1-1970

On the Relation between Hard-Core and Velocity-Dependent Potentials: An Application to the Photonuclear Sum Rules

Paul Finkler
University of Nebraska-Lincoln, pfinkler1@unl.edu

H. S. Valk
Institut für Theoretische Physik der Universität Frankfurt (Main)

Zeitschrift für Physik A: Hadrons and Nuclei 236:2 (Jan 1, 1970), pp. 144-152.

Z. Physik 236, 144—152 (1970)

On the Relation between Hard-Core and Velocity-Dependent Potentials

An Application to the Photonuclear Sum Rules

P. FINKLER

Behlen Laboratory of Physics, University of Nebraska, Lincoln, Nebraska, U.S.A.

H. S. VALK*

Institut für Theoretische Physik der Universität Frankfurt (Main)

Received April 16, 1970

A canonical transformation relating hard-core and velocity-dependent nucleon-nucleon potentials is applied to the Srivastava potential and an equivalent hard-core potential is found. It is shown that the deuteron photonuclear electric-dipole integrated and bremsstrahlung-weighted cross sections resulting from the two equivalent potentials are essentially the same. The reasons for this agreement suggest that differences between the two sets of cross sections may remain small in other nuclei employing this type of potential.

I. Introduction

That canonical transformations exist between hard-core (c) and velocity-dependent (v) nucleon-nucleon potentials having the same energy spectrum has been known for nearly a decade[1,2]. However, the effect of this equivalence on matrix elements involving the interaction of the deuteron with external fields, and, in particular, on the electric-dipole photonuclear sum rules has only recently been the subject of investigation[3]. In that paper Kistler has shown that it is possible to obtain from a given non-exchange hard-core potential a set of equivalent velocity-dependent potentials which yield integrated cross sections ranging from 10% less to 30% greater than the local TRK value associated with the hard-core potential.

Here we consider the closely related, but more restricted, problem of finding that hard-core potential which leads to a given velocity-dependent potential and of comparing both the integrated and the brems-

---

* Currently on leave from the University of Nebraska, Lincoln, Nebraska, U.S.A.

1 Bell, J.S.: Proceedings of the Rutherford Jubilee International Conference, Manchester, 1961, p. 373. London: Heywood and Co.

2 Bell, J.S.: Lecture notes on the many body problem. First Bergen International School of Physics, p. 214, 1961 (C. Fronsdal, ed.). New York: W.A. Benjamin, Inc.

3 Kistler, S.: Z. Physik 223, 447 (1969).

strahlung-weighted cross sections arising from the two potentials. This program is carried out for a model S-state deuteron with the Srivastava potential[4], and it is shown that the two sets of cross sections agree to within fractions of a percent. The reason for this agreement in the deuteron case suggests that the differences between the two sets of cross sections may remain small in other light nuclei employing this type of nucleon-nucleon potential.

II. Formalism

In this section, we derive the expressions needed to effect the transformation from a core to an equivalent velocity-dependent potential. Although the results are not new, and have appeared elsewhere in the literature[1-3], it is convenient in the presentation of the formalism to rederive them here.

Following a procedure adopted by Bohm, Gross, and Bell[1,2], we consider a class of canonical transformations of the form

$$\Psi_v = e^{iS}\Phi_c, \tag{1a}$$

$$H_v = e^{iS} H_c e^{-iS} \tag{1b}$$

where H_v and H_c are the velocity-dependent and hard-core Hamiltonians, Ψ_v and Φ_c are the respective wave functions, and S is a Hermitian operator,

$$S = \sum_{i<j} \frac{1}{2\hbar} \left[(\boldsymbol{p}_i - \boldsymbol{p}_j)\cdot \boldsymbol{f}(\boldsymbol{r}_i - \boldsymbol{r}_j) + \boldsymbol{f}(\boldsymbol{r}_i - \boldsymbol{r}_j)\cdot(\boldsymbol{p}_i - \boldsymbol{p}_j)\right], \tag{2}$$

linear in the relative momenta of nucleons i and j.

