1975

FLUX AND GAUSS' LAW

FLUX AND GAUSS' LAW

INTRODUCTION

Charles Augustine de Coulomb (1736-1806) designed his famous experiment to measure the force relationships between charged bodies: Coulomb's law is the resulting empirical statement. Gauss' law (Karl Friedrich Gauss, 1777-1855), which you will learn in this module, has a more obscure origin. It was originally a mathematical theorem. Scientists in Gauss' nineteenth century were much more inclined than we are today to equate mathematical correctness with physical correctness. When it was realized that Gauss' (mathematical) theorem could be applied to the electric-field concepts of Faraday to produce Gauss' (physical) law, this extension was eagerly accepted. The origins of the law, however, continued and still continue to lie in the domain of pure logic; therefore they may be somewhat inaccessible to you in beginning physics courses. Your texts and this module will use both physical and mathematical arguments and examples to help you achieve a mastery of these ideas and their applications.

PREREQUISITES

Before you begin this module, you should be able to:	Location of Prerequisite Content
*Use vectors to represent certain quantities and add them (needed for Objectives 1 through 4 of this module)	Dimensions and Vector Addition Module
*Perform vector scalar multiplication (needed for Objectives 1 through 4 of this module)	Vector Multiplication Module
*State Coulomb's law (needed for Objective 3 of this module)	Coulomb's Law and the Electric Field Module
*Describe an electric field, conductors, and insulators (needed for Objectives 1 through 4 of this module)	Coulomb's Law and the Electric Field Module

LEARNING OBJECTIVES

After you have mastered the content of this module, you will be able to:

1. Statement of Gauss' law - State Gauss' law and explain all its symbols.

2. Limitations of Gauss' law - Recognize when Gauss' law cannot be used to determine the electric field caused by a static charge distribution, and explain why.

3. Applications of Gauss' law - Use Gauss' law to

   (a) determine the electric field due to certain symmetric charge distributions; or

   (b) determine the net charge inside volumes where the electric field is known everywhere on the surface of the volume.

4. Electric field, charge, and conductors - Given a conductor with a static charge distribution, use the properties of a conductor and/or Gauss' law to

   (a) explain why the electric field is perpendicular to the surface of the conductor;

   (b) explain why the electric field is zero inside the conductor.

   (c) explain why the excess charge is on the surface of the conductor.

TEXT: Frederick J. Bueche, Introduction to Physics for Scientists and Engineers (McGraw-Hill, New York, 1975), second edition

SUGGESTED STUDY PROCEDURE

Read General Comment 1 in this study guide and Sections 19.1 and 19.2 in Chapter 19 of your text. Then read General Comment 2, Section 19.3, and work through Problem A. Read Section 19.4 and work through Illustration 19.1. Read Sections 19.5 and 19.6 and work through Problem D and Illustration 19.2. Then read Sections 19.7 and 19.8, work through Illustrations 19.3 and 19.4 and Problem C. Read General Comments 3 and 4 and work through Problem B before working the Assigned Problems. Try the Practice Test.

BUECHE

Objective Number	Readings	Problems with Solutions		Assigned Problems	Additional Problems
		Study Guide	Text	Study Guide	(Chap. 19)
1	General Comment 1, Secs. 19.1, 19.2, General Comment 2, Sec. 19.3	A			
2	General Comment 3	B			Quest.[a] 5, 7, 11, 12
3	Secs. 19.4, 19.6 to 19.8, General Comment 4	C	Illus.[a] 19.1, 19.2, 19.3, 19.4	E, F, G, H, I	Quest. 3, 6, Probs. 5 to 7, 9, 11, 13 to 15, 17, 18
4	Sec. 19.5	D			

[a]Illus. = Illustration(s). Quest. = Question(s).

TEXT: David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974)

SUGGESTED STUDY PROCEDURE

Read General Comments 1 and 2 in this study guide and Section 24-1 in Chapter 24 of your text before working through Example 1. Then read Section 24-2, and work through Problem A. Read Sections 24-3, 24-4, and 24-5, General Comment 3, and work through Problem B. Then read General Comment 4 and Section 24-6, and work through Examples 2 through 5. Do the Assigned Problems before attempting the Practice Test.

HALLIDAY AND RESNICK

Objective Number	Readings	Problems with Solutions: Study Guide	Problems with Solutions: Text	Assigned Problems: Study Guide	Additional Problems (Chap. 24)
1	General Comment 1, Sec. 24-1, General Comment 2, Sec. 24-2	A	Ex.[a] 1		Quest.[a] 3, 5
2	General Comment 3	B			Quest. 9, 11
3	Sec. 24-3, General Comment 4, Sec. 24-6	C	Ex. 2 to 5	E, F, G, H, I	Quest. 12 to 14, Probs. 8, 9, 12 to 35
4	Sec. 24-4	D			Quest. 7, 8

[a]Ex. = Example(s). Quest = Question(s).

TEXT: Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Mass., 1970), fourth edition

SUGGESTED STUDY PROCEDURE

Read General Comments 1 and 2 in this study guide. Note that we use the symbol Φ for flux instead of ψ as used in your text. Then read Section 25-4 in Chapter 25 of your text and work through Problem A. Read General Comment 3 and work through Problem B; then read Section 24-5 and General Comment 4. Work through Problems C and D before trying the Assigned Problems. Try the Practice Test.

SEARS AND ZEMANSKY

Objective Number	Readings	Problems with Solutions: Study Guide	Problems with Solutions: Text	Assigned Problems: Study Guide	Additional Problems
1	General Comments 1, 2, Sec. 25-4	A			
2	General Comment 3	B			
3	Sec. 25-5, General Comment 4	C	Sec. 25-5	E, F, G, H, I	25-12, 25-13, 25-16, 25-17(except d and e), 25-19, 25-20, 25-21, 25-22
4	Sec. 25-5, General Comment 4	D			

TEXT: Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol. 2

SUGGESTED STUDY PROCEDURE

Read General Comments 1 and 2 in this study guide and Section 24-1 in Chapter 24 of your text. Study Example 24-1. Then read Section 24-2 to p. 489. Read Section 24-3 and work through Problem A; read General Comment 3 and work through Problem B. Then read General Comment 4. Read the rest of Section 24-2 and all of Section 24-4 before working through Examples 24-2, 24-3, and 24-4, and Problem C. Read Section 24-5 and work through Problem D. The main ideas and equations in this module are presented in a summary at the end of Chapter 24 (p. 497). Work the Assigned Problems before trying the Practice Test.

WEIDNER AND SELLS

Objective Number	Readings	Problems with Solutions: Study Guide	Problems with Solutions: Text	Assigned Problems: Study Guide	Additional Problems
1	General Comments 1, 2, Secs. 24-2, 24-3	A	Ex.[a] 24-1		
2	General Comment 3	B			
3	General Comment 4, Secs. 24-2, 24-4	C	Ex. 24-2, 24-3, 24-4	E, F, G, H, I	24-6, 24-8(a), 24-9, 24-10, 24-11, 24-14(a), 24-15, 24-16
4	Sec. 24-5	D			

[a]Ex. = Example(s).

GENERAL COMMENTS

1. Electric Flux

You have seen from your reading that you can picture electric lines of force originating from each positive charge, bending around smoothly, and continuing on until they (possibly) end on some negative charge. You will recall that the direction of the line of force at any point gives the direction of the electric field $\vec{E}$ at that point. We now carry this pictorial representation of the electric field a little further, and specify how many lines shall be drawn. See Figure 1.

This is arbitrary, but if we make the number of lines that originate from or terminate on a charge directly proportional to the amount of charge, we shall be able to count the lines and determine the charge. In the SI system of units we draw $1/\varepsilon_0$ lines for each coulomb of charge:

$$\Phi = (1/\varepsilon_0)q. \tag{1}$$

This quantity Φ is called the electric flux; and in this context the lines of force are often called lines of flux, or flux lines. Note that if q is negative, then Φ is negative, and the flux lines are drawn pointing inward (see Fig. 1).

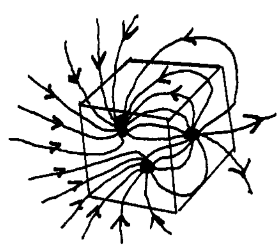


Figure 1

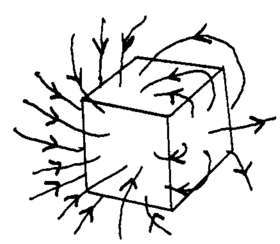


Figure 2

But now suppose you cannot actually see inside the box - that all you can see is the lines of flux leaving the box, as in Figure 2. Can you tell something about the charges contained inside? Yes, indeed, you can count up the total number of lines leaving the box, to find the total flux:

$$\Phi = \sum_i \Phi_i = \frac{1}{\varepsilon_0}\sum_i q_i, \tag{2}$$

where Eq. (1) has been used to get the last equality. That is, the total charge $Q = \Sigma q_i$ contained within any surface S is related to the net amount Φ of flux lines passing outward through the surface by

$$Q = \varepsilon_0\Phi. \tag{3}$$

Note that the "counting up" of flux lines must be done in an algebraic sort of way, in which you add one for each line coming out, and subtract one for each line going in.

