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ON THE SUMMABILITY AND GENERALIZED SUM OF A SERIES OF LEGENDRE POLYNOMIALS*

BY W. C. BRENKE

1. *Introduction.* The results obtained in this paper are as follows.

(A) *The series of Legendre polynomials* $\sum n^p X_n(x)$, *where* p *is a positive integer, is summable* (H, p) *for* $-1 < x < 1$, *and summable* $(H, p+1)$ *for* $-1 \leq x < 1$.

(B) *The generalized sum over the range* $-1 < x < 1$ *is*

$$\sum_1^\infty n^p X_n(x) = - \frac{1}{2(2y)^{p-1/2}} \begin{vmatrix} 2 & 0 & 0 & 0 & \dots & 0 & 1 \\ y & 2y & 0 & 0 & \dots & 0 & 1 \\ A_2^3 & y & 2y & 0 & \dots & 0 & 1 \\ A_3^4 & A_2^4 & y & 2y & \dots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{p-2}^{p-1} & A_{p-3}^{p-1} & \dots & y & & 2y & 1 \\ A_{p-1}^p & A_{p-2}^p & \dots & A_2^p & & y & 1 \end{vmatrix}$$

where

$$y = 1 - x; \quad A_t^p = {}_p C_t + (-1)^t {}_{p-1} C_t; \quad (p > 2).$$

2. *The Cases* $p=0, 1, 2$. We first obtain these results for $p=0, 1, 2$. Let p be a positive integer, $S_{n,p}$ the sum of the first n terms of the series $\sum n^p X_n(x)$, $S_{n,p}^{(p)}$ the p th Hölder mean of $S_{n,p}$, and $S^{(p)}$ the limit of this mean for $n \rightarrow \infty$.

The generating function of the Legendre polynomials gives us at once the sum of the convergent series

$$(1) \quad \sum_1^\infty X_n(x) = S^{(0)} = [2(1-x)]^{-1/2} - 1, \quad (-1 < x < 1).$$

We can readily find $S^{(1)}$ by use of the recursion formula

$$(2) \quad (2m+1)xX_m = (m+1)X_{m+1} + mX_{m-1},$$

* Presented to the Society, November 26, 1932.

which we write in the form

$$2mxX_m = (m + 1)X_{m+1} + (m - 1)X_{m-1} + X_{m-1} - xX_m.$$

Summing from $m = 1$ to $m = n$ we obtain, after some reductions,

$$2(x - 1)S_{n,1} = (1 - x)S_{n,0} + X_0 - X_1 + (n + 1)X_{n+1} - nX_n - X_n.$$

The first mean now gives

$$2(x - 1)S_{n,1}^{(1)} = (1 - x)S_{n,0}^{(1)} + (1 - x) + [(n + 1)X_{n+1} - X_1 - S_{n,0}]/n.$$

For $n \rightarrow \infty$ we then have

$$(1') \quad 2(x - 1)S^{(1)} = (1 - x)S^{(0)} + (1 - x).$$

To obtain $S^{(2)}$ we multiply (2) by m and arrange in the form

$$2m^2xX_m = u(m + 1)X_{m+1} + v(m - 1)X_{m-1} - mxX_m,$$

where u and v are quadratic functions of the indicated arguments. Summing and taking two successive mean values we obtain after some reductions

$$2(x - 1)S_{n,2}^{(2)} = (1 - x)S_{n,1}^{(2)} + S_{n,0}^{(2)} + 1 - (1/n) \sum_{r=1}^n [3S_{r,1} - r(r + 1)X_{r+1} + S_{r,0}]/r.$$

The last term vanishes for $n \rightarrow \infty$. We note also that the existence of the limit of any mean ensures the existence of the same limit for the higher means. Hence for $n \rightarrow \infty$ the preceding equation becomes

$$(1'') \quad 2(x - 1)S^{(2)} = (1 - x)S^{(1)} + S^{(0)} + 1.$$

3. *Proof of (A) and (B).* We shall now show that

$$(3) \quad 2(x - 1)S^{(p)} = (1 - x)S^{(p-1)} + \sum_{t=2}^{p-1} A_t^p S^{(p-t)} + S^{(0)} + 1, \quad (p > 2);$$

provided that $S^{(r)}$ exists for $r = 0, 1, 2, \dots, p - 1$, and A_t^p , expressed in terms of binomial coefficients, is

$$A_t^p = {}_p C_t + (-1)_{p-1}^t C_t.$$

Assuming the existence of the means of order less than p we first prove the following lemma.