It is sufficient for our purposes to restrict the transformation, Eq. (2), to one pair of relative variables, $\boldsymbol{p} = \boldsymbol{p}_1 - \boldsymbol{p}_2$, and $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$, and to choose the function $\boldsymbol{f}(\boldsymbol{r})$ to be along $\boldsymbol{r}$, so that

$$S = (\hbar)^{-1} f^{\frac{1}{2}} p_r f^{\frac{1}{2}}, \tag{3}$$

where $f = (\boldsymbol{r}\cdot\boldsymbol{f})/r$ and $p_r = -i\hbar(d/dr)\,\boldsymbol{r}$. Since S, Eq. (3), is linear in p_r, it follows that an operator function of r, $g(r)$, is transformed into some other function of r, $g^T(r)$, according to

$$g^T(r) = e^{iS} g(r) e^{-iS} = g[\rho(r)] \tag{4}$$

where

$$\rho(r) = e^{iS} r e^{-iS}. \tag{5}$$

In order to evaluate the expression $e^{iS} r e^{-iS}$, we employ the following device: consider a function $G(r)$ whose commutator with S is a con-

4 Srivastava, B.K.: Nuclear Physics 67, 236 (1965).

stant c. Then

$$[iS, G] = fG' = c$$

or

$$G(r) = c\int [f(r)]^{-1}\, dr. \tag{6}$$

But from Eq. (4),

$$G[\rho(r)] = G(r) + c; \tag{7}$$

hence, the function

$$\rho(r) = G^{-1}[G(r) + c] \tag{8}$$

may be found from the inverse function of (7), where the original function, $G(r)$, is given by (6). A more useful form of this solution is obtained by differentiating (7). Thus

$$d\rho/f(\rho) = dr/f(r).$$

or

$$d\rho/dr = f[\rho(r)]/f(r), \tag{9}$$

a result giving the Jacobian of the radial coordinate change (5).

The corresponding transformation of the conjugate variable p_r can be found from the invariance of S, Eq. (3), under the transformation. This leads to

$$p_r^{\mathrm{T}} = [f(r)/f(\rho)]^{\frac{1}{2}}\, p_r [f(r)/f(\rho)]^{\frac{1}{2}}. \tag{10}$$

Applying the above results to the transformation of a Hamiltonian containing a static central hard-core potential, $V(r)$,

$$H_c(\boldsymbol{r}, \boldsymbol{p}) = (2m)^{-1}\, \boldsymbol{p}^2 + V(r) = (2m)^{-1}(p_r^2 + \boldsymbol{L}^2/r^2) + V(r), \tag{11}$$

we arrive at

$$\begin{aligned} H_v(\boldsymbol{r}, \boldsymbol{p}) = (2m)^{-1}\{&\tfrac{1}{2}[\boldsymbol{p}^2\, \Omega(r) + \Omega(r)\, \boldsymbol{p}^2] + \tfrac{1}{4}\hbar^2(\tfrac{1}{4}\Omega'^2/\Omega + \Omega'') \\ &+ [\rho(r)^{-2} - \Omega/r^2]\, \boldsymbol{L}^2\} + V[\rho(r)] \end{aligned} \tag{12}$$

as the equivalent velocity-dependent form. Here we have employed $\boldsymbol{L}$ as the orbital angular momentum operator and have written $\Omega(r) = [f(r)/f(\rho)]^2$, the prime indicating a derivative with respect to r. The relation between (11) and (12) is essentially the same result found by Baker[5] and Kistler[3] in their considerations of the connection between velocity-dependent and hard-core potentials.

By an appropriate choice of $\rho(r)$, one can change the static potential $V(r)$ to one, $V[\rho(r)]$, having no core region. It can be seen from (12), however, that such a change introduces compensatory velocity-dependent effects through $\Omega(r) = (d\rho/dr)^{-2}$.

