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MATRICES WHOSE CHARACTERISTIC EQUATIONS ARE CYCLIC*

BY T. A. PIERCE

One of Sylvester's theorems† on matrices states that if the characteristic equation

$$(1) \quad |M - \lambda I| = f(\lambda) = 0$$

of a square matrix M has the roots $\lambda_1, \lambda_2, \dots, \lambda_n$, then the characteristic equation

$$(2) \quad |\phi(M) - \rho I| = g(\rho) = 0$$

of any integral function of M , namely, $\phi(M)$, has the roots $\rho_i = \phi(\lambda_i), i = 1, 2, \dots, n$. In this note an isomorphism is shown to exist between the algebraic and matrix roots of (1) when this equation is cyclic. Certain consequences of this isomorphism are given. Since

$$\begin{pmatrix} -a & -b & \dots & -k & -l \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

is the matrix which has $\lambda^n + a\lambda^{n-1} + b\lambda^{n-2} + \dots + k\lambda + l = 0$ as its characteristic equation,‡ Sylvester's theorem furnishes a method of effecting the Tschirnhaus transformation $\rho = \phi(\lambda)$ on any equation. When (1) is cyclic an especially interesting type of Tschirnhaus transformation is possible.

Suppose (1) to be a cyclic equation of degree n with the relations $\lambda_{i+1} = \phi(\lambda_i), i = 1, 2, \dots, n$ and $\lambda_{n+j} = \lambda_j$ connecting its roots. For simplicity of notation let ϕ_i be the i th iterated function of ϕ so that $\lambda_2 = \phi(\lambda_1) = \phi_1(\lambda_1), \lambda_3 = \phi(\lambda_2) = \phi(\phi(\lambda_1)) = \phi_2(\lambda_1)$, and in general $\lambda_{i+1} = \phi_i(\lambda_1)$. By Sylvester's theorem the roots of (2) then are

* Presented to the Society, December 30, 1929.

† Sylvester, *Mathematical Papers*, vol. 4, p. 133; Frobenius, *Journal für Mathematik*, vol. 84, p. 11; Dickson, *Algebren und ihre Zahlentheorie*, p. 18.

‡ Wedderburn, *Annals of Mathematics*, vol. 27 (1926), p. 247.

$$\rho_1 = \phi(\lambda_1) = \lambda_2, \rho_2 = \phi(\lambda_2) = \lambda_3, \dots, \rho_n = \phi(\lambda_n) = \lambda_{n+1} = \lambda_1.$$

Thus the roots of (1) and (2) are the same, and M and $\phi(M)$ have the same characteristic equation. Similarly, if we form the characteristic equation of $\phi_i(M)$, we find that its roots are

$$\rho_1 = \phi_i(\lambda_1) = \lambda_{i+1}, \quad \rho_2 = \phi_i(\lambda_2) = \lambda_{i+2}, \dots, \rho_n = \phi_i(\lambda_n) = \lambda_i,$$

where the subscripts are to be reduced modulo n . Hence we have the following theorem.

THEOREM 1. *If the characteristic equation, $f(\lambda)=0$, of the matrix M is cyclic with the relations $\lambda_i=\phi_i(\lambda_1)$, $i=1, 2, \dots, n$, connecting its roots, then the matrices $M, \phi_1(M), \dots, \phi_{n-1}(M)$, all have the same characteristic equation.*

Applying the Hamilton-Cayley theorem we have the following additional theorem.

THEOREM 2. *Under the conditions of Theorem 1 above $f(\lambda)=0$ has its full complement of matrix roots, namely,*

$$M, \phi_1(M), \dots, \phi_{n-1}(M).$$

The matrix roots of this last theorem may be called conjugate* and may be denoted by

$$(3) \quad M_1, M_2 = \phi_1(M_1), \dots, M_n = \phi_{n-1}(M_1).$$

Since these matrices are all functions of one matrix they are commutative under multiplication and division, and since the same cyclic relations exist among the M 's as among the λ 's, we have the following theorem.

THEOREM 3. *The field generated by the roots of any cyclic equation is isomorphic to the field generated by the matrix roots of this same equation.*

Consider now any integral function of the matrices M_1, \dots, M_n , namely $F(M_1, \dots, M_n)$, and form its characteristic equation

$$(4) \quad |F(M_1, \dots, M_n) - \rho I| = 0.$$

Now $F(M_1, \dots, M_n) = G(M_1)$ by virtue of (3), and hence the roots of (4) are by Sylvester's theorem $\rho_i = G(\lambda_i)$, $i=1, 2, \dots, n$. But

* Taber, American Journal of Mathematics, vol. 13, p. 159.

$$\begin{aligned}\rho_1 &= G(\lambda_1) = F[\lambda_1, \phi_1(\lambda_1), \dots, \phi_{n-1}(\lambda_1)] \\ &= F[\lambda_1, \lambda_2, \dots, \lambda_n], \\ \rho_2 &= G(\lambda_2) = F[\lambda_2, \phi_1(\lambda_2), \dots, \phi_{n-1}(\lambda_2)] \\ &= F[\lambda_2, \lambda_3, \dots, \lambda_1],\end{aligned}$$

and in general we find that

$$\rho_i = G(\lambda_i) = F[\lambda_i, \lambda_{i+1}, \dots, \lambda_{i-1}];$$

thus we obtain the theorem which follows:

THEOREM 4. *Let the characteristic equation of M_1 be cyclic and have the roots $\lambda_1, \lambda_2, \dots, \lambda_n$ and the corresponding matrix roots M_1, M_2, \dots, M_n . Then, if $F(M_1, \dots, M_n)$ be an integral function, the roots of the characteristic equation of $F(M_1, \dots, M_n)$ are $F(\lambda_1, \dots, \lambda_n)$ and the quantities obtained from this by permuting $\lambda_1, \lambda_2, \dots, \lambda_n$ cyclically. The matrix roots of this equation are obtained from $F(M_1, \dots, M_n)$ by permuting M_1, M_2, \dots, M_n cyclically. Moreover this equation is cyclic.*

Thus, for example, if $|M_1 - \lambda I| = 0$ is cyclic and has the roots $\lambda_1, \lambda_2, \dots, \lambda_n$, the roots of $|M_1 + M_2 - \rho I| = 0$ are $\rho_1 = \lambda_1 + \lambda_2$, $\rho_2 = \lambda_2 + \lambda_3, \dots, \rho_n = \lambda_n + \lambda_1$. To the first equation the Tschirnhaus transformation $\rho = \lambda + \phi(\lambda)$ has been applied and this transformation affects the roots of the equation in pairs.

If C_m denotes the m th compound of the matrix M it is known* that the roots of the characteristic equation

$$(5) \quad |C_m - \rho I| = 0$$

are the products m at a time of the roots of $|M - \lambda I| = 0$. If the latter is a cyclic equation its roots are all functions of one root and the roots of (5) are therefore all functions of this root. By the isomorphism of Theorem 3 the roots of (5) correspond to matrices all of which are functions of one matrix and hence are commutative under multiplication. Moreover by Theorem 3 these matrices all satisfy (5) so that (5) has its full complement of matrix roots. These matrix roots are, however, in general of order different from the degree of (5).

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* Pascal, *Repertorium der höheren Mathematik*, vol. 2, 1910, p. 139; Whittaker, *Proceedings of the Edinburgh Mathematical Society*, vol. 35 (1916), p. 2.