On the Trigonometric Developments of Certain Doubly Periodic Functions of the Second Kind

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ON THE TRIGONOMETRIC DEVELOPMENTS OF CERTAIN DOUBLY PERIODIC FUNCTIONS OF THE SECOND KIND*

BY M. A. BASOCO

1. *Introduction.* The class of meromorphic functions which satisfy periodicity relations of the form

$$f(z+2\omega_1)=c_1f(z),\quad f(z+2\omega_2)=c_2f(z), \tag{1}$$

where the multipliers c_1 and c_2 are independent of z, and ω_1/ω_2 is a complex number with non-vanishing imaginary part, has been named by Hermite† *doubly periodic of the second kind.* It is possible to make the study of these functions depend on others of the same type, but such that one of the multipliers, say c_1, is unity. In what follows we shall assume, further, that the periods $(2\omega_1, 2\omega_2)$ are $(\pi, \pi\tau)$, where $\tau=a+ib$, $b>0$.

Particular interest is attached to the functions (2) below, which belong to the category just defined. In terms of the Jacobi theta functions they have the form

$$\phi_{\alpha\beta\gamma}(x,y)=\vartheta_1'\,\frac{\vartheta_\alpha(x+y)}{\vartheta_\beta(x)\vartheta_\gamma(y)}, \tag{2}$$

where x, y are independent complex variables, and α, β, γ are certain triads, sixteen in number, which can be selected from the numbers 0, 1, 2, 3. These functions were first discovered by Jacobi‡ and have been studied by Kronecker§, Hermite||, Teixeira¶ and others. More recently, E. T. Bell** has pointed out their importance in connection with certain results in number

* Presented to the Society, April 8, 1932.

† Hermite, Comptes Rendus, vol. 85 (1877), . . . vol. 94 (1882); Annales de l'École Normale Supérieure, (3), vol. 2, 1885, p. 303. Oeuvres, vol. IV, p. 190–200.

‡ Jacobi, Werke, vol. 2, pp. 291–351.

§ Kronecker, Monatsberichte der Berliner Akademie, 1881, pp. 1165–1172; Werke, vol. IV, pp. 309, 318.

|| Loc. cit.; see also Lerch, Acta Mathematica, vol. 12 (1889), pp. 51–55.

¶ Teixeira, Journal für Mathematik, vol. 125 (1901), pp. 301–318.

** E. T. Bell, Transactions of this Society, vol. 22 (1921); Colloquium Series of this Society, vol. 7, p. 88. Giornale di Matematiche, vol. 59 (1921).

theory; in these arithmetical applications, their explicit Fourier expansions play a fundamental rôle.

In what follows, we shall be primarily concerned with deriving the Fourier developments for the squares of the $\phi_{\alpha\beta\gamma}(x, y)$. However, our analysis is perfectly general and the expansion for any positive integral power of these functions may be calculated if desired. It is on account of the possible arithmetic applications that we have carried out the calculations for the squares; these expansions turn out to have a relatively simple structure, while the higher powers are of a far greater complexity and hence may not yield simple and interesting arithmetic results.

The analysis used originates with Teixeira (loc. cit.); however, for our purpose, it was necessary to generalize somewhat certain of his results, as we proceed to indicate.

2. *Extension of Teixeira's Results.* Let $f(z)$ be a doubly periodic function of the second kind with periods equal to π and $\pi\tau$, the corresponding multipliers being unity and c, respectively, where c is independent of z. Suppose that in a fundamental period cell $f(z)$ has k poles, $z=a_r$, $r=1, 2, 3, \cdots, k$. Further, let the order of these poles be m_r respectively, so that in the neighborhood of $z=a_r, f(z)$ has the Laurent expansion

$$(3)\quad f(z) = \frac{A_{m_r}^{(r)}}{(z-a_r)^{m_r}} + \cdots + \frac{A_2^{(r)}}{(z-a_r)^2} + \frac{A_1^{(r)}}{z-a_r} + P(z-a_r).$$

Consider a parallelogram $pqrs$ consisting of $\beta+1$ cells above the real axis and α below. Inside this parallelogram, $f(z)$ has $\alpha+\beta+1$ poles which have the affixes

$$(4)\qquad z = a_r + n\pi\tau = \omega_{r,n},$$

$(r=1, 2, 3 \cdots k; n=-\alpha+1, \cdots, -1, 0, 1, 2, \cdots, \beta)$. The corresponding residues are $c^n A_1^{(r)}$.

