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PACKINGS AND REALIZATIONS OF DEGREE SEQUENCES WITH SPECIFIED
SUBSTRUCTURES

by

Tyler Seacrest

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PACKINGS AND REALIZATIONS OF DEGREE SEQUENCES WITH SPECIFIED SUBSTRUCTURES

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This thesis focuses on the intersection of two classical and fundamental areas in graph theory: graph packing and degree sequences. The question of packing degree sequences lies naturally in this intersection, asking when degree sequences have edge-disjoint realizations on the same vertex set. The most significant result in this area is Kündu's k -Factor Theorem, which characterizes when a degree sequence packs with a constant sequence. We prove a series of results in this spirit, and we particularly search for realizations of degree sequences with edge-disjoint 1-factors.

Perhaps the most fundamental result in degree sequence theory is the Erdős-Gallai Theorem, characterizing when a degree sequence has a realization. After exploring degree sequence packing, we develop several proofs of this famous theorem, connecting it to many other important graph theory concepts.

We are also interested in locating edge-disjoint 1-factors in dense graphs. Before tackling this question, we build on the work of Katerinis to find the largest k such that a graph has a k -factor, where the value of k depends on the minimum degree of the graph. This gives an upper bound on the number of edge-disjoint 1-factors.

The question of finding edge-disjoint 1-factors leads us to a conjecture of Bollobás and Scott about finding spanning balanced bipartite subgraphs with vertices of high degree. We first prove a degree-sequence version of the Bollobás-Scott Conjecture which we apply to the question of edge-disjoint 1-factors. We then generalize and prove an approximate

version of the conjecture, yielding balanced partitions with many edges going to each part. This version has many applications, including finding edge-disjoint 1-factors and edge-disjoint Hamiltonian cycles.

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DEDICATION

To Debbie.

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Chapter 1

Introduction

This thesis focuses on the intersection of two classical and fundamental areas in graph theory: graph packing and degree sequences. The question of packing degree sequences naturally lies in this intersection. Following the spirit of Kundu’s k -Factor Theorem, we prove a series of results about degree sequence packing, and we especially search for realizations of degree sequences with edge-disjoint 1-factors. This leads us to many fruitful areas related to a conjecture of Bollobás and Scott about finding spanning balanced bipartite subgraphs with vertices of high degree. First we prove a degree-sequence version of the Bollobás–Scott Conjecture, and apply the result to the question of edge-disjoint 1-factors. We then generalize an approximate version of the conjecture to find balanced partitions with many edges going to each part. Our theorem has many applications, including finding edge-disjoint Hamiltonian cycles.

Graph packing asks whether two graphs G and H can be drawn on the same set of vertices without edges overlapping. In other words, is G a subgraph of \overline{H} ? A fundamental result in graph packing is due to Catlin [9] and independently Sauer and Spencer [55], who gave a sufficient condition for two graphs G and H to pack based on the maximum degrees $\Delta(G)$ and $\Delta(H)$.

Theorem 1 (Catlin [9], Sauer and Spencer [55]). *If G and H are graphs on n vertices and*

$$2\Delta(G) \cdot \Delta(H) \leq n,$$

then G and H pack.

Catlin [9] and independently Bollobás and Eldridge [3] conjectured a strengthening of the above result that has driven much work in the area.

Conjecture 2 (Bollobas and Eldridge [3], Catlin [9]). *If G and H are graphs on n vertices and*

$$(\Delta(G) + 1) \cdot (\Delta(H) + 1) \leq (n + 1),$$

then G and H pack.

Another classical area of graph theory is the study of degree sequences. The degrees of the vertices of a graph form its *degree sequence*. Given a sequence π , a graph G *realizes* π if π is the degree sequence of G . The most fundamental question in the area is characterizing when a sequence has a realization. A recursive algorithm solution to this was developed by Havel [36] and independently by Hakimi [33].

Theorem 3 (Havel [36], Hakimi [33]). *A sequence $\pi = (d_1 \geq \dots \geq d_n)$ is graphic if and only if the sequence $(d_2 - 1, d_3 - 1, \dots, d_{d_1} - 1, d_{d_1+1}, \dots, d_n)$ is graphic.*

Repeated application of the theorem either yields a sequence with negative numbers (which is not graphic) or an empty sequence (which is graphic). A non-recurrence condition for graphicality was given by Erdős and Gallai [24].

Theorem 4 (Erdős and Gallai [24]). *A sequence $\pi = (d_1 \geq \dots \geq d_n)$ is graphic if and only if*

for all $k = 1, \dots, n$ we have

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min(k, d_i).$$

Since Erdős and Gallai's original argument, this extremely fundamental theorem has had many proofs, including [11], [59], [56], and [62]. In Chapter 6, we give three original proofs of the Erdős-Gallai Theorem, making connections between this theorem and the Gale-Ryser criterion for bigraphic sequences and the Havel-Hakimi theorem.

Combining these two classical areas, we arrive at the the question of packing degree sequences. Given two graphic sequences $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, \dots, b_n)$, do there exist realizations A and B of α and β that pack? Notice this implies a two-step process of first choosing realizations A and B , then then choosing a permutation of the vertices of B so that it lies in \overline{A} . We could instead fix the permutation and require that A and B pack in such a way that vertex i receives A -degree a_i and B -degree b_i . We show in Chapter 3 that the former question is a special case of the latter: if two degree sequences pack, then they pack when α is weakly decreasing and β is weakly increasing. Thus, we focus on the case when the permutation is fixed.

The first result along these lines is due to Kundu [43], who characterized when a sequence packs with a constant sequence.

Theorem 5 (Kundu [43]). *Graphic sequences $\alpha = (a_1, \dots, a_n)$ and $\beta = (k, k, \dots, k)$ pack if and only if the sequence $\alpha + k = (a_1 + k, a_2 + k, \dots, a_n + k)$ is graphic.*

It is usually formulated in terms of finding a realization of a degree sequence with a spanning k -regular subgraph, or k -factor.

Theorem 6 (Kundu [43]). *The graphic sequence $\pi = (d_1, \dots, d_n)$ has a realization containing a k -factor if and only if the sequence $\pi - k = (d_1 - k, d_2 - k, \dots, d_n - k)$ is graphic.*

Lovász [44] proved the result for $k = 1$. Chen [10] gave a beautiful proof of Kundu's Theorem, which we repeat in Chapter 3 since it is an inspiration for many parts of this thesis.

Recently, Dürr, Guíñez and Matamala [20] proved that the problem of deciding whether two sequences pack is NP-complete, even in the case of packing bipartite degree sequences (see Section 3.3 for the definition of bipartite degree sequences). This makes a polynomial characterization extremely unlikely, but makes any special cases or partial results even more interesting. We have several results of this flavor in Chapter 3.

A natural extension to Kundu's theorem would be characterizing when a sequence packs with two constant sequences. This would be a consequence of a conjectured generalization of Kundu's Theorem due to Brualdi [5] and independently Busch, Ferrara, Hartke, Jacobson, Kaul and West [7].

Conjecture 7 (Brualdi [5], Busch et al. [7]). *For n even, let $\pi = (d_1, \dots, d_n)$ and $\pi - k = (d_1 - k, \dots, d_n - k)$ be graphic sequences. Then π has a realization containing k edge-disjoint 1-factors.*

In other words, given the same hypothesis of Kundu's Theorem, they conjecture that one can find a realization with k edge-disjoint 1-factors. This can be phrased as a degree sequence packing question: when does a sequence α pack with k copies of the sequence $(1, 1, \dots, 1)$? Busch et al. proved several special cases, including

Theorem 8 (Busch et al. [7]). *For n even, let $\pi = (d_1, \dots, d_n)$ and $\pi - k = (d_1 - k, \dots, d_n - k)$ be graphic sequences. Then π has a realization containing edge-disjoint copies of a $k - 2$ factor and two 1-factors.*

In Section 3.4, we prove a result in a similar vein where we show π has a realization with edge-disjoint copies of a $k - 1$ factor and a particular 1-factor. Busch et al. also noted that the conjecture must be true for sequences with a large minimum value.

Theorem 9 (Busch et al. [7]). *For n even, let $\pi = (d_1, \dots, d_n)$ be a sequence with minimum element at least $n/2 + k - 2$. Then π has a realization with k edge-disjoint 1-factors.*

In Chapter 4, we prove that if a graphic sequence π has minimum at least $n/2 + 2$, then it has a realization with $n/8$ edge-disjoint 1-factors, and the higher the degree beyond $n/2$ the more 1-factors that are obtained. Our result is stronger than Theorem 9 in all but a few fringe cases.

Our method involves first finding a spanning balanced bipartite subgraph, or a *bisection*. Bollobás and Scott [4] put forth many intriguing partition conjectures, including

Conjecture 10 (Bollobás and Scott [4]). *Every graph G contains a bisection H such that for all vertices v ,*

$$\deg_H(v) \geq \left\lfloor \frac{\deg_G(v)}{2} \right\rfloor.$$

If true, this conjecture would be tight: see Section 4.2. In Chapter 4, we prove a degree sequence version of this conjecture which is very nearly tight. This serves the dual purpose of proving an interesting special case of Conjecture 10 and allowing us to improve Theorem 9.

We are also interested in many of these questions in the non-degree-sequence setting. In Chapter 5, we prove an approximate version of the Bollobás-Scott conjecture using probabilistic methods, proving the result up to an error term. Independently, Bush [8] proved a very similar but slightly weaker result. We also extend this result to finding a (nearly) balanced partition with a prescribed number of parts where each vertex has many neighbors in each part. In addition to progress on the Bollobás-Scott conjecture, we again apply these theorems to the problem of finding edge-disjoint 1-factors, and also to the problem of finding edge-disjoint Hamiltonian cycles.

The question of finding edge-disjoint 1-factors in a fixed dense graph has not received much attention. For our purposes, “dense” means minimum degree at least $n/2$. Two

closely related questions that have received attention are finding the largest k such that a dense graph has a k -factor and finding the largest number of edge-disjoint Hamiltonian cycles in a dense graph. Katerinis [40] and independently Egawa and Enomoto [21] proved the following k -factor result.

Theorem 11 (Katerinis [40], Egawa and Enomoto [21]). *A graph of minimum degree $n/2$ has a k -factor for any*

$$k \leq (n + 5)/4$$

with kn even. This result is best possible in the sense that there exist infinitely many values n and graphs of minimum degree $n/2$ with no k -factor for $k = (n + 6)/4$.

In Chapter 7, we extend this result by showing that one can achieve a k -factor for a much larger value of k if the minimum degree is larger than $n/2$. Independently, Christofides, Kühn, and Osthus [12] proved the same result using similar methods.

Nash-Williams [50] proved

Theorem 12 (Nash-Williams [50]). *If a graph G has minimum degree $n/2$, then G contains at least $\lfloor 5n/224 \rfloor$ edge-disjoint Hamiltonian cycles.*

Nash-Williams conjectured this was far from best possible, and noted that constructions exist showing the best possible value to be approximately $n/8$. Using Szemerédi's Regularity Lemma, Christofides, Kühn, and Osthus [12] proved an approximate version that achieves this upper bound.

Theorem 13 (Christofides et al. [12]). *For every $\epsilon > 0$, if n is sufficiently large and $\delta \geq (1/2 + \epsilon)n$, then G contains $n/8$ Hamiltonian cycles.*

They also showed that if the minimum degree is larger than $n/2$, the graph contains even more Hamiltonian cycles. Using an approximate version of the Bollobás-Scott

Conjecture with any number of parts, we achieve of Theorem 13 that avoids appealing to the Regularity Lemma. This gives both a simpler proof and one that applies to smaller values of n .

Chapter 2

Background

2.1 Graph Theoretic Terms

A *graph* G consists of a set $V(G)$ of *vertices*, and a set $E(G)$ of unordered pairs of vertices called *edges*. If we allow $E(G)$ to be a multiset, G is a *multigraph*. If $E(G)$ is allowed to contain vv , then G is a graph with *loops*. If G is not a multigraph and has no loops, then G is a *simple* graph. Unless otherwise specified, all graphs we consider will be simple. Given $u, v \in V(G)$, if $uv \in E(G)$, we say u is *adjacent* to v or u is a *neighbor* of v . We say the edge uv is *incident* to u and v . If $uv \notin E(G)$, then we call uv a *non-edge*. The graph with vertex set $V(G)$ and edge set all pairs uv such that $uv \notin E(G)$ is called the *complement* of G and is denoted \overline{G} .

Let $n(G)$ denote $|V(G)|$, which is called the *order* of G . Let $e(G)$ denote $|E(G)|$, which is called the *size* of G . If G is understood, we will often just use n to denote the number of vertices of the graph under consideration.

A function $f : V(G) \rightarrow V(H)$ is a *homomorphism* between graphs G and H if $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. Note that the reverse need not be true; that is, $f(u)f(v) \in E(H)$ need not imply $uv \in E(G)$. If f is a bijection and f and f^{-1} are both homomorphisms,

then we say f is an *isomorphism* and G and H are *isomorphic*. This captures the notion of when two graphs are equivalent.

Let $\deg_G(v)$ (or just $\deg(v)$ when the graph is understood) denote the number of edges incident to v , where loops are counted twice and multiple edges between the same pair of vertices are counted multiple times. If v has no incident edges, the v is an *isolated vertex*, and if v is adjacent to all other vertices, then v is a *dominating vertex*. The sequence $(\deg_G(v))_{v \in V(G)}$ is the *degree sequence* of G . A graph where $\deg_G(v) = k$ for all $v \in V(G)$ is called *regular* or *k-regular*. Let $N(v)$ be the *neighborhood* of v , which is the set of all neighbors of v . If $S \subseteq V(G)$, $N(S) = \bigcup_{s \in S} N(s)$.

Given a sequence $\pi = (d_1, \dots, d_n)$, we say that π is *graphic* if there is a graph G such that the degree sequence of G is π . In this case, we say G *realizes* π . For simple graphs, the *complement* of the sequence π is the sequence $(n - 1 - d_1, \dots, n - 1 - d_n)$ and is denoted $\bar{\pi}$. A sequence has *even sum* if $\sum_{i=1}^n d_i$ is even. Having even sum is a necessary condition for π to be graphic, because the degree sum of a graph counts each edge twice and therefore must be even.

A *subgraph* of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. H is *spanning* if $V(H) = V(G)$. H is an *induced subgraph* if $u, v \in V(H)$ and $uv \in E(G)$ implies $uv \in E(H)$. Given $A \subseteq V(G)$, we will write $G[A]$ to denote the unique induced subgraph of G with vertex set A . If H is spanning and k -regular, we say that H is a *k-factor*. A set of edges such that no two edges are incident to the same vertex is a *matching*. If every vertex in $V(G)$ is incident to an edge in the matching, then the matching is *perfect*. Notice the edges of a 1-factor form a perfect matching.

If all vertices of G are isolated, then G is an *empty graph*. If all vertices of G are dominating, then G is a *complete graph*. A set of vertices $I \subseteq V(G)$ is an *independent set* if $G[I]$ is an empty graph. A subgraph H is a *clique* if H is a complete graph. We define K_n to be the complete graph on n vertices.

A *walk* in a graph G is a sequence of vertices $v_1, v_1, v_2, \dots, v_t$ such that $v_i v_{i+1} \in E(G)$ for $i = 1, \dots, t-1$. If edges are not allowed to be repeated, then the walk is a *trail*. If $v_1 = v_t$, then the walk is a *closed walk*, and the trail is a *circuit*.

A *path* is a graph isomorphic to the following graph P on t vertices: let $V(P) = \{1, 2, \dots, t\}$, and let i be adjacent to $i+1$ for $i = 1, \dots, t-1$. The length of the path P is $t-1$. A *cycle* C would be the graph isomorphic to the one obtained by adding the edge between 1 and t to P . Notice that if $t = 1$ or $t = 2$, then C is a graph with either a loop or multiple edges respectively, and in particular is not simple.

We will be especially concerned with alternating trails. An *alternating trail* is a sequence of vertices v_1, v_2, \dots, v_t , where instead of $v_i v_{i+1}$ being an edge in each case, the sequence alternates between edges and non-edges. Furthermore, no edge or non-edge is used twice. Given an alternating trail $T = v_1, \dots, v_t$ in G , we can *switch along* T to form a new graph G' where every non-edge in T becomes an edge of G' , and every edge in T becomes a non-edge of G' . A key property of switching along an alternating trail is that $\deg_{G'}(v_i) = \deg_G(v_i)$ for $i = 2, 3, \dots, t-1$. An *alternating circuit* is an alternating trail where $v_1 = v_t$. A *two-switch* is a switch along an alternating cycle of length 4.

A graph is *connected* if every two vertices are the endpoints of a path. Vertex maximal connected subgraphs are called *connected components* or just *components* of G .

If $V(G)$ can be partitioned into two independent sets X and Y , then G is a *bipartite graph*. We sometimes say that G is a bipartite graph on $X \cup Y$ to indicate a specific partition. If for every $u \in X$ and $v \in Y$, we have $uv \in E(G)$, then G is a *complete bipartite graph*. We use $K_{s,t}$ to denote the complete bipartite graph with part sizes s and t . A matching on a bipartite graph *saturates* X if every vertex in X is matched. If $V(G)$ can be partitioned into sets A and B where $G[A]$ is empty and $G[B]$ is complete, then G is a *split graph*.

A *directed graph* or *digraph* D consists of a vertex set $V(D)$ and ordered pairs of vertices

$E(D)$ called *directed edges*.

See West [68] for a more complete introduction to graph theory.

2.2 Probabilistic Terms

Due to the use of probabilistic methods in Chapter 5, we will define some basic probabilistic terms. Note that some of these definitions are special cases of more general definitions. Let $\mathcal{P}(S)$ denote the power set of a set S . For our purposes, a *probability space* is a finite set Ω of outcomes along with a function $\Pr : \mathcal{P}(\Omega) \rightarrow [0, 1]$ satisfying

- $\Pr(\Omega) = 1$, and
- for disjoint sets $A_1, \dots, A_t \subseteq \Omega$, we have

$$\Pr \left(\bigcup_{i=1}^t A_i \right) = \sum_{i=1}^t \Pr(A_i). \quad (2.1)$$

Elements of $\mathcal{P}(\Omega)$ are *events*, and the value $\Pr(A)$ is the probability that event A occurs. A probability space has *uniform distribution* if for $A \subseteq \Omega$, $\Pr(A) = |A|/|\Omega|$. If $(\Omega_1, \Pr_1), \dots, (\Omega_t, \Pr_t)$ are probability spaces, then we define their *product* (Ω, \Pr) by $\Omega = \prod_{i=1}^t \Omega_i$ and \Pr to be the unique probability function with the property $\Pr((A_1, \dots, A_t)) = \prod_{i=1}^t \Pr_i(A_i)$ for any set $A_i \subseteq \Omega_i$. We will not talk about the product probability space explicitly, but whenever we combine probability spaces *independently*, we mean to take the product.

Often the underlying probability space will be de-emphasized, and we will instead discuss random variables. A *random variable* is a function $X : \Omega \rightarrow \mathbb{R}$. One can perform operations on random variables like any real-valued function. The *expectation* of X is

given by

$$\mathbb{E}X = \sum_{w \in \Omega} X(w) \Pr(w).$$

A critical property of expectation is that it is linear. That is, given random variables X and Y and scalars c and d , we have $\mathbb{E}(cX + dY) = c\mathbb{E}X + d\mathbb{E}Y$.

If X is a random variable from a product space $\prod_{i=1}^t \Omega_i$, we say that X is *independent* of coordinate i if changing the value of coordinate i , while leaving the other coordinates fixed, does not change the value of X . Otherwise, X *depends* on coordinate i . Define the *support* of X , denoted $\text{supp}(X)$, to be those indices i such that X depends on coordinate i . For our purposes, two random variables X and Y are *independent* if $\text{supp}(X) \cap \text{supp}(Y) = \emptyset$. Otherwise, X and Y are *dependent*. We say a collection of random variables X_1, \dots, X_k is *independent* if the random variables are pairwise independent. Note that this last definition does not hold for the usual definition of independence.

Given events A_1, \dots, A_t , we will often want to bound $\Pr(\bigcup_{i=1}^t A_i)$. The *union-sum bound* is

$$\Pr\left(\bigcup_{i=1}^t A_i\right) \leq \sum_{i=1}^t \Pr(A_i),$$

which follows from the Equation 2.1.

See Alon and Spencer [1] for a more complete introduction to probability and its applications in combinatorics.

2.3 Other Terms

Given a real-valued vector $\vec{v} = (v_1, \dots, v_n)$, the *supremum norm* is

$$\|\vec{v}\|_\infty = \max\{v_1, \dots, v_n\}.$$

The *2-norm* or *Euclidean norm* is

$$\|\vec{v}\|_2 = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Given two functions $f(n)$ and $g(n)$, we write $f(n) = O(g(n))$ if there exists a constant C such that $f(n) \leq Cg(n)$ for all n . Intuitively, this means $f(n)$ grows no faster than $g(n)$. We write $f(n) = o(g(n))$ if $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$. Intuitively, this means $f(n)$ grows more slowly than $g(n)$.

Chapter 3

Packing Degree Sequences

3.1 Packing Degree Sequences

Consider the following problem. Suppose you have a sequence α of integers (a_1, \dots, a_n) and another sequence β given by (b_1, \dots, b_n) . Does there exist a pair of simple graphs A and B on vertex set $\{v_1, \dots, v_n\}$ such that $\deg_A(v_i) = a_i$, $\deg_B(v_i) = b_i$, and A and B are edge-disjoint? If so, we will say α and β *pack*.

Degree sequence packing might be more analogous to graph packing if we allowed ourselves more freedom by permuting the numbers in α and β before finding the realizations. The following theorem tells us that if we are allowed this extra freedom, there is one permutation that is the best to choose.

Theorem 14. *Suppose there exist realizations of $\alpha = (a_1 \geq a_2 \geq \dots \geq a_n)$ and $\beta = (b_1 \geq b_2 \geq \dots \geq b_n)$ that pack under some permutation of α and β . Then α and β pack if α is permuted to be weakly decreasing and β is permuted to be weakly increasing.*

Proof. Let A and B be edge-disjoint realizations of the degree sequences α and β on the vertices $\{1, \dots, n\}$. Without loss of generality, we can assume that vertex i has degree a_i

in A . Let σ be a permutation of $\{1, \dots, n\}$ such that vertex i has degree $b_{\sigma(i)}$ in B . Let C be the complement of $A \cup B$. We can think of A, B, C determining a three coloring of the edges of K_n . Thus we have A -colored, B -colored, and C -colored edges.

By construction, as i increases, a_i is weakly decreasing. Our goal is for the $b_{\sigma(i)}$ to be weakly increasing. Contrary to this, suppose there exists $i < j$ such $b_{\sigma(i)} > b_{\sigma(j)}$. Thus, vertex i has higher degree in A and strictly higher degree in B than vertex j . For a vertex $\ell \neq i, j$, we can perform a switching maneuver that maintains the A and B degrees of ℓ but changes the degree on i and j : we switch the coloring of edges $i\ell$ and $j\ell$. By making switches like this, we hope to swap the B -degrees on i and j . That is, our goal is for i to have B -degree $b_{\sigma(j)}$ and j to have B -degree $b_{\sigma(i)}$. By repeating such swaps, we can eventually achieve that the desired realizations from the theorem.

Let $N(X, Y)$ be the set of vertices consisting of each vertex ℓ such that $i\ell$ is an X -colored edge and $j\ell$ is a Y -colored edge. For example, for $\ell \in N(A, C)$, $i\ell$ is an edge in A and $j\ell$ is an edge in neither A nor B (and thus an edge in C). Let $n(X, Y) = |N(X, Y)|$. Thus, $n(B, A)$ is the number of vertices ℓ such that $i\ell$ is B -colored and $j\ell$ is A -colored.

Case 1: $n(A, B) \geq n(B, A)$. In this case, for a certain number of $\ell \in N(B, C)$, switch edges $i\ell$ and $j\ell$. This lowers the B -degree of i and raises the B -degree of j . The number of such switches to make is so that i will have B -degree $b_{\sigma(j)}$ and j will have B -degree $b_{\sigma(i)}$. Hence you must make exactly $b_{\sigma(i)} - b_{\sigma(j)}$ such switches. There are enough vertices in $N(B, C)$ to perform this maneuver because

$$\begin{aligned}
 b_{\sigma(i)} - b_{\sigma(j)} &= n(B, A) + n(B, B) + n(B, C) - n(A, B) - n(B, B) - n(C, B) \\
 &= n(B, A) - n(A, B) + n(B, C) - n(C, B) \\
 &\leq n(B, C)
 \end{aligned}$$

since both $n(B, A) - n(A, B)$ and $-n(C, B)$ are non-positive.

Case 2: $n(A, B) < n(B, A)$. In this case, for all

$$\ell \in N(A, B) \cup N(B, A) \cup N(B, C) \cup N(C, B),$$

switch edges $i\ell$ and $j\ell$. This completely switches all relevant B -edges, so i will then have $b_{\sigma(j)}$ degree in B , and j will then have $b_{\sigma(i)}$ degree in B , as we are hoping for. However, we must also maintain that i has degree a_i in A , and j has degree a_j in A . After these switches, this will not be the case, since we have switched some edges from $N(A, B)$ and $N(B, A)$. Since $n(A, B) < n(B, A)$, vertex i will have gained A -degree $n(B, A) - n(A, B)$ and vertex j will have lost A -degree $n(B, A) - n(A, B)$. We need to fix these A -degrees by switching with respect to some vertices in $N(A, C)$. There will be enough as long as $n(A, C) \geq n(B, A) - n(A, B)$. We see

$$\begin{aligned} 0 \leq a_i - a_j &= n(A, A) + n(A, B) + n(A, C) - n(A, A) - n(B, A) - n(C, A) \\ &= n(A, B) - n(B, A) + n(A, C) - n(C, A) \end{aligned}$$

This implies that $n(B, A) - n(A, B) \leq n(A, C) - n(C, A) \leq n(A, C)$, as desired.

□

Note that we can actually order possible permutations of the realizations with regard to how easy they are to pack. For two permutations on $(1, \dots, n)$, say $\sigma \leq \tau$ if σ is obtained from τ by performing one increasing-to-decreasing transposition. A transposition is the exchange of two not necessarily consecutively labeled elements. For example, if $\sigma : (1, 2, 3, 4, 5) \rightarrow (5, 3, 2, 1, 4)$ and $\tau : (1, 2, 3, 4, 5) \rightarrow (5, 1, 2, 3, 4)$, then $\sigma \leq \tau$. If you take the transitive and reflexive closure of this relation, you get a partial order on the

permutations of $(1, 2, \dots, n)$. Notice that $(1, 2, \dots, n) \rightarrow (1, 2, \dots, n)$ is the maximum element of this order, and $(1, 2, \dots, n) \rightarrow (n, n-1, \dots, 1)$ is the minimal element. This order is the well-studied Bruhat Order.

Now, fix vertex i to have degree a_i in A , and suppose in B , vertex i can have either degree $b_{\sigma(i)}$ or degree $b_{\tau(i)}$. Then if $\sigma \leq \tau$, we have that if the degree sequences pack under τ , then they must pack under σ . We conjecture the converse is true as well, in the following sense.

Conjecture 15. *Given two permutations $\sigma \not\leq \tau$, then there exist graphic degree sequences α and β such that they pack under τ but not under σ .*

3.2 Kundu's Theorem

One can consider Kundu's Theorem to be the first major result in degree sequence packing, even though it is usually stated as a result about potential k -factors. Here we reproduce the following elegant proof by Chen [10] which was inspirational to many parts of this thesis.

Theorem 16 (Kundu [42]). *A graphic sequence $\pi = (d_1, \dots, d_n)$ has a realization containing a k -factor if and only if the sequence $\pi - k = (d_1 - k, \dots, d_n - k)$ is graphic.*

Proof. (Chen [10]) The necessity of the condition is clear: if π has a realization G with a k -factor F , then $G \setminus F$ is a realization of $\pi - k$.

