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Journal of Pure and Applied Algebra 215 (2011) 1492–1501

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

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ARTICLE INFO

Article history:
Received 21 April 2010
Received in revised form 27 July 2010
Available online 4 November 2010
Communicated by A.V. Geramita

MSC: 52B30; 13P25; 13N15

ABSTRACT

In this note we study modules of derivations on collections of linear subspaces in a finite dimensional vector space. The central aim is to generalize the notion of freeness from hyperplane arrangements to subspace arrangements. We call this generalization 'derivation radical'. We classify all coordinate subspace arrangements that are derivation radical and show that certain subspace arrangements of the Braid arrangement are derivation radical. We conclude by proving that under an algebraic condition the subspace arrangement consisting of all codimension c intersections, where c is fixed, of a free hyperplane arrangement are derivation radical.

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1. Introduction

Let V be a vector space of dimension ℓ over a field $\mathbb{K}$ and $S := \mathrm{Sym}(V^*) \cong \mathbb{K}[x_1, \dots, x_\ell]$ be the polynomial ring. Here an arrangement of subspaces $\mathcal{A}$ will mean a finite collection of linear subspaces in V. Suppose that $\mathcal{A} = \{X_1, \dots, X_n\}$ and the defining ideal of X_i is I_i. The radical ideal defining the variety of the union of the subspaces is $I(\mathcal{A}) = \cap I_i$. The main focus here is the module of derivations of $\mathcal{A}$ (also known as the module of logarithmic vector fields along $\mathcal{A}$) defined as

$$D^1(\mathcal{A}) := \{\theta \in \mathrm{Der} | \theta(I(\mathcal{A})) \subseteq I(\mathcal{A})\}$$

where Der is the S-module of $\mathbb{K}$ linear S-derivations. The elements of $\mathrm{Der} \cong \bigoplus_{i=1}^n S \frac{\partial}{\partial x_i}$ can be viewed as polynomial vector fields.

Algebraic properties of subspace arrangements were studied in several recent papers. In [12] Peeva et al. showed that the cohomology of the complex complement of some specific subspace arrangements, there called diagonal arrangements, given by intersections of hyperplanes from the braid hyperplane arrangement $\mathcal{A}_\ell = \{\mathbb{V}(x_i - x_j) | 1 \le i < j \le \ell\}$, can be calculated via algebraic techniques. Li and Li in [8], Lovász and Klietman in [9], and Sidman in [14] all studied generators of the defining ideal of certain types of diagonal subspace arrangements. In [5] Derksen and Sidman prove the Castelnuovo–Mumford regularity of $I(\mathcal{A})$ is less than or equal to $|\mathcal{A}|$. In [4] Derksen examines the Hilbert series of $I(\mathcal{A})$ and studies when it is a combinatorial invariant. A good summary of some of this algebraic work can be found in [15] by Sidman.

Given the literature on algebraic properties of $I(\mathcal{A})$ for $\mathcal{A}$ a subspace arrangement and the vast amount of literature on $D^1(\mathcal{A})$ where $\mathcal{A}$ is a hyperplane arrangement it is somewhat surprising that $D^1(\mathcal{A})$ has received relatively little attention when $\mathcal{A}$ is a subspace arrangement. In [3] Brumatti and Simis study derivations on monomial ideals, which if they are square free are the defining ideals of subspace arrangements given by intersections of coordinate hyperplanes. There Brumatti and Simis present a combinatorially defined generating set and compute bounds on the depth of the derivation module. Very recently Tadesse [17] has shown that the derivations on a monomial ideal I also preserve its multiplier ideals and certain closures of I.

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0022-4049/$ – see front matter. Published by Elsevier B.V.
doi:10.1016/j.jpaa.2010.09.007

A gateway paper to the topic of derivations on subspace arrangements was written by Wiens [18]. He proves that if a subspace arrangement $\mathcal{A}$ contains a subspace of codimension higher than 1 then $D^1(\mathcal{A})$ is not a free S-module. Then Wiens presents generators for $D^1(\mathcal{A})$ where $\mathcal{A}$ is a subspace arrangement given by intersections of hyperplanes in general position.

In this paper we present a class of subspace arrangements generalizing that of free hyperplane arrangements. Towards this aim we define exterior products of derivations on subspace arrangements and study their properties in Section 2. In the same section we define derivation radical subspace arrangements, which generalize free hyperplane arrangements. These are pure c-codimensional subspace arrangements $\mathcal{A}$ such that the defining ideal $I(\mathcal{A})$ is equal to the ideal $M_c(\mathcal{A})$ which is generated by $\ell - c + 1$ minors of the coefficient matrix for a set of generators for $D^1(\mathcal{A})$.

In Section 3 we classify all the coordinate subspace arrangements (whose defining ideals are square free monomial ideals) that are derivation radical. In Section 4 we show that certain diagonal arrangements are derivation radical. Finally in Section 5 we compare the derivation radical property on subspace arrangements to that of free hyperplane arrangements. In doing so, we find that the algebraic information contained in derivations on a subspace arrangement can be very subtle.

In [2] Björner et al. present combinatorial methods to find the generators of an arbitrary subspace arrangement's defining ideal. They focused primarily on the case where the generators can be taken to be products of linear forms. We are interested in a class of subspace arrangements that arise from combinatorics and geometry. In particular, let $\mathcal{A}(c)$ be the collection of all codimension c intersections of a hyperplane arrangement $\mathcal{A}$. Theorem 5.8 states that if $M_c(\mathcal{A})$ has no embedded primes then $\mathcal{A}(c)$ is derivation radical and the generators of the defining ideal $I(\mathcal{A}(c))$ of the subspace arrangement can be easily described by the derivations on the hyperplane arrangement $\mathcal{A}$. The embedded primes condition is surprising but cannot be avoided by assuming that the hyperplane arrangement $\mathcal{A}$ is free, as we show in Example 5.11.

2. Exterior products of derivations

In this section we define higher derivations on a subspace arrangement. Some of this introductory material is taken from [16] and generalized to the more general setting of subspace arrangements. For a general reference on subspace arrangements see [1].

Definition 2.1. The p-th derivation module is $\mathrm{Der}^p := \bigwedge^p \mathrm{Der}$ for $p \geq 1$ and $\mathrm{Der}^0 := S$. So Der^p is an S-module by multiplication. Let $\theta_1, \ldots, \theta_p \in \mathrm{Der}^1$ and $f_1, \ldots, f_p \in S$ then the monomial $\theta_1 \wedge \cdots \wedge \theta_p$ acts on S^p by

$$[\theta_1 \wedge \cdots \wedge \theta_p](f_1, \ldots, f_p) = \sum_{\sigma \in \mathbf{S_p}} (-1)^{sgn(\sigma)} \theta_1(f_{\sigma(1)}) \cdots \theta_p(f_{\sigma(p)})$$

where $\mathbf{S_p}$ is the symmetric group. This extends to an action of Der^p on S^p via linearity.

Note that this definition makes $\theta \in \mathrm{Der}^p$ a derivation in each variable: $\theta(f_1, \ldots, f_k g_k, \ldots, f_p) = f_k \theta(f_1, \ldots, g_k, \ldots, f_p) + g_k \theta(f_1, \ldots, f_k, \ldots, f_p)$.

