2005

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Journal of Actuarial Practice Vol. 12, 2005

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Abstract§

We consider a discrete-time risk model with m ($m \geq 2$) dependent classes of insurance business. The claim processes of these m classes are assumed to follow a multivariate autoregressive time-series model of order 1. Given this claims model, we explore the probability of ultimate ruin assuming exponentially bounded claims. As an example, we use simulations to study the case where there are two business and the underlying losses are of two types: bivariate exponential and bivariate gamma claim distributions.

Key words and phrases: *adjustment coefficient, bivariate exponential distribution, bivariate gamma distribution, discrete-time risk model, multivariate autoregressive model, time series, ultimate ruin probability*

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§The authors are grateful to the referees for their helpful comments and suggestions. This research was supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. HKU 7054/04P). The third author would like to thank the Croucher Foundation for the award of a senior research fellowship.

1 Introduction

For a book of insurance business, it is often assumed that different classes of policies are independent. This assumption, however, may not be justified in many practical situations. For example, a severe car accident may trigger auto-insurance claims as well as medical insurance claims. In recent years, risk models with various dependence structures have been studied by many researchers; for example, Dhaene and Goovaerts (1997), Ambagaspitiya (1998, 1999), Nyrhinen (1998), Wang (1998), Asmussen et al. (1999), Cossette and Marceau (2000), Müller and Pflug (2001), Albrecher and Kantor (2002), Goovaerts and Kaas (2002), Yuen et al. (2002), Picard et al. (2003), and Wu and Yuen (2003).

Actuaries have considered the time-series method as a possible tool to model risk processes. For example, Gerber (1982) investigated the ruin probability by considering the annual gains which form a linear time series. Extensions of his result can be found in Promislow (1991) and Ramsay (1991). Yang and Zhang (2003) studied a risk model with constant interest in which the claim process and the premium process are described by an autoregressive model.

In this paper, we propose a discrete-time risk model with m dependent classes of policies using a time-series approach. Our objective is to investigate the ultimate ruin probability for this model. Specifically, the claim processes of the m classes are described by a multivariate autoregressive model of order 1 (MAR(1)). The MAR(1) model assumes that for each of the m classes, the total claim in a certain period depends not only on the claims occurring in that period, but also on the total claim of its own class and that of other classes in the previous period. Correlation among the claim amounts of the m classes in each period also may be assumed.

Note that Picard et al. (2003) considered a discrete-time model with several interdependent risks in which the claim amounts during successive periods are independent and identically distributed random variables.

The MAR(1) risk model and some basic assumptions are introduced in Section 2. In Section 3, the ultimate ruin probability for the proposed model and its upper bound are investigated. Finally, simulated results in the bivariate case are given in Section 4 to reveal the impact of dependence structure on the ruin probabilities. Some closing comments are given in Section 5.

2 The Model

We now introduce a discrete-time risk model of an insurance portfolio consisting of m dependent classes of insurance policies, where these classes are labeled 1, 2, ..., m. The following assumptions are made:

- Policies are open ended, i.e., they remain in force for an unlimited length of time.
- Each class of policies has its own premiums and claims.
- Premiums are paid at the start of each time period (a period may be a year, quarter, month, etc.) and remain constant throughout the life of the policy.
- The total premium paid in the i^{th} period for the policies in class j is π_j, for $j = 1, 2, \cdots, m$.
- X_{ji} is the total amount of claims **incurred** by the class j policies in the i^{th} period (we only consider exponentially bounded claims).
- We assume that the events causing X_{ji} will cause further claims in the future periods not only in the j^{th} class but also in other classes.
- W_{ji} is the total amount of claims **paid** on behalf of the class j policies in the i^{th} period. It consists of X_{ji} and a linear combination of all the previous claims in all classes (i.e., a linear combination of all X_{hk}s for $h = 1, 2, \cdots, m$ and $k = 1, 2, \cdots, i-1$), and is defined in equation (1).
- If $\mathbf{X}_i = (X_{1i}, X_{2i}, \ldots, X_{mi})'$ denotes the column vector of the m total incurred claims in period i, we assume that $\{\mathbf{X}_1, \mathbf{X}_2, \cdots\}$ is a sequence of independent and identically distributed non-negative random vectors having finite mean and covariance matrix. And, finally
- If $\mathbf{W}_i = (W_{1i}, W_{2i}, \ldots, W_{mi})'$ denotes the column vector of the m total paid claims in period i, we assume that $\{\mathbf{W}_1, \mathbf{W}_2, \ldots\}$ is a sequence of dependent vectors such that they follow a MAR(1) process, i.e., $\mathbf{W}_i$ is given by

$$\mathbf{W}_i = \mathbf{A}\mathbf{W}_{i-1} + \mathbf{X}_i, \tag{1}$$

where $\mathbf{A}$ is a non-negative constant $m \times m$ matrix. Hence, the components of $\mathbf{W}_i$ are correlated.

The model defined by equation (1) may be useful in describing the dependence of several classes of insurance business in some real situations. For example, a natural disaster or a serious fire accident often causes various types of claims, and some of these claims such as the medical and disability ones may last for many periods of time. The suitability of the MAR(1) model for practical purposes is limited, however, partly because the inherent dependence structure affects the marginal distributions and thus a separate statistical estimation of marginals and the degree of dependence is not possible from the given data.

