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Perfect Numbers:

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Perfect Numbers:

Why I've Decided I'm Glad I'm Only A Six,
Because A Ten's Not So Perfect After All!

By
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Preface

While this topic of "Perfect Numbers" was completely new to me (at least to the degree at which it is discussed within this paper,) I found it very intriguing and believe there is still much information and mathematical discovery in it for me. There were many historical points of interest, and I found it difficult to whittle them down to a manageable size for the intent of this paper. Likewise, there were many facts and peculiarities I found interesting and certainly worthy of consideration. However, to maintain anything close to a reasonable length of discussion as outlined in the guidelines for writing this paper, I found these things were best left out. A few of the ideas or facts that I found interesting are listed below. Also, I found I was left with some questions or ideas of my own that I simply did not have time at this point to explore. I've listed those as well as a reminder to myself of things I hope to return to at a later time.

• There are only four perfect numbers between 1 and 3,000,000.

• 652 is the only known non-perfect number whose number of divisors and sum of smaller divisors are perfect. (Divisors: 1 + 2 + 4 + 163 + 326 = 496 (a perfect number) and 1, 2, 4, 163, 326 & 652. There are "6" of them and 6 is perfect.)

• Every even perfect number (except for 6) is the sum of consecutive odd cubes. For example, $28 = 1^3 + 3^3 = 1^3 + (2^{(3+1)/2} - 1)^3$ or $496 = 1^3 + 3^3 + 5^3 + 7^3 = 1^3 + 3^3 + 5^3 + (2^{(5+1)/2} - 1)^3$

• Theories and ideas concerning attributes of odd perfect numbers. (I find it quite interesting that so much can be known about something that may well not exist!)

• Theorems and proofs regarding perfect numbers, the components of the formula used to find perfect numbers, etc. (Much of this was beyond my level of understanding or did not, in my opinion, add to the direction I chose to take this paper. One example, $2^p - 1$ is a Mersenne prime iff p is also prime.)

• Modular arithmetic (I feel there is some connection to, or use for, modular arithmetic involving perfect numbers, the proof they cannot end in two or four, or something. I explored this to some degree, but simply ran out of time. Like many other parts of this research, it is something I hope to return to when time allows.)

(NOTE: The title of this paper makes reference to a popular movie and expression from1979. The movie, "10" references a rating system for how perfect a woman (or man) is on a scale from 1 to 10, with 10 being considered perfect. Discovering the number 10 is not a perfect number but 6 is, I found I might have come upon a way to claim perfection at last!)

The mathematical notation for this paper is somewhat tricky, as is often the case. As a means of clarifying the notation I've chosen, "M_P" denotes a Mersenne prime number is used. The M is for "Mersenne", the subscript p for "prime". It simply means that 2 is raised to the power of a Mersenne prime when it is shown as a superscript (power) and it should be noted that formatting and font issues sometimes cause this notation to become somewhat misleading. As a general rule, the "M_P" together simply means "Mersenne prime" regardless of the subscript p appearing to be a subscript of the term "2^M" as it may seem. Hopefully, this will help avoid confusion between subscripts, superscripts and the use of "P" for perfect numbers and "p" for primes.

A perfect number can be defined as: *A number equal to the sum of its proper divisors.* With this in mind, it seems logical to consider the terms "divisor" and "proper divisor" and the role they play in determining perfect numbers.

Divisor is a term (often called a "factor" of a number) that refers to a number that can be evenly divided into another number. If such a number exists, it is a divisor (or factor) of that number. (For example, 6 has factors of 1, 2, 3 & 6 since all these can evenly divide 6.) Proper divisors include all the divisors of a number (n) excluding that number (n) such that when you divide a number you will get a quotient greater than 1. (Thus, proper divisors of 6 are 1, 2 & 3 and NOT 6 because $6 \div 6$ does not result in a quotient greater than 1.)

Is 1 a proper divisor? My stance on this issue is based on the definition of "proper divisors": The proper divisors of a number are *all* divisors excluding the number itself. By definition, 1 is included as a factor. This used to be considered in terms of "aliquot parts" of a number. (*Aliquot parts* of a number are proper divisors of the number that are smaller than the number. The aliquot parts of six are one, two, and three. While in definition this is exactly the same as "proper divisor", looking at the word "aliquot" makes this a more reasonable explanation. The word *aliquot* joins the Latin *ali* (meaning "other") and *quot* (for "how many.") Together they came to mean a part of something, in this case, a part of the number of which it is a factor. ("How many <u>*other*</u> parts are there?") The "other" meaning of *ali* remains today in words like alias, alibi, and alien; very common "others". The *quot* root remains in the word quotient. While this helps explain why one is a proper divisor, it also describes why the number itself is not a proper divisor: it is not an "other quotient". For the quotient to be "other" (than the number itself) it would include anything that evenly divides the number "other than" that number.

