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Class Notes for Math 871: General Topology, Instructor Jamie Radcliffe

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Topics include: Topological space and continuous functions (bases, the product topology, the box topology, the subspace topology, the quotient topology, the metric topology), connectedness (path connected, locally connected), compactness, completeness, countability, filters, and the fundamental group.

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1 Topological Spaces and Continuous Functions

Topology is the axiomatic study of continuity. We want to study the continuity of functions to and from the spaces $\mathbb{C}, \mathbb{R}^n, C[0,1] = \{f : [0,1] \to \mathbb{R} | f \text{ is continuous}\}, \{0,1\}^{\mathbb{N}}$, the collection of all infinite sequences of 0s and 1s, and $\mathcal{H} = \{(x_i)_1^\infty : x_i \in \mathbb{R}, \sum x_i^2 < \infty\}$, a Hilbert space.

Definition 1.1. *A subset A of $\mathbb{R}^2$ is **open** if for all $x \in A$ there exists $\epsilon > 0$ such that $|y - x| < \epsilon$ implies $y \in A$.*

In particular, the open balls $B_\epsilon(x) := \{y \in \mathbb{R}^2 : |y - x| < \epsilon\}$ are open. This definition of open lets us give a new definition of continuity, other than the basic $\epsilon - \delta$ definition. To do this, first we need to define the preimage of a function: If $f : X \to Y$ and $A \subseteq Y$, then the preimage is $f^{-1}(A) = \{x \in X | f(x) \in A\}$.

Lemma 1.2. *A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous if and only if the preimage of every open set is open.*

Proof. First suppose f is continuous. Consider an open $A \subseteq \mathbb{R}^2$. We want to show $f^{-1}(A)$ is open. Let $x \in f^{-1}(A)$ (if $f^{-1}(A) = \emptyset$, done). As A is open, there exists $\epsilon > 0$ such that $B_\epsilon(f(x)) \subseteq A$. By continuity, there exists $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$, that is, $f(y) \in B_\epsilon(f(x)) \subseteq A$. Thus $y \in f^{-1}(A)$, which implies $B_\delta(x) \subseteq f^{-1}(A)$. Therefore, $f^{-1}(A)$ is open.

Now suppose the preimage of every open set is open. Note that for all x and for all ϵ that $B_\epsilon(x)$ is open. Let $x \in \mathbb{R}^2$ and $\epsilon > 0$ be given. By hypothesis, $f^{-1}(B_\epsilon(f(x))$ is open. So there exists δ such that $B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$, that is, for all y with $|y - x| < \delta$, we have $f(y) \in B_\epsilon(f(x))$, which says $|f(y) - f(x)| < \epsilon$. □

Properties of Open Sets in $\mathbb{R}^2$:

- Arbitrary unions of open sets are open.
- Finite intersections of open sets are open.

We want to abstract this notion to a more general setting. To do so, consider the following definition.

Definition 1.3. *Given a set X, a **topology** on X is a collection $\tau \in \mathcal{P}(X)$ such that*

- *For all $A \subseteq \tau$, we have $\cup_{O \in A} O \in \tau$.*
- *For all $O_1, O_2 \in \tau$, we have $O_1 \cap O_2 \in \tau$.*
- *$\emptyset, X \in \tau$.*

*Given a topology τ on X, we call the sets in τ **open** or $\tau -$ **open** and we call the pair (X, τ) a **topological space**.*

Examples. The following are topologies.

1. The usual topology on $\mathbb{R}^n$.
2. The discrete topology on X, where $\tau = \mathcal{P}(X)$.
3. The indiscrete topology on X, where $\tau = \{\emptyset, X\}$.
4. Let $X = C[0,1]$. Then we can define τ by saying A is open if and only if for all $f \in A$ there exists $\epsilon > 0$ such that for all $g \in X$ with $\sup_{x \in [0,1]} |f(x) - g(x)| < \epsilon$ we have $g \in A$.
5. The Zariski topology on $\mathbb{R}^2$, where $A \subset \mathbb{R}^2$ is open if there exists polynomials $f_1, ..., f_n \in C[x, y]$ such that $A = \{(x, y) | f_i(x, y) = 0, i = 1, ..., n\}^C$.
6. The cofinite topology on X, where $\tau = \{A : |A^C| < \infty\} \cup \{\emptyset\}$.

Examples. The following are continuous functions.

1. If $\mathbb{R}$ is the topological space of the set $\mathbb{R}$ with the usual topology, then $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for all $x \in \mathbb{R}$ and for all $\epsilon > 0$ there exists $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$.
2. If X is any topological space, then $id_X : X \to X$ defined by $x \mapsto x$ is continuous.

3. If X, Y are arbitrary topological spaces and $f : X \to Y$ is constant (that is, for all x, $x \mapsto y_0$), then f is continuous.

Lemma 1.4. *If $f : X \to Y, g : Y \to Z$ are continuous, then so is $g \circ f : X \to Z$.*

Proof. Given $A \in \tau_Z$, $(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A))$. By continuity of g, we have $g^{-1}(A) \in \tau_Y$ and by continuity of f, we have $f^{-1}(g^{-1}(A)) \in \tau_X$. $\square$

Definition 1.5. *A **homeomorphism** from X to Y is a bijection $f : X \to Y$ such that both f and f^{-1} are continuous.*

1.1 Bases

Definition 1.6. *If $\mathcal{B} \subset \mathcal{P}(X)$ is a collection of sets satisfying $X = \cup_{B \in \mathcal{B}} B$ and for all $A, B \in \mathcal{B}$ and all $x \in A \cap B$, there exists $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$ (that is, for all $A, B \in \mathcal{B}$, $A \cap B \in \tau_{\mathcal{B}}$), then **the topology generated by $\mathcal{B}$** is*

$$\tau_{\mathcal{B}} = \{A \subseteq X | \text{ for all } x \in A, \text{ there exists } B \in \mathcal{B} \text{ such that } x \in B \subseteq A\} = \{\cup_{B \in \mathcal{C}} B : \mathcal{C} \subset \mathcal{B}\}.$$

In particular, $\mathcal{B} \subseteq \tau_{\mathcal{B}}$. We call $\mathcal{B}$ a basis for $\tau_{\mathcal{B}}$.

Theorem 1.7. *$\tau_{\mathcal{B}}$ is a topology.*

Examples.

- The usual topology on $\mathbb{R}$ is generated by the basis $\{(x - \epsilon, x + \epsilon) | x \in \mathbb{R}, \epsilon > 0\} = \{(a, b) : a < b\}$.
- The discrete topology on X is generated by $\{\{x\} : x \in X\}$.
- Bases are NOT unique: If τ is a topology, then $\tau = \tau_\tau$.

Theorem 1.8. *If $f : X \to Y$ is a function and the topology on Y is generated by $\mathcal{B}$, then f is continuous if and only if $f^{-1}(B)$ is open for all $B \in \mathcal{B}$.*

Proof. We need only to prove the backward direction. So assume $f^{-1}(B)$ is open for all $B \in \mathcal{B}$. Consider $A \in \tau_{\mathcal{B}}$. Then $A = \cup_{B \in \mathcal{A}} B$ for some $\mathcal{A} \subseteq \mathcal{B}$. Now, $f^{-1}(A) = f^{-1}(\cup_{B \in \mathcal{A}} B) = \cup_{B \in \mathcal{A}} f^{-1}(B)$. As $f^{-1}(B)$ is open, the union is. $\square$

Definition 1.9. *If $\mathcal{S} \subseteq \mathcal{P}(X)$, then **the topology generated by $\mathcal{X}$ as a subbasis** is the topology*

$$\{\textit{arbitrary unions of finite intersections of sets in } \mathcal{S}\}$$

with basis $\{S_1 \cap \cdots \cap S_n | n \geq 0, S_i \in \mathcal{S}, i = 1, .., n\}$. [Note: This is a topology, if we consider $\cap \emptyset = X$].

Theorem 1.10. *If $f : X \to Y$ and a topology on Y is generated by a subbasis $\mathcal{S}$, then f is continuous if and only if $f^{-1}(S) \in \tau_X$ for all $S \in \mathcal{S}$.*

Proof. We need only prove the backward direction. So assume $f^{-1}(S) \in \tau_X$ for all $S \in \mathcal{S}$. If $A = S_1 \cap \cdots \cap S_n$ is a basic open set, then $f^{-1}(A) = \cap_{i=1}^n f^{-1}(S_i) \in \tau_X$. Thus $f^{-1}(A) \in \tau_X$ for all basic open sets A. Thus f is continuous. $\square$

1.2 Product Topology

Suppose X_1 and X_2 are topological spaces. Then $X_1 \times X_2 = \{(x, y) | x \in X_1, y \in X_2\}$. The product topology on $X_1 \times X_2$ has basis $\mathcal{B}_{X_1 \times X_2} = \{U \times V : U \in \tau_{X_1}, V \in \tau_{X_2}\}$.

Example. In $\mathbb{R}^2$, we have two bases: $\mathcal{B}_1 = \{B_\epsilon(x) : x \in \mathbb{R}^2, \epsilon > 0\}$ and $\mathcal{B}_2 = \{U \times V : U, V \text{ are open in } \mathbb{R}\}$. In fact, $\tau_{\mathcal{B}_1} = \tau_{\mathcal{B}_2}$. To prove this fact, note that $\mathcal{B}_1 \subseteq \tau_{\mathcal{B}_2}$ and vice versa.

Theorem 1.11. *$\mathcal{B}_{X_1 \times X_2}$ is a basis.*

Proof. $X_1 \times X_2 = X_1 \times X_2$, were $X_1 \in \tau_{X_1}, X_2 \in \tau_{X_2}$. Thus $X_1 \times X_2 \in \mathcal{B}_{X_1 \times X_2}$. Note that $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}_{X_1 \times X_2}$. $\square$

Lemma 1.12. *If τ is a topology on X and $\mathcal{B} \subseteq \tau$, then $\mathcal{B}$ is a basis for τ if and only if*

1. $\cup_{B \in \mathcal{B}} B \supseteq X$

2. *For all $A \in \tau$ and for all $x \in A$, there exists $B \in \mathcal{B}$ with $x \in B \subseteq A$. (Equivalently, for all $A \in \tau$, $A = \cup_{B \in \mathcal{C}} B$ where $\mathcal{C} \subseteq \mathcal{B}$.*

Proof. Note that for $B, B' \in \mathcal{B}$, $B \cap B' \in \tau$. Hence, for all $x \in B \cap B'$, there exists $C \in \mathcal{B}$ with $x \in C \subseteq B \cap B'$. So $\mathcal{B}$ is a basis for some topology. Since $\mathcal{B} \subseteq \tau$, any union of a subfamily of $\mathcal{B}$ belongs to τ. Thus $\tau_{\mathcal{B}} \subseteq \tau$. Now, condition (2) says $A \subseteq \tau$ implies $A \subseteq \tau_{\mathcal{B}}$. Thus $\tau = \tau_{\mathcal{B}}$. $\square$

Lemma 1.13. *If $\mathcal{B}_1, \mathcal{B}_2$ are bases for the topologies on X_1, X_2, then $\mathcal{B}_{1\times 2} = \{B_1 \times B_2 | B_i \in \mathcal{B}_i\}$ is a basis for $\tau_{X_1 \times X_2}$.*

Proof. Certainly $\mathcal{B}_{1\times 2} \subseteq \mathcal{B}_{X_1 \times X_2} \subseteq \tau_{X_1 \times X_2}$ and $X_1 \times X_2 = (\cup_{B \in \mathcal{B}_1} B) \times (\cup_{B' \in \mathcal{B}_2} B' = \cup B \times B' = \cup_{B \in \mathcal{B}_{1\times 2}} B$. Now suppose $x \in A \in \tau_{X_1 \times X_2}$. Then if $x = (x_1, x_2)$, we see there exists $U_i \in \tau_{X_i}$ such that $(x_1, x_2) \in U_1 \times U_2 \subseteq A$. Since $\mathcal{B}_i$ are bases, there exists $B_i \in \mathcal{B}_i$ such that $x_i \in B_i \subseteq U_i$. Thus $x \in B_1 \times B_2 \subseteq U_1 \times U_2 \subseteq A$. $\square$

Definition 1.14. *The* ***projection maps*** *are defined as $\pi_i : X_1 \times X_2 \to X_i$ where $(x_1, x_2) \mapsto x_i$ for $i = 1, 2$.*

Note π_i is continuous since, for $A \in \tau_{X_1}$, $\pi_1^{-1}(A) = A \times X_2 \in \mathcal{B}_{X_1 \times X_2} \in \tau_{X_1 \times X_2}$ and similarly for π_2. Also, a function $f : Z \to X \times Y$ is continuous if and only if $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

1.3 The Subspace Topology

If X is a topological space and $Y \subseteq X$, then we define the topology on Y by $\tau_Y = \{Y \cap A : A \in \tau_X\}$.

Lemma 1.15. *τ_Y is a topology.*

Examples.

1. Consider $\mathbb{Q} \subseteq \mathbb{R}$. Then $A \subseteq \mathbb{Q}$ is open if $A = A' \cap \mathbb{Q}$ for some open A' in $\mathbb{R}$.

2. Let $Y := [0, 1) \cup (1, 2]$. Then $[0, 1)$ is open in Y. In fact, $[0, 1) = Y \cap (-\infty, 1)$. Also note that Y can be partitioned into two open sets: $[0, 1)$ and $(1, 2]$.

3. $L = \{\frac{1}{n}\}_{n \geq 1} \cup \{0\} \subseteq \mathbb{R}$. Each $\{\frac{1}{n}\}$ is open and each open set around 0 contains $\{\frac{1}{n}\}_{n \geq n_0}$ for some n_0. Notice if $f : L \to \mathbb{R}$ is defined by $\frac{1}{n} \to f_n$ and $0 \mapsto f_0$, then f is continuous if and only if $f_n \to f_0$ as $n \to \infty$.

Remark. Consider the space $\mathbb{N} \cup \{\infty\}$ where the topology has basis $\{\{n\}\}_{n \in \mathbb{N}} \cup \{[n, \infty]\}_{n \in \mathbb{N}}$. Here, we say a sequence (x_n) in a topological space X converges to $\{x\}$ if $f : \mathbb{N} \cup \{\infty\} \to X$ defined by $n \mapsto x_n$ and $\infty \mapsto x$ is continuous.

Lemma 1.16. *If $\mathcal{B}$ is a basis on τ_X and $Y \subseteq X$, then $\{B \cap Y : B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.*

Definition 1.17. *The* ***inclusion map*** *is defined as $i : Y \to X$ where $y \mapsto y$.*

Note. The inclusion map is continuous as for all $A \subseteq X$, $i^{-1}(A) = A \cap Y$. Also, if $f : Z \to Y$ is continuous, then so is $i \circ f : Z \to X$.

Theorem 1.18. *A function $f : Z \to Y$ is continuous if and only if $g = i \circ f : Z \to X$ is continuous (where $i : Y \to X$ is the inclusion map).*

Proof. The forward direction is clear by the continuity of compositions. So suppose g is continuous. Let $A = Y \cap A' \in \tau_Y$ for some $A' \in \tau_X$. Then $f^{-1}(A) = f^{-1}(A' \cap Y) = g^{-1}(A')$. By hypothesis, $f^{-1}(A)$ is open. $\square$

Theorem 1.19.

1. *If $f : X \to Y$ is continuous, then so is $f|_A : A \to Y$ for all $A \subseteq X$.*

2. *If $f : X \to Y$ and $f(x) \subseteq Z \subseteq Y$, then $g : X \to Z$ defined by $x \mapsto f(x)$ is continuous if and only if f is continuous.*

1.4 The Quotient Topology

If $\sim$ is an equivalence relation on a topological space X, then $X/\sim = \{[x] : x \in X\}$ and the **quotient topology** on $X/\sim$ is defined to be $\tau_{X/\sim} = \{A \subseteq X/\sim \mid q^{-1}(A) \text{ is open in } X\}$ where $q : X \to X/\sim$ defined by $p \mapsto [p]$ is the quotient map.

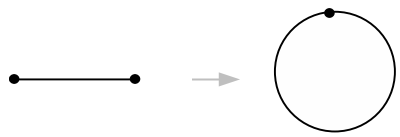
Theorem 1.20. *The quotient topology is in fact a topology. Moreover, $q : X \to X/\sim$ is continuous and a function $f : X/\sim \to Z$ is continuous if and only if $f \circ q$ is continuous.*

Proof. To prove the quotient topology is a topology, note that $q^{-1}(\emptyset) = \emptyset \in \tau_X, q^{-1}(X/\sim) = X \in \tau_X$, for all $A_1, A_2 \in X/\sim$, $q^{-1}(A_1 \cap A_2) = q^{-1}(A_1) \cap q^{-1}(A_2) \in \tau_X$, and $q^{-1}(\cup_{A \in \mathcal{A}} A) = \cup_{A \in \mathcal{A}} q^{-1}(A) \in \tau_X$. For the moreover statement, it is clear that q is continuous and that f continuous implies $f \circ q$ is. So suppose $f \circ q$ is continuous and let $A \in \tau_Z$. Note that $f^{-1}(A)$ is open if and only if $q^{-1}f^{-1}(A)$ is open and of course $q^{-1}f^{-1}(A) = (f \circ q)^{-1}(A)$, which is open. $\square$

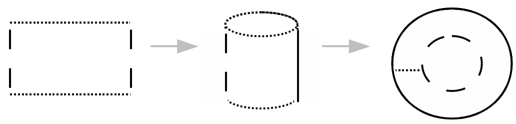
Examples. The following are quotient topologies.

1. $[0, 1]/(0 \sim 1)$

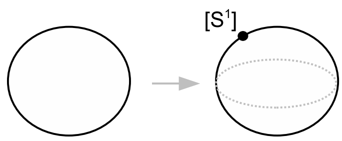
2. $\mathbb{R}/\sim$ where $x \sim y$ if and only if $x - y \in \mathbb{Z}$. This is often written as $\mathbb{R}/\mathbb{Z}$. This is in fact homeomorphic to $[0, 1]/(0 \sim 1)$ and can be thought of as a circle.



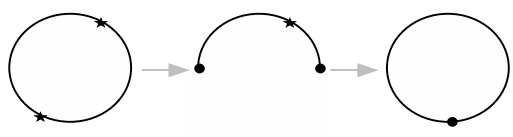
3. $\mathbb{R}^2/\mathbb{Z}^2$, defined similarly to the above. By identifying the top and bottom of the unit square and the two sides, we see that this can be thought of as a torus.



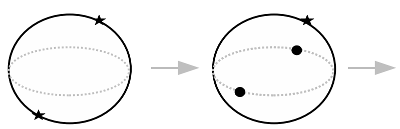
4. Let $D^2 = \{x \in \mathbb{R}^2 : |x| \leq 1\}$ and $\sim$ be generated by $x \sim y$ if and only if $|x| = |y| = 1$. Then $D^2/S^1 := D^2/\sim \cong S^2$.



5. $S^1/(x \sim -x) \cong S^1$.



6. $S^2/(x \sim -x)$ is known as a cross-cap.



1.5 The Metric Topology

Definition 1.21. *A **metric** on a set X is a function $d : X \times X \to \mathbb{R}_{x\geq 0}$ satisfying*

1. *For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.*
2. *For all $x, y \in X$, $d(x, y) = d(y, x)$.*
3. *For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.*

If d is a metric on X, then the metric topology τ_d is the topology with basis $\{B_\epsilon(x)\}_{x\in X,\epsilon>0}$ where $B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}$.

1.6 Topology from the Other Side

Definition 1.22. *A set C in a topological space X is **closed** if C^C is open.*

Facts.

1. $\emptyset, X$ are closed in X.
2. Finite unions of closed sets are closed.
3. Arbitrary intersections of closed sets are closed.

