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Analysis of an Insurance Risk Model with Thinning Dependence and Common Shock

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Abstract§

We consider a continuous-time insurance risk model with m dependent classes of business with dependent claim number processes due to thinning dependence and a common shock. The impact of the dependence is studied via the adjustment coefficient. The case $m = 2$ is investigated analytically for exponential claim distributions and via simulation for non-exponential claim distributions.

Key words and phrases: *adjustment coefficient, by-claim, common shock, main claim, thinning dependence, ultimate ruin probability*

1 Introduction

A traditional assumption in the actuarial literature is independence among classes of policies in a book of insurance business. This assumption, however, may not always reflect the reality. For example, suppose

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a house is damaged by fire. The resulting death claims, medical claims, and claims for household damages cannot be regarded as mutually independent. Over the past decade or so risk models with different types of dependence structures have been studied. For example, Goovaerts and Dhaene (1996) derived a compound Poisson approximation for a correlated aggregate claims distribution. Ambagaspitiya (1998 and 1999) developed methods to compute an aggregate claims distribution with dependent claim-number processes. Cossette and Marceau (2000) studied a discrete-time risk model with the claim number following a Poisson model with common shock. Yuen and Wang (2002), Wu and Yuen (2003), and Wang and Yuen (2005) considered models with thinning dependence in the claim-number processes. Bäuerle and Grübel (2005) used the thinning mechanism and the idea of random shift to construct the dependence structure for a class of multivariate counting processes with Poisson marginals. Macci, Stabile, and Torrisi (2005) utilized the Markov modulated Poisson shot noise process to investigate how the dependence among the claims of different lines in a company affects the Lundberg parameters of the total reserve of the company.

We will consider a continuous-time risk model with m dependent classes of business in which the correlation comes from a so-called thinning dependence and a common shock. The thinning dependence suggests that the claim number of class j up to time t depends not only on the underlying risk in its own class, but also on the risks in other classes with certain probabilities. For each class, claims due to its own underlying risk are called main claims while claims due to the risks in other classes are referred to as by-claims. In addition to the thinning dependence, the claim numbers of all the m classes have a common component due to a common shock that impacts all risks simultaneously. This model will be called Model B throughout the rest of this paper.

Model B can be briefly described as follows: let $N_t^{(jj)}$ be the number of main claims due to the underlying risk in class j which is a Poisson process with intensity λ_j, and $N_t^{(lj)}$ be the number of by-claims in class j due to the main claims in class l for $l \neq j$. It is assumed that the probability of triggering a by-claim in class j due to the main claim in class l is p_{lj} where $0 < p_{lj} < 1$. Therefore, $N_t^{(lj)}$ is a Poisson process with intensity $\lambda_l p_{lj}$. For $N_t^{(jj)}$, it can be treated as the p_{jj}-thinning process with $p_{jj} = 1$.

The number of claims due to the common shock up to time t is denoted by $N_t^{(c)}$ which is also a Poisson process with intensity λ_c. Here, we assume that $N_t^{(jj)}$'s are independent and that $N_t^{(lj)}$'s given $N_t^{(ll)}$ are mutually independent for $l \neq j$. Also, it is assumed that the number of claims due to the common shock $N_t^{(c)}$ is independent of all other claim-number processes. Thus, the claim-number process for class j is given by

$$N_t^{(j)} = \tilde{N}_t^{(j)} + N_t^{(c)},$$

where $\tilde{N}_t^{(j)} = \sum_{l=1}^{m} N_t^{(lj)}$ for $j = 1, 2, \cdots, m$.

Next, let $X_i^{(j)}$ and $\hat{X}_i^{(j)}$ be the claim size of the i^{th} claim in class j that is not due to the common shock and is due to the common shock, respectively. It is assumed that, for all i and j, $X_i^{(j)}$'s and $\hat{X}_i^{(j)}$'s are mutually independent and are also independent of all the claim number processes. For each j, the $X_i^{(j)}$'s and $\hat{X}_i^{(j)}$'s are assumed to have a common distribution $F_j(x)$ with finite mean μ_j and finite variance σ_j^2. The total amount of claims for class j is

$$S_t^{(j)} = \sum_{i=1}^{\tilde{N}_t^{(j)}} X_i^{(j)} + \sum_{i=1}^{N_t^{(c)}} \hat{X}_i^{(j)}, \tag{1}$$

and the aggregate claims process for all classes and risks is S_t, which is given by

$$S_t = \sum_{j=1}^{m} S_t^{(j)}, \tag{2}$$

and the surplus process is defined as

$$U_t = u + \pi t - S_t, \tag{3}$$

where u is the initial surplus and π is the rate of premium.

In what follows we will explore various aspects of this model. In Section 2 we show that the aggregate claims process is a compound Poisson process and derive expressions for the variance and covariances of certain underlying processes. Section 3 uses the adjustment coefficient to compare Model B with other related models. In Section 4 we study the impact of the dependence structure on the probability of ultimate ruin in the cases of exponential and non-exponential claims.

An important and well-known result that will be used is the following (Bowers et al., 1997, Theorem 12.4.1, page 378):

Result 1. *If S_j is a compound Poisson random variable $S_j = X_{j1} + X_{j2} + \cdots + X_{jN_j}$ where the X_{jk}'s ($k = 1, 2, \ldots$) are independent and identically distributed random variables with common distribution function $F_j(x)$*

and N_j is Poisson with mean λ_j, and S_1, S_2, ..., S_m are independent compound Poisson random variables, then $S = S_1 + S_2 + \cdots + S_m$ is also compound Poisson and can be written as $S = Y_1 + Y_2 + \cdots + Y_N$ where N is Poisson with mean λ and the Y_k's ($k = 1, 2, \ldots$) are independent and identically distributed random variables with common distribution function $F_Y(y)$ where

$$\lambda = \sum_{j=1}^{m} \lambda_j \quad \text{and} \quad F_Y(y) = \sum_{j=1}^{m} \frac{\lambda_j}{\lambda} F_j(y).$$

