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Palindromes

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Palindromes

A palindrome is a word or phrase that reads the same forward and backward, such as the word "level" and the phrase "Madam, I'm Adam." Numbers whose digits read the same forward and backward are also called palindromes, such as the numbers 22, 1234321, and 2002. Palindromic phrases can also occur in number form, for example the Universal day of Symmetry: 8:02 P.M. on February 20, 2002. If one were to look at the time on a twenty-four hour clock, it would read 20:02; the date can be read as the twentieth day of the second month, which also represents a 2002. Of course, the year is already a 2002. Putting the whole date together gives 2002 2002 2002, a palindromic number phrase. The word palindrome comes from the Greek word palindromes, meaning "running back again".

All single digits are considered palindromes in a base 10 system: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Two digit palindromes are also easy to find. With the exception of zero, one can use each of the single digits in both the ten's place and one's place to find all of the two digit palindromes: 11, 22, 33, 44, 55, 66, 77, 88, 99. To create the three-digit palindromes, we must have identical, nonzero digits in the one's and hundred's places, but there is no restriction on the digit in the ten's place. The three digit palindromes are 101, 111, 121, 131, 141, 151, 161, 171, 181, 191, 202, 212, 222, . . . One could list all of the three digit palindromes to see how many there are, but a more general formula can be derived to find the number of k-digit palindromes, where k is any whole number greater than one.

In deriving such a formula, one must consider the choices for each position within the number. Clearly, the two outside digits (the first and the last digits) must be the same number and can be any of the digits 1 through 9. Therefore, for any k-digit palindrome, there are nine choices for the first and last digits. Furthermore, for any k digit palindrome, the middle digit (for an odd number of digits) or the middle two (for an even number of digits) must also be the same. These digits may also be chosen from the digits 0 through 9, giving ten choices for the middle position(s). Looking at the chart below, one is better able to visualize a pattern.

Digits in palindrome	Choices of digits in each position: u=units, t=tens, h=hundreds th=thousands, tth = ten thousands, hth= hundred thousands, m=millions, tm=ten millions	Example
2	(t = u) 1 through 9 1 x 9 = 9	t u 1 1
3	(h=u) 1 through 9; (t) 0 through 9 9 x 10 = 90	h t u 1 0 1
4	(th=u) 1 through 9; (h=t) 0 through 9 9 x 10 = 90	th h t u 1 0 0 1
5	(tth=u) 1 through 9; (th=t) 0 through 9; (h) 0 through 9 9 x 10 x 10 = 900	tth th h t u 1 0 2 0 1
6	(hth=u) 1 through 9; (tth=t) 0 through 9; (th=h) 0 through 9; 9 x 10 x 10 = 900	hth tth th h t u 1 0 2 2 0 1
7	(m=u) 1 through 9; (hth=t) 0 through 9; (tth=h) 0 through 9; (th) 0 through 9 9 x 10 x 10 x 10 = 9000	m hth tth th h t u 1 0 2 3 2 0 1
8	(tm=u) 1 through 9; (m=t) 0 through 9;(hth=h) 0 through 9; (tth=th) 0 through 9 9 x 10 x 10 x 10 = 9000	tm m hth tth th h t u 1 0 2 3 3 2 0 1

From the above table, one notices that ninety palindromes can be made with both three digits and four digits. There are nine hundred palindromes using either five or six digits, and both seven and eight digits yield nine-thousand possible palindromes. One can also observe the increasing powers of ten that occurs every two rows. For example, three and four digit palindromes have one 10 multiplied by 9; five and six digit palindromes have a 10^2 multiplied by 9; seven and eight digit palindromes have a 10^3 multiplied by 9, and so on. Regardless of the number of digits in the palindrome, there are only nine choices for the first and last digits, since zero cannot be considered for the first digit. Therefore, the following formula can be used to find the total number of palindromes that can be made with k digits:

When k is even: $10^{(k/2-1)} \times 9$

When k is odd: $10^{((k+1)/2-1)} \times 9$

The following table shows some examples of how the formula works:

Digits in palindrome	Formula Even $10^{(d/2-1)}$ Odd $10^{((d+1)/2-1)}$	Pattern and answer
2	$10^{(2/2-1)}$	$10^0 \times 9 = 9$
3	$10^{((3+1)/2-1)}$	$10^1 \times 9 = 90$
4	$10^{(4/2-1)}$	$10^1 \times 9 = 90$
5	$10^{((5+1)/2-1)}$	$10^2 \times 9 = 900$
6	$10^{(6/2-1)}$	$10^2 \times 9 = 900$
7	$10^{((7+1)/2-1)}$	$10^3 \times 9 = 9000$
8	$10^{(8/2-1)}$	$10^3 \times 9 = 9000$

Numbers can be converted into palindromes using a reverse and add rule. For example take a two digit number that is not a palindrome, say 45. Reverse the digits and add these two numbers together : 45 + 54 = 99. If one is not successful after one step, repeat the steps until a palindrome is reached. For example, 67 + 76 = 143 is not a palindrome, but 143 + 341 = 484 is. All two digit numbers can be converted into a palindrome in a finite number of steps. Most are quite easy to convert; however, there are a few two digit numbers that require a little persistence.

