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On Some Risk-Adjusted Tail-Based Premium Calculation Principles

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Abstract

This paper explores two tail-based premium calculation principles, the tail standard deviation (TSD) premium and the tail conditional expectation (TCE) premium, in their risk-adjusted and unadjusted forms. They are risk-adjusted using so-called distortion functions. We prove that the proportional hazard (PH) risk-adjusted TCE premium is larger than the unadjusted TCE premium. Additionally, given a risk distribution with location and scale parameters, we prove that the PH risk-adjusted TCE premium reduces to the unadjusted TSD premium.

Key words and phrases: *tail conditional expectation, tail standard deviation, distortion function, Wang's premium principle, risk-adjusted tail standard deviation, risk-adjusted tail conditional expectation*

1 Introduction

Let X denote a financial risk, i.e., a non-negative random variable, and let $\mathcal{X}$ denote a set of such risks. A risk measure H is the functional:

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$$H : \mathcal{X} \to [0, \infty], \tag{1}$$

i.e., H provides a measure of the degree of riskiness inherent in $X \in \mathcal{X}$. The quantity $H(X)$ is important in risk management because it may point to the amount of capital needed to be set aside in order to protect against insolvency due to exposure to X. Several types of risk measures exist. The earliest seems to be the Value-at-Risk or VaR (Leavens 1945). More recent ones are the distorted risk measures of Denneberg (1990 and 1994), Wang (1995 and 1996), and Wang, Young, and Panjer (1997).

There is growing interest among insurance and investment experts in the use of the tail conditional expectation (TCE) as a measure of risk because of its desirable properties and its flexibility. To define this premium calculation principle, we suppose X has cumulative distribution function (cdf) $F_X(x)$ and survival function given by $\overline{F}_X(x) = 1 - F_X(x)$. The tail conditional expectation premium calculation principle is defined as

$$\mathrm{TCE}_X\left(x_q\right) = \frac{1}{\overline{F}_X\left(x_q\right)} \int_{x_q}^{\infty} x dF_X(x), \tag{2}$$

subject to $\overline{F}_X\left(x_q\right) > 0$, where, for $0 < q < 1$

$$x_q = \inf\{x | F_X(x) \geq q\} \tag{3}$$

is the Value-at-Risk and is denoted by $VaR_X(q)$. Panjer and Jia (2001) suggest that the tail conditional expectation has some intuitive appeal to actuaries because it represents an expected loss given the loss exceeds a deductible. It should also be noted that the tail conditional expectation is a coherent risk measure in the sense of Artzner et al. (1999). For more tail conditional expectations, see Panjer and Jia (2001), Hürlimann (2001), Landsman and Valdez (2003, 2005), Furman and Landsman (2005a), and Ministre and Hancock (2005).

Once the degree of riskiness is known, there still is the problem of incorporating a risk loading to be added to the net premium. This led Denneberg (1994) and Wang (1996) to develop the following premium calculation principle: For some non-negative random variable X, let g, called a **distortion function**, be an increasing concave function defined on $[0, 1]$ with $g(0) = 0$ and $g(1) = 1$. Wang's premium is given by

$$W_g(X) = \int_0^{\infty} g\left(\overline{F}_X(x)\right) \mathrm{d}x. \tag{4}$$

If, in addition, $g(x) = x^{\rho}$, where $0 < \rho < 1$ is a constant, then W_g is called the proportional hazard (PH) premium principle (Wang 1995). Wang's premium is believed to be a sound premium calculation principle because, for instance, it is convex, comonotonically additive, and preserves first and second stochastic dominance.

It is straightforward to show that TCE_X is a particular case of $W_g(X)$. More precisely, let $I(\mathcal{A})$ be the indicator function of the occurrence of the event $\mathcal{A}$, i.e., $I(\mathcal{A}) = 1$ if $\mathcal{A}$ occurs and 0 otherwise. Then TCE_X is obtained by using the following distortion function in formula (4)

$$g(u) = \frac{u}{1-q} I(u < 1-q) + I(u \geq 1-q). \tag{5}$$

In the sequel we propose the so-called risk-adjusted or distorted version of TCE_X. Our approach is inspired by Denneberg (1994) and Wang (1996). It differs from their approach, however, in that we calculate the tail conditional premium calculation principle of the *risk-adjusted (distorted)* cdf of X rather than its actual cdf. The motivation for using a distorted TCE_X is similar to that discussed in Wang (1996): to obtain a risk-loaded premium.

