The Exponential Function

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Expository Paper

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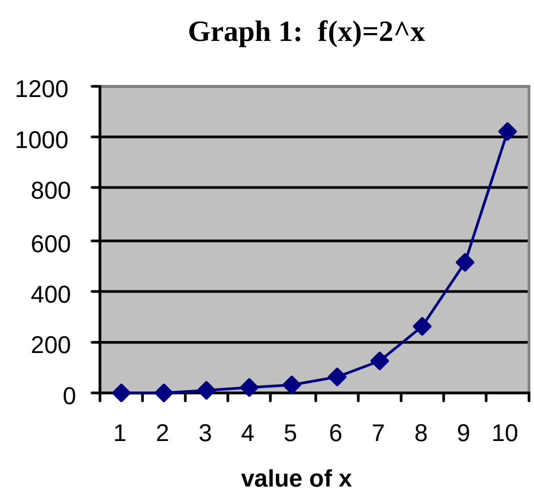
One of the basic principles studied in mathematics is the observation of relationships between two connected quantities. A function is this connecting relationship, typically expressed in a formula that describes how one element from the domain is related to exactly one element located in the range (Lial & Miller, 1975). An exponential function is a function with the basic form $f(x) = a^x$, where a (a fixed base that is a real, positive number) is greater than zero and not equal to 1. The exponential function is not to be confused with the polynomial functions, such as x^2. One way to recognize the difference between the two functions is by the name of the function. Exponential functions are called so because the variable lies within the exponent of the function (Allendoerfer, Oakley, & Kerr, 1977). These functions are often recognized by the fact that their rate of growth is proportional to their value (Bogley & Robson, 1999). This concept of exponential growth has been around much longer than at the dawn of calculus. Evidence of this dates back almost 4,000 years ago on a Mesopotamian clay tablet, which is now on display at the Louvre. The question translated from the stone slab simply asks, "How long will it take for a sum of money to double if invested at 20 percent interest rate compounded annually?"(Aleff, 2005).

The intensity of exponential function study began in the late 17th and early 18th centuries, when great mathematicians, such as Jacob Bernoulli, Leonard Euler, and Isaac Newton began to delve into the depths of exponential and logarithmic functions. Their insights brought about the creation of modern calculus, which in turn gave meaning to many mathematical phenomenon that could only be explained through exponential (and related) functions. Exponential functions are fascinating and widely used in everyday applications. The intent of this paper is to exam the exponential function through research and examples, taking careful note of the special features and distinguishing characteristics of its properties.

Let's begin by looking at a simple exponential function, $f(x) = (c)a^x$, where $y = f(x)$. We see that a is a constant base raised to the x power, and c is some constant number. Here is an example of this function described in the following table and graph:

Chart & Graph for $f(x) = ca^x$, when $c = 1$ and $a = 2$, or $(y = 2^x)$

X	f(x)
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1,024



One of the examples that I use with my own students to express this most basic of exponential functions is to ask them the following: If you had a choice between the following two earnings, which would you choose and why? Option 1 is to earn $100 per day for an entire year. The second choice is to take a checkerboard and to begin on day / square one. You begin by earning one cent on day one. On the next day, the amount in the new checkerboard square doubles the value of the previous days value. So, for example, on day 2 there would be 2 cents, giving you a grand total of 3 cents. This doubling process continues until you run out of checkerboard squares (which there are only 64). Now, which option would you choose and why? Without hesitation or any calculations, students immediately choose option 1 because receiving $100 per day for 365 days sounds like an immediate rags-to-riches story.

My example is adapted from the famous story of the Emperor of China who wanted to thank the inventor of the game of chess by giving him anything he wanted in the kingdom. The inventor only wanted rice, which sounded like a simple gift, but it was said in a particular manner: “I would like one grain of rice on the first square of the chessboard, two grains on the second, four grains on the third and so forth. I would like all of the grains of rice that are placed on the chessboard in this way.” To the emperor, this didn’t sound like much, perhaps a bushel or two of rice, just as my students assuming that placing pennies on a checkerboard in the same manner as the rice wouldn’t have a greater value then $100 per day for 1 year. Let’s look at the growth of this exponential function:

X	1	2	3	4	5	6	7	8	9	10	11
F(x)	1	2	4	8	16	32	64	128	256	512	1024

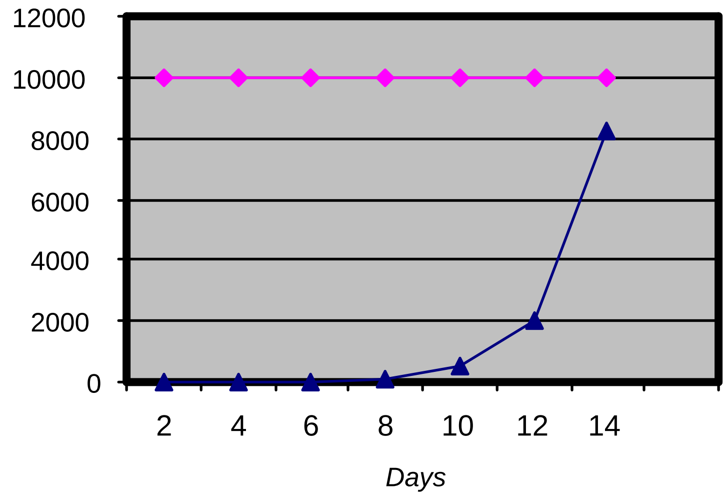
This is similar to the previous function of $f(x) = 2^x$, where a doubling process is witnessed. However, this function is slightly different because the first number in the range is 1, not 2. This function can be stated as $f(x) = 2^{x-1}$. Looking at the first square on the chessboard and placing only one grain of rice on it, signifies the starting point for this exponential growth. This is important because these eventually large numbers will have begun from the most natural of numbers, 1. We can see that this function holds true: $f(1) = 2^{1-1} = 2^0 = 1$, $f(2) = 2^{2-1} = 2^1 = 2$, and so forth. This function allows for the previous table’s data to be used, simply shifting the information down to the next value of x.

According to this table, we only have $20.47 (sum of f(1) through f(11)). At this point, students (and possibly the emperor) still stick with their initial thought of $100 a day for a year. They compare this value to the $1,100 that they could have after 11 days and believe it to be impossible for this doubling process to overtake the linear equation of a set rate of money per day. This exponential function, as do others, may grow slowly at first, but there is a turning point where the exponential function will grow rapidly. Let’s continue the previous chart out a little further:

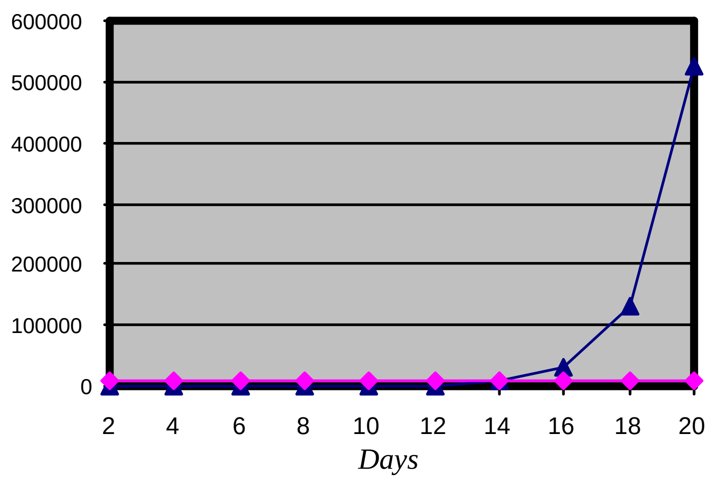
X	12	13	14	15	16	17	18	19	20
f(x)	2048	4096	8192	16384	32768	65536	131,072	262,144	524,288

Students see that on day 15, they now have over $100 per day. According to this table, by day 20 they would have made $5,242.88 on that day alone! Students then begin to realize that they have only gone through less than 1/3 of the checkerboard. Another realization is that each checkerboard square, each function of x, represents a day of earned income. This wealth is accumulating, so they need to add up the value of each checkerboard box! Here are two comparisons between the two graphs $f(x) = 2^{x-1}$, the blue line, and $x = 10{,}000$, where x represents a given day:

Graph 2: First 16 Days



Graph 3: First 20 Days

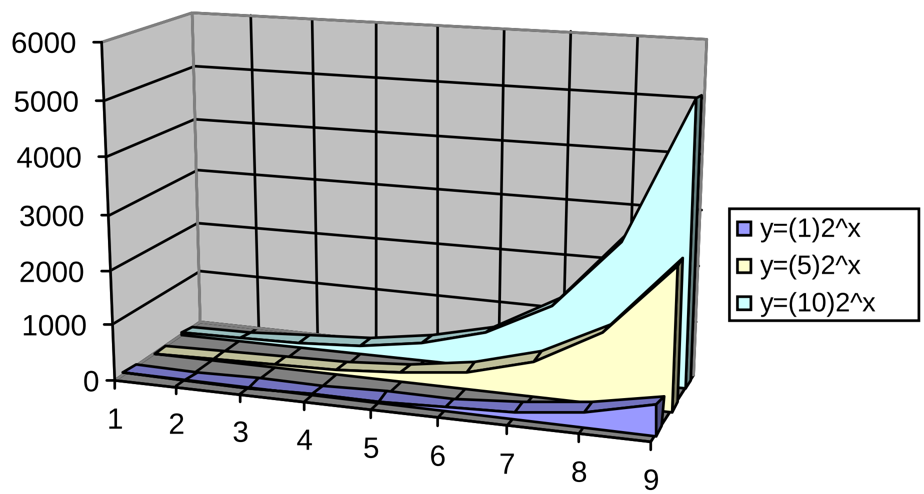


These visuals show students a daily correspondence (at least for the first 20 days) between the two choices given to them. Looking at the first graph, nothing really appears to happen until around day 12. But when looking at the second graph, a noticeable change has occurred and now the exponential function is growing rapidly. Students typically want to change their original answers by this point and want to continue on to see the value of the 64th day. The value of the last checkerboard square is

