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A Modern Approach to Modeling Insurances on Two Lives

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Abstract‡

The analysis of life insurance contracts on two lives using the traditional deterministic approach has been an important part of actuarial education for the past fifty years or more. Recently there has been a shift from this deterministic approach to one using a more modern stochastic approach involving the future lifetime random variable. In this paper we will look at the problem using multiple-state models. In our view this approach allows a deeper analysis than either the traditional or the random future lifetime ones.

Key words and phrases: *multiple-state models, Kolmogorov equations*

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1 Introduction

Insurance for multiple lives is largely confined to those associated with married couples. So, throughout this paper, we consider a married couple consisting of a husband age x and a wife age y at some initial time $t = 0$, which, for notational convenience, is written as (x, y). The time $t = 0$ usually corresponds in practice to the start of an insurance contract. For simplicity we ignore the possibility of divorce.

Traditionally, actuaries have calculated the premiums for joint-life insurance and annuity contracts using the formula for the joint force of mortality, $\mu_{xy}(t)$,

$$\mu_{xy}(t) = \mu_x(t) + \mu_y(t) \tag{1}$$

where $\mu_x(t)$ and $\mu_y(t)$ are the force of mortality for single lives (x) and (y) at ages $x+t$ and $y+t$, respectively; see, for example, Jordan (1967, Chapter 9), Neill (1977, Chapter 7), and Bowers et al. (1997, Chapter 9). The calculation of the premiums for last survivor insurance and annuity contracts then in addition uses the following standard formulas

$$\overline{A}_{\overline{xy}} = \overline{A}_x + \overline{A}_y - \overline{A}_{xy} \quad \text{and} \quad \overline{a}_{\overline{xy}} = \overline{a}_x + \overline{a}_y - \overline{a}_{xy},$$

which relate the last-survivor functions to those for joint-lives and single lives. The deterministic approach of Jordan and Neill and the random approach of Bowers et al. assume that the two lives are statistically independent.

Several authors have studied the impact of dependence between insured lives; see, for example, Carrière and Chan (1986); Carrière (1994); Frees, Carrière, and Valdez (1996); Dhaene and Goovaerts (1997); Frees and Valdez (1998); Denuit and Cornet (1999); and Youn, Shermyakin, and Herman (2002). Of great interest and relevance to us is the paper by Youn, Shermyakin, and Herman (2002), which shows that we can also derive last survivor insurance and last survivor annuity formulas using more general future lifetime random variables.

The object of this paper is to show how we can use multiple-state models to define more precisely the assumptions required for the standard formulas to apply. We also indicate how we might price insurance and annuity contracts where these assumptions do not apply.

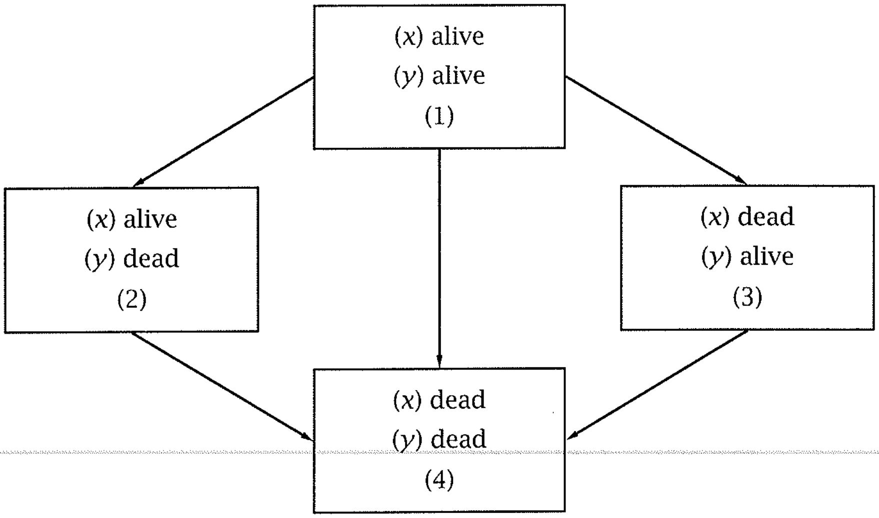

Figure 1: Generalized Mortality Model for Two Lives, (x) and (y)

2 A Model for Two Lives

A well known model for the forces of mortality depending on marital status was proposed by Norberg (1989) as follows:

State 1 = Both husband (x) and wife (y) are alive;
State 2 = Husband (x) is dead and wife (y) is alive;
State 3 = Husband (x) is alive and wife (y) is dead;
State 4 = Both husband (x) and wife (y) are dead.