Now that we've defined perfect numbers (and cleared up the terms used in that definition) we can look at a few perfect numbers as a basis for what it really means for a number to be perfect. As 6 was used in the example above, let's look again at its "proper divisors" and apply them to the definition of perfect numbers.

Divisors of 6 remember are one, two, three and six. *Proper divisors* are one, two and three. Perfect numbers are numbers whose proper divisors have a sum equal to that number. Thus, since $6 = 1 + 2 + 3$ (the proper divisors of 6) this number is perfect. There are 43 known perfect numbers. The first four perfect numbers are:

$6 = 1 + 2 + 3$
$28 = 1 + 2 + 4 + 7 + 14$
$496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248$
$8128 = 1 + 2 + 4 + 8 + 16 + 32 + 64 + 127 + 254 + 508 + 1016 + 2032 + 4064$

As you can see, the jump between perfect numbers is large and continues to get much larger as the list progresses. If you were to guess that the next (or 10^{th} or 20^{th}) perfect numbers would be very large numbers (increasingly so) you would be correct! This helps explain why there are relatively few known perfect numbers when you consider the largest known perfect number would require approximately 4,520 pages of this size to express all its digits! (Note: I actually tested this out! There are 18304103 digits in the 43^{rd} P_N. 4050 digits will fit on a page with these margins. Thus, $18304103 \div 4050 =$ 4519.5316049382716049382716049383 pages.)

Having seen examples of perfect numbers, one might wonder where this idea of "perfect numbers" came from and who first discovered them. How are they found? Why are there so few of them and why are they so difficult to find? A brief history of "the perfect number search" will undoubtedly answer some of these questions.

It is not known specifically who is to be credited with the discovery of the first 4 perfect numbers, but it is known that the ancient Greeks were aware of them. Having been "known" for well over 2000 years, knowledge and understanding of perfect numbers has progressed to the point today that people devote their time and careers to the quest for more knowledge and understanding: A great and vast search for perfect numbers. GIMPS (the Great Internet Mersenne Prime Search,) a computer networking project of over 4500 members, is one such endeavor that has been quite successful.

The mystery and what was often perceived as "the magic" of the numbers known today to be perfect were first coined as "perfect" numbers by the Pythagoreans, or Pythagoras (569BC-475BC) and his disciples. However, the first known recorded mathematical result concerning perfect numbers occurred in 300 BC's *Elements* written by Euclid. In Euclid's *Elements,* he outlines a proposition surmising that a "double proportion" process resulted in perfect numbers. (For example, 28 has proper divisors of 1, 2, 4, 7 and 14. When looking at those which "double" to get the next divisor, 1 (times 2 is) 4 we now have 1 + 2 + 4 = 7. (Since 4 (times 2) is not 7, the "doubling" stops there.) That sum, multiplied by the last divisor that resulted by this doubling (4) results in the perfect number. Thus, 7 x 4 = 28.

Loosely, this looks something like: (the sum) x (the last) = perfect number. In looking further, we find that 1 + 2 + 4 + 8 + 16 = 31 (a prime number) and so 31 x 16 = 496. Incredibly, this too is a perfect number. In modern mathematical form, this can be restated as:

$$1 + 2 + 4 + 8.....+2^{p-1} = 2^p - 1 \text{ or}$$

If, for some $p > 1$, and $2^p - 1$ is prime then $2^{p-1}(2^p - 1)$ is a perfect number.

(Note: At this time, all prime numbers were considered in finding perfect numbers. While the formula remains the same, the terminology has changed from "prime" to "Mersenne prime".)

While in its time this was landmark and amazing (and still is today!), the next significant study of perfect numbers was made in 100 AD by Nicomachus, who presented statements he believed to be true about perfect numbers (though he offered no proof of them) that were taken as "fact" by other mathematicians for many years. This may have led, to some degree, to slowing the progression of perfect number discoveries and eventually, many of those statements were proven to be unfounded.

The Arab mathematicians were also interested in the idea of perfect numbers, and in the late 1100s and early 1200s, one Arab mathematician in particular wrote based on Nicomachus' work. Ismail ibn Fallus produced 10 perfect numbers (which is significant growth in the quest for perfect numbers), but it was later found that 3 of these ten were not perfect numbers. These results, however, went unnoticed by the European mathematics world and were not rediscovered until the mid 15th Century by Regiomentanus during his stay at the University of Vienna, which he left in 1461.

In 1509, Charles de Bovelles published what is believed to be one of the first books devoted specifically to perfect numbers. The 5th perfect number was discovered (again, since clearly Arab Ismail ibn Fallus had found it among his 7) and formally written about in 1458. Not having benefit of e-mail, telephone, interlibrary loan, etc. the 5th perfect number is thus credited to "anonymous."