2. Surface Integral

In Figure 3 is shown a volume enclosed by a surface, much like the volume inside a distorted balloon. The surface can be divided into area elements $d\vec{A}$. You may have done this in your calculus class for regular shapes such as cylinders, spheres, and triangles. You probably have not treated these area elements as vectors, however, but just determined their magnitudes, dA. The direction of $d\vec{A}$ is outward from the volume and perpendicular to the surface.

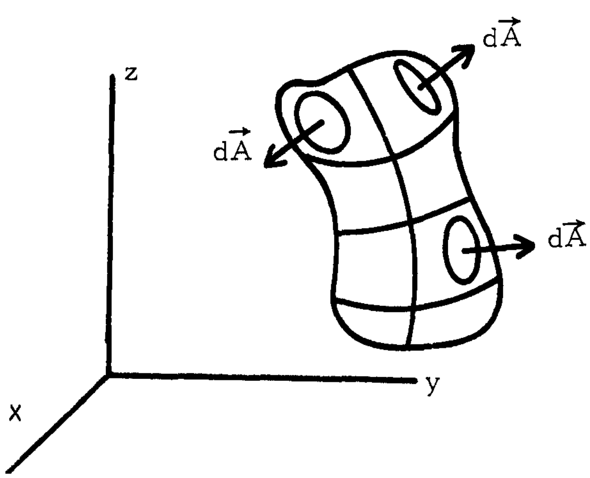


Figure 3

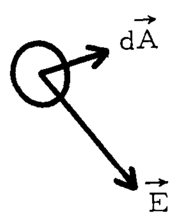


Figure 4

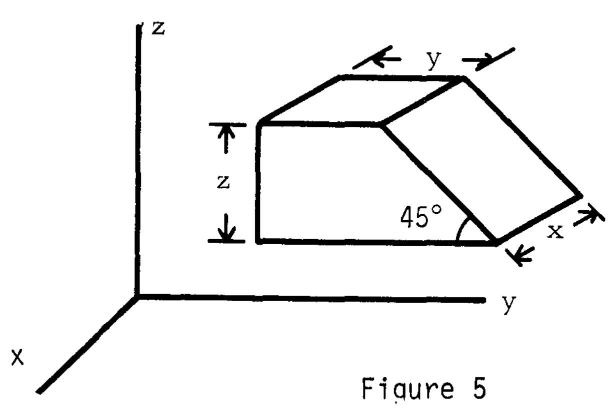


Figure 5

Now suppose you were told the value of the electric field $\vec{E}$ everywhere on the surface. Each surface area element dA would have its own value of $\vec{E}$, as shown in Figure 4. $\vec{E} \cdot d\vec{A}$ is a straightforward scalar multiplication of vectors, and the product will be an infinitesimal scalar quantity;

$$d\Phi = \vec{E} \cdot d\vec{A}.$$

What if someone now asked you to find the total value of Φ for the whole surface of the volume. You would have to add up all the $d\Phi$'s associated with every $d\vec{A}$ on the surface. This sum is written as an integral:

$$\Phi = \int_{\substack{\text{all surface}\\ \text{of volume}}} d\Phi = \int_{\substack{\text{all surface}\\ \text{of volume}}} \vec{E} \cdot d\vec{A}.$$

This integral is called a <u>surface integral</u> and is represented by $\oint \vec{E} \cdot d\vec{A}$ or $\oint \vec{E} \cdot d\vec{S}$ in some texts and by $\int \vec{E} \cdot d\vec{A}$ or $\int \vec{E} \cdot d\vec{S}$ in others. It is important to remember that you must integrate over <u>all</u> of a completely closed surface.

<u>Exercise</u>: Here is a problem to check your ability to perform simple surface integrals: See Figure 5. The electric field in this space is uniform and equals $\vec{E} = E_0\hat{j}$. Calculate Φ for the surface shown in the figure. <u>Hint</u>: There are six sides. Calculate $\int \vec{E} \cdot d\vec{S}$ for each side and then add (scalarly) these answers. Do not forget that the scalar product of perpendicular vectors is zero. (Answers for the three sides shown are at the bottom of this page.)

3. <u>Limitations</u>

Although Gauss' law is always true it is not always useful. The integral

$$\oint \vec{E} \cdot d\vec{A} = \oint E \cos \theta \, dA = q/\varepsilon_0$$

can be solved for $\vec{E}$ only if $E \cos \theta$ can be factored out of the integral. Then you write Gauss' law as

$$E \cos \theta \oint dA = q/\varepsilon_0$$

and solve for E as

$$E = \frac{q/\varepsilon_0}{\cos \theta \oint dA} = \frac{q}{\cos \theta \, A\varepsilon_0} .$$

You can see that in order to factor out, $\vec{E}$ must have the same magnitude and the same direction with respect to every $d\vec{A}$ vector on the surface of integration. However, as indicated in the Exercise in General Comment 2, you might be able to divide this surface into smaller surfaces some of which have $\vec{E}$ perpendicular or parallel to $d\vec{A}$.

4. <u>Use of Gauss' Law to Determine $\vec{E}$ for Symmetric Charge Distributions</u>

When a charge distribution is known and possesses sufficient geometrical symmetry, one can use Gauss' law to deduce the resulting electric field. The procedure for doing so is pretty much the same in all cases and is outlined below.

<u>Answers</u>: $\int_{top} \vec{E} \cdot dA = 0$, $\int_{front} \vec{E} \cdot d\vec{A} = 0$,

$$\int_{\substack{\text{sloping}\\ \text{side}}} \vec{E} \cdot d\vec{A} = (E_0 \cos 45°) \frac{xz}{\sin 45°} = E_0xz.$$

Step 1. Deduce the direction of $\vec{E}$ from the symmetry of the charge distribution and Coulomb's law, e.g., for a spherically symmetric distribution $\vec{E}$ must be radial, i.e., must point away from (or toward) the symmetry center of the distribution.

Step 2. Use the symmetry of the charge distribution to determine the locus of points for which $\vec{E}$ must be constant in magnitude, e.g., for spherical symmetry the magnitude of $\vec{E}$ is necessarily the same at all points on the surface of a sphere concentric about the symmetry center.

Step 3. With Steps 1 and 2 as guides determine a closed (sometimes called Gaussian) surface such that at each point on the surface $\vec{E}$ is either (a) perpendicular to the surface and of constant magnitude $\vec{E}$, or (b) in the plane of the surface, i.e., with no component normal to the surface.

Step 4. Let A be the area of that portion of the Gaussian surface for which $\vec{E}$ is normal and of constant magnitude E. The electric flux for this part of the surface is EA, although the flux for the remaining portion of the surface (if there is any such part) is zero since $\vec{E}$ has no normal component. Thus the surface integral over the Gaussian surface is EA.

Step 5. Now set the electric flux of Step 4 equal to the net charge q enclosed by the Gaussian surface multiplied by $1/\varepsilon_0$, i.e.,

$$EA = 1/\varepsilon_0 q \quad \text{or} \quad E = q/\varepsilon_0 A.$$

Thus at each point on that part of the surface for which $\vec{E}$ is perpendicular the electric field has a magnitude as determined here.

Example 1

Charge is distributed with a uniform density ρ (C/m^3) throughout a long (infinite) cylindrical rod of radius R as in Figure 6. Let r measure the distance from the symmetry axis of the cylinder to a point. Determine $\vec{E}(r)$.

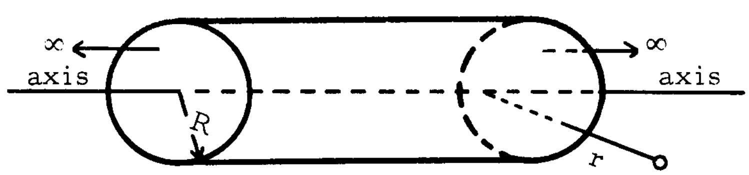


Figure 6

Step 1: From the symmetry of the charge distribution, $\vec{E}$ must be radial, i.e., perpendicular to the symmetry axis (see Fig. 7).

Step 2: The charge symmetry ensures that $\vec{E}$ will have the same magnitude at all points a distance r from the axis (see Fig. 7).

Step 3: The Gaussian surface is a cylinder of length L and radius r concentric with the symmetry axis. On the curved surface $\vec{E}$ is perpendicular (Step 1) and constant in magnitude (Step 2). On the flat ends $\vec{E}$ has no normal component (Step 1). (See Fig. 7.)