2 Tail Standard Deviation (TSD) Premium

The standard deviation premium calculation principle SD_X, is one of the simplest and most popular premium calculation principles used in property and casualty insurance (Bühlmann 1970, Chapter 4), and is given by

$$\mathrm{SD}_X = \mathbb{E}(X) + \lambda\sqrt{\mathbb{V}\mathrm{ar}(X)}.$$

Unfortunately the standard deviation premium principle has a major disadvantage: it overlooks the shape of the risk distribution because it uses only the mean and the variance.

As an alternative to the standard deviation principle, Furman and Landsman (2005b) developed the *tail standard deviation premium calculation principle*[1] (TSD), defined as

$$\mathrm{TSD}_X\left(x_q\right) = \mathrm{TCE}_X\left(x_q\right) + \lambda\sqrt{\mathrm{TV}_X\left(x_q\right)}, \tag{6}$$

[1] The tail standard deviation premium calculation principle was studied by Furman and Landsman (2005b) in the context of elliptical distributions. Unfortunately, all members of the elliptical family are symmetric, while insurance losses are in general modeled by non-negative and positively skewed random quantities.

where

$$\mathrm{TV}_X\left(x_q\right) = \mathbb{V}\mathrm{ar}\left(X|X > x_q\right) \tag{7}$$

and λ is a non-negative constant. The standard deviation premium calculation principle is a particular case of $\mathrm{TSD}_X\left(x_q\right)$, which can be seen by letting $q \to 0$ in equation (6).

We now enumerate certain useful properties that are preserved by the tail standard deviation premium calculation principle. While the first three properties are traditional and explained in Kaas et al. (2001), the fourth has not been studied extensively.

1. *Non-negative loading:*

$$\mathrm{TSD}_X\left(x_q\right) \geq \mathbb{E}\left(X\right).$$

The TSD premium calculation principle is not smaller than the well-known net premium.

2. *Translation invariance:* If c is some constant risk, then

$$\mathrm{TSD}_{X+c}\left(x_q\right) = \mathrm{TSD}_X\left(x_q\right) + c.$$

Increasing the risk by some constant amount c increases the premium by the same amount. Kaas et al. (2001) refer to this property as *consistency*.

3. *Positive homogeneity:* For any risk X and any positive constant β

$$\mathrm{TSD}_{\beta X}\left(x_q\right) = \beta \mathrm{TSD}_X\left(x_q\right).$$

If a company's risk exposure changes proportionally, then its premium must change in the same proportion.

4. *Tail parity:* We call X and Y tail equivalent if some q exists such that $\overline{F}_X\left(t\right) = \overline{F}_Y\left(t\right)$ for every $t \geq x_q$, and then

$$\mathrm{TSD}_X\left(t\right) = \mathrm{TSD}_Y\left(t\right),$$

i.e., TSD depends only on the tail of the distribution. This property is especially useful in the case of reinsurance contracts and policies involving deductibles.

We note that, unlike SD_X, the TSD_X depends on the shape of the distribution of X. The following example illustrates this:

Example 1. Consider two risks X and Y where $\mathbb{E}(X) = \mathbb{E}(Y) = 3$ and $\mathbb{V}\mathrm{ar}(X) = \mathbb{V}\mathrm{ar}(Y) = 15$. Regardless of the shape of the cdf of X and of Y, the standard deviation premium calculation principle yields the same premium for X and Y, i.e., $SD_X = 3 + \lambda\sqrt{15} = SD_Y$. On the other hand, to use the TSD we need the cdf of both X and Y. Suppose X is lognormal and Y is Pareto with cdf F_Y where

$$F_Y(y) = 1 - \left(\frac{\beta}{y+\beta}\right)^{\alpha} \quad \text{and} \quad y_q = \frac{\beta}{(1-q)^{1/\alpha}} - \beta$$

with $\alpha = 5$ and $\beta = 12$. Table 1 shows $\mathrm{TSD}_X\left(x_q\right)$ and $\mathrm{TSD}_Y\left(y_q\right)$ as functions of λ for various values of q.