In one of the first major breakthroughs regarding perfect numbers, Hudalrichus Regius' 1536 discovery of the first prime in the form of $2^{p-1}(2^p - 1)$ that was NOT a perfect number was significant. (He is also semi-credited for the 5^{th} perfect number since he was the first to formally let it be known.) In 1555, J Scheybl wrote of the 6^{th} perfect number, but this was unnoticed until 1977 and thus it had no influence on the historical search for perfect numbers.

The next breakthrough came in the early 1600s when Cataldi found factors up to 800 and a table of all primes up to 750. He also found the 6^{th} perfect number and thus disproved two of Nicomachus' statements that had previously been considered true. (Nicomachus stated that perfect numbers ended in 6 and 8 alternately, and the n^{th} perfect number would have n number of digits. The 6^{th} perfect number $2^{16}(2^{17} - 1) = 8{,}589{,}869{,}056$ should have ended in 8 according to Nicomachus, and should contain 6 digits. Clearly, neither of these are the case.) Using these factors and tables of primes, Cataldi was also able to find the 7^{th} perfect number.

In 1638, Fermat threw his hat into the ring publicly joining the study of perfect numbers and, in his writings to Mersenne regarding propositions of these came what is now known as "Fermat's Little Theorem" as a result of the work done with perfect numbers. Fermat's Little Theorem was effective in proving and/or disproving previous beliefs or ideas regarding perfect numbers.

From this, it would seem, one of the most significant progressions came when Mersenne found claims of his own (later to be known as the Mersenne numbers and Mersenne primes) that fascinated others for a great many years. The connection to perfect numbers and Mersenne Primes has since become the basis for finding perfect numbers, as with each new discovery of a Mersenne prime comes the discovery of a new perfect number. (Note: Throughout this paper, "M_P" will be used to denote a Mersenne prime.)

In 1732, Euler discovered the 8^{th} perfect number. It is amazing to think this was the first new perfect number for 125 years! Though no odd perfect numbers had been found at this point (and in fact, remain to be found!), Euler also proved what form an odd perfect number would have to take. Since no odd perfect number has been found, many believe they simply do no exist. It should be noted that, while significant progress had been made to this point, there were still fallacies and false claims or predictions regarding perfect numbers. Euler's discovery of $2^{30}(2^{31} - 1)$ would remain the largest known perfect number for 150 years.

A note of interest at this point would be the 1811 writings of mathematician Peter Barlow which stated there would never be a larger perfect number found, essentially because he believed no one would find reason or cause to search for it. Of course, Barlow was completely mistaken as in 1876 Lucas found a perfect number $[2^{126}(2^{127} - 1)]$ and made an advance that, when refined in 1930 by Lehmer, became the basis by which computers search for perfect numbers and is still used today. Edouard Lucas worked with Fibonacci numbers extensively in the last half of the 19th century. (In fact, it was Lucas who popularized the name "Fibonacci numbers".) He used properties of Fibonacci numbers in proving the 39-digit Mersenne number $2^{127} - 1$ is prime.

In 1883 Pervusin wrote of the perfect number $2^{60}(2^{61} - 1)$ and in 1911 and 1914, Powers followed with his own perfect numbers of $2^{88}(2^{89} - 1)$ and $2^{100}(2^{101} - 1)$. It should also be noted that $2^{88}(2^{89} - 1)$ is the last perfect number to have been found by hand calculations (1911) and all perfect numbers beyond that point have been found with benefit of technology.

It might be of interest to note in 1952 Robinson found the 13th – 17th Mersenne primes. Also interesting is the finding of the 25th and 26th perfect numbers by high school students in 1978 and 1979. The 39th perfect number wasn't discovered until December of 2001, while the last perfect number to be found to date was discovered in December of 2005.

Two millennia after Euclid, Euler proved that the formula $2^{n-1}(2^n - 1)$ will yield all the even perfect numbers when n is a Mersenne prime. Thus, since a Mersenne prime is found by $(2^n - 1)$, every Mersenne prime will yield a distinct even perfect number as there is a concrete one-to-one relationship between even perfect numbers and Mersenne primes. This result is often referred to as the "Euclid-Euler Theorem". 43 Mersenne primes are currently known, which means there are 43 known perfect numbers. It is still uncertain whether there are infinitely many Mersenne primes and perfect numbers. The search for new Mersenne primes (and thus perfect numbers as well) is the goal some groups (such as GIMPS, who are credited with finding the last 9 Mersenne primes), with a network of interconnected computers running around the clock every day in search of these numbers that have held such intrigue and interest for so many years.

