alpha(R)} X$ for some finitely generated ${}_{\alpha(R)}X$.

Remark. If Γ is a multiplicatively closed set in a commutative ring A and X a finitely generated $\Gamma^{-1}A-$module, then there exists a finitely generated $A-$module Y such that $X \cong \Gamma^{-1}Y$.
Proof. Choose generators $x_1, ..., x_n$ for ${}_{\Gamma^{-1}A}X$ and let Y be the $A-$submodule of X generated by $x_1, ..., x_n$.

Choose ${}_RY$ finitely generated as in the remark so that $\alpha(R) \otimes_R Y \cong X$. Let ${}_SZ = S \otimes_R Y$. We know by Exercise 19.1 that $\alpha(S) \otimes_S Z \cong \alpha(S) \otimes_S N$. Choose any isomorphism $\phi \in \mathrm{Hom}_{\alpha(S)}(\alpha(S) \otimes_S Z, \alpha(S) \otimes_S N) \cong \alpha(S) \otimes_S \mathrm{Hom}_S(Z, N)$. Then $\phi = \frac{1}{q}\psi$ for some $\psi \in \mathrm{Hom}_S(Z, N)$ and an element q not contained in any minimal prime of S. Note that $q\phi : \alpha(S) \otimes_S Z \to \alpha(S) \otimes_S N$ is also an isomorphism as q is a unit of $\alpha(S)$. So we now have an $S-$homomorphism $\psi : Z \to N$ such that $\alpha(S) \otimes \psi : \alpha(S) \otimes_S Z \to \alpha(S) \otimes_S N$ is an isomorphism. Let $K = \ker \psi$ and $C = \operatorname{coker} \psi$. Then we have the following commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & Z \arrow[rr] \arrow[dr] & & N \arrow[r] & C \arrow[r] & 0 \\
 & & & W \arrow[ur] \arrow[dr] & & & \\
 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Note that $\alpha(S) \otimes_S K = 0 = \alpha(S) \otimes_S C$ as $\alpha(S) \otimes \psi$ is an isomorphism. Thus $K_p = 0 = C_p$ for all minimal primes (i.e., all $p \neq n$). This says $\lambda_S(K), \lambda_S(C) < \infty$ so $K = S \otimes_R K$ and $C = S \otimes_R C$. This says $\mathrm{Hom}_R(K, Y)$ has finite length as an $R-$module as $\mathrm{Hom}_R(K, Y)_p = \mathrm{Hom}_{R_p}(K_p, Y_p) = 0$ for all $p \neq n$ as $K_p = 0$ for all $p \neq n$. So, we have the following sequence of isomorphisms

$$\mathrm{Hom}_R(K, Y) \xrightarrow{\cong} S \otimes_R \mathrm{Hom}_R(K, Y) \xrightarrow{\cong} Hom_S(S \otimes_R K, S \otimes_R Y) = \mathrm{Hom}_S(K, Z)$$

as $Z = S \otimes_R Y$. Choose $j \in \mathrm{Hom}_S(K, Z) \cong \mathrm{Hom}_S(K, Y)$. Then there exists $i : K \to Y$ such that $S \otimes_R i = j$. Let $Q = \operatorname{coker} i$. Then $S \otimes_R Q \cong_S \operatorname{coker} j =_S W$ as flatness preserves cokernels. Thus W and C are extended. Now $9 \to W \to N \to C \to 9$ represents an element of $\mathrm{Ext}^1_S(C, W) \cong S \otimes_R \mathrm{Ext}^1_R(C, W)$. Again $\mathrm{Ext}^1_r(C, Q)$ has finite length as $C_p = 0$ for all $p \neq n$. So $S \otimes_R \mathrm{Ext}^1_R(C, Q) = \mathrm{Ext}^1_R(C, Q)$. Thus $0 \to W \to N \to C \to 0$ comes from an element of

$\operatorname{Ext}^1_R(C,Q)$, say $0 \to W \to U \to C \to 0$. Then we have the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W \arrow[r] \arrow[d, equal] & S \otimes_R U \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W \arrow[r] & N \arrow[r] & U \arrow[r] & 0
\end{tikzcd}$$

Thus $N \cong S \otimes_R U$, that is, N is extended. $\square$

19.5 Corollary. *Let R be a one-dimensional local domain. Let $\hat{R}$ be reduced with minimal primes $p_1, ..., p_t$. Let ${}_{\hat{R}}N$ be finitely generated and assume $N_{p_i} \cong \hat{R}_{p_i}^{(r_i)}$ for all i. Then N is extended from R if and only if $r_1 = \cdots = r_t$.*

Proof. Use the following diagram

$$\begin{tikzcd}
 & \hat{R}_{p_1} \times \cdots \times \hat{R}_{p_t} \\
\hat{R} \arrow[ur] & \alpha(R) = K \arrow[u] \\
R \arrow[u] \arrow[ur] &
\end{tikzcd} \qquad (K = \text{ the quotient field })$$

$\square$

Exercise 19.1. (4 points) Let $R \to S$ be a flat homomorphism of commutative Noetherian rings. Then there exists a commutative diagram

$$\begin{tikzcd}
 & \alpha(S) \\
S \arrow[ur] & \alpha(R) \arrow[u] \\
R \arrow[u] \arrow[ur] &
\end{tikzcd}$$

[Hint: Just show the denominators in $\alpha(R)$ are units in $\alpha(S)$. Lemma A.9 in Bruns & Herzog may be helpful.]

§20. A general result on faithfully flat descent.

Let $R \to S$ be a map of commutative rings. The map is *faithfully flat* provided ${}_RS$ is flat and ${}_RX \neq 0$ implies $X \otimes_R S \neq 0$. Exercise 20.1 shows that $R \to S$ faithfully flat also implies $\mathfrak{m}S \neq S$ for any maximal ideal $\mathfrak{m}$.

20.1 Theorem. *Let $R \to S$ be a faithfully flat map of commutative rings. Let M, N be finitely presented R-modules. Then*

$$M \in +(N) \iff [S \otimes_R M] \in +(S \otimes_R N)$$

Proof. One direction is clear. For if $M \in +(N)$ then $M \oplus X \cong N^{(n)}$ for some X and some $n \in \mathbb{N}$. Then $(S \otimes_R M) \oplus (S \otimes_R X) \cong S \otimes_R (M \oplus X) \cong S \otimes_R N^{(n)} \cong (S \otimes_R N)^{(n)}$.

For the other direction assume that $S \otimes_R M | (S \otimes_R N)^{(t)} \cong S \otimes_R N^{(t)}$. Since $+(N) = +(N^{(t)})$ we may assume $t = 1$, i.e. $S \otimes_R M | S \otimes_R N$. This gives the following diagram:

$$\begin{tikzcd} S \otimes_R M \arrow[r, "\alpha"] \arrow[rr, bend right, "1_{S \otimes_R M}"'] & S \otimes_R N \arrow[r, "\Phi"] & S \otimes_R M \end{tikzcd}$$

where $\Phi \in \mathrm{Hom}_S(S \otimes_R N, S \otimes_R M) \cong S \otimes_R \mathrm{Hom}_R(M, N)$. So we can write $\Phi = s_1 \otimes \varphi_1 + \cdots + s_m \otimes \varphi_m$ with $s_i \in S$ and $\varphi_i \in \mathrm{Hom}_R(M, N)$. Set

$$\psi = [\varphi_1 \ldots \varphi_m] : N^{(m)} \to M$$

This gives the following commutative diagram:

$$\begin{tikzcd} S \otimes_R M \arrow[r, "\alpha"] \arrow[drrr, bend right, "1_{S \otimes_R M}"'] & S \otimes_R N \arrow[r, "\begin{bmatrix} s_1 \\ \vdots \\ s_m \end{bmatrix}"] \arrow[drr, "\Phi"] & (S \otimes_R N)^{(m)} \arrow[r, "\cong"] & S \otimes_R N^{(m)} \arrow[d, "1_S \otimes \psi"] \\ & & & S \otimes_R M \end{tikzcd}$$

which shows that $1_S \otimes \psi : S \otimes_R N^{(m)} \to S \otimes_R M$ is a split surjection. By the next lemma ψ is a split surjection and we're done. $\square$

20.2 Lemma. *Let $R \to S$ be a faithfully flat map and M, N R-modules with M finitely presented. If $\psi : N \to M$ is an R-linear map with $1_S \otimes \psi$ a split surjection, then ψ is a split surjection.*

Proof. Choose $\beta : S \otimes_R M \to S \otimes_R N$ such that $(1_S \otimes \psi)\beta = 1_{S \otimes_R M}$. Given any $\sigma \in \mathrm{Hom}_S(S \otimes_R M, S \otimes_R M)$, we have $\beta\sigma : S \otimes_R M \to S \otimes_R N$ and $(1_S \otimes \psi)(\beta\sigma) = \sigma$. This says $1_S \otimes \psi)(\beta\sigma)_* : \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N) \to \mathrm{Hom}_S(S \otimes_R M, S \otimes_R M)$ is surjective. Of course, we have

$$\begin{tikzcd} \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N) \arrow[r] & \mathrm{Hom}_S(S \otimes_R M, S \otimes_R M) \\ S \otimes_R \mathrm{Hom}_R(M, N) \arrow[u, "\cong"] \arrow[r, "1 \otimes \psi_*"] & S \otimes_R \mathrm{Hom}_R(M, N) \arrow[u, "\cong"] \end{tikzcd}$$

By commutativity, we see that $1 \otimes \psi_*$ is surjective. But $1 \otimes \psi_*$ surjective implies ψ_* is surjective since S is faithfully flat. Now choose $\alpha \in \mathrm{Hom}_R(M, N)$ such that $\psi_*(\alpha) = 1_M$. Then $\psi \circ \alpha = 1_M$ implies ψ is split surjective. $\square$

20.3 Example. Let $R = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$ and let $\mathfrak{m}$ be a real maximal ideal of R, i.e. a point on the circle. The ideal $\mathfrak{m}$ will have the form $\mathfrak{m} = \mathfrak{m}_p = (x - a, y - b)$ with $p = (a, b) \in \mathbb{R}^2$ and $a^2 + b^2 = 1$. Exercises 20.2 and 20.3 show that $\mathfrak{m}_p$ is an invertible ideal and $[\mathfrak{m}_p]^{-1} = [\mathfrak{m}_p]$ and $[\mathfrak{m}_p]^{-1} = [\mathfrak{m}_q]$. Thus the class group of R has order $|cl(R)| = 2$ (the nontrivial element is the "Möbius band"). Now let $S = \mathbb{C} \otimes_{\mathbb{R}} R \cong \mathbb{C}[x, y]/(x^2 + y^2 - 1) = \mathbb{C}[u, v]/(uv - 1) \cong \mathbb{C}[u, u^{-1}]$, where $u = x + iy$ and $v = x - iy$. Thus $\mathbb{C} \otimes_{\mathbb{R}} R$ is a PID. For $\mathfrak{m} = \mathfrak{m}_{(0,1)}$ we have $\mathfrak{m} \nmid R$ but $S \otimes_R \mathfrak{m} = (z) \cong S$ for some $z \in S$, i.e. $S \otimes_R \mathfrak{m} | S \otimes_R R$.

20.4 Theorem. *Let (R, m) be a local ring and $R \to S$ a faithfully flat homomorphism with S Noetherian. Let M and N be finitely generated R-modules. Then $M|N$ if and only if $S \otimes_R M | S \otimes_R N$.*

Proof. Faithful flatness implies $mS \neq S$, so there is a maximal ideal $\mathfrak{n}$ of S containing $\mathfrak{m}S$. We can replace $R \to S$ by the flat local homomorphism $R \to S_n$. Changing notation, we may assume that S is local and $R \to S$ is a flat local homomorphism. By Guralnick's theorem, it is enough to show that $M/m^nM \mid N/m^nN$ for all $n \geq 1$.

For any n, we have $A := R/\mathfrak{m}^n \to S/\mathfrak{m}^nS =: B$, a flat local homomorphism, and we have finitely generated B-modules $U := M/\mathfrak{m}^nM$ and $V := N/\mathfrak{m}^nN$ such that $B \otimes_A U | B \otimes_A V$, say, $B \otimes_A U \oplus Z \cong B \otimes_A V$. We want to conclude that $U|V$. We proceed by induction on the length of U (or on the number of indecomposable factors; the argument is the same). We know that $U|V^{(t)}$ for some t. Since A is Artinian we have Krull-Remak-Schmidt for finitely generated A-modules. Therefore if U is indecomposable, we clearly have $U|V$.

If U is not indecomposable, write $U = U' \oplus U''$, a direct sum of modules that are shorter than U. Since $B \otimes_A U' | B \otimes_A V$, we have $U'|V$ by induction, say, $U' \oplus W \cong V$. Then we have $B \otimes_A U' \oplus B \otimes_A U'' \oplus Z \cong B \otimes_A U \oplus Z \cong B \otimes_A V \cong B \otimes_A U' \oplus B \otimes_A W$. Since B is local we can cancel, getting $B \otimes_A U'' | B \otimes_A W$. By induction, $U''|W$, and it follows that $U' \oplus U''|V$, as desired. $\square$

20.5 Exercises.

20.1 (4 points) $R \to S$ is faithfully flat if and only if the map is flat and for each maximal ideal $\mathfrak{m}$, $\mathfrak{m}S \neq S$.

20.2 (5 points) Show that $\mathfrak{m}$ is not principal (use something like the Intermediate Value Theorem).

20.3 (3 points) Show that $\mathfrak{m}_p\mathfrak{m}_q = (f)$ where the line connecting p and q is the vanishing set of f. Also show $\mathfrak{m}_p^2 = (g)$ where $g = 0$ is the tangent line at p. In particular $\mathfrak{m}_{(0,1)}^2 = (y-1)$.

§21. Smooth, Unramified, Étale

21.0 Context. $R \to S$ is a ring homomorphism, R is Noetherian and commutative, and S is finitely generated as an R-algebra ($R[X_1, \ldots, X_n] \twoheadrightarrow S$). For references, see [7] and Sections 17 and 18 of [6].