Table 1

$\mathrm{TSD}_X\left(x_q\right)$ **Versus** $\mathrm{TSD}_Y\left(y_q\right)$

q	x_q	$\mathrm{TSD}_X\left(x_q\right)$	y_q	$\mathrm{TSD}_Y\left(y_q\right)$
0.01	0.1835	$3.0289 + 3.8817\lambda$	0.0241	$3.0302 + 3.8808\lambda$
0.05	0.3603	$3.1446 + 3.9206\lambda$	0.1237	$3.1547 + 3.9129\lambda$
0.10	0.5163	$3.2948 + 3.9744\lambda$	0.2556	$3.3194 + 3.9555\lambda$
0.15	0.6582	$3.4541 + 4.0334\lambda$	0.3965	$3.4956 + 4.0009\lambda$
0.25	0.9420	$3.8081 + 4.1679\lambda$	0.7107	$3.8884 + 4.1024\lambda$
0.50	1.8371	$5.0340 + 4.6385\lambda$	1.7844	$5.2305 + 4.4489\lambda$
0.75	3.5830	$7.4874 + 5.5451\lambda$	3.8341	$7.7926 + 5.1104\lambda$
0.90	6.5365	$11.5637 + 6.9390\lambda$	7.0187	$11.7730 + 6.1382\lambda$
0.99	18.3961	$27.2334 + 11.5717\lambda$	18.1426	$25.6770 + 9.7304\lambda$

The following random variable is useful for describing tail conditional expectation of the risk X.

Definition 1. *Let $X \geq 0$ be a risk with cdf F_X. Assuming the n^{th} moment of X exists, then for $n = 1, 2, \ldots$, we define $X^{(n)}$ as the random variable with cdf given by*

$$F_{X^{(n)}}(x) = \frac{\mathbb{E}\left(X^n I(X \leq x)\right)}{\mathbb{E}\left(X^n\right)} = \frac{1}{\mathbb{E}\left(X^n\right)} \int_0^x y^n dF_X(y). \tag{8}$$

Furman and Landsman (2005a) noted that the tail conditional expectation of X can be expressed in terms of $X^{(1)}$. The following theorem provides a general expression for the TSD premium calculation principle. Note that only the existence of the second moment of X is assumed.

Theorem 1. *Assume that* $\mathbb{E}\left(X^2\right) < \infty$. *The tail standard deviation premium for X is*

$$\mathrm{TSD}_X\left(x_q\right) = \mathbb{E}\left(X\right)\frac{\overline{F}_{X^{(1)}}\left(x_q\right)}{\overline{F}_X\left(x_q\right)} + \lambda\sqrt{\left(\mathbb{E}\left(X^2\right)\frac{\overline{F}_{X^{(2)}}\left(x_q\right)}{\overline{F}_X\left(x_q\right)} - \left(\mathbb{E}\left(X\right)\frac{\overline{F}_{X^{(1)}}\left(x_q\right)}{\overline{F}_X\left(x_q\right)}\right)^2\right)},$$

where λ *is some non-negative constant.*

Proof: The conditional expectation part is

$$\mathbb{E}\left(X|X > x_q\right) = \mathbb{E}\left(X\right)\frac{\overline{F}_{X^{(1)}}\left(x_q\right)}{\overline{F}_X\left(x_q\right)}, \tag{9}$$

as readily follows from the definition of TCE and equation (8). The conditional variance of X may be derived as follows

$$\begin{aligned}\mathbb{V}\mathrm{ar}\left(X|X > x_q\right) &= \frac{1}{\overline{F}_X\left(x_q\right)}\int_{x_q}^{\infty} x^2 dF_X - \mathbb{E}\left(X|X > x_q\right)^2\\ &= \frac{\mathbb{E}\left(X^2\right)}{\overline{F}_X\left(x_q\right)}\int_{x_q}^{\infty}\frac{x^2 dF_X}{\mathbb{E}\left(X^2\right)} - \left(\mathbb{E}\left(X\right)\frac{\overline{F}_{X^{(1)}}\left(x_q\right)}{\overline{F}_X\left(x_q\right)}\right)^2\\ &= \mathbb{E}\left(X^2\right)\frac{\overline{F}_{X^{(2)}}\left(x_q\right)}{\overline{F}_X\left(x_q\right)} - \left(\mathbb{E}\left(X\right)\frac{\overline{F}_{X^{(1)}}\left(x_q\right)}{\overline{F}_X\left(x_q\right)}\right)^2,\end{aligned}$$

which ends the proof.