Let U_n denote the aggregate surplus process of the insurance portfolio at the end of the n^{th} period. As usual, we define the surplus process of class j as

$$U_{jn} = u_j + n\pi_j - \sum_{i=1}^{n} W_{ji},$$

for $n = 1, 2, \ldots$, where u_j is the initial surplus of class j. Thus,

$$U_n = \sum_{j=1}^{m} U_{jn} = u + n\pi - \sum_{i=1}^{n}\sum_{j=1}^{m} W_{ji}, \tag{2}$$

where u and π are the portfolio's aggregate initial reserve and periodic premiums, respectively, i.e.,

$$u = \sum_{j=1}^{m} u_j \quad \text{and} \quad \pi = \sum_{j=1}^{m} \pi_j. \tag{3}$$

For notational convenience, we write $\sum_{j=1}^{m} W_{ji} = \mathbf{1}'_m \mathbf{W}_i$ where $\mathbf{1}_m$ is an m-dimensional column vector of 1.

It is important for the model to be stationary with finite second-order moments. To fulfill this second-order stationarity condition, the eigenvalues of $\mathbf{A}$ must be smaller than 1 in absolute value (see Reinsel 1993). Specifically, all the roots of the characteristic equation of $\mathbf{A}$ (as a function of λ) must be smaller than 1 in absolute value:

$$h(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = 0, \tag{4}$$

where $\mathbf{I}$ is an $m \times m$ identity matrix.

Put $\mathbf{X} = \mathbf{X}_1$. Given initial value $\mathbf{W}_0 = \mathbf{w}$, we see from (1)

$$\mathbb{E}(\mathbf{W}_i) = \frac{\mathbf{I} - \mathbf{A}^i}{\mathbf{I} - \mathbf{A}}\mathbb{E}(\mathbf{X}) + \mathbf{A}^i\mathbf{w},$$

which depends on i. Hence, the MAR(1) process is locally non-stationary. As $i \to \infty$, however, its asymptotic mean becomes

$$\mathbb{E}(\mathbf{W}) = \lim_{i \to \infty} \mathbb{E}(\mathbf{W}_i) = (\mathbf{I} - \mathbf{A})^{-1}\mathbb{E}(\mathbf{X}),$$

which is independent of i. Besides, the covariance of W_{ji} and $W_{j,i+l}$ only depends on lag l, where $l = -i, -i+1, \ldots, n-i$, but not on i. Thus, the MAR(1) process is asymptotically stationary. In this paper, the term stationarity is generally used to mean asymptotic stationarity.

The net-profit condition requires that the aggregate premium should be greater than the expected value of the claims in each time period, that is,

$$\pi > \mathbf{1}_m' \left(\frac{\mathbf{I} - \mathbf{A}^i}{\mathbf{I} - \mathbf{A}} \mathbb{E}(\mathbf{X}) + \mathbf{A}^i \mathbf{w} \right),$$

for all i. Here, we assume in the sense of stationarity that

$$\pi > \mathbf{1}_m' \left((\mathbf{I} - \mathbf{A})^{-1} \mathbb{E}(\mathbf{X}) \right), \tag{5}$$

which is a necessary condition for deriving Theorem 1 given below.

3 The Probability of Ultimate Ruin

Let the time of ruin T be the smallest time at which equation (2) becomes negative, i.e.,

$$T = \min\{n : U_n < 0 | U_0 = u\}.$$

Then, the probability of ultimate ruin given the initial surplus u, the aggregate premium per period π, and the initial claim $\mathbf{W}_0 = \mathbf{w}$ is given by

$$\psi(u, \pi, \mathbf{w}) = \Pr(T < \infty | U_0 = u, \pi, \mathbf{W}_0 = \mathbf{w}). \tag{6}$$

In order to prove the main result of the paper, we need to make use of the following modified surplus process. Define

$$\epsilon_i = (\mathbf{1}_m + \alpha)' \mathbf{X}_i,$$

where

$$\begin{aligned} \alpha' &= \mathbf{1}_m' \mathbf{A} (\mathbf{I} - \mathbf{A})^{-1} \\ &= (\alpha_1, \alpha_2, \cdots, \alpha_m). \end{aligned}$$

It is obvious that $\{\epsilon_1, \epsilon_2, \cdots\}$ is a sequence of independent and identically distributed random variables with finite mean and variance. The modified surplus process $\hat{U}_n$ is then defined as

$$\begin{aligned}\hat{U}_n &= U_n - \boldsymbol{\alpha}'\mathbf{W}_n\\ &= U_{n-1} + \pi - (\mathbf{1}_m + \boldsymbol{\alpha})'\mathbf{W}_n\\ &= U_{n-1} + \pi - (\mathbf{1}_m + \boldsymbol{\alpha})'(\mathbf{A}\mathbf{W}_{n-1} + \mathbf{X}_n)\\ &= U_{n-1} + \pi - (\mathbf{1}_m + \boldsymbol{\alpha})'\mathbf{A}\mathbf{W}_{n-1} - (\mathbf{1}_m + \boldsymbol{\alpha})'\mathbf{X}_n,\end{aligned}$$

with $\hat{U}_0 = \hat{u} = u - \boldsymbol{\alpha}'\mathbf{w}$. By the definition of $\boldsymbol{\alpha}$, we have

$$\boldsymbol{\alpha}'\mathbf{I} = (\mathbf{1}_m + \boldsymbol{\alpha})'\mathbf{A}.$$

This together with the definition of ϵ_n allow us to rewrite the modified surplus as

$$\begin{aligned}\hat{U}_n &= U_{n-1} + \pi - \boldsymbol{\alpha}'\mathbf{W}_{n-1} - \epsilon_n\\ &= \hat{U}_{n-1} + \pi - \epsilon_n.\end{aligned} \tag{7}$$

It can be shown that the condition (5) is equivalent to

$$\pi > \mathbb{E}(\epsilon_1). \tag{8}$$

The total premium per period can be expressed as $\pi = (1+\eta)\mathbb{E}(\epsilon_1)$ where $\eta > 0$ is the relative security loading for the modified surplus process. It is intuitively clear from (7) that ruin is certain if η is negative. We now define the adjustment coefficient R as the smallest positive solution of

$$\mathbb{E}[e^{-R(\pi-\epsilon_1)}] = 1.$$

The adjustment coefficient is assumed to exist for all models considered in this paper.