We're interested in R-algebras C, ideals I of C with $I^2 = 0$, and R-algebra homomorphisms $\bar{\varphi} : S \to C/I$. What liftings are there to $\varphi : S \to C$ making the following diagram commute?

$$\begin{tikzcd} S \arrow[r, "\bar{\varphi}"] \arrow[dr, dashed, "\varphi"] & C/I \\ R \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$

21.1 Definition. With R, S as above, $R \to S$ is $\left\{ \begin{array}{c} \text{smooth} \\ \text{unramified} \\ \text{étale} \end{array} \right\}$ provided for each R-algebra C, each ideal I of C with $I^2 = 0$ and each R-homomorphism $\bar{\varphi} : S \to C/I$, $\left\{ \begin{array}{c} \exists \varphi \\ \exists \text{ at most one } \varphi \\ \exists! \varphi \end{array} \right\}$.

21.2 Remark. *In testing these properties, we can restrict to finitely generated R-algebras C. Why? Write $C = \bigcup_{\vec{\alpha}} C_\alpha$, where the C_α are the finitely generated R-subalgebras of C. Let $I_\alpha = I \cap C_\alpha$. Since S is a finitely generated R-algebra and $R/I = \bigcup_{\vec{\alpha}} R_\alpha/I_\alpha$, $im\bar{\varphi} \subseteq R_\alpha/I_\alpha$ for some α. We can consider the following diagram:*

$$\begin{tikzcd}
S \arrow[rr, "\bar{\varphi}"] \arrow[dr] \arrow[ddr, "\exists"'] & & C/I \\
& C_\alpha/I_\alpha \arrow[ur, hook] & \\
& C_\alpha \arrow[u] \arrow[dr, hook] & \\
R \arrow[uuu] \arrow[rr] & & C \arrow[uuu]
\end{tikzcd}$$

To prove the unramified case, enlarge α if necessary, and we have the following diagram.

$$\begin{tikzcd}
S \arrow[rr, "\bar{\varphi}"] \arrow[dr] \arrow[ddr, bend right] \arrow[ddr, bend left] & & C/I \\
& C_\alpha/I_\alpha \arrow[ur, hook] & \\
& C_\alpha \arrow[u] \arrow[dr, hook] & \\
R \arrow[uuu] \arrow[ur] \arrow[rr] & & C \arrow[uuu]
\end{tikzcd}$$

Now if there are two arrows to C_α, then there must have been two arrows to C_α/I_α.

21.3 Properties.

Transitivity: Let $R \to S \to T$, $R \to S$ finitely generated, $S \to T$ finitely generated. If $R \to S$ and $S \to T$ are smooth/unramified/étale, the same holds for $R \to T$.

Base Change: Suppose $R \to T$ is a ring homomorphism with T Noetherian, and $R \to S$ is smooth/unramified/étale. Then so is $T \to S \otimes_R T$.

Faithfully Flat Descent: Suppose $R \to T$ is faithfully flat and T is Noetherian. Assume $R \to S$ is finitely generated. Then $R \to S$ is unramified if and only if $T \to S \otimes_R T$ is unramified. (This is true for étale as well - later!)

21.4 Standard Étale Homomorphism. Start with R, Noetherian. Let $f \in R[X]$ be monic. Let $g \in R[X]$. Let $S = R[X]/(f)$ (then S is finitely generated and free as an R-module: If $f = X^n + a_{n-1}X^{n-1} + \cdots + a_0$ then S has R basis $\{1, \bar{X}, \ldots, \bar{X}^{n-1}\}$). Assume $f' = \frac{df}{dX}$ is invertible in $S\left[g^{-1}\right]$. Such a homomorphism $R \to S\left[g^{-1}\right]$ is called a *"standard étale homomorphism"*. Note that $R \to S\left[\frac{1}{g}\right]$ is flat. We want to show that $R \to S\left[\frac{1}{g}\right]$ is étale:

$$\begin{tikzcd}
R \arrow[r] \arrow[dr] \arrow[drrr, bend left] & R[X] \arrow[r] \arrow[d] & R[X, \frac{1}{g}] \arrow[d] & \\
& S \arrow[r] & S[\frac{1}{g}] \arrow[r, dashed] \arrow[dr, "\bar{\varphi}"'] & C \arrow[d, "\pi"] \\
& & & C/I
\end{tikzcd}$$

Note: C is an R-algebra, so we have an arrow $R \to C$.

Let $\bar{c} \in C/I$, $\bar{c} = \bar{\varphi}\left(\bar{X}\right)$ where $\bar{X}$ is the image of X in $S\left[\frac{1}{g}\right]$. We want to show that $\exists!\ c \in C$ such that $\pi(c) = \bar{c}$ and $f(c) = 0$. (Notice $\bar{\varphi}(g)$ will be a unit of C/I. Since $I^2 = 0$, any lifting φ will take g to a unit of C.) Choose any $c_0 \in C$ such that $\pi(c_0) = \bar{c}$. Given $e \in I$, using the Taylor expansion about c_0, write $f(c_0 + e) = f(c_0) + ef'(c_0) + q(e)$, where $q \in X^2R[X]$. Note that $e \in I$, so $e^2 = 0$, hence $q(e) = 0$. Now, $f(c_0 + e) = f(c_0) + ef'(c_0)$. Since $f'(c_0)$ is a unit of C/I (as it is the image of a unit from S), there is a unique $e \in I$ such that $f(c_0 + e) = 0$. Namely, $e = -\frac{f(c_0)}{f'(c_0)}$. (Note $f'(c_0)$ is a unit in C since it becomes a unit in C/I.) Thus, we have found a unique $c \in C$, namely $c = c_0 + e$, such that $\pi(c) = \bar{c}$ and $f(c) = 0$, and can use this to get a unique $\varphi : S\left[\frac{1}{g}\right] \to C$ making the diagram commute.

21.5 Definition. Let R be commutative and ${}_RM$ an R-module. A *derivation* $R \to M$ is a function $d : R \to M$ such that $d(a+b) = d(a) + d(b)$ and $d(ab) = ad(b) + bd(a)$. Suppose R is a k-algebra, where k is any commutative ring (not necessarily a field). Then $d : R \to M$ is a $k-$*derivation* provided that it is a derivation and $d(ca) = cd(a)$ for all $c \in k$ and $a \in R$.

Note $d(1) = d(1 \cdot 1) = 1 \cdot d(1) + 1 \cdot d(1) \Rightarrow d(1) = 0$. Therefore, $d(c \cdot 1) = 0$ for all $c \in k$.

21.6 Definition A *universal* $k-$*derivation* is a $k-$derivation $d : R \to N$ such that for any $k-$derivation $d' : R \to M$ there exists a unique $\varphi \in Hom_R(N, M)$ making the following diagram commute.

$$\begin{tikzcd} R \arrow[r, "d"] \arrow[dr, "d'"'] & N \arrow[d, "\exists!\varphi"] \\ & M \end{tikzcd}$$

Let $Der_k(R, M) = \{k\text{-derivations} : R \to M\}$.

Notation: $d : R \to \Omega_{R/k}$ is the universal derivation. ($\Omega_{R/k}$ is called the module of Kähler differentials.) Note that $\varphi \mapsto \varphi \circ d$ defines a natural isomorphism $Hom_R(\Omega_{R/k}, M) \cong Der_k(R, M)$.

21.7 Example. Let $R = k[X, Y]$. Claim: $\Omega_{R/k} = R \oplus R$, where $R \xrightarrow{\delta} R \oplus R$ is defined by $\delta(f) = \left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}\right)$

21.8 Exercises

21.1 (3 points) Show this works! (Hint: Use that if $d : R \to M$ is a k-derivation, then $d(f) = \frac{\partial f}{\partial X}dX + \frac{\partial f}{\partial Y}dY$.)

In general, there exists $\Omega_{R/k}$ whenever $R = k[\underline{X}]$, where $\underline{X}$ is any set of indeterminates (possibly infinite). Each ring R is a quotient $k[\underline{X}]/I$, so we have the picture

$$\begin{tikzcd} k[\underline{X}] \arrow[r, "\delta_{k[\underline{X}]/k}"] \arrow[d] & \Omega_{k[\underline{X}]/k} \arrow[d] \\ R = \frac{k[\underline{X}]}{I} \arrow[r, "\delta_{R/k}"] & \frac{\Omega_{k[\underline{X}]/k}}{R\delta_{k[\underline{X}]/k}(I)} \end{tikzcd}$$

21.2 (2 points) Let $R \twoheadrightarrow S$ (surjective). Then $R \to S$ is unramified.

21.3 (4 points) Let $f \in R$. Then $R \to R\left[\frac{1}{f}\right]$ is étale.

§22. More on k-derivations

Let R be a finitely generated k-algebra, where k is any commutative Noetherian ring. Recall that a universal derivation is a derivation $\delta_{R/k} : R \longrightarrow \Omega_{R/k}$ such that for any derivation $d : R \longrightarrow M$ there exists a unique R-module homomorphism $\varphi : \Omega_{R/k} \longrightarrow M$ making the following diagram commute:

$$\begin{tikzcd} R \arrow[r, "\delta_{R/k}"] \arrow[d, "d"'] & \Omega_{R/k} \arrow[dl, dotted, "\exists!\ \varphi"] \\ M & \end{tikzcd}$$

Letting $\mathrm{Der}_k(R, M) = \{k\text{-derivations} : R \longrightarrow M\}$, we have:

$$\begin{array}{ccc} \mathrm{Der}_k(R, M) & \cong & \mathrm{Hom}(\Omega_{R/k}, M) \\ d & \longrightarrow & \varphi \\ \psi \circ \delta_{R/k} & \leftarrow & \psi \end{array} ,$$

where φ is the map that makes the above diagram commute. Through this isomorphism, we can turn derivations into R-module homomorphisms.

We look now at another approach to constructing $\Omega_{R/k}$. Here k is any commutative ring and R is a k-algebra, not necessarily finitely generated over k. Consider the following canonical exact sequence:

$$0 \longrightarrow I \xrightarrow{\subseteq} R \otimes_k R \xrightarrow{f} R \longrightarrow 0 \ ,$$

where I is the kernel of the map $f : R \otimes_k R \longrightarrow R$ given by $f(a \otimes b) = ab$.

Note that $R \otimes_k R$ is a commutative ring with multiplication given by $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 b_1) \otimes (a_2 b_2)$ (and usual addition).

22.1 Remarks.

(1) I is generated as a left R-module by elements of the form $1 \otimes a - a \otimes 1$.

Proof. Suppose $x = a_1 \otimes b_1 + \cdots + a_n \otimes b_n \in I$. Then $f(x) = a_1 b_1 + \cdots + a_n b_n = 0$. Note that we can write

$$\begin{aligned} x &= (a_1 \otimes 1)(1 \otimes b_1 - b_1 \otimes 1) + \cdots + (a_n \otimes 1)(1 \otimes b_n - b_n \otimes 1) + \\ &\quad + \underbrace{(a_1 \otimes 1)(b_1 \otimes 1) + \cdots + (a_n \otimes 1)(b_n \otimes 1)}_{= (a_1 b_1 + \cdots + a_n b_n) \otimes 1 = 0} \\ &= (a_1 \otimes 1)(1 \otimes b_1 - b_1 \otimes 1) + \cdots + (a_n \otimes 1)(1 \otimes b_n - b_n \otimes 1) \ . \end{aligned}$$

(2) There are two R-module structures on $R \otimes_k R$:

(i) $R \otimes_k R$ is a left R-module via $r(a \otimes b) = (ra) \otimes b$, $r \in R$, and

(ii) $R \otimes_k R$ is a right R-module via $(a \otimes b)r = a \otimes (rb)$, $r \in R$.

These two structures are different, but they agree on I/I^2.

Proof. For each generator of I and for all $r \in R$, we have:

$$\begin{aligned} r(1 \otimes a - a \otimes 1) - (1 \otimes a - a \otimes 1)r &= r \otimes a - ra \otimes 1 - 1 \otimes ra + a \otimes r \\ &= -(1 \otimes r - r \otimes 1)(1 \otimes a - a \otimes 1) \in I^2 \ . \end{aligned}$$

Thus I/I^2 is an R-module **unambiguously**. □

(3) Define a map $\delta : R \longrightarrow I/I^2$ by $\delta(a) = \overline{1 \otimes a - a \otimes 1}$. We leave it as an exercise to show δ is a $k-$derivation. In fact, δ is a universal k-derivation.

Proof. Consider the diagram

$$\begin{tikzcd} R \arrow[r, two heads, "\delta"] \arrow[d, "d"'] & I/I^2 \arrow[dl, dotted, "\exists!\ \varphi"] \\ M & \end{tikzcd}$$

where M is an R-module and $d : R \longrightarrow M$ is a k-derivation. We need to show that there is a unique map $\varphi : I/I^2 \longrightarrow M$. Note that δ is surjective and thus uniqueness follows.

Now define $\psi : R \otimes_k R \longrightarrow M$ by $\psi(a \otimes b) = ad(b)$. Since $d(1) = 0$, $\psi(1 \otimes a - a \otimes 1) = d(a) - ad(1) = d(a)$. Check that $\psi(I^2) = 0$. Then we get a map $\varphi : I/I^2 \longrightarrow M$ satisfying $\varphi(\overline{1 \otimes a - a \otimes 1}) = d(a)$. Hence $\varphi \circ \delta = d$. □

22.2 The Fundamental Exact Sequence.
Consider the sequence

$$k \xrightarrow{f} R \xrightarrow{g} S \ ,$$

of homomorphisms of commutative rings where k is Noetherian, R finitely generated over k and S finitely generated over R. Then the sequence

$$\Omega_{R/k} \otimes_R S \xrightarrow{\alpha} \Omega_{S/k} \xrightarrow{\beta} \Omega_{S/R} \longrightarrow 0 \ ,$$

where

$$\alpha(\delta_{R/k}(a) \otimes b) = b\delta_{S/k}(g(a)) \quad \text{and} \quad \beta(\delta_{S/k}(b)) = \delta_{S/R}(b) \ ,$$

is exact.

Proof. Note that all modules in the sequence above are S-modules and so it is enough to show that the sequence remains exact after applying $\mathrm{Hom}_S(-, M)$, for each R-module M.