An example of the application of Theorem 1 is given below:

Example 2. Given X has a Pareto distribution, i.e., $X \sim \mathrm{Pareto}\left(\alpha, \beta\right)$, with cdf

$$\mathrm{P}_X\left(x|\alpha,\beta\right) = 1 - \left(\frac{\beta}{x}\right)^{\alpha} \quad \text{and} \quad x_q = \frac{\beta}{(1-q)^{1/\alpha}}$$

where $x > \beta > 0$ and $\alpha > 0$. The survival function is $\overline{\mathrm{P}}_X\left(x|\alpha,\beta\right)$. To satisfy the conditions of Theorem 1, it is assumed that $\alpha > 2$. Note that $X^{(n)} \sim \mathrm{Pareto}\left(\alpha - n, \beta\right)$ and TSD is consequently

$$\mathrm{TSD}_X\left(x_q\right) = \mathbb{E}(X)\frac{\overline{\mathrm{P}}_X\left(x_q|\alpha-1,\beta\right)}{\overline{\mathrm{P}}_X\left(x_q|\alpha,\beta\right)} + \lambda\sqrt{\mathbb{E}\left(X^2\right)\frac{\overline{\mathrm{P}}_X\left(x_q|\alpha-2,\beta\right)}{\overline{\mathrm{P}}_X\left(x_q|\alpha,\beta\right)} - \left(\mathbb{E}(X)\frac{\overline{\mathrm{P}}_X\left(x_q|\alpha-1,\beta\right)}{\overline{\mathrm{P}}_X\left(x_q|\alpha,\beta\right)}\right)^2}.$$

It turns out that the formula for the TSD premium simplifies to

$$\mathrm{TSD}_X\left(x_q\right) = \frac{x_q}{\beta}\left(\mathbb{E}(X) + \lambda\sqrt{\mathbb{V}\mathrm{ar}(X)}\right) = \frac{x_q}{\beta}SD_X. \tag{10}$$

Figure 1 shows the tail of the normal, gamma, Pareto, and Weibull densities, each of which has mean 866 and variance 463. Figure 2 reveals a disadvantage of the TCE: it sometimes ignores the tail of the distribution. Figure 2 shows, for instance, that although the Pareto distribution has a heavier tail than the normal, gamma, and Weibull distributions, the classical un-distorted TCE finds the Pareto to be the least risky. Under the undistorted TCE of equation (2), the normal, gamma, and Weibull distributions are shown to bear more risk for relatively small q's than does the Pareto distribution, even though the Pareto has the heaviest tail. On the other hand, Figure 3 shows that TSD finds the Pareto to be the most dangerous distribution once $q \geq 0.5$. Though the standard deviation premium fails to order these risks, Figure 3 shows that the tail standard deviation premium calculation principle orders these risks based on the right tail of the distribution.

3 The Risk-Adjusted TCE Premium

We introduce another method of constructing risk-adjusted TCE, which may be used when one is pessimistic about the size of potential losses and is therefore interested in emphasizing large losses during risk assessment. This method allows for a loading to obtain the so-called risk-adjusted probability distribution of X. Thereafter, the risk-adjusted tail conditional expectation premium calculation principle can be introduced as follows.

Definition 2. *Let g be an increasing concave function on $[0,1]$ with $g(0) = 0$ and $g(1) = 1$. Define $\overline{F}_X^*(x) = g\left(\overline{F}_X(x)\right)$ as the risk-adjusted survival function of X. Then the risk-adjusted tail conditional expectation is defined as*