Theorem 1.23. *If $Y \subseteq X$, then C is closed in Y if and only if $C = D \cap Y$ for some D closed in X.*

Proof. Suppose C is closed in Y. Then $C^C = Y \setminus C$ is open in Y, which says $Y \setminus C = Y \cap A$ for some A open in X. Thus $C = Y \cap (X \setminus A) = Y \cap A^C$ where A^C is closed. On the other hand, if $C = Y \cap D$ where D is closed, then $Y \setminus C = Y \cap D^C$ where D^C is open in X. Thus $Y \setminus C$ is open in Y, which says C is closed in Y. □

Definition 1.24. *A **neighborhood** of a point x in a topological space X is an open set A with $x \in A$.*

Definition 1.25. *The **closure** of a set $A \subseteq X$ is the set $\overline{A} = \{x :$ every neighborhood of x meets $A\}$. Equivalently, $\overline{A}$ is the smallest closed set containing A.*

Note. The smallest closed set containing A is clearly $\cap_{C\in\mathcal{C}}C$ where $\mathcal{C}$ is the set of closed sets containing A.

Theorem 1.26. *The two definitions of closure are indeed equivalent.*

Proof. Note $x \in \cap_{\mathcal{C}}C$ if and only if there does not exist closed C with $A \subseteq C$ but $x \notin C$, which is if and only if there doest not exist an open U such that $x \in Y$ but $A \cap Y = \emptyset$, that is, every neighborhood of x meets A. □

We can similarly define the **interior** of a set A to be $A^O = \{x|$ some neighborhood of x is contained in $A\}$. Equivalently, A^O is the largest open set contained in A, namely $\cup_{U \text{ open}, U\subseteq A}U$.

Note. $(\overline{A})^C = (A^C)^O$.

Theorem 1.27. *If $A \subseteq Y \subseteq X$, then $\overline{A}^Y = (\overline{A}^X) \cap Y$, where $\overline{A}^Y$ denotes the closure of A in Y.*

Proof. Note that $\overline{A}^Y = \cap_{C\subseteq Y \text{ closed}, A\subseteq C}C = \cap_{D\subseteq X \text{ closed}, A\subseteq D}D \cap Y = (\cap D) \cap Y = (\overline{A}^X) \cap Y$. □

Definition 1.28. *We say x is a **limit point** of A if every neighborhood of x meets $A \setminus \{x\}$. Let $A' = \{$limit points of $A\}$.*

Example. Let $A = ((0, 1) \cap \mathbb{Q}) \cup \{5, -1\}$. Notice $\frac{1}{2} \in A \cap A', 0 \in A' \setminus A, 5 \in A \setminus A'$. Clearly, $A' \subseteq \overline{A}$. Indeed, $\overline{A} = A \cup A'$.

Note. A is closed if and only if $A' \subseteq A$ which is if and only if $\overline{A} = A$.

Theorem 1.29. *If $f : X \to Y$ is a map between topological spaces, then TFAE*

1. *f is continuous*
2. *For all $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$*
3. *For all closed $C \subseteq Y$, $f^{-1}(C)$ is closed in X.*

4. For all $x \in X$ and for all neighborhoods U of $f(x)$, there exists a neighborhood V of x such that $f(V) \subseteq U$.

Proof. (1⇒ 2): Suppose f is continuous, but $x \in f(\overline{A}) \setminus \overline{f(A)}$. Then there exists an open neighborhood U of x with $U \cap f(A) = \emptyset$. Now $f^{-1}(U)$ is open. As $x \in f(\overline{A})$, there exists $y \in \overline{A}$ such that $f(y) = x$. But $y \in f^{-1}(U)$, which implies $f^{-1}(U)$ is a neighborhood of y disjoint from A, contradiction.

(2⇒4): Consider $x \in X$ with $f(x) = y$ and U a neighborhood of y. We want to show there exists a neighborhood V of x with $f(V) \subseteq U$. Let $A = f^{-1}(U^C)$. We know $f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{U^C} = U^C$. In particular, $x \notin \overline{A}$. So there exists a neighborhood V of x with $V \cap A = \emptyset$, that is $f(V) \subseteq U$.

(4⇒2): Suppose (4) is true and $A \subseteq X$. Let $f(x) = y$ for some x. If $y \notin \overline{f(A)}$, then there exists a neighborhood U of y with $U \cap f(A) = \emptyset$ and by assumption there exists a neighborhood V of x such that $f(V) \subseteq U$. Then $V \cap A = \emptyset$ implies $x \notin \overline{A}$. Thus $y \notin f(\overline{A})$.

(1⇔3): Note that $f^{-1}(A^C) = [f^{-1}(A)]^C$. As A^C is closed if and only if A is open, this is clear.

(3⇒2): Suppose the preimage of a closed set is closed and let $A \subseteq X$. Then $C = f^{-1}(\overline{f(A)})$ is closed. Of course, $A \subseteq C$ implies $\overline{A} \subseteq C$. Thus $\overline{A} \subseteq f^{-1}(\overline{f(A)})$ which implies $f(\overline{A}) \subseteq \overline{f(A)}$.

(4⇒3): Suppose C is closed. We want to show $f^{-1}(C)$ is closed. We will show for all $x \notin f^{-1}(C)$ that there exists a neighborhood V of x with $V \cap f^{-1}(C) = \emptyset$. We have $f(x) \in C^C$ open. So there exists a neighborhood V of x with $f(V) \subseteq C^C$, that is, $C \subseteq f^{-1}(C^C) = f^{-1}(C)^C$, that is, $V \cap f^{-1}(C) = \emptyset$. □

Lemma 1.30 (**Pasting Lemma**). *Let $f : X \to Y$ and $X = \cup_1^n C_i$, where C_i closed or $X = \cup O_\alpha$, where O_α are open. Then f is continuous if and only if $f|_{C_i}$ is continuous for all i and f is continuous if and only if $f|_{O_\alpha}$ is continuous for all α.*

Proof. Let C be closed in Y. Then $f^{-1}(C) = \cup_{i=1}^n C_i \cap f^{-1}(C) = \cup_{i=1}^n (f|_{C_i})^{-1}(C)$. Now, $(f|_{C_i})^{-1}(C)$ is closed in C_i, that is, there exists D_i closed in X with $C_i \cap D_i = (f|_{C_i})^{-1}(C)$. Of course, C_i, D_i closed in X implies $C_i \cap D_i$ is closed in X. Thus $(f|_{C_i})^{-1}(C)$ is closed in X, which says $f^{-1}(C)$ is closed in X.

Now suppose A is open in Y. Then $f^{-1}(A) = \cup_\alpha f^{-1}(A) \cap O_\alpha = \cup_\alpha (f|_{O_\alpha})^{-1}(A)$. By hypothesis, $(f|_{O_\alpha})^{-1}(A)$ is open in O_α, that is, there exists U_α such that $(f|_{O_\alpha})^{-1}(A) = U_\alpha \cap O_\alpha$ for U_α open in X. Thus $f^{-1}(A)$ is a union $\cup_\alpha (U_\alpha \cap O_\alpha)$ of open sets and therefore is open. □

Note. The essence of the proof is that "closed in closed is closed" and "open in open is open."

Separation Conditions

Definition 1.31. *A topological space X is said to be* $\mathbf{T_1}$ *if $\{x\}$ is closed for all $x \in X$. To be more precise, for all $x, y \in X$ with $y \neq x$, there exists a neighborhood U of y such that $x \notin U$. We say X is* ***Hausdorff*** *if for all $x, y \in X$ with $x \neq y$, there exists a neighborhood U of y and a neighborhood V of x such that $U \cap V = \emptyset$.*

Clearly, Hausdorff implies T_1. However, T_1 does not imply Hausdorff.

1.7 Product Topology Returns

We talked about $X \times Y$ and how the generalization to $X_1 \times \cdots \times X_n$ is straightforward. However, what should the topology on $\mathbb{R}^{\mathbb{N}}$ look like? Note that unlike in the finite case the basis $\mathcal{B} = \{\prod_{i \in A} A_i : A_i \in \tau_{X_i}, i \in I\}$ and the subbasis $\mathcal{S} = \{\pi_{X_i}^{-1}(A_i) : i \in I, A_i \in \tau_{X_i}\}$ typically do not generate the same topology.

Example. Let $X = (\mathbb{R}^{\mathbb{N}}, \tau_{\mathcal{B}})$. Define $f : \mathbb{R} \to X$ by $t \mapsto (t, t, t, ...)$. Then f is not continuous. For, consider $A = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \cdots$, a basic open set. Clearly, $f^{-1}(A) = \{0\}$, which is not open. The topology $\tau_{\mathcal{B}}$ is called the **box topology**.

The example above shows that the box topology is really not what we want the product topology to be. So, we define the **product topology** to be that generated by $\mathcal{S}$. We will show that this is indeed the topology we want, but first we need to following lemma.

Lemma 1.32. *$\pi_j : \prod_{i\in I} X_i \to X_j$ is continuous.*

Proof. For A open in X_j, $\pi_j^{-1}(A)$ is a subbasic set and thus open. $\square$

Theorem 1.33. *If $f : Y \to \prod_{i\in I} X_i$ and $\prod_{i\in I} X_i$ has the product topology, then f is continuous if and only if $\pi_i \circ f$ is continuous for all $i \in I$.*

Proof. For the forward direction, note that $\pi_i \circ f$ is the composition of continuous functions and is thus continuous. For the backward direction, recall that f is continuous if and only if the preimage of every subbasic open set is open. Let $\pi_i^{-1}(A)$ be a subbasic open set. Then $f^{-1}(\pi_i^{-1}(A)) = (\pi_i \circ f)^{-1}(A)$, which is open by the continuity of $\pi_i \circ f$. $\square$

Theorem 1.34. *If X_i is Hausdorff for $i \in I$, then so is $\prod_{i\in I} X_i$.*

Proof. If $(x_i)_I \neq (y_i)_I$, then there exists j such that $x_j \neq y_j$. Let U, V be neighborhoods of x_j, y_j respectively in X_j such that $U \cap V = \emptyset$. Now, $x \in \pi_j^{-1}(U)$ and $y \in \pi_j^{-1}(V)$, both of which are open and $\pi_j^{-1}(U) \cap \pi_j^{-1}(V) = \emptyset$. $\square$

Theorem 1.35. *If $\prod_I A_i \subseteq \prod_I X_i$, then $\overline{\prod_I A_i} = \prod_I \overline{A_i}$.*

Proof. For $\subseteq$, note that $\pi_j(\overline{\prod_I A_i}) \subseteq \overline{\pi_j(\prod_I A_i)} = \overline{A_j}$ as π_j is continuous. For $\supseteq$, suppose $(y_i) \in \prod \overline{A_i}$, that is, $y_i \in \overline{A_i}$ for all i but $(y_i) \notin \overline{\prod A_i}$. Then there exists U open in $\prod X_i$ with $(y_i) \in U$ but $U \cap (\prod A_i) = \emptyset$. Then there exists a basic open set $B = \pi_{i_1}^{-1}(O_1) \cap \cdots \cap \pi_{i_n}^{-1}(O_n)$ where $(y_i) \in B \subseteq U$. As $y_{i_1} \in \overline{A_{i_1}}$, there exists $x_{i_1} \in O_{i_1} \cap A_{i_1}$. Similarly, there exists $x_{i_k} \in O_{i_k} \cap A_{i_k}$ for all $k \leq n$. Let $(x_i) \in \prod X_i$ be any sequence with those values in those coordinates and $x_i \in A_i$ for all other coordinates [note that if $A_i = \emptyset$, then $\overline{\prod A_i} = \overline{\emptyset} = \emptyset = \prod \overline{A_i}$]. Now $(x_i) \in (\prod A_i) \cap B = \emptyset$, a contradiction. $\square$

1.8 The Quotient Topology Returns

Let $f : X \twoheadrightarrow Y$ be any surjective map. Define an equivalence class $\sim_f$ on X by $x \sim_f z$ if and only if $f(x) = f(z)$. Then, there is a one to one correspondence between Y and $X/\sim_f$ defined by $y \leftrightarrow [x]$ where $x \in X$ is such that $f(x) = y$.

Definition 1.36. *We say f is a **quotient map** if the topology on Y is the "same" as the topology on $X/\sim_f$, that is, the canonical bijection is a homeomorphism, that is, for $A \subseteq Y$, A is open if and only if $f^{-1}(A)$ is open.*

Recall the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $(x, y) \mapsto x + y^2$ on homework 2. This is a quotient map.

Theorem 1.37. *$X/\sim_f \cong Y$ by the canonical bijection if and only if f is a quotient map.*

Theorem 1.38. *Let $f : X \twoheadrightarrow Y$ be a surjection.*

1. *f is a quotient map if and only if we have $A \subset Y$ is closed $\Leftrightarrow$ $f^{-1}(A)$ is closed.*

2. *f is a quotient map if and only if f is continuous and for any saturated open set A in X we have $f(A)$ is open.*

3. *If f is open and continuous, then it is a quotient map.*

4. *If f is closed and continuous, then it is a quotient map.*

Proof.
1. A is open in Y if and only if A^C is closed in Y which is if and only if $f^{-1}(A)^C = f^{-1}(A^C)$ is closed in Y, that is $f^{-1}(A)$ is open in X. Similarly, swap "open" and "closed."

2. For the forward direction, suppose f is a quotient map and A a saturated open set. Then $f(A)$ is open as $f^{-1}(f(A)) = A$ is open. For the backward direction if f maps saturated open sets to open sets and is continuous, then A open in Y implies $f^{-1}(A)$ is open in X by continuity and $f^{-1}(A)$ open implies $f(f^{-1}(A)) = A$ is open as $f^{-1}(A)$ saturated and open.

3. If f is open and continuous, then it is a quotient map by (2).

4. If f is closed and continuous and $A \subseteq Y$ has $f^{-1}(A)$ closed, then we conclude $f(f^{-1}(A)) = A$ is closed and we are done by part (1). $\square$

Example. The projection map $\pi_X : X \times Y \twoheadrightarrow X$ is continuous and open. Thus $X \times Y/\sim_{\pi_x} \cong X$.

Theorem 1.39. *If $q : X \to Y$ is a quotient map and $g : X \to Z$ is constant on $q^{-1}(\{y\})$ for all $y \in Y$, then there exists a unique $f : Y \to Z$ such that $f(y) = g(x)$ for all $x \in q^{-1}(\{y\})$, that is, $g = f \circ q$. Also, f is continuous if and only if g is continuous, and f is a quotient map if and only if g is a quotient map.*

$$\begin{tikzcd} X \arrow[d, "q"'] \arrow[dr, "g"] & \\ Y \arrow[r, dashed, "f"] & Z \end{tikzcd}$$

Proof. Certainly, f is well-defined since if $x, x' \in q^{-1}(\{y\})$, then $g(x) = g(x')$. We already know f is continuous if and only if g is continuous. So suppose g is a quotient map. If $A \subseteq Z$ and $f^{-1}(A)$ is open, then $q^{-1}(f^{-1}(A)) = g^{-1}(A)$ is open in X. As g is a quotient map, A is open. Similarly, if f is a quotient and $g^{-1}(A)$ is open in X, then $q^{-1}f^{-1}(A)$ is open and thus $f^{-1}(A)$ is open. Thus A is open. $\square$

1.9 The Metric Topology Returns

Examples.

1. The **discrete metric** on a set X is defined by $d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$ This induces the discrete topology as $\{x\} = B_{\frac{1}{2}}(x)$.
2. The **usual metric** on $\mathbb{R}^n$ is defined as $d_2(x,y) = \sqrt{\sum (x_i - y_i)^2}$.
3. The ℓ_1-**metric** on $\mathbb{R}^n$ is defined as $d_1(x,y) = \sum_1^n |x_i - y_i|$.
4. The $\ell_\infty-$**metric** on $\mathbb{R}^n$ is defined as $d_\infty(x,y) = \max_i |x_i - y_i|$.

   Claim. The above 3 metrics generate the same topology on $\mathbb{R}^n$.

   Proof. To prove that d_2 generates the d_1 metric, for example, it is enough to show that d_2 balls are open under the d_1 metric. It is easy to show $\frac{1}{n} d_1(x,y) \leq d_\infty(x,y) \leq d_2(x,y) \leq d_1(x,y)$. This quickly shows that in fact a ball under one metric is open under another metric.
5. $\mathcal{K} = \{\text{closed, bounded subsets of } \mathbb{R}^2\}$. Given $A \subseteq \mathbb{R}^2$, define $A_\epsilon = \{y | \text{ there exists } x \in A \text{ with } d_2(x,y) < \epsilon\}$. Define the **engulfing number** for A and B to be $e(A,B) = \inf\{\epsilon : B \subseteq A_\epsilon\}$. Note that $e(A,B) = \sup_{y \in B} \inf_{x \in A} d(x,y) = \sup_{y \in B} d(y, A)$. Now, we can define a metric $d_{\mathcal{K}}(A,B) = e(A,B) + e(B,A)$.

1.9.1 Metrics on $\mathbb{R}^{\mathbb{N}}$

Theorem 1.40. *If (X,d) is a metric space, then $\overline{d}(x,y) = \min(d(x,y), 1)$ is a metric on X and induces the same topology as d.*

Proof. Certainly $\overline{d}$ is symmetric and 0 if and only if $x = y$. So we need only show the triangle inequality holds. If $\overline{d}(x,z)$ or $\overline{d}(z,y)$ is 1, done. Otherwise,

$$\overline{d}(x,y) \leq d(x,y) \leq d(x,z) + d(z,y) = \overline{d}(x,z) + \overline{d}(z,y).$$

Thus $\overline{d}$ is a metric. To show it induces the same topology, we want to show every d ball is $\overline{d}-$open and vice versa. Of course, $\overline{d}-$balls are either $d-$balls or X, both of which are $d-$open. If $y \in B_\epsilon^{(d)}(x)$ and we set $\delta = \min(\frac{1}{2}, \epsilon - d(x,y))$, then $B_\delta^{(\overline{d})}(y) = B_\delta^{(d)}(y) \subseteq B_\epsilon^{(d)}(x)$. Thus every $d-$ball is $\overline{d}-$open. $\square$

A standard metric on X^J where (X,d) is a metric space is $d_\infty((x_j)_{j \in J}, (y_j)_{j \in J}) = \sup_j \overline{d}(x_j, y_j) \leq 1$. This metric is called the **uniform metric** and it induces the **uniform topology**.

Examples.

1. $f : \mathbb{R} \to (\mathbb{R}^{\mathbb{N}}, d_\infty)$ defined by $t \mapsto (t,t,t...)$ is continuous.

   Proof. If $0 < \epsilon < 1$, then $B_\epsilon((t,t,t,...,)) \supseteq (t - \frac{\epsilon}{2}, t + \frac{\epsilon}{2}) \times (t - \frac{\epsilon}{2}, t + \frac{\epsilon}{2}) \times \cdots$. So $f^{-1}(B_\epsilon(t,t,...)) \supseteq (t - \frac{\epsilon}{2}, t + \frac{\epsilon}{2})$. $\square$

2. $f : \mathbb{R} \to (\mathbb{R}^{\mathbb{N}}, d_\infty)$ defined by $t \mapsto (1, t, t^2, ...)$ is not continuous.

   Proof. We will show f is not continuous at 2. If $0 < \epsilon < 1$, then $f^{-1}(B_\epsilon(1, 2, 4, 8, ..)) \subseteq \mathbb{R} \cap (2-\epsilon, 2+\epsilon) \cap (t \mapsto t^2)^{-1}(4-\epsilon, 4+\epsilon) \cap \cdots \cap (t \mapsto t^n)^{-1}(2^n - \epsilon, 2^n + \epsilon) \cap \cdots = \{2\}$. As $\{2\}$ is in the preimage, we see it is not open. Thus the uniform topology is not the product topology. □

Let $Z = Y^X$ where Y is a bounded metric space and X a topological space. Then $Z = \{f : X \to Y\}$. A sequence of functions $(f_n)_{n>0}$ with $f_n : X \to Y$ converges to f in (Z, d_∞) if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $d_\infty(f_n, f) < \epsilon$, that is, for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ and $x \in X$ implies $d_\infty(f_n(x), f(x)) < \epsilon$, that is, $f_n \to f$ **uniformly**.