Step 4: The area of the curved part of the Gaussian surface is $A = 2\pi rL$. It is on this part that $\vec{E}$ is parallel to $d\vec{A}$ and constant in magnitude E. Therefore the electric flux for the curved part is

$$\Phi_E = EA = 2\pi rLE.$$

Since $\vec{E}$ is perpendicular to $d\vec{A}$ on the two flat ends of the cylinder, the flux through these two parts is zero. Therefore $2\pi rLE$ is the total flux through the Gaussian surface.

Step 5: See Figure 7. Clearly the charge enclosed by the Gaussian surface depends upon whether $r < R$ or $r > R$. For $r > R$, the surface encloses all the charge in a length L of the rod, namely,

$$q = \rho\pi R^2L \qquad (r > R);$$

but for $r < R$, the surface encloses only that charge inside the Gaussian surface at radius r and length L, namely,

$$q = \rho\pi R^2L \qquad (r < R).$$

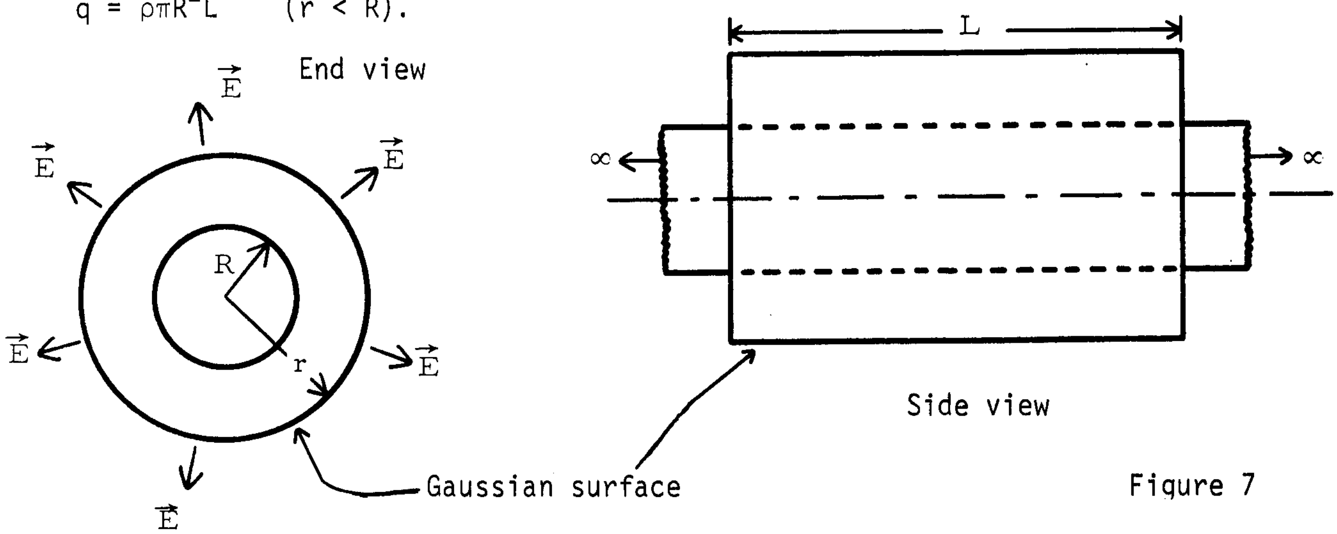


Figure 7

Now using Gauss' law and equating the electric flux to $1/\varepsilon_0$ times the net charge enclosed gives us

$$2\pi rLE = (1/\varepsilon_0)(\rho\pi r^2L) \qquad r < R$$

$$= (1/\varepsilon_0)(\rho\pi R^2L) \qquad r > R,$$

$$E = \rho r/2\varepsilon_0 \qquad r < R$$

$$= \rho R^2/2\varepsilon_0 \qquad r > R.$$

A graph of the magnitude of E versus r is shown in Figure 8. Remember this is just the magnitude. At each point $\vec{E}$ has this magnitude and points radially outward if $\rho > 0$ and radially inward if $\rho < 0$.

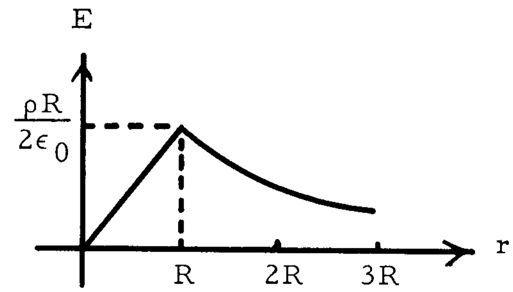


Figure 8

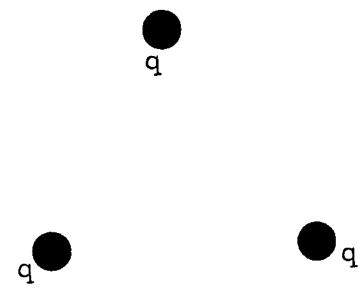


Figure 9

Gauss' law can be used to determine the electric field resulting from highly symmetric charge distributions. To do so, one infers from the charge symmetry a closed surface on which the electric field is either constant in magnitude and perpendicular to the surface, or in the plane of the surface. The electric flux is then easily determined in terms of the magnitude of the field and known geometrical quantities. Equating this flux to $1/\varepsilon_0$ times the known enclosed charge permits a determination of the magnitude of $\vec{E}$ at all points.

PROBLEM SET WITH SOLUTIONS

A(1). State Gauss' law and explain all its symbols.

Solution

Depending on your text's notations, Gauss' law is

$$\oint \vec{E} \cdot d\vec{A} = q/\varepsilon_0 \quad \text{or} \quad \int \vec{E} \cdot d\vec{A} = q/\varepsilon_0$$

$$\text{or} \quad \oint \vec{E} \cdot d\vec{S} = q/\varepsilon_0 \quad \text{or} \quad \int \vec{E} \cdot d\vec{S} = q/\varepsilon_0.$$

Given some volume V enclosed by surface A (or S), containing some net charge q, the surface integral of $\vec{E} \cdot d\vec{A}$ (or $\vec{E} \cdot d\vec{S}$) done over the whole surface equals the enclosed charge divided by ε_0. $\vec{E}$ is the electric field on the surface enclosing the volume; ε_0 is the permittivity of free space.

B(2). The three equal charges shown in Figure 9 are fixed at the points of an equilateral triangle. Explain why Gauss' law is not used to find $\vec{E}$ at any nearby point.

Solution

It is impossible to easily draw a Gaussian surface such that E cos θ is constant on it.

