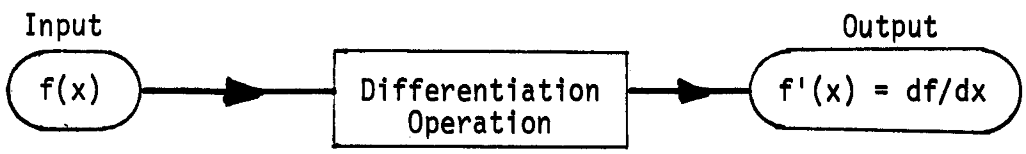
1975

Calculus

CALCULUS

DIFFERENTIATION

Differentiation of a function, say f(x), is a mathematical operation which yields a second function called the derivative of f [symbolized by f'(x) or dy/dx]. This procedure is represented in the diagram, which shows



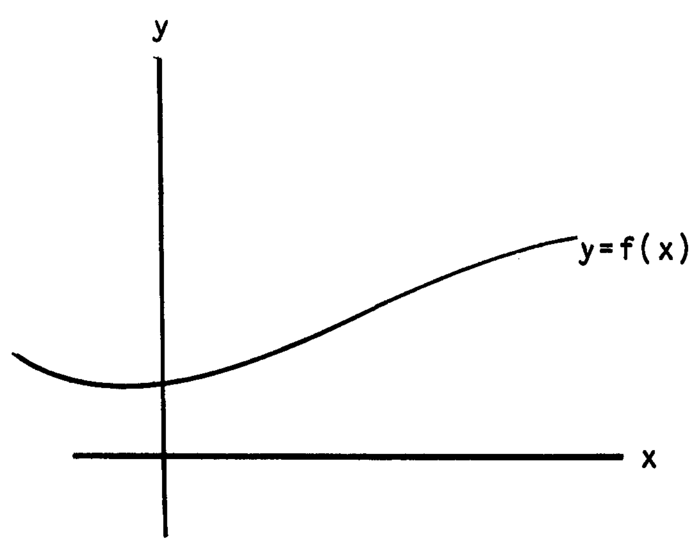
the function f(x) being input to an "analytical machine" that manufactures as output the derivative of f. A detailed mathematical prescription for the differentiation operation is nontrivial. It usually involves a quarter or semester course, which requires time, attention, and effort on the student's part.

Our goal here is to provide you with graphical and intuitive understandings of what information the derivative supplies; and tabular means for determining derivatives.

Graphical Interpretation of the Derivative

A graphical understanding of the derivative will prove useful time and again in your study of physics. So, let's get at it.

A function f(x) is graphed in the xy plane by letting y = f(x), i.e. for each value of x (for which f is defined) there corresponds one value of y obtained from the "rule" y = f(x). This number pair (x,y) plots as a point in the plane. As x changes, this point sweeps out a curve. Such a curve might look like the one shown in the figure on the right.



Question:	What information does the derivative $f'(x)$ provide about this graph?
Answer:	Let x_1 be a specific value of x. The value of the derivative for $x = x_1$ is denoted by $f'(x_1)$ or $(dy/dx)x_1$ and is numerically equal to the slope of the line tangent to the curve $y = f(x)$ at the point (x_1, y_1) where $y_1 = f(x_1)$.

Read this interpretation carefully while studying Figure 1.