Let M be an arbitrary S-module. Apply $\mathrm{Hom}_S(-, M)$ to the sequence above to get:

$$\begin{tikzcd}
\mathrm{Hom}_S(\Omega_{R/k} \otimes_R S, M) \arrow[d, "\cong^{(\star)}"] & \mathrm{Hom}_S(\Omega_{S/k}, M) \arrow[l, "\alpha^*"'] \arrow[d, "\cong"] & \mathrm{Hom}_S(\Omega_{S/R}, M) \arrow[l, "\beta^*"'] \arrow[d, "\cong"] & 0 \arrow[l] \\
\mathrm{Der}_k(R, M) & \mathrm{Der}_k(S, M) \arrow[l] & \mathrm{Der}_R(S, M) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

The isomorphism $(\star)$ comes from the fact that tensor products and Hom are adjoint:

$$\begin{aligned}
\mathrm{Hom}_S(\Omega_{R/k} \otimes_R S, M) &\cong \mathrm{Hom}_R(\Omega_{R/k}, \mathrm{Hom}_S(S, M)) \\
&\cong \mathrm{Hom}_R(\Omega_{R/k}, M) \text{ as } M \cong \mathrm{Hom}_S(S, M) \\
&\cong \mathrm{Der}_k(R, M) \ .
\end{aligned}$$

So both squares commute. The bottom row is easily seen to be exact, and therefore the top one is exact as well. □

22.3 Theorem. *Let R be Noetherian and let $\alpha : R \longrightarrow S$ be a finitely generated R-algebra. Then α is unramified if and only if $\Omega_{S/R} = 0$.*

Proof. $(\Longleftarrow)$ Consider the following diagram

$$\begin{tikzcd}
S \arrow[r] \arrow[dr, "\varphi_i"] & C/I \\
R \arrow[u, "\alpha"] \arrow[r] & C \arrow[u]
\end{tikzcd}$$

where C is a (finitely generated) R-algebra, I is an ideal of C with $I^2 = 0$, and φ_1 and φ_2 are ring homomorphisms making the diagram commute. Define $d : S \longrightarrow I$ by $d(s) = \varphi_1(s) - \varphi_2(s)$. (Note that $d(s) \in I$ since the upper triangle commutes). If $x \in I$ and $s \in S$, then

$$\varphi_1(s)x - \varphi_2(s)x = (\varphi_1(s) - \varphi_2(s))x \in I^2 \quad \Longrightarrow \quad \varphi_1(s)x = \varphi_2(s)x \ .$$

Thus I has a well-defined S-module structure and I is an S-module via $sx := \varphi_1(s)x = \varphi_2(s)x$. Check that $d(\alpha(R)) = 0$ and that $d : S \longrightarrow I$ is a derivation (exercise). Hence $\varphi_1 = \varphi_2$.

($\Longrightarrow$) Consider the sequence

$$0 \longrightarrow J \xrightarrow{\subseteq} S \otimes_R S \longrightarrow S \longrightarrow 0 \ ,$$

where $J := \ker(S \otimes_R S \longrightarrow S)$. We want to show $J = J^2$.

Let $C = \dfrac{S \otimes_R S}{J^2}$, $I = \dfrac{J}{J^2} \leq C$ with $I^2 = 0$. We have:

$$\begin{tikzcd} S \arrow[r, "\overline{\varphi}=1_S"] \arrow[dr, "\varphi_i"] & S \cong C/I \\ R \arrow[u, "\alpha"] \arrow[r] & C = \dfrac{S \otimes_R S}{J^2} \arrow[u, two heads, "\pi"'] \end{tikzcd}$$

Define $\varphi_1 : S \longrightarrow C$ by $\varphi_1(a) = \overline{a \otimes 1}$ $(= a \otimes 1 + J^2)$ and $\varphi_2 : S \longrightarrow C$ by $\varphi_2(a) = \overline{1 \otimes a}$. These two maps make the bottom triangle commute. Check that the diagram commutes. Hence $a \otimes 1 - 1 \otimes a \in J^2$. Since such elements generate J, we conclude that $J = J^2$.

Suppose now that R is Noetherian and S is a finitely generated R-algebra. Then $S \otimes_R S$ is finitely generated as an R-algebra. Hence $S \otimes_R S$ is a Noetherian ring. So $J := \ker(S \otimes_R S \longrightarrow S)$ is a finitely generated ideal of $S \otimes_R S$.

If $R \longrightarrow S$ is unramified, then $J = J^2$ and so J is generated as an ideal of $S \otimes_R S$ by an idempotent (the proof uses the determinant trick). So the sequence

$$0 \longrightarrow J \longrightarrow S \otimes_R S \longrightarrow S \longrightarrow 0$$

splits as $S \otimes_R S$-modules. □

As a consequence, we have the following Proposition.

22.4 Proposition. *Let $R \longrightarrow S$ be unramified and flat. Let N be a finitely generated S-module. Then there is a finitely generated R-module M such that $N \,|\, S \otimes_R M$.*

Proof. Regard N as an R-module (note that N need not be finitely generated as an R-module, unless S is module-finite). Apply $- \otimes_S N$ to the diagonal sequence

$$0 \longrightarrow J \longrightarrow S \otimes_R S \longrightarrow S \longrightarrow 0$$

to get the split exact sequence of left S-modules

$$0 \longrightarrow J \otimes_S N \longrightarrow S \otimes_R N \longrightarrow N \longrightarrow 0 \ .$$

Hence $N \,|\, S \otimes_R N$.

Now write

$$N = \bigcup_{\vec{\alpha}} M_\alpha \ ,$$

where the M_α are finitely generated R-modules. By flatness,

$$S \otimes_R N = \bigcup_{\vec{\alpha}} S \otimes_R M_\alpha \ .$$

So there is α such that $N \,|\, S \otimes_R M_\alpha$. Take $M = M_\alpha$. Then M is a finitely generated R-module, and N is a direct summand of $S \otimes_R M$. □

22.5 Exercises.

22.1 (3 points) Show that δ is a k-derivation.

22.2 (3 points) Prove that $d : S \longrightarrow I$ is an R-derivation.

22.3 (3 points) Find an example of a homomorphism $R \to S$ of Noetherian rings such that $S \otimes_R S$ is not Noetherian.

§23. Fibers

Suppose $R \xrightarrow{\varphi} S$ is a ring homomorphism. Let $Spec(S) \xrightarrow{\varphi^a} Spec(R)$ be defined by

$$\varphi^a(Q) = \varphi^{-1}(Q).$$

We often write $\varphi^{-1}(Q) = Q \cap R$ (even if R does not embed into S). The *fiber* over $P \in Spec(R)$ is

$$(\varphi^a)^{-1}(P) = \{Q \in Spec(S) | Q \cap R = P\}.$$

Let $k(P) = R_P/PR_P$=quotient field of R/P. Then there exists a natural homeomorphism

$$Spec(k(P) \otimes_R S) \longleftrightarrow (\varphi^a)^{-1}(P).$$

Note $k(P) \otimes_R S = S_p/PS_P$ where $S_P := (R-P)^{-1}S$ is called the *fiber ring*. Let $R \longrightarrow S$, $Q \in Spec(S)$, $P = Q \cap R$. We say Q is *isolated in its fiber* provided Q is both minimal and maximal among prime ideals of S lying over P. Equivalently, QS_P/PS_P is maximal and minimal in $Spec(S_P/PS_P)$.

23.1 Zariski's Main Theorem (ZMT). *Let R be a Noetherian ring, $R \xrightarrow{i} S$ finitely generated algebra. Let $Q \in Spec(S)$ and $P = Q \cap R \in Spec(R)$. Let $R' =$ be the integral closure of R in S (the integral closure of $\phi(R)$ in S). Let $Q' = Q \cap R'$. Suppose Q is isolated in its fiber with respect to $R \to S$. Then there exists $f \in R' - Q'$ such that $R'\left[\frac{1}{f}\right] = S\left[\frac{1}{f}\right]$.*

$$\begin{tikzcd}
Q \arrow[r, hook] \arrow[d, dash] & S \\
Q' \arrow[r, hook] \arrow[d, dash] & R' \arrow[u, hook] \\
P \arrow[r, hook] & R \arrow[u]
\end{tikzcd}$$

23.2 Remark. Let $R \to S$ be a finitely generated algebra, R Noetherian. Let $Q \in Spec(S)$ and $P = Q \cap R$. Then Q is isolated in its fiber $\Longleftrightarrow$ S_Q/PS_Q is a finite dimensional k(P)-algebra.

Proof. Let $k = R_P/PR_P$, $A = S_P/PS_P$, the fiber ring. Let $\mathfrak{q} = QA = QS_P/PS_P$. Note that A is a finitely generated k-algebra and that $A_\mathfrak{q}$ is naturally isomorphic to S_Q/PS_Q. The remark now follows from the following proposition. $\square$

23.3 Proposition. *Let k be a field and A be a finitely generated k-algebra. Let $\mathfrak{q} \in Spec(A)$. Then $\mathfrak{q}$ is both minimal and maximal in $Spec(A)$ if and only if $A_\mathfrak{q}$ is a finite dimensional k-algebra.*

Proof. The forward direction is an exercise. For the backward direction, since $A_\mathfrak{q}$ is a finite dimension $k-$algebra, it is Artinian. Thus $\mathfrak{q}$ is minimal in $Spec(A)$ and so we just have to show it is maximal.

Claim. There exists $f \in A - \mathfrak{q}$ such that $A_\mathfrak{q} = A[\frac{1}{f}]$.

Proof. If not, choose $f_i \in A_\mathfrak{q}$ such that the fraction $\dfrac{1}{f_{i+1}} \notin$ the A-subalgebra R_i of $A_\mathfrak{q}$ generated by $\dfrac{1}{f_1}...\dfrac{1}{f_i}$, that is, $\dfrac{1}{f_{i+1}} \notin A\left[\dfrac{1}{f_1...f_i}\right]$. This would give $R_1 \underset{\neq}{\subsetneq} R_2 \underset{\neq}{\subsetneq} ... \underset{\neq}{\subsetneq} A_\mathfrak{q}$ violating finite dimensionality. Thus $A_\mathfrak{q} = A\left[\frac{1}{f}\right]$.

If f is transcendental over k, then $k[\frac{1}{f}]$ is infinite dimensional over k. Since $k[\frac{1}{f}] \subseteq A_{\mathfrak{q}}$, we have a contradiction. Thus, f is algebraic over k. We get $f^n + a_1 f^{n-1} + ... + a_{n-1} f + a_n = 0$, $a_i \in k$. Take n minimal. We have $f(f^{n-1} + a_1 f^{n-2} + ... + a_{n-1}) = -a_n$.

If $a_n \neq 0$, f is a unit (multiply both sides by $-a_n^{-1}$. Then $A_{\mathfrak{q}} = A[\frac{1}{f}] = A$, so $\mathfrak{q}$ is maximal and we are done. So suppose that $a_n = 0$. Write $f^n + a_1 f^{n-1} + ... + a_{n-k} f^k = 0, \quad a_{n-k} \neq 0, \quad k > 0$. Suppose $\mathfrak{q} \subsetneq P \in Spec(A)$. Then $PA[\frac{1}{f}] = PA_q = A_q = A[\frac{1}{f}]$. Therefore $f \in P$. We have

$$f^k(f^{n-k} + a_{n-1} f^{n-k-1} + ... + a_{n-k+1} f + a_{n-k}) = 0 \in \mathfrak{q} \subset P.$$

As $f^k \notin q$ (since $qAq \neq A_q = A[\frac{1}{f}]$), we see $f^{n-k} + a_{n-1} f^{n-k-1} + ... + a_{n-k} \in q$. It follows that $a_{n-k} \in P$ as $f \in P$. But $a_{n-k} \in k - \{0\}$, contradiction. $\square$

22.4 Convention. For the rest of this section, let $R \to S$, R Noetherian, S finitely generated R-algebra. $Q \in Spec(S)$, $P = Q \cap R$.
We say $R \to S$ is *étale near* Q provided there exists an element $b \in S - Q$ such that $R \to S[\frac{1}{b}]$ is étale.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[dl] \\ S[\frac{1}{b}] & \end{tikzcd}$$

"Unramified near Q" is defined similarly.

23.5 Theorem. *Let R be Noetherian, and let $R \to S$ be a finitely generated R-algebra. Let Q be a prime ideal of S, put $P := Q \cap R$ and let $k = R_P/PR_P$, the residue field at P. These are equivalent:*

(1) *$R \to S$ is unramified near Q.*
(2) *There are elements $a \in R - P$ and $b \in S - Q$, a standard étale morphism $R[\frac{1}{a}] \to C$ and a surjective R-algebra homomorphism $\varphi : C \twoheadrightarrow S[\frac{1}{b}]$ such that $\bar{\varphi} : k \otimes_R C \to k \otimes_R S[\frac{1}{b}]$ is an isomorphism, $R[\frac{1}{a}] \to S[\frac{1}{b}]$ is defined, and the following diagram commutes:*

$$\begin{tikzcd} R[\frac{1}{a}] \arrow[r] \arrow[d] & C \arrow[dl, two heads] \\ S[\frac{1}{b}] & \end{tikzcd}$$

Proof. (2) $\implies$ (1) : $R \to R[\frac{1}{a}]$ is always étale (Exercise 21.3) and therefore unramified. Thus, if (2) holds, the map $R \to S[\frac{1}{b}]$ is the composition of two unramified maps and therefore is unramified.

(1) $\implies$ (2) : Just replace "étale" with "unramified" everywhere (except in quoting the result that characterizes étale and unramified extensions of fields) in the following theorem. $\square$

23.6 Theorem. *Let $R \to S$ be Noetherian and let $R \to S$ be a finitely generated R-algebra. Let $Q \in SpecS$ and $P := Q \cap R$. These are equivalent.*

(1) *$R \to S$ is étale near Q.*
(2) *There are elements $a \in R - P$ and $b \in S - Q$ such that $R\left[\frac{1}{a}\right] \to S\left[\frac{1}{b}\right]$ is defined (that is, a maps to a unit of $S\left[\frac{1}{b}\right]$) and standard étale.*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R[\frac{1}{a}] \arrow[dl] \\ S[\frac{1}{b}] & \end{tikzcd}$$

Proof. The proof of (2) $\Rightarrow$ (1) follows exactly from the proof of the previous theorem, where we replace "unramified" with "étale". For the other direction, we proceed as in the following outline.