While it has been conjectured there are infinitely many Mersenne primes, and therefore infinitely many perfect numbers, the odd perfect number eludes us to this day. What is known, however, is if it exists it would have to be beyond the value of 10^{300}. The educated guess remains that there are no odd perfect numbers. Also worth noting is the fact that although 43 Mersenne primes (and perfect numbers) have been discovered, they may not be sequential perfect numbers as not all smaller cases have been ruled out or exhaustively searched, meaning the 43rd known perfect number could well be the 48th perfect number, for example. In the case of perfect numbers, it is interesting to consider the 12th (sequential) perfect number was found before the 9th, 10th and 11th and the 29th was found 5 years after the 30th and three years after the 31st. *(See Appendix A for a more concise timeline.)*

While this brief history still leaves some gaps (certainly not all historical instances of work with perfect numbers can be included though I've tried to include what I found most interesting or significant occurrences,) it does give us a basis for understanding the complications and extensive brainpower involved in getting us to the present day and what is known about "perfect numbers." Clearly, with the advent of modern technology, the search has become somewhat less labor intensive (imagine doing all this work without benefit of calculators or computers!) and announcing new discoveries certainly helps the progress. Historically, "discoveries" (while believed to be "original") occurred several times by more than one person. Had communication among mathematicians been possible and timely, perhaps the advances in the perfect number world would have occurred more rapidly. And, imagine the effects of collaborative efforts had they been possible!

Having researched the beginnings and history of perfect numbers, it became apparent I would need to use some brainpower of my own to further understand these implications. I began by exploring the formula itself for finding perfect numbers. It soon became apparent I needed to look further into the portion of the formula that denotes the Mersenne numbers when $(2^n - 1)$ is a prime. Obviously, this expression does not always result in a prime number, and not every prime number will be found using this expression. So, for certain values of "n", I will arrive at a prime number (known as a member of the Mersenne primes) and for others I will arrive at a composite number, though still a "Mersenne" number. *(See Appendix B for a partial list of Mersenne numbers and Mersenne primes.)*

Finding it helpful to organize my findings in a table of data, the following is a short list of values for the expressions 2^n, $2^n - 1$ and 2^{n-1}. From this table I was able to observe some characteristics of these expressions and how they have come to bear relevance and importance to the search for perfect numbers.

n	2^n	$2^n - 1$	2^{n-1}
1	2	1	1
2	**4**	**3**	**2**
3	**8**	**7**	**4**
4	16	15	8
5	**32**	**31**	**16**
6	64	63	32
7	**128**	**127**	**64**
8	256	255	128
9	512	511	256
20	1,048,576	1,048,575	524,288

Obviously, the Mersenne primes will always be odd since we're subtracting 1 from a multiple of 2. Equally obvious, when multiplied by a multiple of 2, this will result in an even number. If the Mersenne prime ends in a 1, and since the product of that prime and a multiple of two will be a perfect number that ends in 6 or 8, I know the multiplicand must end in a 6 or 8. (If it ends in 3, the multiplicand ends in 2 or 6, if it ends in 7, it must end in 4 or 8, and for 9 it would end in 2 or 4.) Can it end in 9, though?

In looking at the table, it's quite easy to see why the perfect numbers become quite large quite rapidly! It also becomes apparent that 5 and 0 as ending numbers will never be the case for perfect numbers, assuming all perfect numbers are even. Why? Because no prime number ends in 5 or 0 (other than 5 itself), and to arrive at a product that ends in 0, we'd have to multiply a multiple of 2 by a number that ends in 5. Since one of the numbers has to be a Mersenne prime, and neither of them can be by definition of prime, no perfect number will ever end in 5 or 0. Similarly, if a Mersenne prime were to end in a 9, it would have to mean that a power of 2 ended in zero. Since a power of 2 cannot end in zero (this would mean, when it was factored to prime, it would have to contain the prime factor of 5 which it doesn't! It only contains prime factors of twos.), a Mersenne prime cannot end in 9. In fact, all known Mersenne primes beyond 3 end in 7 or 1.

This made me curious about what the multiplicands would have to be if the perfect number were to end in 2 or 4. Looking at possibilities of such numbers, I could quickly eliminate all pairs of multiplicands that were both even numbers since one of them clearly has to be odd to be a Mersenne prime. Thus, the table below shows what the numbers must end in to result in a perfect number:

If perfect # ends in 2			**If perfect # ends in 4**	
M_P ends in:	Power of 2 ends in:		M_P ends in:	Power of 2 ends in:
1	2		1	4
3	4		3	8
7	6		7	2

They say this, but how do we know all known perfect numbers end in 6 or 8? I must confess, in my research I sought a proof for why this is the case and did not find a useful one. What I found was so confusing and made no sense to me or was so completely out of my understanding of mathematics I decided to try to come up with something for myself. While I'm sure this awkward "proof" that perfect numbers cannot end in 2 or 4 is primitive, it was the process and journey I went through to explain it to myself in a way that had meaning in layman's terms. With that in mind I tackled what it would take to for a perfect number to end in 2 or 4, and ultimately found they do indeed end in 6 or 8.