There are several shortcuts one can take to determine how many steps will be necessary to convert a two digit number into a palindrome. The first is to consider: **if a number N is a k-step palindrome (meaning it requires k iterations of the reverse and add rule to turn N into a palindrome) and if the units digit of N is not equal to 0, then the number N' obtained by reversing the digits in N is also a k-step palindrome**. See the example below: (Let t = tens digit of N, where $1 \leq t \leq 9$ and u = units digit of N, where $0 \leq u \leq 9$, and suppose t = 4 and u = 5)

N = 10(t) + u = 10(4) + 5 = 45; 45 + 54 = 99, so 45 is a 1-step palindrome

N’ = 10(u) + t = 10(5) + 4 = 54; 54 + 45 = 99, so 54 is also a 1-step palindrome.

The second shortcut to consider is: **if t ≠ u and t + u is no bigger than 9, then N is a 1-step palindrome**. In the above example, one can see that 4 is not equal to 5 and 4 + 5 = 9, so therefore, it can be converted into a palindrome in 1-step. Another shortcut to remember is: **all two-digit numbers (with t ≠ u) that have the same sum of digits t + u will require the same number of reverse-and-add steps to turn them into palindromes**. Examples of this shortcut would include 19, 28, 37; all have t ≠ u, and the sum of their digits is 10, so we expect them to be the same step-number palindromes:

19 + 91 = 110; 110 + 011 = 121, so 19 is a 2-step palindrome.

28 + 82 = 110; 110 + 011 = 121, so 28 is a 2-step palindrome, also.

37 + 73 = 110; 110 + 011 + 121, so 37 is a 2-step palindrome, also.

Using the above mentioned shortcut and some old fashioned computation, the following table was created to show the number of steps necessary to convert each two digit number into a palindrome.

Steps needed	Sum of tens’ digit And one’s digit (t + u)	Values of N N being a two digit number
0	Already a palindrome	11, 22, 33, 44, 55, 66, 77, 88, 99
1	t + u ≤ 9	10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 23, 24, 25, 26, 27, 30, 31, 32, 34, 35, 36, 40, 41, 42, 43, 45, 50, 51, 52, 53, 54, 60, 61, 62, 63, 70, 71, 72, 80, 81, 90
1	t + u = 11	29, 38, 47, 56, 65, 74, 83, 92
2	t + u = 10	19, 28, 37, 46, 64, 73, 82, 91
2	t + u = 12	39, 48, 57, 75, 84, 93
2	t + u = 13	49, 58, 67, 76, 85, 94
3	t + u = 14	59, 68, 86, 95
4	t + u = 15	69, 78, 87, 96

6	t + u = 16	79, 97
24	t + u = 17	89, 98

(Sally, 2003, pp. 71)

As one can see from the above table, the sum of the digits in the two digit numbers seems to be a key in figuring out the number of steps it will take to convert those two digit numbers into palindromes.

To analyze this a bit closer, let's take a look at each sum t + u. The original two-digit number will be called N and can be written as 10t + u, where t is the tens digit and u is the ones digit satisfying $1 \leq t \leq 9$ and $0 \leq u \leq 9$. We will also require that $t \neq u$, since otherwise, these would be palindromes already. This requirement eliminates the largest possible sum that t + u can be, namely 18. Let's look at the cases where t + u is greater than or equal to 1 and less than or equal to 17. Finding the sum from applying the first step to the numbers is the key to finding how many steps are needed for a number to be converted into a palindrome. If N is the original two digit number and N' is the number made when reversing the digits, then letN_1 be the first step sum: $N_1 = N + N' = 10(t + u) + (t + u)$. At this point, one must separate the sums into two cases.

<u>Case 1</u>: $(t + u \leq 9)$

In this case, t + u is the units digit of N_1 and $t + u \geq 1$. Also, t + u is the tens digit of N_1. Hence, the first step number N_1 is a two digit palindrome. For example: N = 27; N' = 72; $N_1 = 10(2 + 7) + (7 + 2) = 27 + 72 = 99$, a palindrome.

