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Abstract

A top and drop cover is a treaty that can be found on the retrocession market. It offers capacity that can be used either to protect a top layer or a working layer. The former is called a "top" and the latter is called a "drop." Using the traditional collective risk model, we demonstrate the use of a multivariate version of Panjer's algorithm to price this cover. We also compare the premium obtained within the exact model with the premiums obtained either with the Fréchet bounds or with the wrong assumption of independence.

Key words and phrases: *multivariate Panjer's algorithm, excess of loss pricing, dependence, correlation order, stop-loss order, comonotonic risks, Fréchet bounds, supermodular order*

1 Introduction

The traditional collective risk model assumes that an insurance portfolio produces a random number of N positive claims in a year. The claim sizes, $X_1, X_2, \ldots, X_N$, are assumed to be independent and identically distributed positive random variables. The annual aggregate claims S is then given by

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$$S = X_1 + \cdots + X_N.$$

When N belongs to the $(a, b, 0)$ class of counting distributions,[1] i.e., when the probabilities associated with N satisfy

$$\frac{\mathbb{P}[N = n]}{\mathbb{P}[N = n-1]} = a + \frac{b}{n} \quad \text{for} \quad n \geq 1,$$

and the X_is are discrete, then it is easy to obtain the distribution of S using the recursive algorithm due to Panjer (1981).

The assumption of mutual independence of claim sizes in the collective risk model, which makes sense in many situations, offers the advantage of mathematical simplicity. There are situations, however, where the independence assumption needs to be relaxed. Some authors have addressed the problem by imposing upper and lower bounds on the results when some form of stochastic dependence is observed (see Dhaene et al., 2001), while others have attempted to model the dependence (e.g., Frees and Valdez, 1998).

This paper extends the collective risk model to include dependent claims. We distinguish between two models:

Model 1, which considers independent occurrences of the random couple (X, Y), i.e., (X_1, Y_1), (X_2, Y_2),..., (X_N, Y_N), with the X_is and Y_is all independent of the counting random variable N. The bivariate aggregate claim is then defined as

$$(S, T) = (\sum_{i=1}^{N} X_i, \sum_{i=1}^{N} Y_i).$$

The dependence between S and T originates from them sharing the same claim number N as well as from possible correlations between the components of the (X_i, Y_i)s. Sundt (1999) proposed a multivariate extension of Panjer's algorithm, allowing for practical calculations within this multivariate collective risk model.

Model 2, which considers the N independent claim sizes $X_1, X_2, \ldots, X_N$, and M independent claim sizes $Y_1, Y_2, \ldots, Y_M$. We assume a mutual independence between the X_is and the Y_js, as well as with the counting variables N and M. However, N and M may be dependent. The bivariate aggregate claims is then defined as

[1] The $(a, b, 0)$ class of counting distributions contains Poisson ($a = 0$), negative binomial ($a > 0$), and binomial ($a < 0$) distributions.

$$(S,T) = (\sum_{i=1}^{N} X_i, \sum_{i=1}^{M} Y_i).$$

Note that the dependence between S and T now originates only from the dependence between N and M because the claim sizes are mutually independent. Some types of dependence between N and M may be modeled using the trivariate reduction method or by mixing the bivariate Poisson distribution. Walhin and Paris (2000b) and Walhin and Paris (2001) provide some sophisticated bivariate counting models allowing for these calculations.

In another departure from the collective risk model, we distinguish between two types of claims (the extension to more types of claims is trivial): (i) small claims and (ii) large claims. We assume the behavior of small claims may differ significantly from the behavior of large claims. In our models we assume the X_is and the Y_is represent the size of the large and small claims, respectively, while N_l and N_s are the annual number of large and small claims, respectively. The common cdf of the X_is is a limited Pareto distributed with nonnegative parameters A_l, B_l, and α_l, while that of the Y_is is a limited Pareto distribution with nonnegative parameters A_s, B_s, and α_s. Because we assume only two types of claims (large and small), then $B_s = A_l$. The numbers of claims N_l and N_s are assumed to be Poisson distributed with mean λ_l and λ_s, respectively.

A random variable X has a limited Pareto distribution with parameters A, B, and α (which, for notational convenience, can be written as $X \sim \mathrm{Par}(A, B, \alpha)$) if its cdf, F_X, can be written as:

$$F_X(x) = \mathbb{P}[X \le x] = \begin{cases} 0 & \text{if } x < A \\ \dfrac{A^{-\alpha} - x^{-\alpha}}{A^{-\alpha} - B^{-\alpha}} & \text{if } A \le x < B \\ 1 & \text{if } x \ge B. \end{cases} \quad (1)$$

Throughout the rest of this paper, we assume mutual independence between the random variables N_l, N_s, X_i, and Y_i. We consider also the following values for the Poisson and limited Pareto distribution parameters as shown in Table 1.