We will prove sufficiency. Let α be the sequence $\pi - k$ and let β be the complement of the sequence π , which is the sequence $(n-1) - \pi$. We have α is graphic by assumption, and β is graphic since it is the complement of the graphic sequence π . A key property is that $\alpha + \beta$ is the constant sequence $(n-1-k, \dots, n-1-k)$.

We claim that α and β pack. Let A and B be realizations of α and β respectively minimizing the number of edges that they share, which we will call *overlapping edges*. We will think of overlapping edges as multiple edges in the graph $A \cup B$. If there are any overlapping edges, we will proceed to remove at least one such overlap, contradicting minimality and proving that α and β pack.

Let uv be an overlapping edge, if one exists. Since $\deg_A(v) + \deg_B(v) \leq n - 1$, and this sum counts u as a neighbor twice, we know there exists a vertex x not adjacent to v in either A or B . The vertex x has the same number of edges emanating from it as u , and yet x has no edge going to v while u has two edges going to v . Therefore, there must be a vertex y where either

- y has one edge going to x and none to u , or
- y has two edges going to x and at most one going to u .

In the first case, we can perform a two-switch $uvxy$ in one of the graphs A or B to remove the overlap between u and v while creating no new overlaps. In the second case, we can perform the two-switch $uvxy$ which will remove two overlaps (those between uv and xy) and possibly create a third between yv . However, the net number of overlaps is lower. This proves that α and β pack.

Then \bar{B} is a realization of π , and $\bar{B} \setminus \bar{A}$ is a k -factor within \bar{B} . □

Using the same proof, Kundu's theorem can be extended to characterize when a sequence π contains a realization with an *almost regular* subgraph, meaning every vertex has degree k or $k + 1$. We take advantage of this extra wiggle-room in a different way in [Section 3.4](#).

3.3 Kundu for Bipartite Graphs

A *bisequence* is a sequence with a bipartition, such as $\pi = (a_1, \dots, a_m; b_1, \dots, b_n)$. We say π has a realization if there exists a bipartite graph $G = (L, R, E)$ such that $|L| = m$, $|R| = n$, the vertices in L have degrees (a_1, \dots, a_m) , and the vertices in R have degrees (b_1, \dots, b_n) .

With this definition, we can talk about bipartite degree sequence packing, and we can develop an bipartite analog of Kundu's theorem.

Given a graph G and a function $f : V(G) \rightarrow \mathbb{N}$, an *f-factor* of G is a subgraph where vertex v has degree $f(v)$. If G is bipartite with bipartition $L \cup R$ and $f(v) = k$ for all $v \in L$, then f is *left-regular*.

It is probably not surprising that there is an analog of Kundu's Theorem for bipartite graphs. What is surprising is that the f -factor you find need not be entirely regular, but instead needs only to be left-regular.

Proposition 17. *Let $\tau = (a_1, \dots, a_n; b_1, \dots, b_m)$ be a bipartite graphic degree sequence, $L = \{u_1, \dots, u_n\}$, $R = \{v_1, \dots, v_m\}$, and $f : L \cup R \rightarrow \mathbb{N}$ such that f is left-regular. Then there exists a bipartite realization of τ with an f -factor if and only if $\tau - f = (a_1 - f(u_1), \dots, a_n - f(u_n); b_1 - f(v_1), \dots, b_m - f(v_m))$ is bipartite graphic.*

Proof. We follow Chen's proof of Kundu's theorem. The necessity of the condition is clear: if τ has a realization G with a f -factor F , then $G \setminus F$ is a realization of $\tau - f$.

We now prove sufficiency. Let α be the complement degree sequence to τ given by $(m - a_1, \dots, m - a_n; n - b_1, \dots, n - b_m)$. Let $\beta = \tau - f$. Set $k = f(v_i)$, which is the same for any i . Then if α and β have realizations that pack, we have that β lives in the complement of α . That would give a realization of τ with a realization of $\tau - f$ as a subgraph. Taking the complement of $\tau - f$ with respect to τ yields an f -factor.

We need to show that α and β pack. Take two realizations A and B on vertex set $L \cup R$, and do so in a way that minimizes the number of overlapping edges, that is, multi-edges

in the graph $A \cup B$. Suppose, alas, you do have two vertices $u \in L$ and $v \in R$ such that uv is an edge in both A and B . Notice that the degree sequence $\alpha + \beta$ is the constant $m - k$ degree sequence on L . Hence there must be some vertex $x \in L$ such that there is no edge xv in either A or B . But notice that u has two edges coming out, and x has none so far. So there must be a $y \in R$ such that there are more edges between x and y than between u and y (in the multigraph $A \cup B$).

We can now perform a two-switch that will fix the problem between u and v . There are two cases: either there is one edge between u and y , or there are no edges between u and y . If there are no edges between u and y : Without loss of generality, assume that between x and y there is an A -edge. Then do a two-switch in A between u, v, x , and y . If there is an edge between u and y : Without loss of generality, assume that between u and y there is a B -edge. Then there is not an A -edge between u and y , so do a two-switch in α between u, v, x , and y . In the first case, you fix the multi-edge between u and v without introducing any more. In the second case, you fix two multi-edges, uv and xy , while gaining a single multi-edge uy . Hence, we have contradicted the minimality of the number of multi-edges in $A \cup B$. Therefore, our supposition that we had some multi-edge was false, so α and β must pack. \square

3.4 The Easiest 1-Factor

Kundu's Theorem 16 gives when a degree sequence has a realization containing a 1-factor, but does not give any clue as to what 1-factor you obtain. A natural perfect matching to aim for would be the one where the highest degree vertex is matched to the lowest degree vertex, the next highest is matched with the next lowest, and so on. In this section, we prove that any degree sequence with a 1-factor has a realization with this specific 1-factor. In this sense, it is the easiest 1-factor to obtain.

The main work will come from the following proposition, whose proof follows Chen's proof of Kundu's theorem.

Proposition 18. *Let $\pi, \pi - k$ be degree sequences and let M be a partial matching on vertex set V . Then if there exist realizations of π containing M and of $\pi - k$ avoiding M , then there exists a realization G of π such that*

- G contains a k -factor H .
- H contains M .

Proof. We follow Chen's proof of Kundu's theorem. Let $\alpha = \bar{\pi}$ and let $\beta = \pi - k$. Let A and B be realizations of α and β minimizing the number of overlapping edges, with the proviso that both A and B avoid M .

Suppose you do have an overlap. Choose u and v such that uv is a double edge in $A \cup B$, and such that u and v have the greatest number of other double edges emanating from them. Let v_1, \dots, v_s be the other vertices with double edges with u , and let u_1, \dots, u_t be the other ones with double edges with v . There must be $s + 2$ vertices with no edge to u , since u must have a non-edge emanating out of it to begin with, and gets another for each of v, v_1, \dots, v_s . If we eliminate u 's neighbor in M , we still have $s + 1$ candidates. Call them x_1, \dots, x_{s+1} . Similarly, for v , there must be $t + 1$ other vertices y_1, \dots, y_{t+1} with no edge going to v in A or B or M .

Fix an i . For x_i , there must be some y_0 such that there are more edges between x_i and y_0 than y_0 and v . If y_0v is not in M , then we can do a two-switch to reduce the number of overlaps where A and B still avoid M . This contradicts how we chose A and B . Therefore, it must be the case that we must choose y_0 such that $y_0v \in M$. There must be a double edge between x_iy_0 and no edge y_0v , otherwise there would be another candidate for y_0 . However, this is true for all i , but there is only one possible y_0 . Hence we have that y_0x_i is a double edge for all $i = 1, \dots, t + 1$.

By a symmetric argument, we have an x_0 such that ux_0 is in M and ux_i is a double edge for all $i = 1, \dots, s+1$.

Now consider x_0y_0 . If this is not a double edge, then we can perform the following six-switch: $ux_1y_0x_0y_1vu$. This six cycle alternates no edge, double edge, except for x_0y_0 . Since this is not a double edge, it must be missing an edge in A or B . Thus we can do the six cycle switch for whatever graph does not have an edge between x_0 and y_0 . This significantly lowers the number of overlapping edges, still avoids M , and thus contradicts how we chose A and B .

If x_0y_0 is a double edge, then notice that x_0y_0 is a double edge, where x_1, \dots, x_{s+1} are all double edges with y_0 , and y_1, \dots, y_{t+1} are all double edges with x_0 . Therefore x_0y_0 is a double edges with more double edges emanating from them than u and v . This contradicts how we chose u and v .

By this contradiction, we now have that A and B have no overlaps. Thus, \bar{A} is a realization of π that contains a k -factor that contains M . \square

Proposition 19. *Let $\pi = d_1 \geq \dots \geq d_n$ be a degree sequence with a realization with a 1-factor, where v_i gets degree d_i . Then π contains the 1-factor where v_i gets matched with v_{n+1-i} for $i = 1, \dots, n/2$.*

Proof. Consider a realization G of π with a matching F . Set ℓ to be the largest value such that F contains $v_i v_{n+1-i}$ for $i = 1, \dots, \ell$. If $\ell \geq n/2$, then we're done, so assume $\ell < n/2$. Then F does not contain $e = v_{\ell+1} v_{n-\ell}$. We will perform two-switches and then apply to Proposition 18 to extend F to include this next edge, and then we will be done by induction.

Case 1. Suppose that G does not contain e . We see $v_{n-\ell}$ must be matched in F with some vertex u . We know that $u \neq v_1, \dots, v_\ell$ since all those vertices are already matched up. Hence, u must be a vertex "below" $v_{\ell+1}$. Therefore, the neighborhood of $v_{\ell+1}$ dominates

the neighborhood of u in G , meaning there is some x such that $v_{\ell+1}x$ is an edge of G , but xu is not an edge. We can then perform the two-switch $v_{\ell+1}xu v_{n-\ell}$ in G to create G' so that G' contains the edge e . Note that this two-switch did not affect the edges $v_i v_{n+1-i}$ for $i = 1, \dots, \ell$.

Let M be the partial matching $\{v_i v_{n+1-i}\}_{i=1}^{\ell+1}$. Set $H = G - F$. Then we have that G' is a realization of π that contains M , and H is a realization of $\pi - 1$ that avoids M . By Proposition 18, we can create a realization of π with a matching that contains M . Thus M is a matching that contains $v_i v_{n+1-i}$ for $i = 1, \dots, \ell + 1$ as desired.

Case 2. Suppose that G does contain e , but F does not. We have that $H = G - F$ is a realization of $\pi - 1$ that contains e . Now $v_{\ell+1}$ must be matched with some vertex in F , let's say u . We know that $u \neq v_{n-\ell}, \dots, v_n$ since all those vertices are already matched up. Therefore, the neighborhood of u dominates the neighborhood of $v_{n-\ell}$ in H , meaning there is some x such that ux is an edge of H , but $xv_{n-\ell}$ is not an edge. We can then perform the two-switch $v_{\ell+1}uxv_{n-\ell}$ in H to create H' so that H' avoids the edge e . Note that this two-switch did not affect the non-edges $v_i v_{n+1-i}$ for $i = 1, \dots, \ell$.

Let M be the partial matching $\{v_i v_{n+1-i}\}_{i=1}^{\ell+1}$. Then we have that G is a realization of π that contains M , and H' is a realization of $\pi - 1$ that avoids M . By Proposition 18, we can create a realization with a matching that contains M . Thus M is a matching that contains $v_i v_{n+1-i}$ for $i = 1, \dots, \ell + 1$ as desired. \square

Busch et al. [7] strengthened Kundu's theorem by showing that if a degree sequence has a realization with a k -factor, it has a realization with a disjoint $k - 1$ -factor and a 1-factor. (They actually showed more, that you could pull off two 1-factors from the k -factor). We can achieve former result and further strengthen it by specifying the particular 1-factor we are pulling off of the k -factor. We will need the following result of A. R. Rao and S. B. Rao [53].

Lemma 20 (Rao, Rao). *If π is a graphic sequence and $\pi - k$ is a graphic sequence, then $\pi - r$ is a graphic sequence for any $r = 0, 1, \dots, k$.*

Corollary 21. *Let $\pi = d_1 \geq \dots \geq d_n, \pi - k = d_1 - k \geq \dots \geq d_n - k$ be graphic degree sequences, n even. Then there exists a realization of π with a k factor that can be decomposed into a 1-factor and a $k - 1$ -factor. In particular, if vertex v_i gets degree d_i , then the 1-factor in question is the matching v_i with v_{n+1-i} .*

Proof. By Lemma 20, we know that π and $\pi - 1$ are both graphic. Thus there is a realization of π with a 1-factor, hence there is a realization of π with the 1-factor M described in Proposition 19.

By Lemma 20, we know that $\pi - k + 1$ and $\pi - k$ are both graphic. Thus, there is a realization of $\pi - k + 1$ containing M , and thus a realization of $\pi - k$ avoiding M . Hence, by Proposition 18, there is a realization of π containing a k -factor containing M . \square

3.5 Potential Multigraph Packing

A sequence is multigraphic if it is the degree sequence of a multigraph.

James Sellers [57] posed the following question. Suppose you have two multigraphic degree sequences α and β . Do there exist multigraph realizations A and B of α and β on the same set of vertices such that no two vertices have both A -edges and B -edges going between them? While the simple graph version of degree sequence packing is NP-complete [20], there is some evidence to suggest the multigraph case might be easier. For example multigraph realizability is easier than simple graph realizability in that you need only satisfy the first Erdős-Gallai inequality, not all n of them [33]. However, in this section we will see that multigraph degree sequence packing is also NP-complete, and the reduction is more straightforward than in the simple graph case.

The multigraphs we consider have no loops. Let $\pi_1 = (x_1, \dots, x_n)$, $\pi_2 = (y_1, \dots, y_n)$ be sequences of numbers with even sum. A sequence is *multigraphic* if there exists a multigraph G having the sequence as its degree sequence. Hakimi[33] characterized when a sequence is multigraphic:

Theorem 22. (Hakimi) $\pi = (z_1 \geq z_2 \geq \dots \geq z_n)$ is multigraphic if and only if it has even sum and $z_1 \leq \sum_{i=2}^n z_i$.

We say that π_1 and π_2 *multipack* if there exist multigraph realizations G_1 of π_1 and G_2 of π_2 on the same vertex set V such that

1. $v \in V$ gets G_1 degree x_i and G_2 degree y_i for some i , and
2. For every $u, v \in V$, there is not simultaneously an edge between u and v in both G_1 and G_2 .

Let MULTIPACK be the language of all pairs of degree sequences that pack. Let SUBSET SUM be the decision problem to determine if a set of positive integers S has a subset with sum k . SUBSET SUM is known to be NP-complete (see KNAPSACK from Karp's original paper [39]).

Proposition 23. MULTIPACK is reducible to SUBSET SUM.

Proof. Let S be a set, and let k be the target value for the sum of a subset. Let \vec{S} be an ordered list of elements of S . Let N be the sum of all the elements in S . We will assume that $N + k$ is even, though a minor change in the below argument handles the case where $N + k$ is odd. Let

$$\begin{aligned}\pi_1 &= (2(N + 1) + k, \quad N + 1, N + 1, N + 1, N + 1, \vec{S}), \\ \pi_2 &= (2(N + 1) + (N - k), \quad N + 1, N + 1, N + 1, N + 1, \vec{S}).\end{aligned}$$

Notice the degree sum of π_1 and π_2 is even under the assumption $N + k$ is even.

We will show that π_1 and π_2 multipack if and only if S has a subset that sums to k .

(\Leftarrow) First assume that S has a subset that sums to k . Let v be the vertex to receive degree $2(N + 1) + k$ in G_1 and $2(N + 1) + (N - k)$ in G_2 . We will think of G_1 being colored red, and G_2 being colored blue. In the red graph G_1 , attach v to the first two $N + 1$ degree vertices, and to the subset of \vec{S} that sums to k . In the blue graph G_2 , attach v to the second two vertices of $N + 1$ degree, and to the complement of the subset that sums to k (this subset sums to $N - k$). This will exhaust the degree of the first vertex in both graphs.

The nonzero red degrees that are leftover at this point are on a disjoint vertex set from the leftover blue degrees that are nonzero. Therefore, π_1 and π_2 will multipack as long as these two leftover degree sequences are multigraphic. They will be: the largest degree and the second largest degree of both are $N + 1$, and thus satisfy Hakimi's criterion.

(\Rightarrow) Assume now that π_1 and π_2 can multipack. Consider the vertex v that has red degree $2(N + 1) + k$ and blue degree $2(N + 1) + (N - k)$. For every other vertex u , either u and v have red edges between them, or u and v have blue edges between them. Notice for $u \neq v$, the red degree and blue degree of u are the same. Therefore, the most total degree the other vertices can absorb from v is $\sum_{u \neq v} \deg_{red}(u) = 4(N + 1) + N$, which is the total degree of v . Therefore, if you look at a neighbor u of v that receives red edges, v must absorb all of u 's red degree. In order to hit the right amount, v has red edges to exactly two of the vertices of degree $N + 1$. Thus v has red adjacencies to a subset of S that sums to exactly k . \square

We could also ask this same question about degree sequence packing about bipartite multigraphs. However, it is NP-complete in this case as well.

Let $\pi_1 = (x_1, \dots, x_n; w_1, \dots, w_m)$ and $\pi_2 = (y_1, \dots, y_n; z_1, \dots, z_m)$ be sequences, where

$$\sum_{i=1}^n x_i = \sum_{i=1}^m w_i, \quad \sum_{i=1}^n y_i = \sum_{i=1}^m z_i.$$

Under this assumption, π_1 and π_2 represent the degree sequences of bipartite multigraphs.

We say that π_1 and π_2 *bimultipack* if they have realizations G_1 and G_2 such that

1. the i th vertex on the left gets degree x_i in G_1 and y_i in G_2 , and the j th vertex on the right side gets degree w_j in G_1 and z_j in G_2 .
2. For every $u, v \in V$, there is not simultaneously an edge between u and v in both G_1 and G_2 .

The resulting decision problem BIMULTIPACK is also reducible to SUBSET SUM.

Proposition 24. *BIMULTIPACK is reducible to SUBSET SUM.*

Proof. Let S be a set, and let k be the target value for the sum of a subset. Again let \vec{S} be a vector of the elements of S , and let N be the sum of the elements of S . Let

$$\pi_1 = (k, N - k; \vec{S}), \quad \pi_2 = (N - k, k; \vec{S}).$$

We want to show that π_1 and π_2 bimultipack if and only if S has a subset with sum k .

(\Leftarrow) Assume S has a subset S' with sum k . Let v_1 be the vertex of red degree k , blue degree $N - k$, and let v_2 be the vertex of red degree $N - k$, and blue degree k . Thus v_1 and v_2 are the sole vertices on the left hand side. Attach v_1 to the S' with red edges and to the complement of S' with blue edges. Similarly, connect v_2 to S' with blue edges and the complement of S' with red edges. In this way v_1 and v_2 exactly absorb all the degrees of the vertices on the right hand side.

(\Rightarrow) Assume that π_1 and π_2 bimultipack. Since S consists of positive values, every vertex on the right hand side has positive red and blue degree. Therefore, each edge must have red edges going to v_1 or v_2 , and blue edges going to the other one. If you look at the vertices on the right hand side with red edges going to v_1 , the red degree on these vertices must sum to k . This gives a subset of S that sums to k . \square

3.6 Packing with a Threshold Sequence

Threshold graphs are a very interesting class of graphs that relate to many different areas of graph theory. This can be seen by sampling some of the many equivalent definitions. See [46] for more information. G is a *threshold graph* if any of the following hold:

- there exists a weight function $w : V \rightarrow \mathbb{R}$ and a threshold $t \in \mathbb{R}$ such that $u, v \in V(G)$ are adjacent if and only if $w(u) + w(v) \geq t$.
- G can be constructed iteratively by starting with a single vertex, and at each stage adding either an isolated vertex or a dominating vertex.
- G contains a clique K and independent set I with $V(K) \cup V(I) = V(G)$, and the neighborhoods of the vertices in I are nested.
- G is uniquely determined up to labeled isomorphism by its degree sequence.
- G is C_4 , P_4 , $2K_2$ free.
- There is no two-switch that can be preformed in G .

A sequence is a *threshold sequence* if it is a degree sequence of a threshold graph.

In this section we are interested in a question posed to us by Jeremy Martin [47]: if α is a threshold sequence and β is some other sequence, can we easily determine if α

and β pack? Since α determines its realization A uniquely up to labelled isomorphism, this question is equivalent to determining if β is realizable in \overline{A} . This can be done in polynomial time using the f -factor theorem due to Tutte [65], but in this section we see that it is much easier than appealing to this powerful result.

The following lemma will be helpful.

Lemma 25. *Let $\alpha = (a_1 \geq \dots \geq a_n)$ be a threshold degree sequence with realization A on vertex set $\{1, \dots, n\}$, where vertex i gets degree a_i . If $ij \in E(A)$ and $j \geq k$, that implies $ik \in E(A)$.*

Proof. Suppose that $ij \in E(A)$ but $ik \notin E(A)$. Since $a_j \leq a_k$, and k has an edge going to i but j does not, we see there must be some ℓ such that $\ell k \in E(A)$ but $\ell j \notin E(A)$. But then $ij\ell k$ would be a candidate for a two-switch, which is impossible in a threshold graph. \square

Consider now the case where both the threshold sequence α and the sequence β are weakly decreasing. Then there is an analogy to Kundu's k -factor theorem.

Theorem 26. *Let α be the degree sequence of a threshold graph, and let β be another graphic degree sequence. Then α and β pack where both are weakly decreasing if and only if $\alpha + \beta$ is graphic.*

Proof. The forward direction is clear, so we proceed to prove the backwards direction. Let $\alpha = (a_1 \geq \dots \geq a_n)$ be a threshold degree sequence, and $\beta = (b_1 \geq \dots \geq b_n)$ be another degree sequence. Assume $\alpha + \beta$ is graphic.

Let A be the realization of α on $\{1, 2, 3, \dots, n\}$, where vertex i gets degree a_i .

Let B be a realization of β on the same vertex set, again where vertex i gets degree b_i , where B has possible multiedges, loops, or overlaps with A . Let a loop, multiedge, or overlap with A be a *bad edge* of B . Choose B to maximize the the smallest vertex incident to bad edge. In other words, we want the bad edges as far away from the high degree vertices as possible.

Suppose v is this smallest vertex incident to a bad edge, and let u be the other end of that bad edge. It is okay if $u = v$, which happens when the bad edge is a loop. We will want to construct an alternating path that we can switch on and contradict our extremal choice of the realization B . Let $S = \{1, \dots, v\}$. The basic idea is pretty simple, but some case analysis is required to make the argument rigorous.

Case 1: There is a vertex $s \in S$ such that su is a non-edge for both A and B . We know $s \neq v$, since uv is either a loop or has multiple edges in $E(A) \cup E(B)$. Since s has degree at least that of v , and v has edges going to u and s does not, we see there must be a t such that tv is an non-edge but ts is an edge. By Lemma 25, ts must be a B -edge and not an A -edge. Thus $uvts$ is an alternating path which fixes the bad edge in question, contradicting our choice of B .

Case 2: u is adjacent to all of S . The following claim gives us part of our alternating path we are looking for:

Claim 1. *There exist vertices x, x_1, x_2 such that $x_1 \in S$, $x_2 \notin S$, $xx_1 \notin E(A) \cup E(B)$, $xx_2 \in E(A) \cup E(B)$.*

Proof of Claim. For the proof of this claim we will consider only edges in $E(A) \cup E(B)$. We are trying to show there is a vertex x with a non-edge going to S , and an edge going outside of S . Suppose no such x exists.

For any $y \in V$, either y is adjacent to nothing outside of S , or y is adjacent to everything in S . In either case, y is doing everything it can to absorb the degree from the vertices in S . Recall there is a bad edge emanating from v to u . By assumption, u is adjacent to all of S . Thus u is doing everything it can do *and a little bit more* in order to absorb the edges coming out of S . This contradicts the fact that $\alpha + \beta$ is graphic: there will be no way to absorb all the degree of set S without resorting to multiedges. By contradiction, we do have such an x as in the claim. \square

This edge between x and x_2 cannot be in A , since A is threshold. Therefore, this edge is in B .

Case 2.A: Suppose $x_1 = v$. Switch along $uvxx_2u$. This will remove the bad edge from B , but notice ux_2 may already have an A -edge or B -edge, and thus by doing this two-switch, we will introduce a bad edge between u and x_2 . However, this increases the smallest vertex with the bad edge, since $x_2 > v$ and $u > v$. This contradicts our choice for B .

Case 2.B: Suppose $x = v$. There must be some y such that yx_1 is an edge in B , and yu is a non-edge in B (since $b_{x_1} \geq b_u$, and u has an edge going to v that x_1 does not have). In this case, switch along vx_1yuv , which will eliminate the bad edge between u and v . This contradicts our choice for B .

Case 2.C: Neither x_1 nor x is equal to v . First notice that there must be an edge between v and x , otherwise we would be in case 2.A. Furthermore, by Lemma 25, this edge is a B -edge. There must be some y with a B -edge going to x_1 and a non-edge going to v (since $b_{x_1} \geq b_v$, and v has an edge going to x while x_1 does not). Now switch in B along vyx_1xx_2uv . Note that x_2u may be an edge of G or H , and this switch may cause this to become a bad edge. But that doesn't matter, since we have increased the smallest vertex with a bad edge, contradicting our choice of B . \square

Even if it is not the case that both sequences are weakly decreasing, we can still easily check whether a degree sequence and a threshold sequence pack using a Havel-Hakimi-like idea.

Theorem 27. *Let α be a threshold sequence and β be a degree sequence. Let A be the realization of α . A must have a vertex v connected to all other vertices of A of positive degree. Then β packs in the complement of A if and only if it packs in a way such that v is attached using β -edges to the vertices of highest β -degree among those with α -degree zero.*

Proof. Let B be a packing of β in the complement of A . Suppose v is attached to a vertex u but not w , with $\deg_B(u) \leq \deg_B(w)$. There must be a vertex x such that wx is a β -edge, but xu is not a β -edge. Notice that xu cannot be an A -edge, since α is zero for u . Therefore, we can do the two-switch $xuvwx$ to get a realization of β in the complement of A , but where v is attached to w instead of u . \square

To use Theorem 27 to determine when a sequence $\beta = (b_1, \dots, b_n)$ packs with a threshold sequence $\alpha = (a_1 \geq \dots \geq a_n)$, follow this procedure: For the threshold sequence α' formed from α by removing a_1 and subtracting 1 from the largest a_1 entries of α . Also form the sequence β' formed from β by removing b_1 , and subtracting 1 from the b_1 largest entries of β that correspond to zeros in α . The theorem states that α and β pack if and only if the shorter sequences α' and β' pack. By recursively applying this process, eventually the question of whether α and β pack must be resolved one way or the other.

Chapter 4

Potential Bisections of Large Degree¹

4.1 Introduction

A *bisection* of a graph is a balanced bipartite spanning subgraph. Bollobás and Scott [4] conjectured that every graph has a bisection with vertices of large degree.

Conjecture 28 (Bollobás and Scott [4]). *Every graph G has a bisection H such that for every vertex $v \in V(G)$,*

$$\deg_H(v) \geq \frac{\lfloor \deg_G(v) \rfloor}{2}.$$

We prove the following result, which can be seen as a degree sequence version of the Bollobás-Scott Conjecture. A sequence is *graphic* if it is the degree sequence for a finite simple graph.

Theorem 29. *For even n , let $\pi = (d_1, d_2, \dots, d_n)$ be a graphic sequence. Then there exists a realization G of π with a bisection H where for all vertices $v \in V(G)$,*

$$\deg_H(v) \geq \left\lfloor \frac{\deg_G(v) - 1}{2} \right\rfloor.$$

¹This chapter was jointly prepared with Stephen G. Hartke and will appear in J. Graph Theory.

The result is nearly tight; in Section 4.2, we show there are degree sequences such that, for any realization G and any bisection H of G , there exists a vertex v such that $\deg_H(v) \leq \lfloor \deg_G(v)/2 \rfloor$.

In addition to providing support for the Bollobás-Scott Conjecture, Theorem 29 has an application in generalizing Kundu's k -factor Theorem.