<u>Case 2:</u> $(t + u \geq 10)$

Here, when N and N' are added together, carrying a ten or regrouping takes place. The units digit of N_1 is now $t + u - 10$, and a 1 is carried over to the tens place. The tens digit of N_1 is then $t + u + 1 \geq 10$, but since this is greater than 10, it reduces to$t + u + 1 - 10 = t + u - 9$, and a 1 has to be carried over to the hundreds place. The first step sum N_1 is a three digit number that can be written in the form of $N_1 = 100(1) + 10(t + u - 9) + (t + u - 10)$. For example: N = 47 gives N' = 74, and $4 + 7 = 11 \geq 10$. Here, 47 + 74 = 121, so 47 is a one step palindrome. As one can see from the previous example, we had a one-step palindrome where the sum of the tens digit and the ones digit was eleven. Therefore, **N is a one-step palindrome if $t + u \leq 9$ or if $t + u = 11$.**

One can continue in this fashion to check the remaining two digit numbers to see how many steps are needed to convert them into palindromes. There are only a few cases left to check. When $t + u = 10$ and $t \neq u$, we find that $N_1 = 100(1) + 10(t + u - 9) + (t + u - 10)$

$$= 100(1) + 10(1) + (0) = 110 \qquad \text{* substitute 10 for } t + u$$

Step two: $N_2 := N_1 + N'_1 = 110 + 011 + 121$, so N is a two-step palindrome.

Therefore, when $t + u = 10$ and $t \neq u$, the two digit number N can be converted into a two-step palindrome.

There are only a few cases left to check. The remaining two digit numbers can be grouped according to the sum of their ten's digit and their one's digit. As one can see from the table, those with a sum of 12 and 13 are also two-step palindromes. Those two digit numbers with a sum of 14 are three-step palindromes. If the sum of the ten's and unit's digits of any two-digit number is 15, then that number is a four-step palindrome. A six-step palindrome can be made from the two-digit numbers whose digits add up to 16. The last pair of two digit numbers to check is 89 and 98. The sum of their ten's digit and one's digit is 17. It will take 24 steps to convert this pair of numbers into a palindrome.

The reverse and add rule can be used to continue to convert three digit numbers (which are not already palindromes) into palindromes. If one takes on this challenge, start with small three digit numbers and work up toward the larger numbers. Some of the same kind of patterns will apply to three digit numbers that were found with the two digit numbers.

Interestingly, all palindromes with an even number of digits are divisible by 11, thus making these numbers composite, except for 11 itself. Let's see how this works. Take the smallest four digit palindrome 1001 and divide it by 11. We get 91. Take the next 4 digit palindrome 1111 and divide it by 11. The quotient is 101. So, both are divisible by 11. The second one is 110 larger than the first one, and note that 110 is also divisible by 11. Look at the prime factors of 110, which are 2, 5, 11 and of 1001, which are, 7, 11, 13; 11 is a common prime factor of 110 and 1001. Using the distributive property, any four digit palindrome can be written as x(1001) + y(110) where x is some integer between 1 and 9, inclusive, and y is some integer between 0 and 9, inclusive. For example, 6(1001) + 3(110) = 6006 + 330 = 6336 is a palindrome. Since 11 is a factor of both 1001 and 110, we conclude that all four digit palindromes are divisible by 11. Moreover, since 11 is the only prime common to 1001 and 110, it is the only such number with this property.

By looking at the palindromes with an odd number of digits, one, unfortunately, cannot say that they are all composite. But what about being prime? When looking at the first three-digit palindromes that are prime (101, 131, 151, 181, and 191), one might guess that there are a lot of them. However, of the 90 three-digit palindromes, there are only 15 that are prime. Of the 900 five-digit palindromes, 93 are prime. There are 668 prime palindromes with seven digits. Among palindromes with an odd number of digits, the ratio of primes to composites becomes smaller and smaller as the number of digits increases. The largest known prime palindrome, $10^{39026} + 4538354 * 10^{19510} + 1$, has 39,027 digits!