Theorem 1. *For* $u \geq 0$,

$$\psi(u, \pi, \mathbf{w}) = \frac{e^{-R\hat{u}}}{\mathbb{E}[e^{-R\hat{U}_T} \mid T < \infty]}. \tag{9}$$

To prove Theorem 1, one can make use of equations (7) and (8), and then follow the proof of the one-dimensional case given in Bowers et al. (1997). It should be pointed out that Theorem 1 only holds for exponentially bounded claims. The following corollary is easily established:

Corollary 1. *Given that equation (9) holds, we have*

$$\psi(u, \pi, \mathrm{w}) \le \psi^{UB}(u, \pi, \mathrm{w}) = e^{-R\hat{u}}.$$

Proof: As all the α_is of $\boldsymbol{\alpha}$ are non-negative, we have $\hat{U}_T \le U_T < 0$. Therefore, the denominator on the right hand side of (9) is greater than one. This gives us an upper bound $\psi^{UB}(u, \pi, \mathrm{w})$ for the ultimate ruin probability.

4 Simulation Studies: Models and Results

Simulations are used to study the effect of the time-series modeling and the correlation between the current claim amounts on the ruin probabilities in the bivariate case.

4.1 The Models Used

Four discrete-time risk models are used. For notational convenience, we set $W_i = W_{1i}$, $Z_i = W_{2i}$, $X_i = X_{1i}$, and $Y_i = X_{2i}$.

Model 1:

$$\begin{pmatrix} W_i \\ Z_i \end{pmatrix} = \begin{pmatrix} X_i \\ Y_i \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} W_{i-1} \\ Z_{i-1} \end{pmatrix},$$

where a, b, c, and d are non-zero constants, and (X_i, Y_i) follows a bivariate distribution. In this model, the correlation between W_i and Z_i comes from the AR(1) coefficients as well as the correlation of X_i and Y_i.

Model 2:

$$\begin{pmatrix} W_i \\ Z_i \end{pmatrix} = \begin{pmatrix} X_i \\ Y_i \end{pmatrix} + \begin{pmatrix} a' & 0 \\ 0 & d' \end{pmatrix} \begin{pmatrix} W_{i-1} \\ Z_{i-1} \end{pmatrix},$$

where a' and d' are non-zero constants, and (X_i, Y_i) comes from a bivariate distribution. The correlation between W_i and Z_i is solely due to the correlation of X_i and Y_i.

Model 3:

$$\begin{pmatrix} W_i \\ Z_i \end{pmatrix} = \begin{pmatrix} X_i \\ Y_i \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} W_{i-1} \\ Z_{i-1} \end{pmatrix},$$

where a, b, c, and d are non-zero constants, and X_i and Y_i are independent. The correlation between W_i and Z_i comes solely from the AR(1) coefficients.

Model 4:

$$\begin{pmatrix} W_i \\ Z_i \end{pmatrix} = \begin{pmatrix} X_i \\ Y_i \end{pmatrix} + \begin{pmatrix} a' & 0 \\ 0 & d' \end{pmatrix} \begin{pmatrix} W_{i-1} \\ Z_{i-1} \end{pmatrix},$$

where a' and d' are non-zero constants, and X_i and Y_i are independent. In this model, W_i and Z_i are independent.

In order to obtain a consistent comparison across models, X_i and Y_i are set to have equal mean in each of the four models. In order to do a fair comparison, the parameters a, b, c, d, a', and d' are chosen in the way that the asymptotic means of W and Z, $\mathbb{E}[W]$ and $\mathbb{E}[Z]$, in the four models are equal. Thus, we set

$$a' = 1 - \frac{\mathbb{E}(X)}{\mathbb{E}(W)}, \quad \text{and} \quad d' = 1 - \frac{\mathbb{E}(Y)}{\mathbb{E}(Z)}.$$

In our simulation studies, we consider two bivariate distributions for the two types of claims in Models 1 and 2. One is the bivariate exponential distribution while the other is the bivariate gamma distribution. Hence, the claim amounts of the two classes in Models 3 and 4 are generated from the corresponding marginal distributions.