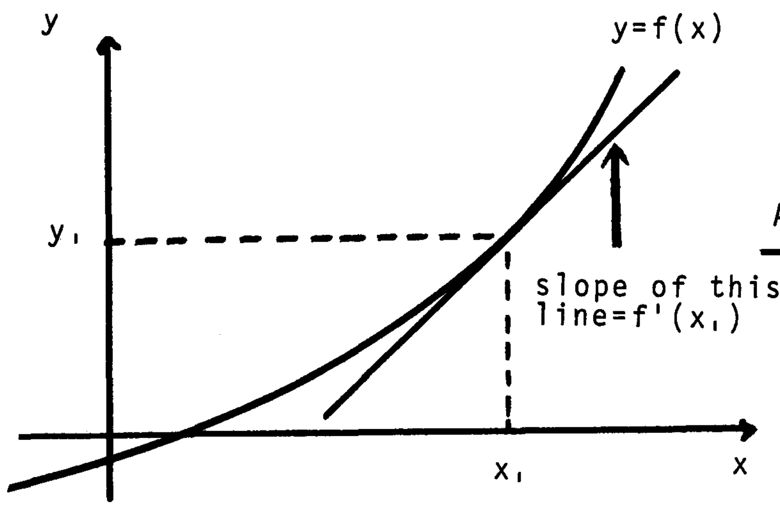


Figure 1

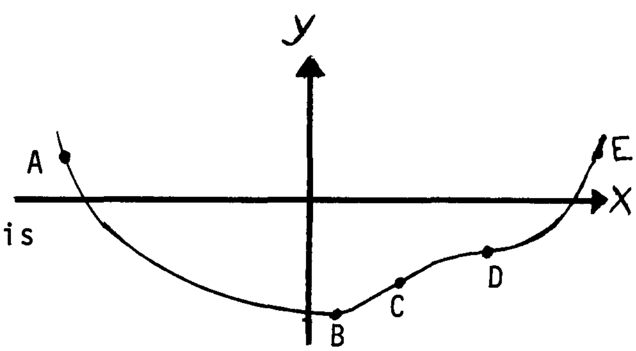


Figure 2

Example 1: In Figure 2, which point or points is:

	Answer
$f' = 0$?	B,D.
$f' > 0$?	C,E.
$f' < 0$?	A.
$\|f'\|$ the greatest?	E.

Each of the answers in Example 1 is obtained by looking at the graph and ascertaining the needed information about the slope of the curve (actually of the tangent line). For example, point B has been drawn at the lowest point on the curve. A tangent line at B is horizontal and therefore has a zero slope. Since the tangent line at A slopes downward (i.e., has a negative slope) $f' < 0$ at A.

Exercise A: By inspecting the graph decide whether to insert >, =, or < in each of the blanks.

(a) f'_A ____ 0

(b) f'_B ____ 0

(c) f'_C ____ 0

(d) f'_D ____ 0

(e) f'_E ____ 0

(f) f'_F ____ 0

(g) f'_G ____ 0

(h) f'_A ____ f'_B

(i) f'_B ____ f'_E

(j) f'_E ____ f'_F

(k) f'_D ____ f'_E

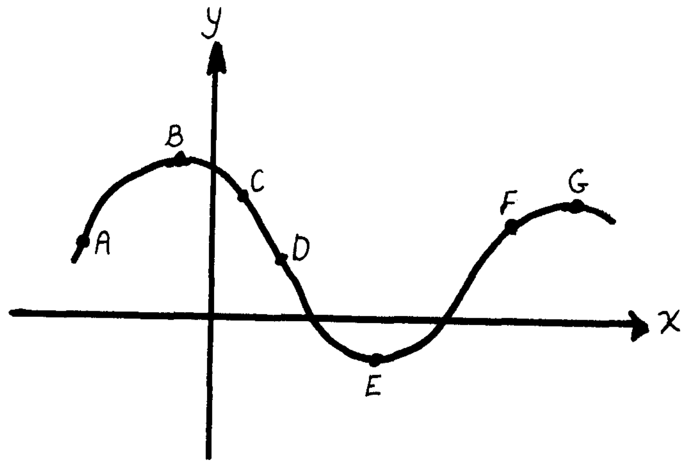
(l) f'_A ____ f'_F

(m) f'_C ____ f'_D

(n) f'_D ____ f'_F

(o) $|f'_D|$ ____ $|f'_F|$

(p) $|f'_D|$ ____ $|f'_B|$

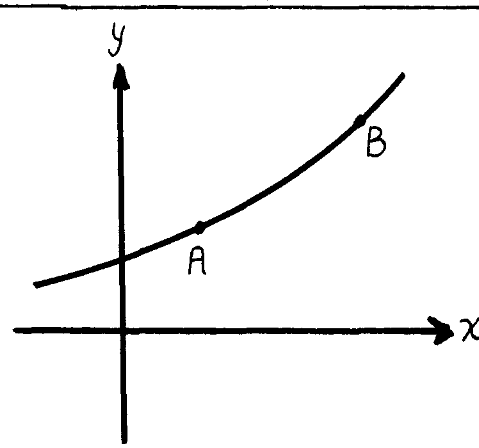


Exercise B: With reference to the graph of Exercise A, complete the table with appropriate words and symbols for the sentence:

At point ____, f is ______ and f' ____ 0.