In this paper, we assume Model 1 holds, as this will allow us to derive specific solutions. We propose a new application of the multivariate extension of the Panjer's algorithm to price the so-called *top and drop* cover. This reinsurance treaty, used primarily for retrocession, includes a top layer and a working layer. There is an obvious stochastic dependence in the model as large claims (affecting the top layer)

Table 1
Parameters

	λ	α	A	B
Large Claims	0.3	0.9	400	1000
Small Claims	2.5	1.4	20	400

necessarily hit the working layer. To use the multivariate extension of the Panjer's algorithm, we discretize the claims size distributions, thus making the derived solutions approximations only. We will compare these solutions to those based on the incorrectly assumed independence hypothesis between S and T, as well as to some upper and lower bounds. These comparisons will be done with theoretical or empirical results.

The rest of the paper is organized as follows. Section 2 provides a brief review of excess of loss reinsurance, and describes two types of top and drop covers within a relatively general collective risk model. Section 3 recalls the multivariate Panjer's algorithm. Section 4 reviews some necessary results on stochastic orderings. Section 5 provides the numerical results and compares them with the case where independence would be incorrectly assumed and with the corresponding Fréchet bounds.

2 Top and Drop Covers

Excess of loss reinsurance is a means to share risks between the ceding insurer (the cedent) and and the reinsurer. The cedent always remains liable for the part of the claim below a given attachment point or deductible P, while the reinsurer offers some capacity between P and the limit $P + L$. So we can write the liability of the excess of loss reinsurer for each claim X_i as

$$R_i = \min(L, \max(0, X_i - P)).$$

In the collective risk model, the aggregate liability of the reinsurer is

$$S_R = R_1 + \cdots + R_N.$$

The reinsurance capacity L may be subject to k reinstatements. If $k = 0$, it means that there is no reinstatement and the reinsurer's liability for the whole period (usually one year) is limited to L, regardless of the number of occurrences. Otherwise, the aggregate capacity is $(k+1)L$. Keep in mind the reinsurer's liability in any occurrence is limited to L, i.e., the aggregate liability of the reinsurer is $\min((k+1)L, S_R)$. In practice, reinstatements can be paid or free. In the present paper we will only discuss the situation where the reinstatements are free.

An annual aggregate deductible (AAD) will reduce the aggregate claims of the reinsurer. A higher AAD should reduce the reinsurance premium. For the general case where there are k reinstatements and an AAD, the annual liability of the reinsurer is $\min((k+1)L, \max(0, S_R - \text{AAD}))$.

It is interesting to see how reinsurance can introduce dependencies in some treaties: for example, the ECOMOR-type treaties involving order statistics (see Thépaut, 1950) or the exotic excess of loss treaty described in Walhin (2002) where some layers inure to the benefit of other layers. Walhin (2002) used a multivariate version of Panjer's algorithm to price that treaty. In Walhin and Paris (2000a) this multivariate version of the Panjer's algorithm is used to study the retained risk of the cedent when it buys excess of loss reinsurance with paid reinstatements. We now describe two treaties.

Treaty 1: Recently Secura has been given the opportunity to examine the following excess of loss cover: in reinsurers' jargon (see below for a translation into formulas), the characteristics of this treaty were

- 200 in excess of 800 (written as 200 **XS** 800)

  AND / OR

- 200 **XS** 200 in the aggregate for each loss exceeding 20 (losses to be aggregated from ground up but with a maximum of 100 each and every loss occurrence).
- No reinstatement granted, i.e., the maximal annual amount to be paid by the reinsurer is 200.

The aim of this treaty is to cover a top layer (200 **XS** 800) that has a very low probability of being hit and, simultaneously, a potential high frequency of small claims.

In mathematical terms, the characteristics of this reinsurance cover can be summarized as follows:

$X_i^{\text{RT}} = \min(200, \max(0, X_i - 800))$, which is the reinsurer's liability for the top part of large claims;

$X_i^{\mathrm{RD}} =$ $\min(100, X_i I_{X_i \geq 20})$, which is the reinsurer's liability for the drop part of large claims;

$Y_i^{\mathrm{RD}} =$ $\min(100, Y_i I_{Y_i \geq 20})$, which is the reinsurer's liability for the drop part of small claims;

$S =$ $X_1^{\mathrm{RT}} + \cdots + X_{N_l}^{\mathrm{RT}}$, which is the reinsurer's aggregate liability for the top part of large claims;

$T =$ $X_1^{\mathrm{RD}} + \cdots + X_{N_l}^{\mathrm{RD}}$, which is the reinsurer's aggregate liability for the drop part of large claims;

$U =$ $Y_1^{\mathrm{RD}} + \cdots + Y_{N_s}^{\mathrm{RD}}$, which is the reinsurer's aggregate liability for the drop part of small claims; and

$$\text{Cover} = \min(200, S + \max(0, T + U - 200))$$

where I_A is the indicator function, i.e., $I_A = 1$ if A is true, $I_A = 0$ otherwise. Note that the choice made for B_s implies that the small claims Y_i do not trigger the top cover.