5 Baker, G. A., Jr.: Phys. Rev. **128**, 1485 (1962).

We can use the transformation (4) to make Eq. (1a) explicit. Thus if we note S as defined above, operates only on the radial part of the wave function, we can write

$$u_v(r)/r = e^{iS} u_c(r)/r \tag{13}$$

or

$$u_v(r) = u_c[\rho(r)]\,\Omega^{-\frac{1}{4}}, \tag{14}$$

where we have assumed the radial functions are normalized according to

$$\int_0^\infty [u_v(r)]^2\,dr = \int_{a_c}^\infty [u_c(\rho)]^2\,d\rho = 1\,.$$

Since $u_c(\rho)$ represents a hard-core function, it must vanish at the core radius $\rho = a_c > 0$. Under (13), we see that this point is carried to the origin of the velocity-dependent system where $\rho(0) = a_c$. It is worth observing here that the transformation (1) may be characterized solely in terms of the function $\Omega(r)$; it is not necessary to specify $f(r)$.

III. Application to Model Potential

We are now in a position to apply the above formalism to a model deuteron Hamiltonian. Because we are more interested in the relative values of the cross sections for equivalent Hamiltonians than in the absolute magnitudes of the cross sections, we need not employ an elaborate description of the nucleon-nucleon interaction. For this reason, it is convenient to use the simple spin-dependent Hamiltonian developed by Srivastava[4] to fit the low-energy $n-p$ data.

It has the standard velocity-dependent form:

$$\begin{aligned} H_v = {} & P_s[(2M)^{-1}(p_r^2\,\Omega^s + \Omega^s p_r^2) - V_0^s \exp(-2r/\beta_0)] + P_s(M)^{-1}\Omega^s L^2/r^2 \\ & + P_t[(2M)^{-1}(p_r^2\,\Omega^t + \Omega^t p_r^2) - V_0^t \exp(-2r/\beta_0)] + P_t(M)^{-1}\Omega^t L^2/r^2, \end{aligned} \tag{15}$$

where

$$\Omega^{s,t} = 1 + V_1^{s,t} \exp[-2r/\beta_1^{s,t}], \tag{16}$$

P_s and P_t are projection operators for the singlet and triplet spin states, M is the nucleon mass, and $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$ is the magnitude of the $n-p$ separation vector. The parameters have the values:

$$\begin{aligned} &V_0^s = 100\,\text{MeV}; \quad V_0^t = 184\,\text{MeV}; \quad V_1^s = 2.0; \quad V_1^t = 1.1 \\ &1/\beta_0 = 0.625\,\text{F}^{-1}; \quad 1/\beta_1^s = 1.4\,\text{F}^{-1}; \quad 1/\beta_1^t = 1.0\,\text{F}^{-1}. \end{aligned} \tag{17}$$

An equivalent radial core Hamiltonian H_c which goes over into (15) for S-states can be found by noting that Ω^s and Ω^t determine the appropriate radial transformation through the relation $\Omega^{s,t}=[d\rho^{s,t}/dr]^{-2}$. Thus

$$\begin{aligned} H_c = p_r^2/M - P_s\Big\{ & V_0^s[x-(x_c^2-x_c)x^{-1}]^{-2\beta_1^s\beta_0^{-1}} + \frac{4\hbar^2}{M(\beta_1^s)^2}(x_c^2-x_c) \\ & \cdot\left[\left(x+\frac{x_c^2-x_c}{x}\right)^2+(x_c^2-x_c)\right]\left[x^2-\left(\frac{x_c^2-x_c}{x}\right)^2\right]^{-2}+V_c^s\Big\} \\ -P_t\Big\{ & V_0^t[y-(y_c^2-y_c)y^{-1}]^{-2\beta_1^t\beta_0^{-1}} + \frac{4\hbar^2}{M(\beta_1^t)^2}(y_c^2-y_c) \\ & \cdot\left[\left(y+\frac{y_c^2-y_c}{y}\right)^2+(y_c^2-y_c)\right]\left[y^2-\left(\frac{y_c^2-y_c}{y}\right)^2\right]^{-2}+V_c^t\Big\} \end{aligned} \tag{18}$$

where

$$V_c^s=\begin{cases}\infty, & r<a_c^s\\ 0, & r\geqq a_c^s;\end{cases}\qquad V_c^t=\begin{cases}\infty, & r<a_c^t\\ 0, & r\geqq a_c^t;\end{cases} \tag{19}$$

$x=\exp(r/\beta_1^s)$; $x_c=\exp(a_c^s/\beta_1^s)$; $y=\exp(r/\beta_1^t)$; and $y_c=\exp(a_c^t/\beta_1^t)$.