Let $\mathcal{A} = \{X_1, \ldots, X_n\}$ be a subspace arrangement in V. The union of these subspaces $\mathcal{V}(\mathcal{A}) = \bigcup_{1 \leq i \leq n} X_i$ is a variety in $\mathbb{K}^\ell$ and its defining ideal is denoted $I(\mathcal{A}) = \{f \in S | f(\mathcal{V}(\mathcal{A})) = 0\}$. For now we do not assume there are no inclusions of elements in $\mathcal{A}$. However, later we will assume that the subspaces in $\mathcal{A}$ all have the same codimension. Now we define the module of derivations on a subspace arrangement.

Definition 2.2. The p-th module of derivations on $\mathcal{A}$ is

$$D^p(\mathcal{A}) = \{\theta \in \mathrm{Der}^p | \theta(I(\mathcal{A}) \times S^{p-1}) \subseteq I(\mathcal{A})\}.$$

If a subspace arrangement $\mathcal{A}$ consists of just one element X then we will write $I(X)$ for its ideal and $D^p(X)$ for its module of derivations.

Remark 2.3. This definition is different from Wiens's Definition 2.1 in [18] which presents $D^1(\mathcal{A})$ as the intersection of all the $D^1(X_i)$ for all $X_i \in \mathcal{A}$. Wiens's definition is arguably more suitable to subspace arrangements. However it is less generalizable to arbitrary varieties or schemes. Actually some of the lemmas below can be easily generalized to arbitrary schemes.

Lemma 2.4. *If $\mathcal{A}$ is a subspace arrangement with no inclusions then $D^p(\mathcal{A}) = \bigcap_{X \in \mathcal{A}} D^p(X)$.*

Proof. Let $\theta \in \bigcap_{X \in \mathcal{A}} D^p(X)$, $s \in I(\mathcal{A})$, and $f_2, \ldots, f_p \in S$. Since $s \in I(X)$ for all X we know $\theta(s, f_2, \ldots, f_p) \in I(X)$ for all X. Hence, $\theta \in D^p(\mathcal{A})$.

Let $\theta \in D^p(\mathcal{A})$. Suppose there exists $Y \in \mathcal{A}$ such that $\theta \notin D^p(Y)$. Then there exists $s \in I(Y)$ and $f_2, \ldots, f_p \in S$ such that $\theta(s, f_2, \ldots, f_p) \notin I(Y)$. Since there are no inclusions $\bigcap_{X \neq Y} I(X) \not\subseteq I(Y)$ and there exists $q \in \bigcap_{X \neq Y} I(X)$ such that $q \notin I(Y)$. But $qs \in I(\mathcal{A})$, $\theta(qs, f_2, \ldots, f_p) \in I(\mathcal{A}) \subseteq I(Y)$, and

$$\theta(qs, f_2, \ldots, f_p) = s\theta(q, f_2 \ldots, f_p) + q\theta(s, f_2, \ldots, f_p).$$

This is a contradiction because $I(Y)$ is a prime ideal, $s\theta(q, f_2 \ldots, f_p) \in I(Y)$, and $q\theta(s, f_2, \ldots, f_p) \notin I(Y)$. Therefore $\theta \in \bigcap_{X \in \mathcal{A}} D^p(X)$. $\square$

For two subspace arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ we say that $\mathcal{A}_1 \subseteq \mathcal{A}_2$ if $X \in \mathcal{A}_1$ implies $X \in \mathcal{A}_2$. Now the next result follows from Lemma 2.4.

Lemma 2.5. *If $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and both arrangements do not have inclusions then $D^p(\mathcal{A}_1) \supseteq D^p(\mathcal{A}_2)$.*

Remark 2.6. Lemma's 2.4 and 2.5 are false for inclusions of subspaces. Let $\ell = 2$, X be the origin, and Y a dimension 1 subspace. We can choose coordinates on V^* such that $I(X) = (x_1, x_2)$ and $I(Y) = (x_1)$. With this basis $D^1(X)$ is generated by $x_1\partial_1, x_2\partial_2, x_1\partial_2$, and $x_2\partial_1$ and $D^1(Y)$ is generated by $x_1\partial_1$ and ∂_2. Hence $D^1(X) \not\subseteq D^1(Y)$ and $D^1(X) \not\supseteq D^1(Y)$.

Definition 2.7. We will say that a subspace arrangement $\mathcal{A}$ is *pure c-codimensional* if $\operatorname{codim}(X_i) = c$ for all $X_i \in \mathcal{A}$.

Let $L(\mathcal{A})$ be the set of all intersections of the elements of $\mathcal{A}$. Then one can order $L(\mathcal{A})$ by reverse inclusion to make it a lattice. The lattice $L(\mathcal{A})$ is an important combinatorial invariant of $\mathcal{A}$.

Definition 2.8. Suppose that $\mathcal{A}$ is a pure k-codimensional subspace arrangement. Given an integer c such that $k \le c \le \operatorname{codim}(\cap\mathcal{A})$ let $\mathcal{A}(c) = \{X \in L(\mathcal{A}) \mid \operatorname{codim}(X) = c\}$ be the codimension c loci of $\mathcal{A}$, which we call the *level c arrangement* of $\mathcal{A}$.

Contrary to Remark 2.6 we will show that when the subspace arrangement is given by intersections of a larger subspace arrangement then Lemma 2.5 holds even in the case of inclusions. This result is also crucial to studying the higher codimension loci of a hyperplane arrangement.

Lemma 2.9. *If $\mathcal{A}$ is a pure subspace arrangement of codimension k then for all c such that $k \le c \le \operatorname{codim}(\cap\mathcal{A})$ we have*

$$D^p(\mathcal{A}) \subseteq D^p(\mathcal{A}(c)).$$

Proof. Let $\theta \in D^p(\mathcal{A})$ and $X \in \mathcal{A}(c)$. Since $k \le c \le \operatorname{codim}(\cap\mathcal{A})$ there exists $Y_s, \ldots, Y_c \in \mathcal{A}$ such that $X = Y_1 \cap \cdots \cap Y_c$. Because $I(X) = \sum I(Y_i)$ Lemma 2.4 says that for all i and $f_t \in S$, $\theta(I(Y_i), f_2, \ldots, f_p) \subseteq I(Y_i) \subseteq I(X)$ and hence $\theta \in D^p(X)$. Since $\mathcal{A}(c)$ has no inclusions Lemma 2.4 again shows that $\theta \in D^1(\mathcal{A}(c))$. □

Lemma 2.10. *If $\mathcal{A}$ is an arrangement of subspaces of an ℓ-dimensional vector space and $I(\mathcal{A})$ is its ideal then $D^\ell(\mathcal{A}) = I(\mathcal{A})\mathrm{Der}^\ell$.*

Proof. For any subspace X the module $D^\ell(X) = I(X)\mathrm{Der}^\ell$. We may assume that $\mathcal{A}$ has no inclusions because the ideal $I(\mathcal{A})$ only contains the information of the subspace arrangement with no inclusions. Then the statement follows from Lemma 2.4. □

The next theorem is a generalization of Saito's criterion (see [13,11]) and provides the basis for the primary definition of this paper. This theorem is known by some experts but because it is crucial to the main definition of this paper and the authors could not find it in the literature we include it and a proof. If $\mathcal{A}$ is an arrangement where all the subspaces have codimension c then $\bigwedge^{\ell-c+1} D^1(\mathcal{A}) \wedge \mathrm{Der}^{c-1} = M_c(\mathcal{A})\mathrm{Der}^\ell$ where $M_c(\mathcal{A})$ is an ideal of S. Note that $M_c(\mathcal{A})$ can also be thought of as the ideal generated by all $\ell - c + 1$ minors of the coefficient matrix of a generating set of $D^1(\mathcal{A})$.