Treaty 2: Another example of top and drop cover is described below:

- 200 **XS** 800

  AND / OR

- 200 **XS** 200 with a global annual aggregate deductible of 400 and unlimited free reinstatements.
- The aim of the treaty is clearly to cover an extra reinstatement on the low layer (which typically would be protected by a classical 200 **XS** 200 with one reinstatement) and/or a top layer (200 **XS** 800).

The reinsurance cover can be described as follows:

$X_i^{\mathrm{RT}} =$ $\min(200, \max(0, X_i - 800))$, which is the reinsurer's liability for the top part of large claims;

$X_i^{\mathrm{RD}} =$ $\min(200, \max(0, X_i - 200))$, which is the reinsurer's liability for the drop part of large claims;

$Y_i^{\mathrm{RD}} =$ $\min(200, \max(0, Y_i - 200))$, which is the reinsurer's liability for the drop part of small claims; and

$$\text{Cover} = \max(0, S + T + U - 400)$$

with S, T, and U described as in Treaty 1.

As a consequence of our choice of distributions for small and large claims, we can simplify the model in two ways:

1)

$$T = \begin{cases} 100N_l & \text{Treaty 1} \\ 200N_l & \text{Treaty 2} \end{cases}$$

which leads to

$$\text{Cover} = \begin{cases} \min(200, S + \max(0, 100N_l + U - 200)) & \text{Treaty 1} \\ \max(0, 200N_l + U - 400) & \text{Treaty 2.} \end{cases}$$

2) Using two independent compound Poisson distributions with limited Pareto distributions for the small and large claims is equivalent to a single compound Poisson with i.i.d. claim sizes that are mixtures of limited Pareto distributions. The new number of claims random variable is $N = N_l + N_s$, which is Poisson with mean $\lambda_l + \lambda_s$, and the new claim sizes are Z_i, which is a mixture of limited Pareto distributions with cdf $F_Z(x)$ given by

$$F_Z(x) = \mathbb{P}[Z_i \le x] = \begin{cases} 0 & \text{if } x < A_s \\ \frac{\lambda_s}{\lambda_s+\lambda_l} \frac{A_s^{-\alpha_s} - x^{-\alpha_s}}{A_s^{-\alpha_s} - A_l^{-\alpha_s}} & \text{if } A_s \le x < B_s = A_l \\ \frac{\lambda_l}{\lambda_s+\lambda_l} \frac{A_l^{-\alpha_l} - x^{-\alpha_l}}{A_l^{-\alpha_l} - B_l^{-\alpha_l}} & \text{if } B_s = A_l \le x < B_l \\ 1 & \text{if } x \ge B_l. \end{cases}$$

We obtain for Treaty 1:

$$\begin{aligned} Z_i^{\mathrm{RT}} &= \min(200, \max(0, Z_i - 800)) \\ Z_i^{\mathrm{RD}} &= \min(100, Z_i I_{Z_i \ge 20}) \\ S &= Z_1^{\mathrm{RT}} + \cdots + Z_N^{\mathrm{RT}} \\ T &= Z_1^{\mathrm{RD}} + \cdots + Z_N^{\mathrm{RD}} \\ \text{Cover} &= \min(200, S + \max(0, T - 200)), \end{aligned}$$

and for Treaty 2:

$$
\begin{aligned}
Z_i^{\mathrm{RT}} &= \min(200, \max(0, Z_i - 800)) \\
Z_i^{\mathrm{RD}} &= \min(200, \max(0, Z_i - 200)) \\
S &= Z_1^{\mathrm{RT}} + \cdots + Z_N^{\mathrm{RT}} \\
T &= Z_1^{\mathrm{RD}} + \cdots + Z_N^{\mathrm{RD}} \\
\text{Cover} &= \max(0, S + T - 400).
\end{aligned}
$$

Though Model 1 yields treaties that can be simplified as above, we will not use these simplifications; rather we use the general formulation in the rest of this paper.

In both treaties, S and T are correlated. We have

- S and T are random sums of non-negative random variables with identical number N of terms.
- The summands, X_i^{RT} and X_i^{RD} are themselves correlated.

This means that even the computation of the pure reinsurance premium $\mathbb{E}[\text{Cover}]$ requires the joint distribution of (S, T). As explained in the introduction, it is possible to obtain this joint distribution by using the multivariate version of the Panjer's algorithm as is explained below.