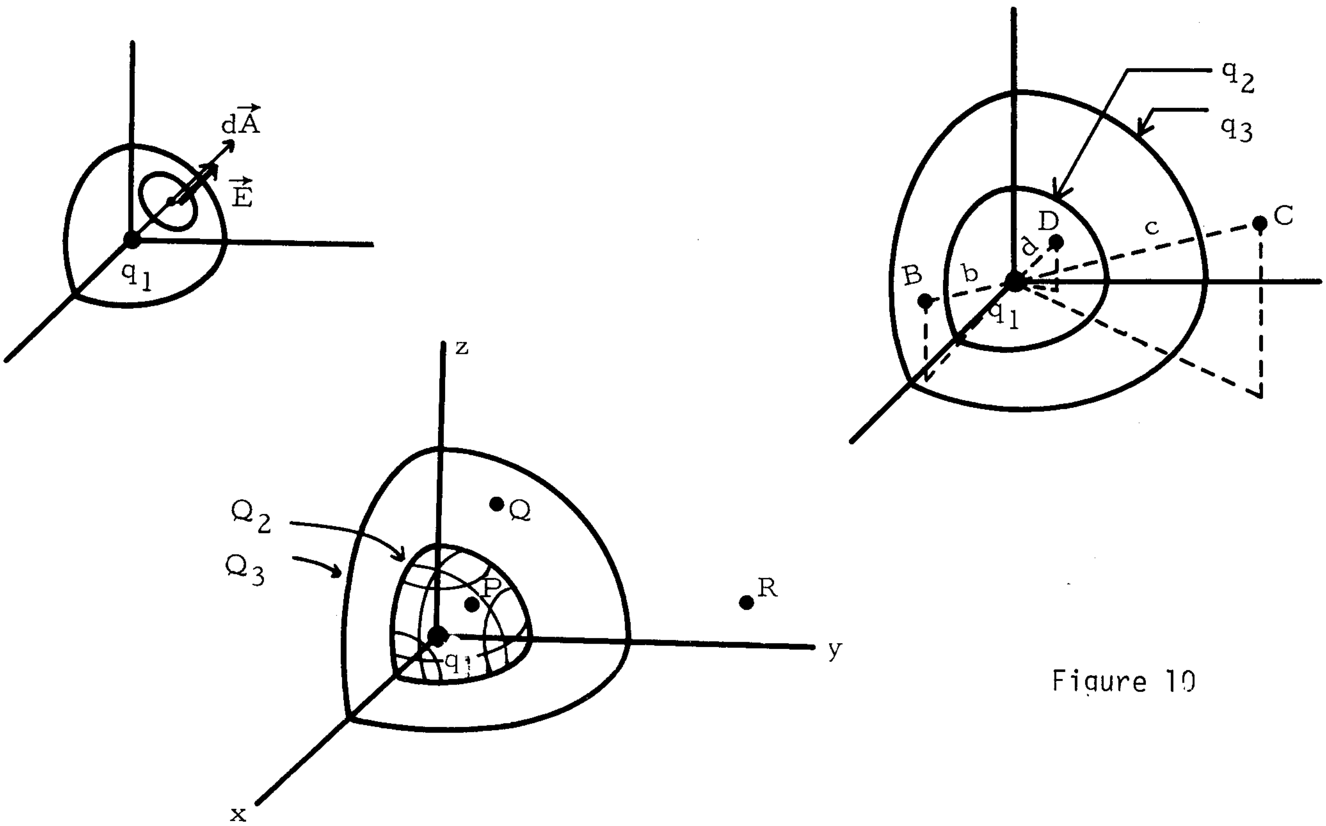


Figure 10

C(3). The point charge q_1 in Figure 10 is surrounded by two charged, spherical, thin metal shells as shown. Point P is inside the inner shell a distance 2.00 m from the origin. Point Q is between the shells a distance 3.00 m from the origin. Point R is outside the shells a distance 4.0 m from the origin.

$q_1 = 3.00 \times 10^{-6}$ C,

$Q_2 = -1.00 \times 10^{-6}$ C,

$Q_3 = -2.00 \times 10^{-6}$ C.

Use Gauss' law to find $\vec{E}$ at P, Q, and R. Show your Gaussian surfaces.

Solution

Point P: A spherical shell of radius 2.00 m will be a good Gaussian surface. Point P is located on this surface, it is the locus of points with constant E, and $\vec{E}$ is parallel to $d\vec{A}$ everywhere on this surface. $\vec{E}$ is outward since q_1 is positive. From

$$\oint \vec{E} \cdot d\vec{A} = q/\varepsilon_0, \qquad \oint \vec{E} \cdot d\vec{A} = E\oint dA = E4\pi a^2.$$

Thus,

$$E = \frac{q_1}{4\pi\varepsilon_0 a^2} = \frac{3.00 \times 10^{-6}\ C}{4\pi(8.9 \times 10^{-12}\ C^2/N\ m^2)(2.00\ m)^2} = 6.7 \times 10^3\ N/C.$$

Point Q: Since the charged shell will not alter the spherical symmetry of this problem, another spherical shell of radius 3.00 m is picked for the Gaussian surface. Now the total charge inside this surface is $(3.00 - 1.00) \times 10^{-6}$ C $= 2.00 \times 10^{-6}$ C. As before,

$$E = \frac{q_1 + Q_2}{4\pi\varepsilon_0 b^2} = \frac{2.00 \times 10^{-6}\ C}{4\pi(8.9 \times 10^{-12}\ C^2/N\ m^2)(3.00\ m)^2} = 2.00 \times 10^3\ N/C.$$

The direction of $\vec{E}$ is again radially outward since the net charge in the Gaussian surface is positive.

Point R: Now the net charge inside a spherical Gaussian surface of radius 4.0 m is zero. $\vec{E}$ cannot be perpendicular to $d\vec{A}$; this would violate the spherical symmetry. Thus $\vec{E} = \vec{0}$.

D(4). (a) Do you need to use Gauss' law to show that $\vec{E}$ is perpendicular to the surface of a conductor with a static charge distribution?
(b) Do you need to use Gauss' law to show that $\vec{E}$ is zero inside a conductor with a static charge distribution?
(c) Do you need to use Gauss' law to show that the excess charge is on the surface of a conductor with static charge distribution?

Solution

(a) No. If $\vec{E}$ were not perpendicular to the surface it would have a component along the surface, and this would cause charge to flow. This would violate our assumption of a static charge distribution.
(b) No. If $\vec{E}$ were not zero inside the conductor, charge would flow, again violating the assumption of static charge distribution.
(c) Yes:

$$\oint \vec{E} \cdot d\vec{S} = q/\varepsilon_0,$$

and since $\vec{E}$ is zero inside the conductor [compare with part (b)], then for a Gaussian surface inside the conductor

$$\oint \vec{E} \cdot d\vec{S} = 0,$$

and q must be zero inside the conductor. The excess charge must be on the surface.

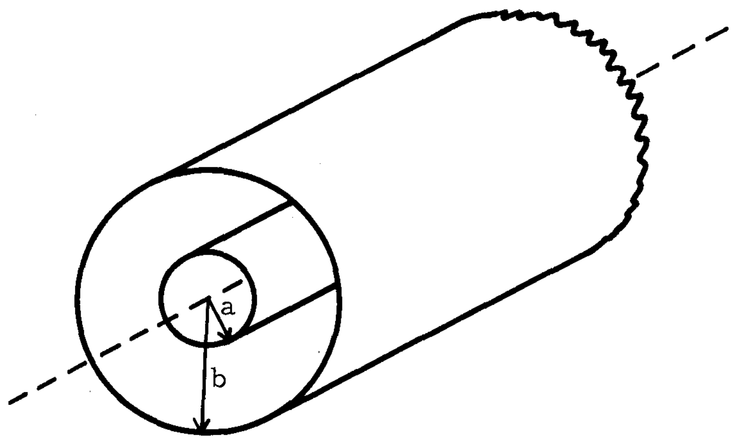


Figure 11

Problems

E(3). Two long statically charged thin coaxial cylinders are shown in Figure 11. The charge densities (in units of coulombs per square meter) have the relationship $\sigma_a/\sigma_b = -b/a$. Use Gauss' law to find $\vec{E}$:

(a) Between the cylinders. Show your choice of Gaussian surface.
(b) Outside the larger cylinder. Show your choice of Gaussian surface.

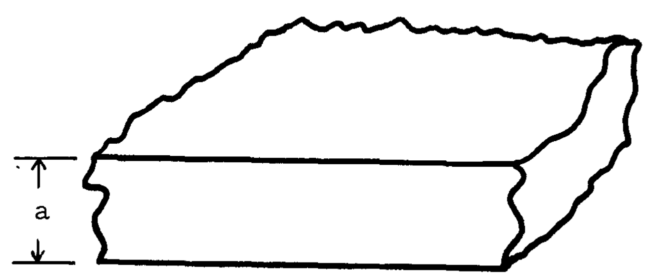


Figure 12

F(3). A large statically charged flat conducting plate is shown in Figure 12. The charge density is σ (in units of coulombs per square meter).
(a) Why is the charge density specified in units of C/m^2 instead of C/m^3?
(b) Use Gauss' law to find $\vec{E}$ outside the plate. Show your choice of Gaussian surface.

G(3). A long cylindrical uniformly charged insulator is shown in Figure 13. Its charge density is ρ (which has units of coulombs per cubic meter). A long, thin uniformly charged wire is coaxial to the cylinder as shown. Its charge density is λ (which has units of coulombs per meter). Use Gauss' law to find:

(a) $\vec{E}$ in the region $r < a$. Show your choice of Gaussian surface.
(b) $\vec{E}$ in the region $a < r < b$. Show your choice of Gaussian surface.
(c) $\vec{E}$ in the region $r > a$. Show you choice of Gaussian surface.

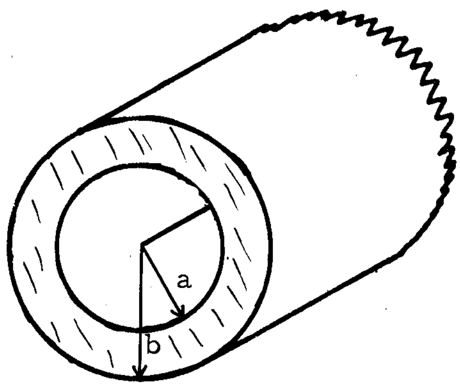


Figure 13

H(3). The infinite slab of insulating material in Figure 14 carries a uniform charge density ρ. There are no other charges in this region of space, so that the field must be symmetric about the plane $y = 0$, that is, $E(y = 0) = 0$. Use Gauss' law to find $E(y)$ for $-a < y < a$. Be sure to indicate clearly the (closed) Gaussian surface you are using. Which way does $\vec{E}$ point if $\rho < 0$?

I(3). A uniformly charged nonconducting sphere has a charge density of 3.00×10^{-12} C/m^3 and a radius of 1.00 m. Use Gauss' law to find $\vec{E}$

(a) 0.50 m from the center of the sphere. Show your choice of Gaussian surface; and

(b) 2.00 m from the center of the sphere. Show your choice of Gaussian surface.