Theorem 2.11. *If $\mathcal{A}$ is a pure c-codimensional arrangement then $\sqrt{M_c(\mathcal{A})} = I(\mathcal{A})$.*

Proof. First we show that $M_c(\mathcal{A}) \subseteq I(\mathcal{A})$. Let $\theta_1, \ldots, \theta_{\ell-c+1} \in D^1(\mathcal{A})$ and $\psi_1, \ldots, \psi_{c-1} \in \mathrm{Der}^1$. Fix coordinates for V^* and then note that $\mathrm{Der}^\ell \cong S(\partial_1 \wedge \cdots \wedge d_\ell)$ so

$$\theta_1 \wedge \cdots \wedge \theta_{\ell-c+1} \wedge \psi_1 \wedge \cdots \wedge \psi_{c-1} = g\partial_1 \wedge \cdots \wedge \partial_\ell.$$

Let $X \in \mathcal{A}$ and suppose that $I(X) = (\alpha_1, \ldots, \alpha_c)$. The forms $\alpha_1, \ldots, \alpha_c$ are linearly independent over $\mathbb{K}$. Choose $\beta_1, \ldots, \beta_{\ell-c}$ linear forms such that $\alpha_1, \ldots, \alpha_c, \beta_1, \ldots, \beta_{\ell-c}$ are linearly independent linear forms. Rename this list as follows: $(\alpha_1, \ldots, \alpha_c, \beta_1, \ldots, \beta_{\ell-c}) = (f_1, \ldots, f_\ell)$. Now

$$\theta_1 \wedge \cdots \wedge \theta_{\ell-c+1} \wedge \psi_1 \wedge \cdots \wedge \psi_{c-1}(\alpha_1, \ldots, \alpha_c, \beta_1, \ldots, \beta_{\ell-c})$$
$$= \sum_{\sigma \in \mathbf{S}_\ell} (-1)^{sgn(\sigma)} \theta_1(f_{\sigma(1)}) \cdots \theta_{\ell-c+1}(f_{\sigma(\ell-c+1)}) \psi_1(f_{\sigma(\ell-c+2)}) \cdots \psi_{c-1}(f_{\sigma(\ell)})$$

is in $I(X)$ because in each term of the sum at least one α_i will be acted on by some θ_j and $\theta_j \in D^1(X)$ by Lemma 2.5. Since

$$\partial_1 \wedge \cdots \wedge \partial_\ell(f_1, \ldots, f_\ell)$$

is a non-zero constant, then

$$\theta_1 \wedge \cdots \wedge \theta_{\ell-c+1} \wedge \psi_1 \wedge \cdots \wedge \psi_{c-1}(f_1, \ldots, f_\ell)$$

is a non-zero multiple of g. Hence $g \in I(X)$ for all $X \in \mathcal{A}$ implies $M_c(\mathcal{A}) \subseteq I(\mathcal{A})$.

We conclude by showing that $I(\mathcal{A}) \subseteq \sqrt{M_c(\mathcal{A})}$. Let $s \in I(\mathcal{A})$ and put $\theta_i = s\partial_i$. Then $\theta_i \in D^1(\mathcal{A})$ and

$$\theta_1 \wedge \cdots \wedge \theta_{\ell-c+1} \wedge \partial_{\ell-c+2} \wedge \cdots \wedge \partial_\ell = s^{\ell-c+1}\partial_1 \wedge \cdots \wedge \partial_\ell \in M_c(\mathcal{A})\mathrm{Der}^\ell.$$

Hence $s \in \sqrt{M_c(\mathcal{A})}$. □

Now we state the main definition of the paper.

Definition 2.12. We say that a pure c-codimensional subspace arrangement $\mathcal{A}$ is *derivation radical* if $M_c(\mathcal{A}) = I(\mathcal{A})$.

If $\mathcal{A}$ is a hyperplane arrangement then $\mathcal{A}$ is free if and only if $\mathcal{A}$ is derivation radical due to Saito's criterion in [13] (also see Theorem 4.19 in [11]). So, if a hyperplane arrangement is not free then it is not derivation radical. Wiens [18] proved that if a subspace arrangement $\mathcal{A}$ has a subspace of codimension higher than 1 then the module $D^1(\mathcal{A})$ is not free. However, many subspace arrangements are derivation radical (see the following example) and these arrangements exhibit properties similar to free hyperplane arrangements.

Example 2.13. Let $\mathcal{A}$ be the collection of coordinate axes in $\mathbb{K}^3$. The defining ideal can be written as

$$I(\mathcal{A}) = (x_1, x_2) \cap (x_1, x_3) \cap (x_2, x_3) = (x_1x_2, x_1x_3, x_2x_3)$$

and has pure codimension $c = 2$. The module of derivations is

$$D(\mathcal{A}) = \langle x_1\partial_1, x_2\partial_2, x_3\partial_3, x_2x_3\partial_1, x_1x_3\partial_2, x_1x_2\partial_3\rangle.$$

Now the ideal $M_2(\mathcal{A})$ is clearly equal to $I(\mathcal{A})$.

One might wonder whether the derivation radical property is preserved when we intersect with a generic hyperplane arrangement. The following example shows that this is not the case.

Example 2.14. Let $\mathcal{A}$ be the Boolean arrangement in $\mathbb{P}^3$: $I(\mathcal{A}) = (x_1x_2x_3x_4)$. The arrangement $\mathcal{A}$ is free and hence derivation radical. Intersecting with a generic plane defined by $L = 0$ in $\mathbb{P}^3$ gives a subspace arrangement $\mathcal{A} \cap L$ consisting of four coplanar lines in $\mathbb{P}^3$. To check whether $\mathcal{A} \cap L$ is derivation radical, we note that $\ell - c + 1 = 4 - 2 + 1 = 3$ so we need to check whether the ideal $I(\mathcal{A} \cap L)$ is generated by the 3×3-minors of the coefficient matrix of a generating set of $D^1(\mathcal{A} \cap L)$. However, it is easy to check that all derivations on $\mathcal{A} \cap L$ must have degree ≥ 1 so the linear form $L \in I(\mathcal{A} \cap L)$ cannot be generated. Thus the subspace arrangement $\mathcal{A} \cap L$ obtained by intersecting with a generic hyperplane is not derivation radical. Moreover, restricting to a generic hyperplane also produces a non-derivation radical subspace arrangement. Thus restricting the previous example $\mathcal{A}$ to a generic plane $L = 0$ gives a non-derivation radical arrangement in the space $L = 0$ since the restriction is a generic hyperplane arrangement of four lines in $\mathbb{P}^2$, which is always non-free and hence not derivation radical.

It is also natural to wonder how derivation radical arrangements behave when combined using various products.

