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Wythoff's Game

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Wythoff’s Game, named after Willem Abraham Wythoff, is a well known game amongst number theorists. Dr. Wythoff, who received a Ph. D. in mathematics from the University of Amsterdam in 1898, described this game in these words: “The game is played by two persons. Two piles of counters are placed on a table, the number of each pile being arbitrary. The players play alternately and either take from one of the piles an arbitrary number of counters or from both piles an *equal* number. The player who takes up the last counter or counters, wins.” This game was previously known in China as *tsyan-shidzi* (“choosing stones”), but was reinvented by Wythoff. Wythoff published a full analysis of this game in 1907.

Nearly a half a century later, around 1960, Rufus P. Isaacs, a mathematician from Johns Hopkins University, created another description of this same game. Completely unaware of Wythoff’s game, Isaacs described the same game in terms of the moves of a chess queen allowed only to travel south, west or southwest on a chessboard. This game was often called “Queen’s Move” or “Cornering the Queen”. In this game the Queen is initially placed in the far right column or in the top row of the chessboard. The player who gets the queen to the lower left corner is the winner.

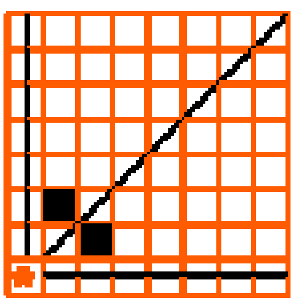
Under the name Last Biscuit, this same game is played by removing cookies from two jars, either from a single jar, or the same number from both jars. Also, this game has been named the Puppies and Kittens Game in *The Teacher's Circle: Finite Games* by Paul Zeitz.

To begin researching this game and its strategies, it only makes sense to play the game in small cases and look for patterns. To gain an understanding of the game I played the game of puppies and kittens with my six year old son. Given the rules, the game had to begin with some number of puppies and a different number of kittens. Mathematically speaking, if one starts with p (puppies) = 0, k (kittens) = 0, or p = k, then by rules of the game player 1 wins. Thus, I started with one puppy (p=1) and two kittens (k=2). In this case, no matter what, player 2 wins. If player 1 takes the puppy, player 2 will win by taking all the kittens. Just the same, if player 1 takes all the kittens, player 2 will win by taking the puppy. If player 1 takes one puppy and one kitten, player 2 wins by taking the remaining kitten. Likewise, if player 1 takes one kitten, player 2 will win by taking the remaining puppy and kitten. This same reasoning applies if you start with two puppies and one kitten; player two will win every time.

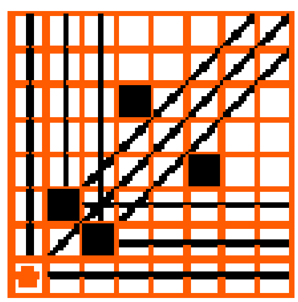
So what happens if p=3 and k=1? Strategically, player 1 is only one move away from p=2, k=1, which I just examined. Player 1 has the winning edge. Player 1 just needs to take one puppy, which forces player two to be the first player to play the p=2, k=1, which is a fatal situation to be in. With this strategy, player 1 can win for k=1 and any number of puppies greater than two. Similarly, player 1 can win for p=1 and any number of kittens greater than two.

This strategy also holds true when taking an equal number of puppies and kittens in a turn. If k=3, p=2 (or vise versa), player 1 would take one from each side forcing player two to be the first player to play k=2, p=1. Similarly, if k=4, p=3, player 1 would take two from each side, again forcing player two to be the first player to play k=2, p=1. In general, player 1 can win if you start k=1, p=k+1 (or p=1, k=p+1) when *k* is greater than or equal to two.

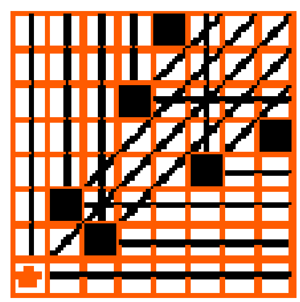
Therefore, in small cases of Puppies and Kittens, the position (2,1) or its reflection (1,2) is safe for the player that leaves the game in this state at the end of his or her turn. These safe positions seem more applicable to the "Queen's Move" on a chessboard in Isaacs's interpretation. Isaac constructed a winning strategy for cornering the queen on boards of unbounded size by letting the cell (square) in the lower left corner, the winning position, be the origin (0, 0) and working backwards. In this way, the chessboard can correspond to the first quadrant in a coordinate plane. If the queen is in the row, column or diagonal containing (0, 0), the person who has the move can win at once. This is represented on illustration A where three straight lines mark these cells.