(I) Reduce to the case (R, P, k) is local. So now (R, P, k) is local, $Q \cap R = P$ (the maximal ideal), and there exists $f \in S - Q$ such that $R \to S\left[\frac{1}{f}\right]$ is étale. (Note: Until the very end, we use only the fact $R \to S$ is unramified near Q.)

(II) Reduce to the case where $R \to S$ is module-finite. By base-change, $k \to k \otimes_R S$ is étale. Now $k \otimes_R S_Q = (S/PS)_Q$, a ring of fractions of $k \otimes S[\frac{1}{f}]$, which is Artinian. By exercise 23.2, $k \otimes_R S[\frac{1}{f}] \to k \otimes_R S_Q$ is surjective and so is unramified. By the fact below, $k \to k \otimes_R S_Q$ is unramified, as $k \to k \otimes_R S[\frac{1}{f}]$ is unramified. Therefore, $k \otimes_R S_Q \cong (S/PS)_Q$ is a finite dimensional k-algebra by the proposition. Therefore Q is isolated in its fiber over P.

Let R' be the integral closure of R in S. By ZMT, there exists $g \in R' - (Q \cap R')$ such that

$$R'\left[\frac{1}{g}\right] = S\left[\frac{1}{g}\right].$$

We will complete the proof in the following section. $\square$

23.7 Fact. Let A be a finitely generated k-algebra, where k is a field. These are equivalent:

(1) $k \to A$ is étale.
(2) $k \to A$ is unramified (as étale = flat + unramified).
(3) $A = k_1 \times ... \times k_t$ where each k_i is a finite, separable field extension of k.

23.8 Exercises.

23.1 (5 points) Prove the forward direction of Proposition 22.3. (You will probably need the version of the Nullstellensatz that says that a field extension that is finitely generated as an algebra is actually a finite-dimensional extension.)

23.2 (3 points) Let Σ be a multiplicative subset of a commutative Artinian Ring A. Then the map $A \to \Sigma^{-1}A$ is surjective.

§24. Completing the proof of Theorems 23.5 and 23.6

From last time, we get the following:

Assume $f \notin Q$ and $R \to S\left[\frac{1}{f}\right]$ étale (unramified). Let S' (R' last time) be the integral closure of $i(R)$ in S and $Q' = Q \cap S'$. By Zariski's Main Theorem, there is an element $g \in S' - Q'$ such that $S'\left[\frac{1}{g}\right] = S\left[\frac{1}{g}\right]$.

From exercise 24.1, S'' is a finitely generated R-module (since it is integral over R, it is module finite). In $S''[\frac{1}{g}] = S[\frac{1}{g}]$, write $f = \frac{h}{g^m}$ where $h \in S''$. Note that $h \notin Q'' := S'' \cap Q$. Invert h to get $S''[\frac{1}{gh}] = S[\frac{1}{fg}]$. Then we have $R \to S[\frac{1}{f}] \to S[\frac{1}{fg}] = S''[\frac{1}{gh}]$ where the first and second maps are étale (unramified) as inverting a single element is always étale (unramified) and therefore $R \to S''[\frac{1}{gh}]$ is étale (unramified).

By exercise 24.2, we may refresh notation by replacing $R \to S$ with $R \to S''$ and Q with Q''. Therefore we have

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & S \arrow[d] \\ & S[\frac{1}{f}] \end{tikzcd}$$

where S is module-finite, (R, P, k) local, $Q \cap R = P$, and $R \to S[\frac{1}{f}]$ is étale (unramified).

Set $\bar{S} = S/PS$, a finite dimensional k-algebra, and $\bar{Q} = Q/PS \in \operatorname{Spec}(\bar{S})$. Then we have $k \to \bar{S} \twoheadrightarrow \bar{S}[\frac{1}{f}] \twoheadrightarrow \bar{S}_{\bar{Q}}$. Since $\bar{S}$ is Artinian, $\bar{Q}$ is a minimal prime. So $k \to \bar{S}_{\bar{Q}}$ is surjective and hence unramified. Then $\bar{S}_{\bar{Q}}$ is a finite direct product of finite separable field extensions of k. As it is local, it is a product of just one. So $\bar{S}_{\bar{Q}} = k(\alpha)$ by the primitive element theorem. Write $\bar{S} = k(\alpha) \times A$, where A is the direct product of the localizations at the other primes. Choose $\sigma \in S$ such that $\bar{\sigma} := \sigma + PS = (\alpha, 0_A) \in \bar{S}$. Let $S' = R[\sigma] \subseteq S, Q' = Q \cap S'$. A little work shows that $S'_{Q'} = S_{Q'}$.

Since S is finitely generated as an S-module, S/S' has closed support, so there is some $f' \in S'$ such that $S'[\frac{1}{f'}] = S[\frac{1}{f}]$. Therefore we can refresh notation by replacing $R \to S, Q$ with $R \to S', Q'$. Now,

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & S = R[\sigma] \arrow[d] \\ & S[\frac{1}{f}] \end{tikzcd}$$

where S is integral and module finite and $R \to S[\frac{r}{f}]$ is étale (unramified). Let $r = \dim_k \bar{S} = \dim_k k[\bar{\sigma}]$. Then $\{1, \sigma, \ldots, \sigma^{r-1}\}$ generates S as an R-module (by NAK since $\{\bar{1}, \bar{\sigma}, \ldots, \bar{\sigma}^{r-1}\}$ generates $\bar{S}$).

Write σ^r as an R-linear combination of $1, \sigma, \ldots, \sigma^{r-1}$ to get a monic polynomial $H \in R[x]$ with $\deg H = r$ such that $H(\sigma) = 0$. We get, after a little work, $a \in R - P, b \in S - Q$, and a commutative diagram

$$\begin{tikzcd} R \arrow[rr] \arrow[d] & & S \arrow[d] \\ R[\frac{1}{a}] \arrow[rr] \arrow[dr] & & S[\frac{1}{b}] \\ & C \arrow[ur, "\phi"] & \end{tikzcd}$$

such that $\bar{\phi} : k \otimes_R C \to k \otimes_R S[\frac{1}{b}]$ is an isomorphism. This completes the proof of Theorem 23.5.

To finish the proof of Theorem 23.6, replace $R \to S$ by $R \to S[\frac{1}{f}]$, so we may assume $R \to S$ is étale. We wish to show that there is an element $c \in C, c \notin \phi^{-1}(Q[\frac{1}{b}])$ such that $C[\frac{1}{c}] \to S[\frac{1}{bc}]$ is an isomorphism.

Let $\mathfrak{q} = \phi^{-1}(Q[\frac{1}{b}]), I = \ker \phi \subseteq \mathfrak{q}$. It's enough to show that $I_{\mathfrak{q}} = 0$. By NAK, it's enough to get $(I/I^2)_{\mathfrak{q}} = 0$. Consider the short exact sequence

$$0 \to I/I^2 \to C/I^2 \to S \to 0$$

Since $R \to S$ is étale, there is a unique $\theta : S \to C/I^2$ making the diagram below commute.

$$\begin{tikzcd} S \arrow[r, "\cong"] \arrow[dr, "\phi"] & C/I \\ R \arrow[u] \arrow[r] & C/I^2 \arrow[u] \end{tikzcd}$$

That is, the short exact sequence is split and therefore remains split exact after applying $k \otimes_R (-)$.

$$0 \to k \otimes_R I/I^2 \to k \otimes_R C/I^2 \xrightarrow{\cong} \bar{S} = k \otimes_R S \to 0$$

Therefore $k \otimes_R I/I^2 = 0$. So $I/I^2 = PI/I^2$ which implies $I/I^2 = \mathfrak{q}(I/I^2)$. Therefore $(I/I^2)_{\mathfrak{q}} = 0$ by NAK, and the proof is complete. $\square$

24.1 Remarks. Under the basic assumptions we've been working with, $R \to S$ is unramified (étale) near Q for all Q if and only if $R \to S$ is unramified (étale).

Proof. For the backward direction, take $f = 1$. For the forward direction, we sill proof each case separately.

Unramified Case. Take $f_1, \ldots, f_n \in S$ with $S = Sf_1 + \cdots + Sf_n$ and $R \to S[\frac{1}{f_i}]$ unramified (étale) for all i (using compactness of $\mathrm{Spec}(S)$). Check that $\Omega_{S[\frac{1}{f}]/R} \cong \Omega_{S/R} \otimes_R S[\frac{1}{f}]$. Therefore $\Omega_{S/R} = 0$ locally, so $\Omega_{S/R} = 0$. Thus S/R is unramified.

Étale Case. We seek to show there is a unique ϕ making the diagram below commute. Notice that by the unramified case, if there is a ϕ it is unique.

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & C/I \\ R \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$

For every i, we get a unique $\phi_i : S[\frac{1}{f_i}] \to C[\frac{1}{f_i}]$ making the appropriate diagrams commute. Considering these over $\mathrm{Spec} S[\frac{1}{f_i}] = D(f_i)$, we have, by the unramified case, that the ϕ_i agree on overlaps and so the maps "glue" together. (Technically, we should reverse the arrows and think of the maps as morphisms of affine schemes. □

24.2 Exercises.

24.1 (3 points) There exists a finitely generated R-subalgebra S'' of S' such that $g \in S''$ and $S''[\frac{1}{g}] = S[\frac{1}{g}]$.

24.2 (5 points) Show that it is sufficient to prove Theorem 23.6 for $R \to S''$ and the prime ideal Q''. (HINT: If $A \to C$ for C as in Theorem 23.5 is standard étale, then so is $A \to C\left[\frac{1}{c}\right]$ for $c \in C$.)

§25. The Henselization.

25.1 Proposition. *Suppose that $R \to S$ is as usual (R noetherian and S a finitely generated R-algebra). Then the following are equivalent:*

(1) *$R \to S$ is étale*
(2) *$R \to S$ is unramified and flat*
(3) *$R \to S$ is flat and $\Omega_{S/R} = 0$*

Proof. $2 \Leftrightarrow 3$ has already been shown in previous sections.

$1 \Rightarrow 2$: Because flatness is a local property and we know that étale implies standard étale near each prime, we're done.

$2 \Rightarrow 1$: By Theorem 23.6 and the Remark 24.1, it's enough to show that $R \to S$ is étale near each prime Q. Fix Q, let $P = Q \cap R$ and put $k := R_P/PR_P$. We know that there exist $a \in R - P$, $b \in S - Q$ and a commutative diagram

$$\begin{tikzcd} & T \arrow[dd, "\phi"] \\ R[\frac{1}{a}] \arrow[ur, "\text{STD étale}"] \arrow[dr] & \\ & S[\frac{1}{b}] \end{tikzcd}$$

such that $\overline{\phi} := k \otimes_R \phi$ is an isomorphism.

For notational simplicity, replace $R \to S$ by $R[\frac{1}{a}] \to S[\frac{1}{b}]$. So we have a commutative diagram:

$$\begin{tikzcd} & T \arrow[dd, "\phi"] \\ R \arrow[ur, "\text{STD étale}"] \arrow[dr] & \\ & S \end{tikzcd}$$

with $\overline{\phi}$ an isomorphism.

By passing to $R_P \to S_P := (R-P)^{-1}S$, we may assume that R is local with maximal ideal P. Let $I = \ker\phi$ and $\mathfrak{q} = \phi^{-1}(Q)$. It will suffice to show that there is an element $t \in T - \mathfrak{q}$ such that $I[t^{-1}] = 0$. Since I is finitely generated it is sufficient to show that $I_{\mathfrak{q}} = 0$. We have an exact sequence

$$0 \longrightarrow I \longrightarrow T \longrightarrow S \longrightarrow 0.$$

Apply $k \otimes_R -$:

$$0 = \mathrm{Tor}_1^R(k, S) \longrightarrow k \otimes_R I \longrightarrow k \otimes_R T \xrightarrow{\overline{\phi}} k \otimes_R S \longrightarrow 0$$

because S is flat. Therefore $k \otimes_R I = 0$ because $\overline{\phi}$ is an isomorphism. So $I = PI$ and, since $PS \subseteq \mathfrak{q}$, $I = \mathfrak{q}I$. By Nakayama's lemma, $I_{\mathfrak{q}} = 0$, as desired. □

25.2 Definition. Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring. Then the *Henselization* of R is a Henselian ring R^h together with a local homomorphism $i : R \to R^h$ that satisfies the following universal property: If S is a Henselian local ring and $R \xrightarrow{\phi} S$ is a local homomorphism (that is, the maximal ideal does not generate S), then there exists a unique local homomorphism ψ making the diagram commute:

$$\begin{tikzcd} R \arrow[rr, "i"] \arrow[dr, "\phi"'] & & R^h \arrow[dl, dashed, "\psi"] \\ & S & \end{tikzcd}$$

By the usual categorical properties, if R^h exists then it is unique up to unique isomorphism.

25.3 Proposition. *Let $(R, \mathfrak{m}, k)$ be a local ring. Then the following are equivalent:*

(1) *R is Henselian*

(2) *If $R \to S$ is étale and $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\phi^{-1}(\mathfrak{p}) = \mathfrak{m}$ and $k = \kappa(\mathfrak{p})$, i.e. the induced map from $R/\mathfrak{m}$ to $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$ is an isomorphism, then $R \to S_{\mathfrak{p}}$ is an isomorphism.*

25.4 Definition. A local homomorphism $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ is *essentially étale* provided there is an étale homomorphism $R \to T$ and a prime $\mathfrak{p} \in \mathrm{Spec}(T)$ such that $T_{\mathfrak{p}} \cong S$ and the following diagram commutes:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & T \arrow[d] \\ S & T_{\mathfrak{p}} \arrow[l, "\cong"'] \end{tikzcd}$$

25.5 Definition. Let $(R, \mathfrak{m}, k)$ be a local ring. An *étale neighborhood* of R is an essentially étale homomorphism $\phi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ such that the induced map $k \to \ell$ is an isomorphism.