Next, consider the auxiliary function given by

$$(5)\qquad \phi(t) \equiv \frac{e^{2it}}{e^{2it}-e^{2iz}} f(t) = \tfrac{1}{2}[1 - i \operatorname{ctn}(t-z)]f(t).$$

If we apply Cauchy's theorem to the contour integral $\int_{(C)}\phi(t)dt$ the contour C being the boundary of the parallelogram $pqrs$, we obtain, after certain reductions* which we shall omit, the follow-

* For more details on the nature of these reductions we refer the reader to some analogous calculations which are indicated in a paper by the writer in this Bulletin, vol. 37 (1931), pp. 117–124.

ing result which is valid in a strip parallel to the axis of reals and bounded by lines of which rq and sp are segments:

$$
(6)\quad \begin{cases} f(z) = \displaystyle\sum_{m=-\infty}^{\infty} C_m e^{2miz} - i \sum_{(r,n)} \overset{(r)}{A_1} c^n (1 + i \operatorname{ctn}(z - \omega_{r,n})) \\ \qquad + \displaystyle\sum_{(r,n)} \sum_{s=1}^{m_r-1} \frac{\overset{(r)}{A}_{s+1}}{s!} c^n D^{(s)}_{\omega_{r,n}} \operatorname{ctn}(z - \omega_{r,n}), \end{cases}
$$

where

$$
(7)\quad \begin{cases} C_m = \dfrac{c^{-\alpha} q^{2m\alpha}}{1 - cq^{-2m}} \displaystyle\sum_{s=1}^{m_r} \sum_{r=1}^{k} \frac{(-1)^s (2i)^s m^{s-1} \overset{(r)}{A_s}}{(s-1)!} e^{-2mia_r}, \\ q = e^{\pi i \tau}, \qquad\qquad (m = 1, 2, 3, \cdots), \\ C_{-m} = \dfrac{c^{1+\beta} q^{2m(1+\beta)}}{1 - cq^{2m}} \displaystyle\sum_{s=1}^{m_r} \sum_{r=1}^{k} \frac{(2i)^s m^{s-1} \overset{(r)}{A_s}}{(s-1)!} e^{2mia_r}, \\ C_0 = \dfrac{2ic^{-\alpha}}{1 - c} \displaystyle\sum_{r=1}^{k} \overset{(r)}{A_1}, \end{cases}
$$

and $D^{(s)}$ is the differential operator of order s, and $\omega_{r,n}$ is the argument $a_r + n\pi\tau$ which is to be substituted after the differentiation has been performed.

3. *Application to the Functions* $\Phi_{\alpha\beta\gamma}(z, v)$. We shall use the notation $\Phi_{\alpha\beta\gamma}(z, v)$ to denote the squares of the $\phi_{\alpha\beta\gamma}(z, v)$ defined by equation (2). These separate into two groups (A) and (B) according as their poles, qua functions of z, are congruent to $\pi\tau/2$ or $\pi\tau$. They all come under the case where $k = 1$ and $m_r = 2$, (that is, there is but one pole $z = a$ in a period cell and its order is two). With these values and also with (α, β) equal to $(0, 0)$ our fundamental formula (6) reduces to the following:

$$
(8)\quad \begin{cases} f(z) = 2iA_1 \left[\displaystyle\sum_{n=0}^{\infty} \frac{e^{2ni(z-a)}}{1 - cq^{-2n}} + \sum_{n=1}^{\infty} \frac{cq^{2n} e^{-2ni(z-a)}}{1 - cq^{2n}} \right] \\ \qquad + 4A_2 \left[\displaystyle\sum_{n=1}^{\infty} \frac{ne^{2ni(z-a)}}{1 - cq^{-2n}} - \sum_{n=1}^{\infty} \frac{ncq^{2n} e^{-2ni(z-a)}}{1 - cq^{2n}} \right] \\ \qquad - iA_1(1 + i \operatorname{ctn}(z - a)) + A_2 \csc^2(z - a). \end{cases}
$$