$2^{63} \approx 9.2233 \times 10^{18}$ pennies. I often have to remind students that this is the value of only 1 square; they still have all of the other monetary values for the other 63 squares! I'm sure that when the emperor discovered the power of this exponential growth, he realized his own inability to comprehend the affects of doubling. Albert Bartlett makes mention of this story, using grains of wheat rather than rice. With his calculations, he tries to put into perspective the amount of wheat that the emperor would be presenting to the inventor: "Since common grains of wheat have a mass of 3.4 grams per hundred, the total mass of wheat needed to repay the debt is 6.27×10^{14} kg which is approximately 500 times the current (1976) annual world-wide harvest of wheat. This apparent modest method…..would require an amount of wheat which is probably larger than the total amount of wheat that has been harvested in the entire history of the earth!" (Bartlett, Fuller, Plano, & Rogers, 2004)

Let's now look at the basic exponential function, $f(x) = (c)a^x$, and what happens as the constant increases. Here is a graph comparing $f(x) = (1) \bullet 2^x$, $f(x) = (5) \bullet 2^x$, $f(x) = (10) \bullet 2^x$:

Graph 4: Changes in the Constant

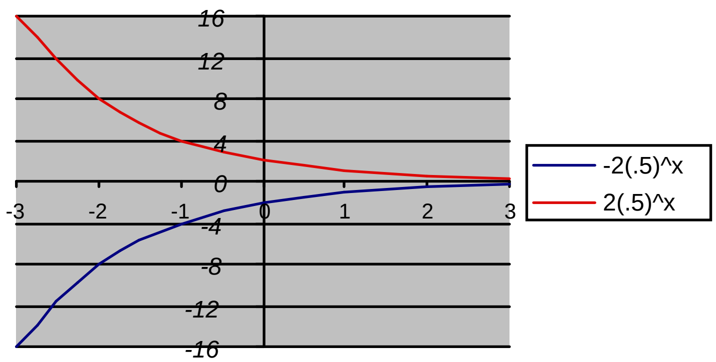


X	1	2	3	4	5	6	7	8	9
$(1) \bullet 2^x$	2	4	8	16	32	64	128	256	512
$(5) \bullet 2^x$	10	20	40	80	160	320	640	1280	2560
$(10) \bullet 2^x$	20	40	80	160	320	640	1280	2560	5120

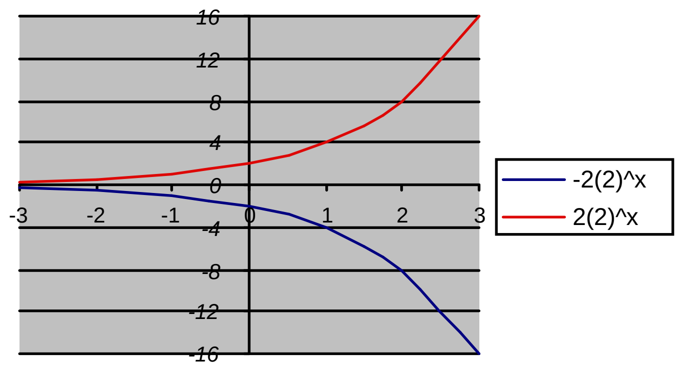
We began with the exponential function from graph 1. We can see from both the chart and graph that the constant does affect the growth of the function, but not exponentially. The reason being is because the constant isn't factored into the function until after the exponential process has occurred. If we observe the values between the three functions, we see that the values from $f(x) = (5) \bullet 2^x$ are five times greater than the original function of $f(x) = (1) \bullet 2^x$. Likewise, the function of $f(x) = (10) \bullet 2^x$ is only twice as large as $f(x) = (5) \bullet 2^x$, because the constant 10 is twice as large as 5. Finally, the function $f(x) = (10) \bullet 2^x$ is 10 times larger then the original function of $f(x) = 2^x$.

We also have to consider the exponential function when multiplying by a negative constant. When multiplying by a negative constant, the function transforms and "flips"

Graph A: Negative Constant (base < 1)



Graph B: Negative Constant (base > 1)