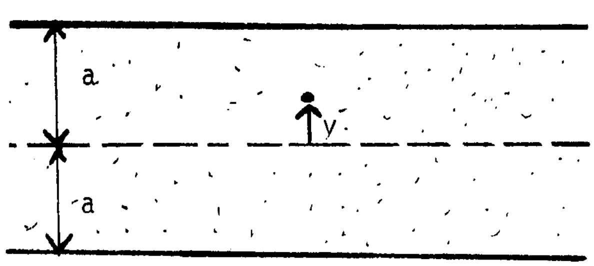


Figure 14

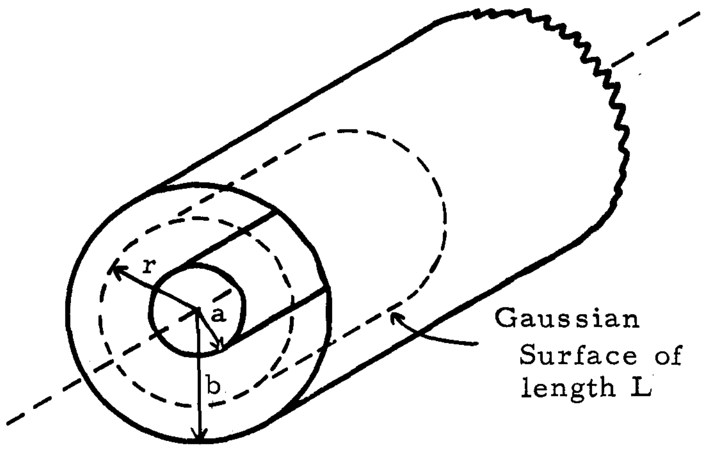


Figure 15

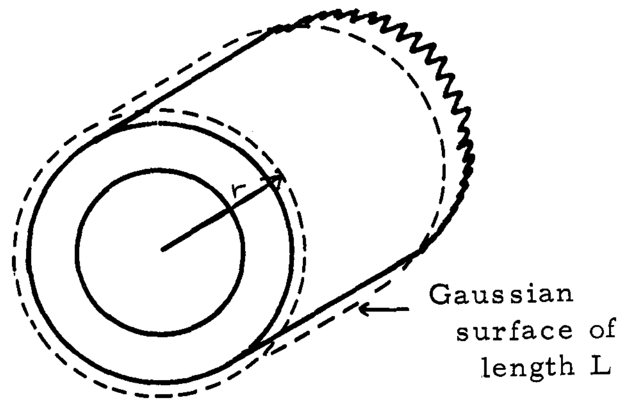


Figure 16

Solutions

E(3). (a) The charge inside the Gaussian surface of length L in Figure 15 is

$$q = 2\pi a L\sigma_a,$$

and the surface integral is

$$\oint \vec{E} \cdot d\vec{A} = \int_{\text{ends}} \vec{E} \cdot d\vec{A} + \int_{\substack{\text{curved}\\ \text{side}}} \vec{E} \cdot d\vec{A} = 0(\vec{E} \perp d\vec{A}) + E2\pi rL.$$

Thus,

$$\vec{E} = 2\pi a L\sigma_a/\varepsilon_0 2\pi rL = a\sigma_a/\varepsilon_0 r; \quad \text{radially outward.}$$

(b) In the Gaussian surface of length L in Figure 16

$$q = 2\pi a\sigma_a L + 2\pi b\sigma_B L = 2\pi L(a\sigma_a + b\sigma_B) = 2\pi L(0).$$

Thus, $\vec{E} = \vec{0}$.

F(3). (a) Inside a statically charged conductor $\vec{E}$ is zero, and the excess charge resides on the surfaces (see Objective 4). In this case there are two equally charged surfaces.
(b) See Figure 17 for two choices for the Gaussian surface. In Figure 17(a), the charge inside the Gaussian surface is $q = \sigma A$. The surface integral is

$$\oint \vec{E} \cdot d\vec{A} = \int_{\text{top}} \vec{E} \cdot d\vec{A} + \int_{\text{sides}} \vec{E} \cdot d\vec{A} + \int_{\text{bottom}} \vec{E} \cdot d\vec{A} = 0(\vec{E} = 0) + 0(\vec{E} \perp d\vec{A}) + EA = EA.$$

Thus, $E = \sigma/\varepsilon_0$, perpendicular to the surface. In Figure 17(b), the charge inside the Gaussian surface is $a = 2\sigma A$. The surface integral is

$$\oint \vec{E} \cdot d\vec{A} = \int_{\text{top}} \vec{E} \cdot d\vec{A} + \int_{\text{sides}} \vec{E} \cdot d\vec{A} + \int_{\text{bottom}} \vec{E} \cdot d\vec{A} = EA + 0(\vec{E} \perp d\vec{A}) + EA = 2\,EA.$$

Thus, $E = \sigma/\varepsilon_0$, perpendicular to the surface.

(a)

(b)

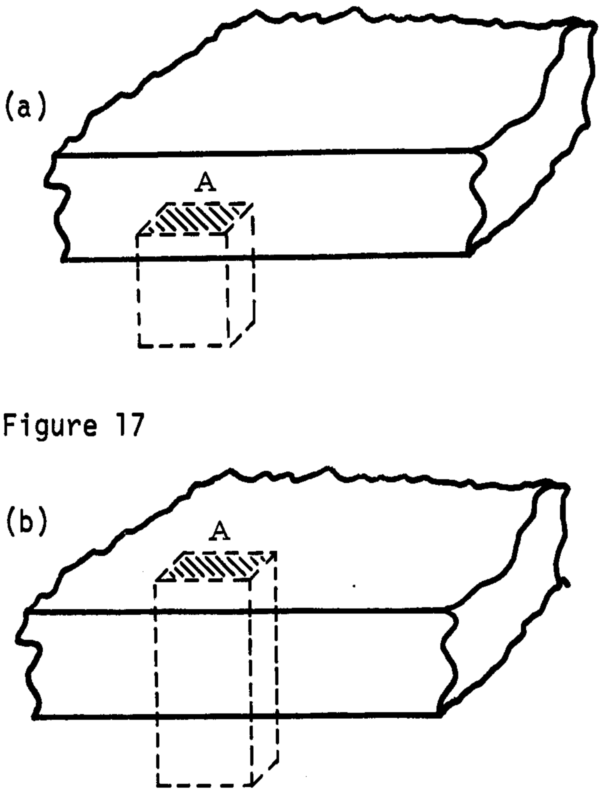


Figure 17

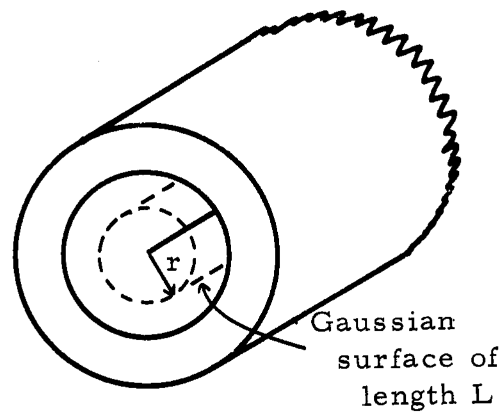


Figure 18

This shows that Gauss' law really works if applied correctly. Note that this result can be applied to any shaped surface if the point where you want to find the electric field is very close to the surface. If you are very close to a surface, it looks flat.

G(3). (a) See Figure 18, using

$$\oint \vec{E} \cdot d\vec{A} = q/\varepsilon_0$$

we find that the charge inside the Gaussian surface is $q = \lambda L$, and

$$\oint \vec{E} \cdot d\vec{A} = \int_{\text{ends}} \vec{E} \cdot d\vec{A} + \int_{\text{curved side}} \vec{E} \cdot d\vec{A} = 0(\vec{E} \perp d\vec{A}) + E2\pi rL.$$

Thus

$$E = \lambda L/\varepsilon_0 2\pi r L = \lambda/2\pi\varepsilon_0 r, \quad \text{radially outward.}$$