We shall now transform this expression into a form suitable for the calculation of the arithmetical expansions of the functions in group (A). To do this, use is made of the relationships

$$(9)\qquad \begin{cases} 1 + i \operatorname{ctn}(z - a) = 2 \sum_{n=0}^{\infty} e^{2ni(z-a)}, \\ \csc^2(z - a) = -4 \sum_{n=1}^{\infty} n e^{2ni(z-a)}, \end{cases}$$

which are valid provided $I(z) > I(a)$. Furthermore, as may be seen from the properties of the theta functions, the $\Phi_{\alpha\beta\gamma}(z, v)$ as functions of z have the multipliers unity and $c = e^{-4iv}$.

Substituting these values in (8) we find, after some reduction,

$$(10)\qquad f(z) = A_1 \psi(z, v, a) + A_2 \overset{(1)}{D_a} \psi(z, v, a),$$

$$(11)\qquad \psi(z, v, a) = e^{-2iv} \sum_{n=-\infty}^{\infty} \frac{e^{2ni(z-a-\pi\tau/2)}}{\sin(2v + n\pi\tau)} .$$

In order to obtain an expansion suitable for use in connection with the functions of group (B), all that is necessary is to rewrite (8) in the form

$$(12)\qquad f(z) = A_1 \zeta(z, v, a) + A_2 \overset{(1)}{D_a} \zeta(z, v, a),$$

$$(13)\qquad \begin{aligned} \zeta(z, v, a) = \operatorname{ctn}(z - a) + \operatorname{ctn} 2v + e^{2iv} \sum_{n=1}^{\infty} \frac{q^n e^{2ni(z-a)}}{\sin(2v + n\pi\tau)} \\ + e^{-2iv} \sum_{n=1}^{\infty} \frac{q^n e^{-2ni(z-a)}}{\sin(2v - n\pi\tau)} . \end{aligned}$$

4. *The Expansions in Group* (A). Consider the function

$$(14)\qquad f(z) \equiv \Phi_{001}(z, v) = \vartheta_1'^2 \frac{\vartheta_0^2(z + v)}{\vartheta_0^2(z)\vartheta_1^2(v)};$$

its periods are π and $\pi\tau$ with multipliers unity and e^{-4iv} respectively. We may select the fundamental period cell so that $a = -\pi\tau/2$ is the affix of the pole to be used in (10). Further, a slight calculation shows that the coefficients A_1 and A_2 of the principal part of its Laurent expansion about $z = -\pi\tau/2$ have the values $A_1 = 2e^{2iv}\vartheta_1'(v)/\vartheta_1(v)$ and $A_2 = e^{2iv}$. Equations (10) and (11) then yield the result

$$\Phi_{001}(z, v) = \vartheta_1'^2 \frac{\vartheta_0^2(z+v)}{\vartheta_0^2(z)\vartheta_1^2(v)} = \frac{2\vartheta_1'(v)}{\vartheta_1(v)} \sum_{n=-\infty}^{\infty} \frac{e^{2niz}}{\sin(2v+n\pi\tau)} - 2i\sum_{n=-\infty}^{\infty} \frac{ne^{2niz}}{\sin(2v+n\pi\tau)}. \tag{15}$$