Point	Increasing, decreasing, not changing	<, =, >
A		
B	not changing	
C		
D		
E		
F		>
G		

Consider the graph. At both points A and B, f is increasing and f' is positive. But more can be said. At B the rate of increase of f is greater than A. This statement is said in derivative language by saying that the derivative is greater at B than at A. We can summarize these remarks with



$$f'_B > f'_A > 0.$$

f increasing more rapidly at B than at A.

f increasing at A.

Clearly f' (or dy/dx) is related to the rate of change of f, and this rate changes from point to point if the slope of $y = f(x)$ is changing as x changes. The following statement (definition) relates (defines) instantaneous rate of change of a function and its derivative.

> If $y = f(x)$, the instantaneous rate of change of y per unit change in x at x_1 is defined to be $f'(x_1)$.

This statement is often abbreviated to say "$f'(x_1)$ is the rate of change of f with respect to x at x_1."

Determining f' by Using the Table

Suppose $f(x) = x^2$. What is $f'(x)$? If you don't happen to know already, then two options are offered here.

Option 1. The formal definition of the derivative gives

$$f'(x) = \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} .$$

Evaluate it. OR

Option 2. Turn to p. 15 of this module entitled "Table of Derivatives." Locate x^2 in the f column and read 2x from the f' column.

Answer: $f'(x) = 2x$.

Now Option 1 is the way you do it when you really want to dig mathematics - and you have a month or two to learn about limits, etc. Guess what we have in mind here?

Let's do another. Let $f(x) = x^9$. Look under the f column. What now? x^9 does not appear as an entry. Oh yes, it does. It appears as x^p. The table gives $f'(x) = px^{p-1}$, or since $p = 9$ in this case, $f'(x) = 9x^8$. Furthermore, $f'(1) = 9 \times 1^8 = 9$. Thus the slope of (the tangent line to) $y = x^9$ at (1,1) is 9; and at $x = 1$ the rate of change of y with respect to x is 9.

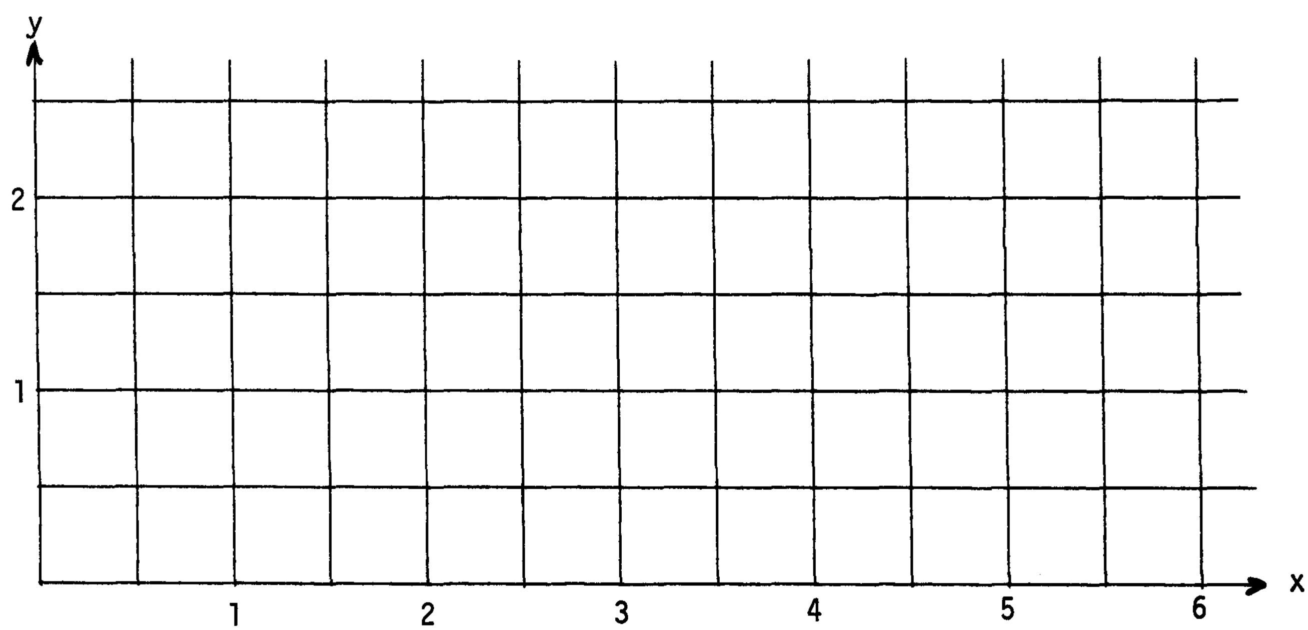
<u>Exercise C:</u> Let $f(x) = \sqrt{x} = x^{1/2}$.