Theorem 30 (Kundu [42]). *The degree sequence $\pi = (d_1, \dots, d_n)$ has a k -factor if and only if both π and $\pi - k = (d_1 - k, \dots, d_n - k)$ are graphic.*

See Chapter 3 for a short and elegant proof of Kundu's k -factor Theorem by Chen [10].

Busch, Ferrara, Hartke, Jacobson, Kaul and West [7] conjectured even more.

Conjecture 31 (Busch et al. [7]). *Let $\pi = (d_1, \dots, d_n)$ and $\pi - k = (d_1 - k, \dots, d_n - k)$ be graphic sequences, where n is even. Then π has a realization containing k edge-disjoint 1-factors.*

To support this conjecture, they prove several special cases, including when $k \leq 3$, and the following.

Theorem 32 (Busch et al. [7]). *Let $\pi = (d_1, \dots, d_n)$ be a graphic sequence, where n is even. If the minimum entry δ in π is at least $n/2$, then there exists a realization of π containing $\delta - n/2 + 1$ edge-disjoint 1-factors.*

In their paper, they present the result slightly differently, obtaining one more 1-factor under the additional assumption that $\pi - k$ is graphic, where k is the number of edge-disjoint 1-factors.

Using Theorem 29 and a result by Csaba [14], we prove a result that is stronger than Theorem 32 except for a few fringe cases, namely when $\delta = \lceil n/2 \rceil, \lceil n/2 \rceil + 1$ and when $\delta \geq n - 12$:

Theorem 33. For even n , let $\pi = (d_1, \dots, d_n)$ be a graphic sequence. If the minimum entry δ in π is at least $n/2 + 2$, then exists a realization of π containing

$$\left\lfloor \frac{\delta - 2 + \sqrt{n(2\delta - n - 4)}}{4} \right\rfloor$$

edge-disjoint 1-factors.

We prove Theorem 33 by finding a regular bisection, which is then easily decomposed into a disjoint union of 1-factors. However, this technique cannot fully prove Conjecture 31. In Section 4.6, we show that there exists an infinite family of degree sequences where, in any realization, the degree of a regular bisection is no more than three-quarters of the degree of the largest k -factor.

4.2 Sharpness of Theorem 29

We now show the result in Theorem 29 is nearly sharp. First we need a few preliminaries. See [68] for standard graph theoretic terminology and definitions not otherwise discussed here.

Definition 34. A sequence $\pi = (d_1, \dots, d_n)$ of nonnegative integers is *weakly decreasing* if $d_1 \geq \dots \geq d_n$. A *degree sequence* of a graph is a sequence consisting of the degrees of the vertices. A sequence is *graphic* if it is the degree sequence of some graph. A sequence is *unigraphic* if it is graphic and any two realizations are isomorphic. A vertex of degree $n - 1$, which is adjacent to all other vertices, is called *dominating*.

Definition 35. A k -factor is a regular spanning subgraph of degree k . For example, a 1-factor is a perfect matching.

Definition 36. For a graph G , a partition of the vertices $V(G)$ into disjoint sets L and R is *balanced* if $|L| = |R|$. A bipartite graph is *balanced* if its partition is balanced. A *bisection* is a balanced bipartite spanning subgraph of G .

Example 37. For n even and $k < n/2$, let π be the sequence

$$(\underbrace{n-1, \dots, n-1}_k, \underbrace{k, \dots, k}_{n-k}).$$

Let G be the join of a complete graph K_k on k vertices and an independent set I_{n-k} on $n-k$ vertices. Then G is a realization of π , and hence π is graphic. Furthermore, G is unigraphic, since the vertices of degree $n-1$ must be dominating, which forces the vertices of degree k to be an independent set.

In addition, any bisection H of G satisfies $\deg_H(v) \leq \lfloor \deg_G(v)/2 \rfloor$ for some vertex v . To see this, consider any partition of the vertices of G into two equal sets L and R . One of L or R must contain at most half of the vertices in the complete graph K_k . Without loss of generality, suppose L contains at most half. Then any vertex v in $I_{n-k} \cap R$, which is nonempty, has degree k in G and degree at most $\lfloor k/2 \rfloor$ in H . Thus, we have an example G where any bisection H satisfies $\deg_H(v) \leq \lfloor \deg_G(v)/2 \rfloor$, and therefore the bound $\deg_H(v) \geq \lfloor (\deg_G(v) - 1)/2 \rfloor$ proven in Theorem 29 is one away from the best possible value.

4.3 The Kleitman-Wang Theorem

Our main tool for proving Theorem 29 is a theorem of Kleitman and Wang. We first need a concept they call “laying off”.

Definition 38. [67] Given $\pi = (d_1, \dots, d_n)$ with $d_1 \geq \dots \geq d_n$, *laying off* d_i means

creating a sequence $\pi' = (d'_1, \dots, d'_{i-1}, d'_{i+1}, \dots, d'_n)$ formed by removing d_i from the list and subtracting 1 from the d_i remaining elements of lowest index.

With this definition, they prove

Theorem 39 (Kleitman–Wang [67]). *For any $i = 1, \dots, n$, the sequence $\pi = (d_1, \dots, d_n)$ is graphic if and only if the sequence π' obtained by laying off d_i is graphic.*

The theorem can be used to inductively find a realization of a sequence π as follows: Form the shorter sequence π' by laying off d_i . Inductively form a realization G' of π' on vertex set $\{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$, where vertex v_j has degree d'_j . We can then construct a realization G of π by adding in a new vertex v_i to G' , where, for j such that $d'_j = d_j - 1$, v_i is adjacent v_j .

A common difficulty in using this result is that after laying off d_i , the sequence π' is not necessarily weakly decreasing. Thus, we modify the definition of laying off d_i slightly.

Definition 40. Let $\pi = (d_1, \dots, d_n)$ with $d_1 \geq \dots \geq d_n$ and let $i \in \{1, \dots, n\}$. Let m be the smallest value among the d_i largest elements in π , not including d_i itself. Let S be the set of indices j , $j \neq i$, such that $d_j > m$. Let T be the set of $d_i - |S|$ largest indices j , $j \neq i$, such that $d_j = m$.

$$\begin{array}{ccccccc} d_1 & & \cdots & & |m & \cdots & m| & & \cdots & & d_n \\ \hline & & S & & | & & | & T & | & & \end{array}$$

Then *laying off d_i with order* means creating the sequence $\pi' = (d'_1, \dots, d'_{i-1}, d'_{i+1}, \dots, d'_n)$ where

$$d'_j = \begin{cases} d_j - 1, & \text{if } j \in S \cup T, \\ d_j, & \text{otherwise.} \end{cases}$$

Notice that if π is weakly decreasing, then the sequence formed by laying off d_i with order is weakly decreasing. It is illuminating to see an example.

Example 41. Let $\pi = (d_1, d_2, d_3, d_4, d_5, d_6) = (5, 3, 3, 3, 3, 1)$ and $i = 5$. If we lay off d_i , we remove d_5 and subtract one from d_1, d_2, d_3 forming the sequence $(4, 2, 2, 3, 1)$. If we lay off d_i with order, we remove d_5 , and subtract one from d_1, d_3, d_4 , resulting in the sequence $(4, 3, 2, 2, 1)$.

The sequence formed by laying off d_i with order is a permutation of the sequence formed by laying off d_i . Since the order does not affect whether a sequence is graphic, the following slightly modified version of the Kleitman–Wang Theorem is true.

Theorem 42. *For any $i = 1, \dots, n$, the sequence $\pi = (d_1, \dots, d_n)$ is graphic if and only if the sequence π' obtained by laying off d_i with order is graphic.*

4.4 Proof of Theorem 29

We will first define the partition we will use in Theorem 29, which depends only on the order of the given weakly-decreasing sequence.

Definition 43. Given a graph on vertex set $\{v_1, \dots, v_n\}$, n even, the *parity bisection* is the bisection given by the partition $L = \{v_1, v_3, \dots, v_{n-1}\}$ and $R = \{v_2, v_4, \dots, v_n\}$. Later, we will consider a graph on vertex set $\{v_1, \dots, v_{n-2}, a, b\}$. In this case, we say $L = \{v_1, v_3, \dots, v_{n-3}\}$, $R = \{v_2, v_4, \dots, v_{n-2}\}$, and the *parity bisection* is the bisection given by $L \cup \{a\}$ and $R \cup \{b\}$.

To simplify our discussion, we make a few other definitions about vertices and edges in different parts of the partition.

Definition 44. An *across-edge* is an edge with one endpoint in L and one endpoint in R . A *same-side-edge* is an edge with both endpoints in the same part. Given a vertex v , an *across-neighbor* of v is a vertex adjacent to v via an across-edge. A *same-side-neighbor* of v is a vertex adjacent to v via a same-side-edge.

Definition 45. Given any indexed set, we say two elements are *consecutive* if their indices differ by one. Given a set of indexed vertices S , a *run* is a maximal subset of S consisting of consecutive vertices.

We will prove the result inductively on the length of the sequence π , using a strengthened inductive hypothesis. Specifically, we will prove

Theorem 46. Let $\pi = (d_1, \dots, d_n)$ be a weakly decreasing graphic sequence, where n is even, and let i be an index such that the value d_i appears at least twice in π . Then there exists a realization G of π with vertex set $\{v_1, \dots, v_n\}$ where $\deg_G(v_j) = d_j$ for all j , such that the parity bisection H satisfies $\deg_H(v_i) \geq \lfloor d_i/2 \rfloor$ and $\deg_H(v_j) \geq \lfloor (d_j - 1)/2 \rfloor$ for all $j \neq i$.

For the index i in Theorem 46, d_i is one of a pair of consecutive equal numbers in π . Let i_1 be the index of the odd-indexed element of the pair, and let i_2 be the index of the even-indexed element.

We first give an overview of the proof of Theorem 46. We first form $\pi' = (d'_1, \dots, d'_n)$ from π by laying off d_{i_1} with order, and next form $\pi'' = (d''_1, \dots, d''_n)$ from π' by laying off d'_{i_2} with order. We also choose an appropriate index ℓ to be used as i when applying the induction. By induction, there is a graph G_1 that realizes π'' , with H_1 its parity bisection. We add new vertices a and b to G_1 , with degrees d_{i_1} and d_{i_2} respectively, forming a new graph G_2 and new bisection H_2 .

In G_1 , every vertex has roughly half of its edges in H_1 . To maintain this property when forming G_2 and H_2 , we need every vertex that gains an edge to gain both one same-side

edge and one across-edge, or gain just one across-edge. However, there may be some “bad” vertices that receive a same-side-edge only. We fix all but perhaps one of the bad vertices using two-switches to create the graph G with bisection H . Furthermore, we can choose the bad vertex to correspond to the index ℓ that we chose before, and hence it will have an extra edge in H by induction. G and H then satisfy the requirements of Theorem 46.

Proof of Theorem 46. The base case is that π is empty or contains all zeros. In that case, the conclusion follows trivially.

We form π' from π by first laying off d_{i_1} with order. We then form π'' from π' by laying off d'_{i_2} with order. We will need to refer to π , π' , and π'' with consistent, consecutive labeling. Therefore, let $\rho = (f_1, \dots, f_{n-2})$ be the sequence π with d_{i_1} and d_{i_2} removed, and re-indexed so that the indices are consecutive (and hence do not skip over i_1 and i_2). Let $\rho' = (f'_1, \dots, f'_{n-2})$ be π' with d'_{i_2} removed, and re-indexed. Finally, let $\rho'' = (f''_1, \dots, f''_{n-2})$ be π'' re-indexed.

Let O be the set of all odd indices j where $f''_j = f_j - 1$, and E the set of all even indices j such that $f''_j = f_j - 1$. If $|E| > |O|$, choose an index $\ell \in E$ such that the value f_ℓ is repeated somewhere in ρ'' . If $|O| > |E|$, choose an index $\ell \in O$ such that the value f_ℓ is repeated somewhere in ρ'' . In both cases $|O| > |E|$ and $|E| > |O|$, the existence of a repeated value will follow from Claim 4, which will be proven later. Finally, if neither of the previous two cases apply, choose ℓ to be any index such that f_ℓ is repeated somewhere in ρ'' , which can be done since any graphic sequence has a repeated value. We are now ready to apply induction.

Stage 1 (Forming G_1). Apply Theorem 46 inductively to construct a realization G_1 on vertex set $\{v_1, \dots, v_{n-2}\}$, where vertex v_j has degree f''_j . In the parity bisection H_1 of G_1 , $\deg_{H_1}(v_\ell) \geq \lfloor \deg_{G_1}(v_\ell)/2 \rfloor$ and $\deg_{H_1}(v_j) \geq \lfloor (\deg_{G_1}(v_j) - 1)/2 \rfloor$ for $j \neq \ell$.

Stage 2 (Forming G_2). Let G_2 be the graph formed by adding vertices a, b to G_1 , such that

- a is adjacent to every v_j such that $f'_j = f_j - 1$,
- b is adjacent to every v_j such that $f''_j = f'_j - 1$,
- and a is adjacent to b if $d'_{i_2} = d_{i_2} - 1$.

Thus a has degree d_{i_1} and b has degree d_{i_2} . Let H_2 be the parity bisection of G_2 .

We consider various sets of vertices.

Definition 47. Let A be the neighborhood of a in G_2 , not including b , and let $A^* \subseteq A$ be those neighbors not adjacent to b . Similarly, let B be the set of vertices attached to b not including a , and let $B^* \subseteq B$ be those neighbors not adjacent to a . Let $A_L = A \cap L$, $A_R = A \cap R$, $A_L^* = A^* \cap L$, $A_R^* = A^* \cap R$, and similarly define B_L , B_R , B_L^* , and B_R^* .

Definition 48. Recall that a run is a maximal subset of consecutively labeled vertices. Vertices in A correspond to elements of π indexed by S and T from Definition 40, and the same is true for B . Thus, A and B each contain at most two runs. If A consists of two runs, let A_1 be the first run and A_2 be the second run. If A consists of one run that contains v_1 , then let A_1 be this run and let A_2 be empty. If A consists of one run that does not contain v_1 , then let A_2 be this run and let A_1 be empty. Similarly define B_1 and B_2 .

Notice that if A_1 is nonempty, it contains v_1 . The same is true for B_1 . Additionally notice that in the degree sequences ρ and ρ' , the degrees corresponding to vertices in A_2 form constant sequences. In ρ' and ρ'' , the degrees corresponding to vertices in B_2 also form constant sequences.

Claim 1. For $1 \leq s \leq t \leq n - 2$, let $Z = \{v_s, v_{s+1}, \dots, v_t\}$.

- (i) If Z is a run in A , then B does not contain $\{v_{s-1}, v_s, \dots, v_t, v_{t+1}\}$.

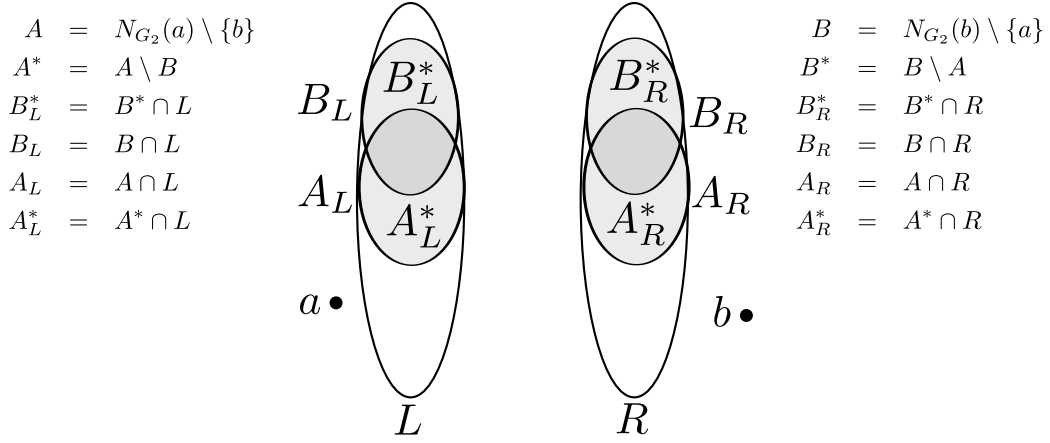


Figure 4.1: Sets of vertices important to the proof

(ii) Similarly, if Z is a run in B , then A does not contain $\{v_{s-1}, v_s, \dots, v_t, v_{t+1}\}$.

Proof. Proof of (i): The run Z is either A_1 or A_2 . If $Z = A_1$, then $v_s = v_1$ and hence B cannot contain v_{s-1} . Therefore, assume $Z = A_2$. Notice f'_{s-1} is at least one more than f'_s , since we subtracted one from f_s when laying off d_i with order. If $\{v_{s-1}, \dots, v_{t+1}\}$ is contained in B , then they must be in run B_1 , since B_2 cannot contain vertices of two distinct degrees in ρ' . B_1 then contains all of A_2 , and since B_1 starts on v_1 it must contain all of A_1 as well. B then contains every vertex of A plus v_{s-1} and v_{t+1} , contradicting that A and B are the same size.

Proof of (ii): Similarly, we see Z must be the run B_2 . Notice that f'_{t+1} must be less than f'_t , or otherwise the run B_2 would contain vertices with higher index than v_t . If $\{v_{s-1}, \dots, v_{t+1}\}$ is contained in A , it must be contained in run A_1 , since A_2 cannot contain vertices of two distinct ρ' degrees. Therefore A contains every vertex of B plus v_{s-1} and v_{t+1} , contradicting that A and B are the same size. □ (Claim 1)

Claim 2. Either A^* consists of at most one run and B^* consists of at most two runs, or B^* consists of at most one run and A^* consists of at most two runs.

Proof. By Claim 1, a run in A contains at most one run of A^* , and a run in B contains at most one run of B^* . Thus, A^* has at most two runs (corresponding to the two runs in A), and B^* has at most two runs (corresponding to the two runs in B). If $|A_1| \geq |B_1|$, then B^* contains no elements of B_1 and hence B^* has at most one run. Similarly, if $|A_1| \leq |B_1|$, A^* must have at most one run. \square (Claim 2)

For a vertex $u \in A \cup B$, we say u is *good* if it is adjacent to both a and b , or if it is adjacent to exactly one of a or b via an across-edge. We say u is *bad* if it is adjacent to exactly one of a and b with a same-side-edge. The problem with a bad vertex c is that it does not maintain the rate of half of its edges going across. That is, if $\deg_{H_1}(c) \geq \left\lfloor (\deg_{G_1}(c) - 1)/2 \right\rfloor$, when we add vertices a and b to form G_2 and H_2 , it is not necessarily true that $\deg_{H_2}(c) \geq \left\lfloor (\deg_{G_2}(c) - 1)/2 \right\rfloor$. Later in Stage 3, we will eliminate bad vertices via two-switches, and we will be able to eliminate all but $||A_L^*| - |B_R^*||$ bad vertices. Hence, it is important to bound this difference.

Claim 3. $||A_L^*| - |B_R^*||$ is at most one.

Proof. Since a and b have the same degree, we know $|A_L^*| + |A_R^*| = |B_L^*| + |B_R^*|$. Let this common sum be t . Given a run Z , $|Z \cap L|$ and $|Z \cap R|$ differ by at most one. From Claim 2, either A^* has at most one run or B^* has at most one run, and the other has at most two runs.

Case 1: A^* has at most one run and B^* has at most two runs. Then $|A_R^*|$ and $|A_L^*|$ differ by at most one and $|B_R^*|$ and $|B_L^*|$ differ by at most two.

Subcase (i): Assume $|A_L^*|$ and $|A_R^*|$ differ by one. Then one of $|A_L^*|$ and $|A_R^*|$ is $(t+1)/2$, and the other is $(t-1)/2$. Since $|A_L^*| + |A_R^*|$ is odd, that means $|B_R^*|$ and $|B_L^*|$ differ by one, and hence one is $(t+1)/2$ and the other is $(t-1)/2$. Therefore $|A_L^*|$ and $|B_R^*|$ differ by at most one.

Subcase (ii): Assume $|A_L^| = |A_R^*|$. Then $|A_L^*| = |A_R^*| = t/2$. $|B_L^*|$ and $|B_R^*|$ may differ by at most two, and hence $|B_R^*|$ is at least $t/2 - 1$ and at most $t/2 + 1$. Therefore $|A_L^*|$ and $|B_R^*|$ differ by at most one.*

Case 2: A^ has at most two runs and B^* has one run. Then $|A_R^*|$ and $|A_L^*|$ differ by at most two and $|B_R^*|$ and $|B_L^*|$ differ by at most one.*

Subcase (i): Assume $|B_L^|$ and $|B_R^*|$ differ by one. Then one of $|B_L^*|$ and $|B_R^*|$ is $(t+1)/2$, and the other is $(t-1)/2$. Since $|B_L^*| + |B_R^*|$ is odd, that means $|A_R^*|$ and $|A_L^*|$ differ by one, and hence one is $(t+1)/2$ and the other is $(t-1)/2$. Therefore $|A_L^*|$ and $|B_R^*|$ differ by at most one.*

Subcase (ii): Assume $|B_L^| = |B_R^*|$. Then $|B_L^*| = |B_R^*| = t/2$. $|A_L^*|$ and $|A_R^*|$ may differ by at most two, and hence $|A_R^*|$ is at least $t/2 - 1$ and at most $t/2 + 1$. Therefore $|A_L^*|$ and $|B_R^*|$ differ by at most one. \square (Claim 3)*

Let $L^* = A_L^* \cup B_L^*$ and let $R^* = A_R^* \cup B_R^*$. If there is a bad vertex after performing two-switches, we could have performed two-switches so that the bad vertex is any vertex in L^* . Similarly, if the bad vertex is in R^* , we can perform two-switches so that it can be any vertex in R^* . Eventually we will want the bad vertex to be one of a pair that are the same degree. The purpose of the next claim is to show that if there is a bad vertex, we will be able to choose the bad vertex so it has the same degree as some other vertex.

Claim 4. *If $|L^*| > |R^*|$, then there exists a vertex in L^* with the same degree as another vertex in G_2 . Similarly, if $|R^*| > |L^*|$, then there exists a vertex in R^* with the same degree as another vertex in G_2 .*

Proof. Case 1: $|L^| > |R^*|$. First we claim that L^* has at least one vertex adjacent to a and one adjacent to b . That is, A_L^* and B_L^* are both nonempty. Since $|L^*| > |R^*|$, either $|A_L^*| > |B_R^*|$ or $|B_L^*| > |A_R^*|$. Since $|A_L^*| + |A_R^*| = |B_L^*| + |B_R^*|$, we get in fact that both*

$|A_L^*| > |B_R^*|$ and $|B_L^*| > |A_R^*|$. In particular, both A_L^* and B_L^* are nonempty. Let $v_{j_1} \in A_L^*$ and let $v_{j_2} \in B_L^*$.

If there are two vertices in L^* of the same degree, then we are done. Even if there are two vertices in L^* whose degrees differ by one, there is a vertex in R with degree equal to one of these two, so again we are done. Therefore, we may assume the degrees of v_{j_1} and v_{j_2} differ by at least two. That is, either $f_{j_2} \geq f_{j_1} + 2$ or $f_{j_1} \geq f_{j_2} + 2$.

Subcase (i): $f_{j_2} \geq f_{j_1} + 2$. Since a is adjacent to v_{j_1} and not v_{j_2} , when laying off d_{i_1} with order, we subtracted one from f_{j_1} but not from f_{j_2} . However, that contradicts the definition of laying off, since $f_{j_2} > f_{j_1}$.

Subcase (ii): $f_{j_1} \geq f_{j_2} + 2$. Since a is adjacent to v_{j_1} and not v_{j_2} , when laying off d_{i_1} with order we subtracted one from f_{j_1} to form f'_{j_1} , and did not change f_{j_2} when forming f'_{j_2} . However, it is still the case that $f'_{j_1} > f'_{j_2}$. Since b is adjacent to v_{j_2} and not v_{j_1} , we subtracted one from f'_{j_2} and not f'_{j_1} . Again, this contradicts the definition of laying off, since $f'_{j_1} > f'_{j_2}$.

In either case, we have a contradiction. Therefore, our assumption that L^* contained vertices that only differed in degree by at least two was incorrect, so it must contain a vertex the same degree as another vertex in G_2 .

Case 2: $|R^*| > |L^*|$. This follows a symmetric argument. First we claim that R^* has at least one vertex adjacent to a and one adjacent to b . That is, A_R^* and B_R^* are both nonempty. Since $|R^*| > |L^*|$, either $|A_R^*| > |B_L^*|$ or $|B_R^*| > |A_L^*|$. Since $|A_L^*| + |A_R^*| = |B_L^*| + |B_R^*|$, we get in fact that both $|A_R^*| > |B_L^*|$ and $|B_R^*| > |A_L^*|$. In particular, both A_R^* and B_R^* are nonempty. Let $v_{j_1} \in A_R^*$ and let $v_{j_2} \in B_R^*$.

Consider the degrees of vertices in R^* . If there are two vertices of the same degree, then we are done. Even if there are two vertices that differ by one, we know that there is a vertex in L equal to one of these two, so again we are done. Therefore we may assume the degrees of v_{j_1} and v_{j_2} differ by at least two. That is, either $f_{j_2} \geq f_{j_1} + 2$ or $f_{j_1} \geq f_{j_2} + 2$.

Subcase (i): $f_{j_2} \geq f_{j_1} + 2$. Since a is adjacent to v_{j_1} and not v_{j_2} , when laying off d_{i_1} with order, we subtracted one from f_{j_1} but not from f_{j_2} . However, that contradicts the definition of laying off, since $f_{j_2} > f_{j_1}$.

Subcase (ii): $f_{j_1} \geq f_{j_2} + 2$. Since a is adjacent to v_{j_1} and not v_{j_2} , when laying off d_{i_1} with order we subtracted one from f_{j_1} to form f'_{j_1} , and did not change f_{j_2} when forming f'_{j_2} . However, it is still the case that $f'_{j_1} > f'_{j_2}$. Since b is adjacent to v_{j_2} and not v_{j_1} , we subtracted one from f'_{j_2} and not f'_{j_1} . Again, this contradicts the definition of laying off, since $f'_{j_1} > f'_{j_2}$.

In either case, we have a contradiction. Therefore, our assumption that R^* contained vertices that only differed in degree by at least two was incorrect, so it must contain a vertex the same degree as another vertex in G_2 . \square (Claim 4)

We will now perform two-switches to eliminate bad vertices.

Stage 3 (Forming G_3). *As long as there is a bad vertex $u \in L$ and a bad vertex $v \in R$, we can perform a two-switch by replacing the edges au and bv with av and bu . Then u and v are no longer bad vertices. Let G_3 be formed from G_2 by removing as many bad vertices as possible.*

By Claim 3, there is at most one bad vertex c in G_3 . We may need one additional two-switch to ensure this remaining bad vertex is the vertex v_ℓ we choose in the beginning. Notice that O is the set of indices of vertices in L^* and E is the set of indices of vertices in R^* .

Stage 4 (Forming G). *If there are no bad vertices left, let $G = G_3$. Otherwise, assume there is one bad vertex c in G_3 . Without loss of generality, assume c is in L^* . Hence $|L^*| > |R^*|$, and by Claim 4, there is a vertex in L^* whose degree is duplicated. Thus, when we chose ℓ , we chose it from O . If $v_\ell = c$, let $G = G_3$. Otherwise, form G from G_3 by replacing edges ac and bv_ℓ with av_ℓ and bc . Now v_ℓ is the bad vertex as desired.*

To agree with the indexing of Theorem 46, reindex the vertices of G so that the vertex set $\{v_1, \dots, v_{n-2}\} \cup \{a, b\}$ becomes $\{v_1, \dots, v_n\}$ with $a = v_{i_1}$ and $b = v_{i_2}$. Let H be the parity bisection.

Next we verify a and b have enough edges in the bisection H .

Claim 5. We have $\deg_H(a) \geq \lfloor \deg_G(a)/2 \rfloor$ and $\deg_H(b) \geq \lfloor \deg_G(b)/2 \rfloor$.