Definition 2.15. Let $\mathcal{A}_i$ be a pure subspace arrangement of codimension c_i in the vector space V_i of dimension ℓ_i ($i = 1, 2$). Define the product

$$\mathcal{A}_1 \times \mathcal{A}_2 = \{X \oplus V_2 : X \in \mathcal{A}_1\} \cup \{V_1 \oplus Y : Y \in \mathcal{A}_2\}.$$

Then $I(\mathcal{A}_1 \times \mathcal{A}_2) = I(\mathcal{A}_1)I(\mathcal{A}_2)$ in a polynomial ring in $\ell_1 + \ell_2$ variables. Define a second product

$$\mathcal{A}_1 \odot \mathcal{A}_2 = \{X \oplus Y : X \in \mathcal{A}_1, Y \in \mathcal{A}_2\}.$$

Then $I(\mathcal{A}_1 \odot \mathcal{A}_2) = I(A_1) + I(A_2)$, where the sum again takes place in a polynomial ring in $\ell_1 + \ell_2$ variables.

Remark 2.16. The product $\times$ may seem less natural than the product $\odot$; however, it is the usual notion of a product used in the hyperplane literature (see Orlik and Terao's Definition 2.13 in [11]). The product $\times$ is a very natural notion of a product because the intersection lattice of the product of two arrangements is isomorphic to the product of the two intersection lattices and the characteristic polynomial of the product is the product of the respective characteristic polynomials.

Remark 2.17. It is easy to see that a subspace arrangement $\mathcal{A}$ contained in the vector space V is derivation radical if and only if $\mathcal{A} \odot \mathbb{K}$ (or equivalently $\mathcal{A} \times \{\emptyset\}$) is derivation radical in the larger vector space $V \oplus \mathbb{K}$. In this vein, it is natural to wonder whether $\mathcal{A} \odot \{0\}$ is derivation radical in $V \oplus \mathbb{K}$ when $\mathcal{A}$ is derivation radical in V — that is, whether the notion of derivation radical is intrinsic to the variety corresponding to $\mathcal{A}$. However, the notion of derivation radical depends on the ambient vector space; for example, let $\mathcal{A}$ be the arrangement of two coordinate axes in $\mathbb{K}^2$. The arrangement $\mathcal{A}$ is clearly derivation radical though the arrangement $\mathcal{A} \odot \{0\}$ fails to be derivation radical in $\mathbb{K}^3$ because the defining ideal contains a linear form that cannot be realized as a 2×2 minor of the appropriate matrix. *Hence we assume for the rest of the paper that all arrangements are not contained in a proper subspace of V.*

It is natural to ask whether $\mathcal{A}_1 \times \mathcal{A}_2$ or $\mathcal{A}_1 \odot \mathcal{A}_2$ are derivation radical subspace arrangements when both $\mathcal{A}_1$ and $\mathcal{A}_2$ are derivation radical. In fact, neither product preserves the derivation radical property (see Examples 3.6 and 3.7).

3. Coordinate subspace arrangements

In this section we focus on those subspace arrangements defined by intersections of coordinate hyperplanes. These are called coordinate arrangements. Let $K \subseteq 2^{[\ell]}$ be a simplicial complex with ℓ vertices and $I_K = \langle x_{i_1} \cdots x_{i_s} | \{i_1, \ldots, i_s\} \notin K\rangle$ its

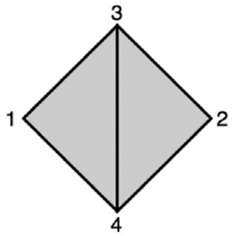

Fig. 1. The simplicial complex with all subsets except those containing $\{1, 2\}$.

Stanley–Reisner ideal in the polynomial ring S. Let max K be the set of all maximal subsets of K. The subspace arrangement associated to K is $\mathcal{A}_K = \{X_\sigma\}_{\sigma \in \max K}$ where $X_\sigma = \{x \in V | x_i = 0 \text{ if } i \notin \sigma\}$. Then the defining ideal of $\mathcal{A}_K$ is I_K.

Remark 3.1. There are coordinate subspace arrangements, intersections of coordinate hyperplanes, that are not $\mathcal{A}_K$ for some simplicial complex K. These are the coordinate subspace arrangements that have all subspaces inside a proper subspace of V. These arrangements cannot be derivation radical see Remark 2.17.

The goal of this section is to characterize the coordinate subspace arrangements that are derivation radical. While one could do this without the use of simplicial complexes we feel that it more clearly illuminates the details.

Definition 3.2. We call a simplicial complex K a *k-skeleton* if its maximal subsets are all the possible subsets of $[\ell]$ of size $k + 1$. A *cone* of a simplicial complex is a new simplicial complex obtained by adding a vertex and inserting this vertex into every non-trivial subset.

We will say that a simplicial complex is a *coning* of a skeleton if it can be constructed from successive cones of a skeleton. By Remark 2.17, coning does not affect the derivation radical property since the arrangement corresponding to a coning of a simplicial complex K can be obtained by the product $\mathcal{A}_K \odot \mathbb{K}$.

Example 3.3. Let $\ell = 4$ and K be the simplicial complex consisting of all subsets of $[\ell]$ except any that contain $\{1, 2\}$. Fig. 1 is a realization of this simplex. It can be viewed as a double cone over the simplicial complex consisting of just two points, which is a 0-skeleton. Hence K is a coning of a skeleton. The ideal is $I_K = (x_1x_2)$. Then the arrangement consists of two hyperplanes. Such arrangements are always free, and hence derivation radical.

Now we state the main theorem of this section which together with Remark 3.1 characterizes all the coordinate subspace arrangements that are derivation radical.

Theorem 3.4. *$\mathcal{A}_K$ is derivation radical if and only if K is a coning over a skeleton of a simplex.*

Proof. If K is a coning of a $(k-2)$-skeleton then there exist $G \subseteq [\ell]$ such that $I(\mathcal{A}_K) = (x_{i_1} \cdots x_{i_k})_{\{i_1,\ldots,i_k\} \in G}$. The codimension of $I(\mathcal{A}_K)$ is $c = |G| - k + 1$ so $|G| - c + 1 = k$. Let $\theta_i = x_i \partial_i$ and note that $\theta_i \in D^1(\mathcal{A}_K)$. Let $W \subseteq [\ell]$ such that $|W| = k$. If $W = \{i_1, \ldots, i_k\}$ then put $\theta_W = \theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$. If $[\ell] \backslash W = \{j_1, \ldots, j_{\ell-k}\}$ then put $\partial_{[\ell]\backslash W} = \partial_{j_1} \wedge \cdots \wedge \partial_{j_{\ell-k}}$. Then for every $W \subseteq [\ell]$ let s_W be the polynomial resulting from the product

$$\theta_W \wedge \partial_{[\ell]\backslash W} = s_W \partial_1 \wedge \cdots \wedge \partial_\ell \in \bigwedge^k D^1(\mathcal{A}_K) \wedge \mathrm{Der}^{\ell-k}.$$

Hence $s_W \in M_c(\mathcal{A}_K)$ and $s_W = x_{i_1} \cdots x_{i_k}$. Thus $I(\mathcal{A}_K) \subseteq M_c(\mathcal{A}_K)$ which implies $\mathcal{A}_K$ is derivation radical.