2 The Aggregate Claims Process

Let $\hat{Y}_i = \sum_{j=1}^{m} \hat{X}_i^{(j)}$ and let

$$\widetilde{S}_t = \sum_{j=1}^{m} \sum_{i=1}^{\widetilde{N}_t^{(j)}} X_i^{(j)} \quad \text{and} \quad S_t^{(c)} = \sum_{i=1}^{N_t^{(c)}} \hat{Y}_i$$

so that the aggregate claims process becomes

$$S_t = \widetilde{S}_t + S_t^{(c)}.$$

Yuen and Wang (2002) proved that $\widetilde{S}_t$ is a compound Poisson process and can be expressed as $\sum_{i=1}^{\widetilde{N}_t} \widetilde{X}_i$ where $\widetilde{N}_t$ is a Poisson process with intensity $\widetilde{\lambda} = \lambda_1 + \lambda_2 + \cdots + \lambda_m$ and $\widetilde{X}_i$'s are independent and identically distributed random variables with distribution $F_{\widetilde{X}}$ being a weighted average of F_j's and their convolutions. From the independence assumptions, it is easy to see that $\widetilde{S}_t$ and $S_t^{(c)}$ are two independent compound Poisson processes. Thus, from Result 1 above, S_t is a compound Poisson process that can be written as $S_t = \sum_{i=1}^{N_t^{(s)}} Z_i$ where $N_t^{(s)}$ is a Poisson process with intensity $\lambda = \sum_{j=1}^{m} \lambda_j + \lambda_c$ and Z_i's are independent and identically distributed random variables with distribution F_Z having moment generating function (mgf) $M_Z(r)$ where

$$M_Z(r) = \sum_{l=1}^{m} \frac{\lambda_l}{\lambda} \left(\prod_{j=1}^{m} (M_j(r) p_{lj} + 1 - p_{lj}) \right) + \frac{\lambda_c}{\lambda} M_{\hat{Y}}(r),$$

$M_j(r)$ is the mgf of the $X_i^{(j)}$'s and $M_{\hat{Y}}(r)$ is the mgf of the $\hat{Y}_i$'s.

For the case $m = 2$, the transformed claim size random variable Z_i can be expressed as

$$Z_i = X_i^{(1)} I(\delta_i = 0) + X_i^{(2)} I(\delta_i = 1) + (X_i^{(1)} + X_i^{(2)}) I(\delta_i = 2), \tag{4}$$

where $I(A)$ is the indicator of the event A, i.e., $I(A) = 1$ if A occurs and 0 otherwise. The underlying probabilities are

$$\mathbb{P}(\delta_i = 0) = \frac{\lambda_1(1-p_{12})}{\lambda}, \quad \mathbb{P}(\delta_i = 1) = \frac{\lambda_2(1-p_{21})}{\lambda}, \quad \text{and}$$
$$\mathbb{P}(\delta_i = 2) = \frac{\lambda_1 p_{12} + \lambda_2 p_{21} + \lambda_c}{\lambda},$$

where $\lambda = \lambda_1 + \lambda_2 + \lambda_c$. The claim size distribution is thus given by

$$\begin{aligned} F_Z(z) = \frac{1}{\lambda}(&\lambda_1(1-p_{12})F_1(z) + \lambda_2(1-p_{21})F_2(z) \\ &+ (\lambda_1 p_{12} + \lambda_2 p_{21} + \lambda_c) F_1 * F_2(z)), \end{aligned} \tag{5}$$

where F_1 and F_2 are the distributions of $X^{(1)}$ and $X^{(2)}$, respectively, and $F_1 * F_2$ represents the convolution of F_1 and F_2.

To study the nature of the dependence structure, it is instructive to derive some statistical properties of Model B. We use the notation $\text{Bin}(n, p)$ to denote a binomial distribution with mean np and variance $np(1-p)$ and $\text{Poi}(\lambda)$ to denote a Poisson variable with mean λ. Clearly $N_t^{(j)}$ is a Poisson process with intensity ${}^{(jc)}\lambda$ where

$${}^{(jc)}\lambda = \sum_{l=1}^{m} p_{lj}\lambda_l + \lambda_c.$$

In addition, for $i \neq j$, as

$$\mathbb{E}[N_t^{(li)} N_t^{(lj)} | N_t^{(ll)}] = p_{li} p_{lj} \mathbb{E}[(N_t^{(ll)})^2]$$

it follows that

$$\begin{aligned} \mathbb{C}\text{ov}(N_t^{(li)}, N_t^{(lj)}) &= p_{li} p_{lj} \mathbb{E}[(N_t^{(ll)})^2] - p_{li} p_{lj} \left(\mathbb{E}(N_t^{(ll)})\right)^2 \\ &= p_{li} p_{lj} \mathbb{V}\text{ar}(N_t^{(ll)}) = p_{li} p_{lj} \lambda_l t. \end{aligned}$$

Also, as $N_t^{(li)}$ and $N_t^{(c)}$ are independent for all i,

$$\begin{aligned}\mathbb{C}\text{ov}(N_t^{(i)}, N_t^{(j)}) &= \mathbb{C}\text{ov}\left(\sum_{l=1}^{m} N_t^{(li)} + N_t^{(c)}, \sum_{l=1}^{m} N_t^{(lj)} + N_t^{(c)}\right)\\ &= \sum_{l=1}^{m} \mathbb{C}\text{ov}(N_t^{(li)}, N_t^{(lj)}) + \mathbb{V}\text{ar}(N_t^{(c)})\\ &= t\,{}^{(ijc)}\lambda,\end{aligned}$$

where

$${}^{(ijc)}\lambda = \sum_{l=1}^{m} p_{li}p_{lj}\lambda_l + \lambda_c.$$

As $S_t^{(j)}$ is a compound Poisson process,

$$\mathbb{E}(S_t^{(j)}) = \mu_j t\left(\sum_{l=1}^{m} p_{lj}\lambda_l + \lambda_c\right), \quad \text{and} \tag{6}$$

$$\mathbb{V}\text{ar}(S_t^{(j)}) = \left(\mu_j^2 + \sigma_j^2\right) t\left(\sum_{l=1}^{m} p_{lj}\lambda_l + \lambda_c\right), \tag{7}$$

while for $i \neq j$,

$$\begin{aligned}\mathbb{C}\text{ov}(S_t^{(i)}, S_t^{(j)}) &= \mu_i\mu_j\left(\sum_{l=1}^{m} p_{li}p_{lj}\mathbb{V}\text{ar}(N_t^{(ll)}) + \mathbb{V}\text{ar}(N_t^{(c)})\right)\\ &= \mu_i\mu_j t\,{}^{(ijc)}\lambda.\end{aligned} \tag{8}$$