Proof. Case 1: Proving $\deg_H(b) \geq \lfloor \deg_G(b)/2 \rfloor$. Recall that B occurs in runs B_1 and B_2 . Notice that the first vertex in B_1 is an across-neighbor for b , since v_1 is in L . Hence, of the vertices in B_1 , at least $\lceil |B_1|/2 \rceil$ are across-neighbors for b . For B_2 , b has at least $\lfloor |B_2|/2 \rfloor$ across-neighbors. Thus, b has at least $\lfloor |B|/2 \rfloor$ across-neighbors, since $\lceil |B_1|/2 \rceil + \lfloor |B_2|/2 \rfloor \geq \lfloor |B|/2 \rfloor$ whenever $|B_1| + |B_2| = |B|$.

Case 2: Proving $\deg_H(a) \geq \lfloor \deg_G(a)/2 \rfloor$. Recall that A occurs in two runs, A_1 and A_2 . This case is not as easy as Case 1, since the first vertex in A_1 is not an across-neighbor. Thus, a might have only $\lfloor |A_1|/2 \rfloor$ across-neighbors in A_1 and $\lfloor |A_2|/2 \rfloor$ across-neighbors in A_2 , accounting for $\lfloor |A_1|/2 \rfloor + \lfloor |A_2|/2 \rfloor \geq \lfloor |A|/2 \rfloor - 1$ edges going across. However, if a two-switch was performed when forming G_3 , a gained an across-neighbor and has at least $\lfloor |A|/2 \rfloor$ across-neighbors. We proceed to show that if there are only $\lfloor |A|/2 \rfloor - 1$ across-neighbors for a in G_2 , then there must be a two-switch performed in Stage 3.

Assume a has only $\lfloor |A|/2 \rfloor - 1$ across-neighbors. Then runs A_1 and A_2 each have an odd number of vertices, and there is one more same-side-neighbor than across-neighbor inside each of A_1 and A_2 . Since A_1 and A_2 are both nonempty and runs are maximal, there is at least one vertex whose index lies between the indices of vertices in A_1 and the indices of vertices in A_2 . We call vertices in this range the *gap* between A_1 and A_2 .

If no two-switches are performed in Stage 3, then either A_L^* is empty or B_R^* is empty. We handle these two cases separately.

Subcase (i): A_L^ is empty.* The lowest and highest indexed vertices of A_2 are in A_L , and since $A_L \subseteq B_L$, b is adjacent in G_2 to the lowest and highest indexed vertices of A_2 . Since the vertices in A_2 all have the same degree in ρ' , b must be adjacent to all of A_2 . In addition, b is adjacent to all vertices with strictly higher ρ' degree, which includes all of A_1 and the gap. Thus, b has larger degree in G_2 than a , a contradiction.

Subcase (ii): B_R^ is empty.* Since $|A| = |B|$ and A_2 is nonempty, B cannot be completely contained in A_1 . Let v be an element of $B \setminus A_1$. Let u be the highest indexed vertex in the gap, and notice $u \in R$.

Subsubcase (a): Suppose $v \in A_2$, or has index higher than any vertex in A_2 . Since u has strictly higher degree in ρ' than v , b must also be adjacent to u . However, this is a vertex in B_R but not A_R , contradicting the supposition B_R^* is empty.

Subsubcase (b): Suppose v falls in the gap. Each of the vertices in the gap has the same ρ' degree. Therefore, if b is adjacent to any vertex in the gap, it is adjacent to the vertex of highest index, which is u . Again, this is a vertex in B_R but not A_R , contradicting the supposition B_R^* is empty. \square (Claim 5)

Having shown a and b have the desired degree in the bisection, we show that most of the other vertices have the desired degree.

Claim 6. For $j \neq \ell, i_1, i_2$,

$$\deg_H(v_j) \geq \left\lfloor \frac{d_j - 1}{2} \right\rfloor.$$

Proof. We have by the induction in Stage 1 that $\deg_{H_1}(v_j) \geq \lfloor (d_j'' - 1)/2 \rfloor$. If $d_j = d_j''$, then we immediately have $\deg_H(v_j) \geq \lfloor (d_j - 1)/2 \rfloor$, as desired. If $d_j'' = d_j - 1$, since v_j is not a bad vertex, this extra edge must be an across-edge, and hence

$$\deg_H(v_j) \geq \left\lfloor \frac{d_j - 2}{2} \right\rfloor + 1 \geq \left\lfloor \frac{d_j - 1}{2} \right\rfloor.$$

If $d_j'' = d_j - 2$, then at least one of the edges to a or b is an across-edge, and hence

$$\deg_H(v_j) \geq \left\lfloor \frac{d_j - 3}{2} \right\rfloor + 1 \geq \left\lfloor \frac{d_j - 1}{2} \right\rfloor.$$

Therefore, v_j has the required degree in H .

□ (Claim 6)

Finally, we show the bad vertex v_ℓ has the desired degree.

Claim 7.

$$\deg_H(v_\ell) \geq \left\lfloor \frac{d_\ell - 1}{2} \right\rfloor.$$

Proof. By the induction in Stage 1, $\deg_{H_1}(v_\ell) \geq \lfloor d_\ell''/2 \rfloor$. If v_ℓ is not a bad vertex in G , then the analysis of Claim 6 applies and v_ℓ will satisfy $\deg_H(v_\ell) \geq \lfloor (d_\ell - 1)/2 \rfloor$. If v_ℓ is a bad vertex, then $d_\ell'' = d_\ell - 1$. Therefore, $\deg_H(v_\ell) \geq \lfloor (d_\ell - 1)/2 \rfloor$ as desired.

□ (Claim 7)

By Claims 5, 6, and 7, every vertex in G has the H -degree required by Theorem 46. This proves Theorem 46 and thus Theorem 29. □

We conjecture that the lower bound from Theorem 29 can be improved to the value given by Example 37.

Conjecture 49. *Let $\pi = (d_1, d_2, \dots, d_n)$ be a graphic sequence, where n is even. Then there exists a realization G of π with a bisection H where for all vertices $v \in V(G)$, we have*

$$\deg_H(v) \geq \left\lfloor \frac{\deg_G(v)}{2} \right\rfloor.$$

4.5 Application to Conjecture 31

Definition 50. Let n be even. Let the *Kundu number* $K(\pi)$ of a graphic sequence π of length n be the largest k such that π has a realization with a k -factor. Equivalently by Kundu's Theorem, this is the largest k such that $\pi - k$ is graphic. Let the *bipartite factor number* $B(\pi)$ of a graphic sequence π be the largest k such that π has a realization G with a bipartite k -factor.

Using Theorem 29, we will show every graphic degree sequence π with large minimum value δ has a large value for $B(\pi)$. We use the following result of Csaba.

Theorem 51 (Csaba [14]). *Let G be a simple balanced bipartite graph on n vertices for n even with minimum degree δ at least $n/4$, and let*

$$\alpha = \frac{2\delta + \sqrt{n(4\delta - n)}}{4}.$$

Then G has an $\lfloor \alpha \rfloor$ -regular spanning subgraph.

This gives us the following theorem.

Theorem 52. *Let π be a graphic sequence with minimum value δ at least $n/2 + 2$. Then*

$$B(\pi) \geq \left\lfloor \frac{\delta - 2 + \sqrt{n(2\delta - n - 4)}}{4} \right\rfloor.$$

Proof. By Theorem 29, there exists a realization G of π with a bisection H of minimum degree at least $\left\lfloor \frac{\delta - 1}{2} \right\rfloor \geq \delta/2 - 1$. Since $\delta \geq n/2 + 2$, we have the minimum degree of H is at least

$$\frac{\delta}{2} - 1 \geq \frac{n/2 + 2}{2} - 1 \geq n/4.$$

Applying Csaba's Theorem to H , we obtain a regular bipartite graph of minimum degree at least

$$\left\lfloor \frac{2(\delta/2 - 1) + \sqrt{n(4(\delta/2 - 1) - n)}}{4} \right\rfloor = \left\lfloor \frac{\delta - 2 + \sqrt{n(2\delta - n - 4)}}{4} \right\rfloor.$$

□

Using the following classical theorem, we are able to split the bipartite factor from Theorem 52 into 1-factors.

Theorem 53 (Marriage Theorem, see [68]). *Every r -regular bipartite graph decomposes into r edge-disjoint 1-factors.*

Theorem 33 then follows from Theorem 52 and the Marriage Theorem.

4.6 Bipartite Number versus Kundu Number

For proving Conjecture 31, it is sufficient to find a bipartite factor, since the Marriage Theorem allows us to split that factor into 1-factors. How large of a bipartite factor can we find among all realizations of a degree sequence of length n ? First note that $B(\pi) \leq n/2$ since $B(\pi)$ is the degree of a regular bipartite graph, while $K(\pi)$ can be as high as $n - 1$. However, restricting $K(\pi)$ to at most $n/2$ we arrive at the question

Open Question 54. What is the largest γ such that, for all graphic sequences π of even length n with $K(\pi) \leq n/2$, we have $B(\pi) \geq \gamma K(\pi)$?

The following example shows that $\gamma \leq 3/4$.

Example 55. Let p be an even positive integer and set $n = p(p + 2)$. Let $\pi = \pi(p)$ be the sequence

$$(\underbrace{n - 1, \dots, n - 1}_p, \underbrace{n - p, \dots, n - p}_{p^2}, \underbrace{2p, \dots, 2p}_p).$$

We will show

Theorem 56. *Given the sequence π in Example 55, $K(\pi) = 2p$ and $B(\pi) = \frac{3}{2}p$. Thus, $B(\pi) = \frac{3}{4}K(\pi)$.*

However, π does have a realization with $K(\pi)$ edge-disjoint 1-factors, and hence satisfies Conjecture 31.

We will prove Theorem 56 as a series of propositions.

Proposition 57. *π is unigraphic.*

Proof. We exhibit a concrete realization G on the vertex set $V = \{v_1, \dots, v_n\}$. Let A be the first p vertices of V , which will have degree $n - 1$; let B be the next p^2 vertices of V , which will have degree $n - p$; let C be the last p vertices of V , which will have degree $2p$. Form G by placing a complete graph on $A \cup B$, and then placing a complete bipartite graph between A and C . Partition B into p equal sets $\{B_c\}_{c \in C}$, and attach $c \in C$ to all of B_c . See Figure 4.2.

We now verify that the degree sequence of G is π . Since the vertices in A are dominating, they have degree $n - 1$. Every vertex in B has p neighbors in A , $p^2 - 1$ neighbors in B , and 1 neighbor in C . Thus, every vertex in B has degree $p^2 + p = n - p$ as desired. Finally, every vertex in C has p neighbors in A , and $\frac{1}{p}p^2 = p$ neighbors in B , and this totals $2p$.

Finally, note that G is, up to isomorphism, the only realization of π . The vertices in A must be dominating, so they are adjacent to all other vertices and themselves. In $G - A$, the p^2 vertices in B have degree p^2 and the p vertices in C have degree p . The graph

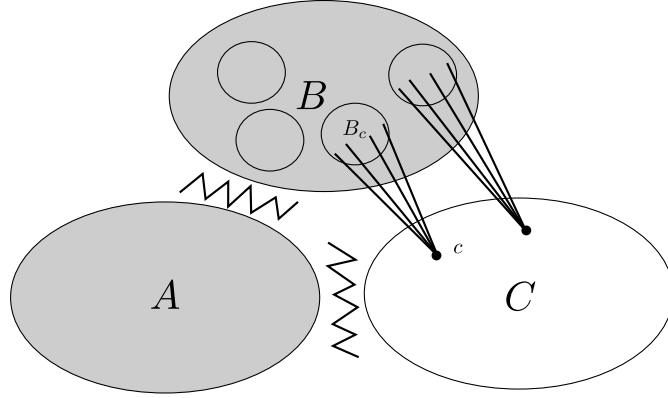


Figure 4.2: The realization of the degree sequence π from Example 55. $A \cup B$ is a complete graph, and there is a complete bipartite graph between A and C . Every vertex $c \in C$ is adjacent to every vertex in its corresponding B_c .

$B \cup C$ must therefore be a split graph, with B a complete graph and C an independent set. In the graph $G - A$ with the complete graph on B removed, the vertices in B each have degree one. Therefore, the vertices in C have disjoint neighborhoods in B that partition B . Thus, up to isomorphism, we arrive at the realization we described before. \square

Proposition 58. $K(\pi) = 2p$.

Proof. First note that $K(\pi) \leq 2p$, since the minimum degree of π is $2p$. Thus, we show that $K(\pi) \geq 2p$. Using Kundu's Theorem, we need only check that $\pi - 2p$ is graphic. In $\pi - 2p$, the last p entries are all zero. The $n - p$ nonzero entries are

$$(\underbrace{p^2 - 1, \dots, p^2 - 1}_p, \underbrace{p^2 - p, \dots, p^2 - p}_{p^2}).$$

To check whether this is graphic, we will use the Erdős–Gallai criterion [24]. Eggleton [22] and later Tripathi and Vijay [63] showed that the Erdős–Gallai Criterion only needs to be checked at the end of runs of degrees with the same value. Thus, for our sequence, we

only need to check one of the inequalities. We need to verify that

$$\sum_{i=1}^p d_i \leq p(p-1) + \sum_{p+1}^n \min\{d_i, p\}.$$

First note that $d_i \geq p$, since $p^2 - p \geq p$ as long as $p \geq 2$. Therefore, we just need to check that

$$\begin{aligned} \sum_{i=1}^p d_i &\leq p(p-1) + \sum_{p+1}^n p, \\ p(p^2 - 1) &\leq p(p-1) + p^2(p), \\ p^3 - p &\leq p^3 + p^2 - p, \end{aligned}$$

which is clearly true. □

Proposition 59. $B(\pi) = \frac{3}{2}p$.

Proof. First we will show that $B(\pi) \leq \frac{3}{2}p$. Let X, Y be the partition of G induced by a bipartite s -factor. We will show $s \leq \frac{3}{2}p$ in two cases.

Case 1: C has vertices in both X and Y . One of $|A \cap X|$ and $|A \cap Y|$ is at most $p/2$ since $|A| = p$. Without loss of generality, assume $|A \cap Y| \leq p/2$. Any vertex in $C \cap X$ has at most $p/2$ neighbors in $A \cap Y$, p neighbors in B , and no neighbors in C . Therefore it has at most $\frac{3}{2}p$ neighbors in Y .

Case 2: C is completely contained in X or Y . Without loss of generality, assume $C \subseteq X$. For any $c \in C$, c has $|A \cap Y| + |B_c \cap Y|$ neighbors in Y . Since the B_c are disjoint, for some $c' \in C$, we have $|B_{c'} \cap Y| \leq |B \cap Y|/p$. If q is $|A \cap Y|$, then c' has at most

$$|A \cap Y| + |B_{c'} \cap Y| \leq |A \cap Y| + |B \cap Y|/p \leq q + \frac{n/2 - q}{p}$$

neighbors in Y . This is maximized when q is as big as possible, which is $|A| = p$.

Therefore, c' has at most

$$q + \frac{n/2 - q}{p} \leq p + \frac{(p^2 + 2p)/2 - p}{p} = \frac{3}{2}p$$

neighbors in Y . This shows $B(\pi) \leq \frac{3}{2}p$.

To show that $B(\pi) \geq \frac{3}{2}p$, we can use a construction very much like Case 1 above. Partition A into two equal pieces A_1 and A_2 , and partition C into two equal pieces C_1 and C_2 . Let $B_1 = \bigcup_{c \in C_2} B_c$ and $B_2 = \bigcup_{c \in C_1} B_c$. Let H be the graph on vertex set $A \cup B \cup C$ consisting of

- all edges from C_1 to B_2 and A_2 ,
- all edges from C_2 to B_1 and A_1 ,
- all edges between A_1 and A_2 ,
- given a bijection $\sigma_1 : A_1 \rightarrow C_1$, edges from $a \in A_1$ to $B_{\sigma_1(a)}$,
- given a bijection $\sigma_2 : A_2 \rightarrow C_2$, edges from $a \in A_2$ to $B_{\sigma_2(a)}$,
- a $(\frac{3}{2}p - 2)$ -regular bipartite subgraph between B_1 and B_2 .

Note that we can find the required $(\frac{3}{2}p - 2)$ -regular bipartite subgraph between B_1 and B_2 since $|B_1| = |B_2|$, and we have a complete bipartite graph between B_1 and B_2 to choose from.

We claim H is a $\frac{3}{2}p$ -regular bipartite graph with partition $X = A_1 \cup B_1 \cup C_1$ and $Y = A_2 \cup B_2 \cup C_2$. We check that every vertex in X has degree $\frac{3}{2}p$, and the vertices in Y will have the same degree by symmetry. The vertices in C_1 have $p/2$ edges to A_2 and p edges to B_2 , and therefore vertices in C_1 have degree $\frac{3}{2}p$. Vertices in A_1 have $p/2$ edges to C_2 and p edges to B , and therefore have degree $\frac{3}{2}p$. Finally, the vertices in B_1 have one edge to A_2 , one edge to C_2 , and $\frac{3}{2}p - 2$ edges to B_2 . This again totals $\frac{3}{2}p$. \square

Chapter 5

Random Balanced Partitions and Edge-Disjoint Hamiltonian Cycles

5.1 Introduction

Our main theorem in this chapter states every graph has a balanced partition of the vertices, so that every vertex has many neighbors in each part.

Theorem 60. *Let G be a graph on n vertices, where $n = pq$. Then there exists a partition of the vertices of G into q parts of size p such that every vertex v has at least $\deg(v)/q - \sqrt{\min\{\deg(v), p\} \cdot \ln(n)}$ neighbors in each part.*

This theorem has many applications, including finding edge-disjoint 1-factors and edge-disjoint Hamiltonian cycles in dense graphs, and finding equitable domatic partitions with many parts. It has connections to discrepancy, especially multi-colored discrepancy, and to a conjecture of Bollobás and Scott.

A *bisection* is a spanning, balanced, bipartite subgraph. The Bollobás–Scott Conjecture says that every graph has a bisection of large degree.

Conjecture 61 (Bollobás and Scott [4]). *Every graph G has a bisection H such that for every vertex $v \in V(G)$,*

$$\deg_H(v) \geq \frac{\lfloor \deg_G(v) \rfloor}{2}.$$

Bush proved that every graph on n vertices contains a bisection of where each vertex v has degree at least $\deg(v)/2 - 4\sqrt{n \ln n}$, if n is sufficiently large [8]. This proves the leading term in the Bollobás–Scott conjecture is correct. Theorem 60 can be seen as a generalization of Bush’s result, where the Bush result is recovered and slightly improved upon by setting $q = 2$.

Nash-Williams [50] proved

Theorem 62 (Nash-Williams [50]). *If $\delta \geq n/2$, then G contains at least $\lfloor 5n/224 \rfloor$ edge-disjoint Hamiltonian cycles.*

He conjectured this was far from best possible, and noted that constructions exist showing the best possible value to be approximately $n/8$. Using the Szemerédi’s Regularity Lemma, Christofides, Kühn, and Osthus [12] proved an approximate version that achieves this upper bound.

Theorem 63 (Christofides et al. [12]). *For every $\alpha > 0$, there is a positive integer n_0 such that every graph on $n \geq n_0$ vertices of minimum degree $\delta \geq (1/2 + \alpha)n$ contains at least*

$$\frac{\delta - n\alpha + \sqrt{2\delta n - n^2}}{4}$$

edge-disjoint Hamiltonian cycles.

The well-known drawback of the regularity lemma is that n needs to be extremely large before the lemma applies, at least an exponential tower of twos of height proportional to $\log_2(1/\epsilon)$ [30].

We achieve the result of Christofides et al. using a simpler proof that avoids appealing to the regularity lemma.

Theorem 64. *Given a graph G of minimum degree $\delta \geq n/2 + O(n^{3/4} \ln n)$, G contains*

$$\frac{\delta - O(n^{7/8} \ln n) + \sqrt{2\delta n - n^2}}{4}$$

edge-disjoint Hamiltonian cycles.

Our method is similar in spirit, with the use of the Regularity Lemma replaced by Theorem 60, but is different in the details.

The Bollobás–Scott Conjecture is related to the well-studied question of discrepancy. Given a collection of subsets of $\{1, \dots, n\}$, the goal is to color the numbers in $\{1, \dots, n\}$ red and blue so that each subset has roughly the same number of blue elements as red elements. More precisely, given a set $A \subseteq \{1, \dots, n\}$ and a function $\sigma : \{1, \dots, n\} \rightarrow \{-1, 1\}$, the *discrepancy* of A is

$$\text{disc}(A) = \left| \sum_{a \in A} \sigma(a) \right|.$$

Given a collection of n sets $\mathcal{S} = \{A_1, \dots, A_n\}$, where each $A_i \subseteq \{1, \dots, n\}$, we wish to minimize the maximum discrepancy of any set in \mathcal{S} . A simple probabilistic argument yields that a discrepancy of at most $\sqrt{n \ln n}$ is achievable. Spencer [60] was able to remove the $\ln n$ factor and replace it with an absolute constant, and furthermore showed this constant could be as low as 6.

When the number of parts q in Theorem 60 is 2, then it can be seen as extending the discrepancy result where the color classes on $\{1, \dots, n\}$ must have equal size.

Doerr and Srivastav [18] extended many discrepancy results to multi-colored discrepancy. Given a collection of subsets of $\{1, \dots, n\}$, the goal is to color the numbers in

$\{1, \dots, n\}$ with many colors so that no subset has too many or too few elements that are a given color. In c -colored discrepancy, we have a function $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, c\}$. Additionally, for every color $i \in \{1, \dots, c\}$ we have a function σ_i given by

$$\sigma_i(a) = \begin{cases} \frac{c-1}{c} & \text{if } \sigma(a) = i \\ -\frac{1}{c} & \text{if } \sigma(a) \neq i. \end{cases}$$

Then the discrepancy of a set $A \subseteq \{1, \dots, n\}$ relative to color i is given by

$$\text{disc}(A, i) = \left| \sum_{a \in A} \sigma_i(a) \right|.$$

Note that this definition is natural in that a set A with $|A|/c$ elements colored red has a red-discrepancy of zero, and if we increase or decrease the number of red elements by a fixed amount ℓ , then the red-discrepancy increases by ℓ . Again, given a collection of m sets $\{A_1, \dots, A_m\}$, the goal is to minimize the discrepancy over all sets and all colors. Extending the simple probabilistic argument shows that the discrepancy can be made smaller than $\sqrt{\frac{1}{2}n \ln(mc)}$.

Theorem 60 can be seen as extending multi-colored discrepancy results to the case where each color class must have equal size.

Doerr and Srivastav [18] also achieved a multicolored discrepancy result similar to Spencer [60]: they showed $\text{disc}(\mathcal{H}, c) \leq O\left(\sqrt{\frac{n}{c} \ln \frac{mc}{n}}\right)$. In Section 5.4, using this Doerr and Srivastav result, we can improve Theorem 60 in some cases. In particular, in the bisection case of $q = 2$ we replace the $\ln(n)$ factor in the error term by an absolute constant.

We can also apply Theorem 60 to prove a partial result to a conjectured generalization of a result of Kundu. Kundu [42] characterized when a graphic sequence π has a realization that contains a k -regular spanning subgraph or k -factor. Brualdi [5] and independently Busch et al. [7] conjectured that every degree sequence with a realization

with a k -factor also has a realization containing k edge-disjoint 1-factors. Partial results by Busch, Ferrara, Hartke, Jacobson, Kaul and West [7] so far have focused on finding as many edge-disjoint 1-factors as possible. Our result gives more edge-disjoint 1-factors than other results in dense graphs.

Theorem 65. *Let G be a graph of minimum degree $\delta \geq n/2 + O(n^{3/4} \ln n)$. Set*

$$\rho = \frac{\delta + \sqrt{2\delta n - n^2}}{2}.$$

Then G contains at least $\rho - O(n^{7/8} \ln(n))$ edge-disjoint 1-factors.

Theorem 60 also applies to the domatic number of a graph. Just as the chromatic number of a graph is the smallest number of parts in any partition of the vertices into independent sets, the *domatic number* is the largest number of parts in any partition of the vertices into dominating sets. A partition into dominating sets is called a *domatic partition*. Feige, Halldórsson, Kortsarz, and Srinivasan [26] proved the strongest known lower bounds in terms of the minimum degree.

Theorem 66 (Feige et al. [26]). *Every graph G of minimum degree δ has domatic number at least*

$$\left(\frac{\delta + 1}{\ln n} \right) (1 + o(1))$$

Their methods were probabilistic in nature. An *equitable domatic partition* is a domatic partition where each set must be the same size, and the largest number of parts in an equitable domatic partition is the *equitable domatic number*. Using Theorem 60, we achieve a similar result to the result of Feige et al. in the case of the equitable domatic number.

5.2 Random Partitions of High Degree

We consider simple graphs, without loops or multiple edges. For any graph G with vertex v , let $N(v)$ denote set of neighbors of v , $\Delta(G)$ denote the largest degree in G , and $\delta(G)$ denote the smallest degree.

To show that a random variable is likely to have values close to its mean, the Chernoff–Hoeffding bound is extremely handy. Not only is it easy to use, but it gives an exponential drop-off in probability as the random variable gets farther away from the mean. The version we will use comes from [19].

Theorem 67 (Chernoff–Hoeffding Bound). *Let $X = \sum_{i \in [n]} X_i$, where the X_i are independent random variables taking values in the interval $[0, 1]$. Then*

$$\Pr[|X - E[X]| > \epsilon] \leq 2e^{-\frac{2\epsilon^2}{n}}.$$

Using the Chernoff–Hoeffding bound, we can show that a particularly chosen multipartite subgraph will be such that every vertex has many neighbors in each part.

Theorem 68. *Let G be a graph on n vertices, where $n = pq$ for $p > 1$. Then there exists a partition of the vertices of G into q parts of size p such that every vertex v has at least $\deg(v)/q - \sqrt{\min\{\deg(v), p\} \cdot \ln(n)}$ neighbors in each part.*

Proof. Let the vertex set be $\{v_{ij}\}_{i \leq p, j \leq q}$, where the labeling of the vertices is arbitrary. We think of the vertices as forming the columns and rows of a matrix M . Let R_1, \dots, R_p be the sets of vertices corresponding to the rows M . Let S_q be the symmetric group on q elements. For each row R_i , randomly, independently, and uniformly choose a permutation $\sigma_i \in S_q$, and permute the entries of R_i with σ_i to form a new matrix M' . Let C_1, \dots, C_q be the columns of M' , each of which has size p .

Let X_{v,C_i} be the random variable indicating the number of v 's neighbors in C_i . We want to calculate the expected value $\mathbb{E}X_{v,C_i}$. We write X_{v,C_i} as

$$X_{v,C_i} = \sum_{j=1}^p |R_i \cap C_j \cap N(v)|.$$

Since each row R_i was permuted independently, X_{v,C_i} is the sum of independent $\{0, 1\}$ random variables. Also note we need not include terms of the sum where $N(v) \cap R_j$ is empty, and hence we can assume X_{v,C_i} is the sum of at most $\min\{p, \deg(v)\}$ such random variables. Let $\ell_v = \min\{p, \deg(v)\}$. We then compute

$$\mathbb{E}X_{v,C_i} = \sum_{j=1}^p \mathbb{E}|C_i \cap R_j \cap N(v)| = \sum_{j=1}^p \frac{|N(v) \cap R_j|}{q} = \frac{\deg(v)}{q}.$$

Let B_{v,C_i} be the bad event that v has fewer than $\deg(v)/q - \sqrt{\ell_v \ln n}$ neighbors in C_i . Since X_{v,C_i} is the sum of independent $\{0, 1\}$ random variables, we can apply the Chernoff-Hoeffding bound, obtaining

$$\Pr[B_{v,C_i}] \leq e^{-2(\sqrt{\ell_v \ln n})^2 / \ell_v} = \frac{1}{n^2}.$$

Notice there are nq bad events. Applying the union-sum bound, the probability at least one of the bad events occurs is at most nq/n^2 , which is less than 1, and hence with nonzero probability C_1, \dots, C_q forms a partition of the vertices where each vertex v has at least $\deg(v)/q - \sqrt{\ell_v \ln n}$ edges to each part. \square

If n is not exactly pq , we obtain a similar result partitioning G into q parts that are almost equal.