Suppose that $\mathcal{A}_K$ is derivation radical and the codimension of $I(\mathcal{A}_K)$ is c. Without loss of generality we may assume that K is not a coning of any simplex. Let $I = (x_{i_1} \cdots x_{i_k})_{\{i_1,\ldots,i_k\} \in [\ell]}$ where $k = \ell - c + 1$ then $I(\mathcal{A}_K) \subseteq I$. We know that the θ_i from above are always contained in $D^1(\mathcal{A}_K)$ for all simplicial complexes. Hence $I \subseteq I(\mathcal{A}_K)$ from the argument above. □

Remark 3.5. Notice that the derivation radical coordinate subspace arrangements correspond to the coordinate subspace arrangements that are level c arrangements $\mathcal{A}(c)$ of a Boolean hyperplane arrangement $\mathcal{A}$ (i.e. the hyperplane arrangement consisting of some of the coordinate hyperplanes). In the next section we will see another example of this form and in the last section we will focus primarily on this topic.

We use Theorem 3.4 to give two examples that show that the product operations defined in Definition 2.15 do not preserve the derivation radical property.

Example 3.6. Let $\mathcal{A}_1$ be the arrangement in $V_1 \cong \mathbb{K}^2$ defined by $I(\mathcal{A}_1) = (x_1, x_2)$ and let $\mathcal{A}_2$ be the arrangement in $V_2 \cong \mathbb{K}^2$ defined by $I(\mathcal{A}_2) = (x_3, x_4)$. Then $\mathcal{A}_1 \times \mathcal{A}_2$ has defining ideal $(x_1, x_2)(x_3, x_4) = (x_1x_3, x_1x_4, x_2x_3, x_2x_4)$ in the ring $\mathbb{K}[x_1, x_2, x_3, x_4]$. Both $\mathcal{A}_1$ and $\mathcal{A}_2$ are derivation radical since they are the coordinate arrangements associated to (-1)-skeletons. However, $\mathcal{A}_1 \times A_2$ is the coordinate arrangement associated to the simplicial complex on 4 vertices that consists of two non-adjacent line segments. This is not a cone and not a skeleton, so the product $\mathcal{A}_1 \times \mathcal{A}_2$ is not derivation radical.

Example 3.7. Let $\mathcal{A}_1$ be the arrangement in $V_1 \cong \mathbb{K}^2$ defined by $I(\mathcal{A}_1) = (x_1x_2)$ and let $\mathcal{A}_2$ be the arrangement in $V_2 \cong \mathbb{K}^2$ defined by $I(\mathcal{A}_2) = (x_3x_4)$. Then $\mathcal{A}_1 \odot \mathcal{A}_2$ has defining ideal (x_1x_2, x_3x_4) in the ring $\mathbb{K}[x_1, x_2, x_3, x_4]$. Both $\mathcal{A}_1$ and $\mathcal{A}_2$ are derivation radical since they are the coordinate arrangements associated to 0-skeletons. However, $\mathcal{A}_1 \odot A_2$ is the coordinate arrangement associated to the simplicial complex on 4 vertices that consists of 4 edges forming a square. This is not a cone and not a skeleton, so $\mathcal{A}_1 \odot \mathcal{A}_2$ is not derivation radical.

4. Braid subspace arrangements

Let $H_{ij} = \{x_i - x_j = 0\}$ then $\mathcal{A}_\ell = \{H_{ij}\}_{1\le i<j\le \ell}$ is the famous Braid arrangement (also known as a Coxeter arrangement of type A). Here we want to study the collection of all codimension c subspaces arising from intersections of hyperplanes from $\mathcal{A}_\ell$ which we will denote $\mathcal{A}_\ell(c) = \{X \in L(\mathcal{A}_\ell) \mid \operatorname{codim}(X) = c\}$. In this section we show that this arrangement is derivation radical and moreover that the defining ideal $I(\mathcal{A}_\ell(c))$ is generated by minors of the derivations on the hyperplane arrangement $\mathcal{A}_\ell$. To do this we need a little combinatorial notation.

It is well known that the intersection lattice $L(\mathcal{A}_\ell)$ is isomorphic to the partition lattice Π_ℓ. For each partition $\pi \in \Pi_\ell$ there corresponds a subspace X_π (the subspace where $x_i = x_j$ if i and j are in the same block of π) and if the number of blocks of π is m then $\operatorname{codim}(X_\pi) = \ell - m$. Then

$$\mathcal{A}_\ell(c) = \{X_\pi \mid \pi \text{ has } \ell - c \text{ blocks}\}.$$

With a fixed partition π and $i, j \in [\ell]$ we say that $i \equiv j$ if i and j are in the same block in π. Let

$$f_\pi = \prod_{i<j;\, i\equiv j} (x_i - x_j).$$

Then in [9] Lovász presents the following result of Kleitman and Lovász (unpublished).

Theorem 4.1. *The ideal $I(\mathcal{A}_\ell(c))$ is generated by all f_π where π has only one non-trivial block and this block has size $c - 1$.*

Now we recall a basis for $D^1(\mathcal{A}_\ell)$ (see [11]). For $0 \le i \le \ell - 1$ let $\theta_i = \sum_{j=1}^{\ell} x_j^i \partial_j$. Then the collection $\{\theta_0, \ldots, \theta_{\ell-1}\}$ is a basis for $D^1(\mathcal{A}_\ell)$. Let $M(\theta_0, \ldots, \theta_{\ell-1}) = (\theta_i(x_j))$ be the coefficient matrix of this basis. Then each upper-most minor of size $\ell - c + 1$ is a Vandermonde determinant and any other minor is a product of variables times a Vandermonde determinant. Hence the ideal of $\ell - c + 1$ minors of $M(\theta_0, \ldots, \theta_{\ell-1})$ is equal to the ideal $M_c(\mathcal{A}_\ell)$. Now the polynomials f_π where π is a partition with only one non-trivial block of size $c + 1$ are equal to the upper-most $\ell - c + 1$ minors of $M(\theta_0, \ldots, \theta_{\ell-1})$. We have established the following result.

Corollary 4.2. *The subspace arrangements $\mathcal{A}_\ell(c)$ are derivation radical. Moreover, $M_c(\mathcal{A}_\ell) = I(\mathcal{A}_\ell(c))$.*

Remark 4.3. Corollary 4.2 shows that the generating set of the ideal $I(\mathcal{A}_\ell(c))$ of the subspace arrangement $\mathcal{A}_\ell(c)$ can be obtained from the derivations on the hyperplane arrangement $\mathcal{A}_\ell$. This presents an attractive way to find generators of the defining ideals of level arrangements of hyperplane arrangements. We study this topic in the next section.

5. Subspace arrangements from hyperplane arrangements

In this section we will assume $\mathcal{A} = \{H_1, \ldots, H_n\}$ is an essential (i.e. $\cap\mathcal{A} = 0$) hyperplane arrangement. Then Saito's criterion (see [13,11]) and Theorem 2.11 imply that $D^1(\mathcal{A})$ is a free S-module (we will also say '$\mathcal{A}$ is free') if and only if $\mathcal{A}$ is derivation radical. Given an integer $1 \le c \le \ell$ recall the notation from Section 2: let $\mathcal{A}(c) = \{X \in L(\mathcal{A}) \mid \operatorname{codim}(X) = c\}$ be the codimension c loci of $\mathcal{A}$. In this case $\mathcal{A}(1) = \mathcal{A}$ and $\mathcal{A}(\ell)$ is the origin. The following lemma is a special case of Lemma 2.9.