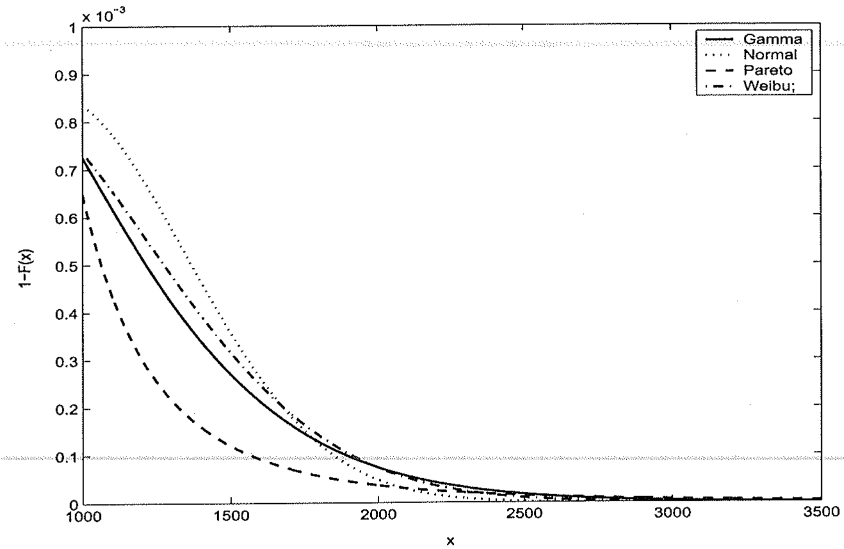

Figure 1: Tail of Normal, Gamma, Pareto, and Weibull PDFs

$$\tau_q^g(X) = \mathbb{E}_{F*}\left(X|X > x_q^*\right), \tag{11}$$

where the expectation is taken with respect to the risk-adjusted cdf $F_X^*(x)$ and $x_q^* = \inf\{x|F_X^*(x) \geq q\}$.

Clearly, $\tau_q^g(X)$ is equal to the non-distorted or risk-free TCE of equation (2) iff $g(u) \equiv u$. $\tau_q^g(X)$ can be expressed in terms of the special distortion function

$$g_q^*(u) = \frac{u}{1-q}I(u < 1-q) + I(u \geq 1-q), \tag{12}$$

which is nondecreasing and concave, as follows

$$\tau_q^g(X) = \int_0^\infty g^*\left(\overline{F}_X^*(x)\right)\mathrm{d}x. \tag{13}$$

It is straightforward to show that $\tau_q^g(X)$ preserves desirable properties such as non-zero loading, sub-additivity, positive homogeneity, translation invariance, layer additivity, and first and second stochastic dominance. We note that the non-zero loading property changes in our context into

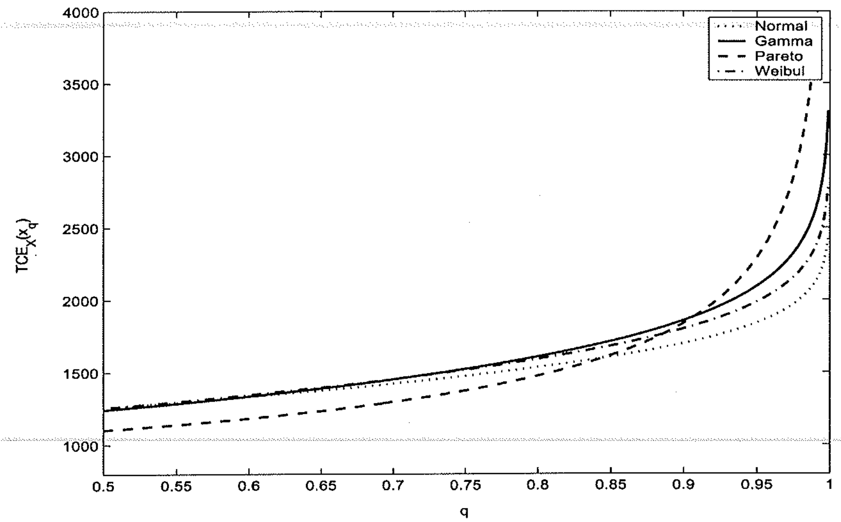

Figure 2: Tail Conditional Expectation for Normal, Gamma, Pareto, and Weibull Risks

$$\tau_q^g(X) \geq E_{F^*}(X) = W_g(X) \geq \mathbb{E}(X).$$

We will now prove that the PH risk-adjusted TCE introduced in equation (11) is not smaller than the undistorted TCE.

Theorem 2. *(Loading property) For $g(u) = u^\rho$, $0 < \rho \leq 1$,*

$$\tau_q^g(X) \geq \mathrm{TCE}_X\left(x_q\right), \tag{14}$$

with equality iff $g(u) = u$, i.e., there is no risk adjustment.