In order to reduce this expression to arithmetical form, we use the following relations, which may easily be proved:

$$\csc(2v+n\pi\tau) = -2i\sum_{n=1}^{\infty} q^{nm}e^{2imv}, \quad (m = 1, 3, 5, 7, \cdots), \tag{16}$$

$$\csc(2v-n\pi\tau) = 2i\sum_{m=1}^{\infty} q^{nm}e^{-2imv}. \tag{17}$$

These hold simultaneously and for all positive integral n in case $\frac{1}{2}I(\pi\tau) < I(v) < \frac{1}{2}I(\pi\tau)$. The expansions sought will be readily obtained from (15) if each sum is replaced by two others in which the indices of summation range from $n=1$ to $n=\infty$, if we pair off corresponding terms and make use of (16) and (17). Thus we obtain, on changing our notation (z, v) to (x, y) in order to be in agreement with current use, the development

$$\text{(I)} \quad \left\{ \begin{aligned} \Phi_{001}(x, y) = \vartheta_1'^2 \frac{\vartheta_0^2(x+y)}{\vartheta_0^2(x)\vartheta_1^2(y)} &= \frac{2\vartheta_1'(y)}{\vartheta_1(y)}\Big[\frac{1}{\sin 2y} \\ &\quad + 4\sum q^n\big(\sum \sin 2(tx+\tau y)\big)\Big] \\ &\quad - 8\sum q^n\big(\sum t\cos 2(tx+\tau y)\big), \end{aligned} \right.$$

in which the summation appearing as the coefficient of q^n is to be extended over all the positive integral divisors t, τ of n, τ being always odd.

Replacing (x, y) successively by $(x, y+\pi/2)$, $(x+\pi/2, y)$, $(x+\pi/2)$, $(y+\pi/2)$ in the preceding, we obtain the following:

$$\text{(II)} \quad \left\{ \begin{aligned} \Phi_{302}(x, y) = \vartheta_1'^2 \frac{\vartheta_3^2(x+y)}{\vartheta_0^2(x)\vartheta_2^2(y)} &= -2\frac{\vartheta_2'(y)}{\vartheta_2(y)}\Big[\frac{1}{\sin 2y} \\ &\quad + 4\sum q^n\big(\sum \sin 2(tx+\tau y)\big)\Big] \\ &\quad + 8\sum q^n\big(\sum t\cos 2(tx+\tau y)\big); \end{aligned} \right.$$

$$\text{(III)}\quad \left\{\begin{aligned} \Phi_{331}(x, y) &= \vartheta_1'^2 \frac{\vartheta_3^2(x+y)}{\vartheta_3^2(x)\vartheta_1^2(y)} = 2\frac{\vartheta_1'(y)}{\vartheta_1(y)}\biggl[\frac{1}{\sin 2y} \\ &\qquad + 4\sum(-1)^n q^n\Bigl(\sum \sin 2(tx+\tau y)\Bigr)\biggr] \\ &\qquad - 8\sum(-1)^n q^n\Bigl(\sum t\cos 2(tx+\tau y)\Bigr); \end{aligned}\right.$$

$$\text{(IV)}\quad \left\{\begin{aligned} \Phi_{032}(x, y) &= \vartheta_1'^2 \frac{\vartheta_0^2(x+y)}{\vartheta_3^2(x)\vartheta_2^2(y)} = -2\frac{\vartheta_2'(y)}{\vartheta_2(y)}\biggl[\frac{1}{\sin 2y} \\ &\qquad + 4\sum(-1)^n q^n\Bigl(\sum \sin 2(tx+\tau y)\Bigr)\biggr] \\ &\qquad + 8\sum(-1)^n q^n\Bigl(\sum t\cos 2(tx+\tau y)\Bigr). \end{aligned}\right.$$