Norberg regarded the future development of the marital status for the couple as a Markov process. We will generalize Norberg's model by assuming the transition intensities depend on the age at which the previous transition occurred, thus removing the Markov property. We also include transition directly from stage 1 to stage 4. Figure 1 illustrates our generalized model.

The following notations are used for $i, j = 1, 2, 3, 4$:

$p_{xy}^{(ii)}(t,s)$ = Probability that the couple (xy) stays in state i for at least t years (i.e., up to time $t+s$) given they entered state i at time s, for $s,t \geq 0$;

$p_{xy}^{(ij)}(t,s)$ = Probability that the couple (xy) is in state j at time $t+s$ given they entered state i at time s, for $s,t \geq 0$; and

$\mu_{xy}^{(ij)}(t,s)dt$ = Probability that the couple (xy) moves from state i to state j in $(t+s, t+s+dt)$ given they entered state i at time s and remained in state i up to time $t+s$, for $s,t \geq 0$ and infinitesimally small dt.

For convenience we define

$$\mu_{xy}^{(i)}(t,s) = \sum_{\substack{j=1 \\ j\neq i}}^{4} \mu_{xy}^{ij}(t,s),$$

which implies

$$\mu_{xy}^{(i)}(t,s) = \begin{cases} \mu_{xy}^{(12)}(t,s) + \mu_{xy}^{(13)}(t,s) + \mu_{xy}^{(14)}(t,s) & \text{if } i = 1; \\ \mu_{xy}^{(24)}(t,s) & \text{if } i = 2; \\ \mu_{xy}^{(34)}(t,s) & \text{if } i = 3. \end{cases}$$

Our model takes into account the empirical observations that, where there is some connection between the two lives, the mortality of one of the pair depends on whether the other is alive or dead and, if the latter, when death occurred.[1] One unusual feature of our model is the inclusion of transitions from state 1 directly to state 4. This allows for the possibility that the two lives die simultaneously in, for example, a car accident or a plane crash.

It is immediately seen that the basic functions needed for joint-life insurances and annuities are

$$\mu_{xy}(t) \equiv \mu_{xy}^{(1)}(t,0)$$
$$_tp_{xy} \equiv p_{xy}^{(11)}(t,0) = e^{-\int_0^t \mu_{xy}^{(1)}(r,0)dr}.$$

Although in practice they do not often arise, we can also use our generalized model to price contingent insurance contracts where a payment

[1]See AES Course Notes, Subject D, Unit 8, Institute of Actuaries, London, 1994.

is made on the death of (x) if that occurs before the death of (y) and vice versa and reversionary annuities. Using standard actuarial notation, for example, we have

$$\overline{A}_{\overset{1}{xy}:\overline{n|}} = \int_0^n v^t p_{xy}^{(11)}(t,0)\mu_{xy}^{(13)}(t,0)\mathrm{d}t,$$

which is the net single premium for a contingent insurance that pays \$1 on the death of (x) if that occurs before the death of (y) and within n years.

In a similar manner, the net single premium for an n-year term insurance contract that pays \$1 if (x) and (y) die simultaneously, i.e., on a transition from state 1 to state 4, is given as:

$$\int_0^n v^t p_{xy}^{(11)}(t,0)\mu_{xy}^{(14)}(t,0)\mathrm{d}t$$

for $n \geq 0$. There is no standard actuarial notation for this, and, to the best of the authors' knowledge, no insurance company offers such a contract.

Given the absence of the Markov property, the traditional Chapman-Kolmogorov equations[2] cannot be used for the transition probabilities. As transitions to previous states are not allowed and there are only four states, however, our analysis can be simplified by assuming the first death occurs at time s and the second death at time $s+t$. Thus, the net single premium for a last survivor n-year term insurance is

$$\begin{aligned}
\overline{A}_{\overset{1}{\overline{xy}}:\overline{n|}} = & \int_0^n v^s p_{xy}^{(11)}(s,0)\mu_{xy}^{(12)}(s,0)\int_0^{n-s} v^t p_{xy}^{(22)}(t,s)\mu_{xy}^{(24)}(t,s)\mathrm{d}t\,\mathrm{d}s \\
&+ \int_0^n v^s p_{xy}^{(11)}(s,0)\mu_{xy}^{(13)}(s,0)\int_0^{n-s} v^t p_{xy}^{(33)}(t,s)\mu_{xy}^{(34)}(t,s)\mathrm{d}t\,\mathrm{d}s \\
&+ \int_0^n v^s p_{xy}^{(11)}(s,0)\mu_{xy}^{(14)}(s,0)\mathrm{d}s,
\end{aligned}$$