A



B



C

The shaded cells in figure A represent "safe" positions. They are "safe" because if you occupy either one when it is your opponents turn, your opponent is forced to move to a cell that enables you to win on the next move. Notice that the shaded cells in figure A are at (2,1) and (1,2), where the starred cell is (0,0). These are the same "safe" positions identified when playing Puppies and Kittens.

Figure B represents the next step. By adding six more lines to mark all the rows, columns and diagonals containing the "safe" cells of (2,1) and (1,2), two more "safe" cells are shown at (3,5) and (5,3). If you occupy either one of these "safe" cells at the end of your turn, your opponent is forced to move, so that on your next move you can either win at once or move to another safe cell closer to (0,0).

Again repeating this procedure in figure C, a third set of "safe" cells is discovered at (7,4) or (4,7). From these illustrations, it is clear that Player 1 can always win by placing the queen on a "safe" cell. The strategy thereafter would be to move to another "safe" cell. If player 1 fails to move to a "safe" cell, player 2 can win by the same strategy. Continuing this process, additional "safe" cells can be located as illustrated in figure D.

The comparisons between Wythoff's game and Queen's Move are obvious. The games are mathematically equivalent even though the context is different. Each cell can be assigned a coordinate pair (x,y). These pairs correspond to the number of counters in piles x and y of Wythoff's game. When the queen moves west, pile x diminishes. When the queen moves south, pile y diminishes. When the queen moves diagonally or southwest, both piles diminish by the same amount. Moving the queen to cell (0,0) is the same as reducing both piles to zero. If the starting piles are "safe", player 2 should win. Player 1 has no choice but to leave an unsafe pair of piles, which the opponent can always reduce to a "safe" pair on his or her next move. If the game begins with unsafe numbers, the first player can always win by reducing the piles to a "safe" pair and continuing to play to "safe" pairs.

From the analysis of the game thus far, it is obvious that winning strategies can be determined by finding "safe" cells. Inevitably however, one must ask, can you mathematically compute where the safe cells or combinations are?