(a) $f'(x) =$ __________________.   (b) $f'(1) =$ _________________.

(c) Rate of change of x with respect to x at $x = 1$ is ____________________.

(d) Graph f for $0 \le x \le 6$. Construct the tangent line to f at (1,1). Slope = ________________. Does this agree with part (b)?

x	0.00	0.50	1.00	1.50	2.00	2.50	3.00	4.0	5.0	6.0
$\sqrt{x}$	0.00	0.71	1.00	1.22	1.41	1.58	1.73	2.00	2.24	2.45



Exercise D: Complete the table.

$f(x)$	$f'(x)$	x_1	$f'(x_1)$
x^2		-1	
x^{-4}		2	
6		1	
$\sin x$		$\pi/2$	
$\cos x$		$\pi/2$	
$\tan x$		$\pi/4$	
$\sin(2x)$		$\pi/2$	
$\cos(\pi x)$		$1/2$	
e^x		0	
e^{-2x}		1	
$\sqrt{x^2 + 16}$		3	
$1/\sqrt{x^2 + 9}$		4	

A Few Important Properties of Derivatives

(Note that both f and g are functions of x.)

$$\frac{d}{dx}(kf) = k\,\frac{df}{dx} \qquad \text{(k is a constant).} \tag{P1}$$

$$\frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx}\ . \tag{P2}$$

$$\frac{d}{dx}(fg) = \frac{df}{dx}\,g + f\,\frac{dg}{dx}\ . \tag{P3}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{(df/dx)g - f(dg/dx)}{g^2}\ . \tag{P4}$$

Example 2

(a) $\frac{d}{dx}(6x^2) \overset{*}{=} 6\,\frac{d}{dx}(x^2) = 6(2x) = 12x.$ (*P1)

(b) $\frac{d}{dx}\,(x^2 + \sin x) \overset{*}{=} \frac{d}{dx}\,(x^2) + \frac{d}{dx}\,(\sin x) = 2x + \cos x.$ (*P2)

(c) $\frac{d}{dx}\,(e^x \cos x) \overset{*}{=} \frac{d}{dx}\,(e^x) \cos x + e^x \frac{d}{dx}\,(\cos x) = e^x \cos x - e^x \sin x.$ (*P3)

(d) $\frac{d}{dx}\,\left(\frac{e^x}{\ln x}\right) \overset{*}{=} \frac{\ln x(de^x/dx) - e^x(d/dx)\,(\ln x)}{(\ln x)^2} = \frac{e^x \ln x - e^x/x}{(\ln x)^2}.$ (*P4)

Exercise E:

(a) $f(x) = 2x^3 - \sin x$: $f'(x) =$ ________________

(b) $g(x) = 3x^2e^{-x}$: $g'(x) =$ ________________

(c) $h(x) = e^x/(x + a)$: $h'(x) =$ ________________

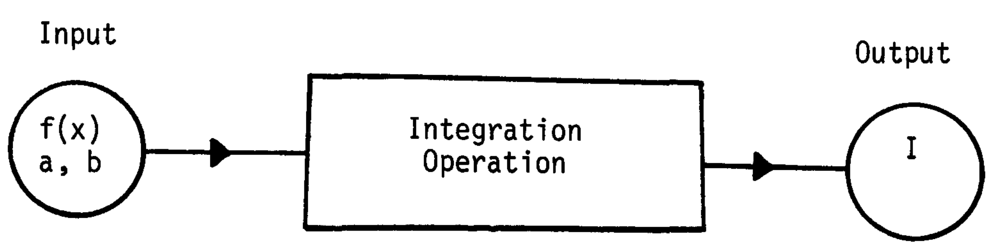
(d) $F(x) = x \sin (\pi x)$: $F'(1) =$ ________________