and the net single premium for a reversionary annuity is

$$\overline{a}_{x|y} = \int_0^\infty v^s p_{xy}^{(11)}(s,0)\mu_{xy}^{(13)}(s,0)\int_0^\infty \overline{a}_{\overline{t|}} p_{xy}^{(33)}(t,s)\mu_{xy}^{(34)}(t,s)\mathrm{d}t\,\mathrm{d}s.$$

[2]See, for example, Cox and Miller (1965, Chapter 4.1) or Taylor and Karlin (1994, Chapter 6.3) for more on Chapman-Kolmogorov equations.

3 Practical Simplifications

One problem with our generalized model is that, in practice, there will be a lack of adequate data to provide estimates of all the transition intensities required by this model. As a result, our model may be impractical to implement. We therefore need to introduce a set of simplifying assumptions that are intended to facilitate estimating these intensities. We will prove that one consequence of our assumptions is that the generalized model will yield results that are consistent with those produced by the independence assumption of the traditional or random future lifetime approaches. To this end we let $T(x)$ and $T(y)$ denote the random future lifetime of (x) and (y), respectively.

Assumption 1. *The events* $\{T(x) \in (t, t+\delta t)\}$ *and* $\{T(y) \in (t, t+\delta t)\}$ *are independent for all* $t \geq 0$ *and* δt *is infinitesimally small.*

Assumption 1 implies that

$$\mathbb{P}\left[T(x) \in (t, t+\delta t\} \bigcap T(y) \in (t, t+\delta t)\right] \\ = \mathbb{P}[T(x) \in (t, t+\delta t)]\,\mathbb{P}[T(y) \in (t, t+\delta t)].$$

Using the transition probabilities we have

$$p_{xy}^{(11)}(t,0)\mu_{xy}^{(14)}(t,0)\delta t = {}_tp_x \mu_x^{(*)}(t)\delta t \times {}_tp_y \mu_y^{(*)}(t)\delta t \tag{2}$$

where ${}_tp_x$ and $\mu_x^{(*)}(t)$ are the marginal survival function and force of mortality of $T(x)$ and ${}_tp_y$ and $\mu_y^{(*)}(t)$ are the marginal survival function and force of mortality of $T(y)$. If we assume that x and y are less than the oldest possible age ω, then these marginal survival function and force of mortality will be positive and finite for $t < \min(\omega - x, \omega - y)$. Dividing both sides of equation (2) by δt and then let $\delta t \to 0$, we obtain

$$\mu_{xy}^{(14)}(t,0) \equiv 0 \quad \text{for} \quad t \geq 0,$$

i.e., transition from state 1 to state 4 is not possible.

For pricing joint-life insurances and annuities we can then use the simplification

$$\mu_{xy}^{(1)}(t,0) = \mu_x^{(*)}(t) + \mu_y^{(*)}(t) \tag{3}$$

which implies

$$p_{xy}^{(11)}(t,0) \equiv {}_tp_x \times {}_tp_y. \tag{4}$$

Thus, we get the traditional actuarial independence assumption of Jordan (1967), Neill (1977), and Bowers et al. (1997).

Assumption 2. *For all $t \geq 0$ and δt is infinitesimally small, the probability that (x) or (y) dies in time period $(t, t+\delta t)$ does not depend on whether the other is alive or dead at t, i.e.,*

$$\mathbb{P}\left[T(x) \in (t, t+\delta t) | \{T(y) \leq t \bigcup T(y) > t\}\right] = \mathbb{P}[T(x) \in (t, t+\delta t)]$$
$$\mathbb{P}\left[T(y) \in (t, t+\delta t) | \{T(x) \leq t \bigcup T(x) > t\}\right] = \mathbb{P}[T(y) \in (t, t+\delta t)].$$

On the basis of Assumption 2 we can state that, for $0 \leq s \leq t$,

$$\mu_{xy}^{(13)}(t, 0) \equiv \mu_{xy}^{(24)}(t-s, s) \equiv \mu_x^{(**)}(t) \tag{5}$$

and

$$\mu_{xy}^{(12)}(t+s, 0) \equiv \mu_{xy}^{(34)}(t, s) \equiv \mu_y^{(**)}(t). \tag{6}$$

Equations (5) and (6) constitute the independence assumption of Youn, Shermyakin, and Herman (2002).