For the entire book of business,

$$\mathbb{E}(S_t) = \sum_{j=1}^{m} \mu_j t\left(\sum_{l=1}^{m} p_{lj}\lambda_l + \lambda_c\right) \quad \text{and}$$

$$\mathbb{V}\text{ar}(S_t) = t\sum_{j=1}^{m}\left(\mu_j^2 + \sigma_j^2\right)\left(\sum_{l=1}^{m} p_{lj}\lambda_l + \lambda_c\right) + 2t\sum_{j=2}^{m}\sum_{i=1}^{m-1} \mu_i\mu_j\,{}^{(ijc)}\lambda.$$

3 Comparisons Using Adjustment Coefficients

We will now investigate the impact of the choice of dependence structure on the probability of ultimate ruin. For simplicity, we only study the case with two dependent classes of business, that is, $m = 2$. The time of ruin, T is defined as $\inf\{t : U_t < 0\}$. If $U_t \geq 0$ for all t, then $T = \infty$. The probability of ultimate ruin given the initial surplus u is

$$\Psi(u) = \mathbb{P}(T < \infty | U_0 = u).$$

A well-known result from classical risk theory (Bowers et al., 1997, Theorem 13.4.1, page 413) is: for $u \geq 0$

$$\Psi(u) = \frac{\mathrm{e}^{-Ru}}{\mathbb{E}\left(\mathrm{e}^{-RU_T} | T < \infty\right)} \leq \mathrm{e}^{-Ru} \tag{9}$$

where R is the adjustment coefficient, which is the smallest positive solution to the equation

$$M_S(r) = \mathrm{e}^{r\pi}$$

as a function of r, and π is the premium rate. When $u = 0$, the compound Poisson model yields

$$\Psi(0) = \frac{1}{1+\eta},$$

where η is relative security loading in π, i.e., $\pi = (1+\eta)\lambda\mathbb{E}(Z)$. Throughout we will assume that $M_j(r)$ exists for $j = 1, 2, \cdots, m$ and that $\eta > 0$. We further assume that the adjustment coefficient for each model considered in this paper also exists. Because of the difficulty in evaluating $\Psi(u)$, the upper bound is often taken as an approximation to $\Psi(u)$. Hence, one may treat the adjustment coefficient as a rough measure of risk in the sense that the smaller the adjustment coefficient, the riskier the model.[1]

Three other compound Poisson risk models (each with $m = 2$) are introduced for comparison with Model B:

Model I: The claim number process for classes 1 and 2, which have the form $N_t^{\mathrm{I}(1)}$ and $N_t^{\mathrm{I}(2)}$, are independent Poisson processes with

[1]For example, suppose you are given two models (1 and 2). If the adjustment coefficient R_1 for model 1 is less than the adjustment coefficient R_2 for model 2, then (in the absence of further information) one may argue that model 1 is more dangerous than model 2 because model 1 may have greater probability of ultimate ruin than model 2.

intensity λ_1^{I} and λ_2^{I}, respectively. The surplus process is $U_t^{\mathrm{I}} = u + \pi t - S_t^{\mathrm{I}}$ where S_t^{I} given by

$$S_t^{\mathrm{I}} = S_t^{\mathrm{I}(1)} + S_t^{\mathrm{I}(2)} = \sum_{i=1}^{N_t^{\mathrm{I}(1)}} X_i^{(1)} + \sum_{i=1}^{N_t^{\mathrm{I}(2)}} X_i^{(2)}.$$

Model A: For class j, $N_t^{\mathrm{A}(j)} = N_t^{(1j)} + N_t^{(2j)}$ (thinning dependence only) with intensity $\lambda_j^{\mathrm{A}} = \lambda_1 p_{1j} + \lambda_2 p_{2j}$ for $j = 1, 2$. The surplus process is $U_t^{\mathrm{A}} = u + \pi t - S_t^{\mathrm{A}}$ where

$$S_t^{\mathrm{A}} = S_t^{\mathrm{A}(1)} + S_t^{\mathrm{A}(2)} = \sum_{i=1}^{N_t^{\mathrm{A}(1)}} X_i^{(1)} + \sum_{i=1}^{N_t^{\mathrm{A}(2)}} X_i^{(2)}.$$

Model C: For class j, $N_t^{\mathrm{C}(j)} = N_t^{(jj)} + N_t^{(c)}$ (common shock only) with intensity $\lambda_j^{\mathrm{C}} = \lambda_j + \lambda_c$ for $j = 1, 2$. The surplus process is $U_t^{\mathrm{C}} = u + \pi t - S_t^{\mathrm{C}}$ where

$$S_t^{\mathrm{C}} = S_t^{\mathrm{C}(1)} + S_t^{\mathrm{C}(2)} = \sum_{i=1}^{N_t^{\mathrm{C}(1)}} X_i^{(1)} + \sum_{i=1}^{N_t^{\mathrm{C}(2)}} X_i^{(2)}.$$