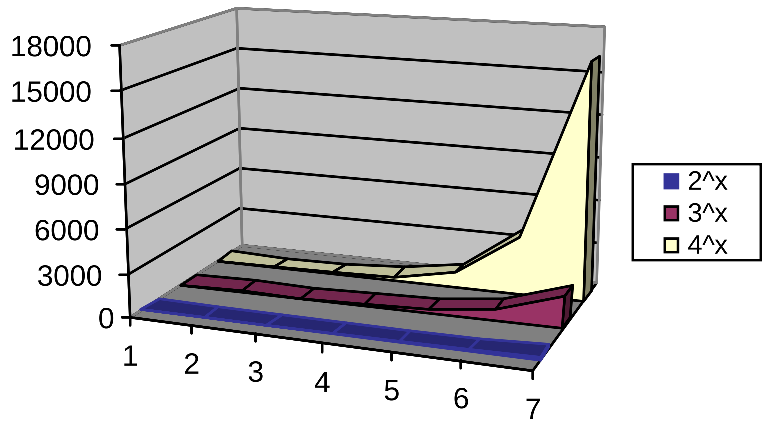


the graph (see the blue line in the graphs above). When a negative constant is multiplied by a base that is greater than 0 and less than 1, we see a result similar to graph A, where the positive x-axis (or y value) becomes the horizontal asymptote. When a negative constant is multiplied by a base that is greater than 1, we see a result similar to graph B, where the negative x-axis (or y value) becomes the horizontal asymptote.

Let's now examine what happens when the constant remains the same for each function, but the base is altered. Here are three functions of the form $f(x) = ca^x$, where c=1 for each:

X	1	2	3	4	5	6	7	8	9
$(1) \bullet 2^x$	2	4	8	16	32	64	128	256	512
$(1) \bullet 3^x$	3	9	27	81	243	729	2,187	6,561	19,683
$(1) \bullet 4^x$	4	16	64	256	1024	4,096	16,384	65,536	262,144

Graph 5: Change in Base value

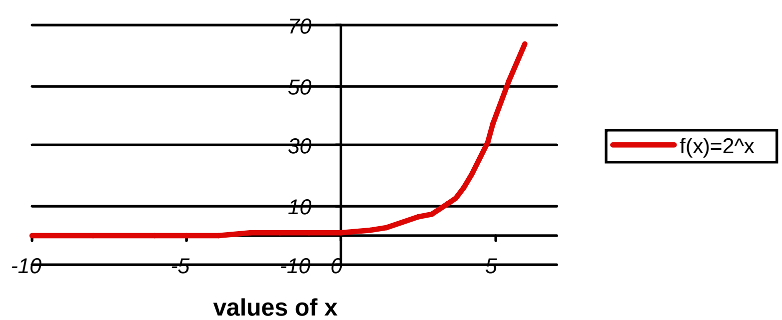


We witness a different type of growth with this change in the exponential function. When the base is changed, the rate of growth is much greater than when manipulating the constant. In the data presented with graph 4, we saw that the growth between different functions remained a constant growth between the values of x (i.e. $f(x) = (10)2^x$ was always twice as large as $f(x) = (5)2^x$). Now we see that the growths between the two functions of $f(x) = 2^x$ and $f(x) = 4^x$ are exponential to one another, or $f(4^x) = f(2^x)^2$, thus the greater the growth rate between the two.

Perhaps this is one reason why Albert Bartlett considers the mathematics of the exponential function the mathematics of growth (Bartlett, Fuller, Plano, & Rogers, 2004). He appears to be a strong advocate for man's understanding of the exponential function, especially in the relevance of the real world. He wrote a series of journals in *The Physics Teacher*, spanning nine parts, trying to educate others in the power and importance of the exponential function. The theme of his first journal / presentation was "The greatest shortcoming of the human race is man's inability to understand the exponential function" (Bartlett, Fuller, Plano, & Rogers, 2004).

So far, all of the examples presented have only included the natural numbers for the exponents in the functions. What happens to the graph when negative exponents are included?

Graph 6: Negative values of x

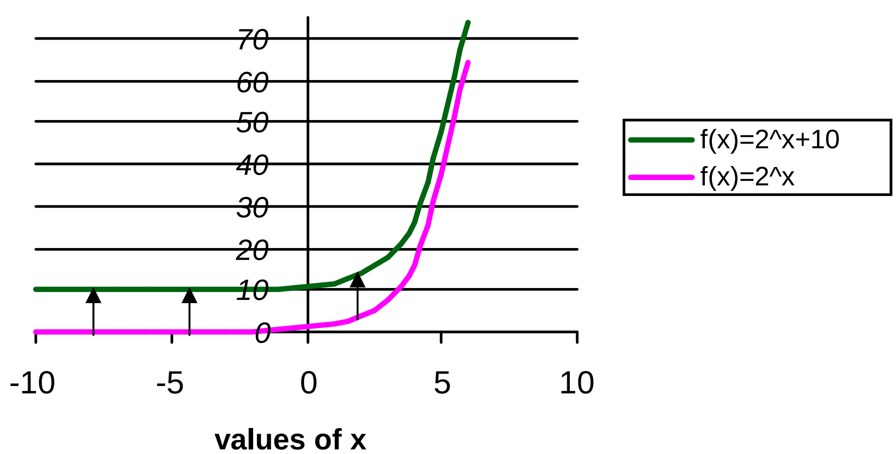


X	-10	-8	-6	-4	-2	0	2	4	6
F(x)	.000977	.003906	.015635	.0625	.25	1	4	16	64

When negative numbers are included in the exponential function, the value will never reach the x-axis, although it gets extremely close. The reason that this happens is because of one of the fundamental properties of exponents: $a^{-x} = \frac{1}{a^x}$. No matter how small of a negative number x is, the value of the function will continue to be a positive real number (all-be-it, very small). This value will never reach or intersect the x-axis, no matter what negative value is placed into the function. However, the smaller the negative number, the closer it gets to the x-axis. The x-axis becomes the horizontal asymptote for any function in the form $f(x) = (c)a^x$.