As a means of proving to myself that a perfect number *cannot* end in two or four, I began looking at the attributes of powers of two. All powers of two follow a pattern of ending in 2, 4, 6 or 8. If the ones value of the exponent is an odd number, the final product will end in 8 or 2. If it is even, the product will end in 4 or 6. Similarly, the pattern of when it ends in "8 or 2" or "4 or 6" is based on alternating values in the tens place. Perhaps it would be helpful to look at a table of powers of two to show this patterning.

Table 1A

Power of 2	**Product**	**Power of 2**	**Product**	**Power of 2**	**Product**	**Power of 2**	**Product**
0	1	10	1024	20	1,048,576	30	1,073,741,824
1	2	11	2048	21	2,097,152	31	2,147,483,648
2	4	12	4096	22	4,194,304	32	4,294,967,296
3	8	13	8192	23	8,388,608	33	8,589,934,592
4	16	14	16,384	24	16,777,216	34	17,179,869,184
5	32	15	32,768	25	33,554,432	35	34,359,738,368
6	64	16	65,536	26	67,108,864	36	68,719,476,736
7	128	17	131,072	27	134,217,728	37	137,438,953,472
8	256	18	262,144	28	268,435,456	38	274,877,906,944
9	512	19	524,288	29	536,870,912	39	549,755,813,888

As Table 1B shows, when the tens value is odd our products end as follows based on the ones values of the exponent (or M_P or M_{P-1}), and similarly the data for even tens values can be seen. The table shows the ending values of each component of our formula $(2^{Mp} – 1)(2^{Mp-1})$ where M_P is a Mersenne prime.

Table 1B

ODD TENS VALUE For M_P			**EVEN TENS VALUE For M_P**		
ODD ones value		**EVEN ones values (M_{P-1})**	**ODD ones value**		**EVEN ones value (M_{P-1})**
If M_P ends in	**$2^{Mp}-1$** ends in	**2^{Mp-1}** ends in	**If M_P** ends in	**$2^{Mp}-1$** ends in	**2^{Mp-1}** ends in
1, 2^{M_P} ends in 8	8 -1 = 7	0 ends in 4	**1,** 2^{M_P} ends in 2	2 – 1 = 1	0 ends in 6
3, 2^{M_P} ends in 2	2 -1 = 1	2 ends in 6	**3,** 2^{M_P} ends in 8	8 – 1 = 7	2 ends in 4
7, 2^{M_P} ends in 2	2 -1 = 1	6 ends in 6	**7,** 2^{M_P} ends in 8	8 – 1 = 7	6 ends in 4
		Note: the ones value is = one less than the ones value of M_P			Note: the ones value is = one less than the ones value of M_P

The following example may help explain the tabled information when used for any value of M_P.

For example, if the Mersenne prime ends in a 1 (say 31), this would mean the value of $2^{M_P} - 1$ would have to end in 7 since the power of two has an odd tens value and an odd ones value. (Note: since we're subtracting a one in the expression, the power of two would end in 8, which means the $2^{M_P} - 1$'s ending value would then be 8 – 1 = 7.) With this in mind, the value of $2^{M_P - 1}$ would have to be 2^{30}. Since the tens value of the exponent is odd and the ones value is 0 (even), this term must end in 4. Thus, this perfect number will end in 8 since *7 times *4 equals 28. (Multiply the ones values)

Now consider the example once again, looking at the M_P value of 31. Obviously, all primes beyond 2 end in an odd number, so when subtracting one from any prime (Mersenne or otherwise) the tens value of that number will not be changed. The smallest value for the ones place will be a 1 for prime numbers, and subtracting 1 from that will leave a zero in the ones place. Thus, the tens value remains the same. In order to multiply the combinations of possible individual ending digits for each component of the formula that would result in the perfect number ending in 2 or 4, we would have to have alternating odd-even tens values generate the ending digit for each component. Since this cannot happen, even perfect numbers will never end in 2 or 4.

With that in mind, I found the following cases had to be true for the values of $2^{M_P} - 1$ and $2^{M_P - 1}$ as shown in the table below:

M_P ends in:	$2^{M_P} - 1$ ends in:	$2^{M_P - 1}$ ends in:
1	7 (odd) or 1 (even)	4 (odd) or 6 (even)
3	7 (odd) or 1 (even)	4 (odd) or 6 (even)
7	7 (odd) or 1 (even)	4 (odd) or 6 (even)

While the table shows there are indeed combinations capable of producing products that end in 2 or 4, it's the manner in which the patterns occur that determine which ending numbers can be multiplied. Obviously, only the "odds" can be multiplied (and conversely only the evens). Therefore, the only possible results for the ending number of even perfect numbers are 8 and 6 since the combinations that result in endings of 2 or 4 would require us to multiply terms that have inconsistent tens values for the Mersenne primes (i.e. an odd tens value multiplied by an even tens value.)