4.2 Bivariate Exponential Distribution

4.2.1 An Overview

Block and Basu (1974) introduced the so-called absolutely continuous bivariate exponential distribution which possesses the loss of memory property. Here, we simply called it the bivariate exponential distribution. Assume that the claim amounts (X, Y) follow the bivariate exponential distribution. With parameters λ_1, λ_2, $\lambda_{12} > 0$, and $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$, the joint distribution function of (X, Y) is defined as

$$\begin{aligned} F(x,y) &= \frac{\lambda}{\lambda_1+\lambda_2} \exp\left(-\lambda_1 x - \lambda_2 y - \lambda_{12}\max(x,y)\right) \\ &\quad - \frac{\lambda_{12}}{\lambda_1+\lambda_2} \exp\left(-\lambda \max(x,y)\right), \end{aligned}$$

for $x, y > 0$. Note that λ_{12} is the key parameter determining the correlation between X and Y and that X and Y are independent when $\lambda_{12} = 0$. Some of the statistical properties derived by Block and Basu (1974) (with minor corrections) are as follows:

$$
\begin{aligned}
\mathbb{E}(X) &= \frac{1}{\lambda_1 + \lambda_{12}} + \frac{\lambda_{12}\lambda_2}{\lambda(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_{12})},\\
\mathbb{E}(Y) &= \frac{1}{\lambda_2 + \lambda_{12}} + \frac{\lambda_{12}\lambda_1}{\lambda(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_{12})},\\
\mathbb{V}\mathrm{ar}(X) &= \frac{1}{(\lambda_1 + \lambda_{12})^2} + \frac{\lambda_{12}\lambda_2(2\lambda_1\lambda + \lambda_{12}\lambda_2)}{\lambda^2(\lambda_1 + \lambda_2)^2(\lambda_1 + \lambda_{12})^2},\\
\mathbb{V}\mathrm{ar}(Y) &= \frac{1}{(\lambda_2 + \lambda_{12})^2} + \frac{\lambda_{12}\lambda_1(2\lambda_2\lambda + \lambda_{12}\lambda_1)}{\lambda^2(\lambda_1 + \lambda_2)^2(\lambda_2 + \lambda_{12})^2},\\
\mathbb{C}\mathrm{ov}(X,Y) &= \frac{(\lambda_1^2 + \lambda_2^2)\lambda_{12}\lambda + \lambda_1\lambda_2\lambda_{12}^2}{\lambda^2(\lambda_1 + \lambda_2)^2(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})},\\
\rho(X,Y) &= \lambda_{12}\Big((\lambda_1^2 + \lambda_2^2)\lambda + \lambda_1\lambda_2\lambda_{12}\Big)\\
&\quad \times \sqrt{\Big((\lambda_1 + \lambda_2)^2(\lambda_1 + \lambda_{12})^2 + \lambda_2(\lambda_2 + 2\lambda_1)\lambda^2\Big)}\\
&\quad \times \sqrt{\Big((\lambda_1 + \lambda_2)^2(\lambda_2 + \lambda_{12})^2 + \lambda_1(\lambda_1 + 2\lambda_2)\lambda^2\Big)},
\end{aligned}
$$

where $\rho(X,Y)$ is the correlation coefficient of X and Y. It is easy to see that X and Y are positively correlated. Block and Basu (1974) also derived some useful properties of the bivariate exponential distribution which allow us to generate (X,Y) easily. The properties include:

1. $\min(X,Y)$ follows exponential distribution with mean λ.
2. The difference $G = X - Y$ has distribution function

$$
F(g) = \begin{cases} \frac{\lambda_1}{\lambda_1+\lambda_2} \exp\left((\lambda_2 + \lambda_{12})g\right), & g \le 0,\\ 1 - \frac{\lambda_2}{\lambda_1+\lambda_2} \exp\left(-(\lambda_1 + \lambda_{12})g\right), & g > 0. \end{cases}
$$

3. $\min(X,Y)$ is independent of G.

Based on these properties, one can generate random variables (X,Y) using the following steps:

Step 1: Generate random variables R_1 and R_2 following uniform $(0,1)$ distribution.

Step 2: If $R_2 < \lambda_1/(\lambda_1 + \lambda_2)$, then $G \le 0$ and

$$
G = \frac{1}{\lambda_2 + \lambda_{12}} \ln\Big(\frac{\lambda_1 + \lambda_2}{\lambda_1} R_2\Big),
$$

else go to Step 5.

Step 3: $X = \frac{-\ln(1-R_1)}{\lambda}$ and $Y = X - G$.

Step 4: Go to Step 1 for a new set of X and Y.

Step 5: $G > 0$ and

$$G = \frac{-1}{\lambda_1 + \lambda_{12}} \ln\Big(\frac{\lambda_1 + \lambda_2}{\lambda_2}(1 - R_2)\Big).$$

Step 6: $Y = (-\ln(1 - R_1))\,/\lambda$ and $X = G + Y$.

Step 7: Go to Step 1 for a new set of X and Y.

4.2.2 Simulation Results

In our simulation studies, we arbitrarily select $a = 0.4$, $b = 0.2$, $c = 0.2$, and $d = 0.4$ for Models 1 and 3. With these parameter values, the solutions of (4) with $m = 2$ are 0.2 and 0.6. For Models 2 and 4, we set $a' = 0.6$ and $d' = 0.6$. Then, both roots of (4) equal 0.6. Therefore, the stationarity condition is satisfied in each of the four models. The parameters of the bivariate exponential distribution are chosen to be $\lambda_1 = \lambda_2 = 0.070466$ and $\lambda_{12} = 0.38486$. Hence, $\mathbb{E}(X) = \mathbb{E}(Y) = 3$ and the asymptotic means of W and Z are $\mathbb{E}(W) = \mathbb{E}(Z) = 7.5$. The correlation coefficient of X and Y is $\rho(X,Y) = 0.3333$. The asymptotic variances $\mathbb{V}\text{ar}(W)$ and $\mathbb{V}\text{ar}(Z)$, the asymptotic covariance $\mathbb{C}\text{ov}(W,Z)$, and the asymptotic correlation coefficient $\rho(W,Z)$ can be calculated using the standard method for a typical stationary MAR(1) model. Further details about the calculation of these values can be found in Reinsel (1993). Their numerical values are summarized in Table 1 and can serve as indicators for the variances, covariances, and correlation coefficients of W_i and Z_i.