Theorem 1.41. *If $f_n \in Z$ and $f_n \xrightarrow{d_\infty} f$ and f_n is continuous for all n, then f is continuous.*

Proof. Suppose $x \in X$ and V a neighborhood of $f(x_0) = y_0$ in Y. WLOG, $V = B_\epsilon(y_0)$ for some $\epsilon > 0$. By uniform convergence, there exists N such that $n \geq N$ implies for all $x \in X$ $d(f_N(x), f(x)) < \frac{\epsilon}{3}$. By continuity, there exists a neighborhood U of x_0 such that $x \in Y$ implies $d(F_N(x), f_N(x_0)) < \frac{\epsilon}{3}$. So for $x \in U$, we see $d(f(x), f(x_0)) \leq d(f(x), f_n(x)) + d(f_n(x), f_n(x_0)) + d(f_n(x_0), f(x_0)) < 3\frac{\epsilon}{3} = \epsilon$. □

Fact. If we define the following metric on $\mathbb{R}^{\mathbb{N}}$: $D(x, y) = \sup_i \frac{\overline{d}(x_i, y_i)}{i}$, where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$, then D induces the product topology.

2 Connectedness

Definition 2.1. *X is **disconnected** if there exists open A, A^C in X with $A, A^C \neq \emptyset$. The pair $\{A, A^C\}$ is called a **disconnection**. Say X is **connected** if it has no disconnection.*

Theorem 2.2. *Let X be a topological space. TFAE*

1. *X is disconnected*
2. *There exists a continuous $f : X \twoheadrightarrow \{0, 1\}$ surjective, where $\{0, 1\}$ has the discrete topology.*
3. *There exists a continuous function $f : X \twoheadrightarrow Y$ surjective where Y is discrete and has ≥ 2 points.*
4. *There exists $A \subseteq X$ with $A \neq \emptyset$ and $BdA = \emptyset$.*

Proof. We have proved (4) $\Leftrightarrow$ (1) as $BdA = \emptyset$ if and only if A is closed and open, that is A, A^C are open. Also (2) $\Rightarrow$ (3) is clear. To prove (3) $\Rightarrow$ (2), pick an arbitrary $B \in Y$ with $B \neq \emptyset, Y$ and define $q : Y \to \{0, 1\}$ by $y \mapsto 1$ if $y \in B$ and $y \mapsto 0$ if $y \notin B$. This is continuous (as Y is discrete) and $q \circ f : X \twoheadrightarrow \{0, 1\}$ is surjective. To prove (1) $\Rightarrow$ (2), define $f : X \to \{0, 1\}$ by $x \mapsto 1$ if $x \in A$ and 0 if $x \in A^C$. Clearly f is continuous and surjective. Lastly, to prove (2) $\Rightarrow$ (1), note if $f : X \twoheadrightarrow \{0, 1\}$ is continuous, then $\{f^{-1}(\{0\}), f^{-1}(\{1\})\}$ is a disconnection. □

Theorem 2.3. *If $f : X \twoheadrightarrow Y$ is continuous and X is connected, then so is Y. [That is, the continuous image of a connected space is connected.]*

Proof. Suppose Y is disconnected. Then there exists $g : Y \twoheadrightarrow \{0, 1\}$. Of course, then $g \circ f : X \twoheadrightarrow \{0, 1\}$ is continuous and surjective, that is, X is disconnected. □

Definition 2.4. *$I \subseteq \mathbb{R}$ is an **interval** if and only if for all $x \leq y \leq z$ with $x, z \in I$, we have $y \in I$.*

Theorem 2.5. *$I \subseteq \mathbb{R}$ is connected if and only if I is an interval.*

Proof. For the forward direction, let $I \subseteq \mathbb{R}$ be connected. If there exists $a, b, c \in R$ with $a < c < b$, $a, b \in I$ but $c \notin I$, then $I \cap (-\infty, c)$ and $I \cap (c, \infty)$ is a disconnection of I, contradiction.

For the backward direction, suppose I is an interval and $I = B \dot{\cup} C$ where B, C are open in I with $x \in B, y \in C$ and WLOG $x < y$. Let $T = \{t : [x, t] \subseteq B\}$ and $z = \sup T$. Then $x \leq z \leq y$. If $z \in B$, then there exists $\epsilon > 0$ such that $(z - \epsilon, z + \epsilon) \subseteq B$ which implies $z \neq \sup T$, a contradiction. If $z \in C$, then there exists $\epsilon > 0$ such that $(z - \epsilon, z + \epsilon) \in C$ which again implies $z \neq \sup T$, a contradiction. □

Corollary 2.6 (Intermediate Value Theorem). *If $f : X \to \mathbb{R}$ is continuous, X is connected, and $a, b \in f(x)$ with $a < b$, then $[a, b] \subseteq f(X)$.*

Proof. We've prove that $f(X)$ is connected and thus $f(X)$ is an interval. By above theorem, done. $\square$

Notes.

- Subspaces of connected spaces are not necessarily connected. For example $\{0, 1\} \subseteq \mathbb{R}$.
- Quotients of connected spaces are connected (as $q : X \twoheadrightarrow X/\sim$ is continuous).

Lemma 2.7. *If $f : X \to \{0, 1\}$ is continuous and $Y \subseteq X$ is connected, then $f|_Y$ is constant.*

Lemma 2.8. *If $Y_\alpha \subseteq X$ is connected and $\cap_I Y_\alpha \neq \emptyset$, then $Y = \cup_I Y_\alpha$ is connected.*

Proof. Suppose $f : Y \to \{0, 1\}$ is continuous. We know $f|_{Y_\alpha}$ is constant for all α. If $\rho \in \cap_I Y_\alpha$, then $f|_{Y_\alpha} \equiv f(\rho)$. So f is constant. $\square$

Theorem 2.9. *If X, Y are connected, then so is $X \times Y$.*

Proof. If $X = \emptyset$, done. Note that for $x \in X, y \in Y$ that $\{x\} \times Y \cong Y$ and $X \times \{y\} \cong X$ are connected. Since they meet, their union is connected. Fix $y_0 \in Y$ and define $T_x = \{x\} \times Y \cup X \times \{y_0\}$. Each T_x is connected and $\cap T_x = X \times \{y_0\}$. Thus $\cup_{x \in X} T_x = X \times Y$ is connected. $\square$

Lemma 2.10. *If $Y \subseteq Z \subseteq \overline{Y}$ are subspaces of X and Y is connected, then so is Z. In particular, Y connected implies $\overline{Y}$ is connected.*

Proof. Suppose $f : Z \to \{0, 1\}$ is continuous. Then $f|_Y$ is continuous and hence constant. Say $f|_Y = 0$. Note $f(Z) = f(\overline{Y}^Z) \subseteq \overline{f(Y)} = \overline{\{0\}} = \{0\}$. $\square$

Theorem 2.11. *If X_α are connected, $\alpha \in I$, then so is $X = \prod_{\alpha \in I} X_\alpha$.*

Proof. If $X_\alpha = \emptyset$, then so is X and we are done. Otherwise, there exists $x = (x_\alpha)_I \in X$ (by the Axiom of Choice). For a finite subset K of I, define $X_K = \{y \in \prod X_\alpha : y_\alpha = x_\alpha, \alpha \notin K\}$. Hence $X_K \cong \prod_{\alpha \in K} X_\alpha$ is connected. Note $\{x\} \in \bigcap_{K \subseteq I, |K| < \infty} X_K$ and so $\bigcup_{K \subseteq I, |K| < \infty} X_K$ is connected.

> *Claim.* $\overline{\cup_{K \subseteq I, |K| < \infty} X_K} = \prod_{\alpha \in I} X_\alpha$.
>
> *Proof.* If $y \in \prod_I X_\alpha$ and N is a basic open neighborhood of y, then $N = \pi_{\alpha_1}^{-1}(O_1) \cap \cdots \cap \pi_{\alpha_n}^{-1}(O_n)$ for some $\alpha_1, ..., \alpha_N \in I, O_i \in \tau_{X_{\alpha_i}}$. Let $K = \{\alpha_1, ..., \alpha_n\}$ and define $z_\alpha = \begin{cases} y_\alpha & \text{if } \alpha \in K, \\ x_\alpha & \text{if } \alpha \notin K. \end{cases}$. Then $z = (z_\alpha) \in N \cap X_K$.
>
> So $y \in \overline{\cup X_K}$.

Thus $\prod X_\alpha$ is connected. $\square$

Definition 2.12. *Let X be a topological space. Define $x \sim y$ if there exists $C \subseteq X$ connected with $x, y \in C$. The* ***(connected) components*** *are defined to be the equivalence classes under $\sim$.*

Theorem 2.13. *$\sim$ is an equivalence relation. The components of X are connected and closed (but* not *necessarily open). If there are a finite number of components, then they are open. Components are maximal connected subspaces. Any connected subspace is contained in some component.*

Proof. If $x \sim y, x \sim z$ then there exists connected C_y, C_z with $x, y \in C_y$ and $x, z \in C_z$. Then $C_y \cup C_z$ is connected and contains y, z. Thus $y \sim z$. Now, $[x]$ is connected as for all $y \in [x]$ there exists C_y connected such that $x, y \in C_y$. So $[x] = \cup_{y \in [x]} C_y$ is connected. If $[x] \subseteq C$ and C is connected, then $z \in [x]$ for all $z \in C$. So $C = [x]$, that is, components are maximal connected sets. To show components are closed, note that $\overline{[x]}$ is connected and $[x] \subseteq \overline{[x]}$, thus they are equal. If the components of X are $C_1, C_2, ..., C_n$, then $C_i = (\cup_{j \neq i} C_j)^C$ is open. Lastly, if $x \in C$ is connected, then $C \subseteq [x]$. $\square$

Example. Let $X = \mathbb{Q}$. Note that $\mathbb{Q} \cap (-\infty, \frac{1}{\pi})$ and $\mathbb{Q} \cap (\frac{1}{\pi}, \infty)$ is a disconnected of any $A \subseteq \mathbb{Q}$ with $0, 1 \in A$.

2.1 Path Connected

Example. The topologists sine curve: Let $G := \{(x, \sin\frac{1}{x}) : x \in (0,1]\}, I = \{0\} \times [-1,1]$ and $S = G \cup I$. Then S is connected as it is the closure of G which is connected as it is the continuous image of the interval $[0,1]$.

Definition 2.14. *A space X is **path connected** if for all $x, y \in X$ there exists a **path** in X from x to y. A **path** from x to y is a continuous function $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.*

Lemma 2.15. *X path connected implies X is connected.*

Proof. Suppose $f : X \to \{0,1\}$ is continuous. Pick $x \in X$. If $y \in X$, there exists $\gamma : [0,1] \to X$ continuous with $\gamma(0) = x, \gamma(1) = y$. So $f \circ \gamma : [0,1] \to \{0,1\}$ is continuous and hence constant. Thus $f(x) = f\gamma(0) = f\gamma(1) = f(y)$. $\square$

Example. Consider the topologists sine curve again. We will show S is in fact not path connected. For suppose γ is a path rom $(0,0)$ to $(1, \sin 1)$. Let $t = \inf\{s : \pi_1(\gamma(r)) > 0 \text{ for all } r > s\}$. WLOG, $t = 0$ (otherwise, we can rescale as $t < 1$). So $\pi(\gamma(0)) = 0$ and $\pi(\gamma(s)) > 0$ for all $s > 0$. Let $\gamma(t) = (x(t), y(t))$ so that we have $x(0) = 0$ and $x(s) > 0$ for all $s > 0$. Since $x(t)$ is continuous, there exists $t_1 > 0$ such that $x(t_1) = \frac{2}{\pi}$, that is, $y(t_1) = 1$. Then there exists $t_2 < \frac{t_1}{2}$ such that $x(t_2) \in (\sin\frac{1}{z})^{-1}(-1)$, that is, $y(t_2) = -1$). Further, we can find $t_3 < \frac{t_2}{2}$ such that $y(t_3) = 1$. Inductively, there exists a sequence $t_n \to 0$ such that $y(t_n) = (-1)^{n+1}$. By continuity, $y(t_n) \to y(0)$, a contradiction as $(-1)^{n+1}$ does not converge.

Definition 2.16. *If X is a topological space, we consider the equivalence relation of being joined by a path. The **path components** of X are the equivalence classes.*

Fact. Path components are path connected and maximal path connected spaces. The continuous image of a path connected space is path connected.

Example. The path components of S are G and I, but G is not closed in S. Thus path components are not necessarily closed.

2.2 Locally connected

Definition 2.17. *A space is **locally connected** if connected open sets form a basis for the topology, that is, for all $O \in \tau$ and $x \in O$, there exists U such that $x \in U \subseteq O$ and U is connected.*

Example. $\mathbb{R}$ is locally connected, but S is not (for example $B_{\frac{1}{4}}(0,0)$ has no connected neighborhood of 0 in it).

Theorem 2.18. *X is locally connected if and only if for all O open in X every component of O is open.*

Proof. For the forward direction, suppose X is locally connected and O is open. Let C be a component of O. If $x \in C$, then by local connectivity there exists a connected neighborhood N with $x \in N \subseteq O$. Since N is connected, its contained in the component C, that is, $x \in N \subseteq C$. Thus C is open.

For the backward direction, if $x \in O$ is some open set, we want to find a connected open N with $x \in N \subseteq O$. Let N be the component of O containing x. $\square$

Corollary 2.19. *If X is locally connected, then components are open.*

Proof. Apply the previous theorem to the open set X. $\square$

One can also talk about **locally path connected**.

3 Compactness

Definition 3.1. *A space X is **compact** if for all $O \subsetneq \tau$ such that $\cup O = X$ (that is, for all open covers) there exists $O_1, .., O_n \in O$ such that $\cup_1^n O_i = X$.*

Examples.

1. $[0,1]$ is compact (proof later)

2. $\mathbb{R}$ is not compact as $\{(-n,n)\}$ is a cover with no finite subcover.

3. $(0,1)$ is not compact as $\{(\frac{1}{n}, 1-\frac{1}{n})\}$ is a cover with no finite subcover.

4. $\{0\} \cup \{\frac{1}{n}\} \subseteq \mathbb{R}$ is compact as any open set containing 0 contains $\{\frac{1}{n}\}_{n \geq n_0}$ for some n_0.

5. If X is finite, then it is compact.

Lemma 3.2. *If $Y \subseteq X$, then Y is compact if and only if for all $\mathcal{O} \subseteq \tau_X$ such that $\cup\mathcal{O} \supseteq Y$, there exists $O_1, ..., O_n \in \mathcal{O}$ such that $\cup_1^n O_i \supseteq Y$.*

Proof. For the backward direction, if $\mathcal{U} \subseteq \tau_Y$ has $\cup\mathcal{U} = Y$, then there exists $\mathcal{O} \subseteq \tau_X$ such that $\mathcal{U} = \{U \cap Y : U \in \mathcal{O}\}$ and $\cup\mathcal{O} \supseteq Y$. So, by hypothesis, there exists $O_1, ..., O_n \in \mathcal{O}$ such that $O_1 \cup \cdots \cup O_n \supseteq Y$. Then defining $U_i = O_i \cap Y$, we get $U_i \in \mathcal{U}$ and $\cup_1^n U_i = Y$. The forward direction is similar. □

Theorem 3.3. *If $f : X \twoheadrightarrow Y$ is continuous and X is compact, then so is Y.*

Proof. If $\mathcal{O} \subseteq \tau_Y$ has $\cup\mathcal{O} = Y$, define $\mathcal{U} = \{f^{-1}(O) : O \in \mathcal{O}\}$. Then $\cup U = X$ which implies there exists $O_1, ..., O_n$ such that $X = \cup_1^n f^{-1}(O_i)$. By surjectivity, $Y = \cup_1^n O_i$. □

Corollary 3.4. *Quotients of compact spaces are compact.*

Theorem 3.5. *$[a,b] \subseteq \mathbb{R}$ is compact.*

Proof. Let $\mathcal{O} \subseteq \tau_{\mathbb{R}}$ have $\cup\mathcal{O} \supseteq [a,b]$. Let $S = \{s : [a,s]$ has a finite subcover of $\mathcal{O}, s \leq b\}$. Then $s \neq \emptyset$ as $a \in S$ and S is bounded above by b. Let $c = \sup S$.

> *Claim.* $c > a$ (unless $b = a$).
>
> *Proof.* There exists $O \in \mathcal{O}$ such that $a \in O$ and $\epsilon > 0$ such that $(a - \epsilon, a + \epsilon) \subseteq O$. So $a + \frac{\epsilon}{2} \in S$.
>
> *Claim.* $c = b$.
>
> *Proof.* If not, then $c < b$. So there exists $O \in \mathcal{O}$ such that $c \in O$. Then there exists $\epsilon > 0$ such that $(c-\epsilon, c+\epsilon) \in O$ and $x \in S \cap (c-\epsilon, c]$. Then $[a,x]$ has a finite subcover $O_1, ..., O_n$. Now, $O \cup \cup_1^n O_i$ is a cover of $[a, c+\frac{\epsilon}{2}]$, a contradiction as c is an upper bound.
>
> *Claim.* $b \in S$.
>
> *Proof.* There exists $O \in \mathcal{O}$ and $\epsilon > 0$ such that $(b-\epsilon, b+\epsilon) \subseteq O \in \mathcal{O}$. Now there exists $x \in S \cap (b - epsilon, b]$. Then, $[a,x]$ has an open cover $O_1, ..., O_n$ which implies $\{O\} \cup \{O_i\}_1^n$ is a finite subcover for $[a,b]$. □

Theorem 3.6. *1. If K is a closed subspace of a compact set X, then K is compact.*

2. If K is a compact subspace of a Hausdorff space X, then its closed in X.

Proof. 1. If $\mathcal{O} \subseteq \tau_X$ has $\cup\mathcal{O} \supseteq K$, then $\mathcal{O} \cup \{K^C\} \subseteq \tau_X$ has union equal to X. So there exists $O_1, ..., O_n \in \mathcal{O} \cup \{K^C\}$ such that $\cup_1^n O_i = X$. Then $\cup(\{O_i\}_1^n \setminus \{K^C\}) \supseteq K$ and is a subcollection of $\mathcal{O}$.

2. Suppose $x \notin K$. Then for all $y \in K$, there exists O_y, U_y disjoint such that O_y is a neighborhood of x and U_y is a neighborhood of y. Then $\cup_y U_y \supseteq K$ and so there exists $y_1, ..., y_n \in K$ such that $\cup_1^n U_{y_i} \supseteq K$. Let $O = \cap_1^n O_{y_i}$. Then $O \cap (\cup_1^n U_{y_i}) = \emptyset$. In particular, $O \cap K = \emptyset$. Thus we have found a neighborhood of x that does not meet K. □

Lemma 3.7 (Tube Lemma). *If Y is compact and $N \subseteq X \times Y$ is open with $\{x_0\} \times Y \subseteq N$, then there exists a neighborhood of x_0 in X (say W) such that $W \times Y \subseteq N$.*

Proof. For all $y \in Y$ there exists $O_y \times U_y \subseteq N$ such that O_y is a neighborhood of $x_0 \in X$ and U_y is a neighborhood of $y \in Y$. Then $\{U_y\}_{y \in Y}$ is an open cover of Y, which implies there exists $y_1, ..., y_n$ such that $Y = \cup_1^n U_{y_i}$. Let $W = \cap_1^n O_{y_i}$. Then $W \times Y \subseteq \cup_1^n O_{y_i} \times U_{y_i} \subseteq N$. □

Theorem 3.8. *If X, Y are compact, then so is $X \times Y$.*

Proof. Suppose $\mathcal{O}$ is an open cover of $X \times Y$. Then for all $x \in X$, $\mathcal{O}$ is an open cover of $\{x\} \times Y$. So there exists n_x and $O_1^{(x)}, .., O_{n_x}^{(x)} \in \mathcal{O}$ such that $\{x\} \times Y \subseteq \cup_1^{n_x} O_i^{(x)}$. Then there exists W_x such that $W_x \times Y \subseteq \cup_1^{n_x} O_i^{(x)}$. By compactness of X, there exists $x_1, ..., x_m$ such that $X = \cup_1^m W_{x_j}$. Thus $X \times Y = (\cup_1^m W_{x_j}) \times Y = \cup_1^m W_{x_j} \times Y \subseteq \cup_1^m \cup_1^{n_{x_j}} O_i^{(x_j)}$. □

Corollary 3.9 (Heine-Borel). *$K \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. For the forward direction, as $\mathbb{R}^n$ is Hausdorff, K is closed. If K is unbounded, then $\cup_1^\infty B_n(\tilde{O})$ (where $\tilde{O}$ is the origin) is an open cover of K with no finite subcover.