We define $\lambda_j^{\mathrm{I}} = \lambda_j + \lambda_c$ for $j = 1, 2$ and choose $\lambda_1 p_{12} = \lambda_2 p_{21} = \lambda_c$ so that $\lambda_j^{\mathrm{I}} = \lambda_j^{\mathrm{A}} = \lambda_j^{\mathrm{C}}$ for $j = 1, 2$ and the three models have the same expected aggregate claims. It is apparent that Model B is more general than Models I, A, and C. The claim number process for Model B is rewritten as

$$N_t^{\mathrm{B}(j)} = N_t^{\mathrm{B}(1j)} + N_t^{\mathrm{B}(2j)} + N_t^{\mathrm{B}(c)},$$

with intensity $\lambda_j^{\mathrm{B}} = \widetilde{\lambda}_1 \widetilde{p}_{1j} + \widetilde{\lambda}_2 \widetilde{p}_{2j} + \widetilde{\lambda}_c$ for $j = 1, 2$. We further assume that Models I, A, C, and B have the same claim size distribution for each of the two classes. To compare Model B to Models I, A, and C, we select the parameters, $\widetilde{p}_{12}$, $\widetilde{p}_{21}$, $\widetilde{\lambda}_1$, $\widetilde{\lambda}_2$, and $\widetilde{\lambda}_c$ such that

$$\lambda_j^{\mathrm{B}} = \lambda_j^{\mathrm{I}} = \lambda_j^{\mathrm{A}} = \lambda_j^{\mathrm{C}}, \tag{10}$$

and hence the four models have the same expected aggregate claims.

We consider the following two cases of Model B:

Case 1: Let $\tilde{p}_{12} = 0.5p_{12}$, $\tilde{p}_{21} = 0.5p_{21}$, $\tilde{\lambda}_1 = \lambda_1$, $\tilde{\lambda}_2 = \lambda_2$, and $\tilde{\lambda}_c = 0.5\lambda_c$, which yields

$$\lambda_1^{\mathrm{B1}} = \lambda_1 + 0.5\lambda_2 p_{21} + 0.5\lambda_c \quad \text{and} \quad \lambda_2^{\mathrm{B1}} = \lambda_2 + 0.5\lambda_1 p_{12} + 0.5\lambda_c,$$

where the superscript 'B1' stands for Case 1 of Model B. The following notations also refer to Case 1 of Model B: $N_t^{\mathrm{B1}(1)}$, $N_t^{\mathrm{B1}(2)}$, S_t^{B1}, $S_t^{\mathrm{B1}(1)}$, $S_t^{\mathrm{B1}(2)}$, and U_t^{B1}.

Case 2: Let $\tilde{p}_{12} = p_{12}$, $\tilde{p}_{21} = p_{21}$ and $\tilde{\lambda}_c = \lambda_c$. From (10), $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are determined by

$$\lambda_1^{\mathrm{B2}} = \tilde{\lambda}_1 + \tilde{\lambda}_2 p_{21} + \lambda_c \quad \text{and} \quad \lambda_2^{\mathrm{B2}} = \tilde{\lambda}_1 p_{12} + \tilde{\lambda}_2 + \lambda_c$$

where the superscript 'B2' stands for Case 2 of Model B. Similarly we use the notation, $N_t^{\mathrm{B2}(1)}$, $N_t^{\mathrm{B2}(2)}$, S_t^{B2}, $S_t^{\mathrm{B2}(1)}$, $S_t^{\mathrm{B2}(2)}$ and U_t^{B2} in this case.

Let the adjustment coefficients of Models I, A, C, B1, and B2 be R_{I}, R_{A}, R_{C}, R_{B1}, and R_{B2}, respectively. Yuen and Wang (2002) showed that

$$R_{\mathrm{A}} < R_{\mathrm{C}} < R_{\mathrm{I}}. \tag{11}$$

In a similar manner we will compare R_{B1} and R_{B2} to R_{A}, R_{C}, and R_{I}. It follows from equation (5) that the claim size distribution of Model B1 is given by

$$\begin{aligned} F_{Z^{\mathrm{B1}}}(z) = \frac{1}{\lambda_{\mathrm{B1}}}&(\lambda_1(1-0.5p_{12})F_1(z) + \lambda_2(1-0.5p_{21})F_2(z) \\ &+ (0.5\lambda_1 p_{12} + 0.5\lambda_2 p_{21} + 0.5\lambda_c)F_1 * F_2(z)), \end{aligned} \tag{12}$$

where $\lambda_{\mathrm{B1}} = \lambda_1 + \lambda_2 + 0.5\lambda_c$. Denote the mgf of the aggregate claims process of Model B1 by $M_{\mathrm{B1}}(r)$. Let $H_{\mathrm{B1}}(r) = M_{\mathrm{B1}}(r) - 1$ and $H_{\mathrm{j}}(r) = M_{\mathrm{j}}(r) - 1$ for $j = 1, 2$. From (12), we have

$$\begin{aligned} \lambda_{\mathrm{B1}} H_{\mathrm{B1}}(r) = {} & \lambda_1 M_1(r) + \lambda_2 M_2(r) + 0.5\lambda_1 p_{12} M_1(r) H_2(r) \\ & + 0.5\lambda_2 p_{21} M_2(r) H_1(r) + 0.5\lambda_c M_1(r) M_2(r) \\ & - \lambda_1 - \lambda_2 - 0.5\lambda_c. \end{aligned} \tag{13}$$

As was mentioned in Yuen and Wang (2002), the aggregate claims processes of Models I, A, and C can be transformed to compound Poisson processes with claim-number processes having intensities $\lambda_{\mathrm{I}} =$

$\lambda_1+\lambda_2+2\lambda_c$, $\lambda_A = \lambda_1+\lambda_2$, and $\lambda_C = \lambda_1+\lambda_2+\lambda_c$, respectively. Of course, the three transformed claim size distributions are different from each other. For details of the forms of the three distributions see Yuen and Wang (2002).

In a manner similar to $M_{B1}(r)$ and $H_{B1}(r)$, we define $M_A(r)$, $M_C(r)$, $H_A(r)$, and $H_C(r)$.