(e) $G(x) = e^{4x} \tan (\pi x)$: $G'(1/4) =$ ________________

(f) $H(x) = e^x\sqrt{x + 9}$: $H'(0) =$ ________________

THE DEFINITE INTEGRAL

The definite integral is a mathematical operation that requires as input a function, say f(x), and two numbers a and b, which are the coordinates of the end points of an interval on the x axis, i.e., $a \le x \le b$. Given this input the definite integral of f on [a,b] yields as output one number, say I. This is symbolized in the figure. Just as with the derivative, a careful prescription of this operation is nontrivial. Again we shall first offer you some graphical understanding as to the meaning of the number I and then give you a procedure (using integral tables) to determine I.



Graphical Interpretation of the Integral

> The definite integral of a function f(x) on the interval $a \le x \le b$ is numerically equal to the area enclosed by the curve $y = f(x)$, the x axis, the line $x = a$, and the line $x = b$.

The shaded area in Figure A is the area determined by the definite integral of f on a,b.

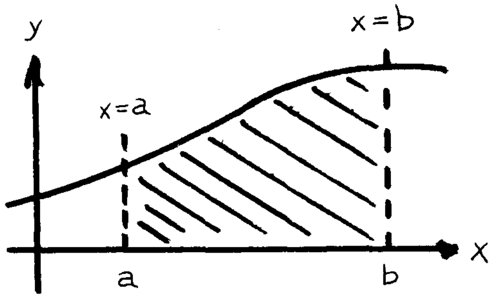


Figure A

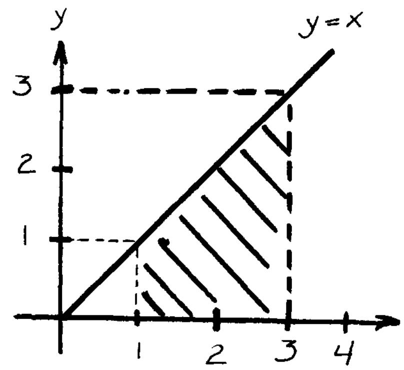


Figure B.

Determination of I

Suppose $f(x) = x$, $a = 1$, and $b = 3$. The area to be found is shown in Figure B. Of course, it's very easy to do so since the figure whose area we seek is a trapezoid. In fact,

$$\text{Area} = (\text{Average height}) \times (\text{base}) = 2 \times 2 = 4.$$

This simple calculation is possible because $y = x$ graphs as a straight line. Almost any other function would not be so trivial.

Here's how you do the integral with the "Table of Integrals" provided in this module (p. 15). Look under the function column f and find x. Read off the antiderivative (indefinite integral) for x, namely, $(1/2)x^2$. Evaluate this function at the upper limit b (= 3 in this case) and subtract the value of this function at the lower limit a (= 1). The result of this calculation is the value of the definite integral. Thus we have

$$I = \frac{1}{2}(3^2) - \frac{1}{2}(1^2) = \frac{9}{2} - \frac{1}{2} = \frac{8}{2} = 4,$$

in exact agreement with our previous calculation.

Let's try another. Let $f(x) = x^2$ from 0 to 2. The area to be determined is shaded in Figure C. From the table the antiderivative for x^2 is listed as $(1/3)x^3$. The desired area is

$$I = \frac{1}{3}(2)^3 - \frac{1}{3}(0)^3 = \frac{8}{3}$$

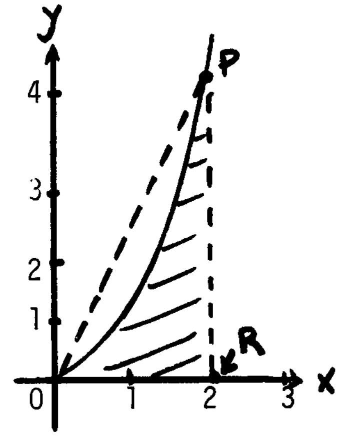


Figure C

Does this look reasonable? The shaded area is less than that of the triangle OPR whose area is (1/2) (base) (height) or (1/2) (2) (4) = 4, which exceeds 8/3 as expected.