The related radial transformations are given by

$$\rho^{s,t}(r)=r+\beta_1^{s,t}\ln\left\{\frac{1+[\Omega^{s,t}(r)]^{\frac{1}{2}}}{2}\right\}. \tag{20}$$

The arbitrariness in determining $\rho(r)$ from (16) has been removed by requiring that $\rho(r)\to r$ as $r\to\infty$; in other words that there be no shift as $r\to\infty$.

The core radii may be derived from the condition $\rho^{s,t}(0)=a_c^{s,t}$, or

$$a_c^{s,t}=\beta_1^{s,t}\ln\{\tfrac{1}{2}[1+(1+V_1^{s,t})^{\frac{1}{2}}]\}. \tag{21}$$

Substituting from (17), we find that

$$x_c=1.366;\quad y_c=1.225;\quad a_c^s=0.223\,\mathrm{F};\quad a_c^t=0.203\,\mathrm{F}. \tag{22}$$

These results are consistent with the negligible spin dependence usually assumed for core radii in more sophisticated nucleon-nucleon potentials[6].

In order to calculate the difference between dipole cross sections in the two cases, we need a wave function which closely simulates the exact triplet core function yet possesses the virtue of being suitable for analytical calculations. We shall use a direct generalization of the

6 See e.g., Hamada, T., Johnston, I.D.: Nucl. Physics 34, 382 (1962).

Hulthèn function,

$$u_c(r)=N\{\exp[-\alpha(r-a_c^t)]+A\exp[-\beta(r-a_c^t)] \\ +B\exp[-\gamma(r-a_c^t)]\}. \tag{23}$$

The six constants entering into (23) are found by imposing the conditions:

(i) $\alpha^2=M\varepsilon/\hbar^2$ where ε is the deuteron binding energy.

(ii) $u_c(a_c^t)=0$, the core condition.

(iii) $\int_{a_c^t}^{\infty}[u_c(r)]^2\,dr=1$, normalization.

(iv) That the potential energy, $\overline{V}(r)$, for which u_c is the exact solution, be finite at a_c^t.

(v) That the long-range part of $\overline{V}(r)$ should agree with the long-range part of the actual core potential. This contains two requirements, one on the range parameter, and one on the well-depth.

The constants generated by this procedure take on the values:

$$\begin{array}{lll} \alpha=0.231\,\mathrm{F}^{-1}; & \beta=1.481\,\mathrm{F}^{-1}; & \gamma=2.385\,\mathrm{F}^{-1} \\ A=-1.61; & B=0.61; & N=0.916. \end{array} \tag{24}$$

With this fit, the model deuteron r. m. s. radius comes out to be $\langle R^2\rangle^{\frac{1}{2}}=2.05$ F, a result not inconsistent with the experimental estimates of ~ 2.0 F for the matter radius[7].

IV. Photonuclear Cross Sections

With the Hamiltonians (18)–(19) and (15)–(17), we are in a position to calculate the electric-dipole integrated (σ_0) and bremsstrahlung-weighted (σ_{-1}) photonuclear cross sections using the approximate core wave function (23)–(24).

In both cases, these cross sections are easily found from the sum-rule relations[8]:

$$\sigma_0=[2\pi^2/(\hbar c)]\langle[D,[H,D]]\rangle, \tag{25}$$

$$\sigma_{-1}=[4\pi^2 e^2/(3\hbar c)]\langle R^2\rangle=[\pi^2 e^2/(3\hbar c)]\langle r^2\rangle, \tag{26}$$

where $D=(e/2)z$ is the deuteron dipole moment, and the other constants have their usual meanings.

7 Herman, R., Hofstadter, R.: High-energy electron scattering tables, p. 62. Stanford: Stanford University Press 1960.

8 Levinger, J.S.: Nuclear photodisintegration, p. 39. London: Oxford University Press 1960.

For the core Hamiltonian (18)–(19), we find

$$\sigma_0(c) = \pi^2 e^2 \hbar/(M c) = 29.9 \text{ MeV} - \text{mb}$$

and $\sigma_{-1}(c) = 4.02$ mb. These are to be compared with the corresponding experimental estimates of $\sigma_0 = 39.7$ MeV-mb and $\sigma_{-1} = 3.8$ mb[9]. The low Thomas-Reiche-Kuhn value of $\sigma_0(c)$ is primarily due to the fact that we have neglected the presence of exchange forces. As we shall see, this neglect simplifies the results without substantially affecting our conclusions.