Lemma 5.1. $D^1(\mathcal{A}) \subseteq D^1(\mathcal{A}(c))$

Recall that $M_c(\mathcal{A})$ is the ideal generated by the $\ell - c + 1$ minors of the generating matrix for $D^1(\mathcal{A})$. Lemma 5.1 and Theorem 2.11 imply $M_c(\mathcal{A}) \subseteq M_c(\mathcal{A}(c)) \subseteq I(\mathcal{A}(c))$. In many cases $M_c(\mathcal{A}) = M_c(\mathcal{A}(c))$; however this is not true in general: see Example 5.12. In any case, we have two chains of ideals

$$0 \subseteq I(\mathcal{A}) \subseteq I(\mathcal{A}(2)) \subseteq \cdots \subseteq I(\mathcal{A}(\ell - 1)) \subseteq I(\mathcal{A}(\ell)) = S_+ \subseteq S$$

and

$$0 \subseteq M_1(\mathcal{A}) \subseteq M_2(\mathcal{A}) \subseteq \cdots \subseteq M_{\ell-1}(\mathcal{A}) \subseteq M_\ell(\mathcal{A}) = S_+ \subseteq S.$$

Note that $I(\mathcal{A}) = (Q)$ and $M_1(\mathcal{A}) \subseteq (Q)$ where Q is the defining polynomial of the hyperplane arrangement $\mathcal{A}$. However, $M_1(\mathcal{A}) = (Q)$ if and only if $D^1(\mathcal{A})$ is free. In this section we will study the difference between the ideals $M_c(\mathcal{A})$ and $I(\mathcal{A}(c))$.

Suppose that $\mathcal{A}$ is free and $D^1(\mathcal{A})$ is generated by $\{\theta_1, \ldots, \theta_\ell\}$. We briefly describe the generators of the ideal $M_c(\mathcal{A})$ in terms of the polynomial Q. For any $\ell - c + 1$-tuple $I = \{i_1, \ldots, i_{\ell-c+1}\} \subseteq [\ell]$ let $C(I) = [\ell]\backslash I$ be its complement and denote $\theta_I = \theta_{i_1} \wedge \cdots \wedge \theta_{i_{\ell-c+1}}$ and $\partial_I = \partial_{i_1} \wedge \cdots \wedge \partial_{i_{\ell-c+1}}$. For any i the derivation $Q\partial_i \in D^1(\mathcal{A})$ and $Q^{c-1}\partial_{C(I)} \in D^{c-1}(\mathcal{A})$. Let $\ell[c-1]$ be the set of $c - 1$ tuples in $[\ell]$. Since $\mathcal{A}$ is free, by Proposition 3.4 of [16], for any $(c - 1)$-tuple T there exist $f_{T,J} \in S$ such that

$$Q^{c-1}\partial_T = \sum_{J\in \ell[c-1]} f_{T,J}\theta_J.$$

For any $(\ell - c + 1)$-tuple I the derivation $\theta_I \wedge Q^{c-1}\partial_T$ is in $D^\ell(\mathcal{A})$ and since it is divisible by Q^{c-1} there is a polynomial $g_{I,T} \in S$ such that

$$\theta_I \wedge Q^{c-1}\partial_T = Q^{c-1} g_{I,T}\partial_{[\ell]}.$$

In fact dividing this equation by Q^{c-1} we see that $g_{I,T}$ are the generators of $M_c(\mathcal{A})$. We also have

$$\theta_I \wedge Q^{c-1}\partial_T = \theta_I \wedge \left(\sum_{J\in \ell[c-1]} f_{T,J}\theta_J\right) = \theta_I \wedge f_{T,C(I)}\theta_{C(I)} = f_{T,C(I)} Q\,\partial_{[\ell]}.$$

Hence

$$Q^{c-1} g_{I,T} = f_{T,C(I)} Q$$

and

$$g_{I,T} = \frac{f_{T,C(I)}}{Q^{c-2}}.$$

Thus, the generators of $M_c(\mathcal{A})$ are the coefficients of the decomposition of the derivation

$$Q\,\partial_T = \sum_{J\in \ell[c-1]} \frac{f_{T,J}}{Q^{c-2}}\theta_J.$$

However these coefficients are still very hard to compute let alone tell whether or not the ideal generated by them is radical. Actually the derivation radical property of a codimension c loci of a free hyperplane arrangement can be very subtle. The rest of the paper is focused on this topic. We first assume that $\mathcal{A}$ is an arbitrary essential hyperplane arrangement (not necessarily free).

Let $\theta \in \mathrm{Der}^1$ and $\{e_1, \ldots, e_\ell\}$ be a basis for V dual to the basis $\{x_1, \ldots, x_\ell\}$ for V^*. We can write $\theta = \sum_{i=1}^{\ell} p_i\partial_i$ where $p_i \in S$. Then for $v \in V$ we define $\theta(v) = \sum_{i=1}^{\ell} p_i(v)e_i$. Notice the ambiguity with a derivation evaluated on a polynomial; we will note the difference whenever there could be confusion. For any $v \in V$ let $\mathcal{A}_v = \{H \in \mathcal{A} \mid v \in H\}$. The following lemma is essentially Proposition 9.4 in [10]; however the language is slightly different so we state it and include a proof.

Lemma 5.2. *For any $v \in V$*

$$D^1(\mathcal{A})(v) = \bigcap_{H\in\mathcal{A}_v} H.$$

Proof. If $\theta \in D^1(\mathcal{A})$ and $H \in \mathcal{A}_v$ then $\theta(\alpha_H)(v) = \alpha_H(\theta(v)) = 0$ since $\alpha_H|\theta(\alpha_H)$. Hence $\theta(v) \in H$ for every $H \in \mathcal{A}_v$ and we have $D^1(\mathcal{A})(v) \subseteq \bigcap_{H\in\mathcal{A}_v} H$.

Proving the opposite inclusion requires a little more machinery. For any $w \in V$ with $w = \sum w_ie_i$ let $\partial_w = \sum w_i\partial_i$. Put

$$\theta_w = \left[\prod_{H\in\mathcal{A}\setminus\mathcal{A}_w} \alpha_H\right]\partial_w.$$

For any $w \in V$ the derivation $\theta_w \in D^1(\mathcal{A})$ and $\theta_w(w) = cw$ where c is a non-zero constant.

Let $\{b_1, \ldots, b_k\}$ be a basis for $\bigcap_{H\in\mathcal{A}_v} H$. Then any $z \in \bigcap_{H\in\mathcal{A}_v} H$ can be written as $z = \sum_{i=1}^{k} z_ib_i$. Also note that $\mathcal{A}_{b_i} \supseteq \mathcal{A}_v$. Hence $\theta_{b_i}(v) = f_ib_i$ for a non-zero constant f_i. Now let

$$\theta = \sum_{i=1}^{k} \frac{z_i}{f_i}\theta_{b_i}.$$

Then $\theta(v) = z$ and we have completed the proof. □

Theorem 2.11 says that the defining ideal of $\mathcal{A}(c)$ is determined by the derivations on $\mathcal{A}(c)$. However, the following theorem shows that the level c arrangements $\mathcal{A}(c)$ of a hyperplane arrangement $\mathcal{A}$ are in fact determined by the derivations on $\mathcal{A}$.