In an entirely similar manner, we may derive from

$$f(z) = \Phi_{100}(z, v) = \vartheta_1'^2 \frac{\vartheta_1^2(z+v)}{\vartheta_0^2(z)\vartheta_0^2(v)}$$

the following developments:

$$\text{(V)}\quad \left\{\begin{aligned} \Phi_{100}(x, y) &= \vartheta_1'^2 \frac{\vartheta_1^2(x+y)}{\vartheta_0^2(x)\vartheta_0^2(y)} = 2\frac{\vartheta_0'(y)}{\vartheta_0(y)}\biggl[\frac{1}{\sin 2y} \\ &\qquad + 4\sum q^n\Bigl(\sum \sin 2(tx+\tau y)\Bigr)\biggr] \\ &\qquad - 8\sum q^n\Bigl(\sum t\cos 2(tx+\tau y)\Bigr); \end{aligned}\right.$$

$$\text{(VI)}\quad \left\{\begin{aligned} \Phi_{203}(x, y) &= \vartheta_1'^2 \frac{\vartheta_2^2(x+y)}{\vartheta_0^2(x)\vartheta_3^2(y)} = -2\frac{\vartheta_3'(y)}{\vartheta_3(y)}\biggl[\frac{1}{\sin 2y} \\ &\qquad + 4\sum q^n\Bigl(\sum \sin 2(tx+\tau y)\Bigr)\biggr] \\ &\qquad + 8\sum q^n\Bigl(\sum t\cos 2(tx+\tau y)\Bigr); \end{aligned}\right.$$

$$\text{(VII)}\quad \left\{\begin{aligned} \Phi_{230}(x, y) &= \vartheta_1'^2 \frac{\vartheta_2^2(x+y)}{\vartheta_3^2(x)\vartheta_0^2(y)} = 2\frac{\vartheta_0'(y)}{\vartheta_0(y)}\biggl[\frac{1}{\sin 2y} \\ &\qquad + 4\sum(-1)^n q^n\Bigl(\sum \sin 2(tx+\tau y)\Bigr)\biggr] \\ &\qquad - 8\sum(-1)^n q^n\Bigl[\sum t\cos 2(tx+\tau y)\Bigr]; \end{aligned}\right.$$

$$\text{(VIII)}\quad \left\{\begin{aligned} \Phi_{133}(x, y) &= \vartheta_1'^2 \frac{\vartheta_1^2(x+y)}{\vartheta_3^2(x)\vartheta_3^2(y)} = -2\frac{\vartheta_3'(y)}{\vartheta_3(y)}\biggl[\frac{1}{\sin 2y} \\ &\qquad + 4\sum(-1)^n q^n\Bigl(\sum \sin 2(tx+\tau y)\Bigr)\biggr] \\ &\qquad + 8\sum(-1)^n q^n\Bigl(\sum t\cos 2(tx+\tau y)\Bigr). \end{aligned}\right.$$

5. *The Expansion in Group* (B). To obtain the Fourier series for the functions in this group we return to formulas (12) and (13). If the series (16) and (17) are used in connection with the expression defining the function $\zeta(z, v, a)$ we obtain the Fourier expansion for this function. Thus,

$$(18)\quad \zeta(z, v, a) = \operatorname{ctn} 2v + \operatorname{ctn}(z - a) + 4 \sum q^n\Big(\sum \sin 2(d(z - a) + \delta v)\Big)$$

where the sum which appears as the coefficient of q^n ranges over all divisors d, δ of n such that δ is even. We may note, in passing, that $\zeta(z, v, a) \equiv \phi_{111}(z-a, 2v)$. Consider, now, the function

$$f(z) \equiv \Phi_{111}(z, v) = \vartheta_1'^2 \frac{\vartheta_1^2(z + v)}{\vartheta_1^2(z)\vartheta_1^2(v)};$$

all that is necessary to obtain the development of $f(z)$ is to calculate the coefficients A_1, A_2 relative to the pole $a=0$. We find that $A_2=1$ and $A_1=2\vartheta_1'(v)/\vartheta_1(v)$. Hence on applying (12) and (18) and changing, as before, the notation (z, v) to (x, y), we have

$$(\mathrm{IX})\quad \left\{\begin{aligned} \Phi_{111}(x, y) &= \vartheta_1'^2 \frac{\vartheta_1^2(x + y)}{\vartheta_1^2(x)\vartheta_1^2(y)} = 2\frac{\vartheta_1'(y)}{\vartheta_1(y)}\Big[\operatorname{ctn} x + \operatorname{ctn} 2y \\ &\quad + 4 \sum q^n\Big(\sum \sin 2(dx + \delta y)\Big)\Big] \\ &\quad + \frac{1}{\sin^2 x} - 8 \sum q^n\Big(\sum d \cos 2(dx + \delta y)\Big). \end{aligned}\right.$$