To arrive at the equivalent expressions for σ_0 and σ_{-1} with Eqs. (12) and (15) we again make use of the sum rule relations (25) and (26). This gives

$$\sigma_0(v) = [\pi^2 e^2 \hbar/(M c)] \{\langle \Omega(r) \rangle_v + \tfrac{2}{3} \langle r^2/\rho^2 - \Omega(r) \rangle_v\}, \tag{27}$$

$$\sigma_{-1}(v) = [\pi^2 e^2/(3 \hbar c)] \langle r^2 \rangle_v. \tag{28}$$

The result (27), which also appears in the paper of Kistler[3], differs from the usual one quoted for velocity-dependent central potentials[10] by the appearance of the term arising from the coefficient of L^2 in Eq. (12). Although this term makes no contribution to the energy for *S*-states, it must be included here for consistency in the transformation. Physically, one would expect the angular-momentum dependence to appear since (27) represents a sum over *all* intermediate states, and (15) is strictly equivalent to (18) only for *S*-states.

In evaluating (27) and (28), we use (14) and convert the expectation values into integrals over ρ. Then

$$\begin{aligned}\sigma_0(v) = {} & [\pi^2 e^2 \hbar/(3M c)] \int_{a_c^\xi}^{\infty} u_c^*(\rho)\, \Omega[r(\rho)]\, u_c(\rho)\, d\rho \\ & + [2\pi^2 e^2 \hbar/(3M c)] \int_{a_c^\xi}^{\infty} u_c^*(\rho) \{[r(\rho)]^2/\rho^2\} u_c(\rho)\, d\rho,\end{aligned} \tag{29}$$

and

$$\sigma_{-1}(v) = [\pi^2 e^2/(3 \hbar c)] \int_{a_c^\xi}^{\infty} u_c^*(\rho) [r(\rho)]^2 u_c(\rho)\, d\rho. \tag{30}$$

These integrals may now be calculated by using (23)–(24) for u_c, and the inverse of (20) for $r(\rho)$. This procedure yields the results

$$[\sigma_0(v) - \sigma_0(c)]/\sigma_0(c) = 0.00075, \tag{31}$$

and

$$[\sigma_{-1}(c) - \sigma_{-1}(v)]/\sigma_{-1}(c) = 0.00095. \tag{32}$$

It is seen that the replacement of the core by an equivalent velocity-dependent Hamiltonian in this case has the effect of reducing the mean-

9 Davey, P. O., Valk, H. S.: Phys. Rev. **156**, 1039 (1967), and references therein.
10 See e.g., Dohnert, L., Rojo, O.: Phys. Rev. **136**, B396 (1964).

square radius by about 0.1% while increasing the integrated cross section by 0.075%, both negligible amounts.

If provision had been made in the Hamiltonians for exchange forces, then the effect of the transformation would be to further increase the ratio (31), leaving (32) unaffected. For example, had we assumed that the attractive static potential in (15) had a Serber exchange character, then the ratio (31) would be 0.0049. In the extreme case where the force were assumed to be wholly Majorana exchange, we would arrive at the maximum value of 0.0075. Thus while it is conceivable that (31) could be altered by as much as a factor of 10 by inclusion of exchange, it would still represent less than a 1% change in the integrated cross section.

The origin of these tiny differences in the ground-state expectation values using the equivalent potentials may be made clearer by writing (27) and (28) more formally as

$$\sigma_0(v)=[2\pi^2/(\hbar c)]\,(\Psi_v, [D,[H_v, D]]\,\Psi_v),$$
$$\sigma_{-1}(v)=[4\pi^2/(\hbar c)]\,(\Psi_v, D^2\,\Psi_v).$$

Substituting from (1a), we find

$$\sigma_0(v)=[2\pi^2/(\hbar c)]\,(\Phi_c, [D',[H_c, D']]\,\Phi_c), \tag{33a}$$
$$\sigma_{-1}(v)=[4\pi^2/(\hbar c)]\,(\Phi_c, D'^2\,\Phi_c). \tag{33b}$$