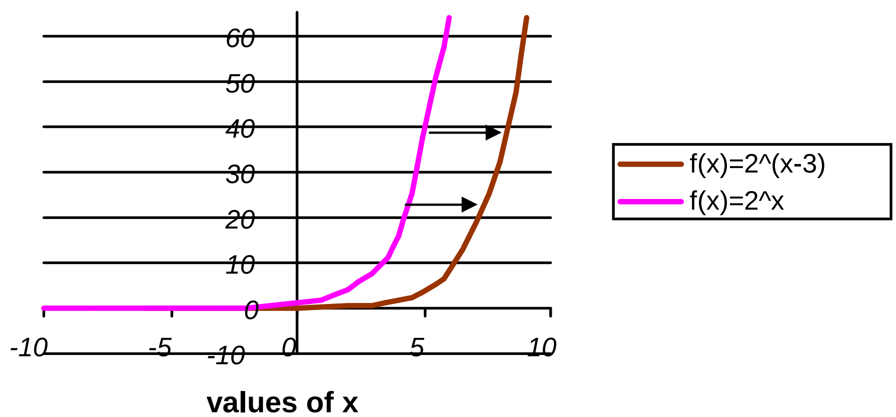
Other changes occur as the exponential function is manipulated. For example, the exponential function can be shifted vertically on the coordinate plane. The graph of f(x) = ca^x + y is shifted vertically by 'y' units from the x-axis. Taking the standard $f(x) = 2^x$ function and changing it to $f(x) = 2^x + 10$, we can see that the graph of the function maintains its general shape, but all points of the graph are shifted up 10 units (see graph 7). This same occurrence will happen if we were to subtract from this function, only the graph would shift down. The number that is added to the function becomes the new horizontal asymptote for the graph.

Graph 7: Vertical Shift of Exponential Graph



Another shift of the exponential graph would be a horizontal shift. Adding or subtracting to the exponent does this. In the example of f(x) = ca^x , by changing it to $f(x) = ca^{x-h}$ the entire graph is shifted horizontally by 'h' units. For example, taking $f(x) = 2^x$ and altering it to $f(x) = 2^{x-3}$, the entire exponential graph is shifted to the right by 3 units (see graph 8). When thinking about the new exponent created when subtracting from the variable, we can now see that it is necessary to have a larger value for x in order to produce the same data from the original function. If adding to the exponent, the graph will shift to the left because now the value of x doesn't need to be as large since it is becoming larger by adding a positive number to it.

Graph 8: Horizontal Shift of Exponential Graph

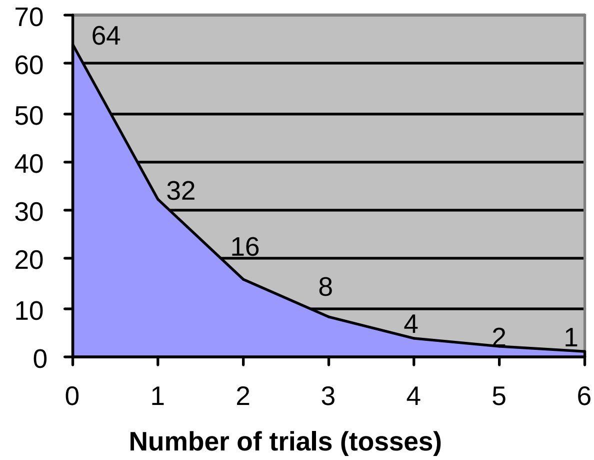


Even though the exponential function is considered the mathematics of growth, it is also extremely relevant to that which decomposes, or decays. There are a couple of different ways to express this function of exponential decay: If $a > 1$, then $f(x) = (c)a^{-x}$. If $0 < a < 1$, then $f(x) = (c)a^{x}$. Both are illustrations of the process of exponential decay. In the first function, $f(x) = (c)a^{-x}$, since a is a base that is larger than 1, multiplying it by a negative exponent with a value of x will decrease the value of the function (recall the property of exponents when using a negative exponent). The second function, $f(x) = (c)a^{x}$, which looks exactly like the function used for exponential growth, decreases in value when a is a fraction or decimal less than 1, yet larger than 0. This makes sense when you think about it; taking a number between 0 and 1, multiplying it by itself will only decrease the end results.