When a palindrome is squared the results are interesting. Let's begin with the single digit palindromes: 1 x 1 = 1; 2 x 2 = 4; 3 x 3 = 9. So far, all produce palindromes, but that is where it ends for the single digit palindromes. The two and three digit palindromes that produce a palindrome when squared are as follows:

11 x 11 = 121 22 x 22 = 484 101 x 101 = 10201 111 x 111 = 12321

121 x 121 = 14641 202 x 202 = 40804 212 x 212 = 44944

Observe that the palindromes in the list above are all combinations of the digits 0, 1, and 2. Although this list appears to be small, there are actually an infinite number of palindromes containing only the digits 0, 1, and 2 whose squares are also palindromes. Take a look at the examples that follow:

$11^2 = 121$, $101^2 = 10201$, $1001^2 = 1002001$, $10001^2 = 100020001$, $100001^2 = 10000200001$, etc.

$22^2 = 484$, $202^2 = 40804$, $2002^2 = 4008004$, $20002^2 = 400080004$, $200002^2 = 40000800004$, etc.

How far will palindromes with only ones as digits continue to have palindrome squares?

11 x 11 = 121
111 x 111 = 12321
1111 x 1111 = 1234321
11111 x 11111 = 123454321
111111 x 111111 = 12345654321
1111111 x 1111111 = 1234567654321
11111111 x 11111111 = 123456787654321
111111111 x 111111111 = 12345678987654321
1111111111 x 1111111111 = 12345679010987654321 This almost works, but not quite.

Unfortunately, the regrouping ruins the pattern.

The world of palindromes continues on in all kinds of directions—it is a kind of recreational math. It is fun to play around with and to see what can be discovered about palindromes. There is even a palindrome conjecture that has yet to be solved. The conjecture states that any number not already a palindrome can be converted into a palindrome by using the reverse and add rule a finite number of times. The world record for applying this to large numbers is in converting the nineteen digit number 1,186,060,307,891,929,990 into a palindrome, which required 261 steps! The number 196 has stumped a lot of people—2 million reverse-and-add steps have been performed on 196, trying to convert it into a palindrome, but without success. Will 2 million and one steps prove to be successful?

As a teacher of seventh grade math, I am excited about introducing my students to the world of palindromes. This topic offers a rich experience in "playing" with numbers. Attached is a lesson plan to use with my seventh graders at the beginning of the year.

Stephanie Fuehrer
Lesson Plan – 7th grade (both regular math and pre-algebra classes)

Palindromes
Objectives:

- The students will be exposed to a subset of numbers they haven't worked with before—they will explore the world of palindromes.
- By the end of the lesson, students will know how many one-, two-, three-, and four digit palindromes are possible. The students will be able to find a pattern or a formula for finding out how many k-digit palindromes exist.
- Students will change two-digit numbers into palindromes using the reverse and add rule. They will keep track of the number of steps it takes to convert each two-digit number into a palindrome. Students will be asked to keep track of their findings on a color-coded chart. This chart will be used by the students to make some observations about palindromes.
- Students will be able to come up with many different palindromes they see in the world outside of math class. (ex.: birthdates, phone numbers, addresses, etc.)

Standards:

*MA 7.1.1 Students will represent and show relationships among rational numbers.
*MA 7.1.3 Students will compute fluently and accurately using appropriate strategies and tools.
*MA 7.4.1 Students will formulate questions that can be addressed with data and then organize, display and analyze the relevant data to answer their questions.
*MA 7.4.2 Students will evaluate predictions and make inferences based on data.

Students will communicate number sense concepts found using palindromes, using multiple representations to explore, reason, and make connections within mathematics and across disciplines.

Introduction:
To introduce palindromes, I will tell my students, "I'm thinking of a kind of number. All of these numbers belong to this set of numbers." I will put up 1, 2, 3, 4, 5, 6, 7, 8, 9. I will have my students brainstorm what they think the set is called and what some of the characteristics of the set. I will write up their ideas on the board. When they are done, I will go through each of their ideas and ask, "if this is the kind of number I am thinking of, what would be some more numbers that would belong to this set?" After we look at each of their ideas, I will say either "yes, these new numbers will work" or "no, they won't work".

I will then put up some two digit palindromes, showing the students more numbers that belong to the set of numbers I am thinking of. We will repeat the steps above of listing their ideas and examples for my numbers. After we complete the above steps, I will put up various numbers and ask my students if they think these numbers belong to the set of numbers I am thinking of. For example: 12321, 10000, 4444, 34543, 6677887766, etc. We will proceed until my students can come up with a definition for palindromes. I will then introduce them to the name palindrome and where it originates.