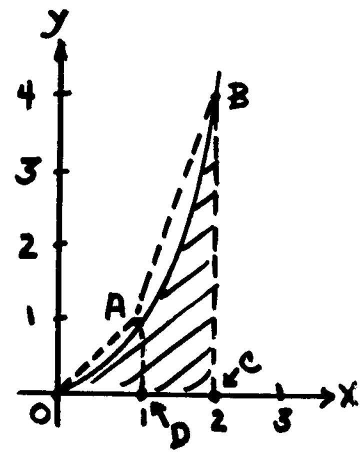


Figure D

We can make a better estimate for the desired area by using the area of the triangle OAD and the trapezoid ABCD (See Figure D). The result for this improved <u>over</u>estimate is

$$\frac{1}{2}(1)\ (1) + \frac{1}{2}(1+4)\ (1) = \frac{1}{2} + \frac{5}{2} = 3,$$

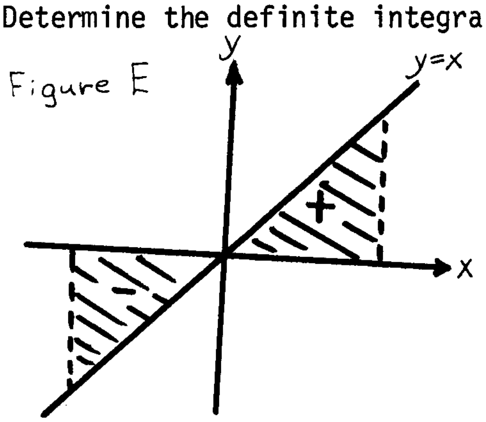
which is indeed less than 4 and 12.5% greater than 8/3, the result from the integral.

<u>Exercise F:</u> Continue this estimating process one more time by dividing the x axis equally four times and calculating the area of the four figures. <u>Answer:</u> $2\frac{3}{4}$.

Essentially, what the integral does is continue this process of dividing the interval [0,2] into increasingly larger number of subintervals. In fact, the integral is the limit of the approximate areas as the number of subintervals approaches infinity. In this sense then, the definite integral can be thought of somewhat intuitively as the sum of many (infinitely many, in fact) terms each of which approaches zero as the total number approaches infinity. We avoid the difficulty of actually doing this by using the table of antiderivatives.

<u>Exercise G:</u> Let $f(x) = x$, $a = -2$, $b = 2$. Determine the definite integral of f on [a,b]. <u>Answer</u>: 0.

Figure E



If you did Exercise G correctly, the result is zero. But how can this be? Here's how. Look at Figure E. Areas below the x axis are treated by the integral as negative. As you can see, the two area in the figure exactly cancel, leaving a net area of D.

Exercise H:

(a) $f(x) = \cos x$, $a = 0$, $b = \pi/2$, I = ________________.

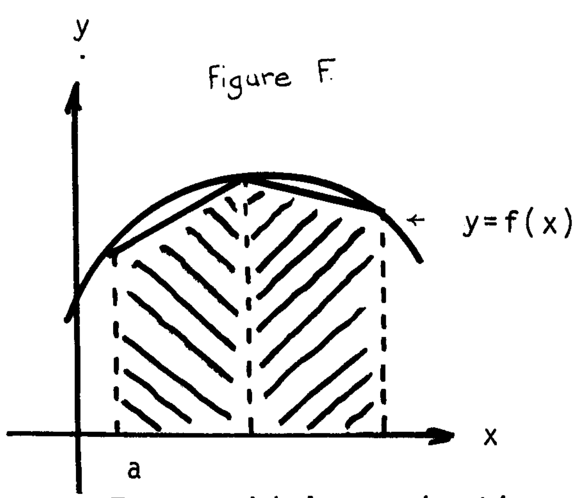
(b) Sketch a graph of cos x and estimate the area determined here by partitioning the interval $[0,\pi/2]$ into two and then three equal parts. Does your result for part (a) appear reasonable?