3 The Multivariate Version of Panjer's Algorithm

Panjer's type algorithms require lattice distributions. Therefore we must first discretize claim amounts. The local one moment matching method (see Gerber, 1982) is a good choice in the sense that it conserves the first moment and is stop-loss conservative, i.e., for any retention, the stop-loss premium calculated with the discretized distribution will be higher than the stop-loss premium calculated with the original distribution. Furthermore, in the case of the limited Pareto distribution ($X \sim \mathrm{Par}(A, B, \alpha)$), it is not difficult to obtain a closed-form of the corresponding lattice distribution. Let us choose a span h and a positive integer m such that $mh = B - A$. It is easy to demonstrate that the probabilities of the lattice version of X, denoted as X_{dis}, with probability function are given by:

$$
f_{X_{dis}}(A + jh) = \frac{2(A + jh)^{1-\alpha} - (A + (j-1)h)^{1-\alpha} - (A + (j+1)h)^{1-\alpha}}{h(1-\alpha)(A^{-\alpha} - B^{-\alpha})}
$$

for $j = 1, \ldots, m-1$, with

$$f_{X_{dis}}(A) = 1 - \frac{\frac{(A+h)^{1-\alpha}}{1-\alpha} - \frac{A^{1-\alpha}}{1-\alpha} - B^{-\alpha}h}{h(A^{-\alpha} - B^{-\alpha})},$$

$$\text{and} \quad f_{X_{dis}}(B) = 1 - f_{X_{dis}}(A) - f_{X_{dis}}(A+h) - \cdots - f_{X_{dis}}(B-h).$$

Now let us turn to the joint distribution of the bivariate random vector

$$(S,T) = (\sum_{i=1}^{N} X_i^{\mathrm{RT}}, \sum_{i=1}^{N} X_i^{\mathrm{RD}}),$$

where $(X_i^{\mathrm{RT}}, X_i^{\mathrm{RD}})$ are independent copies of the lattice random couple $(X^{\mathrm{RT}}, X^{\mathrm{RD}})$. As N is Poisson distributed, Sundt's (1999) multivariate version of the Panjer's algorithm yields

$$\begin{aligned}
f_{S,T}(0,0) &= \Psi_N(f_{X^{\mathrm{RT}},X^{\mathrm{RD}}}(0,0)),\\
f_{S,T}(s,t) &= \sum_{x,y}^{s,t} \frac{\lambda x}{s} f_{S,T}(s-x,t-y) f_{X^{\mathrm{RT}},Y^{\mathrm{RD}}}(x,y) \quad , \quad s \geq 1,\\
f_{S,T}(s,t) &= \sum_{x,y}^{s,t} \frac{\lambda y}{t} f_{S,T}(s-x,t-y) f_{X^{\mathrm{RT}},X^{\mathrm{RD}}}(x,y) \quad , \quad t \geq 1,
\end{aligned}$$

where we use the notation

$$\sum_{x,y}^{s,t} g(x,y) = \sum_{x=0}^{s} \sum_{y=0}^{t} g(x,y) - g(0,0),$$

for any function g and $\Psi_N(u) = \mathbb{E}[u^N] = \exp(\lambda(u-1))$.

4 Some Elements of Stochastic Orderings

In this section, we extensively refer to the seminal paper of Dhaene and Goovaerts (1996) on dependency of risks applied in actuarial science. Some results appear more generally in probability theory, and we will extensively refer to the textbook of Müller and Stoyan (2002).

Stop-Loss Order Stop-loss order allows the actuary to order the risks according to their stop-loss premiums.

Definition 1. *A risk X is said to be smaller in the stop-loss order than a risk Y (written $X \leq_{sl} Y$) whenever one of the following equivalent statements holds true:*

1. $\mathbb{E}[\max(0, X - d)] \leq \mathbb{E}[\max(0, Y - d)]$ *for any nonnegative deductible d; or*
2. $\mathbb{E}[u(X)] \leq \mathbb{E}[v(Y)]$ *for all increasing convex functions u and v, provided these expectations exist.*

The ranking $X \leq_{sl} Y$ implies the stop-loss premiums for X are uniformly smaller than those for Y.

PH-Transform Premium Principle We are interested in calculating premiums with the PH-transform premium principle, introduced by Wang (1996). According to this premium principle, the amount $\Pi_\rho(X)$ charged to cover the risk X is given by

$$\Pi_\rho(X) = \int_0^\infty (1 - F_X(x))^\rho dx,$$

where $0 \leq \rho \leq 1$. In particular when $\rho = 1$, the PH premium reduces to the pure premium. Wang (1996) proved that

$$X \leq_{sl} Y \Rightarrow \Pi_\rho(X) \leq \Pi_\rho(Y) \tag{2}$$

which shows that the PH principle is in accordance with the stop-loss order.

Fréchet Space The concept of Fréchet space emerges when dealing with dependence; it offers the appropriate framework to deal with correlated random variables.

Definition 2. *The bivariate Fréchet space $\mathfrak{R}(F_1, F_2)$ is the class of all bivariate distributions with given marginal cdfs F_1 and F_2.*

For the purpose of this paper, we will consider $\mathfrak{R}(F_1, F_2)$ as a set of random couples.

Correlation Order The correlation order offers a powerful tool to compare the elements of a given Fréchet space.