Let's look at $M_P = 17$ (The tens value is odd, the ones value is odd for M_P but even for M_{P-1}, but the tens value will still be odd!) In order to end up with a 2 or 4 as an end number for a perfect number, the *tens value* of the M_P - 1 would have to change and it never will.

Thus, we know the value of $2^{M_P - 1}$ will stem from this exponent ending in a 6 (7 – 1 = 6) while having an odd tens value. Thus, this term's overall ending digit will be a 6. In order to result in an ending of 2 for the perfect number, this number would have to be multiplied by a number that ends in 7. (Our only choices are 1 or 7, remember.) But, since the tens value is odd and the ones value is 7, we know by table 1B that this value ends in 1 and not 7. This method can be used for all values of M_P, knowing all M_P will end in 1, 3 or 7. To show this with a value of M_P that has an even tens value (let's use 61) we now have $2^{M_P - 1} = 2^{60}$ and $2^{M_P} - 1 = 2^{61} - 1$. 2^{60} has an even tens value for the exponent and 0 for the ones value, which means the overall value of this term will end in 6.
(See Table 1B) $2^{61} - 1$ has an even tens value for the exponent and 1 for the ones value, which means the overall value of this term will end one less that 2, or 1. Clearly, 6 and 1 will not produce an ending number of 2 or 4 for their product.

Now that I've managed to convince myself that every even perfect number will end in 6 or 8, I can move on to exploring other characteristics related to perfect numbers and their attributes.
One of these interesting points was shown fairly early on in my explorations of perfect numbers. This fact, that all perfect numbers are also the sum of all consecutive positive integers starting with 1, is shown as:

$1 + 2 + 3 = 6$
$1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$
$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 .... + 31 = 496$

From this, I could see the last addend is also the M_P used in determining the perfect number. I also realized the other multiplicand in the perfect number formula ($2^{M_P - 1}$, which is simply a power of 2) is exactly half of the M_P when it is increased by one. (This stands to reason since each power of two doubles the previous power of two!) What I recognized next was this string of consecutive integer addends is the manner in which triangular numbers are found. Thus, a formula for finding triangular numbers can be found. (T_N = triangular number P_N = perfect number)

$T_N = 1 + 2 + 3 .... + (n - 2) + (n - 1) + n$ where n is a natural number and the # of terms (addends) which becomes $\boldsymbol{T_N = n \cdot \frac{(n + 1)}{2}}$ (derived in the way of Gauss' pairwise addition strategy)

Keeping in mind the realization that the M_P is equal to the last term in the string of addends, and knowing the last term in the string is equal to n I now know M_P is also equal to n. So, through substitution into the formula I have found:

$T_N = M_P \cdot \frac{(M_P + 1)}{2}$ Clearly, this shows that relationship stated above: One multiplicand in the P_N formula is a M_P while the other is half of that M_P increased by 1.

When considered without benefit of knowing the exact M_P, one could also write this formula as:

$T_N = (2^{M_P} - 1) \frac{(2^{M_P - 1}) + 1}{2}$ as $(2^{M_P} - 1)$ will "calculate" a M_P if a specific one is not used.

We also know each power of 2 doubles the previous power of two, so in multiplying by 2 and performing some simplification of terms, I arrive at a new "variation" of the formula as:

$\boldsymbol{TN = (2^{M_P} - 1) \cdot \frac{(2^{M_P} - 1) + 1}{2}}$ and ultimately $\mathbf{TN = (2^{M_P} - 1) \cdot (2^{M_P} - 1)}$ (The formula for P_N!)

Clearly, this is an indication that all P_N are T_N, while at the same time shows *some* T_N are P_N. Conversely, we can say while all perfect numbers are also triangular not all triangular numbers are also perfect. There are many Triangular Numbers between the Perfect Numbers, so this formula will not determine all T_N. It simply finds those which are also even P_N, which I suppose could be denoted as "T_{PN}" or some such notation.

With this new information, knowing the 3^{rd}, 7^{th}, and 31^{st} T_N are 6, 28 and 496, can we now find the k^{th} T_{PN}? Certainly! k will always be equal to the M_P used to determine the P_N. Thus, the 4^{th} T_{PN} would be derived by using the 4^{th} M_P of 127. Thus, while this is the 4^{th} T_{PN}, it is the 127^{th} T_N.

At this point, while I will offer no examples or proof of such (though it would be quite similar to the previous discussion) it is also known that perfect numbers are hexagonal numbers, as are some triangular numbers. (Hexagonal numbers are always triangular, but some triangular numbers are not hexagonal.) In my mind, this seems logical from a geometrical standpoint since a regular hexagon is a compilation of equilateral triangles. (*Figure1)*

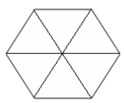
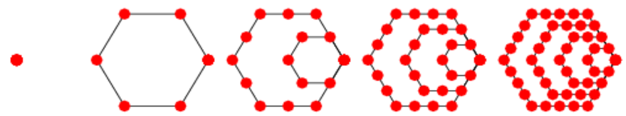


Figure1



1 **6** 15 **28** 45

Figure 2

The diagram above (*Figure 2)* shows the manner in which hexagonal numbers are considered. (Numbers that can be represented by a regular geometric arrangement of equally spaced points in a lattice formation, or rather the number of points in the union of *n* hexagons with partly two common sides, as shown.)