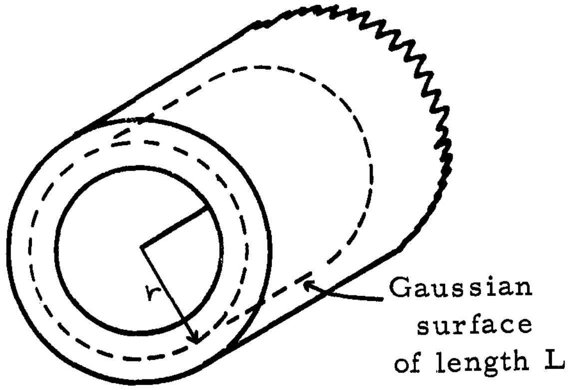


Figure 19

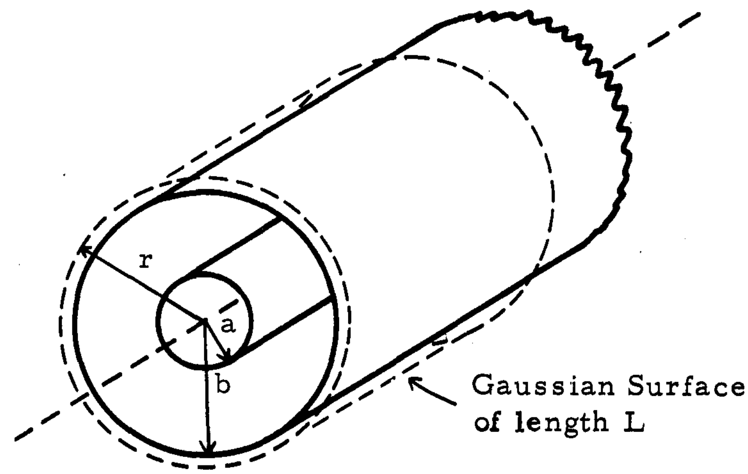


Figure 20

(b) See Figure 19, a Gaussian surface of length L. Now

$$q = \rho V + \lambda L = \rho(\pi r^2 - \pi a^2)L + \lambda L \quad \text{and} \quad \oint \vec{E} \cdot d\vec{A} = 2\pi r L E.$$

Thus,

$$E = \frac{[\rho\pi(r^2 - a^2) + \lambda]L}{2\pi\varepsilon_0 r L} = \frac{\rho\pi(r^2 - a^2) + \lambda}{2\pi\varepsilon_0 r}, \quad \text{radially outward.}$$

(c) See Figure 20, a Gaussian surface of length L, where

$$q = \rho(\pi b^2 - \pi a^2)L + \lambda L \quad \text{and} \quad \oint \vec{E} \cdot d\vec{A} = E 2\pi r L.$$

Thus,

$$E = \frac{\rho\pi(b^2 - a^2) + \lambda}{2\pi\varepsilon_0 r}, \quad \text{radially outward.}$$

H(3). $E(y) = \rho y/\varepsilon_0$, toward the center of the slab.

I(3). (a) $18\pi \times 10^{-3}$ N/C, radially outward. (b) $9\pi \times 10^{-3}$ N/C, radially outward.

ρy :
ε 0 (subscript?)

PRACTICE TEST

1. State Gauss' law and briefly explain all its symbols.

2. Gauss' law is always true but not always useful. Explain why sometimes Gauss' law is a useful tool to determine the electric field caused by a static charge distribution.

3. Use Gauss' law to answer these questions:
(a) If a Gaussian surface encloses zero net charge, does Gauss' law require that $\vec{E} = 0$ at all points on the surface?
(b) If $\vec{E} = 0$ everywhere on a Gaussian surface, is the net charge inside necessarily zero?

4. The top and bottom of the cylindrical can in Figure 21 each have an area of $0.200\ m^2$. In this region of space there is a layer of charge, some of which is inside the can, so that E_B is larger than E_T; but the field points up everywhere. If $E_B = 5.0 \times 10^4$ N/C and $E_T = 2.50 \times 10^4$ N/C, how much charge is contained in the can?

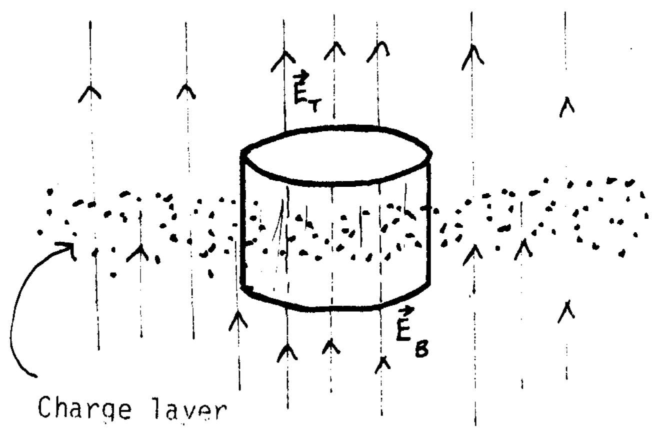


Figure 21

5. You are shown a conductor with a static charge distribution. Use the properties of a conductor and/or Gauss' law to:
(a) Explain why $\vec{E}$ is parallel or antiparallel to $d\vec{A}$ at the surface of the conductor.
(b) Explain why the excess charge lies on the surface of the conductor.

Practice Test Answers

1. $\oint \vec{E} \cdot d\vec{A} = q/\varepsilon_0$ or $\int \vec{E} \cdot d\vec{A} = q/\varepsilon_0$ or $\oint \vec{E} \cdot d\vec{s} = q/\varepsilon_0$ or $\int \vec{E} \cdot d\vec{s} = q/\varepsilon_0$.

The volume in Figure 22 contains a charge q. $d\vec{A}$ is an area element of the surface covering the volume, and $\vec{E}$ is the value of the electric field at that area element.

They are multiplied (scalar product) to determine the component of $\vec{E}$ in the direction of $d\vec{A}$ times dA. The integration must be performed over the complete surface covering the volume. ε_0 is a constant called the permittivity of free space.

2. Gauss' law is useful when E cos θ can be factored out of the integral

$$\oint E \cos\theta \, dA = q/\varepsilon_0 .$$

This can be accomplished when E cos θ is a constant over the surface of integration.

3. (a) No. $\oint \vec{E} \cdot d\vec{A} = 0$ can be true without $\vec{E} = \vec{0}$. (b) Yes.

$$\oint \vec{E} \cdot d\vec{A} = q/\varepsilon_0$$

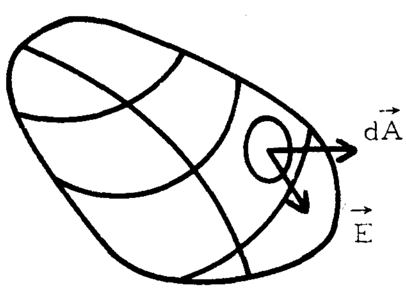
and if $\vec{E} = \vec{0}$ on the surface of integration then the left-hand side of the equation must be zero. Thus the right-hand side must also be zero and q = 0.

4. See Figure 23, where the dotted lines indicate the Gaussian surface.

$$\oint \vec{E} \cdot d\vec{A} = \int_{\text{top}} \vec{E}_T \cdot d\vec{A} + \int_{\substack{\text{curved}\\ \text{side}}} \vec{E} \cdot d\vec{A} + \int_{\text{bottom}} \vec{E}_B \cdot d\vec{A} = E_T A + 0 + (-E_B A) = (E_T - E_B)A = q/\varepsilon_0 ,$$

$$q = \varepsilon_0(E_T - E_B)A = -(8.9 \times 10^{-12}\ C^2/N\ m^2)(2.50 \times 10^4\ N/C)(0.200\ m^2) = -4.4 \times 10^{-8}\ C.$$

Figure 22



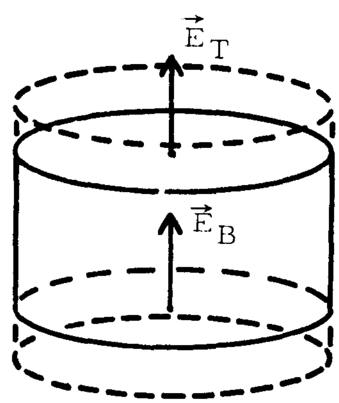


Figure 23

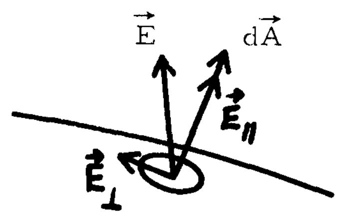


Figure 24

5. (a) See Figure 24. If $\vec{E}$ were not in the direction of $d\vec{A}$ it could be resolved into vectors parallel and perpendicular to $d\vec{A}$. From $\vec{F} = q\vec{E}$, the component of $\vec{E}$ along the surface would produce a force on the charges in the conductor. They would flow and our assumption about a static charge distribution would be false. Thus $\vec{E}$ must be in the direction of $d\vec{A}$.
(b) In the interior of a conductor $\vec{E}$ must be zero. Otherwise there would be charges moving, in contradiction to the assumption of a static charge distribution. Applying Gauss' law to a surface just inside the surface of the conductor shows us that q is zero inside this surface. The excess charge must therefore reside outside the Gaussian surface on the surface of the conductor.

FLUX AND GAUSS' LAW

Date ____________________

Mastery Test   Form A

pass   recycle

1   2   3   4

Name ______________________________   Tutor ______________________________

Use $1/4\pi\varepsilon_0 = 9.0 \times 10^9$ N m^2/C^2 in working these problems.