Here $D'=e^{-iS}De^{iS}$ is that dipole operator which would go into D under Eq. (1a). That is to say, D' is obtained from D by applying, as in (29) and (30), the inverse transformation to that represented by (20). Expanding D' in powers of S, we may write $D'=D+D_R$, where

$$D_R=[-iS, D]+\frac{1}{2!}[-iS,[-iS, D]]+\cdots. \tag{34}$$

Hence Eqs. (33) become

$$\sigma_0(v)=\sigma_0(c)+\Delta\sigma_0,$$
$$\sigma_{-1}(v)=\sigma_{-1}(c)+\Delta\sigma_{-1},$$

where

$$\begin{aligned}\Delta\sigma_0=[2\pi^2/(\hbar c)]\,\{&(\Phi_c, [D_R,[H_c, D]]\,\Phi_c)+(\Phi_c, [D,[H_c, D_R]]\,\Phi_c)\\&+(\Phi_c, [D_R,[H_c, D_R]]\,\Phi_c)\},\end{aligned} \tag{35a}$$

$$\Delta\sigma_{-1}=[4\pi^2/(\hbar c)]\,\{(\Phi_c, D_R D\,\Phi_c)+(\Phi_c, D\,D_R\,\Phi_c)+(\Phi_c, D_R^2\,\Phi_c)\}. \tag{35b}$$

In the present example of the deuteron $D_R\sim\rho-r$ drops to one-third of its maximum value at about 0.5 F from the core, a point at which Φ_c has still not risen to one-third of its maximum. More generally, one can say that the contribution of the transformation, and thus of D_R, is significant primarily in regions where Φ_c is still reasonably small. It follows that expressions like (35) are expected to be much smaller than the corresponding expressions containing D alone.

Eqs. (35) for the differences between velocity-dependent and hard-core cross sections are quite general and may be applied to any nucleus; and to the extent that D_R and Φ_c maintain the same relative behaviour as in the deuteron for the different coordinate pairs, the differences will remain small compared to the respective cross sections. This behaviour is, in turn, dependent on the nature of the potentials (15)−(17) through the transformation function (20). However, the detailed analysis for nuclei with $A > 2$ is less direct because of the appearance of multi-particle terms.

Although we may anticipate that $\Delta\sigma_0$ and $\Delta\sigma_{-1}$ will increase in relative importance as we go to more compact systems, the extreme smallness of these differences for the deuteron implies that they should remain unimportant at least in the lightest nuclei as long as one uses interaction potentials of the type considered here. Furthermore, this implies that once a fit has been achieved to the experimental photo-nuclear cross sections in this region using such a potential, a similar fit can be expected with the equivalent potential. In this regard, the recent work of Lim[11] fitting σ_0 and σ_{-1} in the $1s$ shell nuclei with a modified form of the Srivastava potential would seem to indicate that an equivalent hard-core fit may also be possible. However, this conclusion is far from certain since Lim's analysis does not include the L^2 term of (12) which is required for complete equivalence. While this term does not enter into the binding energy calculation and is expected to make only a small difference to individual phase shifts, it does play a significant role in bringing $\sigma_0(v)$ and $\sigma_0(c)$ into agreement.

One of the authors (H. S. V.) would like to express his appreciation to the University of Nebraska Research Council for the award of a fellowship and to Professor Walter Greiner for the hospitality of the Institute for Theoretical Physics, Frankfurt/Main, where this work was completed. The authors would also like to thank Mr. Karl Roos for his assistance with some of the related calculations, and the Bundesministerium für Wissenschaft und Bildung and the Deutsche Forschungsgemeinschaft for support.

11 Lim, T.K.: Nucl. Physics A**113**, 376 (1968), and references therein.

Prof. Dr. P. Finkler
Behlen Lab. of Physics
University of Nebraska
Lincoln, Nebraska 68508
(U.S.A.)

Prof. Dr. H. S. Valk
Inst. f. Theoret. Physik d. Universität
D-6000 Frankfurt a.M.
Robert Mayer-Str. 8—10
West-Germany