A semi-infinite interfacial crack between two bonded dissimilar elastic strips

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Abstract
This paper is concerned with a semi-infinite interfacial crack between two bonded dissimilar elastic strips with equal thickness. Solutions for the complex stress intensity factor (SIF) and energy release rate (ERR) are obtained in closed form under in-plane deformations. During the procedure, the mixed boundary-value problem is reduced by means of the conformal mapping technique to the standard Riemann–Hilbert problem. In some limiting cases, the present solutions can cover the results found in literature.

Keywords: interfacial crack, strip, Riemann–Hilbert problem, conformal mapping

1 Introduction

Early investigations in the area of interfacial fracture focused mainly on studying the characteristics of asymptotic fields near the crack tip and finding the SIF and ERR of interfacial cracks between two bonded dissimilar half-planes [1–6]. However, interfacial cracks occurring in engineering structures, such as composite laminates and adhesive joints, often bear the crack length comparable to the structure dimensions. In order to evaluate the interaction of structure boundaries and cracks, it is natural to introduce structure dimensions into the crack model. One of the typical crack models discussed by investigators is concerned with a semi-infinite interfacial crack between two bonded dissimilar strips. Its anti-plane case is mathematically equivalent to that of a semi-infinite crack in a homogenous elastic strip, which was solved by Rice, [7], using the method of the J-integral; later, in paper [8], the Wiener–Hopf technique was used, and recently the method of dual integral equations was applied in [9]. In the case of in-plane deformations, a number of researchers contributed significantly to this subject, mainly based on the elementary beam theory. Recent refined work on the basis of the J-integral and asymptotic fields near the crack tip was presented in [10, 11], [12] and [13], where a series of analytic solutions involving various crack configurations has been provided.

To the authors' knowledge, no closed-form solution for the in-plane case has yet been obtained. In this paper, we consider a semi-infinite interfacial crack between two bonded dissimilar elastic strips under in-plane deformations. By using the conformal mapping technique, the present mixed boundary-value problem is reduced to the standard Riemann–Hilbert problem, which is then solved explicitly. Furthermore, the SIFs of some special cases with a semi-infinite crack between two bonded dissimilar half-planes or in a homogeneous strip can be extracted as the limiting cases of the strip dimensions and elastic properties, respectively.

2 Statement of the problem

Consider a material with elastic properties E_1 and v_1 which occupies the upper strip, $0 < y < H$, and a material with elastic properties E_2 and v_2 which occupies the lower strip, $-H < y < 0$. The two materials are bonded along straight-line segments of the x-axis as shown in Figure 1a. In the following, all quantities such as the elastic constants, stresses, and so on, relating to the regions $0 < y < H$ and $-H < y < 0$ will be marked with subscripts 1 and 2, respectively.

Without loss of the generality, we assume the crack surfaces under the action of self-equilibrated forces, the normal traction $p(x)$ and shear traction $q(x)$. The boundary conditions and the stress/displacement continuity along the interface of the bonded strips may be expressed as

$$
\begin{aligned}
&y = \pm H, -\infty < x < +\infty: \quad \sigma_{yy} = \sigma_{xy} = 0,\\
&x = \pm\infty, -H < y < H: \quad \sigma_{xx} = \sigma_{xy} = 0,\\
&y = 0, \ -\infty < x < 0: \quad \sigma_{yy} = p(x), \ \sigma_{xy} = q(x),\\
&y = 0, \ 0 < x < +\infty: \quad \sigma_{yy}(x, 0^+) = \sigma_{yy}(x, 0^-), \quad \sigma_{xy}(x, 0^+) = \sigma_{xy}(x, 0^-),\\
&y = 0, \ 0 < x < +\infty: \quad u_y(x, 0^+) = u_y(x, 0^-), \ u_x(x, 0^+) = u_x(x, 0^-),
\end{aligned}
\tag{1}
$$