For the backward direction, if K is bounded for some $B > 0$, then $K \subseteq [-B, B] \times \cdots [-B, B]$. The right hand side is compact. As K is closed, it must be compact as well. □

Corollary 3.10. *If $f : X \to \mathbb{R}$ is continuous and X compact, then f is bounded and attains its bounds.*

Proof. $f(X)$ compact in $\mathbb{R}$ implies $f(X)$ is closed and bounded. □

Remark. If τ_X is generated by a basis B and every cover by basic open sets has a finite subcover, then X is compact. Perhaps more astounding is the following.

Lemma 3.11 (Alexander's Subbasis Lemma). *If τ_X is generated by a subbasis $\mathcal{S}$ and every open cover by subbasic open sets has a finite subcover, then X is compact.*

Proof. If X is not compact, consider the set

$$\mathbf{O} = \{\mathcal{O} : \mathcal{O} \subseteq \tau \text{ is an open cover with no finite subcover}\}$$

ordered by inclusion. Suppose $\mathbf{C} \subseteq \mathbf{O}$ is a chain. Then $\mathcal{U} = \cup \mathbf{C}$ is an open cover with no finite subcover and is an upper bound. By Zorn's Lemma, $\mathbf{O}$ has a maximal element $\mathcal{M}$.

> *Claim.* $\mathcal{M} \cap \mathcal{S}$ is an open cover (and hence has a finite subcover, a contradiction).
>
> *Proof.* Suppose $x \notin \cup(\mathcal{M} \cap \mathcal{S})$. There exists $\mathcal{O} \in \mathcal{M}$ with $x \in \mathcal{O}$ and $S_1, ..., S_n \in \mathcal{S}$ such that $x \in S_1 \cap \cdots \cap S_n \subseteq \mathcal{O}$. Since $x \notin \cup(\mathcal{M} \cap \mathcal{S})$, we know $S_i \notin \mathcal{M}$ for $i = 1, ..., n$. By maximality, $\mathcal{M} \cup S_i$ has a finite subcover, call it $\mathcal{F}_i \cup \{S_i\}$ where $\mathcal{F}_i \subseteq \mathcal{M}$. Then $\cup \mathcal{F}_i \supseteq S_i^C$ which implies $\mathcal{F} = \cup_1^n \mathcal{F}_i$ covers everything except $S_1 \cap \cdots \cap S_n \subseteq \mathcal{O}$. So $\mathcal{F} \cup \{\mathcal{O}\} \subseteq \mathcal{M}$ is a finite subcover of $\mathcal{M}$, a contradiction.

Thus $\mathcal{M} \cap \mathcal{S}$ is an open cover and thus by hypothesis has a finite subcover. But then, $\mathcal{M}$ has a finite subcover, a contradiction. □

Corollary 3.12 (Tychonoff's Theorem). *If X_i for $i \in I$ are compact topological spaces, then $X = \prod_{i \in I} X_i$ is compact.*

Proof. Let $\mathcal{S} = \{\pi_i^{-1}(O) : O \in \tau_{X_i}, i \in I\}$ be the standard subbasis for τ_X. Consider $\mathcal{O} \subseteq \mathcal{S}$, an open cover. Let $\mathcal{O}_i = \{O \in \tau_{X_i} | \pi_i^{-1}(O) \in \mathcal{O}\}$.

> *Claim.* Some i has $\mathcal{O}_i$ an open cover of X_i.
>
> *Proof.* If not, then for all i there exists $x_i \in X_i$ such that $x_i \notin \cup \mathcal{O}_i$. Let $x = (x_i)_{i \in I}$. Then $x \in S$ for some $S \in \mathcal{O}$ and $S = \pi_{i_0}^{-1}(O)$ for some $i_0 \in I, O \in \tau_{X_{i_0}}$. But then $x_{i_0} \in \cup \mathcal{O}_{i_0}$, a contradiction.

Thus if $X_i = \cup \mathcal{O}_i$, we have $O_1, ..., O_n \in \mathcal{O}_i$ such that $X_i = \cup_1^n O_i$ and $\pi_i^{-1}(O_1), ..., \pi_i^{-1}(O_n) \in \mathcal{O}$ is a finite subcover of $\mathcal{O}$. □

Example. $\{0, 1\}^{\mathbb{N}}$ is compact. This is called the Cantor Space as it is homeomorphic to Cantor's middle third set.

4 Completeness and Compactness in Metric Spaces

Definition 4.1. *A metric space (X, d) is* ***complete*** *if every Cauchy Sequence converges. (A sequence $(x_n)_{n\geq 1}$ is* ***Cauchy*** *if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for $n, m \geq N$.)*

Examples.

1. $\mathbb{R}$ is complete with its usual metric.

2. $\mathbb{R} \cong (0, 1)$ but $(0, 1)$ is *not* complete as $(1 - \frac{1}{n})$ is a Cauchy sequence which does not converge.

3. $\mathbb{Q}$ is not complete. Any sequence (q_n) of rational approximations to π converging in $\mathbb{R}$ to π is Cauchy, but not convergent in $\mathbb{Q}$.

4. $\mathbb{R}^k$ with any of its usual metrics is complete.

   Proof. Suppose (x_n) is a Cauchy sequence in $\mathbb{R}^k$. We have for all $i = 1, ..., k$ that $|\pi_i(x_n) - \pi_i(x_m)| \leq d(x_n, x_m)$ for any $d = d_1, d_2, d_\infty, ...$ and so (x_n) Cauchy implies $(\pi_i(x_n))_{n\geq 1}$ is Cauchy in $\mathbb{R}$. So $\pi_i(x_n) \to z_i$ for $i = 1, ..., k$ since $\mathbb{R}$ is complete. Thus $x_n \to z = (z_i)_1^k$ by Homework 3, #6. □

5. $C[0, 1] = \{f : [0, 1] \to \mathbb{R}, f \text{ is continuous}\}$ with metric $d_\infty(f, g) = \sup_{t\in[0,1]} |f(t) - g(t)|$ is complete.

   Proof. Let (f_n) be a Cauchy Sequence. We have that for all $t \in [0, 1]$ the real sequence $(f_n(t))_{n\geq 1}$ is Cauchy. Thus for some (not necessarily continuous) function f, we see $f_n(t) \to f(t)$. Note $C[0, 1] \subseteq \mathbb{R}^{[0,1]}$ with the uniform metric. So for all $\epsilon > 0$ there exists N such that $n, m \geq N$ implies $|f_n(t) - f_m(t)| < \epsilon$ for all $t \in [0, 1]$. Letting $m \to \infty$, we get $|f_n(t) - f(t)| \leq \epsilon$ for all $t \in [0, 1]$. So $f_n \to f$ uniformly, that is, f is the uniform limit of continuous functions and thus $f \in C[0, 1]$. □

6. If Y is a complete metric space and X a topological space, then Y^X with the uniform metric, $C_Y(X) = \{f : X \to Y | f \text{ is continuous}\}$ with the uniform metric, and $B_Y(X) = \{f : X \to Y | f \text{ is bounded}\}$ with the uniform metric are all complete.

Lemma 4.2. *If Y is a closed subspace of a complete metric space X, then Y is complete.*

Proof. If (y_n) is a Cauchy Sequence in Y, then it is Cauchy in X. Thus there exists $x \in X$ such that $y_n \to x$. Since Y is closed, $x \in Y$. □

Lemma 4.3. *If every Cauchy sequence in X has a convergent subsequence, then X is complete.*

Proof. In fact, any Cauchy sequence with a convergent subsequence converges. If $n_1 < n_2 < \cdots$ and (x_n) is Cauchy with $x_{n_k} \to x$ as $k \to \infty$, then for all $\epsilon > 0$ there exists N_1 such that $k > N_1$ implies $d(x_{n_k}, x) < \frac{\epsilon}{2}$ and there exists N_2 such that $n, m > N_2$ implies $d(x_n, x_m) < \frac{\epsilon}{2}$. Let $N = \max\{n_{N_1}, N_2\}$. Then for $n \geq N$, we have $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. □

Theorem 4.4. *A metric space X is complete if and only if for all sequences $A_1 \supseteq A_2 \supseteq \cdots$ of closed nonempty sets with $diam(A_i) \to 0$, we have $\cap A_i \neq \emptyset$.*

Proof. For the forward direction, choose $a_n \in A_n$. Then $(a_n)_{n\geq 1}$ is a Cauchy sequence as for all ϵ there exists N such that $n \geq N$ implies $diam(A_n) < \frac{\epsilon}{2}$. Now, for all $n, m \geq N$, we have $a_n, a_m \in A_{\min\{n,m\}} \subseteq A_N$. So $d(a_n, a_m) \leq \frac{\epsilon}{2} < \epsilon$. Now, let $a_n \to a$ (as X is complete). We have $(a_n)_{n\geq n_0} \subseteq A_{n_0}$ and so $a \in A_{n_0}$, as it is closed. Thus $a \in \cap_{n\geq 1} A_n$.

For the backward direction, if (a_n) is Cauchy, let $A_m = \overline{\{a_n : n \geq m\}}$. Then $A_1 \supseteq A_2 \supseteq \cdots$, the A_i are closed, and $diam(A_n) \to 0$ as (a_n) are Cauchy. Thus there exists $a \in \cap_{n\geq 1} A_n$. It is easy to get a subsequence $a_{n_k} \to a$. By the lemma, $a_n \to a$. □

Corollary 4.5. *Compact metric spaces are complete.*

Proof. Follows from HW6 #6. □

Definition 4.6. *A topological space*

- *is **limit point compact** if every infinite subset has a limit point.*
- *is **sequentially compact** if every sequence has a convergent subsequence.*
- *is **totally bounded** if for all $\epsilon > 0$ there exists a finite subset $F \subseteq X$ such that $\cup_{x \in F} B_\epsilon(x) = X$ (such a set is called a $\epsilon-$**net**.*
- *has the **Lebesgue number property** if for all open covers $\mathcal{O} \subseteq \tau$ there exists $\epsilon > 0$ such that for all $A \subseteq X$ with $diam(A) < \epsilon$ there exists $O \in \mathcal{O}$ with $A \subseteq O$. (We use this to prove that continuous functions on compact metric spaces are uniformly continuous)*

Lemma 4.7. *If X is sequentially compact then every open cover has a Lebesgue number.*

Proof. Suppose not. Let X be sequentially compact and $\mathcal{O}$ an open cover with no Lebesgue number. Then for all $n \geq 1$ there exists K_n with $diam(K_n) < \frac{1}{n}$ such that there does not exist $O \in \mathcal{O}$ with $K_n \subseteq O$. Pick $x_n \in K_n$ and a subsequence $x_{n_i} \to x$. Then there exists $\epsilon > 0$ and $O \in \mathcal{O}$ such that $B_\epsilon(x) \subseteq O$. Pick i sufficiently large so that $d(x_{n_i}, x) < \frac{\epsilon}{2}$ and $diam(K_{n_i}) < \frac{\epsilon}{2}$. Then for all $y \in K_{n_i}$ we have $d(y, x) < d(y, x_{n_i}) + d(x_{n_i}, x) < \epsilon$, that is, $K_{n_i} \subseteq B_\epsilon(x) \subseteq O$, a contradiction. □

Lemma 4.8. *If X is sequentially compact then X is totally bounded.*

Proof. Suppose X is sequentially compact. Pick $\epsilon > 0$, $x_1 \in X$ and $x_2 \notin B_\epsilon(x_1)$ (if $B_\epsilon(x_1) = X$, done). Continue to pick $x_i \in (\cup_{j<i} B_\epsilon(x_j))^C$ as long as there is such an x_i. Either we find a finite $\epsilon-$net or there exists an infinite sequence $x_1, x_2, ...$ such that the terms are $\epsilon-$separated, that is, $d(x_n, x_m) \geq \epsilon$ for all n, m. Now, let (x_{n_i}) be a convergent subsequence. Then it is Cauchy, but $\epsilon-$separated, a contradiction. □

Theorem 4.9. *If X is a metric space then TFAE*

1. *X is compact*
2. *X is limit point compact*
3. *X is sequentially compact*
4. *every open cover of X has a Lebesgue number and X is totally bounded*
5. *X is complete and totally bounded.*

Proof. $(1 \Rightarrow 2)$ Suppose X is compact and $A \subseteq X$ has no limit points. For all $x \in X$ there exists an open neighborhood $\mathcal{O}_x$ of x such that $O_x \cap A \subseteq \{x\}$. By compactness, there exists $O_{x_1}, ..., O_{x_n}$ such that $\cup O_{x_i} = X$ and $A = \cup_1^n (O_{x_i} \cap A) \subseteq \{x_1, ..., x_n\}$. Thus A is finite.

$(2 \Rightarrow 3)$ Let $(x_n)_1^\infty$ be a sequence in X. If $\{x_n\}_{n \geq 1}$ is finite, then there exists x with $x_n = x$ infinitely often, say for $n_1, n_2, ...$ Then $x_{n_i} \to x$. If $\{x_n\}_{n \geq 1}$ is infinite, then it has a limit point x. So there exists n_1 such that $x_{n_1} \in B_1(x) \setminus \{x\}$, $n_2 > n_1$ such that $x_{n_2} \in B_{\frac{1}{2}}(x) \setminus \{x\}$ and similarly for all k there exists $n_k > n_{k-1}$ such that $x_{n_k} \in B_{\frac{1}{k}}(x) \setminus \{x\}$. Clearly, $x_{n_k} \to x$.

$(3 \Rightarrow 4)$ Follows from Lemmas.

$(4 \Rightarrow 1)$ Let X have the Lebesgue number property and be totally bounded. Let $\mathcal{O}$ be an open cover of X. Let ϵ be a Lebesgue number for $\mathcal{O}$ and let $x_1, ..., x_n$ be an $\frac{\epsilon}{3}-$net. Then the Lebesgue number property says there exists $O_k \in \mathcal{O}$ such that $B_{\frac{\epsilon}{3}}(x_k) \subseteq O_k$. So $X = \cup B_{\frac{\epsilon}{3}}(x_k) \subseteq \cup O_k \subseteq X$, that is, $O_1, ..., O_n$ form a finite subcover.

$(1 \Rightarrow 5)$ We have seen compact implies complete. Given $\epsilon > 0$ we see $\{B_\epsilon(x) : x \in X\}$ is an open cover. So there exists $x_1, ..., x_n$ such that $X = \cup_1^n B_\epsilon(x_i)$.

(5 ⇒ 3) Let $(x_n)_{n\geq 1}$ be a sequence in X. Let J_1 be a finite 1-net. We have $X = \cup_{y\in J_1} B_1(y)$. Then there exists $y_1 \in J_1$ such that there are infinitely many $x_n \in B_1(y_1)$, say $n = n_{1,1} < n_{1,2} < \cdots$. Inductively, define $n_{k,1} < n_{k,2} < \cdots$ as follows: Let J_k be a finite $\frac{1}{k}$−net and find $y_k \in J_k$ and an infinite subsequence $n_{k,1} < n_{k,2} < \cdots$ of $n_{k-1,1} < n_{k-1,2} < \cdots$ with $x_{n_{k,i}} \in B_{\frac{1}{k}}(y_k)$. Then $x_{n_{k,i}} \in B_1(y_1) \cap B_{\frac{1}{2}}(y_2) \cap \cdots B_{\frac{1}{k}}(y_k)$. Now, define $n_i = n_{i,i}$. Then (x_{n_i}) is Cauchy since for $i, \ell \geq k$ we have x_{n_i} and x_{n_ℓ} both belong to $B_{\frac{1}{k}}(y_k)$. So $x_{n_i} \to x$ by the completeness of X. □

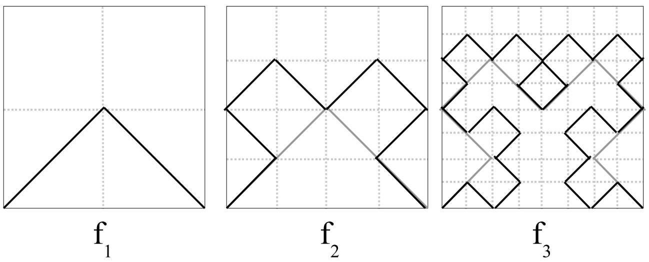
Theorem 4.10. *Suppose $f : X \to Y$ is continuous where X is a compact metric space and Y a metric space. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \epsilon$ (i.e., f is uniformly continuous)*

Proof. Consider $\mathcal{O} = \{f^{-1}(B_{\frac{\epsilon}{2}}(y)) : y \in Y\}$. Let δ be a Lebesgue number for $\mathcal{O}$. Then $d(x, x') < \delta$ implies there exists $O \in \mathcal{O}$ with $\{x, x'\} \subseteq O$, that is, $f(x), f(x') \in B_{\frac{\epsilon}{2}}(y)$ for some $y \in Y$. Thus $d(f(x), f(y)) < \epsilon$. □

4.1 Space Filling Curves

Theorem 4.11. *Let $I = [0, 1]$. There exists a continuous surjection from $I \to I^2$.*

Proof. Define a sequence of continuous maps on $I \to I^2$ as follows:



Give I^2 the ℓ_∞ metric: $d((x, y), (x', y')) = \max\{|x - x'|, |y - y'|\}$. On $C(I, I^2)$, take the sup metric: $d_\infty(f, g) = \sup_{t\in I} d(f(t), g(t))$. We've shown $C(I, I^2)$ is complete with respect to d_∞.

Claim. $d_\infty(f_n, f_{n+1}) \leq 2^{-n}$.

Proof. For $t \in \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right]$, both $f_n(t)$ and $f_{n+1}(t)$ are in some subsquare of I^2 with diameter $\leq 2^{-n}$.

For $m \geq n$, we have $d_\infty(f_n, f_m) \leq \sum_{i=n}^{m-1} 2^{-i} \leq \sum_{i=n}^{\infty} 2^{-i} \leq 2^{-n+1} \to 0$. Thus, by completeness, there exists $f \in C(I, I^2)$ such that $f_n \to f$.

Claim. f is surjective.