One algebra level representation of exponential decay is the M&M© tossing game (www.pbs.org). Students begin with a bag of '*c*' M&M's. Students then shake the bag and pour the candy onto the top of their table. Students first remove the M&M's that are not showing the symbol "m", and then record the number of M&M's that are showing the symbol "m" in their data table. Only the candies that were counted are placed back into the bag and the process is repeated until there are fewer than 10 candies left. It is very important that students have a number greater than zero in their last entry on their data table. Students then work, through class discussion and teacher-led questioning, on how to represent this process with an exponential function for decay. Students use the formula $y = c(1-r)^{x}$, where c represents the constant, the initial amount of M&M's, x is the number that represents the toss / pouring step, and r represents a decay rate (which is often a percent in decimal form). Students use their graphing calculators to generate scatter plots of their data. The intent is for students to recognize that the shape of the plot appears to be exponential. Finally, students attempt to calculate an exponential model for their data, seeing if it fits the data that was collected.

Theoretically, the rate (r) should be 50%, because each M&M has one side that is marked with the symbol 'm' and one side that is not labeled. Each side has the same probability of showing when the bags are poured onto the desks. The intent is to take the original function of $y = c(1-r)^x$ and get to $f(x) = c(.5)^x$. An example of this process is graphed below, starting with 64 M&M's

Graph 9: M&M exponential decay

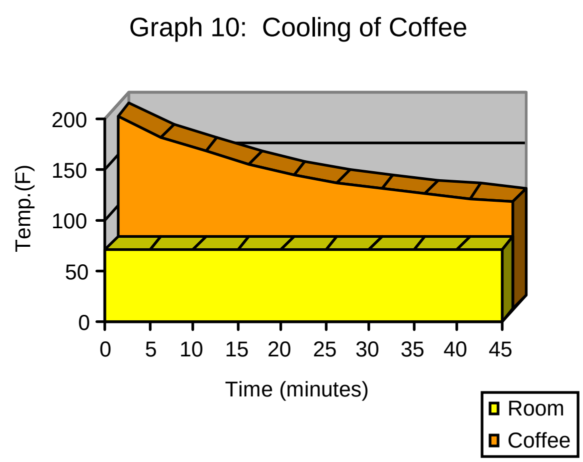


The importance of not continuing this process until there are zero M&M's left is to understand the process of exponential decay, but using real numbers, not necessarily whole numbers. The positive x-axis, in this example, becomes the horizontal asymptote, where the exponential function gets closer and closer with each pouring of the M&M bag. My example uses "nice" numbers. As I think of the function for exponential decay, these numbers appear every March when the NCAA© basketball tournament begins. Starting with 64 teams, after the first round only 32 are left (or half). With each round only 50% of the teams remain (16, 8, 4,2) until after the sixth and final round, only one team remains.

Some real-life applications of exponential decay involve the cooling of items in relation to their environments. During this current semester, we (my cohort and I) have been using exponential decay to describe the cooling process of a dead body. In this course, we have spent time studying Newton's Law of Cooling, which states that the rate of change of the temperature of an object is proportional to the difference between the object's temperature and the room temperature. If we looked at this mathematically, it would be set up as the rate of change of temperature equals some constant times the difference between the object's temperature and the environment's temperature. The decay function that we used was $T_{(t)} = ce^{-rt} + t_{(room)}$. Here we are introduced to the number *e* as a base for the first time, which I will discuss later. In this function, *c* is a constant determined by the difference between the initial temperature of the object and the temperature of the environment, *r* represents the rate of decay, and *t* represents the elapsed time from the initial temperature of the object. The use of $t_{(room)}$ is the temperature of the environment (such as a room) in which the object is cooling. The object's temperature is cooling exponentially towards the temperature of the room, yet not reaching the temperature of the room. Here again, we see that by adding a number to the exponential expression part of the function, we are moving the asymptote to something different then the x-axis.

Let's look at an example that I did involving the cooling of a cup of hot coffee in a room. First, temperatures are collected at 5-minute intervals for the cup of coffee, which is cooling in an environment with room temperature $71.5^{\circ}F$. What we know is that once we take the coffee and pour it into the cup, it will cool exponentially, towards the temperature of the room. Here is the data and graph that was collected for the first 35 minutes of cooling:

Time	Temp	r (rate)
0	190	-
5	168	.041
10	154	.036
15	142	.035
20	132.5	.033
25	124.5	.033
30	119	.030
35	114	.029



The rate, *r*, which is a constant for the cooling process, appears to have different values at different cooling times. Finding *r* is a critical piece for understanding the exponential decay of temperature. Here is an example of calculating *r* using the data from the table (when time is at 5 minutes):

The initial function is stated as $118.5e^{-r(5)}+71.5=168$. 118.5 is calculated by taking 190 (coffee temp.) – 71.5 (room temp.). The base of e is used, where *r* represents the rate. 71.5 is added to the exponential expression because the term we find using the exponential expression needs to be elevated above the room temperature, since the x-axis, or 0, is not the asymptote of our graph.