From equations (3.7) and (3.8) of Yuen and Wang (2002), we have

$$\begin{aligned} \lambda_A H_A(r) &= \lambda_1 M_1(r) + \lambda_2 M_2(r) + \lambda_1 p_{12} M_1(r) H_2(r) \\ &\quad + \lambda_2 p_{21} M_2(r) H_1(r) - \lambda_1 - \lambda_2, \quad \text{and} \qquad (14) \\ \lambda_C H_C(r) &= \lambda_1 M_1(r) + \lambda_2 M_2(r) + \lambda_c M_1(r) M_2(r) \\ &\quad - \lambda_1 - \lambda_2 - \lambda_c. \qquad (15) \end{aligned}$$

Hence equations (14), (15), and (13) yield

$$\begin{aligned} \lambda_{B1} H_{B1}(r) &= \lambda_A H_A(r) - 0.5\lambda_c H_1(r) H_2(r) \\ &= \lambda_C H_C(r) + 0.5\lambda_c H_1(r) H_2(r). \end{aligned} \tag{16}$$

As $H_1(r)$ and $H_2(r)$ are greater than zero for $r > 0$, one can conclude that $\lambda_A H_A(r) > \lambda_{B1} H_{B1}(r) > \lambda_C H_C(r)$ for $r > 0$. This means that $\lambda_A H_A(r)$ ($\lambda_{B1} H_{B1}(r)$) intercepts the straight line $r\pi$ before $\lambda_{B1} H_{B1}(r)$ ($\lambda_C H_C(r)$) does. Therefore equations (16) and (11) imply that

$$R_A < R_{B1} < R_C < R_I. \tag{17}$$

We next consider Model B2 with the claim size distribution

$$\begin{aligned} F_{Z^{B2}}(z) = \frac{1}{\lambda_{B2}}(\tilde{\lambda}_1(1-p_{12})F_1(z) + \tilde{\lambda}_2(1-p_{21})F_2(z) \\ + (\tilde{\lambda}_1 p_{12} + \tilde{\lambda}_2 p_{21} + \lambda_c) F_1 * F_2(z)), \end{aligned} \tag{18}$$

where $\lambda_{B2} = \tilde{\lambda}_1 + \tilde{\lambda}_2 + \lambda_c$. Analogous to equation (13), we obtain

$$\begin{aligned} \lambda_{B2} H_{B2}(r) = \tilde{\lambda}_1(1-p_{12})M_1(r) + \tilde{\lambda}_2(1-p_{21})M_2(r) \\ + (\tilde{\lambda}_1 p_{12} + \tilde{\lambda}_2 p_{21} + \lambda_c) M_1(r) M_2(r) - \tilde{\lambda}_1 - \tilde{\lambda}_2 - \lambda_c, \end{aligned} \tag{19}$$

from equation (18). As equation (10) implies that

$$\lambda_1 = \tilde{\lambda}_1 + \tilde{\lambda}_2 p_{21} \qquad \text{and} \qquad \lambda_2 = \tilde{\lambda}_2 + \tilde{\lambda}_1 p_{12},$$

equation (19) can be rewritten as

$$\begin{aligned}\lambda_{B2}H_{B2}(r) = \lambda_1 M_1(r) + \lambda_2 M_2(r) + \lambda_c M_1(r)M_2(r)\\ + (\tilde{\lambda}_1 p_{12} + \tilde{\lambda}_2 p_{21})H_1(r)H_2(r) - \lambda_1 - \lambda_2 - \lambda_c.\end{aligned} \tag{20}$$

Using equations (15) and (20), we get

$$\lambda_{B2}H_{B2}(r) = \lambda_C H_C(r) + (\tilde{\lambda}_1 p_{12} + \tilde{\lambda}_2 p_{21})H_1(r)H_2(r). \tag{21}$$

Furthermore,

$$\lambda_A H_A(r) = \lambda_C H_C(r) + \lambda_c H_1(r)H_2(r), \tag{22}$$

because of equation (16) and

$$\lambda_c = \lambda_1 p_{12} = (\tilde{\lambda}_1 + \tilde{\lambda}_2 p_{21})p_{12} < \tilde{\lambda}_1 p_{12} + \tilde{\lambda}_2 p_{21}. \tag{23}$$

It follows from equations (21), (22), and (23) that

$$\lambda_{B2}H_{B2}(r) > \lambda_A H_A(r). \tag{24}$$

Similar to the derivation of (17), we reach

$$R_{B2} < R_A < R_{B1} < R_C < R_I, \tag{25}$$

due to (17) and (24). Inequality (25) can be easily explained by comparing the covariances of the two claim-number processes of the five models:

$$\begin{aligned}&\mathbb{C}\mathrm{ov}(N_t^{I(1)}, N_t^{I(2)}) = 0,\\ &< \mathbb{C}\mathrm{ov}(N_t^{C(1)}, N_t^{C(2)}) = \lambda_c t,\\ &< \mathbb{C}\mathrm{ov}(N_t^{B1(1)}, N_t^{B1(2)}) = 0.5t(\lambda_1 p_{12} + \lambda_2 p_{21} + \lambda_c) = 1.5\lambda_c t,\\ &< \mathbb{C}\mathrm{ov}(N_t^{A(1)}, N_t^{A(2)}) = (\lambda_1 p_{12} + \lambda_2 p_{21})t = 2\lambda_c t,\\ &< \mathbb{C}\mathrm{ov}(N_t^{B2(1)}, N_t^{B2(2)}) = (\tilde{\lambda}_1 p_{12} + \tilde{\lambda}_2 p_{21} + \lambda_c)t > 2\lambda_c t.\end{aligned}$$

As the correlation in Model A (thinning dependence only) is much stronger than that in Model C (common shock only), it is natural to expect that the impact of dependence in Model B (a mixture of thinning dependence and common shock) is somewhat smaller than that in Model A. Our results, however, show that it is not always the case. In fact, different sets of parameter values of Model B may lead to conflicting results. Hence, in order to obtain an accurate assessment of the underlying risk, selection of dependence structure and determination of parameter values are equally important.

4 Ultimate Ruin Probabilities

We will compare the ultimate ruin probabilities, $\Psi^{\mathrm{I}}(u)$, $\Psi^{\mathrm{C}}(u)$, $\Psi^{\mathrm{B1}}(u)$, $\Psi^{\mathrm{A}}(u)$, and $\Psi^{\mathrm{B2}}(u)$, of Models I, C, B1, A, and B2, respectively.