Table 1
Asymptotic Variances, Covariances and Correlation Coefficients for Bivariate Exponential distribution

Model	$\mathbb{V}\text{ar}(W)$	$\mathbb{V}\text{ar}(Z)$	$\mathbb{C}\text{ov}(W,Z)$	$\rho(W,Z)$
1	10.0482	10.0482	5.0238	0.5000
2	11.3043	11.3043	3.7677	0.3333
3	9.4203	9.4203	1.8841	0.2000
4	11.3043	11.3043	0.0000	0.0000

The relative security loading η is set to be 0.05, so the constant total premium per period is $\pi = 15.75$. The initial values are $u = 10$ and $w = z = 0$. The number of simulations used for computing the results is 10,000, and the sample size is 100. We first study the finite-time ruin probability which is defined as

$$\psi_N(u, \pi, w, z) = \Pr(T \leq N | U_0 = u, \pi, W_0 = w, Z_0 = z).$$

The results are shown in Table 2 with standard errors in parentheses. It is observed that as N increases, the finite-time ruin probabilities for the four models increase. As $N \to \infty$, with other parameters fixed, the values of $\psi_N(u, \pi, w, z)$ approach the ultimate ruin probability $\psi(u, \pi, w, z)$.

We also compare the results across the four models with the same length of period. The values of the finite-time ruin probability for Model 1 are greater than those for Model 2 simply because Model 1 has a higher degree of dependence which leads to a higher asymptotic correlation coefficient $\rho(W, Z)$. With the same argument, the finite-time ruin probabilities for Model 3 are greater than those for Model 4. Moreover, the finite-time ruin probabilities for Model 1 and Model 2 are higher than those for Model 3 and Model 4, respectively, because the correlation between X and Y introduces additional dependence in the former two models.

From Table 2, we see that the values of $\psi_N(u, \pi, w, z)$ for $N = 1,800$ and $N = 2,000$ are very close. Therefore, the value of $\psi_N(u, \pi, w, z)$ with $N = 1,800$ can be treated as a good approximation of the ultimate ruin probability in the following numerical studies.

Simulation studies are further carried out to investigate how the ultimate ruin probability is affected by the value of the initial surplus u. For $\pi = 15.75$ and $w = z = 0$, the ultimate ruin probabilities for various values of u are summarized in Table 3. With a larger initial surplus, the approximated values of the ultimate ruin probability become smaller. It is also noticed that as the value of u increases, the standard error for the estimated upper bound also increases. This can be easily explained by the form of the upper bound, that is, $\psi^{UB}(u, \pi, w, z) = \exp(-Ru)$. In words, a small deviation of the simulated R from the mean has a relatively much larger effect on the upper bound with a large value of u. The relation between the relative security loading and the ultimate ruin probability also is examined. Table 4 summarizes the results with $u = 10$ and $w = z = 0$. As η increases, the ultimate ruin probabilities decrease very quickly.

Finally, in view of Theorem 1, we discuss the empirical behavior of the adjustment coefficient R as a function of the model parameters. From Tables 3 and 4, we see that R increases as u or π increases. In general, for a given set of (u, π), ψ^{UB} decreases steadily as the degree of dependence decreases from Model 1 to Model 4, and hence R is not too sensitive to the model change (that is, the change in the degree of dependence). If we fix a model and change the correlation between X and Y, empirical evidence also shows that R decreases in a rather uniform manner as the correlation between X and Y increases.

4.3 Bivariate Gamma Distribution

4.3.1 An Overview

Johnson and Kotz (1972) constructed a multivariate gamma distribution from independent random variables $H_0, H_1, \ldots, H_m$ where H_j follows standard gamma distributions with parameters θ_j ($\theta_j > 0$) for $j = 1, 2, \cdots, m$. Here, we only consider the bivariate case. Let $X = H_0 + H_1$ and $Y = H_0 + H_2$. Then, the claim amounts (X, Y) have a bivariate gamma distribution with joint density

$$f(x,y) = \frac{e^{-(x+y)}}{\Gamma(\theta_0)\Gamma(\theta_1)\Gamma(\theta_2)} \int_0^{\min(x,y)} z^{\theta_0-1}(x-z)^{\theta_1-1}(y-z)^{\theta_2-1}e^{z}dz,$$

with $\mathbb{E}(X) = \mathbb{V}\mathrm{ar}(X) = \theta_0 + \theta_1$, $\mathbb{E}(Y) = \mathbb{V}\mathrm{ar}(Y) = \theta_0 + \theta_2$, $\mathbb{C}\mathrm{ov}(X, Y) = \theta_0$, and

$$\rho(X,Y) = \frac{\theta_0}{\sqrt{(\theta_0+\theta_1)(\theta_0+\theta_2)}}.$$

It is clear that X and Y are positively correlated. Hence, the bivariate gamma random variables (X, Y) can be generated using the following steps:

Step 1: Generate H_0, H_1, and H_2 from standard gamma distributions with means θ_0, θ_1, and θ_2, respectively.

Step 2: $X = H_0 + H_1$ and $Y = H_0 + H_2$.

Step 3: Go to Step 1 for a new set of X and Y.