1. State Gauss' law and explain all its symbols.

2. For the following charged conductors EITHER

   Sketch the conductor and the Gaussian surface you would choose to find the electric field outside the conductor, OR

   Explain why Gauss' law cannot easily be used to find $\vec{E}$ outside the conductor.

   (a) a charged sphere,
   (b) a very long charged cylinder,
   (c) a very long charged wire,
   (d) two nearby point charges,
   (e) a large, flat, charged surface,
   (f) a charged cube.

3. What is the net charge on a statically charged conducting sphere of 2.00 m radius if $\vec{E}$ is 15.0×10^9 N/C in the radial direction toward the center of the sphere at a distance 3.00 m from the center of the sphere. Use Gauss' law.

4. Given a conductor with a static charge distribution:
   (a) Use the properties of a conductor and Gauss' law to explain why $\vec{E}$ is discontinuous across the surface of the conductor (why $\vec{E}$ changes abruptly from just inside to just outside the surface of the conductor).
   (b) Use the properties of a conductor to explain why $\vec{E}$ is perpendicular to the surface of the conductor.

FLUX AND GAUSS' LAW

Date ________________

Mastery Test   Form B

pass   recycle

1   2   3   4

Name ______________________________   Tutor ____________________________

Use $1/4\pi\varepsilon_0 = 9.0 \times 10^9$ N m^2/C^2 in working these problems.

1. State Gauss' law and explain all its symbols.

2. Explain why Gauss' law is not always a useful tool with which to determine the electric field.

3. The long uniformly charged circular rod in Figure 1 with a radius of 2.00 m has a constant charge density of

   $\rho = (1/\pi) \times 10^{-12}$ C/m^3.

   (a) Use Gauss' law to find $\vec{E}$ at a radius of 1.00 m from the axis of the rod. Show your choice of a Gaussian surface.
   (b) Use Gauss' law to find $\vec{E}$ at a radius of 3.00 m from the axis of the rod. Show your choice of a Gaussian surface.

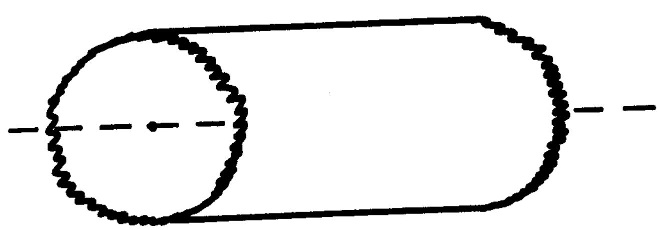


Figure 1

4. Given a conductor with a static charge distribution, use the properties of a conductor and/or Gauss' law to:
   (a) Explain why $\vec{E}$ is perpendicular to the surface of the conductor.
   (b) Explain why $\vec{E}$ is zero inside the conductor.
   (c) Explain why the excess charge is on the surface of the conductor.

FLUX AND GAUSS' LAW

Date ____________

Mastery Test Form C

pass recycle

1 2 3 4

Name ______________________ Tutor ______________________

Use $1/4\pi\varepsilon_0 = 9.0 \times 10^9$ N m^2/C^2 in working these problems.

1. State Gauss' law and explain all its symbols.

2. For the following charged conductors EITHER

   Sketch the conductor and the Gaussian surface you would choose to find $\vec{E}$ outside the conductor, OR

   Explain why Gauss' law cannot easily be used to find $\vec{E}$ outside the conductor.

   (a) a charged disk,
   (b) a charged sphere,
   (c) a long, hollow cylinder,
   (d) three equal charges at the points of an equilateral triangle,
   (e) a large, thick, flat plate.

3. A large flat conducting plate is 0.50 m thick. It is statically charged, and the surface charge density σ is 2.50×10^{-15} C/m^2. Use Gauss' law to find $\vec{E}$ 1.50 m from the upper surface of the plate.

4. Given a conductor with a static charge distribution:
   (a) Use the properties of a conductor and Gauss' law to explain why $\vec{E}$ is discontinuous across the surface of the conductor (why $\vec{E}$ changes abruptly from just inside to just outside the surface of the conductor).
   (b) Use the properties of a conductor to explain why $\vec{E}$ is perpendicular to the surface of the conductor.

MASTERY TEST GRADING KEY - Form A

1. What To Look For: q causes $\vec{E}$. Integration is over a closed surface. q is inside the surface of integration. It's OK to write out $\vec{E} \cdot d\vec{A}$ as $E \cos \theta \, dA$, but now θ must be explained.

Solution:

$\oint \vec{E} \cdot d\vec{A} = q/\varepsilon_0$ or $\int \vec{E} \cdot d\vec{A} = q/\varepsilon_0$ or $\oint \vec{E} \cdot d\vec{S} = q/\varepsilon_0$ or $\int \vec{E} \cdot d\vec{S} = q/\varepsilon_0$.

Given some volume V enclosed by surface A (or S), and which contains some net charge q, the surface integral of $\vec{E} \cdot d\vec{A}$ (or $\vec{E} \cdot d\vec{S}$) over the whole surface equals the enclosed charge divided by ε_0. $\vec{E}$ is the electric field at area element $d\vec{A}$, and ε_0 is the permittivity of free space.

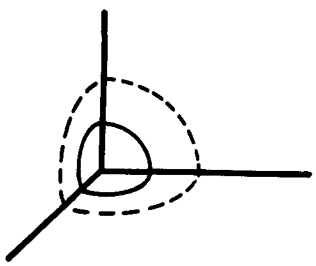


Figure 27

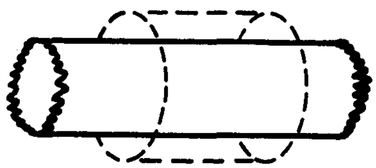


Figure 28

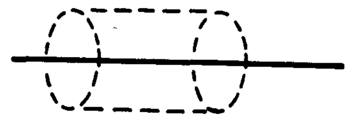


Figure 29

2. What To Look For: (a) See Figure 27, where the dotted lines indicate the Gaussian surface.
(b) See Figure 28, where the dotted lines indicate the Gaussian surface.
(c) See Figure 29, where the dotted lines indicate the Gaussian surface.
(d) Cannot easily draw a Gaussian surface over which E cos θ is constant.
(e) See Figure 30, where the dotted lines indicate the Gaussian surface.
(f) Cannot easily draw a Gaussian surface over which E cos θ is constant. The corners give trouble.

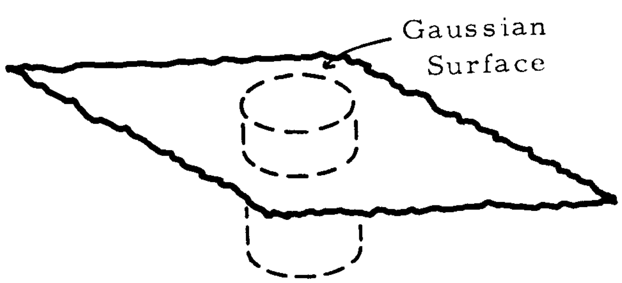


Figure 30

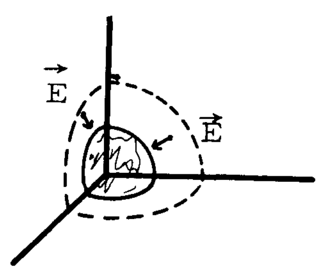


Figure 31

3. What To Look For: Correct Gaussian surface charge is negative.

Solution: See Figure 31, where the dotted lines indicate the Gaussian surface of radius 3.00 m. The spherical symmetry of the charge distribution causes spherical symmetry in $\vec{E}$.

$$q/\varepsilon_0 = \vec{E} \cdot d\vec{A} = -EA = -E4\pi r^2,$$

$$q = -\varepsilon_0 E4\pi r^2 = \frac{(15.0 \times 10^9\ \mathrm{N/C})(9.0\ \mathrm{m}^2)}{9.0 \times 10^9\ \mathrm{N\ m^2/C^2}} = 15.0\ \mathrm{C}.$$

4. Solution: (a) Inside the conductor $\vec{E} = 0$. The static charge distribution of the movable charges indicates that $\vec{F}$ and therefore $\vec{E}$ is zero inside. Outside the conductor $\vec{E} \neq 0$. Gauss' law says that if q is not zero then $\vec{E}$ is not zero. Thus $\vec{E}$ changes from $\vec{0}$ to $\vec{E}$ as you go from inside to outside a conductor with static charge distribution.
(b) If $\vec{E}$ were not perpendicular to the surface there would be an electric field along the surface. This would contradict the static charge assumption.

MASTERY TEST GRADING KEY - Form B

1. What To Look For: q causes $\vec{E}$. Integration is over a closed surface. q is inside the surface of integration. It's okay to write out $\vec{E} \cdot d\vec{A}$ as $E \cos \theta \, dA$, but now θ must be explained.