4.1 Exponential Claims

Here we assume that the claim amounts $X_i^{(j)}$ follow an exponential distribution with $F_j(z) = 1 - e^{-\theta_j z}$ for $j = 1, 2$. It is easy to check that

$$F_1 * F_2(z) = \frac{\theta_2}{\theta_2 - \theta_1} F_1(z) + \frac{\theta_1}{\theta_1 - \theta_2} F_2(z).$$

Hence, $F_Z(z)$ of (5) becomes

$$F_Z(z) = \frac{1}{\lambda}\left(\left(\lambda_1(1-p_{12}) + \frac{\theta_2(\lambda_1 p_{12} + \lambda_2 p_{21} + \lambda_c)}{\theta_2 - \theta_1}\right) F_1(z) + \left(\lambda_2(1-p_{21}) + \frac{\theta_1(\lambda_1 p_{12} + \lambda_2 p_{21} + \lambda_c)}{\theta_1 - \theta_2}\right) F_2(z)\right),$$

which is a mixed exponential distribution. In this case, the method introduced by Gerber (1979, Chapter 8, pages 116–118) allows us to calculate the exact value of $\Psi(u)$.

Let $\mathbb{E}(X_i^{(1)}) = \mu_1 = 1$ and $\mathbb{E}(X_i^{(2)}) = \mu_2 = 3$. We set $\lambda_1 = 5$, $\lambda_2 = 3$, $\lambda_c = 2$, $p_{12} = 2/5$, and $p_{21} = 2/3$ so that $\widetilde{\lambda}_1 = 45/11$ and $\widetilde{\lambda}_2 = 15/11$ in Model B2. The expected aggregate claims per unit time is 22 in each model. The constant premium rate is arbitrarily chosen as 24.2 with $\eta = 0.1$. The means and variances of the claim numbers and the aggregate claims are summarized as follows:

	Mean	Variance
$N_t^{\mathrm{I}(1)}, N_t^{\mathrm{A}(1)}, N_t^{\mathrm{B1}(1)}, N_t^{\mathrm{B2}(1)}, N_t^{\mathrm{C}(1)}$	$7t$	$7t$
$N_t^{\mathrm{I}(2)}, N_t^{\mathrm{A}(2)}, N_t^{\mathrm{B1}(2)}, N_t^{\mathrm{B2}(2)}, N_t^{\mathrm{C}(2)}$	$5t$	$5t$
$S_t^{\mathrm{I}(1)}, S_t^{\mathrm{A}(1)}, S_t^{\mathrm{B1}(1)}, S_t^{\mathrm{B2}(1)}, S_t^{\mathrm{C}(1)}$	$7t$	$14t$
$S_t^{\mathrm{I}(2)}, S_t^{\mathrm{A}(2)}, S_t^{\mathrm{B1}(2)}, S_t^{\mathrm{B2}(2)}, S_t^{\mathrm{C}(2)}$	$15t$	$90t$

Using the formulas developed in Section 2, the correlation coefficients (ρ) between the claim numbers and between the aggregate claims for each model are:

$$\rho(N_t^{A(1)}, N_t^{A(2)}) = 0.676, \quad \rho(N_t^{B1(1)}, N_t^{B1(2)}) = 0.507,$$
$$\rho(N_t^{B2(1)}, N_t^{B2(2)}) = 0.768, \quad \rho(N_t^{C(1)}, N_t^{C(2)}) = 0.338,$$
$$\rho(S_t^{A(1)}, S_t^{A(2)}) = 0.338, \quad \rho(S_t^{B1(1)}, S_t^{B1(2)}) = 0.254,$$
$$\rho(S_t^{B2(1)}, S_t^{B2(2)}) = 0.384, \quad \rho(S_t^{C(1)}, S_t^{C(2)}) = 0.169.$$

Table 1 displays the ultimate ruin probabilities for the five models. Notice that these results can be ordered as follows:

$$\Psi^{I}(u) < \Psi^{C}(u) < \Psi^{B1}(u) < \Psi^{A}(u) < \Psi^{B2}(u), \tag{26}$$

which is consistent with equation (25).

Table 1

$\Psi(u)$ **for Exponential Claims and** $\eta = 0.10$

u	$\Psi^{I}(u)$	$\Psi^{A}(u)$	$\Psi^{B1}(u)$	$\Psi^{B2}(u)$	$\Psi^{C}(u)$
0	0.9091	0.9091	0.9091	0.9091	0.9091
10	0.6128	0.6642	0.6527	0.6701	0.6403
30	0.2871	0.3559	0.3399	0.3644	0.3231
50	0.1346	0.1907	0.1770	0.1982	0.1630
70	0.0631	0.1022	0.0922	0.1078	0.0822
90	0.0295	0.0548	0.0480	0.0586	0.0415
110	0.0138	0.0294	0.0250	0.0319	0.0209
130	0.0065	0.0157	0.0130	0.0173	0.0106
150	0.0030	0.0084	0.0068	0.0094	0.0053
200	0.0005	0.0018	0.0013	0.0021	0.0010

4.2 Non-Exponential Claims

As it is generally difficult to obtain explicit expressions for the ultimate ruin probability for a compound Poisson model when the claim amounts are not exponential, we use simulations to get approximations for $\Psi(u)$ for non-exponential claim size distributions. We use two pairs of claim size distributions: (i) gamma and Weibull, and (ii) lognormal and Weibull distributions. In both cases, the parameters in the claim number processes are chosen to be $\lambda_1^{I} = 7$, $\lambda_2^{I} = 6$, $\lambda_1 = 5$, $\lambda_2 = 4$,

$\lambda_c = 2$, $p_{12} = 0.4$, and $p_{21} = 0.5$, which yields $\tilde{\lambda}_1 = 3.75$ and $\tilde{\lambda}_2 = 2.5$ in Model B2.