Definition 3. *If (X_1, X_2) and (Y_1, Y_2) are elements of $\mathfrak{R}(F_1, F_2)$, we say that (X_1, X_2) is less correlated than (Y_1, Y_2), written $(X_1, X_2) \leq_c (Y_1, Y_2)$, if*

$$\mathbb{C}ov(f(X_1), g(X_2)) \leq \mathbb{C}ov(f(Y_1), g(Y_2)),$$

for all non-decreasing functions f and g for which the covariances exist.

The intuitive meaning of a ranking $(X_1, X_2) \leq_c (Y_1, Y_2)$ is that (X_1, X_2) is "less positively dependent" than (Y_1, Y_2).

The correlation order enjoys a number of convenient mathematical properties, some of which are reviewed below. These properties of correlation order are found in Müller and Stoyan (2002):

P1 Let (X_1, X_2) and (Y_1, Y_2) be elements of $\mathfrak{R}(F_1, F_2)$, then the following statements are equivalent:

(i) $(X_1, X_2) \leq_c (Y_1, Y_2)$, and

(ii) $F_{X_1,X_2}(x_1, x_2) \leq F_{Y_1,Y_2}(x_1, x_2) \quad , \quad \forall x_1, x_2 \geq 0.$

P2 Let (U_1, U_2) and (V_1, V_2) be elements of $\mathfrak{R}(F_1, F_2)$, and let (R_1, R_2) be a random vector independent of both (U_1, U_2) and (V_1, V_2). It follows that

$$(U_1, U_2) \leq_c (V_1, V_2) \Rightarrow (U_1 + R_1, U_2 + R_2) \leq_c (V_1 + R_1, V_2 + R_2).$$

P3 Suppose (W, X) and (Y, Z) are elements of $\mathfrak{R}(F_1, F_2)$. Let (W_i, X_i) and (Y_i, Z_i) be independent copies of (W, X) and (Y, Z), respectively, such that $(W, X) \leq_c (Y, Z)$, and let N be a nonnegative counting random variable independent of (W, X) and (Y, Z). It follows that $(S_W, S_X) \leq_c (S_Y, S_Z)$ where $(S_W, S_X) = (\sum_{i=1}^N W_i, \sum_{i=1}^N X_i)$ and $(S_Y, S_Z) = (\sum_{i=1}^N Y_i, \sum_{i=1}^N Z_i)$.

P4 $(W, X) \leq_c (Y, Z)$ implies $(f(W), g(X)) \leq_c (f(Y), g(Z))$ for all increasing functions f and g.

P5 $(W, X) \leq_c (Y, Z)$ implies $W + X \leq_{sl} Y + Z$, i.e., correlation order implies stop-loss order of the sum of the elements.

Using properties P4 and P5, we immediately obtain the following result:

Result 1. *Let (X_1, X_2) and (Y_1, Y_2) be two elements of $\mathfrak{R}(F_1, F_2)$. Then $(X_1, X_2) \leq_c (Y_1, Y_2)$ implies*

$$\max(0, X_1 - a) + \max(0, X_2 - b) \leq_{sl} \max(0, Y_1 - a) + \max(0, Y_2 - b)$$

for all $a, b \geq 0$.

Positive Quadrant Dependence An important concept used in later in this paper is the concept of positive quadrant dependence.

Definition 4. *Let $(X_1, X_2) \in \mathfrak{R}(F_1, F_2)$, and let $(X_1^{\perp}, X_2^{\perp})$ be the independent version of (X_1, X_2), i.e., $(X_1^{\perp}, X_2^{\perp}) \in \mathfrak{R}(F_1, F_2)$ and $X_1^{\perp}$ is independent of $X_2^{\perp}$. Then (X_1, X_2) is said to be positively quadrant dependent if one of the following equivalent statements holds:*

(i) $F_{X_1}(x_1)F_{X_2}(x_2) \leq F_{X_1,X_2}(x_1, x_2), \quad x_1, x_2 \geq 0.$

(ii) $(X_1^{\perp}, X_2^{\perp}) \leq_c (X_1, X_2)$.

(iii) $\mathbb{C}ov(f(X_1), g(X_2)) \geq 0$ *for all non-decreasing functions f and g.*

See, e.g., Dhaene and Goovaerts (1996) for a proof of these equivalences.

The following result will be useful for the applications in reinsurance.

Result 2. *Let X_i and Y_i be independent copies of the non-negative random variables X and Y. Let us assume that X, Y, and N are mutually independent and define*

$$S = X_1 + \cdots + X_N \text{ and } T = Y_1 + \cdots + Y_N.$$

Then (S, T) is positively quadrant dependent.