   Solution:

   $\oint \vec{E} \cdot d\vec{A} = q/\varepsilon_0$ or $\int \vec{E} \cdot d\vec{A} = q/\varepsilon_0$ or $\oint \vec{E} \cdot d\vec{S} = q/\varepsilon_0$ or $\int \vec{E} \cdot d\vec{S} = q/\varepsilon_0$.

   Given some volume V enclosed by surface A (or S), and which contains some net charge q, the surface integral of $\vec{E} \cdot d\vec{A}$ (or $\vec{E} \cdot d\vec{S}$) done over the whole surface equals the enclosed charge divided by ε_0. $\vec{E}$ is the electric field at area element $d\vec{A}$, and ε_0 is the permittivity of free space.

2. Solution: It is not always possible to draw a Gaussian surface everywhere on which $E \cos \theta$ is the same, and also of which you know the area.

3. What To Look For: (a) Correct Gaussian surface. Correct charge inside Gaussian surface. Direction of $\vec{E}$.

   Solution: (a) See Figure 32, where the dotted lines indicate the Gaussian surface. The charge inside the Gaussian surface is $q = \rho\pi r^2 \ell$. The surface integral is

   $\int \vec{E} \cdot d\vec{A} + \int \vec{E} \cdot d\vec{A} + \int \vec{E} \cdot d\vec{A}$.

   On the ends $\vec{E} \perp d\vec{A}$:

   $0 + E2\pi r\ell + 0$.

   Thus

   $$E = \rho r/2\varepsilon_0 = \frac{(1/\pi \times 10^{-12} \text{ C/m}^3)(1.00 \text{ m})(4\pi)(9.0 \times 10^9 \text{ N m}^2/\text{C}^2)}{2}$$

   $= 1.80 \times 10^{-2}$ N/C, radially outwards.

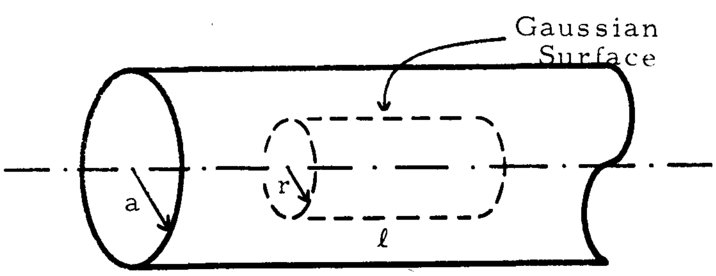


Figure 32

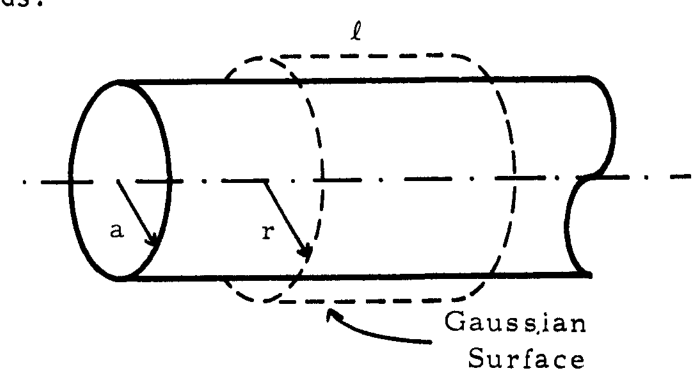


Figure 33

(b) See Figure 33, where the dotted lines indicate the Gaussian surface. Now the charge inside the Gaussian surface is $q = \rho\pi a^2 \ell$. The surface integral is treated as before to give

$$E = \rho a^2/2\varepsilon_0 r = \frac{(1/\pi \times 10^{-12}\ \mathrm{C/m^3})(2.00\ \mathrm{m})^2(4\pi)(9.0 \times 10^9\ \mathrm{N\ m^2/C})}{2(3.0\ \mathrm{m})}$$

$$= 2.4 \times 10^{-2}\ \mathrm{N/C},\ \text{radially outward.}$$

4. Solution: (a) If $\vec{E}$ were not perpendicular to the surface, there would be some electric field along the surface of the conductor, and charges would flow. This would contradict the static charge assumption.
(b) If $\vec{E}$ were not zero inside the conductor, charges would flow. This contradicts the static charge assumption.
(c) Use the results of (b) in Gauss' law. Draw a Gaussian surface just inside the surface of the conductor. The charge inside this surface is zero. Therefore the charge must be on the surface of the conductor.

MASTERY TEST GRADING KEY - Form C

1. What To Look For: q causes $\vec{E}$. Integration is over a closed surface. q is inside the surface of integration. It's OK to write out $\vec{E} \cdot d\vec{A}$ as E cos θ dA, but now θ must be explained.

   Solution:

   $\oint \vec{E} \cdot d\vec{A} = q/\varepsilon_0$ or $\int \vec{E} \cdot d\vec{A} = q/\varepsilon_0$ or $\oint \vec{E} \cdot d\vec{S} = q/\varepsilon_0$ or $\int \vec{E} \cdot d\vec{S} = q/\varepsilon_0$.

   Given some volume V enclosed by surface $\vec{A}$ (or $\vec{S}$), and which contains some net charge q, the surface integral of $\vec{E} \cdot d\vec{A}$ (or $\vec{E} \cdot d\vec{S}$) done over the whole surface equals the enclosed charge divided by ε_0. $\vec{E}$ is the electric field at area element dA, and ε_0 is the permittivity of free space.

2. Solution: (a) Cannot easily draw a Gaussian surface over which E cos θ is constant. Corners give trouble.
   (b) See Figure 34, wherein the dotted lines indicate the Gaussian surface.
   (c) See Figure 35, wherein the dotted lines indicate the Gaussian surface.
   (d) Cannot easily draw a Gaussian surface over which E cos θ is constant.
   (e) See Figure 36, wherein the dotted lines indicate the Gaussian surface.

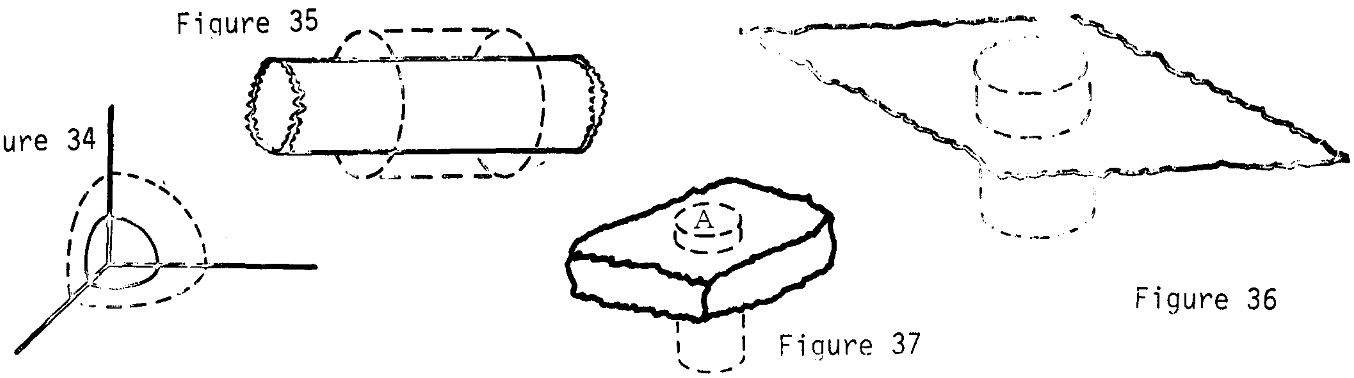


Figure 34 Figure 35 Figure 36 Figure 37

3. Solution: See Figure 37, where the dotted lines indicate the Gaussian surface. The charge inside the Gaussian surface is q = 2πA. The surface integral is

   $$\oint \vec{E} \cdot d\vec{A} = \int \vec{E} \cdot d\vec{A} + \int \vec{E} \cdot d\vec{A} + \int \vec{E} \cdot d\vec{A} = EA + 0 + EO = 2EA.$$

   Thus $E = \sigma/\varepsilon_0$ which does not depend on the distance from the plate.

   $$E = (2.50 \times 10^{-15}\ C/m^2)(4\pi)(9.0 \times 10^9\ N\ m^2/C^2) = 28.3 \times 10^{-5}\ N/C$$

   outward perpendicular to the plate's surface.

4. Solution: (a) Inside the conductor $\vec{E} = 0$. The static charge distribution of the movable charges indicate that F and therefore E is zero inside. Outside the conductor $\vec{E} \neq 0$. Gauss' law says that if q is not zero then E is not zero. Thus $\vec{E}$ changes from 0 to $\vec{E}$ as you go from inside to outside a conductor with static charge distribution. If $\vec{E}$ were not perpendicular to the surface there would be an electric field and moving charge along the surface. This would contradict the static charge assumption.