Proof. Pick $x \in I^2$. We will show $x \in \overline{f(I)}$. Given $\epsilon > 0$ pick N sufficiently large such that $2^{-N} < \frac{\epsilon}{2}$ and $d_\infty(f_N, f) < \frac{\epsilon}{2}$. Then there exists $t \in I$ such that $d(f_N(t), x) \leq 2^{-N}$ (the N^{th} function visits all blocks of size 2^{-N}) and so $d(f(t), x) \leq d(f(t), f_N(t)) + d(f_N(t), x) < \epsilon$. Of course, $f(I)$ is compact and hence closed. Thus $x \in f(I)$. □

4.2 Compactification

Theorem 4.12. *If X is any metric space, then there exists a complete metric space $Y \supseteq X$ with $\overline{X} = Y$.*

Proof. Let $Y' = B(X, \mathbb{R})$, the space of all bounded continuous functions from $X \to \mathbb{R}$, with the sup metric $d_\infty(f, g) = \sup_{x\in X} |f(x) - g(x)|$. Then Y' is complete. Fix $x_0 \in X$. Given $a \in X$, define $\phi_a : X \to \mathbb{R}$ by $x \mapsto d(x, a) - d(x, x_0)$. Note $|\phi_a(x)| \leq d(x_0, a)$ for all x, that is, $\phi_a \in B(X, \mathbb{R})$.

Claim. $d_\infty(\phi_a, \phi_b) = d_X(a, b)$

Proof. For $x \in X$ we have $|\phi_a(x) - \phi_b(x)| = |d(x, a) - d(x, b)| \leq d(a, b)$. Of course, $|\phi_a(a) - \phi_b(a)| = d(a, b)$.

The set $\tilde{X} = \{\phi_a : a \in X\} \subseteq Y'$ is an isometric copy of X. Now $\overline{X} \subsetneq Y'$. So define $Y := \overline{X}$. Then $\overline{X}$ is closed in $B(X, \mathbb{R})$ and hence is complete. □

Recall that if X was a noncompact Hausdorff locally compact topological space, then X_∞ with $X \subseteq X_\infty$ is such that X_∞ is compact, Hausdorff and $\overline{X} = X_\infty$. This was a **compactification** of X. There are in fact many compactifications. However, there is only one completion.

4.3 Countability and Separation Axioms

Definition 4.13. *A topological space X is **first countable** if for all $x \in X$ there exists a countable collection $\{O_i\}_{i\in\mathbb{N}}$ of neighborhoods of x such that for all $O \in \tau$ with $x \in O$ there exists i such that $x \in O_i \subseteq O$, that is, X has a countable neighborhood basis at each point.*

Example. If X is a metric space, take $O_i = B_{\frac{1}{i}}(x)$.

Definition 4.14. *A topological space is **second countable** if the topology has a countable basis. A topological space X is **separable** if there exists countable $A \subseteq X$ with $\overline{A} = X$.*

Theorem 4.15. *If X is 1^{st} countable, then*

1. *For all $A \subseteq X$, $x \in \overline{A}$ if and only if there exists $a_n \in A$ such that $a_n \to x$.*

2. *$f : X \to Y$ is continuous if and only if for all $x_n \to x$ in X we have $f(x_n) \to f(x)$.*

Proof. 1. The backward implication is true in all topological spaces. For the forward, pick a countable neighborhood basis $\{N_i\}_{i\geq 1}$ of x. WLOG $N_1 \supseteq N_2 \supseteq \cdots$. Then for all n there exists $a_n \in N_n \cap A$. If O is a neighborhood of x, there exists k such that $N_k \subseteq O$. Then for all $n \geq k$ we have $a_n \in N_n \subseteq N_k \subseteq O$, that is, $a_n \to x$.

2. The forward implication is always true. For the backward, suppose f satisfies the condition on sequences. If $A \subseteq X$, it is enough to show $f(\overline{A}) \subseteq \overline{f(A)}$. If $x \in \overline{A}$, there exists $a_n \to x$ with $a_n \in A$ by part a. By hypothesis, $f(a_n) \to f(x)$, that is, $f(x) \in \overline{f(A)}$. □

Lemma 4.16. *If X has a countable subbasis, then X is 2^{nd} countable.*

Proof. If $\mathcal{S}$ is a countable subbasis then $\mathcal{B} = \{S_1 \cap \cdots \cap S_n | n \geq 0, S_i \in \mathcal{S}\}$ is a countable basis. □

Theorem 4.17. *Both 1^{st} and 2^{nd} countability is preserved under taking subspaces and countable products.*

Proof. If $Y \subseteq X$ and $\{N_i\}_{i\geq 1}$ is a neighborhood basis at x, then $\{N_i \cap Y\}_{i\geq 1}$ is a neighborhood basis at x in Y. Similarly, if $\mathcal{B}$ is a countable basis for τ_X then $\{B \cap Y : B \in \mathcal{B}\}$ is a countable basis for τ_Y.

If $X = \prod X_i$ is a product of 2^{nd} countable spaces, then if $\mathcal{B}_i$ is a countable basis for X_i then $\mathcal{S} = \{\pi_i^{-1}(O) : i \in \mathbb{N}, O \in \mathcal{B}_i\}$ is a countable subbasis for τ_X. If $x \in X$ and $\mathcal{N}_i$ is a countable neighborhood basis at $\pi_i(x) \in X_i$, then $\mathcal{N} = \{\pi_{i_1}^{-1}(N_1) \cap \cdots \cap \pi_{i_k}^{-1}(N_k) : i_1, ..., i_k \in \mathbb{N}, N_j \in \mathcal{N}_{i_j}\}$ is a countable neighborhood basis at x. □

Theorem 4.18. *If a topological space X is 2^{nd} countable, then it is separable.*

Proof. Let $\mathcal{B}$ be a countable basis for τ_X. For all $B \in \mathcal{B}$ pick $x_B \in B$. Then $D = \{x_B : B \in \mathcal{B}\}$ is countable. Then if O is open and $O \neq \emptyset$, say $x \in O$, then there exists $B \in \mathcal{B}$ with $x \in B \subseteq O$. Then $x_B \in O$ and $\overline{D} = X$. □

Examples.

- $\mathbb{R}$ is 1^{st} countable, 2^{nd} countable, and separable.

- $\mathbb{R}^{\mathbb{N}}$ is a countable product of $\mathbb{R}$ and is thus 1^{st} and 2^{nd} countable. It is also separable (consider $D = \{(x_n) : x_n \in \mathbb{Q},$ there exists N such that $x_n = 0$ for all $n \geq N\}$).

- $(\mathbb{R}^{\mathbb{N}}, d_\infty)$ (the uniform topology) is a metric space and thus 1^{st} countable. It is not 2^{nd} countable nor separable, however. $S = \{0,1\}^{\mathbb{N}}$ has $d(x,y) = 1$ for all $x \neq y \in S$. So $\mathbb{R}^{\mathbb{N}}$ has an uncountable collection of disjoint open sets $\{B_{\frac{1}{2}}(x) : x \in S\}$.

- Let $\mathbb{R}_\ell$ be $\mathbb{R}$ with basis $\mathcal{B} = \{[a,b) : a, b \in \mathbb{R}\}$. If $x \in \mathbb{R}_\ell$, then $\mathcal{N} = \{(x, x + \frac{1}{n}) : n \geq 1\}$ is a countable neighborhood basis for x. So $\mathbb{R}_\ell$ is 1^{st} countable. $\mathbb{Q}$ is a countable dense subset and so $\mathbb{R}_\ell$ is separable. However, $\mathbb{R}_\ell$ is not 2^{nd} countable. To see this, let $O_x = [x, x+1)$. If $\mathcal{B}$ is any basis, then for each O_x there exists $B_x \in \mathcal{B}$ with $x \in B_x \subseteq O_x$. For all x, we see $x = \inf B_x$ and so $x \neq y$ implies $B_x \neq B_y$. So $\mathcal{B}$ is uncountable.

Definition 4.19. *A topological space X is **regular** if points are closed and for all $x \in X$ and $A \subseteq X$ closed with $x \notin A$, there exists open O, U with $x \in O, A \subseteq U$ and $O \cap U = \emptyset$. A topological space is **normal** if points are closed and for all $A, B \subseteq X$ closed with $A \cap B = \emptyset$, there exists open O, U with $A \subseteq O, B \subseteq U$ and $O \cap U = \emptyset$.*

Alternatively, one could define regularity as having the property that for all $x \in O$ open there exists a neighborhood U of x with $\overline{U} \subseteq O$. Similarly, X is normal if for all closed A with $A \subseteq O$ open there exists U open with $A \subseteq U \subseteq \overline{U} \subseteq O$.

Theorem 4.20. *Hausdorffness and regularity are closed under taking subspaces and arbitrary products.*

Proof. We have already shown the result for Hausdorff spaces. So suppose $Y \subseteq X$, A is closed in Y and $x \in Y \setminus A$. There exists $B \subseteq X$ with $A = Y \cap B$. In particular, $x \notin B$. Then there exists O, U open in X with $x \in O, A \subseteq\subseteq BU$ and $O \cap U = \emptyset$. Now $O' = O \cap Y$ and $U' = U \cap Y$ separate x from A in Y.

Suppose $X = \prod_{\alpha \in I} X_\alpha$ and each X_α is regular. Recall $\overline{\prod A_\alpha} = \prod \overline{A_\alpha}$. So points are closed. Suppose now $x \in O$ is open in X. Consider a basic open set B with $x \in B \subseteq O$. Then $B = \cap_1^n \pi_{\alpha_i}^{-1}(O_i)$ for $O_i \in \tau_{X_{\alpha_i}}$. By regularity of X_{α_i}, there exists V_i with $x_{\alpha_i} \in V_i$ and $\overline{V_i} \subseteq O_i$. Setting $V = \prod_{\alpha \in I} V_\alpha$ where $V_\alpha = X_\alpha$ if $\alpha \notin \{\alpha_1, ..., \alpha_n\}$, we have $x \in V$ and $\overline{V} = \prod_{\alpha \in I} \overline{V_\alpha} \subseteq \prod_{\alpha \in I} O_\alpha \subseteq B \subseteq O$. □

Example. $\mathbb{R}_\ell$ is normal: If A, B are closed in $\mathbb{R}_\ell$ with $A \cap B = \emptyset$, then for all $a \in A$ there exists $\epsilon_a > 0$ such that $[a, a + \epsilon_a) \cap B = \emptyset$. Similarly, for all $b \in B$ there exists $\epsilon_b > 0$ such that $[b, b + \epsilon_b) \cap A = \emptyset$. Let $O = \cup_{a \in A}[a, a + \epsilon_a)$ and $U = \cup_{b \in B}[b, b + \epsilon_b)$. Then $A \subseteq O$ and $B \subseteq U$.

Claim. $O \cap U = \emptyset$.

Proof. Suppose $x \in O \cap U$ so $x \in [a, a + \epsilon_a) \cap [b, b + \epsilon_b)$ for some $a \in A, b \in B$. WLOG $a < b$. Then $b < a + \epsilon_A$ which says $b \in [a, a + \epsilon_a)$, a contradiction.

Example. The Sorgenfrey Plane, $\mathbb{R}_\ell^2$ is not normal: Suppose it were. Consider $L = \{(x, -x) : x \in \mathbb{R}\}$. Then L is closed (anything not in L is not in $\overline{L}$) and discrete in the subspace topology (any $(x, -x)$ is open as it $([x, x+1] \times [-x, -x+1]) \cap L$). Then for any $A \subseteq L$ we have $A, L \setminus A$ are disjoint closed sets. For each $A \subseteq L$ with $A \neq \emptyset, L$, there exists U_A, V_A open in $\mathbb{R}_\ell^2$ with $A \subseteq U_A, L \setminus A \subseteq V_A$ and $U_A \cap V_A = \emptyset$. Define $\theta : \mathcal{P}(L) \to \mathcal{P}(\mathbb{Q}^2)$ by $A \mapsto U_A \cap \mathbb{Q}^2$, $\emptyset \mapsto \emptyset$, and $L \mapsto \mathbb{Q}^2$.

Claim. θ is injective.

Proof. Suppose $A, B \subseteq L$ with $A \neq B$. Then WLOG there exists $x \in A \setminus B$. So $x \in U_A \cap V_B$ and thus $U_A \cap V_B$ is a nonempty open subset of $\mathbb{R}_\ell^2$ and so contains a part of $\mathbb{Q}_\ell^2$. Thus $U_A \setminus U_B \neq \emptyset$, that is, $U_A \neq U_B$.

Note $\mathcal{P}(L)$ has cardinality 2^c where $c = |\mathbb{R}|$ and $\mathcal{P}(\mathbb{Q}^2)$ has cardinality $2^{\aleph_0} = c$. Clearly, this is impossible.

Theorem 4.21. *Metric spaces are normal.*

Proof. If A, B are disjoint subsets of a metric space X, then for all $x \in A$ there exists $\epsilon(x)$ such that $B_{\epsilon(x)}(x) \cap B = \emptyset$ and for all $y \in B$ there exists $\epsilon(y)$ such that $B_{\epsilon(y)}(y) \cap A = \emptyset$. Suppose $B_{\frac{\epsilon(x)}{2}}(x) \cap B_{\frac{\epsilon(y)}{2}}(y) \neq \emptyset$. Then WLOG $\epsilon(y) \leq \epsilon(x)$ and $d(x, y) < \frac{\epsilon(x)}{2} + \frac{\epsilon(y)}{2} \leq \epsilon(x)$ a contradiction. Thus $U = \cup_{x \in A} B_{\frac{\epsilon(x)}{2}}(x)$ and $V = \cup_{y \in B} B_{\frac{\epsilon(y)}{2}}(y)$ are open disjoint neighborhoods of A and B respectively. □

Theorem 4.22. *Compact Hausdorff spaces are normal.*

Proof. Homework 6 #2. □

Theorem 4.23. *Regular 2^{nd} countable spaces are normal.*

Proof. Suppose X is regular and 2^{nd} countable with countable basis $\mathcal{B}$. Let A, B be disjoint closed sets. For all $a \in A$ let U_a be an open neighborhood of a such that $U_a \cap B = \emptyset$. Then regularity implies there exists a neighborhood V_a of a such that $V_a \subseteq \overline{V_a} \subset U_a$. By 2^{nd} countability, there exists $B_a \subset V_a$ with $x \in B_a \in \mathcal{B}$. List all the B_a's for all a's chosen in this way as $O_1, O_2, ...$ Similarly, list all of the B_b's as $N_1, N_2, ...$ so that $\overline{N_i} \cap A = \emptyset$. Now, let $O_k' = O_k \setminus \cup_{j=1}^k \overline{N_j}$ and $N_k' = N_k \setminus \cup_{j=1}^k \overline{O_j}$. Define $O = \cup_1^\infty O_k'$ and $N = \cup_1^\infty N_k'$. Then $A \subset O, B \subset N$ and $O \cap N = \emptyset$ as $O_j' \cap N_k' = \emptyset$ for all j, k. □

Alternative proof that metric spaces are normal:

Proof. Suppose X is a metric space and $A, B \subseteq X$ are closed and disjoint. Define $f(x) = \dfrac{d(x,A)}{d(x,A)+d(x,B)}$. Then $f : X \to \mathbb{R}$ is continuous with the property that $f|_A = 0$ and $f|_B = 1$. Then $U = f^{-1}(-\infty, \frac{1}{3})$ and $V = f^{-1}(\frac{2}{3}, \infty)$ are disjoint open sets separating A and B. $\square$

Theorem 4.24 (**Urysohn's Lemma**). *If X is normal and $A, B \subseteq X$ are disjoint and closed, then there exists a continuous $f : X \to \mathbb{R}$ with $f|_A = 0$ and $f|_B = 1$.*

Proof. Let $P = \mathbb{Q} \cap [0,1]$. Enumerate as $1, 0, p_3, p_4, \ldots$ We will construct $\{U_p\}_{p \in P}$ with the property that $p < q$ implies $\overline{U_p} \subseteq U_q$. Let $U_1 = B^C$. By normality, there exists U_0 with $A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1$. Suppose we have contructed $\{U_p\}_{p \in F}$ where F is some initial segment of our enumeration. Say r is the next rational. Let p, q be the immediate predecessor and successor of r in F. By normality, there exists U_r such that $\overline{U_p} \subseteq U_r \subseteq \overline{U_r} \subseteq U_q$.

Extend the definition of $\{U_p\}$ to $U_p = \emptyset$ if $p < 0$ and $U_p = X$ if $p > 1$. Now define $S(x) = \{p : x \in U_p\}$ and $f(x) = \inf_{p \in S(x)} p$. This is well-defined. Furthermore, notice

- If $x \in \overline{U_r}$ then $f(x) \leq r$ as $S(x) \supseteq (r, \infty) \cap \mathbb{Q}$.
- If $x \notin U_r$, then $f(x) \geq r$ as $S(x) \subseteq (r, \infty) \cap \mathbb{Q}$.

Suppose $f(x) = \alpha \in \mathbb{R}$ and $\epsilon > 0$. Then there exists rationals p, r with $a - \epsilon < p < \alpha < r < \alpha + \epsilon$.

> *Claim.* $U_r \setminus \overline{U_p}$ is a neighborhood of x contained in $f^{-1}(\alpha - \epsilon, \alpha + \epsilon)$.
>
> *Proof.* $U_r \setminus \overline{U_p}$ is open. Further $x \in U_r$ as otherwise $f(x) \geq r$ and $x \notin \overline{U_p}$ as otherwise $f(x) \leq p$. So $x \in U_r \setminus \overline{U_p}$. Similarly, for all $y \in U_r \setminus \overline{U_p}$, we see $p \leq f(y) \leq r$ which implies $y \in f^{-1}(a - \epsilon, a + \epsilon)$.

Now $x \in A$ implies $x \in U_0$ and so $f(x) = 0$. Also $x \in B$ implies $x \notin U_1$ so $f(x) \geq 1$. Of course, $S(x) \supseteq (1, \infty) \cap \mathbb{Q}$ and so $f(x) = 1$. $\square$

Definition 4.25. *A topological space X is **completely regular** if points are closed and for all $x \in X, B \subseteq X$ closed with $x \notin B$ there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x) = 0$ and $f|_B \equiv 1$.*

Theorem 4.26. *Complete regularity is preserved by taking subspaces and products.*

Proof. If $X \subseteq Y$ and Y is completely regular with $x \in X, B \subseteq X$ closed and $x \notin B$, then there exists $B' \subseteq Y$ with B' closed and $B = B' \cap X$. In particular, $x \notin B'$. By complete regularity of Y there exists $g : Y \to \mathbb{R}$ with $g(x) = 0$ and $g|_{B'} \equiv 1$. Let $f = g|_X$.

Suppose now $X = \prod X_i$ and each X_i is completely regular. Then $x \in X, B \subseteq X$ is closed with $x \notin B$. We produce $f : X \to \mathbb{R}$ which is 1 at x and 0 on B. There exists a basic open neighborhood $U = \prod_{i_1}^{-1}(U_1) \cap \cdots \cap \prod_{i_n}^{-1}(U_n)$ of x. For $j = 1, \ldots, n$ there exists $\phi_j : X_j \to \mathbb{R}$ which is 1 at $\pi_{i_j}(x)$ and 0 on U_j^C. Let $f(y) = \prod_1^n \phi_j \circ \pi_{i_j}(y)$. This has $f(x) = 1$ and $f|_{U^C} \equiv 0$. $\square$

Example. $\mathbb{R}_\ell^2$ is completely regular since $\mathbb{R}_\ell$ is normal and hence completely regular. However $\mathbb{R}_\ell^2$ is not normal (proved earlier). Note that there are spaces which are regular but not completely regular.