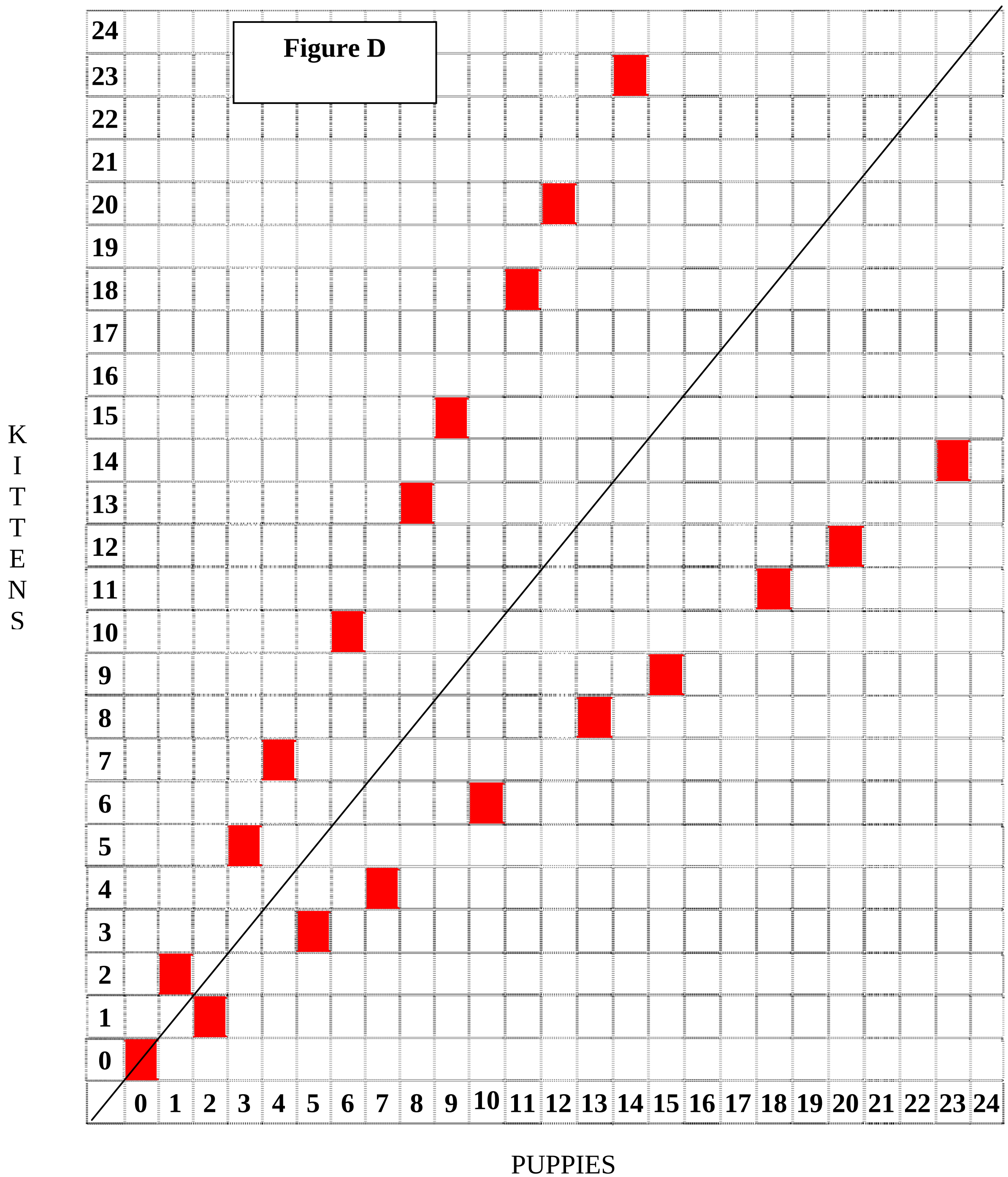


Figure D
KITTENS
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
PUPPIES

By looking at figure D, it can be observed that the order of the two numbers in a "safe" pair is not important. This condition says that if a position is safe, then its mirror image when reflected across the main diagonal is also safe, i.e. the collection of safe pairs is symmetric with respect to the main diagonal. Ordered pairs (*a*, *b*) and (*b*, *a*) are symmetric about the main diagonal; they have the same coordinate numbers, one pair being the reverse order of the others. Therefore, when arranging the safe pairs in a table, one only needs to focus on the ordered pairs (cells) above the diagonal that correspond to "safe" locations, with *n* representing position in the sequence, *A* representing the sequence of the top numbers of the safe pairs, and *B* representing the sequence of the bottom numbers of the safe pairs.

The Sequence of Safe Pairs from Figure D											
n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
A	0	1	3	4	6	8	9	11	12	14	…
B	0	2	5	7	10	13	15	18	20	23	…

In studying this table, many numerical patterns can be observed in the sequences. The first pattern being that each *B* value is the sum of it's *A* value and its position number *n*. The second pattern is when an *A* value is added to its *B* value, the sum is an *A* value that appears in the *A* sequence at a position number equal to *B*. For example, 4 + 7 = 11, and the 7^{th} number of the *A* sequence is 11. Using this idea, 6 + 10 = 16, so when *n* = 10, sequence *A* is 16.

The sequences can be generated by a recursive pattern. To begin assign 1 as the top number (the *A* value) of the first safe pair. Add this to its position number to obtain 2 as the bottom number. The top number of the next pair is the smallest integer not previously used, which, in this case, is 3. The corresponding bottom number is then 5, which is the sum of 3 and its position number, 2. For the top of the third pair, we need to find the smallest positive integer not yet used, which, in this case, is 4. The sum of 4 and 3 is then 7, which is the bottom number. The top of the 4th pair would be 6 (the only positive integer not used yet). Below it goes 10, the sum of 6 and 4. Continuing in this way will generate all safe pairs for Wythoff's game. It appears (and will later be demonstrated) that every positive integer must appear once and only once somewhere in the two sequences. Using this procedure, we can generate additional "safe" pairs which strategically would guarantee the next player a win in this game.

The Sequence of Safe Pairs in Wythoff's Game

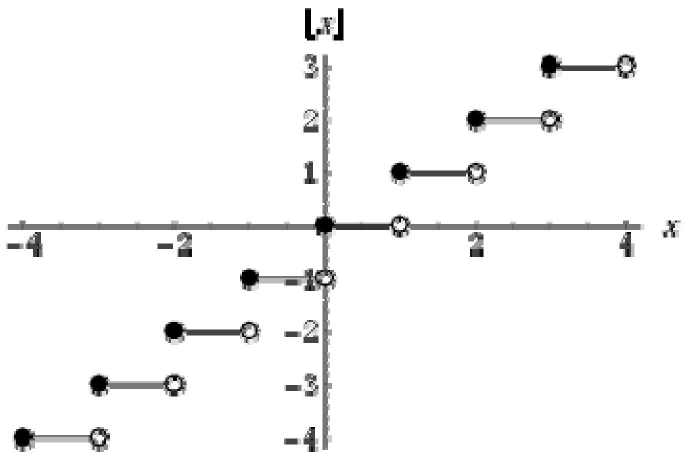
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
A	0	1	3	4	6	8	9	11	12	14	16	17	19	21	22	24	25	27	29	30	32	33	35	37	38
B	0	2	5	7	10	13	15	18	20	23	26	28	31	34	36	39	41	44	47	49	52	54	57	60	62