Proposition 25.3 says that R is Henselian if and only if every étale neighborhood is an isomorphism.

25.6 Remark. If $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ is essentially étale then $\mathfrak{m}S = \mathfrak{n}$. The reason is that by base change $k \to S/\mathfrak{m}S$ is étale. Therefore $S/\mathfrak{m}S$ is a finite product of separable field extensions of k. But it's local, so it's a field. Therefore $\mathfrak{m}S = \mathfrak{n}$.

25.7 Theorem. *Let $(S_i, \mathfrak{n}_i, \ell_i)$ be essentially étale over $(R, \mathfrak{m}, k)$, where $i = 1, 2$. Suppose that there are local homomorphisms α and β that make the following diagram commute:*

$$\begin{tikzcd} & S \arrow[dr, shift left, "\alpha"] \arrow[dr, shift right, "\beta"'] & \\ R \arrow[ur] \arrow[rr] & & S_2 \end{tikzcd}$$

If $\overline{\alpha} = \overline{\beta} : \ell_1 \to \ell_2$ then $\alpha = \beta$.

Note that

$$\begin{tikzcd} & \ell_1 \arrow[dd, "\overline{\alpha}=\overline{\beta}"] \\ k \arrow[ur] \arrow[dr] & \\ & \ell_2 \end{tikzcd}$$

commutes. If $k = \ell_i$ then the following diagram commutes:

$$\begin{tikzcd} & \ell_1 \arrow[dd, shift right, "\overline{\alpha}"'] \arrow[dd, shift left, "\overline{\alpha}"] \\ k \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & \\ & \ell_2 \end{tikzcd}$$

This forces $\overline{\alpha} = \overline{\beta}$.

25.8 Corollary. *Let R, S_1, S_2 be as above and assume S_i are étale neighborhoods of R. Then there is at most one local homomorphism $\alpha : S_1 \to S_2$ making the following diagram commute:*

$$\begin{tikzcd} R \arrow[rr] \arrow[dr] & & S_1 \arrow[dl, "\alpha"] \\ & S_2 & \end{tikzcd}$$

Moreover, if there exist α and β with $S_1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} S_2$ *, then $\beta\alpha = 1_{S_1}$ and $\alpha\beta = 1_{S_2}$.*

Consider isomorphism classes of étale neighborhoods of $(R, \mathfrak{m}, k)$. Then

$$(R \to S_1) \cong (R \to S_2)$$

means that there exists α and β with $S_1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} S_2$. We say that $[R \to S_1] \le [R \to S_2]$ if there is a commutative diagram

$$\begin{tikzcd} S_1 \arrow[rr, "\alpha"] & & S_2 \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

It turns out that it's not hard to see there is only a set of isomorphism classes (instead of a proper class). This set is directed: we can obtain a commutative diagram

$$\begin{tikzcd} & S_1 \arrow[dr] & \\ R \arrow[ur] \arrow[dr] & & T \\ & S_2 \arrow[ur] & \end{tikzcd},$$

with $T = (S_1 \otimes_R S_2)_{\mathfrak{n}}$ and $\mathfrak{n} = \ker[S_1 \otimes_R S_2 \to k \otimes_R k \cong k]$.

Taking the direct limit of this set we obtain the Henselization of R.

§26. Comparing CM Type with a Ring and its Henselization

From last time we have

$$R^h = \varinjlim_i R_i$$

where $R \to R_i$ are étale neighborhoods.

$$\begin{tikzcd} & R_1 \arrow[dd, dashed, "\exists \text{ at most one map}"] \\ R \arrow[ur] \arrow[dr] & \\ & R_2 \end{tikzcd}$$

Say that $R_1 \leq R_2$ if $\exists$ a map $R_1 \to R_2$ making the diagram above commute. This relation is transitive and antisymmetric.

Given any R_1 and R_2 we always have the commutative diagram

$$\begin{tikzcd} & R_1 \arrow[dr] & \\ R \arrow[ur] \arrow[r] & R_2 \arrow[r] & T = (R_1 \otimes_R R_2)_n \end{tikzcd}$$

where $n = ker(R_1 \otimes_R R_2 \to k \otimes_R k = k)$.

All the maps are flat. (R^h, m^h, k) is local (Noetherian) with the same residue field as R. $R \to R^h$ is flat since the direct limit of flat modules is flat. Also $mR^h = m^h$. Now R^h is Henselian (a local ring is Henselian provided it has no proper étale neighborhoods). The Henselization has the universal property that if $R \to H$ is a local homomorphism and H is Henselian, then there is a unique local homomorphism making the following diagram commute:

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & R^h \arrow[d, dashed, "\exists!"] \\ & H \end{tikzcd}$$

In particular, taking $H = \hat{R}$, we get a natural map $R^h \to \hat{R}$, which turns out to be injective. Moreover, $\hat{R} = \widehat{R^h}$. If R is countable, so is R^h, but $\hat{R}$ is never countable unless R is Artinian.

What we are looking for are results that allow us to transfer finite CM type from R to R^h.

26.1 Theorem *Let $(R, \mathfrak{n}) \to (S, \mathfrak{n})$ be a flat local homomorphism with the property that for each finitely generated R-module L, $S \otimes_R L$ is a MCM S-module if and only if L is a MCM R-module. (By Theorems 2.1.7 and A.11 of Bruns & Herzog, this holds whenever the closed fiber $S/\mathfrak{m}S$ is Artinian, e.g., when S is the Henselization or the completion.) If S has finite CM type then so does R.*

Proof. Let $A = \{$ indecomposable MCM S-modules Z s.t. $\exists$ MCM ${}_RX$ with $Z \oplus W \cong S \otimes_R X\}$. Let $Z_1, \ldots, Z_t$ be a complete list of representatives of isomorphism classes in A. $\forall i$ write $Z_i \oplus W_i \cong S \otimes_R X_i$ where X_i is a MCM R-module. Let $X = \bigoplus_{i=1}^t X_i$. Given a MCM R-module L, we can write $S \otimes_R L$ as a direct sum of copies of the Z_i, say $S \otimes_R L = Z_1^{a_1} \oplus \cdots \oplus Z_t^{a_t}$. Define $b = \max\{a_1, ..., a_t\}$. Add $M = Z_1^{b-a_1} \oplus \cdots \oplus Z_t^{b-a_t} \oplus Q_1^b \oplus \cdots \oplus Q_t^b$ to both sides to get

$$S \otimes_R L \oplus M \cong Z_1^b \oplus W_1^b \oplus \cdots \oplus Z_t^b \oplus Q_t^b \cong S \otimes_R (X_1^b \oplus \cdots \oplus X_t^b) = S \otimes_R X^b.$$

Therefore $S \otimes_R L \in +(S \otimes_R X)$.

From a theorem found in §20, $L \in +(X)$. We have that $+(X)$ embeds as a full submonoid of $\mathbb{N}^{(u)}$ for some u. Therefore, $+(X)$ has only finitely many indecomposables and there exists only finitely many indecomposable L's. $\square$

We now know that if R^h has finite CM type then so does R. Our goal is to prove the converse:

If R has finite CM type then so does R^h.

It is enough to show that for each MCM R^h-module N $\exists$ a MCM R-module M s.t. $N|R^h \otimes_R M$.

Start with ${}_{R^h}N$, MCM. Let $S = R^h$. We have a presentation for N: $S^{(n)} \xrightarrow{\Phi} S^{(m)} \to N \to 0$. Choose an étale neighborhood R_i such that all entries of Φ "come from" R_i (since S is the direct limit of the R_i). Write $\Phi = [s_{ab}]$, and choose i "large enough" so that there exist $r_{ab} \in R_i$ with $r_{ab} \to s_{ab}$ via $R_i \to S$. Let $\phi = [r_{ab}]$. Define $M = coker(\phi)$. We have a presentation $R_i^{(n)} \xrightarrow{\phi} R_i^{(m)} \to M \to 0$. Then $N = S \otimes_R M$, and M is MCM as an R_i-module.

Refresh Notation: Let $(R, m, k) \to (S, n, k)$ be an étale neighborhood. We have a MCM S-module ${}_SN$ and we want a MCM R-module ${}_RM$ such that ${}_SN|S \otimes_R M$. We showed before that $\exists$ f.g. R-module M such that ${}_SN|S \otimes_R M$.

(Sketch of how this works: We have an $S \otimes_R S$ split exact sequence $0 \to I \to S \otimes_R S \to S \to 0$. Apply $\_ \otimes_S N$ to get an S-split surjection $S \otimes_R N \to N \to 0$. Thus ${}_SN|S \otimes_R N$. By writing ${}_RN$ as a direct limit of finitely generated R-modules, we can get ${}_SN$ as a direct summand of $S \otimes_R M$ for some finitely generated R-module M.)

But there is no reason to believe ${}_RM$ is MCM. The idea is to show that the MCM S-module N is a d^{th} syzygy where $d = dim(S) = dim(R)$. If we can do this, we'll have an exact sequence $0 \to N \to S^{(n_{d-1})} \to \cdots \to S^{(n_0)} \to X \to 0$. As above, we have ${}_SX|S \otimes_R Y$ for some f.g. ${}_RY$. Let Z be the d^{th} syzygy of ${}_RY$. Then Z is MCM. Then we get ${}_SN|S \otimes_R (Z \oplus free)$.

26.2 Theorem. ([Evans & Griffith, "Syzygies"]): *Let (R,m,k) be CM. Assume R_P is Gorenstein $\forall$ prime $P \neq m$. Then every MCM R-module is a d^{th} syzygy, where $d = \dim(R)$.*

26.3 Theorem. *Let (R, m, k) be a local CM ring with finite CM type. Then R_P is regular $\forall P \neq m$.*

26.4 Lemma. (Huneke & Leuschke) *Let*

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "i"] \arrow[d, "r"] & K \arrow[r] \arrow[d, "f"] & M \arrow[r] \arrow[d, "="] & 0 & (\zeta) \\ 0 \arrow[r] & N \arrow[r, "j"] & L \arrow[r] & M \arrow[r] & 0 & (r\zeta) \end{tikzcd}$$

be an exact commutative diagram of f.g. modules over a local ring (R, m, k) where $r \in m$. Assume $K \cong L$. Then $\zeta = 0$ (that is, the sequence splits).

Proof. We have an exact sequence

$$0 \longrightarrow N \xrightarrow{\left[\begin{smallmatrix} r \\ i \end{smallmatrix}\right]} N \oplus K \xrightarrow{[j\ \ -f]} L \longrightarrow 0$$

Check easily that this is exact. Since $N \oplus K \cong N \oplus L$, Miyata's Theorem implies that the sequence splits. (Miyata's Theorem says that if you have a short exact sequence of fintely presented modules over a commutative ring and the center module is isomorphic to the direct sum of the outside modules then the short exact sequence splits.) Apply $Ext^1_R(M, -)$ to get a split injection

$$\begin{tikzcd} Ext^1_R(M,N) \arrow[r, "\left[\begin{smallmatrix} r \\ i \end{smallmatrix}\right]"] \arrow[dr, "\left[\begin{smallmatrix} r \\ i_* \end{smallmatrix}\right]"'] & Ext^1_R(M, N \oplus K) \arrow[d, "\cong"] \\ & Ext^1_R(M,N) \oplus Ext^1_R(M,K) \end{tikzcd}$$

with $\zeta \mapsto (r\zeta, i_*(\zeta))$ Now apply $Hom_R(M, -)$ to (ζ), getting an exact sequence

$$\longrightarrow Hom_R(M,M) \longrightarrow Ext^1_R(M,N) \xrightarrow{i_*} Ext^1_R(M,K) \longrightarrow .$$

The connecting homomorphism takes 1_M to $\pm\zeta$, so $i_*(\zeta) = 0$.

Let h split the injection $\left[\begin{smallmatrix} r \\ i_* \end{smallmatrix}\right]$ above. Then $\zeta = h(r\zeta, 0) = rh(\zeta, 0)$. Substituting $rh(\zeta, 0)$ for ζ in the right-hand side of the formula $\zeta = rh(\zeta, 0)$, we get $\zeta = rh(rh(\zeta, 0), 0) = r^2 h(h(\zeta, 0), 0)$. Continuing merrily, we get $\zeta \in r^n Ext^1_R(M, N)$ for all $n \geq 1$. By the Krull Intersection Theorem, $\zeta = 0$. $\square$

§27. The Universal Push-Forward and Serre's Condition

Let R be a Noetherian ring and M a finitely generated $R-$module. Recall that M has finite length if and only if $M_p = 0$ for every non-maximal prime ideal p. If $(R, \mathfrak{m})$ is local, then M has finite length if and only if $\mathfrak{m}^h M = 0$ for some integer $h \geq 1$.

27.1 Theorem. *Let (R, m) be local and M, N finitely generated $R-$modules. Suppose, for some positive integer h, there exist only h isomorphism classes of modules K for which there exists a short exact sequence*

$$0 \to N \to K \to M \to 0.$$

Then $\mathfrak{m}^h Ext^1_R(M, N) = 0$. In particular, $Ext^1_R(M, N)$ has finite length as an $R-$module.

Proof. Let $\xi \in Ext^1_R(M, N)$, and let $r_1, \ldots, r_h \in m$. It will suffice to show that $r_1 \cdots r_h \xi = 0$. Let $\xi_0 = \xi$, and for $1 \leq n \leq h$ let $\xi_n = r_1 \cdots r_n \xi$. We will show that $\xi_a = 0$ for some $a = 0, \ldots, h-1$. (Then $r_1 \cdots r_h \xi = r_{a+1} \cdots r_h \xi_a = 0$, and we'll be done.) Let K_n be the middle module in a short exact sequence representing the element $\xi_n \in \mathrm{Ext}^1_R(M, N)$. In the list $K_0, \ldots, K_h$ there must be a repeated isomorphism class, say, $K_a \cong K_b$, where $0 \leq a < b \leq h$. Then $r_{a+1} \cdots r_b \in \mathfrak{m}$, and $\xi_b = r_{a+1} \cdots r_b \xi_a$. By Lemma 26.4, $\xi_a = 0$. $\square$

The following important result was proved by Auslander [A] in the complete case and by Huneke and Leuschke [HL] in general.