If Assumptions 1 and 2 jointly apply, it is clear from the description of our generalized model that we must have

$$\mu_x^{(*)}(t) \equiv \mu_x^{(**)}(t) \quad \text{and} \quad \mu_y^{(*)}(t) \equiv \mu_y^{(**)}(t).$$

We assert that Assumptions 1 and 2 are both necessary if we are to have independence of the two lives. Assumption 1 clearly does not imply Assumption 2; see Youn, Shermyakin, and Herman (2002). Assuming that only Assumption 2 applies does not lead to any contradictions. Assumptions 1 and 2 are sufficient in themselves to derive the simple formulas used in practice. They require the estimation of transition intensities $\mu_x^{(**)}(t)$ and $\mu_y^{(**)}(t)$, which relate to joint lives, i.e., lives that have taken out a contract jointly.

In situations where there are not adequate experience data on joint lives, it is usual to make use also of the following assumption:

Assumption 3. *The mortality of each of the individual lives (x) and (y) in the pair (x, y) is identical to that of the single lives (x) and (y), respectively.*

Assumptions 3 is a different type of assumption from Assumptions 1 and 2, which are concerned with the relationship between the mortality of the two joint lives. Assumption 3 simply equates the numerical values of the transition intensities for each of the joint lives to those for individual lives, where we give the phrase "individual lives" the meaning commonly applied to it in life insurance mortality investigations. This allows us to replace the joint life intensities by those relating to individual lives, i.e.,

$$\mu_x^{(**)}(t) \equiv \mu_x(t) \quad \text{and} \quad \mu_y^{(**)}(t) \equiv \mu_y(t). \tag{7}$$

In practice reliable estimates of the transition intensities for individual lives are almost always available.

Applying all three assumptions[3] to our generalized model we obtain Norberg's (1989) Markov model with transition intensities depending only on current age. The Chapman-Kolmogorov forward equations are

$$\begin{aligned}
\frac{\partial p_{xy}^{(11)}(t,0)}{\partial t} &= -\left(\mu_x(t) + \mu_y(t)\right) p_{xy}^{(11)}(t,0) \\
\frac{\partial p_{xy}^{(12)}(t,0)}{\partial t} &= p_{xy}^{(11)}(t,0)\mu_y(t) - p_{xy}^{(11)}(t,0)\mu_x(t) \\
\frac{\partial p_{xy}^{(13)}(t,0)}{\partial t} &= p_{xy}^{(11)}(t,0)\mu_x(t) - p_{xy}^{(13)}(t,0)\mu_y(t)
\end{aligned}$$

which yield the solutions

$$\begin{aligned}
p_{xy}^{(11)}(t,0) &= e^{-\int_0^t (\mu_x(s)+\mu_y(s))ds} = {}_tp_{xy} \\
p_{xy}^{(12)}(t,0) &= e^{-\int_0^t \mu_x(s)ds}\left(1 - e^{-\int_0^t \mu_y(s)ds}\right) = {}_tp_x\,{}_tq_y \\
p_{xy}^{(13)}(t,0) &= e^{-\int_0^t \mu_y(s)ds}\left(1 - e^{-\int_0^t \mu_x(s)ds}\right) = {}_tp_y\,{}_tq_x.
\end{aligned}$$

It follows that the net single premium of the last survivor annuity is:

$$\begin{aligned}
\overline{a}_{\overline{xy}} &= \int_0^\infty v^t \left[p_{xy}^{(11)}(t,0) + p_{xy}^{(12)}(t,0) + p_{xy}^{(13)}(t,0)\right] dt \\
&= \int_0^\infty v^t \left[{}_tp_x + {}_tp_y - {}_tp_{xy}\right] dt \\
&= \overline{a}_x + \overline{a}_y - \overline{a}_{xy}.
\end{aligned}$$

[3] If we do not make use of Assumption 3, the derivation is the same, but with the joint life intensities replacing those relating to individual lives.

4 Closing Comments

We have indicated how we can price life insurances involving two lives using a generalized multi-state model. By introducing a set of clearly defined assumptions we have shown that using our model we can also derive the standard formulas traditionally used for pricing joint-life and last-survivor contracts. These assumptions are unrealistic, however. Thus, if they are used in practice, care must be taken in deciding whether any premiums calculated using these formulas are adequate.

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