Using this equation, $118.5e^{-r(5)}+71.5=168$, we can simplify by subtracting 71.5 from both sides and dividing both by 118.5. This leaves us with $e^{-r(5)}=.8143459916$. In order to find the rate, we will need to use the inverse function of the exponential function, or the natural logarithm. Here we see that $\ln(e^{-r(5)})=\ln(.8143459916)$, which simplifies to $-5r=-.2053699522$. By dividing both sides by –5, we finally arrive at $r=.0410739904$, or about .041. This was done for each of the time intervals. This rate should have been consistent, but Newton's Law of Cooling is only an approximation. Likewise, there are other factors, such as measurement reading errors and inconsistency in the environmental temperature that could cause this rate to fluctuate. Looking at the different rates calculated for this data, a rate was selected that was the best representation for this data. The rate selected was r=.033, since this number happens to be both the mean and the median of all the rates that were calculated (up to the first 45 minutes). So the final formula for the cooling of this cup of coffee was determined to be: $T_{(t)} = 118.5e^{-.033(t)} + 71.5$, where $T_{(t)}$ is the temperature of the coffee at a particular time after the beginning cooling process.

Knowing this exponential decay function can now help calculate the time when the cup of coffee will reach a specific temperature. For example, if a cup of coffee is $120^{\circ}F$, we can determine approximately how long the coffee has been cooling, if it met the previous conditions (room temp is $71.5^{\circ}F$ and the initial temperature of the coffee was $190^{\circ}F$). Let's try it!

$$120 = 118.5e^{-.033(t)} + 71.5$$
$$48.5 = 118.5e^{-.033(t)}$$
$$.4092827004 = e^{-.033(t)}$$
$$\ln(.4092827004) = \ln(e^{-.033(t)})$$
$$-.8933491626 = -.033t$$
$$t \approx 27 \text{ minutes}$$

So with the given environmental conditions, and finding an approximate rate for the cooling conditions, a time of 27 minutes will be needed to cool the coffee to $120^{\circ}F$. This same process can be used for the natural cooling of any object, but realizing that the rate of cooling is different and needs to be calculated for each object. Another example of exponential decay relates to radioactive decay, or the breakdown of radioactive elements. In each example, the exponential decay begins quickly and then levels off as it approaches the asymptote (i.e. room temperature), never quite reaching this point.

Perhaps the most widely used real-life application for exponential growth is compound interest. Compound interest simply means the interest that is earned over a period of time, where previous interest is added to the principal and interest continues to grow. A formula that represents this concept is $V = P(1+\frac{r}{n})^{nt}$, where V is the value of the account, *P* is the initial principal, *r* is the interest rate expressed as a fraction or decimal, *n* is the number of times per year that the interest is compounded, and *t* is the number of years invested. Assuming that there are no additions made to the account or withdrawals, we can view this as an exponential growth function. Let's look at an example of compounding interest yearly for a couple of years at 8% interest and an initial principal of $1000. After one year, the initial $1000 is still there, but now an additional 8% of that $1000, or $80, is included for a total account value of $1,080! Looking back at the original formula, $V = P(1+\frac{r}{n})^{nt}$, the value of the account equals the initial principal times 1.08. This number accounts for the initial principal and the 8% additional interest. If this is left in the account for another year, the new principal of $1,080 will still be there, but now there is an additional 8% of this, or 1,080+(.08x1,080). The new value of the account after 2 years is $1,166.40. Looking at it exponentially, we see that $V = 1{,}000(1.08) + 1{,}000(1.08)(.08)$, which the first term (1,000 x 1.08) is the amount after the first year and the second term that is added on to this (1,000 x 1.08 x .08) is the interest for the second year. This is also written as 1000(1.08)(1.08) or $1000(1.08)^2$. After t years, the value of the account, compounded annually, could be expressed as $V = 1000(1.08)^t$. The growth of the account is exponential because interest becomes principal and then the interest is earning interest.

If an account earns interest quarterly (4 times per year), then the same account mentioned above will see a value increase in their account. Let's look at this example. For quarter 1, we use the formula $V = P(1+\frac{r}{n})^{nt}$, or $1000(1+\frac{.08}{4})$, which equals $1,020. This seems to be right on track with the annual compounding. However, after the first quarter, this interest is placed into the account and becomes principal. When calculating the second quarter we now use $1020(1+\frac{.08}{4})$, which is $1,040.40. After the third quarter there is $1,061.21 and after one year the accounts value is $1,082.43! By compounding quarterly the value earned an additional $2.43.

If the same account was compounded monthly, we would use the following formula to calculate its value after one year: $1000(1+\frac{.08}{12})^{12}$, or $1,083.00. Compounded daily for an entire year would be $1000(1+\frac{.08}{365})^{365}$, or approximately $1,083.28. To calculate at a continuous rate the following equation is used: $Value = Pe^{rt}$.