27.2 Theorem. *Let (R, m) be a CM ring with finite CM type. Then R is an isolated singularity, that is, R_p is a regular local ring for all non-maximal prime ideals p.*

Proof. Let $p \neq m$ be a prime ideal. To show that R_p is a regular local ring, it is enough to show that the residue field R_p/pR_p has finite projective dimension as an R_p-module. Let d $= \dim(R)$. Resolve R/p over R:

$$\begin{tikzcd}
 & 0 \arrow[dr] & 0 & & & & & & \\
 & & M \arrow[u] \arrow[dr] & & & & & & \\
\cdots \arrow[r] & F_{d+1} \arrow[r] \arrow[dr] & F_d \arrow[r] \arrow[u] & F_{d-1} \arrow[r] & \cdots \arrow[r] & F_0 \arrow[r] & F_0 \arrow[r] & R/p \arrow[r] & 0 \\
 & & N \arrow[u] \arrow[dr] & & & & & & \\
 & & 0 \arrow[u] & 0 & & & & &
\end{tikzcd}$$

where each sequence is exact. Therefore M and N are MCM. Hence, there exist only finitely many isomorphism classes of modules K such that there exists an exact sequence $0 \to N \to K \to M \to 0$, since such a K would also be MCM. By Theorem 27.1, $E := Ext^i_R(M, N)$ has finite length, so $E_p = 0$. Since Ext commutes with localizations, $0 \to N_p \to (F_d)_p \to M_p \to 0$ splits. Thus M_p is free as an R_p module. Therefore we have the resolution $0 \to M_p \to (F_{d-1})_p \to \cdots \to (F_0)_p \to (R/p)_p \to 0$. So $Pd_{R_p}(R_p/pR_p) = Pd_{R_p}((R/p)_p)$ is finite, and we are done. $\square$

Now, we are moving towards showing that a CM ring (R, m) has finite CM type if and only if R^h has finite CM type. We have already shown the backwards direction, so we may focus on the forwards direction. To do this, we

will want to show that if $R \to S$ is an étale neighborhood of R, then every MCM $S-$module is a d^{th} syzygy, where $d = \dim(S)$.
Fact: In this situation, if R is an isolated singularity, then so is S. This follows from the fact that if R_p is Gorenstein for all $p \neq m$, then S has this property as well.

27.3 Theorem. *Suppose that (R, m) is local, has dimension d and is CM, and that R_p is Gorenstein for all $p \neq m$. Then every MCM module is a d^{th} syzygy.*

The proof of Theorem 27.3 requires the following lemmas and a construction.

27.4 Definition. A finitely generated R-module M satisfies *Serre's condition* (S_n), provided

(1) M_p is a MCM R-module if $\dim R_p \leq n$, and
(2) $\operatorname{depth}_R M_p \geq n$ if $\dim R_p > n$.

More succinctly, $\operatorname{depth}_{R_p} M_p \geq \min\{n, \dim R_p\}$. (Notice that $(S_n) \Rightarrow (S_{n-1})$.)

27.5 Lemma. *Suppose $_RM$ is finitely generated and satisfies (S_1) and R_P is Gorenstein for each minimal prime P. The canonical map $M \to M^{**}$ is one-to-one. (Bass(1960's) referred to this condition as "torsionless".)*

Proof. Look at the sequence
$$0 \to K \to M \to M^{**}$$
If P is a minimal prime, then $M_P \cong M_P^{**}$ is an isomorphism, since R_P is Gorenstein. Therefore $K_P = 0$. If $K \neq 0$, there exists $Q \in \operatorname{Ass} K$. Then $\operatorname{depth} K_Q = 0$, so Q is not a minimal prime ideal of R. Therefore $\operatorname{depth} M_Q \geq 1$, by (S_1). This contradicts $K_Q \hookrightarrow M_Q$. $\square$

27.6 Definition. We define the *Universal Push-forward* whenever M satisfies (S_1) and R_P is Gorenstein for all minimal primes P.

Resolve M^* minimally:
$$0 \to K \to F \to M^* \to 0$$
We define the push-forward M_1 by the following commutative diagram:

(1)
$$\begin{tikzcd}
 & & & & 0 & & \\
 & & & M_1 \arrow[ur] & & & \\
0 \arrow[r] & M^{**} \arrow[r] & F^* \arrow[r] \arrow[ur] & K^* \arrow[r] & \operatorname{Ext}^1_R(M^*, R) \arrow[r] & 0 \\
 & M \arrow[ur] \arrow[urr] & & & & & \\
0 \arrow[ur] & & & & & &
\end{tikzcd}$$

where each sequence is exact.

27.7 Lemma. *Let $1 \leq n \leq d$. Assume R satisfies (S_{n-1}), M satisfies (S_n), and R_P is Gorenstein for all primes P with $\dim R_P \leq n-1$. Then M_1 satisfies (S_{n-1}).*

Proof. Let $\dim R_P \leq n-1$. We will show that $(M_1)_P$ is MCM.

Since R_P is Gorenstein, $M_P \overset{\cong}{\hookrightarrow} M_P^{**}$. Also M_P^* is MCM and $\operatorname{Ext}^1_{R_P}(M_P^*, R_P) = 0$. Finally, K_P is MCM, from the first exact sequence.

Examining (1) localized at P shows that $(M_1)_P \cong K_P$ so $(M_1)_P$ is MCM.

Next suppose $\dim R_P \geq n-1$. Thus we have

$$0 \longrightarrow M_P \longrightarrow F_P^* \longrightarrow (M_1)_P \longrightarrow 0$$

As $\operatorname{depth} M_P \geq n$ and $\operatorname{depth} F_P^* \geq n-1$, the depth lemma implies $\operatorname{depth}(M_1)_P \geq n-1$. □

Theorem 27.3 then follows from the first assertion in the following theorem:

27.8 Theorem. *Let $(R, \mathfrak{m})$ be local, and let $1 \leq n \leq d := \dim R$.*

- *Assume R_P is Gorenstein for each prime ideal P with $\operatorname{ht} P \leq n-1$, and that R satisfies (S_{n-1}). If ${}_RM$ satisfies (S_n), then M is an nth syzygy.*
- *Assume R satisfies (S_n). If M is an nth syzygy, then ${}_RM$ satisfies (S_n).*

The following remark shows that the hypothesis, in Theorem 27.3, that R be Gorenstein on the punctured spectrum is a consequence of the conclusion (at least for CM rings with canonical module):

27.9 Remark. Let $(R, \mathfrak{m})$ be CM, $\dim R = d$, and assume R has a canonical module ω_R. (Then ω is MCM.) If ω is a dth syzygy, then R_P is Gorenstein for all primes $P \neq \mathfrak{m}$.

Proof. For some M finitely generated, we have an exact sequence, with F_i free:

$$0 \to \omega \to F_{d-1} \to F_{d-2} \to \cdots \to F_1 \to F_0 \to M \to 0$$

Let $P \neq \mathfrak{m}$. Then $\dim R_P < d$, ω_P is a canonical module for R_P, and we have the exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \omega_P \arrow[r] & (F_{d-1})_P \arrow[r] \arrow[dr] & (F_{d-2})_P \arrow[r] & \cdots \arrow[r] & M_P \arrow[r] & 0 \\
 & & & K \arrow[u] \arrow[dr] & & & \\
 & & & 0 \arrow[u] & 0 & &
\end{tikzcd}$$

□

Note K is a $(d-1)$st syzygy so it is MCM. Therefore the exact sequence

$$0 \to \omega_P \to (F_{d-1}) \to K \to 0$$

splits (Exercise 27.3). Therefore ω_P is free, and $\omega_R \cong R_P$. Thus R_P is Gorenstein. Putting all of this stuff together, we have finally proved the following theorem:

27.10 Theorem. *Let R be a CM local ring. Then R has finite CM type if and only if the Henselization R^h has finite CM type.*

Q: Suppose R is a d-dimensional CM local ring and every MCM R-module is a d^{th} syzygy. Is R necessarily Gorenstein on the punctured spectrum?

27.11 Exercises.

27.1 (2 points) Let (R, m) be local, and M a finitely generated $R-$module. Show that M has finite length if and only if for all $r \in m$ and for all $x \in M$, there exists an integer n such that $r^n x = 0$.

27.2 (6 points) Prove Theorem 27.8.

27.3 (3 points) Prove the sequence at the end of Remark 27.9 splits, using only things we've done in class

§28. Gorenstein rings with finite CM Type: Wrap-Up

Let $(R, \mathfrak{m}, k)$ be a complete local CM ring containing a field. Assume $d := \dim(R) > 0$. If R is Gorenstein and has finite CM type, we know that R is a hypersurface: $R = k[[X_0, \dots, X_d]]/(f)$. If k is algebraically closed and of characteristic 0, we know that R is an ADE-singularity: Using the Weierstraß Preparation Theorem and change of coordinates, we can put f into the form $g + X_2^2 + \dots X_d^2$, where $g \in k[[X_0, X_1]$ defines a one-dimensional ADE-singularity (see the list at the beginning of §7). Conversely, every ADE-singularity is Gorenstein and has finite CM type.

If k is algebraically closed and of positive characteristic, normal forms have been worked out for hypersurfaces of finite CM type, though the equations are rather complicated, particularly in characteristic 2. Also, in characteristic 2 one must use the "iterated double branched cover" $R^{\#\#} := k[[U, V]]/(f + UV)$, and this requires classifying both one- and two-dimensional simple singularities before getting the induction going. Anyway, all of this has been worked out, so we shall concern ourselves with the following questions: What if R is not complete? And what if k is not algebraically closed?

Our best results will be for Gorenstein rings, since even for rings of the form $\mathbb{C}[[X_1, \dots, X_n]]/I$, the non-Gorenstein rings of finite CM type have been classified only up to dimension 2 (and it's unknown whether there are *any* of dimension ≥ 4).

At this point, we seem to need something like excellence (or the weaker condition that R be a "G-ring" (cf. [M, p. 156])) in order to conclude that the completion $\hat{R}$ has an isolated singularity. (We need this so that we can apply Elkik's theorem [E].) We know that R has an isolated singularity if R has finite CM type, so the following will do (see, e.g., [W, Lemma 2.7], for a proof):

28.1 Theorem. *Assume $(R, \mathfrak{m})$ is a G-ring. Then R has an isolated singularity if and only if $\hat{R}$ has an isolated singularity.*

Again, excellent rings are G-rings (by definition, cf. [M], p. 260). Rings occurring in nature tend to be excellent. More precisely: Fields are excellent. $\mathbb{Z}$ is excellent. Complete rings are excellent. Finitely generated algebras over excellent rings are excellent. Localizations of excellent rings are excellent. (What's left? Actually, there's a fairly harmless-looking non-excellent ring in [LW]. It has finite CM type (for the rather silly reason that it has *no* MCM modules!), but $\hat{R}$ does not have finite CM type. Of course R is not CM.)

28.2 Theorem. *Let $(R, \mathfrak{m})$ be a CM local G-ring. These are equivalent:*

(1) *R has finite CM type.*
(2) *R^h has finite CM type.*
(3) *$\hat{R}$ has finite CM type.*

Proof. We know that (1) and (2) are equivalent (Theorem 27.11) and that (3) implies (1) (cf. Theorem 0.1 of §26). Assume (1) and (2). Then R has an isolated singularity by Theorem 27.2. By Theorem 28.1, $\hat{R}$ has an isolated singularity. Let M be any MCM $\hat{R}$-module. For each non-maximal prime ideal P of $\hat{R}$, M_P is a MCM R_P-module and therefore is free (since $\hat{R}_P$ is a regular local ring). A theorem due to René Elkik [E] now says that M is extended from the Henselization R^{h}. Since R^{h} has finite CM type and every $\hat{R}$-module is extended from R^{h}, it follows that $\hat{R}$ has finite CM type. $\square$

This gives us the following characterization:

28.3 Theorem. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local G-ring containing a field. Assume $\dim(R) \geq 1$ and k is algebraically closed. Then R has finite CM type if and only if $\hat{R}$ is an ADE-singularity.*

It would be nice to get rid of the G-ring assumption. To do so, it would suffice to prove the following conjecture (which I find plausible):

28.4 Conjecture. *Let $(R, \mathfrak{m})$ be a local ring with finite CM type. Then $\hat{R}$ has an isolated singularity.*

Now we see what can be done about the assumption that k is algebraically closed. If k is imperfect, there are lots of open questions. The approach we will outline below, simply passing to the algebraic closure, does not work, as the following example shows (cf. [W, Example 3.4]):

28.5 Example. Let k be an imperfect field of characteristic 2, and choose $\alpha \in k - k^2$. Put $f = X^2 + \alpha Y^2$, and let $R = k[[X, Y]]/(f)$. Then R is a one-dimensional complete local domain of multiplicity 2, so it has finite CM type (cf. §5.2). Let K be the algebraic closure of R. Then $K[[X, Y]]/(f)$ is Cohen-Macaulay and has a non-zero nilpotent element, to wit, $x + \sqrt{\alpha}y$. By Corollarly 1.10, $K[[X, Y]]/(f)$ does *not* have finite CM type.

Suppose now that $(R, \mathfrak{m}, k)$ is a complete local Gorenstein ring containing a field, and assume k is perfect. If R is a candidate for finite CM type, we may assume $R = k[[X_0, \ldots, X_d]]/(f)$ for some non-zero $f \in (X_0, \ldots, X_d)$. Let K be an algebraic closure of k, and note that K is the directed union of the finite étale extensions $k \hookrightarrow E$, where E ranges over the finite algebraic extensions of k contained in K. Therefore $K \otimes_k R$ is the direct union of the finite étale extensions $E \otimes_k R = E[[X_0, \ldots, X_d]]/(f)$. The argument in the proof of Theorem 0.1 of §26 shows that R has finite CM type if and only if $K \otimes_k R$ has finite CM type. The completion of $K \otimes_k R$ is $R^\star := K[[X_0, \ldots, X_d]]/(f)$, and $K \otimes_k R$ is excellent by [SG, (5.3)]. Therefore Theorem 28.2 tells us that $K \otimes_k R$ has finite CM type if and only if $R^\star$ has finite CM type.