From this, the following are deduced:

$$(\mathrm{X})\quad \left\{\begin{aligned} \Phi_{221}(x, y) &= \vartheta_1'^2 \frac{\vartheta_2^2(x + y)}{\vartheta_2^2(x)\vartheta_1^2(y)} = 2\frac{\vartheta_1'(y)}{\vartheta_1(y)}\Big[- \tan x + \operatorname{ctn} 2y \\ &\quad + 4 \sum q^n\Big(\sum (-1)^d \sin 2(dx + \delta y)\Big)\Big] \\ &\quad + \frac{1}{\cos^2 x} - 8\sum q^n\Big(\sum(-1)^d d\cos 2(dx+\delta y)\Big); \end{aligned}\right.$$

$$(\mathrm{XI})\quad \left\{\begin{aligned} \Phi_{212}(x, y) &= \vartheta_2'^2 \frac{\vartheta_2^2(x + y)}{\vartheta_1^2(x)\vartheta_2^2(y)} = 2\frac{\vartheta_2'(y)}{\vartheta_2(y)}\Big[\operatorname{ctn} x + \operatorname{ctn} 2y \\ &\quad + 4 \sum q^n\Big(\sum \sin 2(dx + \delta y)\Big)\Big] \\ &\quad + \frac{1}{\sin^2 x} - 8 \sum q^n\Big(\sum d \cos 2(dx + \delta y)\Big); \end{aligned}\right.$$

$$
\text{(XII)}\quad \begin{cases} \Phi_{122}(x,\,y) = \vartheta_1'^2 \dfrac{\vartheta_1^2(x+y)}{\vartheta_2^2(x)\vartheta_2^2(y)} = 2\dfrac{\vartheta_2'(y)}{\vartheta_2(y)}[-\tan x + \operatorname{ctn} 2y \\ \qquad + 4\sum q^n(\sum(-1)^d \sin 2(dx+\delta y))] \\ \qquad + \dfrac{1}{\cos^2 x} - 8\sum q^n(\sum d(-1)^d \cos 2(dx+\delta y)). \end{cases}
$$

In an analogous manner, if we let

$$
f(z) = \Phi_{010}(z,\,v) = \vartheta_1'^2 \frac{\vartheta_0^2(z+v)}{\vartheta_1^2(z)\vartheta_0^2(v)},
$$

we obtain the following:

$$
\text{(XIII)}\quad \begin{cases} \Phi_{010}(x,\,y) = \vartheta_1'^2 \dfrac{\vartheta_0^2(x+y)}{\vartheta_1^2(x)\vartheta_0^2(y)} = 2\dfrac{\vartheta_0'(y)}{\vartheta_0(y)}[\operatorname{ctn} x + \operatorname{ctn} 2y \\ \qquad + 4\sum q^n(\sum \sin 2(dx+\delta y))] \\ \qquad + \dfrac{1}{\sin^2 x} - 8\sum q^n(\sum d \cos 2(dx+\delta y)); \end{cases}
$$

$$
\text{(XIV)}\quad \begin{cases} \Phi_{320}(x,\,y) = \vartheta_1'^2 \dfrac{\vartheta_3^2(x+y)}{\vartheta_2^2(x)\vartheta_0^2(y)} = 2\dfrac{\vartheta_0'(y)}{\vartheta_0(y)}[-\tan x + \operatorname{ctn} 2y \\ \qquad + 4\sum q^n(\sum(-1)^d \sin 2(dx+\delta y))] \\ \qquad + \dfrac{1}{\cos^2 x} - 8\sum q^n(\sum(-1)^d d \cos 2(dx+\delta y)); \end{cases}
$$