We now define the N-year ruin probability as

$$\Psi_N(u) = \mathbb{P}(T \le N | U_0 = u), \tag{27}$$

which, for large N, will be used as an approximation to $\Psi(u)$. It turns out that $N = 1,000$ is large enough to give reasonably accurate estimates of $\Psi(u)$. Also, the number of simulated realizations (sample paths) used is 1,000.[2] Our simulations are based on the fact that the claim interarrival times follow an exponential distribution.

Based on equation (4), the transformed claim amounts Z_i's can be generated using the following steps:

Step 1: Generate U from the uniform (0,1) distribution.

Step 2: If $U < \lambda_1(1 - p_{12})/\lambda$ or $U > (\lambda_1(1 - p_{12}) + \lambda_2(1 - p_{21}))/\lambda$, then generate $\hat{Y}_1$ from the distribution of $X_i^{(1)}$, else set $\hat{Y}_1 = 0$.

Step 3: If $U \ge \lambda_1(1 - p_{12})/\lambda$, then generate $\hat{Y}_2$ from the distribution of $X_i^{(2)}$, else set $\hat{Y}_2 = 0$.

Step 4: $Z = \hat{Y}_1 + \hat{Y}_2$.

Step 5: Return to Step 1 for another simulated Z value.

We provide two examples of the simulations.

Example 1 Consider the case where $X_i^{(1)}$ and $X_i^{(2)}$ are gamma and Weibull random variables, respectively, with pdfs

$$f_1(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, \quad \text{and} \tag{28}$$

$$f_2(x) = \frac{\tau (x/\omega)^\tau \exp(-(x/\omega)^\tau)}{x}, \tag{29}$$

with $\alpha = 0.5$, $\beta = 6$, $\omega = 1.5$, and $\tau = 0.5$. Thus, $\mathbb{E}(X_i^{(1)}) = \mathbb{E}(X_i^{(2)}) = 3$, $\mathbb{V}\text{ar}(X_i^{(1)}) = 18$,, and $\mathbb{V}\text{ar}(X_i^{(2)}) = 45$. The expected aggregate claims per unit time for each of the five models is 39 and the premium rate is set to be $k = 46.8$ with $\eta = 0.2$. The means and variances of the claim numbers and the aggregate claims are as follows:

[2]There are several sophisticated simulation methods for estimating $\Psi(u)$ for compound Poisson models such as using the Pollaczeck-Khinchine formula and importance sampling. We refer the readers to Asmussen (2000) and references therein for details of many of these methods.

	Mean	Variance
$N_t^{I(1)}, N_t^{A(1)}, N_t^{B1(1)}, N_t^{B2(1)}, N_t^{C(1)}$	$7t$	$7t$
$N_t^{I(2)}, N_t^{A(2)}, N_t^{B1(2)}, N_t^{B2(2)}, N_t^{C(2)}$	$6t$	$6t$
$S_t^{I(1)}, S_t^{A(1)}, S_t^{B1(1)}, S_t^{B2(1)}, S_t^{C(1)}$	$21t$	$189t$
$S_t^{I(2)}, S_t^{A(2)}, S_t^{B1(2)}, S_t^{B2(2)}, S_t^{C(2)}$	$18t$	$324t$

Table 2 displays estimates of $\Psi_N(20)$for various values of N. The standard errors of the estimates are shown in parentheses.[3] Notice that $\Psi_N(20)$ appears to be constant for $N \geq 1000$. Therefore, the approximation $\Psi(u) \approx \Psi_{1000}(u)$ is used in Tables 3 through 5.

Table 2

$\Psi_N(20)$ for Gamma and Weibull Claims and η = 0.20

N	$\Psi_N^I(u)$	$\Psi_N^A(u)$	$\Psi_N^{B1}(u)$	$\Psi_N^{B2}(u)$	$\Psi_N^C(u)$
200	0.4372	0.5243	0.5047	0.5456	0.4939
	(0.0209)	(0.0233)	(0.0216)	(0.0325)	(0.0179)
400	0.4376	0.5245	0.5048	0.5460	0.4941
	(0.0210)	(0.0234)	(0.0217)	(0.0325)	(0.0180)
600	0.4376	0.5246	0.5048	0.5463	0.4941
	(0.0210)	(0.0234)	(0.0217)	(0.0325)	(0.0180)
800	0.4376	0.5246	0.5048	0.5465	0.4941
	(0.0210)	(0.0234)	(0.0217)	(0.0325)	(0.0180)
1,000	0.4376	0.5246	0.5048	0.5466	0.4941
	(0.0210)	(0.0234)	(0.0217)	(0.0324)	(0.0180)
1,200	0.4376	0.5246	0.5048	0.5466	0.4941
	(0.0210)	(0.0234)	(0.0217)	(0.0324)	(0.0180)

Notes: Standard errors of estimates are shown in parentheses.

Table 3 shows the estimates $\hat{\Psi}^I(u)$, $\hat{\Psi}^C(u)$, $\hat{\Psi}^{B1}(u)$, $\hat{\Psi}^A(u)$, and $\hat{\Psi}^{B2}(u)$ for various values of u with $N = 1,000$. In line with what we observe in Section 4.1, the ultimate ruin probabilities for each value of u can be arranged in the same order as equation (26).

Example 2 Here $\ln(X_i^{(1)}) \sim N(\mu, \sigma^2)$ (i.e., $X_i^{(1)}$ is lognormal) with μ = 0.434044 and σ = 1.1528816 while $X_i^{(2)}$ has pdf given in equation (29) with ω = 0.902703 and τ = 0.4. Then, $E(X_i^{(1)}) = E(X_i^{(2)}) = 3$,

[3]Note that all standard errors shown in parentheses in Tables 2 to 5 are calculated from 100 simulated values of $\Psi_N(u)$.