Corollary 69. *Let G be a graph on n vertices, q a positive integer less than n , $p = \lfloor n/q \rfloor$. Then there exists a partition of the vertices of G into q parts, each of size p or $p + 1$, such that every*

vertex v has at least $\deg(v)/q - \sqrt{\min\{\deg(v), p+1\} \cdot \ln(n+q)}$ neighbors in each part.

Proof. Form G' by adding $(p+1)q - n$ isolated vertices to G . Notice that G' has exactly $(p+1)q$ vertices. We will now apply Theorem 60 to G' , but recall that in the proof of that theorem, we arranged the vertices into a matrix arbitrarily. Here we ensure that all the newly added isolated vertices are in the same row. After randomly permuting inside each row, every column will have at most one of these new isolated vertices.

By Theorem 60, every vertex v in G' will have at least

$$\deg(v)/q - \sqrt{\min\{\deg(v), p+1\} \cdot \ln((p+1)q)}$$

neighbors in each column of the matrix of vertices. Since each column contains at most one of the isolated vertices, by removing these vertices, we will have a partition of G into q parts of size p or $p+1$, where each vertex has at least $\deg(v)/q - \sqrt{\min\{\deg(v), p+1\} \cdot \ln(n+q)}$ neighbors in each part. \square

The error term in the result is a factor of at most $3q^2\sqrt{\ln n}$ away from being tight, as the following examples of graphs show. The proof below is modified from the proof of a similar result in discrepancy theory; see Spencer [60] and Doerr and Srivastav [18].

Theorem 70. *For infinitely many n , there exists a graph G on n vertices such that any partition of G into q parts contains a part P and vertex v such that v has less than $\deg(v)/q - \frac{1}{2}\sqrt{n/q^3}$ neighbors in P .*

Proof. Let H be a symmetric Hadamard matrix. That is, H is a symmetric matrix with $\{-1, 1\}$ entries whose columns h_1, \dots, h_n are pairwise orthogonal. Symmetric Hadamard matrices are known to exist for all powers of 2 via Sylvester's construction [37]. By multiplying rows and columns by -1 , we can assume that h_1 consists entirely of 1's. Let A be the corresponding $\{0, 1\}$ matrix; that is, if J is the all ones matrix, then $A = \frac{1}{2}(H + J)$.

Set $\bar{A} = J - A$. Let G be the graph whose adjacency matrix is A , with vertices v_1, \dots, v_n corresponding to the rows or columns of A . Notice that G has some loops, but we will remedy this issue later on.

Partition the vertices of G into q parts, and let P be a part with at most $\lfloor (n-1)/q - 1 \rfloor$ vertices that does not contain v_1 . Let χ be a vector of length n with $(q-1)/q$ in positions corresponding to elements of P , and $-1/q$ everywhere else. Then $A\chi$ is a vector indicating the “discrepancy” of the number of neighbors each vertex has inside of P . That is, $A\chi$ indicates for each v the number of neighbors more or less than $\deg(v)/q$ inside of P . If $\|\cdot\|_\infty$ is the supremum norm, then $\|A\chi\|_\infty$ indicates the maximum of the discrepancy of how many neighbors a vertex can have outside P . We will proceed by giving a lower bound for this discrepancy.

First, we know $\|A\chi\|_\infty \geq \frac{1}{\sqrt{n}}\|A\chi\|_2$ by standard relationships among norms. Also, $A\chi = \sum_{i=1}^n \frac{1}{2}(\chi_i h_i + \chi_i h_1)$. If we set $\lambda = 2\chi_1 + \sum_{i=2}^n \chi_i$, we get $A\chi = \lambda h_1 + \sum_{i=2}^n \frac{1}{2}\chi_i h_i$. Thus we can compute

$$\begin{aligned}
\|A\chi\|_2 &= \left\| \lambda h_1 + \sum_{i=2}^n \frac{1}{2}\chi_i h_i \right\|_2 = \sqrt{\lambda^2 \|h_1\|_2^2 + \sum_{i=2}^n \chi_i^2 \left\| \frac{1}{2}h_i \right\|_2^2} \\
&\geq \sqrt{\sum_{i=2}^n \chi_i^2 \left\| \frac{1}{2}h_i \right\|_2^2} = \frac{\sqrt{n}}{2} \sqrt{\sum_{i=2}^n \chi_i^2} = \frac{\sqrt{n}}{2} \sqrt{\sum_{i \in P} \frac{(q-1)^2}{q^2} + \sum_{i \notin P} \frac{1}{q^2}} \\
&\geq \frac{\sqrt{n}}{2} \sqrt{\frac{(n-1)(q-1)^2}{q^2(q-1)} + \left(n-1 - \frac{n-1}{q-1}\right) \frac{1}{q^2}} \\
&\geq \frac{\sqrt{n}}{2} \sqrt{\frac{(n-1)(q-1)}{q^2}}.
\end{aligned}$$

Therefore, there exists a vertex v with fewer than $\deg(v)/q - \frac{1}{2}\sqrt{\frac{(n-1)(q-1)}{q^2}}$ neighbors in a part P or more than $\deg(v)/q + \frac{1}{2}\sqrt{\frac{(n-1)(q-1)}{q^2}}$ neighbors in some part P . If the

former case holds, then the result follows, so assume some vertex v has more than $\deg(v)/q + \frac{1}{2}\sqrt{\frac{(n-1)(q-1)}{q^2}}$ neighbors in P .

Since v has at least $\deg(v)/q + \frac{1}{2}\sqrt{\frac{(n-1)(q-1)}{q^2}}$ neighbors in P , v has at most $\frac{q-1}{q}\deg(v) - \frac{1}{2}\sqrt{\frac{(n-1)(q-1)}{q^2}}$ neighbors in the other $q-1$ parts. By the Pigeonhole Principle, there is some part with at most $\frac{1}{q}\deg(v) - \frac{1}{2}\sqrt{\frac{n}{q^3}}$ neighbors of v . \square

5.3 Local Lemma Version

In this section we make use of the Lovász Local Lemma [1].

Lemma 71 (Lovász Local Lemma [1]). *Let $\{B_v\}$ be a set of events such that each B_v is independent from all but d of the B_u (including B_v itself) and $\Pr[B_v] \leq p$ for all v . Suppose*

$$epd \leq 1,$$

where e is the base of the natural logarithm. Then there is a positive probability none of the B_v occur.

Using the Local Lemma, we can remove the dependence on n and replace it with a dependence on Δ and q .

Theorem 72. *Let G be any graph on $n = pq$ vertices for $p > 1$, and let $\Delta = \Delta(G)$. Then there exists a partition of the vertices of G into q parts of size p such that every vertex v has at least $\deg(v)/q - \sqrt{\deg(v) \ln(\sqrt{e}\Delta q)}$ neighbors in each part.*

Proof. Let M' be the random matrix described in Theorem 60 with rows R_1, \dots, R_p and columns C_1, \dots, C_q . If X_{v,C_i} is the number of neighbors v has in C_i , then again X_{v,C_i} is the sum of independent $\{0, 1\}$ random variables with mean $\deg(v)/q$.

Let B_{v,C_i} be the bad event that v has fewer than $\deg(v)/q - \sqrt{\deg(v)(1/2) \ln(e\Delta^2 q^2)}$ edges to C_i . Applying the Chernoff–Hoeffding bound, we have

$$\Pr[B_{v,C_i}] \leq e^{-2\left(\sqrt{\deg(v)(1/2) \ln(e\Delta^2 q^2)}\right)^2 / \deg(v)} = \frac{1}{e\Delta^2 q^2}.$$

We need to determine how many other bad events B_{v,C_i} depends on to apply the Local Lemma. Consider two vertices v and w . If $N(v)$ and $N(w)$ lie in disjoint rows, then B_{v,C_i} and B_{w,C_j} do not depend on each other, since they are determined by disjoint, independent permutations. For vertex v , it has neighbors in at most Δ rows. Thus, there are at most Δq vertices in these rows. Each of these have up to Δ neighbors, and hence there are at most $\Delta^2 q$ vertices w such that w has a neighbor in a row with a neighbor of v . Finally, for each of these w , there are q bad events B_{w,C_j} . Hence, B_{v,C_i} depends on at most $\Delta^2 q^2$ other bad events (including itself). We can set $d = \Delta^2 q^2$ for applying the Local Lemma. We have that

$$epd = e \frac{1}{e\Delta^2 q^2} \Delta^2 q^2 = 1.$$

Applying the Local Lemma, we see the probability none of the bad events occur is positive, and hence Q_1, \dots, Q_q forms a partition of the vertices where each vertex v has at least $\deg(v)/q - \sqrt{\deg(v)(1/2) \ln(e\Delta^2 q^2)} = \deg(v)/q - \sqrt{\deg(v) \ln(\sqrt{e}\Delta q)}$ edges to each part. \square

5.4 Using Results from Discrepancy

Recall with multicolored discrepancy, the goal is to color the elements of a ground set so that no subset from a collection of m subsets has too many or too few elements of any particular color. To avoid confusion over the meaning of n , we will change our ground set from the usual $\{1, \dots, n\}$ to $\{1, \dots, t\}$. When the number of colors is two, this reduces

to normal discrepancy, and Spencer [60] proved a well known result that there exists a coloring within a discrepancy of at most $O(\sqrt{t})$, which improves basic probabilistic bounds by removing a $\ln(t)$ factor. Doerr and Srivastav proved the following bound that was analogous to the result by Spencer.

Theorem 73 (Doerr and Srivastav [18]).

$$\text{disc}(\mathcal{H}, c) \leq O\left(\sqrt{\frac{t}{c} \ln\left(\frac{mc}{t}\right)}\right)$$

We can use this to obtain a result about balanced partitions of high degree.

Theorem 74. *There exists an absolute constant K with the following property: Let G be a graph on n vertices, where $n = pq$ for $p > 1$. Then there exists a partition of the vertices of G into q parts of size p such that every vertex v has at least $\deg(v)/q - K\sqrt{n \ln(q)}$ neighbors in each part.*

Proof. Form the matrix M from the proof of Theorem 60. Recall this matrix has p rows and q columns. Let $A_{v,i}$ be the rows containing vertices from $N(v) \cap C_i$. Then there are qn sets of rows $A_{v,i}$. Setting $t = p$, $c = q$, and $m = qn$, we apply Theorem 73, and obtain a coloring of the rows of M with q colors where the discrepancy is at most $O\left(\sqrt{\frac{p}{q} \ln q^3}\right) = O\left(\sqrt{\frac{p}{q} \ln q}\right)$. Now, if R_i receives color j , perform a cyclic shift of the row R_i , where the shift is j units to the right. This forms a new matrix M' .

Now consider a vertex v and column C_i . How many neighbors does v have in C_i ? We need to think about the q sets $A_{v,j}$ associated with v . We know each of these sets had discrepancy at most $O\left(\sqrt{\frac{p}{q} \ln q}\right)$ under the coloring of the rows. Therefore, of the vertices in $A_{v,j} \cap C_j$ that ended up in C_i , these have discrepancy relative to $A_{v,j}$ at most $O\left(\sqrt{\frac{p}{q} \ln q}\right)$. In other words, we have a range of $|A_{v,j}|/q \pm O\left(\sqrt{\frac{p}{q} \ln q}\right)$ neighbors of v that started in C_j end up in C_i . Summing over all j , the total number of neighbors of v

that end up in C_i is at least $\deg(v)/q - O\left(q\sqrt{\frac{p}{q}\ln q}\right)$, which is $\deg(v)/q - O(n\ln q)$, as desired. \square

Note that this theorem is an improvement over Theorem 60 when q is a constant relative to n .

5.5 Equitable Domatic Colorings

Recall that analogous to the chromatic number, which is the size of the partition with fewest parts into independent sets, the *domatic number* is the partition with most number of parts into dominating sets, and that such a partition is a *domatic partition*.

A partition is *balanced* if the size of any two parts differ by at most 1. An equitable k -coloring of a graph is a balanced partition into independent sets. An *equitable domatic k -coloring* is a balanced partition into k dominating sets. We define the *equitable domatic number* to be the largest k such that G has an equitable domatic k -coloring.

Ideally, we would like some sort of analog to a deep theorem due to Hajnal and Szemerédi concerning the equitable chromatic number.

Theorem 75 (Hajnal and Szemerédi [32]). *Every graph of maximum degree Δ has a equitable coloring using at most Δ colors.*

In this section we consider how high the minimum degree of a graph needs to be to guarantee an equitable domatic k -coloring.

As a simple corollary to the Hajnal-Szemerédi theorem, we obtain the following result.

Corollary 76. *If G is a graph with $\delta(G) \geq \frac{k-1}{k}n$, then G has an equitable domatic k -coloring.*

Proof. Consider \overline{G} . It has maximum degree $\left\lfloor n - 1 - \frac{k-1}{k}n \right\rfloor = \lfloor n/k \rfloor - 1$. By the Hajnal-Szemerédi Theorem, there exists a partition of the vertices into $A_1, \dots, A_{\lfloor n/k \rfloor}$ with A_i

independent in \overline{G} and $||A_i| - |A_j|| \leq 1$. This means each A_i has size k or $k + 1$. In G , each A_i is a clique.

Now form sets V_1, \dots, V_k by placing one vertex from each A_i into each V_i . If there are vertices left over, distribute them as equally as possible. Each V_i will be dominating, since every vertex will have a neighbor from its clique A_j in each V_i . Thus G has an equitable domatic k -coloring. \square

However, the requirement $\delta(G) \geq \frac{k-1}{k}n$ is very strong. Can we weaken the condition $\delta(G) \geq \frac{k-1}{k}n$ and still get the result to hold? If we want an equitable domatic 2-coloring, the Corollary requires $\delta(G) \geq \frac{n}{2}$. However, a result of Bush [8] shows that $\delta \geq 2$ suffices. In particular, Bush [8] shows that every simple graph has a spanning, balanced, bipartite subgraph of minimum degree 1. Here we give a shorter proof of this fact.

Theorem 77. *For any graph G such that $\delta(G) \geq 2$, G has an equitable domatic 2-coloring.*

Proof. We are asked to partition G into two dominating sets A and B with $|A| = |B|$ or $|A| = |B| + 1$. In other words, we want every vertex has one neighbor in the other partition.

Start with a maximum matching M of G , and place one endpoint from each edge in the matching into A_1 , and the other endpoint into B_1 . The vertices covered by the matching now have one neighbor in the other partition, so we just need to consider the set of vertices C not in the matching. Notice C forms an independent set, since otherwise we could increase the size of the matching M . Every vertex in C is adjacent to at least two vertices in M . Let A_2 and B_2 be a partition of C , such that every vertex in A_2 has a neighbor in B_1 , and every vertex in B_2 has a neighbor in A_1 . Outlined below is a procedure to “even out” A_2 and B_2 by making the larger set smaller by one, and the smaller set larger by one, while maintaining the property that every vertex has at least

one neighbor across. By repeating this procedure, we will be able to guarantee that $|A_2|$ is within one of $|B_2|$.

Without loss of generality, assume $|A_2| > |B_2|$. If there is a vertex in A_2 with neighbors in both A_1 and B_1 , we can move that vertex to B_2 to even out the size of the two sets.

Suppose every vertex in A_2 has all its neighbors in B_1 . Choose an arbitrary $x \in A_2$, let v be a neighbor of x in B_1 , and let $u \in A_1$ be the vertex matched with v under the matching M . Perform the following list of swaps: place u into B_1 , v into A_1 , and x into B_2 .

These swaps have evened out A_2 and B_2 , but is it still true that every vertex in A_2 has a neighbor in B_1 , and every vertex in B_2 has a neighbor in A_1 ? Consider $y \in A_2$. We know y has at least two neighbors, and before the swaps they were both in B_1 . By switching around u and v , y may have lost one neighbor in B_1 , but not both. Therefore it must still be the case that every vertex in A_2 has a neighbor in B_1 .

Consider $y \in B_2$. If y does not have a neighbor in A_1 after the switch, then y is adjacent to u . But that's impossible, since if we remove uv from M and add in yu and xv , we get a bigger matching, a contradiction. Therefore every vertex in B_2 has a neighbor in B_1 . Thus we have successfully evened out $|A_2|$ and $|B_2|$.

By repeating this procedure, we can make it so $|A_2|$ and $|B_2|$ are within one of each other. After this process, $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ is the desired partition. \square

If we apply Corollary 69, we can achieve equitable domatic partitions with many more parts, provided we have high enough minimum degree.

Theorem 78. *Every graph G on n vertices of minimum degree $\delta > \sqrt{k(n+k) \ln(n+k)}$ has a equitable domatic coloring of size k .*

Proof. Fix $k > 1$, and let G be a graph of minimum degree $\sqrt{k(n+k) \ln(n+k)}$ and set $q = k$. By Theorem 60, there exists partition of G into q nearly equal sets each of

size $\lfloor n/q \rfloor$ or $\lceil n/q \rceil$ such that every vertex has at least $\deg(v)/q - \sqrt{\lceil n/q \rceil \ln(n+q)}$ neighbors in each part. Notice that

$$\begin{aligned} \deg(v)/q - \sqrt{\lceil n/q \rceil \ln(n+q)} &\geq \delta/k - \sqrt{(n/k+1) \ln(n+k)} \\ &> \frac{\sqrt{k(n+k) \ln(n+k)}}{k} - \sqrt{(n/k+1) \ln(n+k)} \\ &= 0. \end{aligned}$$

Thus, every vertex has at least one neighbor in each part. Hence the partition of size k is domatic and equitable, as desired. \square

Notice that Theorem 78 can be turned around and stated as every graph G has a equitable domatic coloring of size at least $\delta^2/((2n) \ln(2n)) - 1$. In comparison to the result of Feige et al. [27] which states that every graph has a (possibly non-equitable) domatic coloring of size $\delta/(\ln n) - o(n)$, the equitable result is smaller by a factor of approximately $\delta/(2n)$.

5.6 Many Edge-Disjoint 1-Factors

Using Theorem 60, we can find many edge-disjoint perfect matchings in a graph. We use the following result of Csaba.

Theorem 79 (Csaba [14]). *Let G be a simple balanced bipartite graph on $2n$ vertices with minimum degree at least $n/2$. Let $\delta = \delta(G)/n$, and let*

$$\rho = \frac{\delta + \sqrt{2\delta - 1}}{2}.$$

Then G has a $\lfloor \rho n \rfloor$ -regular spanning subgraph.

Using the following classical theorem, we can split the regular bipartite graph from Csaba's Theorem into 1-factors.

Theorem 80 (Marriage Theorem, see [68]). *Every r -regular bipartite graph decomposes into r edge-disjoint 1-factors.*

We also use another classical theorem.

Theorem 81 (See [68]). *For q even, the complete graph on q vertices decomposes into $q - 1$ edge-disjoint 1-factors.*

We now prove a result showing that a graph of high minimum degree contains many edge-disjoint 1-factors. We will outline the general scheme for proving this result, which will be followed again later as we improve upon this result in Sections 5.7 and 5.8.

Scheme 82 (Finding Edge-Disjoint 1-Factors). Given a graph G of high minimum degree, the general scheme that we follow to show it has many edge-disjoint 1-factors is as follows.

1. Using Theorem 60, we form a partition P_1, \dots, P_q such that every vertex has many neighbors in each part.
2. We think of P_1, \dots, P_q as forming the vertices of a complete host graph H . Thus, H is isomorphic to K_q .
3. We form a multiset of 1-factors in H . These 1-factors of H will be in exact 1-to-1 correspondence with the edge-disjoint 1-factors we will obtain in G . However, the 1-factors in H need not be edge disjoint, and in fact might be copies of the same 1-factor. However, we do require that no edge of H is used in too many 1-factors.
4. Given a pair of parts P_i and P_j , using Csaba's Theorem and the Marriage Theorem we obtain many edge-disjoint perfect matchings between P_i and P_j .

5. Using the perfect matchings between each pair of parts, we take the 1-factors of H and convert them to 1-factors of G .

The simplest use of this scheme yields the following theorem.

Theorem 83. *Let G be a graph on n vertices, where $n = pq$ and q is even. If G has minimum degree $\delta = \delta(G)$ at least $n/2 + q\sqrt{p \ln n}$, then G contains $(n - p)/4$ edge-disjoint 1-factors.*

Proof. Using Theorem 60, we partition the vertices of G into q equal parts P_1, \dots, P_q , each of size p , so that every vertex has degree at least $\delta/q - \sqrt{p \ln n}$ to each part. Using $\delta \geq n/2 + q\sqrt{p \ln n}$, we see there are at least

$$n/(2q) + \sqrt{p \ln n} - \sqrt{p \ln n} = p/2$$

edges to each part. We apply Csaba's theorem to obtain a $(p/4)$ -regular subgraph between every pair of parts. By the Marriage theorem, each subgraph between a pair of parts consists of $p/4$ edge-disjoint 1-factors. For $i < j$, let $m_1^{ij}, \dots, m_{p/4}^{ij}$ be the matchings between P_i and P_j .

Let M_1, \dots, M_{q-1} be edge-disjoint 1-factors on the complete graph on vertex set $\{P_1, \dots, P_q\}$. For each M_ℓ we produce $p/4$ 1-factors in G in the following way: consider all the edges in all the matchings m_ℓ^{ij} for all i, j such that $P_i P_j \in M_\ell$. Putting these edges together yields a matching in G , as in Figure 5.1. This yields a total of $p/4$ matchings for each M_i . Thus, we have a total of $(q - 1)(p/4) = (n - p)/4$ edge-disjoint matchings in G . □

Notice this is most powerful if p and q are both large. For example, if n is an even perfect square and we set $p = q = \sqrt{n}$, then we achieve $n/4 - \sqrt{n}/4$ edge-disjoint 1-factors whenever G has a minimum degree of $n/2 + n^{3/4}\sqrt{\ln n}$.

We can apply this towards a partial result of the conjectured Kundu generalization.

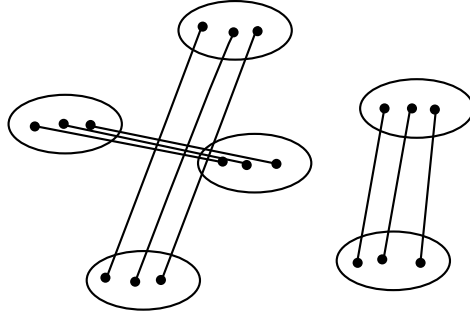


Figure 5.1: This example shows how matchings between parts can be put together into a matching of the entire graph (not all edges are shown). Here we have $q = 6$ and $p = 3$.

Corollary 84. *Given a degree sequence π of length n , where $n = pq$ for q even, then if π has minimum entry $n/2 + q\sqrt{p \ln n}$, there exists a realization of π (and indeed, every realization of π) with $(n - p)/4$ edge-disjoint 1-factors.*

It is interesting to note that, if $q \geq 4$, this result is stronger than our result from Chapter 4.

If δ from Theorem 83 is strictly larger than $n/2 + q\sqrt{p \ln n}$, then we can obtain even more 1-factors as δ increases.

Theorem 85. *Any graph G on n vertices, where $n = pq$, where q is even, of minimum degree $\delta = \delta(G)$ at least $n/2 + q\sqrt{p \ln n}$ contains*

$$\left\lfloor \frac{\delta/q - \sqrt{p \ln n} + \sqrt{2p\delta/q - \sqrt{p^3 \ln n} - p^2}}{2} \right\rfloor (q - 1)$$

edge-disjoint 1-factors.

The proof is identical to the proof of Theorem 83, except we use the full power of Csaba's theorem, yielding more 1-factors between each pair of parts P_i and P_j .

To highlight the differences between Theorem 83 and Theorem 85, an example is helpful.

Example 86. Let G be a graph on n vertices, where $n = pq$ for $q = 4$, and n is sufficiently large. Suppose further G has minimum degree $3n/4$. Theorem 83 gives $3n/16$ edge-disjoint 1-factors. Theorem 85 gives more than $7n/16$ edge-disjoint 1-factors.

5.7 More on Edge-Disjoint 1-Factors

As a consequence of Theorem 83, we obtained nearly $n/4$ many edge-disjoint 1-factors when n was a perfect square. In this section, we remove the requirement that n be a perfect square. We will follow Scheme 82, but instead of applying Theorem 60, we will use Corollary 69. We will also need several modifications to account for all the parts are not all the same size.

First, we need to choose p and q close to \sqrt{n} such that pq is close to n .

Lemma 87. *Given any positive integer n , there exists an integer p where $p \geq \sqrt{n}$ such that if $q = \lfloor n/p \rfloor$, then*

- q is even,
- $p - \sqrt{n} < \sqrt{2}n^{1/4} + 4$,
- $\sqrt{n} - q < \sqrt{2}n^{1/4} + 4$, and
- $n - pq < 4\sqrt{2}n^{1/4} + 8$.

In particular, a set of size n is can be partitioned into parts with sizes p and $p + 1$ such that there are at most $2\sqrt{2}n^{1/4} + 5$ parts of size $p + 1$.

Proof. Let $t = \lceil \sqrt{n} \rceil$. Let s^2 be the smallest perfect square such that $t^2 - s^2 \leq n$ and $t - s$ is even. Set $p = t + s$ and $q = t - s$. We know that $t < \sqrt{n} + 1$. We also have $(t - 1)^2 \leq n$, which implies $t^2 - n \leq 2t - 1$. We also know $(s - 2)^2$ is too small for $t^2 - (s - 2)^2$ to be less

than n , hence $(s-2)^2 \leq t^2 - n \leq 2t - 1$. Hence $s \leq \sqrt{2t-1} + 2 \leq \sqrt{2\sqrt{n}+1} + 2$. Both $p - \sqrt{n}$ and $\sqrt{n} - q$ are at most $s + 1 \leq \sqrt{2\sqrt{n}+1} + 3 \leq \sqrt{2}n^{1/4} + 4$. Since $pq = t^2 - s^2$ is less than n , but $t^2 - (s-2)^2$ is greater than n , we see $n - pq$ is at most $4s - 4$, and hence $n - pq$ is at most $4\sqrt{2}n^{1/4} + 8$. \square

When we use Theorem 60 to break a graph into nearly balanced parts, some of the parts may be one larger than others. We will need the following variation of Csaba's Theorem which applies when the parts are not exactly the same size.

Lemma 88. *Let G be a bipartite graph on $X \cup Y$ with $|X| = n + 1$, $|Y| = n$, such that vertices in X have minimum degree δ , and vertices in Y have minimum degree $\delta + 1$ for $\delta \geq n/2$. Let $k \leq \frac{\delta + \sqrt{2\delta n - n^2}}{2}$ and let $f : V(G) \rightarrow \mathbb{N}$ satisfy*

$$f(v) \leq k \quad \text{if } v \in X$$

$$f(v) = k \quad \text{if } v \in Y.$$

$$\text{and } \sum_{v \in X} f(v) = \sum_{v \in Y} f(v).$$

Then G has an f -factor H . Furthermore, the edges of H can be split up into matchings that saturate Y .

Proof. Suppose the lemma is not true. Then there are some functions f satisfying the requirements of the lemma such that G has no f -factor. Among these, choose the f to minimize the number of vertices $x \in X$ such that $f(x) < k$. Order the vertices of $X = \{x_0, \dots, x_n\}$ such that $f(x_0) \geq f(x_1) \geq \dots \geq f(x_n)$. Choose a b -factor B of G maximizing t such that

- $b(y) = k$ for $y \in Y$,
- $b(x_i) = f(x_i)$ for $i < t$,

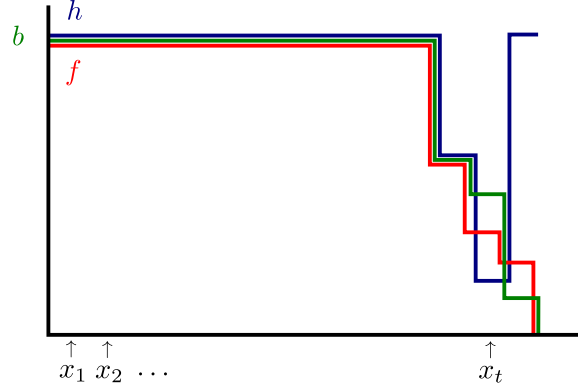


Figure 5.2: This is a visualization of the three sequences f , b , and h . For $i = 1, 2, \dots, n$, the height of the three lines represents $f(x_i)$, $b(x_i)$, and $h(x_i)$ respectively. Notice that $f(x_t)$ is between the two values $b(x_t)$ and $h(x_t)$. This allows us to modify b to form b' so that $b'(x_t)$ is closer to f than b is, contradicting the choice of b .