Proof: From Definition 2, the un-distorted TCE can be expressed as:

$$\mathrm{TCE}_X\left(x_q\right) = \int_0^\infty g_q^*\left(\overline{F}_X(x)\right) \mathrm{d}x,$$

where $g_q^*(u)$ is given in equation (12). As $g(u) = u^\rho$, it follows that $g(u/v) = g(u)/g(v)$, we get

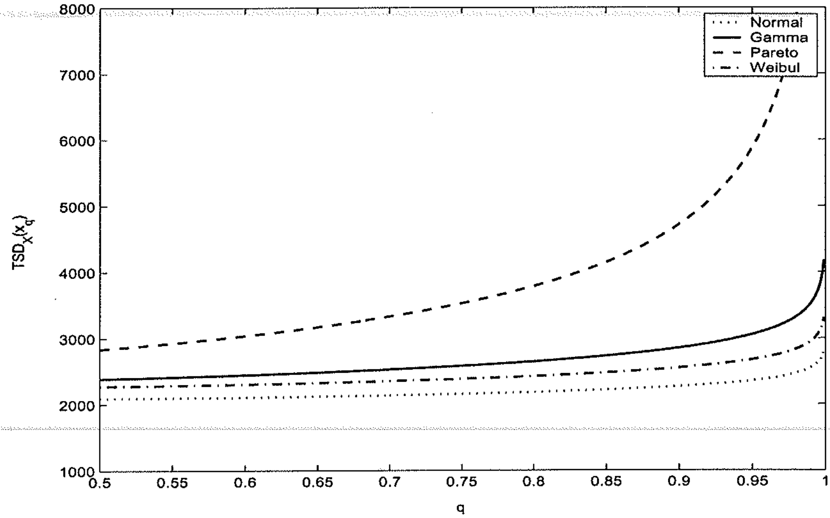

Figure 3: Tail Standard Deviation for Normal, Gamma, Pareto, and Weibull with $\lambda = 3$

$$
\begin{aligned}
g_q^*(g(u)) &= \frac{g(u)}{1-q} I(g(u) < 1-q) + I(g(u) \geq 1-q) \\
&= g\left(\frac{u}{g^{-1}(1-q)} I(u < g^{-1}(1-q)) + I(u \geq g^{-1}(1-q))\right) \\
&= g\left(g_{\tilde{q}}^*(u)\right).
\end{aligned}
$$

Here

$$\tilde{q} = 1 - g^{-1}(1-q) \geq q \tag{15}$$

because g is concave implies g^{-1} is convex. Further, taking into account the concavity of g again

$$
\begin{aligned}
\tau_q^g(X) &= \int_0^\infty g_q^*\left(\overline{F}_X^*(x)\right) \mathrm{d}x = \int_0^\infty g\left(g_{\tilde{q}}^*\left(\overline{F}_X(x)\right)\right) \mathrm{d}x \\
&\geq \int_0^\infty g_{\tilde{q}}^*\left(\overline{F}_X(x)\right) \mathrm{d}x = \mathrm{TCE}_X\left(x_{\tilde{q}}\right) \geq \mathrm{TCE}_X\left(x_q\right),
\end{aligned}
$$

which ends the proof.

We must emphasize that, according to Theorem 2, there is no direct interchange between g and g^*, i.e.,

$$g_q^*\left(g\left(u\right)\right) \neq g\left(g_q^*\left(u\right)\right).$$

Loosely speaking, this inequality implies that the risk-adjusted TCE cannot be obtained by first calculating the undistorted TCE and then applying the desirable distortion to it. For this reason, the computation of $\tau_q^g\left(X\right)$ from equation (11) is generally complicated. It is noted that the difficulty involved depends on the complexity of the corresponding quantile function

$$x_q = Q_X\left(q\right) = F_X^{-1}\left(q\right) = \inf\left\{x | F\left(x\right) \geq q\right\}.$$

The next example sheds some light on this issue.

Example 3. Let $X \sim$ Pareto (α, β) as in Example 2. The PH risk-adjusted survival function is

$$\left(\overline{\mathrm{P}}_X\left(x | \alpha, \beta\right)\right)^{\rho} = \left(\frac{\beta}{x}\right)^{\tilde{\alpha}} = \overline{\mathrm{P}}_X\left(x | \tilde{\alpha}, \beta\right),$$

where $\tilde{\alpha} = \alpha\rho$. Consequently the risk-adjusted TCE premium calculation principle, if it exists, is given by

$$\tau_q^g\left(X\right) = \frac{\tilde{\alpha}\beta}{\left(\tilde{\alpha} - 1\right)} \frac{\overline{\mathrm{P}}_X\left(x_q | \tilde{\alpha} - 1, \beta\right)}{\overline{\mathrm{P}}_X\left(x_q | \tilde{\alpha}, \beta\right)}.$$

Figure 4 demonstrates the inverse relation between the PH coefficient ρ and the PH risk-adjusted TCE. It implies that smaller ρ parameters lead to higher risk-adjusted TCE values. A relatively small confidence parameter, $q = 0.7$, was chosen in order to emphasize again that a tail-based risk measure such as TCE can find a distribution with a lighter tail to be more dangerous than one having higher probabilities of rare events.