(c) $f(x) = \cos x$, $a = 0$, $b = \pi$, I = ________________.

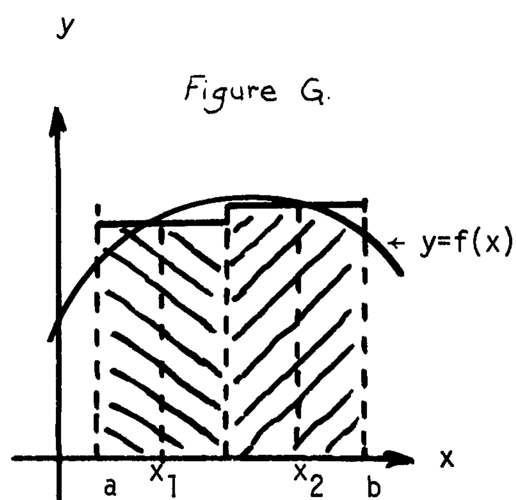
(d) Explain your result for part (c).

Estimating the Value of a Definite Integral

You may find yourself needing to evaluate a definite integral of a function that is not in your table. You have already seen one way to estimate the integral by partitioning the integration interval a,b into N equal subintervals and approximating the area by a set of N trapezoids. This technique is depicted graphically in Figure F below for N = 2.



N=2 Trapezoid Approximation



N=2 Rectangle Approximation

A second technique is shown in Figure G. Here the function is approximated by N rectangles. The height of each rectangle is the value of the function at the midpoint of the corresponding subinterval. From the figure you can see that the N = 2 approximation for I is then

$$I \simeq f(x_1)\,\Delta x + f(x_2)\,\Delta x = \sum_{k=1}^{2} f(x_k)\,\Delta x,$$

where x_k is the value of x at the center of the k-th subinterval. For an N partition then,

$$I \simeq \sum_{k=1}^{N} f(x_k)\,\Delta x.$$

As N increases the rectangular approximation gets closer and closer to the area being sought. In fact, the definite integral can be defined by

$$I = \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k)\,\Delta x.$$

This suggests the usual notation for a definite integral, which is

$$\int_a^b f(x)\,dx.$$

The symbol $\int$ is called an integral sign. It symbolizes both the summation sign Σ and the limiting process. The values of a and b are included so as to indicate the end points of the integration interval. Finally, dx symbolizes the Δx in the sum. We shall use this notation for the integral. To be sure you understand, here is the appropriate way to write the integrals with the results we have obtained so far.

Function	Interval	Integral	Value
x	$[1,3]$	$\int_1^3 x\,dx =$	4
x^2	$[0,2]$	$\int_0^2 x^2\,dx =$	8/3
x	$[-2,2]$	$\int_{-2}^2 x\,dx =$	0
$\cos x$	$[0,\pi/2]$	$\int_0^{\pi/2} \cos x\,dx =$	1
$\cos x$	$[0,\pi]$	$\int_0^{\pi} \cos x\,dx =$	0

One more comment before you go to work: Most integral tables will indicate the antiderivative for f(x) by $\int f(x)\,dx$, i.e., the definite integral without limits. For example,

$$\int x\,dx = \frac{1}{2}x^2,$$

so that

$$\int_1^3 x\,dx = \frac{1}{2}(3)^2 - \frac{1}{2}(1)^2 = \frac{9}{2} - \frac{1}{2} = 4,$$

as we found earlier.