- $b(x_t) \geq f(x_t)$,
- t is maximized, and
- the quantity $b(x_t) - f(x_t)$ is minimized for the given t .

Note that by Csaba's theorem, we can find such a b -factor for $b(x_i) = k$ for $i < n$ and $b(x_n) = 0$, and hence b is well-defined. By assumption, $b(x_t) - f(x_t) \geq 1$. Hence there is some $\ell > t$ such that $f(x_\ell) - b(x_\ell) \geq 1$. Consider an h -factor H satisfying

$$h(x_i) = \begin{cases} f(x_i) & \text{if } i < t \\ k & \text{if } i > t \\ f(x_t) - \sum_{i=t+1}^{n+1} (k - f(x_i)) & \text{if } i = t. \end{cases}$$

Such an h -factor exists, since it has fewer non- k entries than f . Let $D(B)$ be the collection of edges in B but not in H , and let $D(H)$ be the collection of edges in H but not in B . Let P be a maximal alternating trail starting on an edge in $D(B)$ incident to x_t , and then alternating between edges in $D(B)$ and edges in $D(H)$. Notice that such an edge

exists since $b(x_t) > h(x_t)$. This will be an alternating trail in the graph $D(B) \cup D(H)$. In this graph, vertices in Y have equal b -degree and h -degree, so P does not end in Y . Nor does P end on x_0, \dots, x_{t-1} , since each of these also have equal number of incident B -edges and H -edges. Hence P ends on x_i for $i > t$. Switching along this alternating path will modify B into a new graph B' , which is a b' -factor such that

- $b'(y) = k$ for $y \in Y$,
- $b'(x_i) = f(x_i)$ for $i < t$,
- $b'(x_t) \geq f(x_t)$, and
- the quantity $b'(x_t) - f(x_t)$ is smaller than $b(x_t) - f(x_t)$,

which is a contradiction of the choice of b .

Now we know there exists such an f -factor F . We now wish to partition F into matchings that saturate Y . For any vertex $x \in X$ such that $f(x) < k$, consider the graph F' consisting of $F - x$. Let $Y' \subseteq Y$, and consider $N = N_{H'}(Y')$. In H' , there are at least $|Y'|k - f(x)$ edges leaving Y' . The set N has at most $k|N|$ edges entering it. If $|N| < |Y'|$, then Y' has at least $k - f(x)$ more edges leaving than N has entering, a contradiction. Hence $|N| \geq |Y'|$, and by Hall's Matching Theorem, there exists a matching saturating Y that avoids x . We can keep removing matchings in this way until F is totally partitioned. □

As in Theorem 83, we will use a 1-factorization of K_t as a part of the 1-factorization of the whole graph. However, in this case we will need to throw out a small number of 1-factors that include edges in a pre-specified set. The following lemma tells us that we will still have enough edge-disjoint 1-factors.

Lemma 89. *Let t be even, and let A be a set of vertices in K_t with $|A| = a \geq 2$. Then there exist $t - 2a + 2$ edge-disjoint 1-factors in K_t with no edges completely contained inside of A .*

Proof. In a standard 1-factorization of K_t , we start by drawing K_t in the plane by arranging $t - 1$ of the vertices as the vertices of a regular polygon, and placing the last vertex in the middle. Edges thus become line segments, each of which has a slope. The set of all edges with a certain slope s , plus the one edge from the center vertex with perpendicular slope, form a 1-factor. 1-factors of this type partition the edges, so this forms a 1-factorization. Let \mathcal{F} be this set of $(t - 1)$ 1-factors.

If $a \geq t/2$, then the result is trivial, so assume $a < t/2$. Since the arrangement of the vertices was arbitrary, without loss of generality, assume the vertices of A are consecutive vertices along the polygon, v_1, \dots, v_a . We will then simply take \mathcal{F} and remove any 1-factor with an edge inside of A . How many 1-factors are we removing? Suppose F is a 1-factor that is to be removed. Hence $v_i v_j$ is an edge inside of A that belongs to F . Notice if $|i - j| > 2$, then we can find another edge $v_{i+1} v_{j-1}$ that is also inside A and belongs to F , since the edges $v_i v_j$ and $v_{i+1} v_{j-1}$ have the same slope. Without loss of generality, we can assume that $|i - j| = 1$ or $|i - j| = 2$. However, there are only $a - 1$ such edges if $|i - j| = 1$, and $a - 2$ such edges if $|i - j| = 2$. Therefore, there are only $2a - 3$ 1-factors being removed. Thus the total number of 1-factors remaining is $(t - 1) - (2a - 3) = t - 2a + 2$. \square

We now put these ingredients together to prove the theorem.

Theorem 90. *Let G be a graph of minimum degree $\delta \geq n/2 + O(n^{3/4} \ln n)$. Set*

$$\rho = \frac{\delta + \sqrt{2\delta n - n^2}}{2}.$$

Then G contains at least $\rho - O(n^{7/8} \ln(n))$ edge-disjoint 1-factors.

Proof. Choose p and q as in Lemma 87. Hence, p and q are both within $2\sqrt{2}n^{1/4} + 4$ of $n^{1/2}$, and q is even. Let P_1, \dots, P_q be a decomposition as in Corollary 69. Let A be the set of parts of size $p + 1$, and B be all other parts. Let H be the complete graph on vertex set $A \cup B$. By Lemma 87, $|A| \leq 4\sqrt{2}n^{1/4} + 8$. By Lemma 89, there exists a set of 1-factors in H with $q - 2|A| + 2 \geq q - 8\sqrt{2}n^{1/4} - 14$ edge-disjoint 1-factors with no edges in A . Take $\rho/q - O(n^{3/8} \ln n)$ copies of each of these to form a set of 1-factors in H and call it \mathcal{F} . Since $\rho \leq n$ and we can assume $q \geq n^{1/2}/2$, we see ρ/q is at most $2n^{1/2}$. This gives at least $\rho - O(n^{7/8} \ln(n))$ 1-factors in \mathcal{F} .

By Corollary 69, between every pair of parts in B there exists a minimum degree $\delta/q - \sqrt{p \ln(n+q)} \geq \delta/q - 3n^{1/4} \ln(n)$ bipartite subgraph. By Csaba's Theorem, there is a k -factor between this pair of parts for

$$k = \frac{\delta/q - 3n^{1/4} \ln(n) + \sqrt{2(\delta/q - 3n^{1/4} \ln(n))(n/q) - (n/q)^2}}{2} \geq \rho/q - O(n^{3/8} \ln(n)).$$

Each of these split into $\rho/q - O(n^{3/8} \ln n)$ edge-disjoint 1-factors.

Between any pair of parts in A , there are $p\delta/q + 2p \geq \delta$ edges between them. Notice also that $|A|$ is even, since q is even and n is even, then $|A| = n - pq$ must also be even. Take an arbitrary matching of the parts in A , and for each F of the 1-factors in \mathcal{F} we will choose a collection of edges E_F , one between each pair of parts in A . We can make these choices arbitrarily.

For each $F \in \mathcal{F}$, we will now associate a distinct 1-factor F^* of the larger graph G . If F contains an edge between parts $P_i, P_j \in B$, then choose one of the 1-factors between the two parts. If F contains an edge between $P_i \in A$ and $P_j \in B$, then choose a matching between P_i and P_j that saturates B and avoids the edge from E_F inside P_i (we can do this by Lemma 88). Finally, place all the edges from E_F into the matching. This process yields a 1-factor of G , completing the proof. \square

5.8 Hamiltonian Cycles

We will now find many edge-disjoint Hamiltonian cycles in a graph of high minimum degree. We will again follow Scheme 82, but instead of finding 1-factors of the host graph H , we will use Hamiltonian cycles of the host graph H , which we will be able to translate to Hamiltonian cycles of the larger graph G .

We will use the following generalization of Dirac's Theorem due to Ghouila-Houri [29].

Theorem 91 (Ghouila-Houri [29]). *Let D be a digraph on n vertices such that the minimum in degree plus the minimum out degree is at least n . Then D contains a directed Hamiltonian cycle.*

This implies the following result about Hamiltonian cycles in bipartite graphs.

Lemma 92. *Let G be a bipartite graph with each part size n of minimum degree $n/2 + 1$, and let M be a perfect matching of G . Then M can be extended into a Hamiltonian cycle.*

Proof. Suppose G has bipartition $L \cup R$. For each vertex $x \in L$, let x' be the vertex in R that is matched with x in M . Let each $xx' \in M$ form the vertices of a digraph D . For each edge $x'y \in G$, place an edge from xx' to yy' in D . We have chosen D so that G is the split of the digraph D . Applying Ghouila-Houri Theorem to D , we obtain a directed Hamiltonian cycle. The vertices of this Hamiltonian cycle in D , in order of the cycle, have the form $x_1x'_1, x_2x'_2, \dots, x_nx'_n$. Since there is a directed edge from $x_ix'_i$ to $x_{i+1}x'_{i+1}$ in D , we have an edge between x'_i and x_{i+1} in G . Hence the edges $x_1x'_1, x'_1x_2, \dots, x_nx'_n, x'_nx_1$ form a Hamiltonian cycle in G . \square

Following Scheme 82, we need to find Hamiltonian cycles in the host graph H which will be in 1-to-1 correspondence with the Hamiltonian cycles we will obtain in G . Given a multiset of Hamiltonian cycles $\mathcal{F} = \{F_i\}$ of a graph G , and given some edge $e \in E(H)$, define $N(\mathcal{F}, e)$ to be the number of Hamiltonian cycles in \mathcal{F} containing e . Note that \mathcal{F}

need not contain edge-disjoint cycles, and in fact may contain multiple copies of identical Hamiltonian cycles.

We will use a multiplicative form of the Chernoff-Hoeffding bound from [19]:

Theorem 93 (Chernoff–Hoeffding Bound, see [19]). *Let $X = \sum_{i \in [n]} X_i$, where the X_i are independent random variables taking values in the interval $[0, 1]$. Then for $0 < \delta < 1$,*

$$\Pr(X > (1 + \delta)\mu) < e^{-\mu\delta^2/3}.$$

Lemma 94. *Fix $H = K_t$, and a, b be integers such that $a \leq 3\sqrt{t}$ and $t^{3/4} \leq b \leq t^{3/4} + 2$. Let $A, B, C \subseteq V(H)$ a partition such that $|A| = a$, $|B| = b$. For any $t/4 \leq \alpha \leq t$, there exists a multiset \mathcal{F} of $\alpha t/2 - 2t^{7/4}$ Hamiltonian cycles such that*

1. *for $e \in E(H)$, $N(\mathcal{F}, e) \leq \alpha + \sqrt{6t \ln t}$,*
2. *for $e \in E(H[A])$, $N(\mathcal{F}, e) = 0$,*
3. *for $e \in E(H[B])$, $N(\mathcal{F}, e) \leq \sqrt{t}$,*
4. *and every $F \in \mathcal{F}$ has exactly one edge in $E(H[B])$.*

Proof. Let $c = |C|$, and assume c is odd (otherwise we can just add a vertex from C to B). It is well known [2] that the complete graph on c vertices can be decomposed into $(c - 1)/2$ edge-disjoint Hamiltonian cycles. Let this set of Hamiltonian cycles on C be \mathcal{C} . We form the multiset \mathcal{D} by repeating each cycle of \mathcal{C} exactly α times. Thus \mathcal{D} is a set of at least $\alpha t/2 - 2t^{7/4}$ Hamiltonian cycles on $H[C]$. We just need to extend these cycles to cover the entire graph H .

We assign each cycle $F \in \mathcal{D}$ an edge e_F from $H[B]$ so that no edge of $H[B]$ is used more than \sqrt{t} times. Since there are at least $(t^{3/2} - t^{3/4})/2$ edges in B we can account for $(t^2 - t^{5/4})/2$ Hamiltonian cycles, which is larger than the number of in \mathcal{D} .

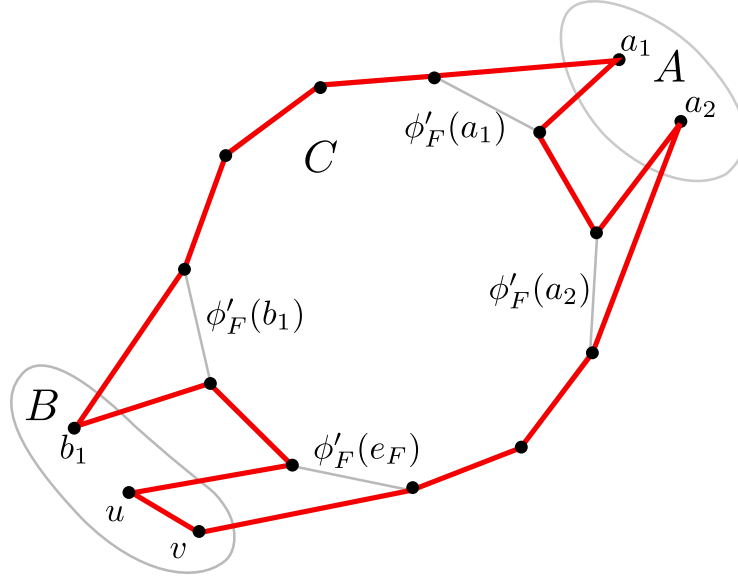


Figure 5.3: Shown is a Hamiltonian cycle in H formed by extending a Hamiltonian cycle in $H[C]$.

Fix a cycle $F \in \mathcal{D}$, and let $e_F = uv$. We will extend F to cover the vertices in $A \cup B$ using labels on the edges of F to guide the extension. Let $L_F = A \cup (B - \{u, v\}) \cup \{e_F\}$ be the *label set* of F . Let ϕ_F be an arbitrary injection from L_F to $\{0, 1, \dots, c-1\}$. Choose a random variable i_F that takes values uniformly from $\{0, 1, \dots, c-1\}$. Finally, define ϕ'_F such that for $a \in L_F$, $\phi'_F(a) = \phi_F(a) + i_F \pmod{c}$. Let $\{e_0, \dots, e_{c-1}\}$ be the edges of F . Then we extend F by using edge $e_{\phi'_F(a)}$ to take a detour through $a \in L_F$ as in Figure 5.3. In particular, for a vertex in $a \in L_F$, we remove edge $e_{\phi'_F(a)} = xy$ and replace it with the edges xa and ay . For the edge $e_F \in L_F$, we remove $e_{\phi'_F(e_F)} = xy$ and replace it with the edges xu , uv , and vy . Extend each cycle $F \in \mathcal{D}$ independently in this way to create a new multiset of Hamiltonian cycles \mathcal{F} .

The Hamiltonian cycles \mathcal{F} satisfy conditions 2, 3, and 4 of the theorem statement by construction, and satisfy condition 1 for all but perhaps the edges between $A \cup B$ and C . We will show that there is an assignment of the random variables i_F so that \mathcal{F} satisfies condition 1 for these edges as well. The probability any edge e between $A \cup B$ and C is

used in a given $F \in \mathcal{F}$ is at most $2/c$. Note that $N(\mathcal{F}, e)$ is a random variable counting the number of Hamiltonian cycles that use edge e in \mathcal{F} . We see $\mathbb{E}N(\mathcal{F}, e) \leq \frac{2}{c}|\mathcal{F}| = \frac{c-1}{c}\alpha < \alpha$. For a given edge e , the event that F uses e is independent the even that another cycle in \mathcal{F} uses edge e . For every edge e between $A \cup B$ and C , we have the bad event that $N(\mathcal{F}, e) \geq \alpha + \sqrt{6\alpha \ln t}$. We want to show that the probability of these bad events is low. Applying Theorem 93, we get

$$\begin{aligned} \Pr(N(\mathcal{F}, e) \geq \alpha + \sqrt{8\alpha \ln t}) &= \Pr\left(N(\mathcal{F}, e) \geq \left(1 + \sqrt{6 \ln t / \alpha}\right) \alpha\right) \\ &\leq e^{-\alpha \cdot (\sqrt{6 \ln t / \alpha})^2 / 3} \\ &\leq \frac{1}{t^2}. \end{aligned}$$

Since $0 \leq |A| \leq 3\sqrt{t}$, $t^{3/4} \leq |B| \leq t^{3/4} + 2$, we have at most

$$\begin{aligned} |C|(|A| + |B|) &= (t - t^{3/4})(t^{3/4} + 2 + 3\sqrt{t}) \\ &= t^{7/4} + 2t^{3/2} - 3t^{5/4} + 2t - 2t^{3/4} \end{aligned}$$

edges between $A \cup B$ and C . By the union-sum bound, the probability any bad event occurs is at most $t^{-1/4} + 2t^{-1/2} - 3t^{-3/4} + 2t^{-1} - 2t^{-5/4}$. This is strictly less than 1 for all values of t . \square

Now we are ready to prove the theorem.

Theorem 95. *Let G be a graph of minimum degree $\delta \geq n/2 + O(n^{3/4} \ln(n))$. Set*

$$\rho = \frac{\delta + \sqrt{2\delta n - n^2}}{4}.$$

Then G contains at least $\rho - O(n^{7/8} \ln n)$ edge-disjoint Hamiltonian cycles.

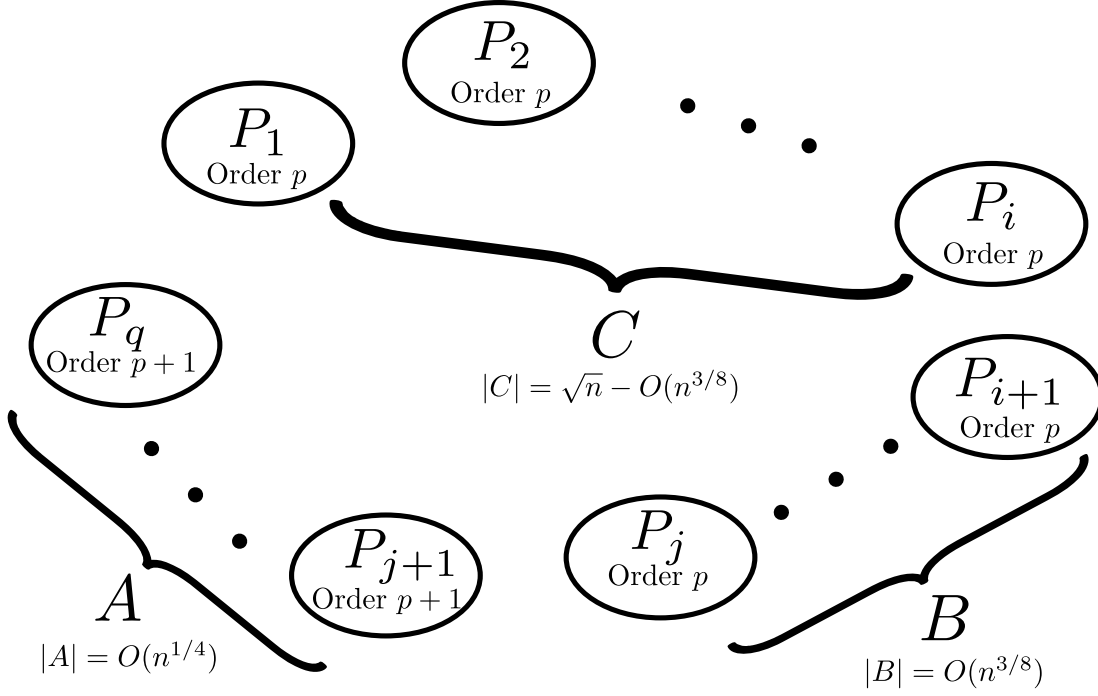


Figure 5.4: G is partitioned into sets of size p or $p + 1$, and these parts make up three groups, A , B , C as shown. The number below each group is the number of parts in that group.

Proof. We will follow Scheme 82, except we will use Hamiltonian cycles in place of 1-factors. First we will use Corollary 69 to break the graph into parts, where each vertex has many neighbors in each part. Choose p and q as in Lemma 87. Hence, p and q are both within $3n^{1/4}$ of $n^{1/2}$. Let P_1, \dots, P_q be a decomposition as in Corollary 69. Let A be the parts that have size $p + 1$, which has size $n - pq \leq 3n^{1/4}$, and let B be an arbitrary disjoint set of at least $n^{3/8}$ parts and at most $n^{3/8} + 2$ parts. Let C contain all other parts; see Figure 5.4. Choose the size of B to be such that C has an odd number of parts. Between any two parts is a bipartite graph of minimum degree $\delta/q - O(n^{1/4} \ln(n))$. By Csaba's Theorem, between any two parts in $B \cup C$ we can achieve a regular bipartite subgraph of degree at least $2\rho/q - O(n^{3/8} \ln(n))$.

We can think of these parts as being vertices of a host graph H equal to the complete

graph on t vertices. We will want to take Hamiltonian cycles in the host graph H and convert them to Hamiltonian cycles of the larger graph G .

By Lemma 94 with $\alpha = 2\rho/q - O(n^{3/8} \ln n)$ and $t = q$, there exists a set of Hamiltonian cycles \mathcal{F} on the complete graph on vertex set $A \cup B \cup C$ satisfying the conditions of the lemma. Our goal is to convert the cycles in \mathcal{F} into edge-disjoint Hamiltonian cycles \mathcal{H} of G . To do so, fix a Hamiltonian cycle $F \in \mathcal{F}$. We will describe what edges make up the Hamiltonian cycle F^* in G by associating a set of edges for each $e \in F$. If e goes between P_i and P_j in H , the set of edges for F^* will roughly consist of disjoint matching between P_i and P_j . The specifics of how to accomplish this will be given in three steps.

Step 1: e is within C or between C and B . By Lemma 94, there are at most $2\rho/q - O(n^{3/8} \ln n)$ cycles in \mathcal{F} that use edge e . Between P_i and P_j , there are at least $2\rho/q - O(n^{3/8} \ln n)$ edge-disjoint perfect matchings between P_i and P_j . We arbitrarily assign a perfect matching between P_i and P_j to F^* to each $F \in \mathcal{F}$ that uses e .

Step 2: e is between A and C . Assume edge e corresponds to $P_i \in C$ and $P_j \in A$, and suppose the cycle F continues via edge e' , corresponding to $P_j \in A$ and $P_k \in C$. Assigning a matching to e between P_i and P_j is not as easy in this case, since P_j has size $p + 1$ and P_i has size p . We will use Lemma 88 to find edge disjoint matchings that almost saturate P_j , and use P_j 's internal edges to cover the remaining vertex as in Figure.

By Corollary 69, the degree sum of $G[P_j]$ is at least

$$\begin{aligned} |P_j|(\delta/q - O(n^{1/4} \ln(n))) &= (p+1)(\delta/q - O(n^{1/4} \ln(n))) \\ &\geq \delta - O(n^{3/4} \ln n). \end{aligned}$$

Therefore, there are more edges within P_j than there are cycles in \mathcal{F} . Arbitrarily orient

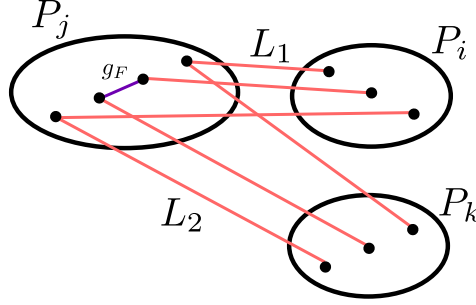


Figure 5.5: We route the Hamiltonian cycle F^* through the part of size $p + 1$ using an internal edge.

each cycle $F \in \mathcal{F}$. For each $F \in \mathcal{F}$, associate either a unique edge g_F within P_j , and arbitrarily direct g_F . Let W be the set of edges $\{g_F\}_{F \in \mathcal{F}}$. For $e \in E(H)$ incident to P_j , let W_e be the set of edges g_F of W where F uses edge e .

For each vertex $v \in P_j$, let $w_{e,v}$ be the number of edges g_F going in the same direction as F incident to v in W_e . By “going in the same direction”, we mean to count those edges going out of v if e is oriented into P_j , and count those edges coming into v if e is oriented out of F . For each Hamiltonian cycle F that use e , there is exactly one g_F , and hence $\sum_{v \in P_j} w_{e,v} = N(\mathcal{F}, e)$. Let f be the function that is constant $N(\mathcal{F}, e)$ on P_i , and is the function $N(\mathcal{F}, e) - w_{e,v}$ on P_j . Then f satisfies the requirements of Lemma 88, and hence we can find an f -factor that splits up into 1-factors. For a given $F \in \mathcal{F}$, there is a 1-factor L_1 between P_i and P_j that avoids the appropriate endpoint of g_F . Choose L_1 to associate with $e \in F$, and choose a 1-factor L_2 that avoids the other endpoint of g_F to associate with e' . In expanding F to F^* , we expand edges e and e' to edges $L_1 \cup L_2 \cup \{g_F\}$.

Step 3: e is within B . In this case, P_i and P_j again have the same size p , so we can avoid the complications of step 2. However, if we use an arbitrary perfect matching between P_i and P_j to expand e , the resulting F^* will not necessarily be a Hamiltonian cycle, but instead be only a 2-factor, albeit a 2-factor where each cycle goes through each

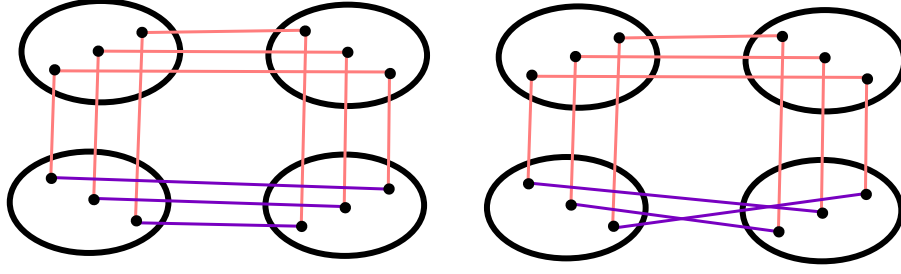


Figure 5.6: A 2-factor can be converted into a Hamiltonian cycle by simply choosing a particular matching between one pair of parts.

part. Instead, we will choose a specific perfect matching that will complete a Hamiltonian cycle. The F^* graph we have formed in Steps 1 and 2 consists of p vertex-disjoint paths from P_i to P_j . We can think of these paths as forming a perfect matching M between P_i and P_j . By Lemma 92, if the minimum degree between P_i and P_j is at least $p/2 + 1$, there is some other perfect matching M' between P_i and P_j that completes M into a Hamiltonian cycle. Choose this M' to expand e . Since the minimum degree between P_i and P_j is at least $p/2 + \sqrt{p}$, we can keep pull off at least $\sqrt{q} \leq \sqrt{p}$ matchings M' before the minimum degree of edges we can use between P_i and P_j falls below $p/2 + \sqrt{p}$. However, by Lemma 94, there are only \sqrt{q} cycles F that use e , so we have enough matchings to expand every cycle F that uses e . This complete F^* into a Hamiltonian cycle of G .

Thus, we have $|\mathcal{F}|$ edge-disjoint Hamiltonian cycles in G . We see $|\mathcal{H}| = \alpha \cdot t/2 - O(t^{7/4})$. Using $\alpha = 2\rho/q - O(n^{3/8} \ln n)$ and $t = q$, we have at least

$$(2\rho/q - O(n^{3/8} \ln n))(q/2) - O(n^{7/8}) \geq \rho - O(n^{7/8} \ln n)$$

edge-disjoint Hamiltonian cycles as desired.