Theorem 4.27 (**Embedding Theorem**). *Let X be a topological space in which points are closed, $\mathcal{F}$ a family of continuous functions $X \to \mathbb{R}$ such that for all $x \in X$ and U a neighborhood of x there exists $f \in \mathcal{F}$ with $f(x) > 0$ and $f|_{U^C} \equiv 0$. Then there exists a homeomorphism X into a subspace of $\mathbb{R}^{\mathcal{F}}$.*

Proof. Define $F : X \to \mathbb{R}^{\mathcal{F}}$ by $x \mapsto (f(x))_{f \in \mathcal{F}}$. We need only show $F : X \to F(X)$ is a homeomorphism. It is continuous if and only if each coordinate is continuous, but this is true by hypothesis. It is injective as points are closed, so if $x \neq y$ there exists U neighborhood of x with $y \notin U$ and $f \in \mathcal{F}$ with $f(x) > 0$ and $f(y) = 0$, i.e., $F(x) \neq F(y)$. Now let $U \subseteq X$ be open and $\alpha \in F(U)$. Pick $x \in U$ with $\alpha = F(x)$. By hypothesis, there exists $f \in \mathcal{F}$ with $f(x) > 0$ and $f|_{U^C} = 0$. Let $V = \pi_f^{-1}((0, \infty)) \cap F(X)$. Want $\alpha \in V \subseteq F(U)$. Suppose $\beta \in V$ with $\beta = F(y)$. Since $\beta \in \pi_f^{-1}((0, \infty))$ we have $F(y) > 0$ which implies $y \notin U^c$, that is, $y \in U$ and $V \subseteq F(U)$. $\square$

Corollary 4.28. *X is completely regular if and only if $X \hookrightarrow [0,1]^J$ for some set J.*

Proof. Take $J = C(X, \mathbb{R})$. $\square$

Theorem 4.29 (Urysohn's Metrization Theorem). *If X is second countable and regular (and thus normal), then X is metrizable.*

Proof. Recall that $\mathbb{R}^{\mathbb{N}}$ is metrizable as $d_H(x,y) = \sup_i \frac{d(x_i,y_i)}{i}$ induces the product topology. So we will embed X in $\mathbb{R}^{\mathbb{N}}$. Let $\mathcal{B}$ be a countable basis for τ_X. For $B, C \in \mathcal{B}$ with $\overline{B} \subseteq C$ there exists continuous $g_{B,C} : X \to \mathbb{R}$ such that $g|_{\overline{B}} = 1$ and $g|_{C^C} = 0$ by Urysohn's Lemma. Let $G = \{g_{B,C} : B, C \in \mathcal{B}, \overline{B} \subseteq C\}$.

> *Claim.* G satisfies the hypotheses of the embedding theorem.
>
> *Proof.* Suppose $x \in X$ and U a neighborhood of x. Then there exists $C \in \mathcal{B}$ with $x \in C \subseteq U$ and by regularity there exists an open V with $x \in V \subseteq \overline{V} \subseteq C \subseteq U$. Then, there exists $B \in \mathcal{B}$ with $x \in B \subseteq V$. So $x \in B \subseteq \overline{B} \subseteq C$. Thus $g_{B,C}(x) = 1$ and $g_{B,C}|_{U^C} = 0$.

By the embedding theorem, $X \hookrightarrow \mathbb{R}^G \cong \mathbb{R}^{\mathbb{N}}$ as G is countable. □

Theorem 4.30 (Tietze Extension Theorem). *If X is normal, $A \subseteq X$ closed and $f : A \to \mathbb{R}$ is continuous, then there exists $\hat{f} : X \to \mathbb{R}$ continuous with $\hat{f}|_A = f$. Indeed if $f : A \to [a,b]$ then there exists $\hat{f} : X \to [a,b]$ with $\hat{f}|_A = f$.*

Proof. We will prove the second statement. WLOG $f : A \to [-1,1]$.

> *Claim.* If $f : A \to [-r,r]$ then there exists $g : X \to [-\frac{r}{3}, \frac{r}{3}]$ such that $|f(a) - g(a)| \le \frac{2r}{3}$ for all $a \in A$.
>
> *Proof.* Let $B = f^{-1}([-r, -\frac{r}{3}])$ and $C = f^{-1}([\frac{r}{3}, r])$. Then B and C are closed in a closed set and thus closed. By Urysohn's Lemma, there exists $g : X \to [-\frac{r}{3}, \frac{r}{3}]$ with $g|_B = -\frac{r}{3}$ and $g|_C = \frac{r}{3}$. Thus g has the property that $|f(a) - g(a)| \le \frac{2r}{3}$ for all $a \in A$.

By the claim, there exists $g_1 : X \to [-\frac{1}{3}, \frac{1}{3}]$ with $|f(a) - g_1(a)| \le \frac{2}{3}$ for all $a \in A$. Lets consider $f - g_1 : A \to [-\frac{2}{3}, \frac{2}{3}]$. There exists $g_2 : X \to [-\frac{2}{9}, \frac{2}{9}]$ with $|f(a) - g_1(a) - g_2(a)| \le \left(\frac{2}{3}\right)^2$. Having constructed $g_1, ..., g_n : X \to \mathbb{R}$ with $|g_i(x)| \le \frac{1}{3}\left(\frac{2}{3}\right)^{i-1}$ and $|f(a) - \sum_1^n g_i(a)| \le \left(\frac{2}{3}\right)^n$, by the lemma there exists $g_{n+1} : X \to [-\frac{1}{3}\left(\frac{2}{3}\right)^n, \frac{1}{3}\left(\frac{2}{3}\right)^n]$ with $|f(a) - \sum_1^{n+1} g_i(a)| \le \left(\frac{2}{3}\right)^{n+1}$. By induction, we have an infinite sequence $(g_i)_{i \ge 1}$.

Define $g = \sum_1^\infty g_i(x)$. The sequence of partial sums $(s_n(x))_{n \ge 1}$ is Cauchy for all x as $|s_n(x) - s_m(x)| = |\sum_{n+1}^m g_i(x)| \le \sum_{n+1}^m |g_i(x)| \le \sum_{n+1}^\infty \frac{1}{3}\left(\frac{2}{3}\right)^{n+1} \to 0$. Letting $m \to \infty$, we see $|s_n(x) - g(x)| \le \sum_{n+1}^\infty \frac{1}{3}\left(\frac{2}{3}\right)^i$. As the RHS does not depend on x, we see $s_n(x) \to g(x)$ uniformly. Thus g is continuous and $|f(a) - g(a)| \le \lim_{n\to\infty}\left(\frac{2}{3}\right)^n$, that is, $f = g|_A$. Also $|g(x)| < \sum_1^\infty \frac{1}{3}\left(\frac{2}{3}\right)^i = 1$.

Suppose now $f : A \to \mathbb{R}$. There exists a homeomorphism $h : \mathbb{R} \to (-1,1)$ and so $h \circ f : A \to (-1,1) \subseteq [-1,1]$. So there exists $g : X \to [-1,1]$ with $g|_A = h \circ f$. Let $D := g^{-1}(\{-1,1\})$. Then D is closed and disjoint from A. By normality there exists a continuous function $\phi : X \to [0,1]$ with $\phi|_D = 0$ and $\phi|_A = 1$. Now $g'(x) := \phi(x)g(x)$ has $g'|_D = 0$ and $g'|_A = g|_A$. So $h^{-1}g' : X \to \mathbb{R}$ is a continuous extension of f. □

4.4 Compactifications

Definition 4.31. *A* ***compactification*** *of a topological space X is a topological space Y which is compact, Hausdorff, and $X \subseteq Y$ (i.e., the subspace topology is the original topology). In addition, we require $\overline{X} = Y$.*

Lemma 4.32. *If X has a compactification then it is completely regular.*

Proof. As $X \subseteq Y$ and Y is completely regular, done. □

Examples.

1. Consider $(-1,1) \xrightarrow{\cong} S^1 \setminus \{-1\} \subseteq S^1$ defined by $t \mapsto e^{i\pi t}$. This is a (1 point) compactification of $(-1,1)$.

2. $(-1,1) \subseteq [-1,1]$ is another compactification of $(-1,1)$

3. Let $D^2 = \{z \in \mathbb{C} : |z| \le 1\}$. Then $B_1(0) \subseteq D^2$ is a compactification. Also $B_1(0) \subseteq (B_1(0))_\infty \cong S^2$, the sphere (Recall $(\cdot)_\infty$ is the one-point compactification).

4. Consider $(0,1) \hookrightarrow [0,1] \times [-1,1]$ defined by $t \mapsto (t, \sin\frac{1}{t})$. Let $S = \{(t, \sin\frac{1}{t} | t \in (0,1)\}$. Then $\overline{S}$, the topologist's sine curve, is a compactification of $(0,1)$. Note the function $f : t \mapsto \sin\frac{1}{t}$ has a continuous extension to $\overline{S}$ (i.e., a continuous function $\hat{f} : \overline{S} \to [-1,1]$ such that $\hat{f}|_S = f$), namely $\hat{f} = \pi_2$, the projection on the second coordinate.

Theorem 4.33. *If X is completely regular, then there exists a compactification βX (called the* ***Stone-Cěch Compactification****) such that every bounded continuous function $f : X \to \mathbb{R}$ has a continuous extension $\hat{f} : \beta X \to \mathbb{R}$ with $\hat{f}|_X = f$.*

Proof. Let $\mathcal{F} = \{f : X \to \mathbb{R} : f$ is continuous, bounded$\}$ and $I_f = [\inf_{x\in X} f(x), \sup_{x\in X} f(x)]$. Embed X into $\prod_{f\in\mathcal{F}} I_f$ by $\beta : X \to \prod_{f\in\mathcal{F}} I_f$ defined via $x \mapsto (f(x))_{f\in\mathcal{F}}$. We know $\prod I_f$ is compact by Tychonoff and β is an embedding. By the Embedding Theorem $\beta(X) \cong X$ and so $\beta X = \overline{\beta(X)}$ is a compactification of X. Given $f : X \to \mathbb{R}$ continuous, bounded, then $\pi_f|_{\beta(x)} = f$. So $\pi_f : \beta X \to \mathbb{R}$ is the required continuous extension of f. $\square$

Theorem 4.34. *If we have $X \subseteq Y$ such that $\overline{X} = Y$ and every bounded continuous function $X \to \mathbb{R}$ can be extended to Y then every continuous function $f : X \to Z$ with Z compact and Hausdorff can be extended to Y.*

Proof. Z compact Hausdorff implies it is completely regular and so $Z \cong \tilde{Z} \subseteq [0, 1]^J$ for some index set J. Suppose $Z \subset [0, 1]^J$. Then $f_j = \pi_j f : X \to [0, 1]$ is continuous for all $j \in J$ and there exists $\hat{f}_j : Y \to [0, 1]$ continuous with $\hat{f}_j|_X = f_j$. Define $\hat{f}(y) = (\hat{f}_j(y))_{j\in J}$. Clearly $\hat{f}$ is continuous and $\hat{f}|_X = f$. Notice $\hat{f}(Y) = \hat{f}(\overline{X}) = \overline{\hat{f}(X)} = \overline{f(X)} \subseteq \overline{Z} = Z$. $\square$

Lemma 4.35. *If $f : A \to Z$ is continuous and Z Hausdorff, then there exists at most one continuous extension of f to $\overline{A}$.*

Proof. Suppose $g, h : \overline{A} \to Z$ are continuous extensions of f with $g \neq h$. Then $g(x) \neq h(x)$ for some x and by Hausdorffness there exists U, V disjoint open sets with $g(x) \in U$ and $h(x) \in V$. Now $g^{-1}(U)$ and $g^{-1}(V)$ are neighborhoods of X and so $g^{-1}(U) \cap h^{-1}(V)$ is. Then there exists $a \in A \cap (g^{-1}(U) \cap h^{-1}(V))$ since $x \in \overline{A}$. Thus $f(a) = g(a) \in U$ and $f(a) = h(a) \in V$, a contradiction. $\square$

Corollary 4.36. *βX is unique. If Y, Y' are two compactifications of X for which the extension property of the theorem holds, then $Y \cong Y'$.*

Proof. Since $i : X \to Y$ is a continuous function to a compact Hausdorff space, there exists $\widehat{i} : Y' \to Y$ continuous with $\widehat{i}|_X = i$. Similarly there exists $\widehat{i'} : Y \to Y'$ continuous with $\widehat{i'}|_X = i$. Then $\widehat{i'} \circ \widehat{i} : Y' \to Y'$ extends $\widehat{i'} \circ i = i'$. By Id'_Y is the (unique) extension of i' to Y'. Thus $\widehat{i'} \circ \widehat{i} = Id$ and similarly $\widehat{i} \circ \widehat{i'} = Id$. Thus $Y \cong Y'$. $\square$

4.5 Filters

Definition 4.37. *A* ***filter*** *on X is a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ such that*

1. *If $A \in \mathcal{F}, B \supseteq A$, then $B \in \mathcal{F}$*
2. *$\emptyset \notin \mathcal{F}$*
3. *$A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$.*

Examples.

1. $\mathcal{F} = \{A \subseteq \mathbb{N} : A^C$ is finite$\}$ is called the **cofinite filter**.
2. If $x \in X$, then $\mathcal{F} = \{A \subseteq X : x \in A\}$ is a filter.
3. If X is a topological space, $x \in X$, then $\mathcal{N}_x = \{A \subseteq X : A$ contains a neighborhood of $X\}$ is called the **neighborhood filter at** x.
4. Suppose $(x_i)_1^\infty$ is a sequence in X. Then $\mathcal{F} = \{A \subseteq X :$ there exists N such that $i \geq N$ implies $x_i \in A\}$ is a filter. If $x_i \to x$, then $\mathcal{N}_x \subseteq \mathcal{F}$.

Definition 4.38. *A* ***filter base*** *$\mathcal{B} \subseteq \mathcal{P}(X)$ is a collection satisfying*

1. *$\emptyset \notin \mathcal{B}$*
2. *$A, B \in \mathcal{B}$ implies there exists $C \in \mathcal{B}$ with $A \cap B \supseteq C$.*

Lemma 4.39. *If $\mathcal{B}$ is a nonempty filter base, then there exists a filter $\mathcal{F}_\mathcal{B} = \{A \subseteq X|$ there exists $B \in \mathcal{B}$ with $A \supseteq B\}$.*

Examples.

1. If X a topological space, $x \in X$, then $\mathcal{B}_x = \{$ neighborhoods of $x\}$ is a filter base. We see $\mathcal{N}_X = \mathcal{F}_{B_x}$.
2. $\mathcal{B} = \{B\}$ for some $B \neq \emptyset$. This is a filter base and $\mathcal{F}_{\mathcal{B}} = \{A \subseteq X | A \supseteq B\}$.
3. If $\mathcal{F}$ is a filter on X and $f : X \to Y$, then $\{f(A) : A \in \mathcal{F}\}$ is a filter base. Certainly $f(A) \cap f(A') \supseteq f(A \cap A')$.

Definition 4.40. *If $\mathcal{F}$ is a filter on X, $f : X \to Y$, define $f(\mathcal{F})$ to be the filter generated by $\{f(A) : A \in \mathcal{F}\}$.*

Definition 4.41. *If X is a topological space and $\mathcal{F}$ a filter on x, say $\mathcal{F} \to x$ if $\mathcal{N}_x \subseteq \mathcal{F}$.*

Theorem 4.42. *Let X, Y be topological spaces and $f : X \to Y$. Then f is continuous if and only if for all $\mathcal{F} \to x$ in X, we have $f(\mathcal{F}) \to f(x) =: y$.*

Proof. For the forward direction, we want to show that a set A with $y \in U \subseteq A$ (U open) is in $f(\mathcal{F})$. Well $f^{-1}(U)$ is a neighborhood of x which implies $f^{-1}(U) \subseteq \mathcal{N}_x \subseteq \mathcal{F}$. So $A \supseteq U \supseteq f(f^{-1}(U)) \in f(\mathcal{F})$.

For the backward direction, let U be open in Y and $x \in f^{-1}(U)$. We know $\mathcal{N}_x \to x$ and so $f(\mathcal{N}_x) \to f(x)$. This implies $\mathcal{N}_{f(x)} \subseteq f(\mathcal{N}_x)$. In particular, $U \in f(\mathcal{N}_x)$. So there exists a neighborhood O of x with $f(O) \subseteq U$, that is, $O \subseteq f^{-1}(U)$. ☐

Definition 4.43. *A filter $\mathcal{U}$ is an **ultrafilter** if for all filters $\mathcal{F}$ on X with $\mathcal{F} \supseteq U$, we have $\mathcal{F} = U$.*

Definition 4.44. *Two filters $\mathcal{F}, \mathcal{G}$ are **compatible** if for all $A \in \mathcal{F}, B \in \mathcal{G}$, we have $A \cap B \neq \emptyset$.*

Lemma 4.45. *If $\mathcal{F}$ and $\mathcal{G}$ are compatible filters then $\mathcal{B} = \{A \cap B : A \in \mathcal{F}, B \in \mathcal{G}\}$ is a filter base and we define $\mathcal{F} \vee \mathcal{G} = \mathcal{F}_{\mathcal{B}}$.*

Notice that $\mathcal{F}, \mathcal{G} \subseteq \mathcal{F} \vee \mathcal{G}$. Also, any filter containing $\mathcal{F}$ and $\mathcal{G}$ also contains $\mathcal{F} \vee \mathcal{G}$.

Lemma 4.46. *If U is an ultra filter and $\mathcal{G}$ is compatible with U, then $\mathcal{G} \subseteq U$.*

Theorem 4.47. *Ultrafilters exist.*

Proof. Consider $P = \{$filters on $X\}$ ordered by inclusion. If $\mathcal{C} \subseteq P$ is a chain of filters in P, then $\cup\mathcal{C}$ is an upper bound for $\mathcal{C}$ since it is a filter. By Zorn's Lemma, done. Indeed for all $\mathcal{F} \in P$, there exists an ultrafilter U with $\mathcal{F} \subseteq U$. ☐

Theorem 4.48. *U a filter is an ultrafilter if and only if for all $A \subseteq X$ either $A \in U$ or $A^C \in U$.*

Proof. For the backward direction, if $W \supsetneq U$, then there exists A with $A, A^C \in W$ which implies W is not a filter.

For the forward direction, consider $A \subseteq X$. If $A \cap B = \emptyset$ for some $B \in U$ then $A^C \supseteq B$ implies $A^C \in U$. Otherwise, $A \cap B \neq \emptyset$ for all $B \in U$. Then $\mathcal{F}_{\{A\}}$ is compatible with U, which implies $\mathcal{F}_{\{A\}} \subseteq U$. So $A \in U$. ☐

Theorem 4.49. *If $f : X \to Y$ and $\mathcal{U}$ is an ultrafilter on X then $f(\mathcal{U})$ is an ultrafilter on Y.*

Proof. Suppose $A \notin f(\mathcal{U})$. Then for all $B \in \mathcal{U}$ we have $A \supseteq f(B)$, that is, $A^C \cap f(B) \neq \emptyset$. Similarly if $A^C \notin f(\mathcal{U})$ then $A \cap f(B) \neq \emptyset$ for all $B \in \mathcal{U}$. But exactly one of $f^{-1}(A)$ and $f^{-1}(A^C)$ is in $\mathcal{U}$, a contradiction. ☐

Lemma 4.50.
1. *If $\mathcal{F} \to x$ then $x \in \overline{A}$ for all $A \in \mathcal{F}$.*
2. *If $\mathcal{U}$ is an ultrafilter and $x \in \overline{A}$ for all $A \in \mathcal{U}$, then $\mathcal{U} \to x$.*

Proof.
1. If $x \notin \overline{A}$ then there exists a neighborhood U of x with $U \cap A = \emptyset$. But by hypothesis $U, A \in \mathcal{F}$, a contradiction.
2. If $x \in \overline{A}$ for all $A \in \mathcal{U}$ then for all $B \in \mathcal{N}_x$ and $A \in \mathcal{U}$ we have $A \cap B \neq \emptyset$. So $\mathcal{N}_x$ is compatible with $\mathcal{U}$, which implies $\mathcal{N}_x \subseteq \mathcal{U}$ as $\mathcal{U}$ is an ultrafilter. Thus $\mathcal{U} \to x$.

☐

Theorem 4.51.
1. *A topological space X is Hausdorff if and only if every ultrafilter has ≤ 1 limit.*
2. *A topological space X is compact if and only if every ultrafilter as ≥ 1 limit.*

Proof.
1. For the forward direction, we will in fact show all filters have ≤ 1 limit. Suppose $\mathcal{F} \to x, y$ and U, V are disjoint neighborhoods of x, y respectively. Then $U, V \in \mathcal{F}$ but $U \cap V = \emptyset$, a contradiction.