4 Risk-Adjusted TCE and TSD Premiums

Christofides (1998) conjectured that the PH premium principle reduces to the SD_X premium principle for distributions with constant skewness. Young (1999) showed that this conjecture is generally false

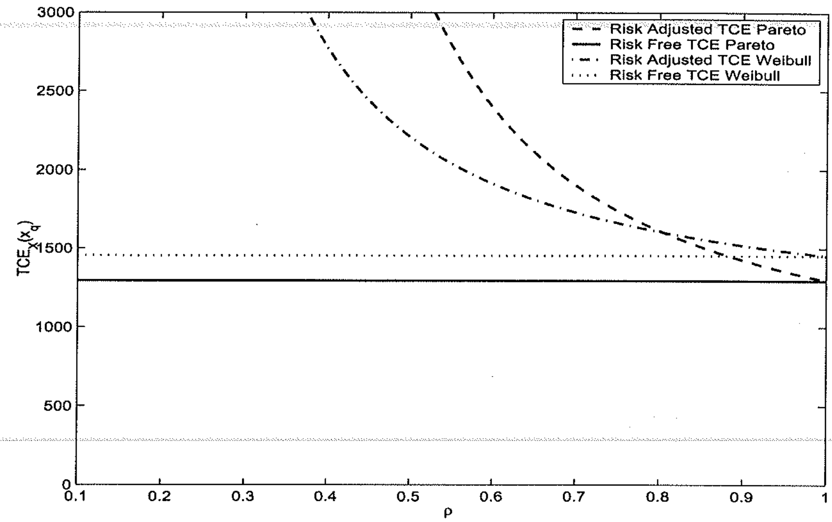

Figure 4: Distorted Versus Classical TCE for Pareto and Weibull Distributions with Mean 866 and Variance 463, and $q = 0.7$

except for location-scale families[2] and a few other families of distributions.

Let $\mathcal{L}$ denote the set of two parameter members of the location-scale family of distributions. We will now prove that the loading property of the PH risk-adjusted TCE (Theorem 2) allows it to be reduced to the tail standard deviation premium calculation principle for members of $\mathcal{L}$. Note that one parametric scale families can be considered members of $\mathcal{L}$ with $\mu = 0$. Therefore, the results of Theorem 3 also apply to them.

Theorem 3. *Given $F_X \in \mathcal{L}$ and a PH distortion function $g(u) = u^\rho$, $0 < \rho \leq 1$, the risk-adjusted TCE premium reduces to the TSD premium.*

Proof: As $F_X \in \mathcal{L}$, it is clear that $1 - g^*\left(\overline{F}_X^*(x)\right) = 1 - (g^*(g(\bar{F}_X)) \in \mathcal{L}$, and from equation (13) it immediately follows that $\tau_q^g(X)$ is scale and translation invariant. Therefore, if $X = \mu + \sigma Z$, then

[2]A random variable X with cdf $F_X(x|\mu, \sigma)$ is said to belong to a location-scale family with location parameter μ ($-\infty < \mu < \infty$) and scale parameter σ ($\sigma > 0$) if $F_X(x|\mu, \sigma) = F_X((x - \mu)/\sigma|0, 1)$. Examples of location-scale families include the normal, student-t, and logistic distributions.

$$\tau_q^g(X) = \tau_q^g(\mu + \sigma Z) = \mu + \sigma \tau_q^g(Z). \tag{16}$$

The TSD premium calculation principle for X is given by

$$\begin{aligned} \mathrm{TSD}_X\left(x_q\right) &= \mathbb{E}\left(X|X > x_q\right) + \lambda\sqrt{\mathbb{V}\mathrm{ar}\left(X|X > x_q\right)} \\ &= \mu + \sigma\left(\mathbb{E}\left(Z|Z > z_q\right) + \lambda\sqrt{\mathbb{V}\mathrm{ar}\left(Z|Z > z_q\right)}\right). \end{aligned} \tag{17}$$

Comparing the equations (17) and (16), the constant λ becomes

$$\lambda = \frac{\tau_q^g(Z) - \mathbb{E}\left(Z|Z > z_q\right)}{\sqrt{\mathbb{V}\mathrm{ar}\left(Z|Z > z_q\right)}}, \tag{18}$$

for some fixed q and ρ. It should be emphasized that Theorem 2 guarantees that (18) is non-negative, and hence TSD is risk-loaded. As λ is independent of μ and σ, the theorem is proved.