□

We note that Theorem 95 is sharp except for the error terms. In Construction 7.5, we give a graph G of minimum degree δ with no k -factor for $k > 2\rho$, and hence in particular cannot contain more than ρ Hamiltonian cycles.

5.9 Open Questions

A consequence of our method that seems difficult to avoid is requiring the minimum degree of a graph to be beyond $n/2$ by an error term before finding edge-disjoint 1-factors or edge-disjoint Hamiltonian cycles. If the minimum degree is exactly $n/2$ or very near $n/2$, what can be said?

Question 96. Given a constant c and a graph G on n vertices with $n/2 \leq \delta(G) \leq n/2 + c$, how many edge-disjoint-1-factors must G contain? How many edge-disjoint Hamiltonian cycles does G contain?

An upper bound is $(n + 6)/4$ given by the construction due to Katerinis [40]. A lower bound is given by the theorem of Nash-Williams [50] stating that every such graph has $\lfloor 5n/224 \rfloor$ edge-disjoint Hamiltonian cycles.

We would be very interested in any progress on the the conjecture of Brualdi [5] and Busch et al. [7].

Conjecture 97 (Brualdi [5], Busch et al. [7]). *Every graphic sequence π with a realization with a k -factor also has a realization containing k edge-disjoint 1-factors.*

We would also be interested in further progress on the conjecture of Bollobás and Scott.

Conjecture 98 (Bollobás and Scott [4]). *Every graph G has a bisection H where, for every vertex v ,*

$$\deg_H(v) \geq \left\lfloor \frac{\deg_G(v)}{2} \right\rfloor.$$

Chapter 6

Three Proofs of the Erdős-Gallai

Theorem

The degrees of the vertices of a graph form its *degree sequence*. Given a sequence $\pi = (d_1 \geq \dots \geq d_n)$, a graph G *realizes* π if π is the degree sequence of G . The most fundamental question in the area is whether a sequence of nonnegative integers has a realization. Erdős and Gallai [24] proved that a sequence is graphic if and only if the sequence satisfies a series of n inequalities.

Theorem 99 (Erdős and Gallai [24]). *A sequence $\pi = (d_1 \geq \dots \geq d_n)$ is graphic if and only if for all $k = 1, \dots, n$,*

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min(k, d_i).$$

Necessity of the Erdős-Gallai condition is straight-forward but instructive for understanding the inequalities.

Proof. (Necessity of Erdős-Gallai) Suppose π is graphic with realization G . Let v_i be the vertex with degree d_i . We view the edges of G as actually two directed edges, oriented in opposite directions. Set $X = \{v_1, \dots, v_k\}$.

Consider counting the outgoing edges from X . On the one hand, this is exactly $\sum_{i=1}^k d_i$ since vertex v_i has d_i outgoing edges.

We can count these same edges as incoming edges. Each of the vertices v_1, \dots, v_k can receive at most $k - 1$ edges from X , which totals $k(k - 1)$ edges. Any vertex v_i outside of X can receive at most $\min\{d_i, k\}$ edges from X . Hence the number of edges counted by $\sum_{i=1}^k d_i$ is at most $k(k - 1) + \sum_{i=k+1}^n \min\{d_i, k\}$. \square

The more interesting part is proving sufficiency of the condition. Since Erdős and Gallai's original argument, this has been proven many different ways, including [11], [59], [56], and [62]. In this chapter, we give three original proofs of the condition.

The first proof uses two main ingredients: a famous theorem due to Gale and Ryser characterizing when a bisequence is bigraphic, and the Ryser Criterion [59], which connects the bipartite and non-bipartite versions of graphicness. While Sierksma and Hoogeveen [59] showed that the Ryser Criterion and the Erdős-Gallai condition were equivalent through a long chain of implications including many other equivalent conditions, we give a new proof of the Ryser Criterion and show how it can be applied to prove Erdős-Gallai more directly.

The second proof is inspired by the well-known Havel-Hakimi Theorem, which gives a way to inductively test whether a sequence is graphic, and if so build a realization. While connecting these two classical theorems seems like a natural idea, we know of no similar proof appearing in the literature.

The third proof is based on first finding a realization with possible loops and multiple edges, and then removing these conflicts.

6.1 Proof One: (from the Gale-Ryser Theorem via the Ryser Criterion)

In this section we prove Erdős-Gallai from a similar but easier to prove statement for bipartite degree sequences, the Gale-Ryser criterion for bipartite graph realizability.

A *bisequence* is a sequence grouped into two parts, such as $\pi = (a_1, \dots, a_m; b_1, \dots, b_n)$. We say π has a realization if there exists a bipartite graph $G = (L, R, E)$ such that $|L| = m$, $|R| = n$, the vertices in L have degrees (a_1, \dots, a_m) , and the vertices in R have degrees (b_1, \dots, b_n) . An obvious necessary condition for a sequence π to be bigraphic is for $\sum_{i=1}^n a_i = \sum_{i=1}^m b_i$.

Theorem 100. *Let π be a bisequence $(a_1, \dots, a_m; b_1, \dots, b_n)$ with $\sum_{i=1}^n a_i = \sum_{i=1}^m b_i$. Set $X = \{1, \dots, m\}$ and $Y = \{1, \dots, n\}$. Then π is bigraphic if and only if for all $S \subseteq X$ we have*

$$\sum_{i \in S} a_i \leq \sum_{i \in Y} \min\{|S|, b_i\}.$$

Proof. Any bisequence with common sum has a multigraph realization, so let G be a bipartite multigraph on vertex set $X \cup Y$ that realizes π but minimizes the number of multiple edges. For every pair $x \in X$ and $y \in Y$, either there is at least one edge between x and y , or there is none, and if there is none we call that a *non-edge*. If there are no multiple edges, then G is the realization we are after. Suppose, alas, there is a multi-edge between $s_0 \in X$ and $t_0 \in Y$. Our goal is to remove one of the multi-edges between s_0 and t_0 by finding an alternating path between s_0 and t_0 , that starts on a non-edge, alternates between edges and non-edges, and ends on a non-edge.

Let S be the set of vertices $s \in X$ such that there is an alternating path from s_0 to s that ends in an edge (and thus that starts with a non-edge). Place s_0 in this set as well.

We claim that either there is an alternating path available to remove the multi-edge between s_0 and t_0 , or we have violated the Gale-Ryser criterion using set S .

Decompose Y into sets U and V where U consists of all vertices adjacent to all of S , and V is $Y \setminus U$. If $t_0 \in V$, then there is a non-edge going from some $s \in S$ to t_0 . But there is already an alternating path P from s_0 to s ending in a edge. If P contains t_0 , then we have an alternating path from s_0 to t_0 as desired. If P does not contain t_0 , then we can complete P into the desired alternating path by adding the edge st_0 at the end.

Hence assume $t_0 \in U$. Now for any vertex $v \in V$, its entire neighborhood must be in S , since otherwise a neighbor not in S would have been placed into S . Thus

$$\begin{aligned}
 \sum_{x \in S} \deg(x) &= \sum_{y \in Y} \deg_S(y) \\
 &= \deg_S(t_0) + \sum_{v \in V} \deg_S(v) + \sum_{u \in U} \deg_S(u) \\
 &> |S| + \sum_{v \in V} \deg(v) + \sum_{u \in U} |S| \\
 &\geq \sum_{y \in Y} \min\{\deg(y), |S|\},
 \end{aligned}$$

which contradicts the Gale-Ryser inequality. □

The proof uses an arbitrary set S in which the Gale-Ryser inequalities must be verified, but it is clear that the most difficult sets contain the largest degrees. Hence we get

Corollary 101 (Gale [28], Ryser [54]). *Let $\pi = (a_1, \dots, a_n; b_1, \dots, b_m)$ be a bisequence with $\sum_{i=1}^n a_i = \sum_{i=1}^m b_i$. Then π is bigraphic if and only if for all $k = 1, \dots, n$ we have*

$$\sum_{i=1}^k a_i \leq \sum_{i=1}^m \min\{k, b_i\}.$$

In proving Erdős-Gallai, we will need to verify the Gale-Ryser inequalities for a mirrored bisequence. In a *mirrored* bisequence, $|X| = |Y|$ and the prescribed degrees in

X are identical to the prescribed degrees in Y . One need not check all of the Gale-Ryser inequalities: you need only check the ones in the “Durfee Square”, those indices k such that $d_k \geq k$. For a degree sequence $\pi = d_1 \geq \dots \geq d_n$, let the *Durfee Square number* m to denote the largest index such that $d_m \geq m$.

Proposition 102. *Let $\pi = (d_1 \geq \dots \geq d_n; d_1 \geq \dots \geq d_n)$ be a mirrored bisequence. Then π is graphic if and only if*

$$\sum_{i=1}^k d_i \leq \sum_{i=1}^n \min(k, d_k)$$

for all k less than or equal to the Durfee Square number.

When the Durfee Square result is proven for Erdős-Gallai as in Lemma 106, the $(k - 1)$ st Erdős-Gallai inequality was used by induction, and the rest of the steps were fairly straight-forward. If we try the same trick we see

$$\begin{aligned} \sum_{i=1}^k d_i &= \sum_{i=1}^{k-1} d_i + d_k \\ &\leq \sum_{i=1}^n \min(d_i, k-1) + d_k \\ &= \sum_{i=1}^n \min(d_i, k) - \ell + d_k. \end{aligned}$$

At this point we would like to declare that $-\ell + d_k$ is negative, and the desired result would follow. However, there is no reason to think that $-\ell + d_k$ is negative. Instead of inducting using the $(k - 1)$ st inequality, we will induct using an inequality earlier in the sequence.

Proof. Assume the Gale-Ryser inequalities are true for values less than k , and assume $d_k < k$. Let ℓ be the largest index such that $d_\ell \geq k$ (note that ℓ is strictly less than k). If no such ℓ exists, set $\ell = 0$. We want to show that $\sum_{i=1}^k d_i = \sum_{i=1}^n \min(d_i, k)$. We will induct

using the ℓ th Gale-Ryser inequality, as so:

$$\sum_{i=1}^k d_i = \sum_{i=1}^{\ell} d_i + \sum_{i=\ell+1}^k d_i \leq \sum_{i=1}^n \min(d_i, \ell) + \sum_{i=\ell+1}^k d_i.$$

Applying some basic manipulations to this expression, we see

$$\begin{aligned} \sum_{i=1}^k d_i &\leq \sum_{i=1}^n \min(d_i, \ell) + \sum_{i=\ell+1}^k d_i \\ &= \sum_{i=1}^k \min(d_i, \ell) + \sum_{i=k+1}^n \min(d_i, \ell) + \sum_{i=\ell+1}^k d_i \\ &\leq \sum_{i=1}^k \ell + \sum_{i=k+1}^n \min(d_i, k) + \sum_{i=\ell+1}^k d_i \\ &= \sum_{i=1}^{\ell} k + \sum_{i=\ell+1}^k d_i + \sum_{i=k+1}^n \min(d_i, k) \end{aligned}$$

Notice in the range $\{1, \dots, \ell\}$ that $k \leq d_i$, and hence we can replace k with $\min(d_i, k)$ without consequence. Similarly, in the range $\{\ell + 1, \dots, k\}$ we have $d_i \leq k$, so in this range we can replace d_i with $\min(d_i, k)$ without consequence. Our calculation then becomes

$$\begin{aligned} \sum_{i=1}^k d_i &\leq \sum_{i=1}^{\ell} k + \sum_{i=\ell+1}^k d_i + \sum_{i=k+1}^n \min(d_i, k) \\ &= \sum_{i=1}^{\ell} \min(d_i, k) + \sum_{i=\ell+1}^k \min(d_i, k) + \sum_{i=k+1}^n \min(d_i, k) \\ &= \sum_{i=1}^n \min(d_i, k), \end{aligned}$$

as desired. □

The connection we will use between the Gale-Ryser and Erdős-Gallai Theorems will be the Ryser Criterion. Let $\pi = (d_1 \geq \dots \geq d_n)$ be a sequence. Let m be the Durfee

Square number. Define \tilde{d}_i to be

$$\tilde{d}_i = \begin{cases} d_i + 1, & \text{if } i \leq m, \\ d_i, & \text{if } i > m. \end{cases}$$

The following is a new proof of the Ryser Criterion.

Theorem 103 (Ryser Criterion [59]). *A sequence $\pi = (d_1, \dots, d_n)$ with even sum is graphic if and only if the sequence $\tau = (\tilde{d}_1, \dots, \tilde{d}_n; \tilde{d}_1, \dots, \tilde{d}_n)$ is bigraphic.*

Proof. (\Leftarrow) Let G be a realization of π on vertex set v_1, \dots, v_n , where vertex v_i has degree d_i . Consider the bipartite graph $H = G \times K_2$, $V(K_2) = \{1, 2\}$, where \times is the graph product where $(u_1, v_1)(u_2, v_2) \in E(A \times B)$ if and only if $u_1 u_2 \in A$ and $v_1 v_2 \in B$. Add all edges to H between the vertices of the form $(v_i, 1)$ and $(v_i, 2)$ for $i \leq m$. Then H is a bipartite realization of τ .

(\Rightarrow) Let H be a bipartite realization of $(\tilde{d}_1, \dots, \tilde{d}_n; \tilde{d}_1, \dots, \tilde{d}_n)$. Let $X = \{x_1, \dots, x_n\}$ be one part and $Y = \{y_1, \dots, y_n\}$ be the other, where both x_i and y_i have degree \tilde{d}_i in H .

We say an edge $x_i y_j$ is *mirrored* if $x_j y_i$ is also an edge. The proof will proceed in two steps. First, we will ensure that all edges are mirrored. Then we will ensure that all edges of the form $x_i y_i$ are included if and only if $i \leq m$, the Durfee Square number. By then taking a graph on vertex set $\{v_1, \dots, v_k\}$ with edges of the form $v_i v_j$ if and only if there is an edge between $x_i y_j$ and $x_j y_i$, we will then have the desired realization of π .

Step 1: There exists a realization of τ such that all edges are mirrored.

Suppose H has the fewest number of non-mirrored edges of any realization. If $x_i y_j$ is an edge of H that is not mirrored, let the edge $x_i y_j$ along with the non-edge $x_j y_i$ be a *mismatched pair*. Let A be the collection of all mismatched pairs. Choose a vertex x_i with at least one non-mirrored edge. Starting at x_i , start a trail C on A alternating edges and

non-edges. Since every vertex in A has an equal number of incident edges and non-edges, eventually such a walk must get back to x_i , and thus C can be closed to a circuit.

Consider the number of times C visits y_i , the vertex opposite x_i . If C never visits y_i , then we can swap along the alternating circuit C , which will fix at least one mismatched pair without creating any new mismatched pairs. This would contradict our assumption H had the fewest non-mirrored edges to begin with.

Suppose C visits y_i more than once. Then by removing a sub-circuit starting and ending at y_i , we can shorten the circuit C so that it visits y_i exactly once.

So suppose C visits y_i exactly once. In H , $x_i y_i$ is either an edge or a non-edge. Suppose that $x_i y_i$ is an edge. Notice that the circuit that leaves x_i on a non-edge must enter y_i on a non-edge, since the graph is bipartite and the circuit is alternating. Thus there exists an alternating circuit C' that follows C from x_i to y_i , and then comes back to x_i via the edge $x_i y_i$. Swapping along C' fixes at least one edge, again a contradiction. If $x_i y_i$ is a non-edge, we form a similar C' , but instead of using the path in C leaving x_i and entering y_i using non-edges, instead we use the path leaving x_i and entering y_i using edges. In all cases, we get contradictions.

Step 2: There exists a realization of τ , with every edge mirrored, such that $x_i y_i$ is included if and only if $i \leq m$.

We say that $x_i y_i$ is *bad* if it does not satisfy $x_i y_i \in E(H)$ if and only if $i \leq m$. If there are no bad edges, then Step 2 is finished, so assume there is at least one bad edge $x_i y_i$. We will again use edge swaps to make it so $x_i y_i$ is not a bad edge, without introducing any other bad edges. Repeated application will give the desired realization of τ .

Claim: The number of bad edges is at least two. Let t be the number of edges in H of the form $x_i y_i$. How many edges are in the graph? One one hand, counting from the left degrees, this is $\sum_{i=1}^n \tilde{d}_i = m + \sum_{i=1}^n d_i$. On the other hand, this is t plus the number of edges which come in mirrored pairs. Since $\sum_{i=1}^n d_i$ is even and the number of edges in

mirrored pairs is even, t and m have the same parity.

Suppose $x_i y_i$ and $x_j y_j$ are both bad.

Case 1: $i \leq m, j > m$. Then x_i dominates x_j , and y_i dominates y_j , and thus this gives a six-cycle switch.

Case 2: $i \leq m, j \leq m$. It must be the case that there is some y_ℓ for $\ell > m$ such that x_i is adjacent to y_ℓ (or otherwise i would not be less than or equal to m). If $x_\ell y_\ell$ is bad, then we are in Case 1, so assume $x_\ell y_\ell$ is not a bad edge. Then two-switch $x_i y_i$ with $x_\ell y_\ell$. Now $x_\ell y_\ell$ is bad, and we are in Case 1.

Case 3: $i > m, j > m$. It must be the case that there is some y_ℓ for $\ell \leq m$ such that x_i is not adjacent to y_ℓ (or otherwise i would not be greater than m). If $x_\ell y_\ell$ is bad, then we are in Case 1, so assume $x_\ell y_\ell$ is not a bad edge. Then two-switch $x_i y_i$ with $x_\ell y_\ell$, and we are back in Case 1.

This finishes Step 2. As discussed before, we can now take the realization of τ with every edge mirrored, such that $x_i y_i$ is included if and only if $i \leq m$, and form a realization of π by placing an edge between v_i and v_j if there is an edge between x_i and y_j in the realization of τ .

□

The Erdős-Gallai Theorem can now be seen as a consequence of the Gale-Ryser Theorem, Proposition 102, and the Ryser Criterion. In fact, we get the generalization of Erdős-Gallai showing that you need only check the first m inequalities.

Corollary 104 (Erdős-Gallai). *Let $\pi = d_1 \geq \dots \geq d_n$ be a sequence with even sum. Then π is graphic if and only if for all $k = 1, \dots, m$,*

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min(d_i, k).$$

Proof. Suppose these inequalities hold for $k = 1, \dots, m$.

Define \tilde{d}_i as in the previous theorem, and consider the bipartite sequence $(\tilde{d}_1, \dots, \tilde{d}_n; \tilde{d}_1, \dots, \tilde{d}_n)$. If this sequence is bigraphic, then the original sequence is graphic. We check Gale-Ryser for values of k in the Durfee Square. Choose a k from $\{1, \dots, m\}$. We see

$$\begin{aligned}
 \sum_{i=1}^k \tilde{d}_i &= \sum_{i=1}^k d_i + k \\
 &\leq k(k-1) + \sum_{i=k+1}^n \min(d_i, k) + k \\
 &= k^2 + \sum_{i=k+1}^n \min(d_i, k) + k - k \\
 &= \sum_{i=1}^k k + \sum_{i=k+1}^n \min(d_i, k) \\
 &= \sum_{i=1}^n \min(d_i, k) \\
 &\leq \sum_{i=1}^n \min(\tilde{d}_i, k)
 \end{aligned}$$

By Gale-Ryser, the bisequence is graphic, and by the Ryser Criterion π is as well. \square

6.2 Proof Two (Havel-Hakimi Method)

The next proof is inspired by the Havel-Hakimi Theorem. Havel [36] and independently by Hakimi [33] showed

Theorem 105 (Havel [36], Hakimi [33]). *A sequence $\pi = (d_1 \geq \dots \geq d_n)$ is graphic if and only if the sequence $(d_2 - 1, d_3 - 1, \dots, d_{d_1} - 1, d_{d_1+1}, \dots, d_n)$ is graphic.*

Note that the following proof does not use the Havel-Hakimi theorem explicitly, and in fact we will obtain the Havel-Hakimi theorem as a corollary.

We need the following standard lemma, which shows that only the first m Erdős-Gallai inequalities imply the others.

Lemma 106. *Let $\pi = (d_1 \geq d_2 \geq \dots \geq d_n)$ be a sequence. Then the Erdős-Gallai inequalities hold for all k if they hold for all $k \leq m$.*

Proof. Choose a k such that $d_k < k$, and by induction assume the $(k-1)$ st Erdős-Gallai inequality holds. Using the $(k-1)$ st inequality, we see:

$$\begin{aligned} \sum_{i=1}^k d_i &= \sum_{i=1}^{k-1} d_i + d_k \\ &\leq (k-1)(k-2) + \sum_{i=k}^n \min(k-1, d_i) + d_k \end{aligned}$$

Note that $d_k < k$ implies $d_k \leq k-1$. We then have

$$\begin{aligned} \sum_{i=1}^k d_i &\leq (k-1)(k-2) + \sum_{i=k}^n \min(k-1, d_i) + d_k \\ &= k(k-1) - 2(k-1) + \sum_{i=k}^n \min(k-1, d_i) + d_k \\ &= k(k-1) - 2(k-1) + \sum_{i=k+1}^n \min(k-1, d_i) + 2d_k \\ &\leq k(k-1) + \sum_{i=k+1}^n \min(k, d_i). \end{aligned}$$

□

The Erdős-Gallai Theorem will be a direct consequence of the following.

Theorem 107. *Let $\pi = (d_1 \geq d_2 \geq \dots \geq d_n)$ be a sequence with even sum and let $\tilde{\pi} = (\tilde{d}_2, \tilde{d}_3, \dots, \tilde{d}_n)$ be the sequence obtained from π by removing d_1 and subtracting 1 from the d_1 entries of highest value, and reordering. If π satisfies the Erdős-Gallai inequalities, then so does $\tilde{\pi}$.*

Proof. We will proceed by assuming π satisfies the Erdős-Gallai inequalities, and proving $\tilde{\pi}$ also satisfies the inequalities. By the previous lemma, we need only check the inequalities for $\tilde{\pi}$ for those values k in the Durfee Square. Since $\tilde{\pi}$ is indexed starting at 2, a value k in the Durfee Square is a k such that $\tilde{d}_k \geq k - 1$. Let v_1 be the vertex to receive degree d_1 , and let v_i be the vertex to receive degree \tilde{d}_i .

Fix a k such that $\tilde{d}_k \geq k - 1$. We need to verify the k th Erdős-Gallai inequality for the sequence $\tilde{d}_2, \dots, \tilde{d}_n$. That is, we need to check

$$\sum_{i=2}^k \tilde{d}_i \leq (k-1)(k-2) + \sum_{i=k+1}^n \min(k-1, \tilde{d}_i).$$

Case 1: Suppose $\tilde{d}_2 = d_1$. In this case, π starts with a large block of values that are all the same. When subtracting 1 from the d_1 entries of highest value, we finish part of the way through this large block. The Durfee Square number m for $\tilde{\pi}$, in this case, is before the end of this initial block of values that were originally equal to d_1 . Thus, there are at least $d_1 + 1$ values at least $k - 1$ in $\tilde{\pi}$. This gives at least $d_1 + 1 - (k - 1)$ values at least $k - 1$ outside the first $k - 1$ entries. We compute

$$\begin{aligned} \sum_{i=2}^k \tilde{d}_i &\leq d_1(k-1) \\ &= (k-1)(k-2) + (d_1 - (k-2))(k-1) \\ &= (k-1)(k-2) + (d_1 + 1 - (k-1))(k-1) \\ &\leq (k-1)(k-2) + \sum_{i=k+1}^{d_1+2} \min(k-1, \tilde{d}_i) \\ &\leq (k-1)(k-2) + \sum_{i=k+1}^n \min(k-1, \tilde{d}_i). \end{aligned}$$

as desired.

Case 2: We have $\tilde{d}_2 < d_1$ and there are at least d_1 entries in $\tilde{\pi}$ of value at least $k - 1$. Thus,

there are at least $d_1 - (k - 1)$ values at least $k - 1$ outside the first $k - 1$ entries. We compute

$$\begin{aligned}
\sum_{i=2}^k \tilde{d}_i &\leq (d_1 - 1)(k - 1) \\
&= (k - 1)(k - 2) + (d_1 - (k - 1))(k - 1) \\
&\leq (k - 1)(k - 2) + \sum_{i=k+1}^{d_1+1} \min(k - 1, \tilde{d}_i) \\
&\leq (k - 1)(k - 2) + \sum_{i=k+1}^n \min(k - 1, \tilde{d}_i).
\end{aligned}$$

as desired.

If there are not d_1 entries in $\tilde{\pi}$ of value at least $k - 1$, that means $\tilde{d}_{d_1+1} < k - 1$.

Case 3: We have $\tilde{d}_\ell < k - 1$ where $\ell = d_1 + 1$. Consider $\sum_{i=k+1}^n \min(k - 1, \tilde{d}_i)$ with respect to $\sum_{j=k+1}^n \min(k, d_j)$. If \tilde{d}_i came from d_j , then $\min(k - 1, \tilde{d}_i)$ will be at most one less than $\min(k, d_j)$. For it to be one less, either $\tilde{d}_i \geq k - 1$ (in which case $i \leq \ell$), or in the case $\tilde{d}_i = d_j - 1$ (in which case $i \leq \ell$). So $\min(k - 1, \tilde{d}_i)$ is one less than $\min(k, d_j)$ only when $i \leq \ell$. For $i \geq k + 1$, this happens at most $d_1 - (k - 1)$ times. With this in mind, we compute

$$\begin{aligned}
\sum_{i=2}^k \tilde{d}_i &= \sum_{i=1}^k d_i - d_1 - (k - 1) \\
&\leq k(k - 1) + \sum_{i=k+1}^n \min(k, d_i) - d_1 - (k - 1) \\
&= (k - 1)(k - 2) + \sum_{i=k+1}^n \min(k, d_i) - d_1 + (k - 1) \\
&\leq (k - 1)(k - 2) + \sum_{i=k+1}^n \min(k - 1, \tilde{d}_i)
\end{aligned}$$

as desired. □

We now obtain the Erdős-Gallai Theorem.

Corollary 108 (Erdős-Gallai [24]). *Let $\pi = d_1 \geq \dots \geq d_n$ be a sequence with even sum. Then π is graphic if and only if for all $k = 1, \dots, n$,*

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min(d_i, k).$$

Proof. Proof by induction on n . If $n = 1$, then the inequality for $k = 1$ says that $d_1 \leq 0$, and hence will have a realization given by a single isolated vertex.

For $n > 1$, assume all graphic sequences of length $n - 1$ satisfy the Erdős-Gallai Theorem, and assume π is a sequence of length n . Since the inequalities hold for π , by Theorem 107, they hold for $\tilde{\pi}$ obtained by removing d_1 and subtracting 1 from the highest d_1 entries. Since $\tilde{\pi}$ has length $n - 1$, it has a realization G' . If we add a vertex to G' with adjacencies to the appropriate vertices, we will have a realization G of π as desired. \square

We also get an alternate proof to the Havel-Hakimi Theorem.

Corollary 109 (Havel-Hakimi). *The sequence π is graphic if and only if $\tilde{\pi}$ is graphic, where $\tilde{\pi}$ is obtained from π by removing d_1 and subtracting 1 from the highest d_1 entries.*

Proof. Assume $\tilde{\pi}$ is graphic. We get that π is graphic by taking a realization G' of $\tilde{\pi}$ and adding a vertex with the appropriate adjacencies.

Assume π is graphic. Then π satisfies the Erdős-Gallai inequalities. By Theorem 107, $\tilde{\pi}$ also satisfies the Erdős-Gallai inequalities. Hence $\tilde{\pi}$ is graphic. \square

6.3 Proof Three (Direct)

We now give a direct proof of the Erdős-Gallai Theorem starting from a multigraph with loops realization of the degree sequence, and then inductively removing the loops and

multiple edges.