Theorem 5.3. *If $\mathcal{A}$ is any essential hyperplane arrangement then $\sqrt{M_c(\mathcal{A})} = I(\mathcal{A}(c))$.*

Proof. Fix $1 \le c \le \ell$. We will show that the solution sets of the two ideals are equal. So, let Z_M and Z_I be the solution sets of the ideals $M_c(\mathcal{A})$ and $I(\mathcal{A}(c))$ respectively. Lemma 5.1 states that $D^1(\mathcal{A}) \subseteq D^1(\mathcal{A}(c))$, hence $M_c(\mathcal{A}) \subseteq M_c(\mathcal{A}(c))$. Since $M_c(\mathcal{A}(c)) \subseteq I(\mathcal{A}(c))$ we have that $M_c(\mathcal{A}) \subseteq I(\mathcal{A}(c))$ and $Z_M \supseteq Z_I$.

Choose bases for V and V^* as before. Let $\{\theta_1, \ldots, \theta_t\}$ be a generating set for $D^1(\mathcal{A})$. We call

$$M(\theta_1, \ldots, \theta_t) = (\theta_j(x_i))$$

a generating matrix for $D^1(\mathcal{A})$. For all ordered $(\ell-c+1)$-tuples $I=(i_1,\dots,i_{\ell-c+1})\in[\ell]^{\ell-c+1}$ and all ordered $(c-1)$-tuples $J=(j_1,\dots,j_{c-1})\in[\ell]^{c-1}$ let $g_{I,J}$ be the polynomial arising from

$$\theta_I\wedge\partial_J=\theta_{i_1}\wedge\cdots\wedge\theta_{i_{\ell-c+1}}\wedge\partial_{j_1}\wedge\cdots\wedge\partial_{j_{c-1}}=g_{I,J}\partial_{[\ell]}.$$

This polynomial is equal to the determinant of the $\ell-c+1\times\ell-c+1$ minor of $M(\theta_1,\dots,\theta_t)$ with rows given by I and columns given by $C(J)=[\ell]\backslash J$. Hence $M_c(\mathcal{A})$ can also be characterized as the ideal generated by all $\ell-c+1\times\ell-c+1$ minors of a generating matrix for $D^1(\mathcal{A})$.

Let $v\in Z_M$. This means that any $\ell-c+1\times\ell-c+1$ minor of a generating matrix for $D^1(\mathcal{A})$ evaluated at v is zero. Hence for all $\theta_{i_1},\dots,\theta_{i_{\ell-c+1}}\in D^1(\mathcal{A})$ the vectors $\theta_{i_1}(v),\dots,\theta_{i_{\ell-c+1}}(v)$ are linearly dependent. Now suppose that $v\notin Z_I$. This means that $\dim\left(\bigcap_{H\in\mathcal{A}_v}H\right)>\ell-c$. However Lemma 5.2 provides us with a contradiction because $\bigcap_{H\in\mathcal{A}_v}H=D^1(\mathcal{A})(v)$ and from the above the $\dim(D^1(\mathcal{A})(v))\le\ell-c$. □

For $X\in L(\mathcal{A})$ let $\mathcal{A}_X=\{H\in\mathcal{A}\mid X\subseteq H\}$.

Lemma 5.4. *If $\mathcal{A}$ is any arrangement and $X\in L(\mathcal{A})$ then $M_c(\mathcal{A})_{I(X)}=M_c(\mathcal{A}_X)_{I(X)}$ for $1\le c\le\ell$.*

Proof. Since $D^1(\mathcal{A})\subseteq D^1(\mathcal{A}_X)$ we have $M_c(\mathcal{A})\subseteq M_c(\mathcal{A}_X)$ and so $M_c(\mathcal{A})_{I(X)}\subseteq M_c(\mathcal{A}_X)_{I(X)}$. We show the opposite inclusion. Let $r\in M_c(\mathcal{A}_X)_{I(X)}$. Then there exists $p\in M_c(\mathcal{A}_X)$ and $q\notin I(X)$ such that $r=\frac{p}{q}$. Since $p\in M_c(\mathcal{A}_X)$ there exists $\theta_1,\dots,\theta_{\ell-c+1}\in D^1(\mathcal{A}_X)$ and $\psi_1,\dots,\psi_{c-1}\in\mathrm{Der}^1$ such that $\theta_1\wedge\cdots\wedge\theta_{\ell-c+1}\wedge\psi_1\wedge\cdots\wedge\psi_{c-1}=p\partial_{[\ell]}$. Let

$$\bar{Q}=\prod_{H\in\mathcal{A}\backslash\mathcal{A}_X}\alpha_H$$

be the defining polynomial of $\mathcal{A}\backslash\mathcal{A}_X$ and $\bar{\theta}_i=\bar{Q}\theta_i$. Then $\bar{\theta}_i\in D^1(\mathcal{A})$ for all i and $\bar{Q}\notin I(X)$. Since

$$\bar{\theta}_1\wedge\cdots\wedge\bar{\theta}_{\ell-c+1}\wedge\psi_1\wedge\cdots\wedge\psi_{c-1}=\bar{Q}^{\ell-c+1}p\partial_{[\ell]}$$

the polynomial $\bar{Q}^{\ell-c+1}p\in M_c(\mathcal{A})$. Then

$$r=\frac{\bar{Q}^{\ell-c+1}p}{\bar{Q}^{\ell-c+1}q}\in M_c(\mathcal{A})_{I(X)}.\quad\square$$

Remark 5.5. Lemma 5.4 shows that the functor M_c is a local functor in the sense of [16]. This Lemma can also be proved using the fact that $D^1(\mathcal{A})$ is a local functor as in [16] but here we do not need this terminology.

We have shown that the localization at $I(X)$ just depends on the hyperplanes containing X. If X is a codimension c intersection of hyperplanes in $\mathcal{A}$ then $M_c(\mathcal{A})_{I(X)}$ just depends on the level c arrangement $\mathcal{A}(c)$.