4.3.2 Simulation Results

Similar to the simulation studies in Section 4.2, we set $a = 0.4$, $b = 0.2$, $c = 0.2$, and $d = 0.4$ for Models 1 and 3 and $a' = d' = 0.6$ for Models 2 and 4. These parameter values imply that all the four models satisfy the stationarity condition. The parameters of the bivariate gamma distribution are arbitrarily selected as $\theta_0 = 1$, $\theta_1 = 2$, and $\theta_2 = 2$ so that the means of X and Y and the asymptotic means of W and Z are the same as those in Section 4.2, that is, $\mathbb{E}(X) = \mathbb{E}(Y) = 3$ and $\mathbb{E}(W) = \mathbb{E}(Z) = 7.5$. The correlation coefficient of X and Y is also 0.3333. The asymptotic variances $\mathbb{V}\text{ar}(W)$ and $\mathbb{V}\text{ar}(Z)$, the asymptotic covariance $\mathbb{C}\text{ov}(W,Z)$, and the asymptotic correlation coefficient $\rho(W,Z)$ are shown in Table 5.

Again, we let $\eta = 0.05$, $\pi = 15.75$, $u = 10$, and $w = z = 0$. The number of simulations and the sample size are also 10,000 and 100, respectively. Table 6 presents the finite-time ruin probability $\psi_N(u,\pi,w,z)$ for various values of N. As expected, the observations made in Section 4.2 from Table 2 also hold in this case. The values in Table 2, however, are generally higher than those in Table 6. It is mainly due to the fact that the asymptotic variances $\mathbb{V}\text{ar}(W)$ and $\mathbb{V}\text{ar}(Z)$ are larger in Section 4.2 (although the asymptotic means and the asymptotic correlation coefficients are the same in both sections).

As shown in Table 6, the finite-time ruin probabilities with $N = 1{,}000$ and $N = 1{,}500$ are the same. Therefore, we use $N = 1{,}000$ to obtain approximations of the ultimate ruin probabilities. Table 7 displays the ultimate ruin probabilities for $u = 5, 10, 15, 20, 25, 30, 50$ with $\pi = 15.75$ and $w = z = 0$ while Table 8 shows the ultimate ruin probabilities for eight values of η with $u = 10$ and $w = z = 0$. Not surprisingly, the patterns of Tables 7 and 8 are more or less parallel to those of Tables 3 and 4, respectively. Also, the empirical behavior of R in this case is similar to that in Section 4.2.

Table 2
Finite-Time Ruin Probabilities $\psi_N(u, \pi, w, z)$ and Upper Bounds for Ultimate Ruin Probabilities $\psi^{UB}(u, \pi, w, z)$ with $u = 10$, $\pi = 15.75$ and $w = z = 0$ for Bivariate Exponential Distribution

	Model 1	Model 2	Model 3	Model 4
N	ψ_N	ψ_N	ψ_N	ψ_N
50	0.3328	0.3187	0.2610	0.2394
	(0.0108)	(0.0146)	(0.0098)	(0.0098)
100	0.4148	0.4007	0.3416	0.3217
	(0.0098)	(0.0143)	(0.0080)	(0.0091)
150	0.4518	0.4362	0.3780	0.3592
	(0.0087)	(0.0138)	(0.0069)	(0.0086)
200	0.4708	0.4558	0.3982	0.3794
	(0.0081)	(0.0135)	(0.0066)	(0.0081)
500	0.5046	0.4918	0.4347	0.4154
	(0.0075)	(0.0128)	(0.0060)	(0.0075)
800	0.5105	0.4982	0.4407	0.4213
	(0.0073)	(0.0127)	(0.0060)	(0.0074)
1,000	0.5118	0.4997	0.4420	0.4226
	(0.0073)	(0.0126)	(0.0060)	(0.0074)
1,200	0.5123	0.5003	0.4426	0.4231
	(0.0073)	(0.0126)	(0.0059)	(0.0074)
1,500	0.5134	0.5011	0.4435	0.4241
	(0.0072)	(0.0125)	(0.0059)	(0.0074)
1,600	0.5136	0.5012	0.4436	0.4242
	(0.0072)	(0.0125)	(0.0059)	(0.0074)
1,800	0.5137	0.5013	0.4437	0.4244
	(0.0072)	(0.0125)	(0.0059)	(0.0074)
2,000	0.5137	0.5014	0.4438	0.4244
	(0.0072)	(0.0125)	(0.0059)	(0.0074)
	ψ^{UB}	ψ^{UB}	ψ^{UB}	ψ^{UB}
	0.8914	0.8833	0.8750	0.8717
	(0.0620)	(0.0723)	(0.0672)	(0.0653)

Table 3
Ultimate Ruin Probabilities $\psi(u, \pi, w, z)$ and their Upper Bounds $\psi^{UB}(u, \pi, w, z)$ with $\pi = 15.75$ and $w = z = 0$ for Bivariate Exponential Distribution