Putting all of this together, we have the following:

28.6 Theorem. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local G-ring of positive dimension. Assume k is perfect. Then R has finite CM type if and only if $\hat{R}^\star$ is an ADE-singularity.*

§29. Auslander-Reiten Sequences (AKA AR- or almost-split sequences)

29.1 Lemma.[Yoshino, 1.22] *Let $f : N \to M$ be an R-homomorphism, and suppose $N = \bigoplus_{i=1}^{n} N_i$. Consider the diagram*

$$\begin{array}{ccc} N & \xrightarrow{f} & M \\ {\scriptstyle u_j}\uparrow & & \downarrow{\scriptstyle p_j} \\ N_j & \xrightarrow{f_j} & \dfrac{M}{f(\bigoplus_{i\neq j} N_i)} \end{array}$$

where j is fixed, u_j is the natural injection, p_j is also natural, and $f_j := p_j f u_j$. If each f_j is a split monomorphism, so is f.

Notation: For the remainder of the course, $(R, \mathfrak{m}, k)$ is a local, CM, Henselian ring. Define

$$\operatorname{Ind}(R) := \{\text{isomorphism classes } [M] \text{ of indecomposable MCM } R\text{-modules } M\}$$

. Suppose $[M] \in \operatorname{Ind}(R)$. Define

$$\mathcal{S}(M) := \bigcup_{[N] \in \operatorname{Ind}(R)} (\operatorname{Ext}^1_R(M, N) \setminus \{0\}).$$

If $s \in \mathcal{S}(M)$,

$$(s) \qquad 0 \longrightarrow N_s \longrightarrow E_s \longrightarrow M \longrightarrow 0$$

is nonsplit.

29.2 Proposition. $\mathcal{S}(M) \neq \emptyset \iff M \ncong R$.

Proof. The forward direction is easy. For the backward direction, resolve:

$$0 \to N \xrightarrow{\subseteq} F \to M \to 0,$$

with F free and finitely generated. This is not split, or else M is free. Write $N = \bigoplus_i N_i$, where the N_i are indecomposable. We have, for all j, an exact sequence:

$$(\xi_j) \qquad 0 \longrightarrow N_j \longrightarrow \frac{F}{\bigoplus_{i \neq j} N_i} \longrightarrow M \longrightarrow 0.$$

By the lemma, some ξ_j is not split (else all are split, so the sequence is split). $\square$

We now define a partial order on $\mathcal{S}(M)$. We say $s \geq t$ if there exists a commutative diagram:

$$\begin{tikzcd} (s) & 0 \arrow[r] & N_s \arrow[r] \arrow[d, "f"] & E_s \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\ (t) & 0 \arrow[r] & N_t \arrow[r] & E_t \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

(This is actually a pushout, and in this case, we have $t = fs$.)

Recall that if $(\xi) : 0 \to A \to B \to C \to 0$ in $\operatorname{Ext}^1_R(C, A)$ and $A \xrightarrow{f} X$, we get $f\xi$ in $\operatorname{Ext}^1_R(C, X)$ via a pushout:

$$\begin{tikzcd} (\xi) & 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ (f\xi) & 0 \arrow[r] & X \arrow[r] & Z \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

If also $X \xrightarrow{g} Y$, then $g(f\xi) = (gf)\xi$. (In particular, this makes $\operatorname{Ext}^1_R(C, A)$ into a left $\operatorname{End}_R(A)$-module.) Thus $\geq$ is reflexive and transitive. Also, we'll say

$$s \sim t \iff \text{there exists an isomorphism } f \text{ as above.}$$

This relation is obviously reflexive and transitive. By the five-lemma, it is symmetric as well, so it's an equivalence relation. We'll show that $s \geq t$ and $t \geq s$ implies $s \sim t$. Suppose we have $f : N_s \to N_t$ showing that $s \geq t$ and $g : N_t \to N_s$ showing $t \geq s$. Then we have $gf : N_s \to N_s$ showing that $s \geq s$. If we can show that this forces gf to be an isomorphism, then by symmetry fg will be an isomorphism, too, forcing f and g to be isomorphisms, giving $s \sim t$.

29.3 Lemma. *Suppose $s \in \mathcal{S}(M)$ and $h \in \operatorname{End}_R(N_s)$ such that $hs = s$. Then h is an automorphism. More generally, if $s \sim t$ in $\mathcal{S}(M)$ and $h : N_s \to N_t$ is such that $hs = t$, then h is an isomorphism.*

Proof. We'll prove the first assertion. (The second follows easily.) Suppose h is not an isomorphism. Since $\operatorname{End}_R(N_s)$ is "local" (in the noncommutative sense), then $h \in J$, the Jacobson radical. We have $(1 - h)s = 0$, but as $1 - h$ is invertible, $s = 0$, a contradiction with $s \in \mathcal{S}(M)$. $\square$

Now, we have that $\mathcal{S}(M)/\sim$ is a poset, but in fact $\mathcal{S}(M)/\sim$ is downward directed. Suppose $s, t \in \mathcal{S}(M)$. We seek u such that $s \geq u$ and $t \geq u$. With

$$(s) \qquad 0 \longrightarrow N_s \longrightarrow E_s \xrightarrow{p} M \longrightarrow 0$$

$$(t) \qquad 0 \longrightarrow N_t \longrightarrow E_t \xrightarrow{q} M \longrightarrow 0$$

let $E = E_s \oplus E_t$, $N = \ker(E_s \oplus E_t \xrightarrow{[p\ q]} M)$. Then $0 \to N \xrightarrow{f} E \xrightarrow{[p\ q]} M \to 0$ is exact, and, by the next exercise, it does not split.

Now decompose $N = \bigoplus_i N_i$, and let $E_j = E/f(\bigoplus_{i \neq j} N_i)$. Then

$$0 \to N_j \to E_j \to M \to 0$$

is exact for all j. By the lemma, at least one is unsplit. Then we have a commutative exact diagram

$$\begin{tikzcd}
(s) & 0 \arrow[r] & N_s \arrow[r] \arrow[d] & E_s \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
& 0 \arrow[r] & N \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
(u) & 0 \arrow[r] & N_j \arrow[r] & E_j \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

so that $s \geq u$. By symmetry, $t \geq u$.

29.4 Definition. An *AR-sequence ending in* M is an element of $\mathcal{S}(M)$ whose $\sim$ class is minimal in $\mathcal{S}(M)/\sim$. It is a short exact sequence

$$(s) \qquad 0 \longrightarrow N_s \longrightarrow E_s \xrightarrow{p} M \longrightarrow 0$$

such that $s \in \mathcal{S}(M)$, and if $s \geq t$, then $s \sim t$.

Note that if s is an AR-sequence ending in M and $t \in \mathcal{S}(M)$, then $t \geq s$. To see this, use downward directedness to get u such that $t \geq u, s \geq u$. Then $s \sim u$, so $t \geq s$. This highlights an important property: in a directed set, minimal elements are mini*mum* elements. From this we see any 2 AR-sequences ending in M must be $\sim$.

29.5 Theorem. *Let $[M] \in \mathrm{Ind}(R)$. There exists an AR-sequence ending in M if and only if $M \not\cong R$ and M_P is R_P-free for all $P \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$.*

29.6 Remark. Suppose R is an isolated singularity. Then, for each $P \neq \mathfrak{m}$, R_P is a regular local ring. If M is a MCM R-module, then M_P is a MCM R_P-module, so M_P is free over R_P. Therefore there exists an AR-sequence ending in M for each $[M] \in \mathrm{Ind}(R) \setminus \{[R]\}$.

29.7 Exercises.

29.1 (3 points) Prove Lemma 29.1.

29.2 (4 points) Let $E \xrightarrow{p} M$ and $F \xrightarrow{q} M$ be R-homomorphisms, and assume M is indecomposable and finitely generated. If $E \oplus F \xrightarrow{[p\ q]} M$ is a split surjection, then either p or q is a split surjection. [You'll need the assumption that R is Henselian.]

§30. AR Sequences

Let$(R, \mathfrak{m})$, CM and Henselian. Assume R has a canonical module ω_R.

30.1 Theorem. *Let $[M] \in \mathrm{Ind}(R) := \{$isomorphism classes $[M]$ of indecomposable MCM $R-$ modules$\}$. Assume $M \not\cong R$. Then there exists an AR$-$sequence ending in $M \iff M_P$ is R_P-free for all $P \neq \mathfrak{m}$.*

Proof. We will only prove "$\Rightarrow$ ". We will show that if R has finite CM type then there exists an $AR-$sequence ending in M whenever $[M] \in \mathrm{Ind}(R)$.

($\Rightarrow$) Let $u: \; 0 \longrightarrow N_u \longrightarrow E_u \longrightarrow M \longrightarrow 0$ be the $A-R$ sequence ending in M. Resolve M, getting a short exact sequence $0 \longrightarrow L \longrightarrow F \longrightarrow M \longrightarrow 0$, where F is free and finitely generated. Suppose there exists $P \neq \mathfrak{m}$ such that M_P is not free. Then

$$0 \longrightarrow L_P \longrightarrow F_P \longrightarrow M_P \longrightarrow 0$$

is unsplit. Therefore $\mathrm{Ext}^1_R(M, L)_P \neq 0$. Choose an indecomposable direct summand N of L such that $\mathrm{Ext}^1_R(M, N)_P \neq 0$. There exists $s \in \mathrm{Ext}^1_R(M, N)$ such that $\frac{s}{1} \neq 0$ in $\mathrm{Ext}^1_R(M, N)_P$. Then $(0 :_R s) \subseteq P$. Choose $r \in \mathfrak{m} - P$. Then

$r^n \notin P\ \forall n$, so $\forall n\ r^n s \neq 0$. This gives the following diagram where f_n exists as u is an $AR-$sequence.

$$\begin{tikzcd}
s: & 0 \arrow[r] & N_s = N \arrow[r] \arrow[d, "r^n"'] & E_s \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "="] & 0 \in \mathcal{S}(M) \\
t := r^n s: & 0 \arrow[r] & N \arrow[r] \arrow[d, "f_n"'] & E_t \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "="] & 0 \\
u: & 0 \arrow[r] & N_u \arrow[r] & E_u \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Then $u = f_n r^n s = r^n f_n s$ where $f_n s \in Ext^1_R(M, N_u)$. Therefore $u \in r^n Ext^1_R(M, N_u)\ \forall n$. Thus by NAK $u = 0$ which is a contradiction. $\square$

30.2 Lemma. *Let N_i be finitely generated modules, all indecomposable. Let*

$$N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} N_3 \longrightarrow \cdots \longrightarrow N_{t-1} \xrightarrow{f_{t-1}} N_t$$

be a sequence of non-isomorphisms. Then $h := f_{t-1} \dots f_1$ is not an isomorphism.

Proof. Suppose h is an isomorphism. Then $N_{t-1} \xrightarrow{f_{t-1}} N_t$ is a split surjection:

$$\begin{tikzcd}
N_t \arrow[r] \arrow[rrr, bend right, "1_{N_t}"'] & N_1 \arrow[r, "f_{t-2}\cdots f_1"] & N_{t-1} \arrow[r, "f_{t-1}"] & N_t
\end{tikzcd}$$

Therefore, $N_t \mid N_{t-1}$ and hence $N_t \cong N_{t-1}$ as they are indecomposables. Then, since f_{t-1} is a surjection, it has to be an isomorphism, which is a contradiction. $\square$

30.3 Theorem. $(R, \mathfrak{m})$ *Henselian, CM. Assume R has finite CM type. Let $[M] \in \mathrm{Ind}(R)$, and assume $M \ncong R$. Then there exists an AR-sequence ending in M.*

Proof. If there is not an AR-sequence ending in M, we get, in $\mathcal{S}(M)$, $s_1 > s_2 > s_3 > \dots$ where "$s > t$" means $s \geq t$ but $s \nsim t$.

$$\begin{tikzcd}
s_i: & 0 \arrow[r] & N_{s_i} \arrow[r] \arrow[d, "f_i"'] & E_{s_i} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "="] & 0 \\
s_{i+1}: & 0 \arrow[r] & N_{s_{i+1}} \arrow[r] & E_{s_{i+1}} \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Since R has finite CM type and the N_{s_i} are indecomposable MCM modules, there exists an infinite sequence $i_1 < i_2 < i_3 < \dots$ such that $N_{s_{i_1}} \cong N_{s_{i_2}} \cong N_{s_{i_3}} \cong \dots$

Let $N_j = N_{s_{i_j}}$ and let g_j be the product of the various f_i connecting N_j to N_{j+1}. By the lemma, (as none of the f_i are isomorphisms as $s_i \nsim s_{i+1}$) $g_j : N_j \to N_{j+1}$ is not an isomorphism. Put $E_j := E_{s_{i_j}}$. Let $N = N_1$ and choose, $\forall j$, an isomorphism $\varphi_j : N \xrightarrow{\cong} N_j$. With $h_j := \varphi^{-1} g_j \varphi$, the following diagram commutes:

$$\begin{tikzcd}
N \arrow[r, "\varphi", "\cong"'] \arrow[d, "\exists ! h_j"] \arrow[rr, bend left] & N_j \arrow[r] \arrow[d, "g_j"] & E_j \arrow[d] \\
N \arrow[r, "\varphi", "\cong"'] \arrow[rr, bend right] & N_{j+1} \arrow[r] & E_{j+1}
\end{tikzcd}$$

Forgetting the middle column, we have, $\forall j$, an exact commutative diagram:

$$\begin{tikzcd} \xi_j: & 0 \arrow[r] & N \arrow[r] \arrow[d] & E_j \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "="] & 0 \\ \xi_{j+1}: & 0 \arrow[r] & N \arrow[r] & E_{j+1} \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

Let $A = \operatorname{End}_R(N)$, $J = J(A)$. Since A is "local" and since g_j is not an isomorphism, $g_j \in J\ \forall j$. Since A is finitely generated as an R-module, $A/\mathfrak{m}A$ is a finite dimensional k-algebra. Therefore $A/\mathfrak{m}A$ is an Artinian ring.