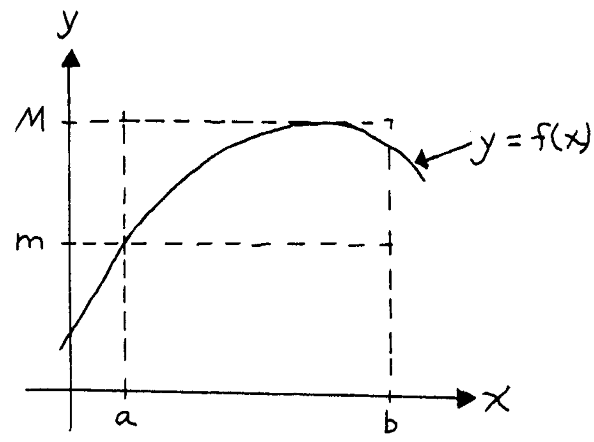


Figure H

One More Comment on Estimating Integrals

Consider the definite integral suggested by Figure H. A valuable technique for getting bounds on a definite integral is suggested. Let m be the minimum value of f on [a,b] and M the maximum value of f on [a,b]. Consider the rectangle of height m and base (b - a). Its area is necessarily less than the area determined by

$$\int_a^b f(x)\,dx.$$

Similarily the area M(b - a) is greater than that of the integral. Thus,

$$m(b - a) \le \int_a^b f(x)\,dx \le M(b - a).$$

Note: The equality signs are included in this equality to take care of the case where f is constant on the interval. Then m = M = f(x) and

$$m(b - a) = \int_a^b f(x)\,dx = M(b - a).$$

Exercise I: Evaluate the following definite integrals. Also determine m and M and determine bounds on the integral. [You may need to sketch a graph of f(x) to determine m and M].

Integral		m(b - a)	M(b - a)
$\int_1^2 x^2\,dx =$	______	$1(1) = 1$	______
$\int_{-1}^2 x\,dx =$	______	______	______
$\int_0^{\pi/4} \cos x\,dx =$	______	______	______
$\int_0^1 e^x\,dx =$	______	______	______
$\int_0^{16} \sqrt{x+9}\,dx =$	______	______	______
$\int_0^3 \frac{x\,dx}{\sqrt{x^2+16}} =$	______	______	______
$\int_0^2 \frac{dx}{x+2} =$	______	______	______
$\int_{-1}^1 e^{-x}\,dx =$	______	$e^{-1}(2) \simeq 0.3$	$e(2) \simeq 5.4$

Remember in each case you should check to see that

$$m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a).$$

Two More Important Properties of the Integral

The following properties of integrals are frequently required:

$$\int_a^b k\,f(x)\,dx = k \int_a^b f(x)\,dx \quad \text{(k is a constant),} \qquad \text{(P1)}$$

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx. \qquad (P2)$$

Example 3

(a) $\int_0^3 4x\, dx = 4 \int_0^3 x\, dx$ [by (P1)]

$= 4\ (3^2/2 - 0^2/2)$

$= 18.$

(b) $\int_0^1 (x + x^2)\, dx = \int_0^1 x\, dx + \int_0^1 x^2\, dx$ [by (P2)]

$= (\frac{1^2}{2} - \frac{0^2}{2}) + (\frac{1^3}{3} - \frac{0^3}{3})$

$= \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$

Exercise J:

(a) $\int_1^2 (x^2 - x)\, dx =$ ____________________.

(b) $\int_0^{\pi/2} (\sin x + \cos x)\, dx =$ ____________________.

(c) $\int_0^3 (x^2 + e^{-x})\, dx =$ ____________________.

(d) $\int_0^1 (x + \sin \pi x)\, dx =$ ____________________.

TABLE OF DERIVATIVES

f(function)	f'(derivative of function)
.	.
.	.
.	.
x^{-3}	$-3/x^4$
x^{-2}	$-2/x^3$
x^{-1}	$-1/x^2$
constant	0
x	1
x^2	$2x$
x^3	$3x^2$
.	.
.	.
.	.
x^p	px^{p-1}
sin x	cos x
cos x	-sin x
e^x	e^x
e^{ax}	ae^{ax}
ln x	x^{-1}

TABLE OF INTEGRALS

f(function)	∫f(antiderivative)
.	.
.	.
.	.
x^{-3}	$-x^{-2}/2$
x^{-2}	$-x^{-1}$
x^{-1}	ln x
constant (c)	cx
x	$x^2/2$
x^2	$x^3/3$
x^3	$x^4/4$
.	.
.	.
.	.
x^p	$x^{p+1}/(p+1)$
sin x	-cos x
cos x	sin x
e^x	e^x
e^{ax}	$1/a\ e^{ax}$