One other interesting property of perfect numbers is that the sum of the reciprocals of *all* the divisors (not just the proper divisors) of any perfect number is 2. Is it a coincidence that perfect numbers rely on powers of two to be formed? For example, for $P_N = 28$,

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{7}+\frac{1}{14}+\frac{1}{28}=2$$

The question was posed regarding the Fibonacci numbers and square numbers and if perfect numbers can also be Fibonacci and or square numbers. While there are four triangular numbers (1, 3, 21 & 55) that are also Fibonacci, none of these are "perfect". And clearly, for a number to be "square" it requires two multiplicands of equal value. As this will never be the case for $(2^{M_P} - 1)\ (2^{M_P - 1})$, perfect numbers will never be square numbers.

While this is merely a tip of the iceberg in what is to be known about perfect and triangular numbers, Mersenne numbers and primes, and "perfect triangular numbers", it is a basis for initial understanding of these relationships. In researching this topic, I found the more I learned the more there was to learn! One could easily fill as many pages on any one aspect of perfect numbers and their attributes relational to other number classifications. These, however, far exceed the intent of this discussion. Perhaps it is something I will (for the interest is certainly there!) revisit when such limitations (and time restrictions) are not present.

The journey into the world of perfect numbers can be as vast as the traveler desires it to be, as is the case with any mathematical journey. There is mystery and intrigue that, from thousands of years ago, exists today. One of the greatest mysteries in all of mathematics pertains to perfect numbers, after all! Are there odd perfect numbers? Prior to my experiences with Math in the Middle, I wouldn't have cared! It is my hope that this question will be answered within my lifetime so I can rejoice and celebrate it as a part of this most interesting journey, for now I can appreciate the value and dedication the discovery of one would represent!

Appendix A: Perfect Number Timeline

	$(2^{M_P - 1})(2^{M_P} - 1)$	# of Digits in P_N	Year of Discovery	Discoverer (credited)
1	$2^{1}(2^{2}-1)$	1	Unknown	Unknown
2	$2^{2}(2^{3}-1)$	2	Unknown	Unknown
3	$2^{4}(2^{5}-1)$	3	Unknown	Unknown
4	$2^{6}(2^{7}-1)$	4	Unknown	Unknown
5	$2^{12}(2^{13}-1)$	8	1456*	**disputed*
6	$2^{16}(2^{17}-1)$	10	1588	Cataldi
7	$2^{18}(2^{19}-1)$	12	1588	Cataldi
8	$2^{30}(2^{31}-1)$	19	1772	Euler
9	$2^{60}(2^{61}-1)$	37	1883	Pervushin
10	$2^{88}(2^{89}-1)$	54	1911	Powers
11	$2^{106}(2^{107}-1)$	65	1914	Powers
12	$2^{126}(2^{127}-1)$	77	1876	Lucas
13	$2^{520}(2^{521}-1)$	314	1952	Robinson
14	$2^{606}(2^{607}-1)$	366	1952	Robinson
15	$2^{1278}(2^{1279}-1)$	770	1952	Robinson
16	$2^{2202}(2^{2203}-1)$	1327	1952	Robinson
17	$2^{2280}(2^{2281}-1)$	1373	1952	Robinson
18	$2^{3216}(2^{3217}-1)$	1937	1957	Riesel
19	$2^{4252}(2^{4253}-1)$	2561	1961	Hurwitz
20	$2^{4422}(2^{4423}-1)$	2663	1961	Hurwitz
21	$2^{9688}(2^{9689}-1)$	5834	1963	Gillies
22	$2^{9940}(2^{9941}-1)$	5985	1963	Gillies
23	$2^{11212}(2^{11213}-1)$	6751	1963	Gillies
24	$2^{19936}(2^{19937}-1)$	12003	1971	Tuckerman
25	$2^{21700}(2^{21701}-1)$	13066	1978	Noll & Nickel
26	$2^{23208}(2^{23209}-1)$	13973	1979	Noll
27	$2^{44496}(2^{44497}-1)$	26790	1979	Nelson & Slowinski
28	$2^{86242}(2^{86243}-1)$	51924	1982	Slowinski
29	$2^{110502}(2^{110503}-1)$	66530	1988	Colquitt & Welsh
30	$2^{132048}(2^{132049}-1)$	79502	1983	Slowinski
31	$2^{216090}(2^{216091}-1)$	130100	1985	Slowinski
32	$2^{756838}(2^{756839}-1)$	455663	1992	Slowinski & Gage et al.
33	$2^{859432}(2^{859433}-1)$	517430	1994	Slowinski & Gage
34	$2^{1257786}(2^{1257787}-1)$	757263	1996	Slowinski & Gage
35*	$2^{1398268}(2^{1398269}-1)$	841842	1996	Armengaud, Woltman,
36*	$2^{2976220}(2^{2976221}-1)$	1791864	1997	Spence, Woltman,
37*	$2^{3021376}(2^{3021377}-1)$	1819050	1998	Clarkson, Woltman, Kurowski
38*	$2^{6972592}(2^{6972593}-1)$	4197919	1999	Hajratwala, Woltman, Kurowski
??*	$2^{13466916}(2^{13466917}-1)$	8107892	2001	Cameron, Woltman, Kurowski
??*	$2^{20996010}(2^{20996011}-1)$	12640858	2003	Shafer, Woltman, Kurowski
??*	$2^{24036582}(2^{24036583}-1)$	14471465	2004	Findley, Woltman, Kurowski
??*	$2^{25964950}(2^{25964951}-1)$	15632458	2005	Nowak, Woltman, Kurowski
??*	$2^{30402456}(2^{30402457}-1)$	18304103	2005	Cooper, Boone, Woltman, Kurowski