   For the backward direction, suppose x, y do not have disjoint neighborhoods. Then $\mathcal{N}_x$ and $\mathcal{N}_y$ are compatible. Then $\mathcal{N}_x \vee \mathcal{N}_y$ is a filter and lives in some ultrafilter $\mathcal{U}$. Then $\mathcal{U} \to x, y$, a contradiction.

2. For the forward direction, let $\mathcal{U}$ be an ultrafilter. Consider $\{\overline{A} : A \in \mathcal{U}\}$. This has the finite intersection property and so $\cap_{A\in\mathcal{U}}\overline{A} \neq \emptyset$. If $x \in \cap_{A\in\mathcal{U}}\overline{A}$, then $\mathcal{U} \to x$ by the lemma.

   For the backward direction, suppose $\mathcal{O}$ is an open cover of X with no finite subcover. Let $\mathcal{B} = \{O_1^C \cap \cdots \cap O_n^C : O_i \in \mathcal{O}\}$. This is a filter base. Let $\mathcal{U}$ be an ultrafilter with $\mathcal{U} \supseteq \mathcal{F}_{\mathcal{B}}$. We have $\mathcal{U} \to x$ for some $x \in X$. So there exists $O \in \mathcal{O}$ with $x \in O$ and $O \in \mathcal{N}_x \subseteq U$. So we have $O^C, O \in \mathcal{U}$ (as $O^C \in \mathcal{B}$), a contradiction. □

Theorem 4.52. *$\mathcal{F}$ a filter on $\prod_{i\in I} X_i$ has $\mathcal{F} \to x = (x_i)_{i\in I}$ if and only if $\pi_i(\mathcal{F}) \to x_i$ for all $i \in I$.*

Proof. For the forward direction, π_i continuous implies $\pi_i(\mathcal{F}) \to \pi_i(x) = x_i$. For the backward direction, let O be a neighborhood of x in $\prod X_i$. We know $x \in \pi_{i_1}^{-1}(O_1) \cap \cdots \cap \pi_{i_n}^{-1}(O_n) \subseteq O$ for some open sets $O_j \in \pi_j(\mathcal{F})$ for all $j = 1, ..., n$. Then there exists $F_1, ..., F_n \in \mathcal{F}$ with $O_j \supseteq \pi_{i_j}(F_j)$. We know $F_1 \cap \cdots \cap F_n \subseteq \cap_1^n \pi_{i_j}^{-1}(O_j)$. So $\cap_1^n \pi_{i_j}^{-1}(O_j) \in \mathcal{F}$. Thus $O \in \mathcal{F}$, that is, $\mathcal{F} \to x$. □

Theorem 4.53 (**Tychonoff's Theorem**)**.** *X_i compact for all $i \in I$ implies $X = \prod X_i$ is compact.*

Proof. Let $\mathcal{U}$ be an ultrafilter on X. So $\pi_i(\mathcal{U})$ is an ultrafilter for all $i \in I$. So $\pi_i(\mathcal{U}) \to x_i$ for some $x_i \in X$. By the previous theorem $\mathcal{U} \to (x_i)_{i\in I}$, that is, every ultrafilter converges and thus X is compact. □

Given a topological space X let $uf(X) = \{\mathcal{U} | \mathcal{U}$ is an ultrafilter on $X\}$. Notice $uf(X)$ contains the **principal ultrafilters** $\mathcal{F}_{\{x\}} = \{A \subseteq X | x \in A\}$ for all $x \in X$. We can put a topology on $uf(X)$ as follows:

- Given $A \subseteq X$ define $S_A = \{\mathcal{U} \in uf(X) : A \in \mathcal{U}\}$ and $\mathcal{B} = \{S_A : A \subseteq X\}$. Note that $\mathcal{B}$ is a basis for the topology on $uf(X)$ as $S_x = uf(X)$ and $S_A \cap S_B = S_{A\cap B}$.

Lemma 4.54. *Any bounded continuous function $f : X \to \mathbb{R}$ has a continuous extension $\tilde{f} : uf(X) \to \mathbb{R}$.*

Proof. Define $\tilde{f}(\mathcal{U}) = \lim f(\mathcal{U})$. The limit exists as $f(\mathcal{U})$ is an ultrafilter on the compact set $[-M, M] \subseteq \mathbb{R}$ for some M. Note that $X \hookrightarrow uf(X)$ via $x \mapsto \mathcal{F}_{\{x\}}$. See $\tilde{f}(\mathcal{F}_{\{x\}}) = \lim f(\mathcal{F}_{\{x\}}) = \lim \mathcal{F}_{\{f(x)\}} = f(x)$. To show continuity, suppose $r \in O$ for some open O in $\mathbb{R}$ and $\tilde{f}(\mathcal{U}) = r$. Now there exists a neighborhood V of $\mathbb{R}$ with $\overline{V} \subseteq \mathcal{U}$. Notice that $\mathcal{U} \in S_{f^{-1}(V)}$ because $f(\mathcal{U}) \to r$. Then $V \in f(\mathcal{U})$ which implies $V \supseteq f(A)$ for some $A \in \mathcal{U}$ and thus $f^{-1}(V) \supseteq A$, that is, $f^{-1}(V) \in \mathcal{U}$. Now suppose $\mathcal{V}$ is an ultrafilter in $S_{f^{-1}(V)}$, so $f^{-1}(V) \in \mathcal{V}$, which implies $f(f^{-1}(V)) \in f(\mathcal{V})$, that is, $V \in f(\mathcal{V})$. We know $\lim f(\mathcal{V}) \in \overline{V} \subseteq U$, that is, $S_{f^{-1}(V)} \subseteq \tilde{f}^{-1}(U)$. Thus $\tilde{f}$ is continuous at $\mathcal{U}$. □

Theorem 4.55. *If X is discrete then $uf(X) = \beta X$.*

Proof. Note $S_{\{x\}} = \{\mathcal{F}_{\{x\}}\}$. So the induced topology on X from $uf(X)$ is discrete. Need to show $uf(X)$ is compact, Hausdorff and X is dense in $uf(X)$. Note $uf(X) \subseteq \{0,1\}^{\mathcal{P}(X)}$. Certainly this is Hausdorff but also compact as $uf(X)$ is closed in $\{0,1\}^{\mathcal{P}(X)}$: it is the intersection of

$$\begin{gathered}
\pi_{\emptyset}^{-1}(0), \\
\pi_A^{-1}(0) \cup \left[\pi_A^{-1}(1) \cap \pi_B^{-1}(1)\right], \text{ for } A \subseteq B, \\
\pi_A^{-1}(0) \cup \pi_B^{-1}(0) \cup \left[\pi_A^{-1}(1) \cap \pi_B^{-1}(1) \cap \pi_{A\cap B}^{-1}(1)\right], \text{ for } A, B \subseteq X, \\
\left[\pi_A^{-1}(1) \cap \pi_{A^C}^{-1}(0)\right] \cup \left[\pi_A^{-1}(0) \cap \pi_{A^C}^{-1}(1)\right] \text{ for } A \subseteq X
\end{gathered}$$

Now we need $\overline{X} = uf(X)$. Well consider a basic neighborhood S_A of $\mathcal{U}$. For all $a \in A$, we have $\mathcal{F}_{\{a\}} \in S_A$, that is, $S_A \cap X \neq \emptyset$. So $uf(X) = \beta X$. □

5 The Fundamental Group

We would like to get better at distinguishing "different" topological spaces. To do so, we need more invariants. Our first is a group called the **fundamental group**.

Definition 5.1. *Given two continuous functions $f_0, f_1 : X \to Y$ we write $f_0 \simeq f_1$ and say f_0 is **homotopic** to f_1 if there exists a continuous function $F : [0,1] \times X \to Y$ such that for all $x \in X$ $F(0, X) = f_0(x)$ and $F(1, x) = f_1(x)$.*

Definition 5.2. *Two paths $f_0, f_1 : I \to X$ with $f_0(0) = x_0 = f_1(0)$ and $f_0(1) = x_1 = f_1(1)$ are* ***path homotopic****, written $f_0 \simeq_p f_1$, if there exists $f : I \times I \to X$ such that for all $t \in I$ $F(0,t) = f_0(t), F(1,t) = f_1(t)$ and for all $s \in I$ $F(s,0) = x_0$ and $F(s,1) = x_1$.*

Theorem 5.3. *Both $\simeq$ and $\simeq_p$ are equivalence relations.*

Proof. Suppose $f \simeq g \simeq h$ for $f, g, h : X \to Y$. Let F be the function that takes f to g and G the function that takes g to h. Let

$$H(s,x) = \begin{cases} F(2s,x) & 0 \le s \le \frac{1}{2}, \\ G(2s-1,x) & \frac{1}{2} \le s \le 1. \end{cases}$$

This is continuous by the pasting lemma. Also if F, G are path homotopies, then H is a path homotopy.

If $f \simeq g$ by F, define $F' : I \times X \to Y$ by $F'(s,x) = F(1-s,x)$. This is a homotopy making $g \simeq f$. If F was a path homotopy, so would F'.

If $f : X \to Y$ then $F : I \times X \to Y$ given by $F(s,x) = f(x)$ is a homotopy. □

Theorem 5.4. *If $f_0, f_1 : X \to U \subseteq \mathbb{R}^n$ and U is convex, then $f_0 \simeq f_1$. Moreover, if f_0, f_1 are both paths from x_0 to x then $f_0 \simeq_p f_1$.*

Proof. Define $F : I \times X \to U$ by $F(s,x) = (1-s)f_0(x) + sf_1(x)$. This is clearly a homotopy from f_0 to f_1. If f_0, f_1 are paths from x_0 to x_1 then f is a path homotopy. □

<u>The path groupoid</u>

Given an x_0 to x_1 path f and a x_1 to x_2 path g in Y we define $f \cdot g$ by $f \cdot g(t) = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$ This is an x_0 to x_2 path.

Definition 5.5. *If $f : I \to X$ is a path then we write $[f]$ for the path homotopy equivalence class of f. For $f, g : I \to X$ with $f(1) = g(0)$ we define the operation $[f] \cdot [g] = [f \cdot g]$ on the equivalence classes.*

Lemma 5.6. *The operation on equivalence classes is well-defined.*

Proof. If $f \simeq_p f'$ and $g \simeq_p g'$ then $f'(1) = g'(0)$. One can also see $f' \cdot g' \simeq_p f \cdot g$. □

Lemma 5.7. *If $f_0, f_1 : X \to Y$ are (path) homotopic and $g : Y \to Z$ is a continuous map, then $g \circ f_0$ and $g \circ f_1$ are (path) homotopic.*

Proof. If F is the map that takes f_0 to f_1, then $g \circ F$ is the map that takes $g \circ f_0$ to $g \circ f_1$. □

Definition 5.8. *If X is a topological space and $x \in X$ then $e_x : I \to X$ defined by $t \mapsto x$ is the* ***constant path at*** *x. If $f : I \to X$ is a path, we let $\overline{f} : I \to X$ be defined by $t \mapsto f(1-t)$, the same path but opposite arrow.*

Theorem 5.9.

1. *If either side of the following equation is defined then so is the other and they are equal:*

$$[f] \cdot ([g] \cdot [h]) = ([f] \cdot [g]) \cdot [h]$$

2. *If f is a path from x to y then $[e_x] \cdot [f] = [f]$ and $[f] \cdot [e_y] = [f]$.*

3. *If f is a path from x to y then $[f] \cdot [\overline{f}] = [e_x]$ and $[\overline{f}] \cdot [f] = [e_y]$.*

Proof.

2. Consider the paths $e_0 : I \to I$ and $i : I \to I$ defined by $t \mapsto t$. Then I is a convex subset of $\mathbb{R}$ and so $e_0 \cdot i \simeq i$. Thus $e_x \circ f \simeq_p f \circ (e_0 \cdot i) \simeq_p f \circ i = f$.

3. Let $t : I \to I$ be the function $t = i \cdot \bar{i}$. Then $t \simeq_p e_0$ and so $f \cdot \overline{f} = f \circ (i \cdot \bar{i}) \simeq_p f \circ e_0 = e_x$ and $\overline{f} \cdot f = f \circ (\bar{i} \cdot i) \simeq_p \overline{f} \circ e_0 = e_y$.

1. Define the path $\gamma : I \to I$ to be

$$\gamma = \begin{cases} 2t & 0 \le t \le \frac{1}{4} \\ t + \frac{1}{4} & \frac{1}{4} \le t \le \frac{1}{2} \\ \frac{1}{2}t + \frac{1}{2} & \frac{1}{2} \le t \le 1 \end{cases}.$$

Then $(f \cdot (g \cdot h)) \circ \gamma = (f \cdot g) \cdot h$. Since $\gamma \simeq i$, we see $(f \cdot g) \cdot h = (f \cdot (g \cdot h)) \circ \gamma \simeq_p ((f \cdot (g \cdot h)) \circ i = f \cdot (g \cdot h)$. □

Definition 5.10. *If X is a topological space and $x_0 \in X$ then $\pi_1(X, x_0) = \{[f] | f : I \to X \text{ has } f(0) = f(1) = x_0\}$ is the collection of all homotopy classes of loops in X based at x_0. This is a group under concatenation with identity $[e_0]$ and $[f]^{-1} = [\overline{f}]$.*

Example. If $U \subseteq \mathbb{R}^n$ is convex and $x_0 \in U$, then $\pi_1(U, x_0) = \{[e_{x_0}]\}$.

Theorem 5.11. *If $\alpha : I \to X$ is a path in X with $\alpha(0) = x_0$ and $\alpha(1) = x_1$ and we define $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by $[f] \mapsto [\overline{\alpha}] \cdot [f] \cdot [\alpha]$, then $\hat{\alpha}$ is an isomorphism.*

Proof. First note that it is a homomorphism as

$$\begin{aligned} \hat{\alpha}([f] \cdot [g]) &= [\overline{\alpha}] \cdot [f] \cdot [g] \cdot [\alpha] \\ &= [\overline{\alpha}] \cdot [f] \cdot [e_{x_0}] \cdot [g] \cdot [\alpha] \\ &= [\overline{\alpha}] \cdot [f] \cdot [\alpha] \cdot [\overline{\alpha}] \cdot [g] \cdot [\alpha] \\ &= \hat{\alpha}([f]) \cdot \hat{\alpha}([g]). \end{aligned}$$

To show it is an isomorphism, we will show $\hat{\alpha}^{-1} = \hat{\overline{\alpha}}$. For $[f] \in \pi_1(X, x_0)$, we see

$$\hat{\overline{\alpha}} \cdot \hat{\alpha}([f]) = [\alpha] \cdot [\overline{\alpha}] \cdot [f] \cdot [\alpha] \cdot [\overline{\alpha}] = [e_{x_0}] \cdot [f] \cdot [e_{x_0}] = [f]$$

and similarly $\hat{\alpha} \cdot \hat{\overline{\alpha}}([f]) = [f]$. □

Corollary 5.12. *If X is path connected then for all $x_0, x_1 \in X$ we have $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

Definition 5.13. *If $f : X \to Y$ and $f(x_0) = y_0$ we write $f : (X, x_0) \to (Y, y_0)$ and we define $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $[\gamma] \mapsto [f \circ \gamma]$.*

Theorem 5.14. *If $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$, then $g_* \circ f_* = (g \circ f)_*$ and $id_*^X = id^{\pi_1(X, x_0)}$.*

Proof. Note that $g_* \circ f_*([\gamma]) = [g \circ f \circ \gamma] = (g \circ f)_*([\gamma])$ and $id_*([\gamma]) = [\gamma]$. □

In particular, if $f : (X, x_0) \to (Y, y_0)$ is a homeomorphism then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

Example. A space is called **simply connected** if its path connected and $\pi_1(X)$ is trivial. In a simply connected space, two paths are path homotopic if and only if they have the same endpoints.

Proof. Suppose α, β are paths from x to y. Then

$$[\alpha] = \underbrace{[\alpha] \cdot [\overline{\beta}]}_{\in \pi_1(X, x) = \{[e_x]\}} \cdot [\beta] = [e_x] \cdot [\beta] = [\beta].$$

□

Example. $\pi_1(S') \cong \mathbb{Z}$. There are three ways to approach this problem.

1. The analysts will say $S' \subseteq \mathbb{C}$ and a path γ in S' has a winding number $n(\gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z} dz$.
2. The engineers would construct an apparatus with a crank and ruler. Turning the crank one full revolution would make the marker go from 0 to 1 on the ruler and turning it one full revolution in the opposite direction would make it turn from 0 to -1. This gives the same winding number as in the analysts approach.
3. The topologists would wind the real line up like a slinky and define a projection map $\pi : \mathbb{R} \to S'$ by $x \mapsto (\cos(2\pi x), \sin(2\pi x))$. This is called a **covering map,** that is, it has the property that for all $x \in S'$ there exists a neighborhood U of x such that $\pi^{-1}(U)$ is a disjoint union of open sets in $\mathbb{R}$, each homeomorphic by a restriction map of $\pi \to U$.

Remark. If $\pi : E \to B$ is a covering map then

1. π is open

   Proof. If U is open in B and $y = \pi(z) \in \pi(U)$ with $z \in U$, then there exists a neighborhood V of y such that V is a **good neighborhood**, that is, one as in the definition of a covering map. Now there exists α such that $z \in V_\alpha$, one of the parts of $\pi^{-1}(V)$. So $z \in V_\alpha \cap U \subseteq V_\alpha$ and $\pi(V_\alpha \cap U)$ is open in B since $\pi|_{V_\alpha}$ is a homeomorphism. □

2. $\pi^{-1}(y)$ is discrete in E for all $y \in B$.

Proof. If V is good neighborhood of y, then each V_α contains exactly one point of $\pi^{-1}(y)$. $\square$

Definition 5.15. *If $\pi : E \to B$ is a covering map and $f : X \to B$ is continuous then a **lift** of f is any continuous map $\tilde{f} : X \to E$ with $f = \pi \circ \tilde{f}$.*

Lemma 5.16. *If $\pi : (E, e_0) \to (B, b_0)$ is a covering map, then for all paths $\gamma : I \to B$ with $\gamma(0) = b_0$, there exists a unique map $\tilde{\gamma} : I \to E$ lifting γ with $\tilde{\gamma}(0) = e_0$.*

Proof. Let $\mathcal{O}$ be a covering of B by good neighborhoods. Then $\{\gamma^{-1}(U) : U \in \mathcal{O}\}$ is an open cover of I and hence has a Lebesgue number. Thus we can decompose I into subintervals $[t_i, t_{i+1}]$ where $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ such that $\gamma([t_i, t_{i+1}])$ is contained in some good neighborhood for $i = 0, ..., n-1$. We will build $\tilde{\gamma}$ inductively. If $\gamma([0, t]) \subseteq V$ for some good neighborhood and $\pi^{-1}(V) = \cup_{\alpha \in I} V_\alpha$, suppose $e_0 \in V_0$ (one of the V_α's). Define $\tilde{\gamma}|_{[0,t]} = (\pi|_{V_0})^{-1} \circ \gamma$ (as $\pi|_{V_0}$ is a homeomorphism, this makes sense). Having defined $\tilde{\gamma}$ on $[0, t_i]$, do the same for $[t_i, t_{i+1}]$: If $\gamma([t_i, t_{i+1}]) \subseteq U$ for some good neighborhood and $\gamma(\tilde{t_i}) = x \in U_0$ for one of the parts of $\pi^{-1}(U)$, define $\tilde{\gamma}|_{[t_i,t_{i+1}]} = (\pi|_{U_0})^{-1} \circ \gamma$. Then $\tilde{\gamma}$ defined this way is continuous by the Pasting Lemma and is a lift of γ by construction.