Being a mathematician, Wythoff was not satisfied with only the recursive patterns mentioned above. He is credited for generating the two sequences non-recursively. In 1907 he published a full analysis of the game where he wrote that he pulled this discovery "out of a hat". Wythoff discovered that the numbers in sequence *A* are simply multiples of the *golden ratio* rounded down to integers.

What is the golden ratio? The Golden Ratio, denoted by ϕ, is an irrational number. It can be defined as the number which is equal to its own reciprocal plus one: $\phi = 1/\phi + 1$. Multiplying both sides of this same equation by the Golden Ratio derives the interesting property that the square of the Golden Ratio is equal to the number itself plus one: $\phi^2 = \phi + 1$.

Since that equation can be written as $\phi^2 - \phi - 1 = 0$, the value of the Golden Ratio can be derived from the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. In this case $a = 1$, $b = -1$, and $c = -1$, which then implies that $\phi = \frac{1 \pm \sqrt{5}}{2}$. This is approximately either 1.618033989 or -0.618033989. The first number is usually regarded as the Golden Ratio.

Rounding down integers can be realized as the floor function. The floor function f(x)=$\lfloor x \rfloor$, also called the greatest integer function, gives the largest integer less than or equal to x. Formally this means: $\lfloor x \rfloor = \max \{n \in Z : n \leq x\}$. Therefore, $\lfloor 3.7 \rfloor = 3$, $\lfloor 11.5 \rfloor = 11$, and $\lfloor -1.4 \rfloor = -2$. A graph of the floor function is shown below.



Wythoff's discovery implies that the values of the sequence (*A*) are given by $\lfloor \phi \cdot n \rfloor$. Referring to the pattern noticed earlier that each *B* value is the sum of it's *A* value and its position number *n*, the sequence (*B*) is given by $\lfloor \phi \cdot n \rfloor + n$. A table representing $\phi \cdot n$, $\lfloor \phi \cdot n \rfloor$ and $\lfloor \phi \cdot n \rfloor + n$ is below.

The Sequence of Safe Pairs in Wythoff's Game											
n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**…**	n^{th}
$\phi \cdot n$	0	1.618	3.236	4.854	6.472	8.090	9.708	11.326	12.944	…	$\phi \cdot n$
$\lfloor \phi \cdot n \rfloor$	0	1	3	4	6	8	9	11	12	…	$\lfloor \phi \cdot n \rfloor$
$\lfloor \phi \cdot n \rfloor + n$	0	2	5	7	10	13	15	18	20	…	$\lfloor \phi \cdot n \rfloor + n$

It has also been discovered that the two Wythoff sequences are complementary Beatty sequences. The Beatty Theorem states:

If α and β are irrational and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then

$\{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \ldots\}$ and $\{\lfloor \beta \rfloor, \lfloor 2\beta \rfloor, \lfloor 3\beta \rfloor, \ldots\}$ are disjoint and their union is all integers.

In Wythoff's game, α represents ϕ and β represent ϕ^2. We know ϕ is irrational, and since $\phi^2 = \phi + 1$, ϕ^2 is also irrational. Furthermore, we need to show that $\frac{1}{\phi} + \frac{1}{\phi^2} = 1$. This is true because $\phi + 1 = \phi^2$ as stated by the property of squaring the golden ratio. By dividing this equation by ϕ^2, we see that $\frac{\phi + 1 = \phi^2}{\phi^2} = \frac{1}{\phi} + \frac{1}{\phi^2} = 1$.

The conclusion of Beatty's Theorem indicates that the multiples of α (or in this case ϕ) rounded down form one sequence of positive integers and multiples of β (or in this case ϕ^2) rounded down form another sequence of positive integers. These sequences have no terms in common, in other words they have no overlap. For example, the number 3 cannot appear in both sequences.

Beatty's Theorem supports the pattern noticed earlier where every positive integer must appear once and only once somewhere in the two sequences.