$$
\text{(XV)}\quad \begin{cases} \Phi_{313}(x,\,y) = \vartheta_1'^2 \dfrac{\vartheta_3^2(x+y)}{\vartheta_1^2(x)\vartheta_3^2(y)} = 2\dfrac{\vartheta_3'(y)}{\vartheta_3(y)}[\operatorname{ctn} x + \operatorname{ctn} 2y \\ \qquad + 4\sum q^n(\sum \sin 2(dx+\delta y))] \\ \qquad + \dfrac{1}{\sin^2 x} - 8\sum q^n(\sum d \cos 2(dx+\delta y)); \end{cases}
$$

$$
\text{(XVI)}\quad \begin{cases} \Phi_{023}(x,\,y) = \vartheta_1'^2 \dfrac{\vartheta_0^2(x+y)}{\vartheta_2^2(x)\vartheta_3^2(y)} = 2\dfrac{\vartheta_3'(y)}{\vartheta_3(y)}[-\tan x + \operatorname{ctn} 2y \\ \qquad + 4\sum q^n(\sum(-1)^d \sin 2(dx+\delta y))] \\ \qquad + \dfrac{1}{\cos^2 x} - 8\sum q^n(\sum(-1)^d d \cos 2(dx+\delta y)). \end{cases}
$$

6. *Conclusion.* The developments obtained in what precedes contain terms which are multiplied by expressions of the form $\vartheta_\alpha'(y)/\vartheta_\alpha(y)$; the arithmetized trigonometric expansions for these are well known* and may be substituted in our developments; when the multiplication and reduction are carried out the corresponding coefficient of the general power of q will then be a finite sum extended over the solutions of certain quaternary quadratic forms.

It should, perhaps, be noticed that our series lead in one direction to the expansions for the squares of the Jacobian elliptic functions, while in another they give rise to several doubly periodic functions of the *third kind.* Thus, for example, if in expansion (I) we let y tend to the value $\pi/2$ we obtain the development for the square of the delta amplitude function:

$$(19)\qquad \vartheta_0^2\vartheta_3^2\,\frac{\vartheta_3^2(x)}{\vartheta_0^2(x)} = -\frac{\vartheta_2''}{\vartheta_2} + 8\sum q^n\Big(\sum t\cos 2tx\Big),$$

where

$$-\frac{\vartheta_2''}{\vartheta_2} = 1 + 8\sum\frac{q^{2n}}{(1+q^{2n})^2}\,.$$

Again, if in (I) we put $x=-y$ we get an expansion for $\vartheta_1'^2\vartheta_0^2/(\vartheta_0^2(x)\vartheta_1^2(x))$. If in (V) we put $x=y$ and apply the transformation of the second order we obtain a development for $\vartheta_1'\vartheta_1(x)\vartheta_2(x)\vartheta_3(x)/\vartheta_0^3(x)$. These examples suffice to indicate the possibilities in this direction.

Finally, it should be noted that the methods here used will not yield the expansions for the remaining forty-eight $\Phi_{\alpha\beta\gamma}(x,y)$. These could be obtained in a slightly modified form by multiplying each of the sixteen functions given above by a suitable elliptic function such as (19) above.†

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* Bell, Messenger of Mathematics, vol. 54 (1924), p. 169. Reference should also be made to certain reduction formulas in his paper in the Giornale di Matematiche (loc. cit.).

† In this connection we may refer to a Chicago thesis (1930) by D. A. F. Robinson, which is to appear in the Proceedings of the Royal Society of Canada. In this work, expansions for the forty-eight $\phi_{\alpha\beta\gamma}(x, y)$ in modified form are given. The methods there used could, doubtless, be applied to obtain the remaining $\Phi_{\alpha\beta\gamma}(x, y)$.