Suppose now $\tilde{\gamma}, \tilde{\gamma}'$ are two lifts of γ with $\tilde{\gamma}(0) = e_0 = \tilde{\gamma}'(0)$. Suppose they agree on $[0, t_i]$ for some $i = 0, ..., n$ (certainly true for $i = 0$). On $[t_i, t_{i+1}]$, there is a good neighborhood of V with $\gamma([t_i, t_{i+1}]) \subseteq V$ and $\tilde{\gamma}(t_i) = \tilde{\gamma}'(t_i) \in V_0$, one of the parts of $\pi^{-1}(V)$. We have $\tilde{\gamma}([t_i, t_{i+1}]), \tilde{\gamma}'([t_i, t_{i+1}]) \subseteq \pi^{-1}(V)$ which has a separation $\{V_0, \pi^{-1}(V) \setminus V_0\}$. Since $[t_i, t_{i+1}]$ is connected, we see $\tilde{\gamma}([t_i, t_{i+1}]), \tilde{\gamma}'([t_i, t_{i+1}]) \subseteq V_0$. So $\pi|_{V_0} \circ \tilde{\gamma}|_{[t_i,t_{i+1}]} = \pi|_{V_0} \circ \tilde{\gamma}'|_{[t_i,t_{i+1}]}$. As $\pi|_{V_0}$ is invertible, we see $\tilde{\gamma}|_{[t_i,t_{i+1}]} = \tilde{\gamma}'|_{[t_i,t_{i+1}]}$. By induction, $\tilde{\gamma} = \tilde{\gamma}'$. $\square$

Lemma 5.17 (Homotopy Lifting). *If $\pi : E \to B$ is a covering map and $\pi(e_0) = b_0$ and $F : I \times I \to B$ is a (path) homotopy with $F(0, 0) = b_0$, then there exists a unique lift $\tilde{F} : I \times I \to E$ with $\tilde{F}(0, 0) = e_0$ which is a (path) homotopy.*

Proof. Let $f(t) = F(0, t), \gamma(s) = F(s, 0)$ and $\tilde{f}, \tilde{\gamma}$ be the unique lifts of f, γ starting at e_0. If $\mathcal{O}$ is an open cover of B by good neighborhoods then there exists partitions $0 = t_0 < t_1 < \cdots < t_n = 1$ and $0 = s_0 < s_1 < \cdots < s_m = 1$ such that for all i, j there exists $0 \in \mathcal{O}$ such that $F([s_i, s_{i+1}] \times [t_j, t_{j+1}]) \subseteq O$. Order the set of rectangles $\{[s_i, s_{i+1}] \times [t_j, t_{j+1}]\}$ by lexicographic order on (i, j). We will prove by induction that there exists a unique lift on the union of any initial segment of these rectangles. At every stage we know F (our rectangle) is contained in a good neighborhood V in B and we have values already determined on Γ (the left and top edges of the rectangle). Since Γ is connected, we have $\tilde{F(\Gamma)} \subseteq V_0$, where V_0 is one of the parts of $\pi^{-1}(V)$. If we lift F to $\tilde{F}$ we require $\pi \circ \tilde{F} = F$. Since the rectangle is connected we would have for any lifting extending the previous values $\tilde{F}$(our box) $\subseteq V_0$. Then $\pi|_{V_0} \circ \tilde{F} = F$ but this serves to define $\tilde{F} = (\pi|_{V_0})^{-1} \circ F$. The total $\tilde{F}$ is continuous by the Pasting Lemma. $\square$

Corollary 5.18. *If $\pi : (E, e_0) \to (B, b_0)$ is a covering map and $f, g : I \to B$ are path homotopic with $f(0) = g(0) = b_0$, then the unique lifts $\tilde{f}, \tilde{g}$ starting at e_0 are path homotopic, in particular, $\tilde{f}(1) = \tilde{g}(1)$.*

Theorem 5.19. $\pi_1(S') \simeq \mathbb{Z}$.

Proof. Consider $\pi_1(S', (1, 0))$ and the covering map $\pi : (\mathbb{R}, 0) \to (S', (1, 0))$ by $x \mapsto (\cos 2\pi x, \sin 2\pi x)$. Let $\phi : \pi_1(S', (1, 0)) \to \mathbb{Q}$ by $[f] \mapsto \tilde{f}(1)$ where $\tilde{f}$ is the unique lift of f with $\tilde{f}(0) = 0$. By the corollary, ϕ is well-defined.

Claim. ϕ is injective.

Proof. Suppose we have $\phi[f] = \phi[g]$, that is, $\tilde{f}(1) = \tilde{g}(1)$. Since $\mathbb{Q}$ is simply connected, $\tilde{f} \simeq_p \tilde{g}$ so $g = \pi \circ \tilde{f} \simeq_p \pi \circ \tilde{g} = g$. So $[f] = [g]$.

Claim. ϕ is surjective.

Proof. Let $\tilde{w}_n(t) = nt$. Then $w_n = \pi \circ \tilde{w}_n$ is a loop in S', and w_n lifts to $\tilde{w}_n$. So $\phi[w_n] = n$.

Claim. ϕ is a homomorphism.

Proof. Suppose f, g are loops in S' at $(1, 0)$ and $\phi([f]) = n, \phi([g]) = m$. Let $\tilde{f}, \tilde{g}$ be the standard lifts and define $\tilde{g}'(t) = \tilde{g}(t) + n$. We have $f \tilde{\cdot} g = \tilde{f} \cdot \tilde{g}$. Since clearly the right hand side is continuous (by the Pasting Lemma), starts at O and has $\pi \circ (\tilde{f} \cdot \tilde{g}') = f \cdot g$. We have $\phi([f] \cdot [g]) = (\tilde{f} \cdot \tilde{g}')(1) = m + n$. $\square$

Definition 5.20. *If $\pi : (E, e_0) \to (B, b_0)$ is a covering map then the functor $\phi : \pi_1(B, b_0) \to \pi^{-1}(b_0)$ defined by $[f] \mapsto \tilde{f}(1)$ is called the **lifting correspondence**.*

Corollary 5.21. *1. If E is path connected, then ϕ is surjective.*

2. If E is simply connected, then ϕ is bijective.

Examples.

- $\pi : \mathbb{R}^2 \to S' \times S'$ defined by $(x, y) \mapsto ((\cos 2\pi x, \sin 2\pi x), (\cos 2\pi y, \sin 2\pi y))$ is a covering map of the torus by the plane. The lifting correspondence is a bijection $\pi_1(S' \times S') \to \mathbb{Z} \times \mathbb{Z}$. In fact, it is an isomorphism.
- The real projective plan $\mathbb{RP}^2 = S^2/(x \sim -x)$. The quotient map $\pi : S^2 \to S^2/(x \sim -x)$ is a covering map. In particular, since S^2 is simply connected, $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$.

Some Applications

Definition 5.22. *If $A \subseteq X$ a topological space we say that $r : X \to A$ is a **retraction** if $r|_A = id_A$.*

Example. The central band of the möbius band is a retract of the band.

Lemma 5.23. *If A is a retract of X and $i : A \hookrightarrow X$ is the inclusion map then $i_* : \pi_1(A, a) \to \pi_1(X, a)$ is injective for all $a \in A$.*

Proof. $\mathrm{id}_A = r \circ i : (A, a) \to (A, a)$. So $(\mathrm{id}_A)_* = \mathrm{id}_{\pi_1(A,a)} = r_* \circ i_* : \pi_1(A, a) \to \pi_1(A, a)$. Since i_* is invertible on the left, it is injective. $\square$

Theorem 5.24. *There is no retraction $D^2 \to S'$.*

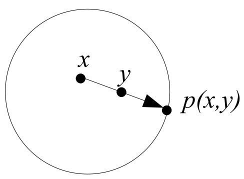
Proof. Recall $D^2 = \{z \in \mathbb{Q}^2 : |z| \leq 1\}$ is convex. As $i_* : \pi_1(S') \to \pi_1(D^2)$ is $i_* : \mathbb{Z} \to 0$, it is not injective. $\square$

Theorem 5.25 (Fundamental Theorem of Algebra). *If $p \in \mathbb{C}[x]$ is nonconstant, then there exists $z \in \mathbb{C}$ such that $p(z) = 0$.*

Proof. Define f_s, a loop in S' for $s \in [0, \infty)$, by $f_s(t) = \frac{p(se^{e\pi it})/p(s)}{|p(se^{2\pi it})|/|p(s)|}$. If p is never 0, this is always defined. All these paths are path homotopic. In particular, they are all homotopic to $f_0 = e_1$. WLOG, $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$. Pick $s_0 > \sum_1^n |a_i|$ with $s_0 \geq 1$. If $|z| = s_0$ we have $|z^n| > \sum_1^n |a_i| s_0^{n-i} \geq |\sum_1^n a_i z^{n-i}|$. Thus on $\{|z| = s_0\}$ the polynomial $p^\alpha = z^n + \alpha \sum a_i z^{n-i}$ for $\alpha \in [0, 1]$ has no zeros. Define $f_{s_0} = \frac{p^\alpha(s_0 e^{2\pi it})/p^\alpha(s_0)}{|p^\alpha(s_0 e^{2\pi it})|/|p^\alpha(s_0)|}$. We have $f_0 \simeq_p f_{s_0} = f_{s_0}^1 \simeq_p f_{s_0}^0$ where $f_{s_0}^0 = e^{2\pi int}$. The lift of this is $t \mapsto nt$ which finishes at n. So $0 = n$. Thus a polynomial with no zeros is constant. $\square$

Theorem 5.26 (Brower Fixed Point Theorem for D^2). *Let $D^2 = \{x \in \mathbb{R}^2 : ||x|| \leq 1\}$. If $f : D^2 \to D^2$ is continuous, then there exists a fixed point of f.*

Proof. Define a map $p : D^2 \times D^2 \setminus \Delta \to S' \subseteq D^2$ where $\Delta = \{(x, x) | x \in D^2\}$ by



If $f : D^2 \to D^2$ is continuous and has no fixed point, then define $f(x) = p(f(x), x)$. Then f is continuous and for $x \in S'$ we have $f(x) = x$, that is, f is a retraction of D^2 onto S' and by the previous lemma, no such beast exists. $\square$

Theorem 5.27. *For $n \geq 2$, $\pi_1(S^n) = 0$ (the group with one element).*

Proof. Recall $S^n \setminus \{point\} \cong \mathbb{R}^n$ as S^n is the one-point compactification. Consider a loop $f : I \to S^n$ and pick $y \notin f(I)$. Then $f : I \to S^n \setminus \{y\} \cong \mathbb{R}^n$. Therefore $f \simeq_p e_{f(0)}$. We will prove every path is path homotopic to one omitting at least one point. Suppose f is based at x_0 and $x \neq x_0$ and B is a neighborhood of x missing x_0 (that is, $x_0 \notin \overline{B}$). Lets pick B nicely homomorphic to D^n. Consider $f^{-1}(B) = \cup(a_i, b_i)$, a countable (possibly finite) union of disjoint intervals. Note $f^{-1}(\{x\})$ is compact and covered by $\cup(a_i, b_i)$. Thus there exists a finite number of intervals covering $f^{-1}(\{x\})$. For each such interval, we have $f([a_i, b_i]) \subseteq \overline{B}$ and $f(a_i), f(b_i) \in \overline{B} \setminus B$. Path homotop each segment $f|_{[a_i, b_i]}$ to a path avoiding x (in order). (We can do this as B is simply connected). $\square$

Lemma 5.28. *If $h : I \to S'$ is continuous and has $h(t+\frac{1}{2}) = -h(t)$ for all $t \in [0, \frac{1}{2}]$, then $[h]$ is an odd element of $\pi_1(S', h(0))$.*

Proof. WLOG $h(0) = (1,0)$. Consider the lift of h starting at 0 and call it $\tilde{h}$. Then $\tilde{h}(t+\frac{1}{2}) - \tilde{h}(t) \in \{\frac{q}{2} : q \text{ is an odd integer}\}$. As the left hand side is continuous from I which is connected and the right hand side is a discrete space, the function is constant. So there exists q_0, an odd integer, with $\tilde{h}(t+\frac{1}{2}) - \tilde{h}(t) = \frac{q_0}{2}$ for all $t \in [0, \frac{1}{2}]$. Thus $\tilde{h}(1) = \frac{q_0}{2} + \frac{q_0}{2} = q_0$. □

Theorem 5.29 (Borsuk-Ulan Theorem for $n = 2$). *If $f : S^2 \to \mathbb{R}^2$ is continuous there exists $x \in S^2$ with $f(-x) = f(x)$.*

Proof. Suppose $f : S^2 \to \mathbb{R}^2$ is continuous and $f(x) \neq f(-x)$ for all $x \in S^2$. Define $g : S^2 \to S^2$ by $g(x) = \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$. Let ν be the equatorial loop in S^2. Consider $h = g \circ \nu : I \to s^1$. This has $h(t+\frac{1}{2}) = -h(t)$ for all $t \in [0, \frac{1}{2}]$. By the lemma, $[h] = q$ for some odd integer q. Also $[h] = g_*[\nu] \in \pi_1(S_2, \nu(0))$, which implies $[h] = [e_{\nu(0)}]$ as G_* is a homomorphism, a contradiction. □

Theorem 5.30. *X, Y path connected implies $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.*

Proof. Pick $x_0 \in X, y_0 \in Y$. Define $\phi : \pi_1(X, x_0) \times \pi_1(Y, y_0) \to \pi_1(X \times Y, (x_0, y_0))$ by $[\gamma], [\nu] \mapsto [(\gamma, \nu)]$ and $\psi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ by $[\gamma] \mapsto [\pi_x \circ \gamma], [\pi_Y \circ \gamma]$. There are well-defined and are inverses of each other. □

Corollary 5.31. $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{Z}$.

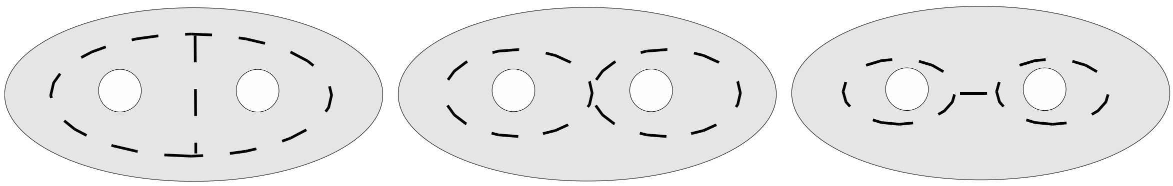
Proof. $\mathbb{R}^2 \setminus \{0\} \cong S^1 \times (0, \infty)$ by using polar coordinates. As $\pi_1((0,\infty))$ is trivial, done by the theorem. □

Corollary 5.32. $\mathbb{R}^2 \ncong \mathbb{R}^n$ *for* $n \geq 3$.

Proof. $\pi_1(\mathbb{R}^n \setminus \{\text{point}\}) \cong \pi_1(S^{n-1} \times (0, \infty)) = 0$. Done by previous corollary. □

Definition 5.33. *$A \subseteq X$ is a* ***deformation retract*** *of X if there exists a homotopy $R : X \times I \to X$ between $r_0 = 1_X$ and r_1 a retraction $X \to A$ such that $R(a, s) = a$ for all $a \in A, s \in I$, that is, there exists a retraction $r : X \to A$ such that with $i : A \hookrightarrow X$ the canonical inclusion we have $r \circ i = 1_A$ and $i \circ r \simeq_A 1_X$ ($\simeq_A$ means homotopy relative to A, so it is the identity on A).*

Examples. The following are three examples of deformation retracts. In all cases, X is given by the shaded region and A by the dotted line.



Definition 5.34. *Spaces X and Y are* ***homotopy equivalent*** *if there exists continuous maps $\phi : X \to Y$ and $\psi : Y \to X$ such that $\phi \circ \psi \simeq 1_Y$ and $\psi \circ \phi \simeq 1_X$.*

Definition 5.35. *A* ***base point preserving homotopy*** *from (X, x_0) to (Y, y_0) is a map $F : X \times Y \to Y$ such that $F(x_0, s) = y_0$ for all $s \in I$.*

Lemma 5.36. *If ϕ_0 is homotopic to ϕ_1 where $\phi_0, \phi_1 : (X, x_0) \to (Y, y_0)$ by a base point preserving homotopy Φ, then $(\phi_0)_* = (\phi_1)_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.*

Proof. $[\phi_0 \circ f] = [\phi_1 \circ f]$ through $\Phi \circ f$. □

If A is a deformation retract of X and $a_0 \in A$ then $\pi_1(A, a_0) \cong \pi_1(X, a_0)$. Since we have $i_* : \pi_1(A, a_0) \to \pi_1(X, a_0), r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ and $r_* \circ i_* = (r \circ i)_* = (1_A)_* = 1_{\pi_1(A, a_0)}$ but $i_* \circ r_* = (i \circ r)_* = (1_X)_* = 1_{\pi_1(X, x_0)}$.

Lemma 5.37. *If $\Phi : X \times I \to Y$ is a homotopy from ϕ_0 to ϕ_1 and α is the path traced out by $x_0 \subseteq X$, that is, $\alpha(s) = \phi(x_0, s)$ for $s \in I$, the the following diagram commutes:*

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "(\phi_0)_*"] \arrow[dr, "(\phi_1)_*"'] & \pi_1(Y, \phi_0(x_0)) \arrow[d, "\hat{\alpha}"] \\ & \pi_1(Y, \phi_1(x_0)) \end{tikzcd}$$

Proof. Define $\alpha_s : I \to Y$ by $t \mapsto \alpha(1-s+st)$. This is the tail end of α traversed at the right speed so that it takes time 1 to complete. Let $\phi_s : X \to Y$ be defined by $x \mapsto \Phi(x,s)$. We have $\overline{\alpha_0} \cdot (\phi_1 \circ \gamma) \cdot \alpha_0 \simeq_p \overline{\alpha_1} \cdot (\phi_0 \circ \gamma) \cdot \alpha_1$ through $\overline{\alpha_s} \cdot (\phi_s \circ \gamma) \cdot \alpha_s$. So

$$\phi_1 \circ \gamma \simeq_p e_{\phi_1(x_0)} \cdot (\phi_1 \circ \gamma) \cdot e_{\phi_1(x_0)} = \overline{\alpha_0} \cdot (\phi_1 \circ \gamma) \cdot \alpha_0 \simeq_p \overline{\alpha_1} \cdot (\phi_0 \circ \gamma) \cdot \alpha_1 = \overline{\alpha} \cdot (\phi_0 \circ \gamma) \cdot \alpha.$$

So $(\phi_1)_*(\gamma) = [\overline{\alpha}] \cdot [\phi_0 \circ \gamma] \cdot [\alpha] = \hat{\alpha}((\phi_0)_*[\gamma])$. $\square$

Theorem 5.38. *If $\phi : X \to Y$ is a homotopy equivalence then $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$ is an isomorphism for all $x_0 \in X$.*

Proof. Let ψ be a homotopy inverse to ϕ. Consider

$$\pi_1(X, x_0) \xrightarrow{\phi_*} \pi_1(Y, \phi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi\phi(x_0)) \xrightarrow{\phi_*} \pi_1(Y, \phi\psi\phi(x_0)).$$

We have the first pairwise composition $\psi_* \circ \phi_*$ is an isomorphism. Since by the previous lemma $\psi \circ \phi$ is homotopic to 1_X, there exists α with $\psi_* \circ \phi_* = (\psi \circ \phi)_* = \hat{\alpha} \circ (1_X)_*$. In particular, ϕ_* is injective. Similarly, $\phi_* \circ \psi_*$ is also an isomorphism and thus ψ_* is injective. As $\psi_* \circ \phi_*$ is surjective, we have ϕ_* is surjective. Thus ϕ_* is an isomorphism. $\square$