Lemma 5.6. *If $\mathcal{A}$ is any essential hyperplane arrangement, $1\le c\le\ell$, and $X\in L(\mathcal{A})_c=\mathcal{A}(c)$ then $M_c(\mathcal{A})_{I(X)}=I(\mathcal{A}(c))_{I(X)}$ in the ring $S_{I(X)}$.*

Proof. Suppose that we choose coordinates on V^* such that $I(X)=(x_1,\dots,x_c)$. Then for all $c+1\le i\le\ell$, $\partial_i\in D^1(\mathcal{A}_X)$. Let $d_i=\partial_1\wedge\cdots\wedge\partial_{i-1}\wedge\partial_{i+1}\wedge\cdots\wedge\partial_\ell$ and let $\theta_E=\sum_{i=1}^{\ell}x_i\partial_i$ be the usual Euler derivation. Then for $1\le i\le c$, $\theta_E\wedge d_i\in M_c(\mathcal{A}_X)$ and $\theta_E\wedge d_i=x_i\partial_i$. This means that $x_i\in M_c(\mathcal{A}_X)$ for $1\le i\le c$. Hence $I(X)\subseteq M_c(\mathcal{A}_X)$. But by Lemma 5.1 $M_c(\mathcal{A}_X)\subseteq I(\mathcal{A}_X(c))$ and $I(\mathcal{A}_X(c))=I(X)$ so $M_c(\mathcal{A}_X)=I(X)$. By Lemma 5.4 we have $M_c(\mathcal{A})_{I(X)}=M_c(\mathcal{A}_X)_{I(X)}$. Since X is codimension c the ideals $I(\mathcal{A}(c))_{I(X)}$ and $I(X)_{I(X)}$ are equal in the localized ring $S_{I(X)}$. We conclude that $M_c(\mathcal{A})_{I(X)}=M_c(\mathcal{A}_X)_{I(X)}=I(X)_{I(X)}=I(\mathcal{A}(c))_{I(X)}$. □

Remark 5.7. Lemma 5.6 shows that the minimal primary components of $M_c(\mathcal{A})$ are prime. Moreover since $M_c(\mathcal{A})\subseteq M_c(\mathcal{A}(c))$ it shows the same for $M_c(\mathcal{A}(c))$. Hence the obstruction to $\mathcal{A}(c)$ being derivation radical is the existence of embedded components in $M_c(\mathcal{A})$.

Theorem 5.8. *If $\mathcal{A}$ is an essential hyperplane arrangement such that $M_c(\mathcal{A})$ has no embedded primes then $M_c(\mathcal{A})=I(\mathcal{A}(c))$. In particular, if $M_c(\mathcal{A})$ has no embedded primes then $\mathcal{A}(c)$ is derivation radical.*

Proof. Theorem 5.3 shows that $M_c(\mathcal{A})$ and $I(\mathcal{A}(c))$ have the same minimal associated primes, which are codimension c. Couple this with the assumption that $M_c(\mathcal{A})$ has no embedded associated primes and we have that

$$\mathrm{Ass}(M_c(\mathcal{A}))=\mathrm{Ass}(I(\mathcal{A}(c)))=\{I(X)|X\in\mathcal{A}(c)\}.$$

Lemma 5.6 states that $M_c(\mathcal{A})_{I(X)}=I(\mathcal{A}(c))_{I(X)}$ for all $X\in\mathcal{A}(c)$. Hence we have that for any associated prime P of $M_c(\mathcal{A})$ and $I(\mathcal{A}(c))$ the localization $M_c(\mathcal{A})_P=I(\mathcal{A}(c))_P$. Therefore $M_c(\mathcal{A})=I(\mathcal{A}(c))$. □

Remark 5.9. We give a brief interpretation of Theorem 5.8 for the case when $c=1$. The minimal prime of $M_1(\mathcal{A})$ is the ideal of the defining polynomial Q and any embedded prime will have more than one generator. The upshot is that $\mathcal{A}$ is a free hyperplane arrangement if and only if $M_1(\mathcal{A})$ is a principal ideal.

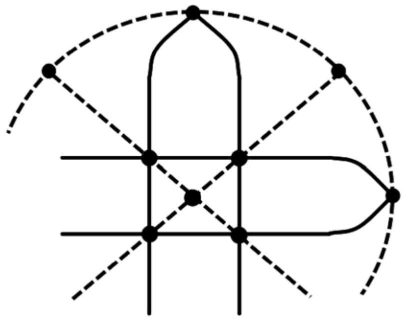

Fig. 2. Projectivized picture of the non-Fano arrangement.

Remark 5.10. The hypothesis of Theorem 5.8 is unfortunately unavoidable. The reader well-studied in *free* hyperplane arrangements might want to guess that the hypothesis of Theorem 5.8 could be replaced with "$\mathcal{A}$ is a free hyperplane arrangement". However, this is not true as the following example illustrates. Moreover, this example shows that even when $\mathcal{A}$ is free $\mathcal{A}(2)$ need not be derivation radical.

Example 5.11. Let $\ell = 3$, $c = 2$, and $\mathcal{A}$ be the hyperplane arrangement defined by the polynomial $Q(\mathcal{A}) = xyz(x-y)(x-z)(y-z)(x+y-z)$. This arrangement is referred to as the non-Fano arrangement in the literature because of the form of its associated matroid. The module of derivations $D^1(\mathcal{A})$ of this arrangement is a free module with three generators of degree 1,3, and 3. Hence the minimal degree generators of $M_2(\mathcal{A})$ have degree 4. However, $\mathcal{A}(2)$ consists of nine points, hence there is a degree 3 polynomial in the ideal $I(\mathcal{A}(2))$. Thus $M_2(\mathcal{A}) \neq I(\mathcal{A}(2))$ even though $\mathcal{A}$ is free. In this case the maximal ideal is an embedded prime of the ideal $M_2(\mathcal{A})$. Furthermore, a Macaulay 2 (see [7]) calculation shows that $M_2(\mathcal{A}(2)) = M_2(\mathcal{A})$, hence $M_2(\mathcal{A}(2)) \neq I(\mathcal{A}(2))$ and $\mathcal{A}(2)$ is not derivation radical. Fig. 2 is a projective picture of $\mathcal{A}$. The dotted lines are in the arrangement, but they also constitute the unique cubic that contains the nine codimension 2 intersection points.

The next example shows that $M_c(\mathcal{A})$ can have many embedded primes and that $M_c(\mathcal{A})$ can be different than $M_c(\mathcal{A}(c))$ where $\mathcal{A}$ is a hyperplane arrangement.

Example 5.12. Let $\ell = 4$, $c = 2$, and $\mathcal{A}$ be the hyperplane arrangement defined by the polynomial

$$Q(\mathcal{A}) = \prod_{a,b,c,d \in \{0,1\}} ax + by + cz + dw$$

where of course not all a, b, c, and d are zero. This arrangement is the restriction of the famous counterexample to Orlik's conjecture found by Edelman and Reiner in [6]. Using the computer algebra system Macaulay 2 (see [7]) one can show that $M_2(\mathcal{A}(2)) = M_2(\mathcal{A}) \neq I(\mathcal{A}(2))$ and $M_3(\mathcal{A}(3)) \neq M_3(\mathcal{A})$. Moreover Macaulay 2 states that the embedded associated primes of $I(\mathcal{A}(2))$, which is a codimension 2 ideal, consists of the maximal ideal (x, y, z, w) and all of the codimension 3 ideals that are intersections of 5 hyperplanes:

$$\{(w, y+z, x), (w, z, y), (w, z, x), (w, z, x+y), (w, y, x),$$
$$(w, y, x+z), (z, y+w, x), (z, y, x), (z, y, x+w), (z+w, y, x)\}.$$

Acknowledgements

The authors would like to thank Sergey Yuzvinsky for numerous discussions and help with these topics. The authors also want to thank Hiroaki Terao and Takuro Abe for many discussions and helpful suggestions. The authors are also grateful for discussions with Masahiko Yoshinaga, Hal Schenck, Graham Denham, Uli Walther, and Mathias Schulze. The second author has been supported by NSF grant # 0600893, the NSF Japan program, and the Office of Naval Research.

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