Table 3

$\Psi_{1000}(u)$ for Gamma and Weibull Claims and $\eta = 0.20$

u	$\hat{\Psi}^{\mathrm{I}}(u)$	$\hat{\Psi}^{\mathrm{A}}(u)$	$\hat{\Psi}^{\mathrm{B1}}(u)$	$\hat{\Psi}^{\mathrm{B2}}(u)$	$\hat{\Psi}^{\mathrm{C}}(u)$
20	0.4376	0.5246	0.5048	0.5466	0.4941
	(0.0210)	(0.0234)	(0.0217)	(0.0324)	(0.0180)
30	0.3323	0.4267	0.4078	0.4524	0.3938
	(0.0196)	(0.0226)	(0.0203)	(0.0313)	(0.0179)
40	0.2591	0.3490	0.3326	0.3788	0.3166
	(0.0190)	(0.0206)	(0.0181)	(0.0293)	(0.0170)
50	0.2058	0.2881	0.2722	0.3182	0.2575
	(0.0163)	(0.0183)	(0.0160)	(0.0272)	(0.0152)
60	0.1646	0.2377	0.2249	0.2684	0.2101
	(0.0144)	(0.0171)	(0.0146)	(0.0264)	(0.0149)
70	0.1338	0.1982	0.1876	0.2275	0.1727
	(0.0133)	(0.0161)	(0.0125)	(0.0237)	(0.0141)
80	0.1106	0.1655	0.1561	0.1933	0.1436
	(0.0122)	(0.0141)	(0.0119)	(0.0232)	(0.0133)

Notes: Standard errors of estimates are shown in parentheses.

$\mathrm{Var}(X_i^{(1)}) = 25$, and $\mathrm{Var}(X_i^{(1)}) = 88.78$. Like Example 1, the expected aggregate claims per unit time is 39 for each model and $k = 46.8$. The means and variances of various quantities are given below:

	Mean	Variance
$N_t^{\mathrm{I}(1)}, N_t^{\mathrm{A}(1)}, N_t^{\mathrm{B1}(1)}, N_t^{\mathrm{B2}(1)}, N_t^{\mathrm{C}(1)}$	$7t$	$7t$
$N_t^{\mathrm{I}(2)}, N_t^{\mathrm{A}(2)}, N_t^{\mathrm{B1}(2)}, N_t^{\mathrm{B2}(2)}, N_t^{\mathrm{C}(2)}$	$6t$	$6t$
$S_t^{\mathrm{I}(1)}, S_t^{\mathrm{A}(1)}, S_t^{\mathrm{B1}(1)}, S_t^{\mathrm{B2}(1)}, S_t^{\mathrm{C}(1)}$	$21t$	$238t$
$S_t^{\mathrm{I}(2)}, S_t^{\mathrm{A}(2)}, S_t^{\mathrm{B1}(2)}, S_t^{\mathrm{B2}(2)}, S_t^{\mathrm{C}(2)}$	$18t$	$586.71t$

Table 4 displays estimates of $\Psi_N(20)$; we use $\Psi(u) \approx \Psi_{1000}(u)$ as $\Psi_N(20)$ again appears to be constant for $N \geq 1000$. Estimates of $\Psi(u)$ with different values of u are shown in Table 5. Not surprisingly, the results in Table 5 exhibit a pattern similar to those in Table 3. The results in Table 5 are generally higher than those in Table 3 mainly because the claim distributions used in Table 5 have heavier tails that make the model in Example 2 riskier.

In closing, the results shown in this section illustrate the important fact that modeling the dependence structure and estimating the parameter values are equally important in assessing the underlying risk.

Table 4

$\Psi_N(20)$ **for Lognormal and Weibull Claims and** $\eta = 0.20$

N	$\Psi_N^{\mathrm{I}}(u)$	$\Psi_N^{\mathrm{A}}(u)$	$\Psi_N^{\mathrm{B1}}(u)$	$\Psi_N^{\mathrm{B2}}(u)$	$\Psi_N^{\mathrm{C}}(u)$
200	0.4785	0.5353	0.5269	0.5548	0.5003
	(0.0269)	(0.0189)	(0.0193)	(0.0448)	(0.0262)
400	0.4787	0.5354	0.5270	0.5553	0.5010
	(0.0270)	(0.0188)	(0.0192)	(0.0444)	(0.0262)
600	0.4787	0.5354	0.5270	0.5554	0.5012
	(0.0270)	(0.0188)	(0.0192)	(0.0445)	(0.0262)
800	0.4787	0.5354	0.5270	0.5555	0.5013
	(0.0270)	(0.0188)	(0.0192)	(0.0445)	(0.0262)
1,000	0.4787	0.5354	0.5270	0.5555	0.5013
	(0.0270)	(0.0188)	(0.0192)	(0.0445)	(0.0262)

Notes: Standard errors of estimates are shown in parentheses.

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Table 5

$\Psi_{1000}(u)$ for Lognormal and Weibull Claims and $\eta = 0.20$

u	$\widehat{\Psi}^{I}(u)$	$\widehat{\Psi}^{A}(u)$	$\widehat{\Psi}^{B1}(u)$	$\widehat{\Psi}^{B2}(u)$	$\widehat{\Psi}^{C}(u)$
20	0.4787	0.5354	0.5270	0.5555	0.5013
	(0.0270)	(0.0188)	(0.0192)	(0.0445)	(0.0262)
30	0.4008	0.4533	0.4454	0.4750	0.4256
	(0.0248)	(0.0192)	(0.0205)	(0.0437)	(0.0263)
40	0.3370	0.3892	0.3806	0.4104	0.3635
	(0.0247)	(0.0194)	(0.0205)	(0.0414)	(0.0262)
50	0.2878	0.3361	0.3286	0.3564	0.3142
	(0.0257)	(0.0193)	(0.0190)	(0.0400)	(0.0258)
60	0.2470	0.2914	0.2837	0.3108	0.2742
	(0.0238)	(0.0183)	(0.0176)	(0.0371)	(0.0266)
70	0.2142	0.2542	0.2467	0.2720	0.2401
	(0.0227)	(0.0177)	(0.0168)	(0.0350)	(0.0266)
80	0.1857	0.2221	0.2153	0.2386	0.2120
	(0.0218)	(0.0161)	(0.0163)	(0.0322)	(0.0258)

Notes: Standard errors of estimates are shown in parentheses.

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