LEMMA. *The means of order p of $n^p X_n$ and of $(n+1)^p X_{n+1}$ have the limit zero for $n \rightarrow \infty$, when $-1 < x < 1$.*

To prove this we multiply (2) by m^{p-1} , express the coefficients of X_{m+1} and X_{m-1} on the right in powers of $m+1$ and $m-1$, respectively, sum from $m=1$ to $m=n$, and write the result in the form

$$(4) \quad 2(x-1)S_{n,p} = (1-x)S_{n,p-1} + \sum_{t=2}^{p-1} A_t^p S_{n,p-t} + S_{n,0} \\ + X_0 + n^{p-1}(n+1)X_{n+1} - (n+1)^p X_n.$$

Successive application of this formula enables us to write

$$2(x-1)S_{n,p} = w_n + u_p(n+1)X_{n+1} + v_p(n)X_n,$$

where $w_n = A_p S_{n,0} + B_p X_0$, A_p and B_p being independent of n , and $u_p(n+1)$, $v_p(n)$ are polynomials of degree p in the indicated arguments. Denoting the mean of order k of $n^p X_n$ by $M_{n,p}^{(k)}$, we have

$$M_{n,p}^{(1)} = S_{n,p}/n,$$

which, by use of the preceding equation, we may write in the form

$$2(x-1)M_{n,p}^{(1)} = w_n^{(1)} + u_{p-1}(n+1)X_{n+1} + v_{p-1}(n)X_n,$$

where $w_n^{(1)} = (w_n + C_p X_{n+1} + D_p X_n)/n$, C_p and D_p being independent of n , and where the coefficients of the last two terms are polynomials of the indicated degree and arguments. We note that $w_n^{(1)}$ and all of its means $\rightarrow 0$ when $n \rightarrow \infty$. Proceeding in this way we have

$$2(x-1)M_{n,p}^{(2)} = w_n^{(2)} + u_{p-2}(n+1)X_{n+1} + v_{p-2}(n)X_n,$$

and finally

$$2(x-1)M_{n,p}^{(p)} = w_n^{(p)} + u_0(n+1)X_{n+1} + v_0(n)X_n,$$

where each $w_n^{(r)}$ and its means $\rightarrow 0$ when $n \rightarrow \infty$. The means of $(n+1)^p X_{n+1}$ may be treated similarly.

COROLLARY 1. *The mean of order p of $u_p(n+1)X_{n+1}$ and of $v_p(n)X_n$ approach zero when $n \rightarrow \infty$, for $-1 < x < 1$.*

COROLLARY 2. *For $x = -1$ the mean of order $p+1$ of each of the preceding expressions vanishes when $n \rightarrow \infty$. This follows from $X_n(-1) = (-1)^n$.*

Now (3) is obtained at once by taking the limit of the mean of order p of (4). But the values of $S^{(0)}$, $S^{(1)}$, $S^{(2)}$ already found show that (3) holds for $p = 3$. Hence it holds for positive integral values of $p > 2$.

The result under (B) is obtained by expressing each $S^{(r)}$, ($r = 1, 2, \dots, p$), in terms of the sums of lower order by use of (1'), (1''), (3) and solving this system of equations for $S^{(p)}$.

When $x = -1$, Corollary 2 shows that the series $\sum (-1)^n n^p$ is summable ($H, p+1$) and a new form is obtained for its sum by putting $y = 2$ in the formula under (B).

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ERRATA

This Bulletin, volume 37 (1931), pages 759–765:

On page 759, line 12, for $P_1 \equiv P, P_2, \dots$ read $P \equiv P_1, P_2, \dots$.

On page 764, line 9, second parenthesis, for (x_1x_2, x_2, x_1x_4) read (x_1x_2, x_2^2, x_1x_4) .

On page 764, line 8 from the bottom, second parenthesis, for $(z_2, z_3, \epsilon_3z_4)$ read $(\epsilon z_2, z_3, \epsilon^3z_4)$.

W. R. HUTCHERSON

This Bulletin, volume 39 (1933), p. 589:

Lines 13–14, omit the words: *one point of inflexion*; and add the sentence: *A point of inflexion lies at infinity on each bisector of the angles formed by adjacent cuspidal tangents.*

D. C. DUNCAN