	Model 1		Model 2		Model 3		Model 4	
u	ψ	ψ^{UB}	ψ	ψ^{UB}	ψ	ψ^{UB}	ψ	ψ^{UB}
5	0.5452	0.9436	0.5323	0.9391	0.4728	0.9347	0.4524	0.9330
	(0.0070)	(0.0329)	(0.0135)	(0.0387)	(0.0057)	(0.0360)	(0.0078)	(0.0350)
10	0.5137	0.8914	0.5013	0.8833	0.4437	0.8750	0.4244	0.8717
	(0.0072)	(0.0620)	(0.0125)	(0.0723)	(0.0059)	(0.0672)	(0.0074)	(0.0653)
30	0.4057	0.7186	0.3963	0.7030	0.3453	0.6817	0.3297	0.6736
	(0.0075)	(0.1486)	(0.0101)	(0.1682)	(0.0063)	(0.1554)	(0.0074)	(0.1518)
50	0.3210	0.5902	0.3137	0.5735	0.2692	0.5432	0.2564	0.5321
	(0.0072)	(0.2013)	(0.0080)	(0.2237)	(0.0063)	(0.2048)	(0.0067)	(0.2007)
70	0.2541	0.4934	0.2490	0.4786	0.2105	0.4422	0.2002	0.4294
	(0.0066)	(0.2333)	(0.0067)	(0.2561)	(0.0059)	(0.2320)	(0.0058)	(0.2280)
90	0.2013	0.4193	0.1973	0.4077	0.1644	0.3673	0.1563	0.3537
	(0.0057)	(0.2526)	(0.0056)	(0.2751)	(0.0052)	(0.2461)	(0.0050)	(0.2427)
110	0.1596	0.3619	0.1567	0.3538	0.1283	0.3107	0.1218	0.2971
	(0.0057)	(0.2640)	(0.0045)	(0.2862)	(0.0049)	(0.2527)	(0.0046)	(0.2500)
150	0.0998	0.2809	0.0989	0.2792	0.0785	0.2335	0.0742	0.2208
	(0.0043)	(0.2746)	(0.0030)	(0.2956)	(0.0038)	(0.2539)	(0.0033)	(0.2534)
200	0.0556	0.2183	0.0556	0.2228	0.0422	0.1762	0.0401	0.1656
	(0.0028)	(0.2780)	(0.0021)	(0.2970)	(0.0024)	(0.2468)	(0.0023)	(0.2495)

Table 4
Ultimate Ruin Probabilities $\psi(u,\pi,w,z)$ and their Upper Bounds $\psi^{UB}(u,\pi,w,z)$ with $u = 10$ and $w = z = 0$ for Bivariate Exponential Distribution

	Model 1		Model 2		Model 3		Model 4	
η	ψ	ψ^{UB}	ψ	ψ^{UB}	ψ	ψ^{UB}	ψ	ψ^{UB}
0.01	0.8488	0.9599	0.8347	0.9482	0.8175	0.9485	0.8111	0.9486
	(0.0024)	(0.0465)	(0.0052)	(0.0574)	(0.0031)	(0.0524)	(0.0025)	(0.0509)
0.03	0.6696	0.9282	0.6565	0.9174	0.6124	0.9139	0.5973	0.9111
	(0.0043)	(0.0575)	(0.0094)	(0.0673)	(0.0042)	(0.0639)	(0.0047)	(0.0608)
0.05	0.5137	0.8914	0.5013	0.8833	0.4437	0.8750	0.4244	0.8717
	(0.0072)	(0.0620)	(0.0125)	(0.0723)	(0.0059)	(0.0672)	(0.0074)	(0.0653)
0.08	0.3488	0.8389	0.3362	0.8314	0.2768	0.8184	0.2584	0.8156
	(0.0099)	(0.0648)	(0.0128)	(0.0712)	(0.0084)	(0.0657)	(0.0093)	(0.0671)
0.1	0.2701	0.8065	0.2593	0.7992	0.2032	0.7840	0.1867	0.7811
	(0.0110)	(0.0643)	(0.0120)	(0.0684)	(0.0093)	(0.0638)	(0.0094)	(0.0658)
0.15	0.1442	0.7355	0.1386	0.7297	0.0962	0.7096	0.0849	0.7062
	(0.0108)	(0.0611)	(0.0092)	(0.0631)	(0.0084)	(0.0593)	(0.0075)	(0.0625)
0.2	0.0782	0.6775	0.0762	0.6728	0.0468	0.6487	0.0400	0.6448
	(0.0095)	(0.0596)	(0.0068)	(0.0601)	(0.0066)	(0.0571)	(0.0053)	(0.0608)
0.3	0.0240	0.5883	0.0249	0.5853	0.0116	0.5546	0.0096	0.5503
	(0.0052)	(0.0595)	(0.0035)	(0.0584)	(0.0029)	(0.0570)	(0.0023)	(0.0606)

Table 5
Asymptotic Variances, Covariances and Correlation Coefficients for Bivariate Gamma Distribution

Model	$\mathbb{V}\text{ar}(W)$	$\mathbb{V}\text{ar}(Z)$	$\mathbb{C}\text{ov}(W,Z)$	$\rho(W,Z)$
1	4.1667	4.1667	2.0833	0.5000
2	4.6875	4.6875	1.5625	0.3333
3	3.9062	3.9062	0.7812	0.2000
4	4.6875	4.6875	0.0000	0.0000

Table 6
Finite-Time Ruin Probabilities $\psi_N(u,\pi,w,z)$ and Upper Bounds for Ultimate Probabilities $\psi^{UB}(u,\pi,w,z)$ with $u = 10$, $\pi = 15.75$ and $w = z = 0$ for Bivariate Gamma Distribution