If $x \in \mathfrak{m}A$, then $\mathfrak{m}A + A(1+x) = A$. By NAK, $A(1+x) = A$. Therefore, $\mathfrak{m}A \subseteq J$ and $J/\mathfrak{m}A$ is the Jacobson Radical of $A/\mathfrak{m}A$. Therefore, as the Jacobson radical of an Artinian ring is nilpotent, there exists v such that $J^v \subseteq \mathfrak{m}A$. Let $\mathcal{E} = \operatorname{Ext}^1_R(M,N)$. Since R has finite CM type, R_p is a RLR $\forall p \neq \mathfrak{m}$. Therefore M_p is R_p-free, $\forall p \neq \mathfrak{m}$ and $\mathcal{E}_p \neq 0$, $\forall p \neq \mathfrak{m}$, i.e, $\mathcal{E}$ has finite length as R-module. So there exists q such that $\mathfrak{m}^q\mathcal{E} = 0$. Then $\xi_{vq+1} \in J^{vq}\mathcal{E} \subseteq \mathfrak{m}^q A\mathcal{E} = \mathfrak{m}^q\mathcal{E} = 0$ and hence ξ_{vq+1} splits, a contradiction. □

Lifting Property of AR-Sequences

$$\begin{tikzcd} & & & & L \arrow[dl, dotted, "\exists f"'] \arrow[d, "q"] & \\ (s)\ 0 \arrow[r] & N_s \arrow[r] & & E_s \arrow[r, "p"] & M \arrow[r] & 0 \end{tikzcd}$$

Here (s) is an AR-sequence, L is MCM, and q is a non-split surjection.

30.4 Theorem *Let*

$$0 \longrightarrow N_s \longrightarrow E_s \xrightarrow{p} M \longrightarrow 0$$

be an AR-sequence, let L be a MCM, and let $q : L \to M$ be a non-split surjection. Then there exists $f : L \to E_s$ such that $pf = q$. Conversely, if

$$0 \longrightarrow N \longrightarrow E \xrightarrow{p} M \longrightarrow 0$$

is exact, p non-split, M and N indecomposable and it has the lifting property, then it is an AR-sequence.

Proof. We'll prove only the first assertion.

$$\begin{tikzcd} & & L \arrow[dr, "q"] & & \\ 0 \arrow[r] & N_s \arrow[r] & E_s \arrow[r, "p"] & M \arrow[r] & 0 \end{tikzcd}$$

Look at the exact sequence

$$0 \to Q \hookrightarrow E_s \oplus L \xrightarrow{[p\ q]} M \to 0.$$

By Exercise 29.2, $[p\ q]$ is not a split surjection. Let $Q = \bigoplus_i Q_i$, where each Q_i is indecomposable. For each j, we get an exact sequence

$$0 \to Q_j \to \frac{E_s \oplus L}{\oplus_{i \neq j} Q_i} \to M \to 0.$$

Some j gives an element of $\mathcal{S}(M)$. Since $(s) \geq (t)$, we get the following diagram, which provides the lifting f:

$$
\begin{tikzcd}
 & & L \arrow[d, "\begin{bmatrix} 0 \\ 1 \end{bmatrix}"'] \arrow[dr, bend left, "q"] \arrow[ddd, bend right=60, "f"'] & & \\
0 \arrow[r] & Q \arrow[r] \arrow[d] & E_s \oplus L \arrow[r, "{[pq]}"] \arrow[d, "natural"'] & M \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & Q_j \arrow[r] \arrow[d] & \dfrac{E_s \oplus L}{\oplus_i Q_i} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & N_s \arrow[r] & E_s \arrow[r] & M \arrow[r] & 0
\end{tikzcd}
$$

$\square$

FACT:
Suppose $0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$ is the AR-sequence ending in M. Then $N = (\mathrm{Syz}^d_R(\mathrm{tr}(M)))\check{}$, where $(-)\check{} = \mathrm{Hom}_R(-, \omega_R)$. What is $\mathrm{tr}(M)$? Choose a presentation

$$F_1 \xrightarrow{\varphi} F_0 \longrightarrow M \longrightarrow 0 \ ,$$

where F_0 and F_1 are finitely generated and free. Then $\mathrm{tr}(M)$ is the cokernel of the transpose matrix φ^*. Thus $\mathrm{tr}(M)$ is defined by the following exact sequence:

$$0 \longrightarrow M^* \longrightarrow F_0^* \xrightarrow{\varphi^*} F_1^* \longrightarrow \mathrm{tr}(M) \longrightarrow 0$$

If $d = 2$, then $\mathrm{Syz}^2_R(\mathrm{tr}(M)) = M^*$.

§31. Duality in Relation to AR-sequences

Let $(R, \mathfrak{m}, k)$ be a Henselian local CM ring and assume R admits a canonical module ω_R. Let $(\_\_)\check{} := \mathrm{Hom}_R(\_\_, \omega_R)$. Recall that this gives a perfect duality on MCM $R-$modules, $R\check{} \cong \omega_R$, hence $\omega_R\check{} = \mathrm{Hom}_R(\omega_R, \omega_R) \cong R$.

Given $[N] \in \mathrm{Ind}(R)$, let $\mathcal{T}(N)$ consist of the equivalence classes of extensions

$$(s) \qquad 0 \longrightarrow N \longrightarrow E_s \longrightarrow M_s \longrightarrow 0$$

where M is an indecomposable MCM module and (s) is not split.

31.1 Definition. "$s \geq t$" if and only if there exists a commutative diagram

$$
\begin{tikzcd}
(s) & 0 \arrow[r] & N \arrow[r] & E_s \arrow[r] & M_s \arrow[r] & 0 \\
(t) & 0 \arrow[r] & N \arrow[r] \arrow[u, "="] & E_t \arrow[r] \arrow[u] & M_t \arrow[r] \arrow[u] & 0
\end{tikzcd}
$$

Say $s \sim t$ if and only if in the above CD f can be chosen to be an isomorphism.

This makes $\mathcal{T}(N)/\sim$ into a directed set.

31.2 Definition. "The" *RA-sequence starting at* N is the unique minimal element of $\mathcal{T}(M)$ if it exists.

31.3 Theorem. *Given a MCM $R-$module N, an RA-sequence starting at N always exist provided R has finite CM type and $N \neq \omega_R$.*

Proof. Note that $N^{\check{}} \ncong R$, so there exists an AR-sequence $0 \longrightarrow A \longrightarrow B \longrightarrow N^{\check{}} \longrightarrow 0$. Then $0 \longrightarrow N \longrightarrow B^{\check{}} \longrightarrow A^{\check{}} \longrightarrow 0$ is an RA-sequence. □

Note $s \geq t$ in $\mathcal{T}(M)$ $\iff$ $s^{\check{}} \geq t^{\check{}}$ in $\mathcal{S}(M)$.

31.4 Theorem. *Let* (s) $\quad 0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$ *be an exact sequence of $R-$modules. Then, s is an AR-sequence if and only if s is an RA-sequence.*

Proof. For the forward direction, assume (s) is an AR-sequence. Then (s) is in $\mathcal{T}(N)$. Suppose in addition that (t) is in $\mathcal{T}(N)$ and $(s) \geq (t)$. We show that $(s) \sim (t)$ in $\mathcal{T}(N)$. Consider the following diagram

$$\begin{tikzcd}
(s) & 0 \arrow[r] & N \arrow[r, "q"] & E \arrow[r, "p"] & M \arrow[r] & 0 \\
(t) & 0 \arrow[r] & N \arrow[u, "="] \arrow[r, "b"'] & G_t \arrow[u, "f"] \arrow[ul, dashed, "\varphi"] \arrow[r, "a"'] & M_t \arrow[u, "g"'] \arrow[ul, dashed, "\theta"] \arrow[r] & 0
\end{tikzcd}$$

We want to show that g is an isomorphism. Then so will f by the Five Lemma and $s \leq t$. So suppose g is not an isomorphism. Since M and M_t are indecomposable modules, g cannot be a split surjection. By the lifting property of AR-sequences there exists $\theta: \ M_t \longrightarrow E$ such that $p\,\theta = g$. Note that from commutativity of the diagram $p\,(f - \theta a) = pf - p\,\theta a = pf - ga = 0$. Thus $\mathrm{im}(f - \theta a) \subseteq \mathrm{im}\, q$, so there exists $\varphi: \ G_t \longrightarrow N$ such that $q\varphi = f - \theta a$. Then $q\varphi b = (f - \theta a)b = fb - \theta ab = fb = q\,1_N$. Thus $\varphi b = 1_N$ as q is injective, contradicting that (t) is not split exact.

The reverse implication follows by duality. □

31.5 Definition. If (s) $\quad 0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$ is an AR-sequence we define the *AR-translate of* M, denoted $\tau(M)$, to be N. Similarly $\tau^{-1}(N) = M$.

31.6 Definition. Let $[M], [N] \in \mathrm{Ind}(R)$. Define

$$(M,N)_1 := \{ \text{ the non-isomorphisms in } \mathrm{Hom}_R(M,N)\}$$

and

$(M,N)_2 := \{f \in \mathrm{Hom}_R(M,N) |\ \exists$ a factorization $\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "p"'] & N \\ Q \arrow[ur, "q"'] & \end{tikzcd}$ such that Q is MCM,

p is not a split injection and q is not a split surjection$\}$.

31.7 Remark.

(1) $(M,N)_1 := \begin{cases} \mathrm{Hom}_R(M,N)\ , & \text{if } M \ncong N \\ \mathrm{J(End_R(M))}, & \text{if } M \cong N \end{cases}$

(2) $(M,N)_2$ is an $R-$submodule of $(M,N)_1$ and $\mathfrak{m}(M,N)_1 \subseteq (M,N)_2$.

Proof. Let $f \in (M,N)_1$ and $r \in \mathfrak{m}$. Then rf factors as

$$\begin{tikzcd} M \arrow[r, "rf"] \arrow[d, "f"'] & N \\ N \arrow[ur, "\cdot r"'] & \end{tikzcd}$$

By Nakayama $N \xrightarrow{\cdot r} N$ is not a (split) surjection and, as f is not an isomorphism, f is not a split injection. Thus $\frac{(M,N)_1}{(M,N)_2}$ is a finite dimensional $k-$vector space. $\square$

31.8 Definition. A homomorphism $M \xrightarrow{f} N$ is *irreducible* provided $f \in (M,N)_1 \backslash (M,N)_2$. Let $Irr(M,N) := (M,N)_1 \backslash (M,N)_2$ and $irr(M,N) := \dim_k \dfrac{(M,N)_1}{(M,N)_2}$.

From now on, we assume k is algebraically closed. Then $\mathrm{End}_R(M)/\mathrm{J}(\mathrm{End_R(M)})$ is a division ring which is a finitely generated algebra over k and hence a field extension. Thus $\frac{\mathrm{End}_R(M)}{\mathrm{J}(\mathrm{End_R(M)})} = k$.

AR QUIVERS

To construct the AR quivers of a ring R start with a vertex for each isomorphsim class of indecomposable MCM $R-$module. Draw n arrows from $[M]$ to $[N]$ when $irr(M,N) = n$. Draw a dotted line between $[M]$ and $[\tau(M)]$. Note that having no arrow from $[M]$ to $[N]$ means $\nexists$ irreducible homomorphism from M to N.

31.9 Theorem. *Let k be algebraically closed, R Henselian. Let M, N, L be indecomposable MCM modules. Let $0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$ be an AR-sequence, then $irr(N,L) = irr(L,M)$ =multiplicity of L in the decomposition of E as a direct sum of indecomposables.*

31.10 Examples.

(1) Refer to Yoshino for the AR quiver of the (E_6) singularity $R = \frac{k[[x,y]]}{(x^3+y^4)}$. Note that R is Gorenstein and hence $\omega_R \cong R$. The module M_2 is the ideal (x^2, y^2). The arrows from M_2 to X and back show that there is an AR sequence $0 \to M_2 \to X \to M_2 \to 0$. In particular, X is an indecomposable MCM R-module of rank 2.

Of course R is isomorphic (via change of variable) to $k[[x,y]]/(x^3 - y^4) \cong k[[t^3, t^4]] =: R$. The endomorphism ring of the maximal ideal of $k[[t^2, t^3]]$ is the overring $S := k[[t^3, y^4, t^5]]$ (which is not Gorenstein). By the argument in Proposition 1 of §7, every indecomposable MCM R-module other than R is actually an S-module. Therefore the AR quiver for S is obtained from the one for R by erasing R and the arrows into and out of R.

(2) Consider the 2-dimensional Veronese ring $R = k\left[\left[x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5\right]\right]$. The AR quiver for R (and the more general case where 5 is replaced by any integer $n \geq 2$) is drawn at the end of the chapter in Yoshino's book. This is the ring of invariants of the cyclic group of order five acting on $S := k[[x,y]]$ by multiplying each variable by ζ_5, a primitive fifth root of unity. (We assume that the characteristic of k is different from 5.) From our work on two-dimensional rings of finite CM type, we know that R is a normal domain and that the indecomposable maximal Cohen-Macaulay R-modules are the indecomposable R-direct summands of S. Clearly $S = R \oplus J_1 \oplus \cdots \oplus J_4$, where J_s is the R-submodule of S generated by the monomials of degree s in x and y. (With $I := (x^5, x^4y)$, we see that $j_s \cong I^s$ for $1 \leq s \leq 4$.) The canonical module is J_3 (cf. "Indecomposable Gorenstein modules of odd rank" by C. Rotthaus, D. Weston and R. Wiegand, J. Algebra (~1999). The vertices of the AR quiver of R are the numbers $0, \ldots, 4$ corresponding to $J_0 = R$ and $J_1, \ldots, J_4$. There's a dotted line connecting i and $i-2 \pmod 5$, for $i = 1,2,3,4$ (but no dotted line between 0 and 3). (In particular, $\tau(\omega) = J_1$ and $\tau(J_2) = R$.) There are two arrows from i to $i+1 \pmod 5$. Thus, for example, there is an AR sequence $0 \to J_1 \to J_2 \oplus J_2 \to J_3 \to 0$. It is more than coincidental that the J_i are the non-isomorphic divisorial (= reflexive) ideals of R, and that the muliplication in the divisor class group corresponds to addition modulo 5. Thus in the exact sequence above we confirm the well-known fact that $1 + 3 = 2 + 2$.

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