Proof: f and g are non-decreasing functions. By the decomposition formula of the covariance, we have

$$\begin{aligned}\mathbb{C}ov(f(S), g(T)) &= \mathbb{E}(\mathbb{C}ov(f(S), g(T)|N)) \\ &\quad + \mathbb{C}ov(\mathbb{E}(f(S)|N), \mathbb{E}(g(T)|N)).\end{aligned}$$

The first term of the right part of the equality vanishes because the covariance between independent random variables is 0. For the second term, it is clear that the expectations are increasing functions of N (because the summands in S and T are assumed to be positive) and therefore the second term can be rewritten as $\mathbb{C}ov(u(N), v(N))$, where u and v are non-decreasing functions. This covariance is clearly non-negative, which closes the proof.

Comonotonicity The concept of comonotonicity generalizes perfect correlation. Comonotonic random variables are functionally (and not necessarily linearly) dependent. For a reference in actuarial science, see e.g., Wang and Dhaene (1998).

Definition 5. *Two risks X and Y are said to be comonotonic if*

(i) Their joint cdf satisfies $F_{X,Y}(x,y) = \min(F_X(x), F_Y(y))$ for any $x, y \geq 0$, or, equivalently,

(ii) There exists a random variable Z and non-decreasing functions u and v on $\mathbb{R}$ such that (X,Y) is distributed as $(u(Z), v(Z))$.

By construction, the couples (X_i^{RT}, X_i^{RD}) are comonotonic.

Fréchet's Theorem An interesting result that is related to the concept of comonotonicity is the following theorem, due to Fréchet (1951) and Hoeffding (1940). It gives the extremal elements of any Fréchet space with respect to $\leq_c$.

Theorem 1 (Fréchet). *Let $(X_1, X_2) \in \mathfrak{R}(F_1, F_2)$, then*

$$(F_1^{-1}(U), F_2^{-1}(1-U)) \leq_c (X_1, X_2) \leq_c (F_1^{-1}(U), F_2^{-1}(U))$$

with U uniformly distributed over (0,1), or, equivalently, in terms of distribution functions, the inequalities

$$\max[F_1(x_1)+F_2(x_2)-1; 0] \leq F_{X_1,X_2}(x_1,x_2) \leq \min[F_1(x_1); F_2(x_2)]$$

hold for any $x_1, x_2 \in \mathbb{R}$.

5 Numerical Results

As mentioned in Section 3, all continuous random variables are discretized using the local one moment matching method with a discretization step $h = 10$. Thus, all random variables in this section are the discrete version of the original random variable.

5.1 Treaty 1

Table 2 shows some interesting characteristics of the claims. Note that the Pearson's correlation coefficient between S^{RT} and T^{RD} is estimated at 0.35. The pure premium for this cover is

$$\begin{aligned}\mathbb{E}[\text{Cover}] &= \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty} f_{S,T}(s,t) f_U(u) \min(200, s + \max(0, t + u - 200)) \\ &= 20.519.\end{aligned}$$

Table 2
Means and Variances for Treaty 1

	X^{RD}	X^{RT}	Y^{RD}	S^{RT}	T^{RD}	U^{RD}
Mean	100	16.14	42.87	30	4.84	107.17
Variance	0	1817	631.72	3000	623.4	6173.89

We easily can obtain upper and lower bounds. Let

$$F_{S^{min},T^{min}}(x_1,x_2) = \max[F_S(x_1)+F_T(x_2)-1;0], \quad \text{and}$$
$$F_{S^{max},T^{max}}(x_1,x_2) = \min[F_S(x_1),F_T(x_2)].$$

Using Theorem 1 in connection with P1 we have

$$(S^{min},T^{min}) \leq_c (S,T) \leq_c (S^{max},T^{max}).$$

Using P2, we have

$$(S^{min},T^{min}+U) \leq_c (S,T+U) \leq_c (S^{max},T^{max}+U).$$

Using Result 1, we have

$$\begin{aligned}\mathbb{E}[\max(0;S^{min}&+\max(0;T^{min}+U-200)-200)]\\ &\leq \mathbb{E}[\max(0;S+\max(0;T+U-200)-200)]\\ &\leq \mathbb{E}[\max(0;S^{max}+\max(0;T^{max}+U-200)-200)],\end{aligned}$$

which is equivalent to

$$\begin{aligned}\mathbb{E}[\min(200;S^{max}&+\max(0;T^{max}+U-200))]\\ &\leq \mathbb{E}[\min(200;S+\max(0;T+U-200))]\\ &\leq \mathbb{E}[\min(200;S^{min}+\max(0;T^{min}+U-200))].\end{aligned}$$

The numerical bounds are $19.469 \leq 20.519 \leq 21.279$.