	Model 1	Model 2	Model 3	Model 4
N	ψ_N	ψ_N	ψ_N	ψ_N
50	0.2174	0.1522	0.0964	0.0720
	(0.0053)	(0.0092)	(0.0039)	(0.0026)
100	0.2724	0.2058	0.1344	0.1077
	(0.0051)	(0.0094)	(0.0042)	(0.0031)
150	0.2931	0.2263	0.1489	0.1218
	(0.0052)	(0.0095)	(0.0044)	(0.0030)
200	0.3024	0.2360	0.1556	0.1286
	(0.0052)	(0.0096)	(0.0044)	(0.0031)
500	0.3140	0.2480	0.1618	0.1346
	(0.0052)	(0.0097)	(0.0045)	(0.0031)
800	0.3149	0.2489	0.1621	0.1349
	(0.0051)	(0.0097)	(0.0045)	(0.0031)
1,000	0.3150)	0.2490	0.1622	0.1349
	(0.0051)	(0.0096)	(0.0045)	(0.0031)
1,500	0.3150	0.2490	0.1622	0.1349
	(0.0051)	(0.0096)	(0.0045)	(0.0031)
	ψ^{UB}	ψ^{UB}	ψ^{UB}	ψ^{UB}
	0.7555	0.7508	0.6997	0.6961
	(0.0997)	(0.0912)	(0.0845)	(0.0907)

Table 7

Ultimate Ruin Probabilities $\psi(u,\pi,w,z)$ and their Upper Bounds $\psi^{UB}(u,\pi,w,z)$ with $\pi = 15.75$ and $w = z = 0$ for Bivariate Gamma Distribution

	Model 1		Model 2		Model 3		Model 4	
u	ψ	ψ^{UB}	ψ	ψ^{UB}	ψ	ψ^{UB}	ψ	ψ^{UB}
5	0.3643	0.8673	0.2866	0.8649	0.1963	0.8349	0.1632	0.8325
	(0.0058)	(0.0578)	(0.0108)	(0.0529)	(0.0050)	(0.0505)	(0.0033)	(0.0544)
10	0.3150	0.7555	0.2490	0.7508	0.1622	0.6997	0.1349	0.6961
	(0.0051)	(0.0997)	(0.0096)	(0.0912)	(0.0045)	(0.0845)	(0.0031)	(0.0907)
15	0.2729	0.6610	0.2165	0.6541	0.1340	0.5884	0.1118	0.5845
	(0.0050)	(0.1299)	(0.0083)	(0.1185)	(0.0039)	(0.1065)	(0.0028)	(0.1140)
20	0.2366	0.5807	0.1883	0.5720	0.1110	0.4967	0.0925	0.4928
	(0.0046)	(0.1513)	(0.0073)	(0.1375)	(0.0035)	(0.1199)	(0.0024)	(0.1280)
25	0.2056	0.5123	0.1637	0.5019	0.0921	0.4207	0.0764	0.4172
	(0.0042)	(0.1663)	(0.0063)	(0.1503)	(0.0031)	(0.1270)	(0.0021)	(0.1354)
30	0.1785	0.4538	0.1424	0.4419	0.0763	0.3576	0.0632	0.3546
	(0.0037)	(0.1763)	(0.0053)	(0.1584)	(0.0029)	(0.1298)	(0.0018)	(0.1382)
50	0.1011	0.2901	0.0814	0.2745	0.0360	0.1931	0.0292	0.1924
	(0.0027)	(0.1886)	(0.0032)	(0.1640)	(0.0017)	(0.1184)	(0.0014)	(0.1267)

Table 8

Ultimate Ruin Probabilities $\psi(u, \pi, w, z)$ and their Upper Bounds $\psi^{UB}(u, \pi, w, z)$ with $u = 10$ and $w = z = 0$ Bivariate Gamma Distribution

η	Model 1		Model 2		Model 3		Model 4	
	ψ	ψ^{UB}	ψ	ψ^{UB}	ψ	ψ^{UB}	ψ	ψ^{UB}
0.01	0.7516	0.9173	0.7003	0.9142	0.6482	0.9193	0.6215	0.9103
	(0.0036)	(0.0898)	(0.0078)	(0.0868)	(0.0057)	(0.0806)	(0.0031)	(0.0890)
0.03	0.4988	0.8378	0.4286	0.8341	0.3331	0.8084	0.2992	0.8029
	(0.0055)	(0.1029)	(0.0103)	(0.0988)	(0.0052)	(0.0943)	(0.0035)	(0.1023)
0.05	0.3150	0.7555	0.2490	0.7508	0.1622	0.6997	0.1349	0.6961
	(0.0051)	(0.0997)	(0.0096)	(0.0912)	(0.0045)	(0.0845)	(0.0031)	(0.0907)
0.08	0.1587	0.6491	0.1118	0.6451	0.0560	0.5695	0.0410	0.5676
	(0.0048)	(0.0870)	(0.0063)	(0.0770)	(0.0026)	(0.0716)	(0.0019)	(0.0743)
0.1	0.1012	0.5903	0.0667	0.5868	0.0282	0.5002	0.0186	0.4992
	(0.0039)	(0.0805)	(0.0047)	(0.0700)	(0.0018)	(0.0654)	(0.0013)	(0.0661)
0.15	0.0332	0.4750	0.0193	0.4724	0.0053	0.3701	0.0030	0.3707
	(0.0025)	(0.0695)	(0.0024)	(0.0588)	(0.0008)	(0.0556)	(0.0005)	(0.0523)
0.2	0.0110	0.3913	0.0058	0.3892	0.0010	0.2814	0.0004	0.2831
	(0.0014)	(0.0634)	(0.0011)	(0.0534)	(0.0003)	(0.0506)	(0.0002)	(0.0446)
0.3	0.0014	0.2794	0.0006	0.2781	0.0000	0.1725	0.0000	0.1753
	(0.0004)	(0.0584)	(0.0003)	(0.0508)	(0.0000)	(0.0460)	(0.0000)	(0.0371)

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