Note that when $q \to 0$ and therefore $z_q \to -\infty$, equation (18) reduces to the result of Young (1999), i.e.,

$$\lambda = \frac{W_g(Z) - \mathbb{E}(Z)}{\sqrt{\mathbb{V}\mathrm{ar}(Z)}}. \tag{19}$$

The coefficient λ, which actually determines the contribution of the risk loading, depends on q. Figure 5 implies that in the case of Pareto risks, λ is an increasing function in q and a decreasing function in ρ. In other words, a higher level of conservatism demands more significant risk loading, which seems rational.

5 Closing Comments

Though determining the risk loading for premiums is vitally important to actuaries, there is not single principle that is accepted to determine the appropriate risk load to charge. The most popular principle in use is the standard deviation (SD) premium calculation principle. We propose two basic tail-based premium calculation principles that are analogous to the SD principle: the tail standard deviation (TSD) premium calculation principle and the risk-adjusted tail conditional expectation (TCE) premium calculation principle. As both principles result in excess of the mean loss, they have a built-in risk loading.

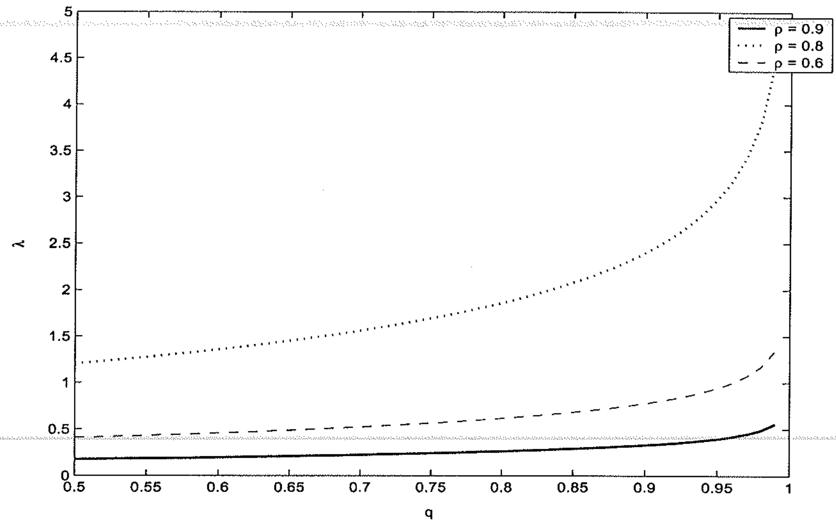

Figure 5: λ Versus q for Various Values of ρ in the Pareto Case

The premiums resulting from the risk-adjusted TCE and TSD principles depend on two parameters λ and q. What is known is that as $q \to 0$ the TSD premium converges to the well-known SD premium. Also, regulators may be interested in premiums where q is relatively large, thereby producing large premiums. Unfortunately there is little guidance on how one selects λ and q, thus further research is needed in this area.

An interesting ordering of risk appears in Table 1. Though the Pareto distribution has a heavier tail than the log-normal distribution (e.g., Klugman, Panjer and Willmot, 2004, Chapter 4.3), classical TCE, which is the TSD with $\lambda = 0$, orders these two distributions properly (i.e., charges a larger premium for the Pareto risk) for $q < 0.99$. On the other hand, the TSD produces a larger TV_X (steeper slope) for the lognormal than for the Pareto, which appears to be counter intuitive. Further research is needed.

Another problem is that the conditions of Theorem 1 are somewhat restrictive. For instance, one can, in theory, face a risk with an infinite variance, as in modeling catastrophic risks (Embrechts, Kluppelberg, and Mikosch, 1997), so that neither the tail standard deviation principle nor the standard deviation principle is applicable. For such risks finding a premium functional may be difficult, but may be a fruitful subject for future research.

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