It is possible to improve the upper bound. Let $(X^{\text{RT},\perp},X^{\text{RD},\perp})$ be the independent version of $(X^{\text{RT}},X^{\text{RD}})$. We define

$$S^{\perp} = X_1^{\text{RT},\perp}+\cdots+X_N^{\text{RT},\perp},$$
$$T^{\perp} = X_1^{\text{RD},\perp}+\cdots+X_N^{\text{RD},\perp}.$$

Clearly X^{RT} and X^{RD} are comonotonic random variables. We then have

$$F_{X^{\mathrm{RT},\perp},X^{\mathrm{RD},\perp}}(x_1,x_2) \le \min(F_{X^{\mathrm{RT}}}(x_1),F_{X^{\mathrm{RD}}}(x_2)) = F_{X^{\mathrm{RT}},X^{\mathrm{RD}}}(x_1,x_2), \quad \forall x_1,x_2 \ge 0.$$

Therefore, $(X^{\mathrm{RT},\perp},X^{\mathrm{RD},\perp}) \le_c (X^{\mathrm{RT}},X^{\mathrm{RD}})$. Using P3 we have $(S^{\perp},T^{\perp}) \le_c (S,T)$. $S^{\perp}$ and $T^{\perp}$ are dependent, however, because they involve the same number of summands. Therefore, let us define the independent versions of $(S^{\perp},T^{\perp})$ as $(S^{\perp\perp},T^{\perp\perp})$. Using Result 2 we have $(S^{\perp\perp},T^{\perp\perp}) \le_c (S^{\perp},T^{\perp})$. By transitivity, we then obtain $(S^{\perp\perp},T^{\perp\perp}) \le_c (S,T)$. Using P2 we have $(S^{\perp\perp},T^{\perp\perp}+U) \le_c (S,T+U)$. Using Result 1 we have

$$\mathbb{E}[\max(0;S^{\perp\perp}+\max(0;T^{\perp\perp}+U-200)-200)] \le \mathbb{E}[\max(0;S+\max(0;T+U-200)-200)],$$

which is equivalent to

$$\mathbb{E}[\min(200;S+\max(0;T+U-200))] \le \mathbb{E}[\min(200;S^{\perp\perp}+\max(0;T^{\perp\perp}+U-200))].$$

Numerically, we have $20.519 \le 21.131$. A summary of the results is shown in Table 3.

Table 3

Pure Premiums for Treaty 1

Fréchet Lower Bound	19.469
Exact Result	20.519
Independent Case	21.131
Fréchet Upper Bound	21.279

It is also interesting to analyze other moments of the cover, or premiums obtained by the PH-transform premium principle. They are given in Table 4.

Unfortunately, the other moments, as well as the premiums obtained with the PH-transform premium principle, are not ordered anymore.

5.2 Treaty 2

Some preliminary statistics are displayed in Table 5. The correlation between S^{RT} and T^{RD} is 0.35. The pure premium for our cover is:

Table 4
Moments and PH-Transform
Premium Principle for Treaty 1

	Exact	Independent
$\mathbb{E}[\text{Cover}]$	20.519	21.131
$\mathbb{E}[\text{Cover}^2]$	265.04	261.32
$\mathbb{E}[\text{Cover}^3]$	4124.3	3915.8
$\mathbb{E}[\text{Cover}^4]$	70331.0	64760.0

ρ	Π_ρ (Exact)	Π_ρ (Independent)
1.00	20.519	21.131
0.75	34.898	35.420
0.50	60.786	61.034
0.25	108.71	108.55

$$\mathbb{E}\text{Cover} = \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty} f_{S,T}(s,t) f_U(u) \max(0, s+t+u-400) = 2.252.$$

Upper and lower bounds can be obtained using Theorem 1 in connection with P1 to give

$$(S^{min}, T^{min}) \leq_c (S,T) \leq_c (S^{max}, T^{max}).$$

Using P2, we have

$$(S^{min}, T^{min} + U) \leq_c (S, T+U) \leq_c (S^{max}, T^{max} + U).$$

Using Result 1, we have

Table 5
Means and Variances for Treaty 2

	X^{RD}	X^{RT}	Y^{RD}	S^{RT}	T^{RD}	U^{RD}
Means	200	16.14	1.83	600	4.84	4.57
Variances	0	1817.63	206.31	12000	623.4	524.15

$$
\begin{aligned}
\mathbb{E}[\max(0; S^{min} + T^{min} + U - 400)] \\
&\le \mathbb{E}[\max(0; S + T + U - 400)] \\
&\le \mathbb{E}[\max(0; S^{max} + T^{max} + U - 400)].
\end{aligned}
$$

These numerical bounds are $0.952 \le 2.252 \le 5.471$.