* NOTE : #s 35-"43" are credited to GIMPS (et al) Also, the 39th – 43rd P_N are not proven to be sequential and are thus not numbered as such. The 5th P_N was noted to be discovered by "anonymous" in several sources, but literature shows several who "discovered" it so I elect to refer to this as "disputed." Appendix A information was obtained from http://amicable.homepage.dk/perfect.htm and other web sources.

Appendix B

Mersenne Numbers (found by: $2^p - 1$)

$2^1 - 1 = 2 - 1 = 1$
$2^2 - 1 = 4 - 1 = 3$*
$2^3 - 1 = 8 - 1 = 7$*
$2^4 - 1 = 16 - 1 = 15$

**Note: When "p" is prime, the result of* 2^P-1 *will be a Mersenne prime.*

Value of p	Mersenne Number **(Mersenne Primes in bold and italics)**
1	1
2	***3***
3	***7***
4	15
5	***31***
6	63
7	***127***
8	255
9	511
10	1023
11	2047
12	4095
13	***8191***
14	16,383
15	32,767
16	65,535
17	***131,071***
18	262,143
19	***524,287***
20	1,048,577
21	2,097,151
22	4,194,303
23	8,388,607
24	16,777,215
25	33,554,431
26	67,108,865
27	134,217,727
28	268,435,455
29	536,870,911
30	1,073,741,823
31	***2,147,483,647***

References & Credits

Andrews, G. “Number Theory”, Dover Publications, NY 1994.

Conway, J. H. and Guy, R. K. "Perfect Numbers." In *The Book of Numbers.* New York: Springer-Verlag, pp. 136-137, 1996.

Eaton, C. F. "Perfect Number in Terms of Triangular Numbers." Solution to Problem 1482. *Math. Mag.* 69, 308-309, 1996.

GIMPS. The Great Internet Prime Search Home Page: http://www.mersenne.org

Glenn Learning Technologies Project, http://www.grc.nasa.gov/WWW/K-12/Numbers/Math/Mathematical_Thinking/prfctno.htm

Gupta, Shyam Sunder “Curious Properties of Triangular Numbers.” *Science Reporter*, Sept. 1987, India http://www.grc.nasa.gov/WWW/K-12/Numbers/Math/Mathematical_Thinking/prfctno.htm

Heng, O.K. “The Perfect Number Journey: Lessons on Number Patterns and Properties of Numbers.” http://home1.pacific.net.sg/~novelway/MEW2/lesson1.html

MacTutor: “History of Mathematics.” http://www-history.mcs.st-andrews.ac.uk/HistTopics/Perfect_numbers.html

Pe, Joseph L., “On a Generalization of Perfect Numbers.” *The Journal of Recreational Mathematics* 31(3)

Pedersen, J.M. http://amicable.homepage.dk/perfect.htmFascinating. Last update: 25-Dec-2005

Smith, H. J. "Perfect Numbers." http://www.geocities.com/hjsmithh/Perfect.html.

The Math Forum: AskDr.Math™©1994-2006 http://mathforum.org/dr.math/

Trotter, T. Jr. "Some Identities for the Triangular Numbers." *Journal of Recreational Mathematics* 6, 128-135, 1973. http://www.trottermath.net/numthry/trident.html.

Weisstein, E.W., “Perfect Number.” From MathWorld: A Wolfram Web Resource. http://mathworld.wolfram.com/PerfectNumber.html

Wells, David. The Penguin Dictionary of Curious and Interesting Numbers. Penguin USA (P); Reprint edition (November 27, 1986)