Using the notation in Beatty's theorem the n^{th} safe combination is $(\lfloor \phi \cdot n \rfloor, \lfloor \phi^2 n \rfloor)$. This is represented in a table below.

The Sequence of Safe Pairs in Wythoff's Game using Beatty's Theorem											
n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**…**	n^{th}
$\phi \cdot n$	0	1.618	3.236	4.854	6.472	8.090	9.708	11.326	12.944	…	$\phi \cdot n$
$\lfloor \phi \cdot n \rfloor$	0	1	3	4	6	8	9	11	12	…	$\lfloor \phi \cdot n \rfloor$
$\phi^2 n$	0	2.618	5.236	7.854	10.472	13.090	15.708	18.326	20.944	…	$\phi^2 n$
$\lfloor \phi^2 n \rfloor$	0	2	5	7	10	13	15	18	20	…	$\lfloor \phi^2 n \rfloor$

For these reasons, most publications of Wythoff's Game state the n^{th} safe combination more formally as $(x, \ x + n)$, where $x = \lfloor \phi \cdot n \rfloor$, with ϕ the golden ratio and $\lfloor x \rfloor$ the floor function. It is also true that $x + n = \lfloor \phi^2 n \rfloor$ so that the safe combinations are more formally written as $(\lfloor \phi \cdot n \rfloor, \lfloor \phi^2 n \rfloor)$.

Suggested Lesson Plan for Wythoff's Game

Is it possible to design a lesson for middle school students that involves learning about Wythoff's Game? While I am confident that middle school students can play the game, they would probably have to (using Wythoff's words) pull the mathematics "out of a hat". In most middle school curricula (where algebra isn't taught until 9^{th} grade), students are not introduced to the golden ratio, quadratics, or the floor function, but with an above average group of 8^{th} graders these topics could be meaningful. The mathematics of this game would simply be a discovery of patterns for students which could be extended to a chessboard and to the first quadrant of a coordinate plane

Students could begin a lesson over Wythoff's game the same way that I began research on this topic, by merely playing the game. The game could be introduced as "Puppies and Kittens" (or possibly a more appropriate middle school name like "Guppies and "Minnows"). After several minutes of play, the teacher could facilitate a discussion of what students have observed about the game; features such as starting the game with a different number of counters in each pile, and starting the game with no piles equal to zero if you don't want player 1 to win on the first turn, etc.

The teacher would then have students continue to play with questions such as these in mind:

- Is there a magic number of counters left on the table that guarantees a win?
- Can you find a magic number of counters that guarantees obtaining the first magic number?
- Does going first guarantee winning?

- Does taking a particular number of counters from the large group guarantee a win? From the small group? From both groups?
- What are some strategies for winning?

After playing for an adequate length of time to discover patterns, the teacher would again lead a group discussion regarding the students' observations. Hopefully at this point, students have noticed "magic numbers" such as (2,1) and/or (3,5) etc.

At this point, the teacher could introduce the same game as "Cornering the Queen". The title "Cornering the Queen" gives the students a better visual of the game than "Queen's Move", and this venue also makes the strategy easier to visualize, allowing students to map out the "safe" or "magic" combinations. Again, the teacher could lead a discussion in how "Cornering the Queen" is really the same as "Puppies and Kittens". The teacher could plot the "magic numbers" discovered by the students on a chessboard, have students demonstrate how and why these numbers are "safe" or "magic", and then have students extend the chessboard to the first quadrant of a coordinate plane to determine additional safe combinations. Hopefully students would begin to notice patterns and symmetry on their graphs.

As a whole group, the class could then make a table of the safe combinations that were discovered. Students could then be assigned (as homework) the task of studying the table, journaling about any noticed numerical patterns, and extending the table with additional safe/magic combinations using their pattern rules. The teacher could have students share their patterns the next day.

At this point, depending on the ability of the group, the teacher could share the pattern of multiples of the golden ratio, etc. Having students discover the "safe" combinations of the game is developmentally appropriate for all middle school students. This activity would be a fun and engaging method in meeting Nebraska's mathematics standard on algebraic patterns and rules. Standard 8.6.3 states: *"By the end of 8th grade, students will describe and represent relations using tables, graphs and rules."* Students will definitely recognize and describe patterns when deriving the mathematics of Wythoff's game.

Overall, Wythoff's Game is a fun and intriguing game that can be enjoyed by all ages. Whether playing under the title "Puppies and Kittens", "Queen's Move" or Wythoff's Game, William A. Wythoff is credited for the original invention of the theory behind this entertaining game.

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