Though we are now able to improve the lower bound, the process is, unfortunately, less interesting than in the previous example. $X^{\mathrm{RT},\perp}$ and $X^{\mathrm{RD},\perp}$ are the independent versions of X^{RT} and X^{RD}; that is $X^{\mathrm{RT},\perp}$ and $X^{\mathrm{RD},\perp}$ are independent and have the same distributions as X^{RT} and X^{RD}, respectively. Let us define

$$
\begin{aligned}
S^{\perp} &= X_1^{\mathrm{RT},\perp} + \cdots + X_N^{\mathrm{RT},\perp}, \\
T^{\perp} &= X_1^{\mathrm{RD},\perp} + \cdots + X_N^{\mathrm{RD},\perp}.
\end{aligned}
$$

It is clear, from their construction, that X^{RT} and X^{RD} are comonotonic random variables. We then have

$$
\begin{aligned}
F_{X^{\mathrm{RT},\perp}, X^{\mathrm{RD},\perp}}(x_1, x_2) &\le \min(F_{X^{\mathrm{RT}}}(x_1), F_{X^{\mathrm{RD}}}(x_2)) \\
&= F_{X^{\mathrm{RT}}, X^{\mathrm{RD}}}(x_1, x_2)
\end{aligned}
$$

for any $x_1, x_2 \ge 0$. Therefore,

$$
(X^{\mathrm{RT},\perp}, X^{\mathrm{RD},\perp}) \le_c (X^{\mathrm{RT}}, X^{\mathrm{RD}}).
$$

Using P3, we have $(S^{\perp}, T^{\perp}) \le_c (S, T)$. As $S^{\perp}$ and $T^{\perp}$ are dependent, however, there is little interest in working with this random vector.

We define the independent versions of $S^{\perp}$ and $T^{\perp}$ as $S^{\perp\perp}$ and $T^{\perp\perp}$, respectively. Using Result 2, $(S^{\perp\perp}, T^{\perp\perp}) \le_c (S^{\perp}, T^{\perp})$. By transitivity, we obtain $(S^{\perp\perp}, T^{\perp\perp}) \le_c (S, T)$. Using P2, we have $(S^{\perp\perp}, T^{\perp\perp} + U) \le_c (S, T + U)$. Using Result 1, we have

$$
\mathbb{E}[\max(0; S^{\perp\perp} + T^{\perp\perp} + U - 400)] \le \mathbb{E}[\max(0; S + T + U - 400)].
$$

Numerically, we have $1.153 \le 2.252$. These results are summarized in Table 6. Contrary to Treaty 1, we are now able to compare other moments and premiums obtained by the PH-transform premium principle. Numerical results are summarized in Table 7. In this case the characterization of the stop-loss order and equation (2) are directly applicable because we have $\mathrm{Cover}^{\perp} \le_{sl} \mathrm{Cover}$.

Table 6
Treaty 2: Pure Premium

Fréchet Lower Bound	0.952
Independent Case	1.153
Exact Result	2.252
Fréchet Upper Bound	5.471

Table 7
Moments and PH-Transform Premium Principle for Treaty 2

	Exact	Independent
$\mathbb{E}[\text{Cover}]$	2.252	1.152
$\mathbb{E}[\text{Cover}^2]$	486.9	242.3
$\mathbb{E}[\text{Cover}^3]$	140198.0	61874.0
$\mathbb{E}[\text{Cover}^4]$	51084848.0	19062223.0

ρ	Π_ρ (Exact)	Π_ρ (Independent)
1.00	2.252	1.152
0.75	7.815	4.611
0.50	30.22	20.60
0.25	170.04	143.41

5.3 Treaty 2bis

Typically, a reinsurer will not offer an unlimited cover, at least for property business. Therefore, the cover of Treaty 2 should be limited in practice and could read

$$\text{Cover} = \min(400, \max(0, S + T + U - 400)),$$

which we call Treaty 2bis.

Pricing this realistic cover thus requires exact computations because of our inability to show that the derived bounds remain valid. We obtain the following results in Table 8. For this example, the bounds remain valid. This result is probably due to the very low probability of exhausting the cover, a fact that is confirmed by observing that the pure premium is the same (at least with three decimal digits) in both cases.

Table 8
Pure premiums for Treaty 2bis

Fréchet Lower Bound	0.952
Independent Case	1.153
Exact Result	2.252
Fréchet Upper Bound	5.471

6 Conclusion

By assuming the reinsurance cover is a function of a comonotonic random vector, we have shown how it is possible to obtain bounds for the pure premium. In particular, we observed that for Treaty 1, the wrong hypothesis of independence provides an upper bound for the pure premium of the treaty. This happens when the cover of the treaty is limited and when the comonotonic random variables are expressed as an excess of the same underlying random variable. Unfortunately, we have found in one case that the other moments of the cover are no longer ordered, i.e., even if we can prove that the first moment under the wrong hypothesis of independence is larger than the first moment under the exact hypothesis of independence, this property is not true for higher moments. In addition, we do not have a theoretical result on these orders.

In a second example we show that the wrong hypothesis of independence was not conservative, which shows the following consequence of not working with the exact model when it is known: if you work with the wrong model, you compute wrong premiums, which are too low when compared to exact premiums. Furthermore, the upper and lower Fréchet bounds may be quite far from the exact result, as shown in Treaty 2.

The theoretical results derived in this paper were based on a two dimensional paradigm. However these results can be extended to higher dimensions by using the supermodular ordering. Further research is being pursued in this area.

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