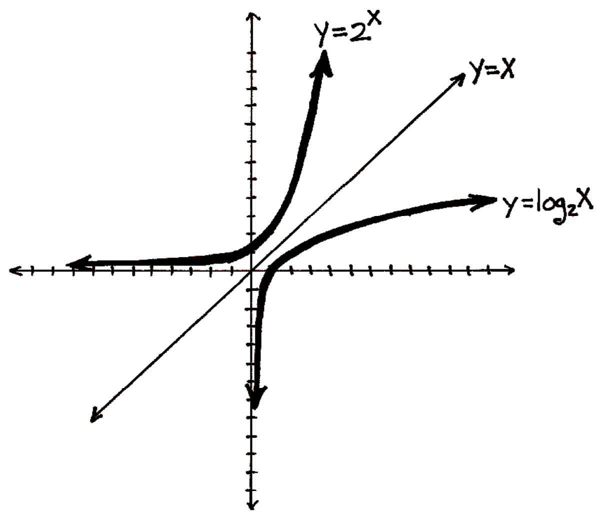
Again, this value of *e* appears. What is it and why is it used so frequently as a base in the exponential function? Let's use the example of compound interest to understand the number *e* a little better. Assume that the initial principal of the account is $1 and the interest rate (r) is also equal to 1 (which is 100%). Let's look at the value of the account after 1 year for different compounding methods.

If compounded annually	$1(1+\frac{1}{1})^{1}$, or 2
If compounded quarterly	$1(1+\frac{1}{4})^{4}$, or 2.44140625
If compounded daily	$1(1+\frac{1}{365})^{365}$, or 2.714567482
If compounded by the minute	$1(1+\frac{1}{525{,}600})^{525{,}600}$, or 2.718279215
If compounded by the second	$1(1+\frac{1}{31{,}536{,}000})^{31{,}536{,}000}$, or 2.718282473

It appears that as we compound more frequently, we are reaching a value that is close to 2.71828… Actually, as we observe what is happening in each of the previous compound equations, even though the value inside of the parentheses, $1+\frac{1}{n}$, is getting smaller, it is being raised to a larger power of n. The formula $(1+\frac{1}{n})^{n}$, as $n \to \infty$, becomes one of the definitions for the number *e*.

The number *e* is named in honor of Leonhard Euler, who spent great amounts of time exploring its properties, although John Napier used it to develop his conceptual understanding and development of logarithms almost a century before Euler. Some believe that *e* received its notation simply because it stood for "exponential", while others believe that the letters *a*-d had already been given mathematical meanings, so *e* was the next obvious choice (Aleff, 2005). Whatever, or whoever, this interesting irrational number is named for, it plays an important role as a base value for the exponential function. The value of *e* is often used as a base in economic analysis and other situations involving natural growth and decay. It is perhaps the most important irrational number in mathematics, with the exception of π.

While researching the exponential function, other topics appeared numerous times, such as the number *e*. Another one of these topics is the logarithmic function, also known as the inverse function of the exponential function. Let's look at the graph of $f(x)=2^x$ and its reflection across the line made by y=x.



Looking at this graph, the line y=x is a mirror line for the graph of $y=2^x$, an exponential function. The line created across this reflection is the inverse function of the exponential function, or the logarithm. This logarithm is $y=\log_2 x$. We can see that with the log function, the vertical asymptote is located at the y-axis, and its x-intercept is at (1,0). In contrast, the exponential function of 2^x has a horizontal asymptote of the x-axis and a y-intercept at (0,1). The line y=x is also considered the perpendicular bisector of the line segments created by joining inverse points from the logarithmic and exponential functions (Allendoerfer, Oakley, & Kerr, 1977). For example, the line segment created by connecting point (0,1) which is located on the exponential line graph and point (1,0) which is located on the logarithm line graph, is bisected by the line y=x. This is true for the other inverse points: (1,2) & (2,1), (2,4) & (4, 2), (3,8) & (8,3), as well as any reflected points found on the lines. Comparing the two functions, we could say that a logarithm is an exponent (Larson & Hostetler, 1979). Some examples that show the relationship between exponents and logarithms are:

Logarithmic Form		*Exponential Form*
$\log_2 8=3$	⟶	$2^3=8$
$\log_{10}1000=3$	⟶	$10^3=1000$
$\log_a x=b$	⟶	$a^b=x$ (If $a>0$ and $a\neq 1$)

(Larson & Hostetler, 1979)

Here, the number *e* appears again because of its limit concept (as it approaches infinity) and its convenience as a universal base. A logarithm with a base of e is defined as the natural logarithm of a number, with notation as such: $\log_e x=\ln x$. Both the exponential and the logarithmic functions are the foundation for the study of calculus.

There appears to be much to study and understand in the world of exponential functions and its applications to our world. I can now look back and recognize that the Math in the Middle courses have always had topics related to the exponential function. It seems intriguing that such a concept appears so frequently in different topics within mathematics. I look forward to understanding their properties and relationships more intimately during my summer courses in the Math in the Middle program.

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