1st Activity – Students will be able to give examples of palindromes.
I will ask my students to name some numbers that are palindromes. I will put them up on the board. I will ask them to think of places numbers are used – addresses, amounts, dates, phone number, etc. We will use these places to explore how they could be palindromes. This discussion will lead into talking about palindrome dates (today's date, birth dates, the last palindrome year – at what exact time to the nearest minute of that year was time a palindrome? (8:02 P.M., the 20th day of February, 2002)). The students may need help with the time – on a 24-hour clock, and naming the date before the month. When will the next palindrome year be? Can we ever have a palindrome with the time, date, and year again? Why or why not?

2nd Activity – Identify a pattern to tell how many palindromes there are of a certain digit length.
I will put up a chart on how many palindromes of a certain digit length exist. We will start with one digit, then two digits, then three digits. I will have them work with partners on the three digit ones to see if they can discover a pattern and find them all. I will ask them to share their methods or discoveries with the rest of the class. *Here is where, as a teacher, I need to be ready to go with my students' ideas and extend them toward a formula or lead them to some discoveries with place value and choices for each digit.

3rd Activity – Next, the students will be introduced to the reverse and add rule to forming a palindrome from a non-palindrome number. I will ask for a two digit number and demonstrate for the students how to change it into a palindrome by reversing the digits and then adding that new number with the one they gave me. We will count the steps completed to make a palindrome. We will do several examples together. I will pass out a chart of the numbers 0-99 for them to color code as they change each two digit number into a palindrome. Before they begin, I will ask them to make a prediction – "Can all two digit numbers be changed to palindromes using this reverse and add rule?" I will encourage them to make some observations as they work through the numbers. The students will jot down anything they notice on another sheet of paper or the back of their chart. (They may work in pairs if they like.)
When the students are done, we will share their charts and their observations.
Some things I want the students to notice –

- A number and its reverse take the same number of steps to become a palindrome.
- There are more numbers that only take one step – the tens digit and units digit don't equal each other and their sum is nine.
- 89 and 98 require the most steps. Why do you think this is true?

Next, we will try a few non-palindrome, three-digit numbers with this same procedure. We will try applying some of the above observations to see if they still hold true for three-digit numbers. I will ask them to predict if there are some three-digit numbers that will never become palindromes and why do they think this. I will then challenge them to try 196. After allowing them some time to work on this number, I will share with them that it is not known whether or not every three-digit number is a k-step palindrome, for some k." In other words, we don't know if every three-digit number can be changed into a palindrome with the reverse and add rule. I will also tell them that someone has done 2,000,000 steps on 196, without producing a palindrome!" (I might wait until the next day to share this with them.)

I would also like to start a chart for the students to compile their knowledge of palindromes. As this lesson closes, I will encourage and challenge my students to do a little research on palindromes to see what they can find. They can share their information by writing it up on our chart. The students can also write up examples of palindromes they find in the "real world".

Name: ________________________________
Palindromes

Definition:
____

Write down your birthday - _______________________________ (Ex: October 6, 1960)
Using numbers for the month and day or day and month with either this year or the year you were born. _______________________________________ (Ex: 10062009 or 10061960 or 06102009 or 06101960 - No, my birthday is not a palindrome.)

Is your birthday a palindrome? _________

List all of the . . .
One digit palindromes:

Two digit palindromes: **Total**: _______

Three digit palindromes: **Total**: _______

Four digit palindromes: **Total**: ________

Patterns I noticed:

Name: ____

Reverse and Add Rule for Finding Palindromes

To show how many steps it takes for each two-digit number to become a palindrome, use a color coding system. Pick a different color for each number of steps. Work through each two-digit number using the reverse and add rule, counting the steps needed for it to become a palindrome. Color each number the appropriate color according to your chart.

☐ Already a palindrome ☐ 3 step palindrome ☐ ____ step palindrome

☐ 1 step palindrome ☐ 4 step palindrome ☐ ____ step palindrome

☐ 2 step palindrome ☐ 5 step palindrome ☐ ____ step palindrome

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
10	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**
20	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**
30	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**
40	**41**	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**
50	**51**	**52**	**53**	**54**	**55**	**56**	**57**	**58**	**59**
60	**61**	**62**	**63**	**64**	**65**	**66**	**67**	**68**	**69**
70	**71**	**72**	**73**	**74**	**75**	**76**	**77**	**78**	**79**
80	**81**	**82**	**83**	**84**	**85**	**86**	**87**	**88**	**89**
90	**91**	**92**	**93**	**94**	**95**	**96**	**97**	**98**	**99**

Try using the reverse and add rule with a couple of three digit numbers that are not palindromes. How many steps did it take for them to become palindromes?

Do you notice anything similar with the two-digit numbers you tried? What?

Challenge: After you have completed a few of your own choice and answered the questions above, try the number **196.**

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