Theorem 110. *Let $\pi = d_1 \geq d_2 \geq \dots \geq d_n$ be a sequence with even sum. Then π is graphic if and only if for all $k = 1, \dots, n$ we have*

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min\{k, d_i\}.$$

Proof. If there exists a realization of π without multiple edges or loops, then we are done. Therefore, let G be a multigraph with loops realization of π on vertex set $V = \{v_1, \dots, v_n\}$ maximizing the smallest index k such that v_k is incident to a multi-edge or loop. Subject to maximizing k , also minimize the number of multiple edges incident to v_k . Let v_ℓ be the other endpoint of a multi-edge or loop incident to v_k , and note it is possible $k = \ell$. Here, if there are t edges between v_k and v_ℓ , we count that as $t - 1$ multiple edges, and each loop at v_k counts as 2 multiple edges.

Let $u \neq v_k$ be any other vertex of the graph. If u is adjacent to v_i for $i = 1, \dots, k$, we say u fills v_k . If u is not adjacent to v_i for all $i > k$, then we say u is absorbed by v_k . If u neither fills v_k nor is absorbed by v_k , we call u a pivot vertex.

Let G' be the graph obtained from G by deleting the multiple edge or loop at v_k . Given two vertices v_i and v_j with $i > j$ and $\deg_{G'}(v_i) > \deg_{G'}(v_j)$, there must be a vertex z such that there are strictly more edges between $v_i z$ than $z v_j$. If we swap an edge between $v_i z$ for a non-edge between $z v_j$, we do not increase the smallest index with multiple edges or increase the number of multiple edges incident to v_k . By making that edge swap, we are therefore not introducing any problems. We will call such a z a gateway between v_i and v_j . If either $v_i = v_j$ or both $i > j$ and $\deg_{G'}(v_i) > \deg_{G'}(v_j)$, we say that there is a gateway from v_i to v_j , since we can extend alternating paths from v_i to v_j (where in the case $x = y$ this extension is trivial).

First we claim that for any vertex v_j for $j \leq \ell$, v_j fills v_k : suppose v_j has a non-edge to

v_z for $z \leq k$. Since either $v_j = v_\ell$ or $\deg_{G'}(v_j) > \deg_{G'}(v_\ell)$, there is a gateway between v_j and v_ℓ . Similarly, since either $v_z = v_k$ or $\deg_{G'}(v_z) > \deg_{G'}(v_k)$, there is a gateway between v_z and v_k . Therefore, if we flip along the alternating path $v_k v_z v_j v_\ell v_k$, we can eliminate the multiple edge or loop between v_k and v_ℓ , without adding any loops or multiple edges anywhere else. This lowers the number of multiple edges incident to v_k , a contradiction.

Next notice that if there are no pivot vertices, then the k th Erdős-Gallai inequality is violated: on the one hand, if you count edges coming from vertices indexed k or smaller, you get $\sum_{i=1}^k d_i$. On the other hand, looking at the other endpoints of these edges, you have at least a complete graph on vertices $1, \dots, k$, which gives a count of at least $k(k-1)$. On vertices indexed $k+1, \dots, n$, they either fill v_k and thus absorb at least k edges, or are absorbed by v_k and absorb d_i edges. This is true for all vertices except v_ℓ , which absorbs at least $k+1$ edges. Thus you get

$$\sum_{i=1}^k d_i \geq k(k-1) + \sum_{k+1}^n \min(k, d_i) + 1$$

contradicting the k th Erdős-Gallai inequality.

Therefore, there is a pivot vertex u with a non-edge going to v_i for $i \leq k$, and an edge going to v_j for $j > k$. Either $v_i = v_k$ or $\deg_{G'}(v_i) > \deg_{G'}(v_k)$, so there is a gateway from v_i to v_k . By switching along the alternating path $v_k v_\ell v_j u v_i v_k$, we remove the multiple edge between v_k and v_ℓ , and perhaps put in a multiple edge only between v_ℓ and v_j . This lowers the number of multiple edges incident to v_k , a contradiction. \square

Chapter 7

Minimum Degree and the Existence of a k -Factor¹

7.1 Introduction

The classic result of Dirac [17] states that any simple graph on $n \geq 3$ vertices with minimum degree at least $n/2$ has a Hamiltonian cycle. This result has been generalized in a number of ways, including where the minimum degree being larger than $n/2$ gives extra structure beyond a Hamiltonian cycle. For example, Pósa (in the case $k = 2$, see Erdős [23]) and Seymour [58] conjectured that a graph of minimum degree $\frac{k}{k+1}n$ contains the k th power of a Hamiltonian cycle, and Komlós, Sárközy, and Szemerédi [41] proved an approximate version of this conjecture.

Katerinis [40] proved that a graph with minimum degree at least $n/2$ has a large k -factor.

Theorem 111 (Katerinis [40]). *Let G be a simple graph on $n \geq 2$ vertices with minimum degree at least $n/2$. Then G contains a k -factor for any $k \leq \frac{n+5}{4}$ such that kn is even.*

¹This chapter was prepared jointly with Stephen G. Hartke and Ryan Martin.

In this note, we show that if the minimum degree is larger than $n/2$, then a denser k -factor exists. Our main theorem is as follows:

Theorem 112. *Let G be a graph with n vertices, let $\delta = \delta(G)$, and suppose $\delta \geq n/2$. Set*

$$\rho = \frac{\delta + \sqrt{2\delta n - n^2 + 8}}{2}.$$

Then G has a k -factor for any $k < \rho$ such that kn is even. In particular, G has a k -factor for some $k \geq \rho - 2$.

Note that for $\delta = n/2$, one obtains a k factor for any k up to $\frac{n+2\sqrt{8}}{4} \geq \frac{n+5}{4}$, so Theorem 112 generalizes Katerinis' result. As δ gets closer to n , we see that k approaches n as well. See Figure 7.1.

Furthermore, we show that Theorem 112 is almost best possible.

Theorem 113. *Let n and δ be positive integers such that $n/2 \leq \delta \leq n - 1$. There exists a graph G on n vertices with minimum degree δ such that if*

$$k > \frac{\delta + \sqrt{2\delta n - n^2}}{2} + \frac{4}{\sqrt{2\delta n - n^2} + 4}$$

then G has no k -factor.

The case of $\delta = n/2$ was established by Katerinis [40]. Summarizing Theorems 112 and 113 for when $\delta > n/2$,

Corollary 114. *Given positive integers n and δ such that $n/2 < \delta \leq n - 1$, then every graph G on n vertices with minimum degree at least δ has a k -factor for any k such that kn is even and $k < \rho$. Furthermore, there exists a graph for which there is no k -factor for $k > \rho + 2/\sqrt{n+8}$.*

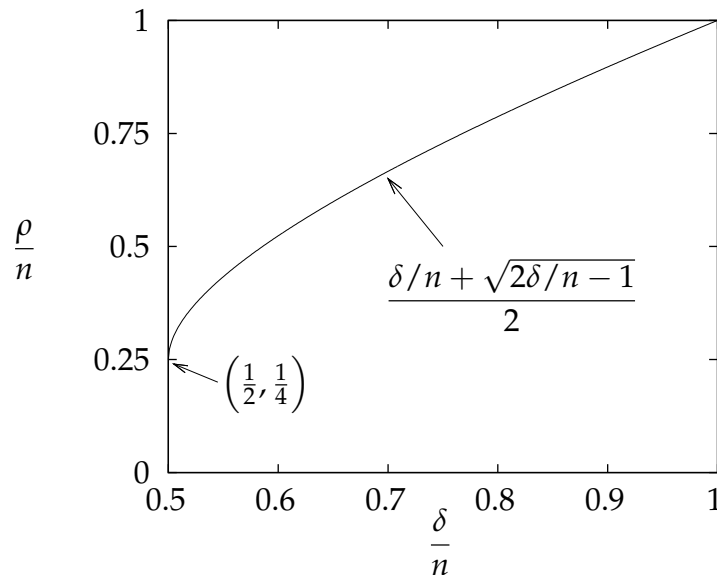


Figure 7.1: This plot shows how large a k -factor exists (if kn is even) for a given minimum degree.

Thus, the only values of k , with kn even, where we do not know if every graph on n vertices with minimum degree δ contains a k factor is when the value k lands in the range $[\rho, \rho + 2/\sqrt{n+8}]$. Hence our bounds are off by at most an additive constant of 1.

The rest of the note is laid out as follows: Section 7.2 gives some motivation and origins to this problem. Section 7.3 states the main tools we will use to prove the theorems. Section 7.4 is the proof of Theorem 112, modeled on the ideas of Katerinis' result. Section 7.5 is the proof of Theorem 113, and section 7.6 is the proof of Corollary 114.

After finishing our work, we were subsequently informed that our main result was proven in a manuscript by D. Christofides, D. Kühn and D. Osthus [12] that is, as yet, unpublished.

7.2 Motivation

This problem was originally motivated by the so-called multipartite version of the Hajnal-Szemerédi Theorem. A version of Hajnal and Szemerédi's famous theorem [32] is as follows:

Theorem 115 (Hajnal-Szemerédi). *Let $k \geq 2$ be a positive integer. If G is a simple graph on n vertices minimum degree at least $(k-1)n/k$, then G contains a subgraph consisting of $\lfloor n/k \rfloor$ vertex-disjoint copies of K_k .*

(The case for $k = 2$ is a consequence of Dirac's Theorem and $k = 3$ was originally proved by Corrádi and Hajnal [13].)

Theorem 115 is best possible in general. For the example where n is divisible by k , we may consider a complete k -partite graph with one class of size $n/k - 1$, another of size $n/k + 1$ and the rest of size n/k . Such a graph has minimum degree $(k-1)n/k - 1$ but no subgraph consisting of n/k vertex disjoint copies of K_k .

The multipartite version is as follows:

Conjecture 116. *Let $k \geq 2$ be a positive integer. If G is a k -partite graph with N vertices in each part and each vertex is adjacent to at least $(k-1)N/k$ vertices in each of the other parts, then either G contains a subgraph consisting of N vertex-disjoint copies of K_k or kN is odd, N is a multiple of k and G is isomorphic to a single graph with the property that each vertex is adjacent to exactly $(k-1)N/k$ vertices in each of the other parts.*

For $k = 2$, it is merely a consequence of König-Hall. For N sufficiently large, the conjecture has been proven for $k = 3, 4$ [45, 48] and various Erdős-Stone-type generalizations have been proven [72, 49]. Csaba and Mydlarz [15] were able to prove an approximate version. To wit:

Theorem 117 (Csaba-Mydlarz). *Let $q \geq 5$ be a positive integer. If G is a q -partite graph with N vertices in each part and each vertex is adjacent to at least $(k - 1)N/k$ vertices in each of the other parts, where $k = q + O(\log q)$, then G contains a subgraph consisting of N vertex-disjoint copies of K_k , as long as N is sufficiently large.*

The main tool they use is a lemma of Csaba [14]:

Theorem 118 (Csaba). *Let G be a bipartite graph where each part has size n , with minimum degree $\delta = \delta(G)$, and assume $\delta \geq n/2$. Set*

$$\rho = \frac{\delta + \sqrt{2\delta n - n^2}}{2}.$$

Then G contains a $\lfloor \rho \rfloor$ -factor.

Theorem 112 is, therefore, the general graph version of Theorem 118.

7.3 Main tools

A major tool in finding k -factors is Tutte's *f -factor theorem*. Given a graph G with a non-negative function $f : V(G) \rightarrow \mathbb{Z}$, G has an f -factor if G has a spanning subgraph in which vertex v has degree $f(v)$.²

Tutte [64, 65] gave necessary and sufficient conditions that characterized when a graph has a f -factor. Here, for a set of vertices S we write $f(S)$ to denote $\sum_{v \in S} f(v)$. For two sets of vertices $S, T \subset V(G)$, $e(S, T)$ denotes the number of edges with one endpoint in S and the other endpoint in T . For disjoint S and T , the function $q(S, T)$ denotes the number of components C of $G \setminus S \setminus T$ such that $f(C) + e(T, C)$ is odd.

²In order to clarify the terminology, we observe that a k -factor is the same as an f -factor if f is the constant function equal to k . Whether the prefix of “factor” is a function or an integer will be clear from the context.

Theorem 119 (Tutte's f -factor theorem). *A graph G has a f -factor if and only if for all disjoint $S, T \subseteq V(G)$, we have*

$$q(S, T) + f(T) - \sum_{v \in T} \deg_{G \setminus S}(v) \leq f(S).$$

We are looking for a k -factor, so we set $f(v) = k$ for all $v \in V(G)$. In that case, Tutte's theorem becomes

Corollary 120. *A graph G has a k -factor if and only if for all disjoint $S, T \subseteq V(G)$, we have*

$$q(S, T) + k|T| - \sum_{v \in T} \deg_{G \setminus S}(v) \leq k|S|.$$

Furthermore, Tutte [64] showed Proposition 121, which we will use.

Proposition 121. *For any graph G on n vertices, if kn is even, then $q(S, T) + k|T| - \sum_{v \in T} \deg_{G \setminus S}(v) \equiv k|S| \pmod{2}$.*

We will also use a lemma of Katerinis [40], which we reprove for completeness. Here, for any subset of vertices X in a graph, $c(X)$ denotes the number of connected components of the subgraph induced by X .

Lemma 122 (Katerinis [40]). *If G is a Hamiltonian graph and $X \subseteq V(G)$, then $c(G - X) \leq |X|$.*

Proof. Let H be a Hamiltonian cycle of G . One can verify that $c(H - X) \leq |X|$, since removing the first vertex leaves one component, and any additional vertices removed creates at most one new component. Since H is a subgraph of G , we have $c(G - X) \leq |X|$. \square

All graphs considered in this note are Hamiltonian because we require that the minimum degree is at least $n/2$. Hamiltonicity is thus a direct result of Dirac's theorem.

Finally, we need the following lemma to verify that Theorem 112 is not claiming that the graph has a k -factor for $k > \delta$, and furthermore that when $k = \delta$ the condition $k < \rho$ forces G to have a k -factor.

Lemma 123. *Let G, n, δ, ρ be as in Theorem 112. If k is an integer such that kn is even and $\delta \leq k < \rho$, then $k = \delta$ and G has a k -factor.*

Proof. If we write $k = \delta + t$ for some nonnegative integer t , we see

$$\begin{aligned} \delta + t &< \frac{\delta + \sqrt{2\delta n - n^2 + 8}}{2} \\ \delta^2 + 4\delta t + 4t^2 &< 2\delta n - n^2 + 8 \\ n^2 - 2\delta n + \delta^2 + 4\delta t + 4t^2 &< 8 \\ (n - \delta)^2 + 4\delta t + 4t^2 &< 8 \end{aligned}$$

Since $\delta \geq 1$, we have a contradiction if $t > 0$. Hence $t = 0$ which gives $k = \delta$. The calculation above gives $(n - \delta)^2 < 8$, which means $\delta = n - 1$ or $\delta = n - 2$. If $\delta = n - 1 = k$, then G itself is a k -factor. If $\delta = n - 2 = k$, then kn being even implies that both k and n are even. In this case, G is a complete graph minus a partial matching. But since n is even, we can extend this partial matching to a full matching. What remains of G outside this matching is a k -factor. \square

7.4 Proof of Theorem 112

Fix a $k > 0$ such that kn is even and $k < \rho$. Suppose G does not contain a k -factor. Then by Tutte's f -factor theorem, for f identically k , there exist disjoint $S, T \subseteq V(G)$ such that

$$q(S, T) + k|T| - \sum_{v \in T} \deg_{G \setminus S}(v) > k|S|. \quad (7.1)$$

Since kn is even, the left side has the same parity as the right side by Proposition 121, and we have

$$q(S, T) + k|T| - \sum_{v \in T} \deg_{G \setminus S}(v) \geq k|S| + 2. \quad (7.2)$$

Set $W = G \setminus (S \cup T)$. Notice that we have $q(S, T) \leq c(W)$. Therefore, we can further reduce (7.2) to

$$c(W) + k|T| - \sum_{v \in T} \deg_{G \setminus S}(v) \geq k|S| + 2. \quad (7.3)$$

We break the proof into 4 cases, establishing a contradiction in each.

Case 1: $T = \emptyset$.

We have $|T| = 0$, then $W = G \setminus S$ and $\sum_{v \in T} \deg_{G \setminus S}(v) = 0$, so (7.3) becomes $c(G \setminus S) \geq k|S| + 2$. Since G is Hamiltonian by Dirac's theorem, we can apply Lemma 122 to see $c(G - S) \leq |S|$. However, this contradicts $c(G \setminus S) \geq k|S| + 2$ since $k \geq 1$. This concludes Case 1.

Set $\mu = \min_{x \in T} \{\deg_{G \setminus S}(x)\}$. We have $\mu|T| \leq \sum_{v \in T} \deg_{G \setminus S}(v)$, and therefore $k|T| - \sum_{v \in T} \deg_{G \setminus S}(v) \leq (k - \mu)|T|$. Thus (7.3) becomes

$$c(W) + (k - \mu)|T| \geq k|S| + 2. \quad (7.4)$$

We now have three cases based on where the value of μ falls relative to k .

Case 2: $T \neq \emptyset$, $\mu \geq k + 1$.

By (7.4), we have $c(W) - |T| \geq k|S| + 2$. By Lemma 122, we have $c(W) = c(G - (S \cup T)) \leq |S \cup T|$. Thus, we get $|S \cup T| - |T| \geq k|S| + 2$, which becomes $|S| \geq k|S| + 2$, which contradicts $k \geq 1$.

Case 3: $T \neq \emptyset, \mu = k$

From (7.4), we have

$$c(W) \geq k|S| + 2.$$

We use the trivial bound $c(W) \leq |W|$. Furthermore, let us bound $|W| = n - |S| - |T| \leq n - |S| - 1$. Thus (7.4) becomes

$$n - |S| - 1 \geq k|S| + 2. \quad (7.5)$$

Note that a vertex $x \in T$ such that $\mu = \deg_{G-S}(x)$ has at most $|S| + \mu$ neighbors in G , and we therefore have $|S| + \mu \geq \delta$ or $|S| \geq \delta - \mu$. From (7.5) we have

$$n - 1 - \delta + k \geq k\delta - k^2 + 2,$$

which gives

$$k^2 - \delta k + n - \delta - 3 \geq 0. \quad (7.6)$$

Solving (7.6) using the quadratic formula yields either

$$k \geq \frac{\delta + \sqrt{\delta^2 - 4(n - \delta - 3)}}{2}$$

or

$$k \leq \frac{\delta - \sqrt{\delta^2 - 4(n - \delta - 3)}}{2}.$$

We can verify the second solution is less than 1 since substituting $k = 1$ into (7.6) yields a negative value. Consider the first solution. We may assume $k \leq \delta - 1$ because of

Lemma 123. Therefore, we have

$$\begin{aligned}\delta - 1 &\geq \frac{\delta + \sqrt{\delta^2 - 4(n - \delta - 3)}}{2} \\ \delta - 2 &\geq \sqrt{\delta^2 - 4n + 4\delta + 12} \\ \delta^2 - 4\delta + 4 &\geq \delta^2 - 4n + 4\delta + 12 \\ n &\geq 2\delta + 2,\end{aligned}$$

which contradicts $\delta \geq n/2$.

Case 4: $T \neq \emptyset, 0 \leq \mu \leq k - 1$.

From (7.4), we have

$$c(W) + (k - \mu)|T| \geq k|S| + 2.$$

We replace $|T|$ with $n - |S| - |W|$. Again, we use the trivial bound $c(W) \leq |W|$. Thus we have

$$|W| + (k - \mu)(n - |S| - |W|) \geq k|S| + 2.$$

Recall $|S| \geq \delta - k$. Therefore,

$$\begin{aligned}|W| + (k - \mu)(n - (\delta - \mu) - |W|) &\geq k(\delta - \mu) + 2 \\ (k - \mu)(n + \mu - \delta) + k(\mu - \delta) - 2 &\geq (k - 1 - \mu)|W| \geq 0 \\ -\mu^2 + (\delta + 2k - n)\mu + (nk - 2\delta k - 2) &\geq 0\end{aligned}\tag{7.7}$$

Consider (7.7) as a quadratic inequality in the variable μ . Since $0 \leq \mu$, the left-hand side achieves its maximum at $\mu^* = \max\{0, (\delta + 2k - n)/2\}$. Thus, we have a contradiction unless (7.7) is satisfied when $\mu = \mu^*$.

If $\mu^* = 0$, then (7.7) reduces to $k(n - 2\delta) \geq 2$, which contradicts $n - 2\delta \leq 0$. If

$\mu^* = (\delta + 2k - n)/2$ and $\mu^* \geq 0$, then

$$k \geq \frac{n - \delta}{2}. \quad (7.8)$$

Setting $\mu = \mu^*$ in (7.7) gives

$$\begin{aligned} (\delta + 2k - n)^2/4 + (nk - 2\delta k - 2) &\geq 0 \\ k^2 - k\delta + \delta^2/4 - \delta n/2 + n^2/4 - 2 &\geq 0 \end{aligned}$$

Solving this quadratic, we obtain roots

$$\frac{\delta \pm \sqrt{\delta^2 + 2\delta n - \delta^2 - n^2 + 8}}{2}.$$

This gives either

$$k \geq \frac{\delta + \sqrt{2\delta n - n^2 + 8}}{2},$$

or

$$k \leq \frac{\delta - \sqrt{2\delta n - n^2 + 8}}{2}. \quad (7.9)$$

In the former, we have a contradiction to our assumption $k < \frac{\delta + \sqrt{2\delta n - n^2 + 8}}{2}$. In the latter,

we have a contradiction, because by combining (7.9) and (7.8), we have

$$\begin{aligned}
 \frac{\delta - \sqrt{2\delta n - n^2 + 8}}{2} &\geq \frac{n - \delta}{2} \\
 \delta - \sqrt{2\delta n - n^2 + 8} &\geq n - \delta \\
 2\delta - n &\geq \sqrt{2\delta n - n^2 + 8} \\
 4\delta^2 - 4n\delta + n^2 &\geq 2\delta n - n^2 + 8 \\
 4\delta^2 - 6n\delta + 2n^2 &\geq 8 \\
 (n - 2\delta)(n - \delta) &\geq 4.
 \end{aligned}$$

This is impossible, however, since $n - 2\delta \leq 0$ and $n - \delta$ is positive.

Thus we have shown that there exists a k -factor for the largest k where kn is even and $k < \rho$. The smallest this k can be is $\rho - 2$.

This concludes the proof of Theorem 112.

7.5 Proof of Theorem 113

Construction. Given positive integers such that $n/2 \leq \delta \leq n - 1$. Construct G such that $V(G) = S \cup T$ with $S \cap T = \emptyset$.

If $\delta = n/2$, then choose $\sigma = 2 \left\lceil \frac{n-4}{4} \right\rceil$. If $\delta > n/2$, then choose σ to be the largest integer such that $(n - \sigma)(\delta - \sigma)$ is even and

$$\sigma \leq \frac{n - \sqrt{2\delta n - n^2}}{2}.$$

Let $G[S]$ be an independent set of order σ , $G[T]$ be a $(\delta - \sigma)$ -regular graph of order $n - \sigma$ and the bipartite graph $G[S, T]$ be complete.

For this construction to work, we need $(n - \sigma)(\delta - \sigma)$ to be even, so that $G[S]$ to have a $(\delta - \sigma)$ -factor.

Let H be a k -factor of G for some k . In H , there are exactly $k\sigma$ edges leaving S because $G[S]$ is an independent set. Thus, some vertex v in T must receive at most $k\sigma/(n - \sigma)$ edges from S . Since v has only $\delta - \sigma$ edges within T , we see that, in H ,

$$\deg_H(v) \leq \frac{k\sigma}{n - \sigma} + (\delta - \sigma).$$

Since $\deg_H(v) = k$, we have $\frac{k\sigma}{n - \sigma} + (\delta - \sigma) \geq k$. Solving this inequality for k , we have

$$k \leq \frac{(n - \sigma)(\delta - \sigma)}{n - 2\sigma}. \quad (7.10)$$

Note that Theorem 119 can also be used to derive (7.10).

If $\delta = n/2$, then set $\sigma = \lceil (n - 4)/4 \rceil$, and (7.10) gives

$$k \leq \frac{(n - \sigma)(n/2 - \sigma)}{n - 2\sigma} = \frac{n - \sigma}{2} = \frac{n}{2} - \left\lceil \frac{n - 4}{4} \right\rceil = \left\lfloor \frac{n + 4}{4} \right\rfloor.$$

Observe that this construction matches Katerinis' bound because when $\delta = n/2$, and hence n is even, $\lfloor (n + 4)/4 \rfloor = \lfloor (n + 5)/4 \rfloor$.

If $\delta > n/2$, then we set $\sigma = \frac{n - \sqrt{2\delta n - n^2}}{2} - \epsilon$, where $\epsilon \in [0, 2)$ is just some small quantity to make σ an integer of the right parity (recall σ needs to be a certain parity in order for $(n - \sigma)(\delta - \sigma)$ to be even). The minimum degree of the construction is in fact δ , since the vertices in T were designed to have degree δ , and the vertices in S have degree $n - \sigma$. To

see that $n - \sigma \geq \delta$,

$$\begin{aligned}
 n - \sigma &\geq n - \left(\frac{n}{2} - \frac{\sqrt{2\delta n - n^2}}{2} \right) \\
 &= \delta + \frac{n - 2\delta}{2} + \frac{\sqrt{2\delta n - n^2}}{2} \\
 &= \delta + \frac{\sqrt{2\delta n - n^2}}{2n} \left(n - \sqrt{2\delta n - n^2} \right) \geq \delta.
 \end{aligned}$$

Now that we have verified the validity of the construction, we can return to (7.10):

$$\begin{aligned}
 k &\leq \frac{(\delta - \sigma)(n - \sigma)}{n - 2\sigma} \\
 &= \frac{\left(\delta - \left(n - \sqrt{2\delta n - n^2} - 2\epsilon \right) / 2 \right) \left(n - \left(n - \sqrt{2\delta n - n^2} - 2\epsilon \right) / 2 \right)}{n - \left(n - \sqrt{2\delta n - n^2} - 2\epsilon \right)} \\
 &= \frac{\left(2\delta - n + \sqrt{2\delta n - n^2} + 2\epsilon \right) \left(n + \sqrt{2\delta n - n^2} + 2\epsilon \right)}{4\sqrt{2\delta n - n^2} + 8\epsilon} \\
 &= \frac{4\delta n - 2n^2 + (2\delta + 4\epsilon)\sqrt{2\delta n - n^2} + 4\delta\epsilon + 4\epsilon^2}{4\sqrt{2\delta n - n^2} + 8\epsilon} \\
 &= \frac{\left(2\delta + 2\sqrt{2\delta n - n^2} \right) \left(\sqrt{2\delta n - n^2} + 2\epsilon \right) + 4\epsilon^2}{4\sqrt{2\delta n - n^2} + 8\epsilon} \\
 &= \frac{\delta + \sqrt{2\delta n - n^2}}{2} + \frac{\epsilon^2}{\sqrt{2\delta n - n^2} + 2\epsilon} \\
 &\leq \frac{\delta + \sqrt{2\delta n - n^2}}{2} + \frac{4}{\sqrt{2\delta n - n^2} + 4}
 \end{aligned}$$

This concludes the proof of Theorem 113.

7.6 Proof of Corollary 114

The construction of the graph G from Theorem 113 requires that

$$\begin{aligned}
 k &\leq \frac{\delta + \sqrt{2\delta n - n^2}}{2} + \frac{4}{\sqrt{2\delta n - n^2} + 4} \\
 &= \rho + \frac{\sqrt{2\delta n - n^2} - \sqrt{2\delta n - n^2 + 8}}{2} + \frac{4}{\sqrt{2\delta n - n^2} + 4} \\
 &= \rho + \frac{-8}{2(\sqrt{2\delta n - n^2} + \sqrt{2\delta n - n^2 + 8})} + \frac{4}{\sqrt{2\delta n - n^2} + 4} \\
 &\leq \rho + \frac{-2}{\sqrt{2\delta n - n^2} + 8} + \frac{4}{\sqrt{2\delta n - n^2} + 8} \\
 &\leq \rho + \frac{2}{\sqrt{n+8}},
 \end{aligned}$$

where the last inequality uses the fact that $\delta > n/2$. This establishes Corollary 114.

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