We will derive the solution by reducing this mixed boundary-value problem into a kind of Riemann–Hilbert problem, which has been discussed extensively by Muskhelishvili, [14], and used for finding the solutions of collinear interfacial cracks by England, [2], Erdogan, [3], Rice and Sih [4], and recently in [15]. Two Dundurs' parameters, [16], will be used to simplify the derivation, which are defined as follows:

$$
\alpha = \frac{\Gamma(\kappa_2+1)-(\kappa_1+1)}{\Gamma(\kappa_2+1)+(\kappa_1+1)}, \quad \beta = \frac{\Gamma(\kappa_2-1)-(\kappa_1-1)}{\Gamma(\kappa_2-1)+(\kappa_1-1)}
\tag{2}
$$

where $\Gamma = \mu_1/\mu_2$, $\kappa_j = 3-4v_j$ for plane strain and $(3-v_j)/(1+v_j)$ for plane stress, μ_j ($j = 1, 2$) is the material shear modulus, and v_j ($j = 1, 2$) is the Poisson's ratio. The physically admissible values of α and β are restricted to a parallelogram enclosed by $\alpha = \pm 1$ and $\alpha - 4\beta = \pm 1$ in the (α, β)-plane, and the two parameters measure the elastic dissimilarity of two materials in the sense that both vanish when the dissimilarity does. There are two other bimaterial parameters, Σ, the stiffness, and ε, the oscillatory index, relating to α and β, respectively, by

$$
\Sigma = \frac{c_2}{c_1} = \frac{1+\alpha}{1-\alpha}, \quad \varepsilon = \frac{1}{2\pi}\ln\frac{1-\beta}{1+\beta},
\tag{3}
$$

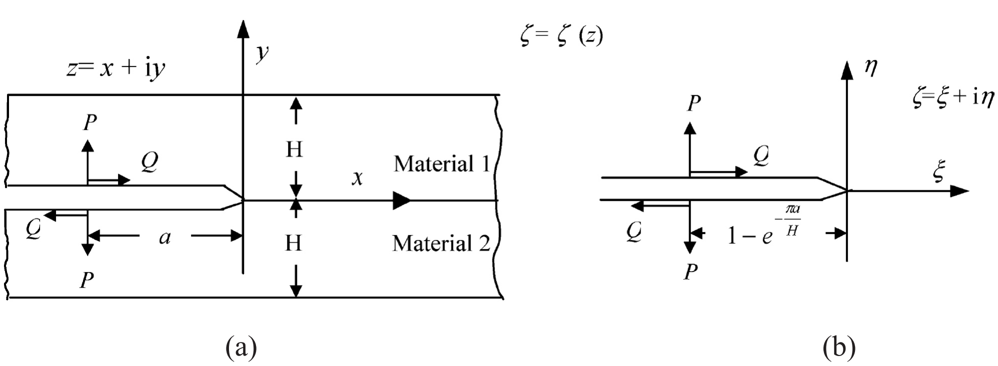

Figure 1a, b. A semi-infinite interfacial crack in a bimaterial system: **a)** two bonded dissimilar strips, **b)** two bonded dissimilar half-planes where $c_j = (\kappa_j + 1)/\mu_j$, $(j = 1, 2)$, is a measure of the compliance of a material. Thus, α can be further interpreted as a measure of the dissimilarity in the stiffness of the two materials, with material 1 stiffer than 2 as $\alpha > 0$, while material 1 is relatively compliant as $\alpha < 0$.

For an interfacial crack, the stress field at the crack tip bears a $r^{-1/2+i\varepsilon}$-type singularity. The oscillatory feature of elastic field near the crack tip occurs due to the nonzero value of ε. Generally, ε is very small. In this paper, since our purpose is to find the closed-form solution, no special attention is paid to such effects as crack faces contact due to the nonzero ε, which were discussed by Comninou, [17–19], Rice [6] and others. This phenomenon causes confusion about the definitions of the corresponding fracture parameters. By introducing an intrinsic material length scale l_0, Rice, [6], defined a complex SIF with the same unit as the classic SIF of cracks in homogenous materials, $\mathrm{K} = K_1 + iK_2$. The magnitude of K does not vary with different choices of l_0 while its phase angle may change. With the complex SIF, the tractions in the interface at distance r ahead of the crack tip are expressed as

$$\sigma_{yy} + i\sigma_{xy} = \frac{\mathbf{K}}{\sqrt{2\pi r}}\left(\frac{r}{l_0}\right)^{i\varepsilon} , \tag{4}$$

the relative crack-face displacements at distance r behind the crack tip are

$$\delta_y + i\delta_x = \frac{c_1 + c_2}{2\sqrt{2\pi}(1 + i\varepsilon)\cosh(\pi\varepsilon)}\mathbf{K}\sqrt{r}\left(\frac{r}{l_0}\right)^{i\varepsilon} , \tag{5}$$

and the energy release rate is evaluated as

$$G = \frac{c_1 + c_2}{16\cosh^2(\pi\varepsilon)}|\mathbf{K}|^2 . \tag{6}$$

Now let us consider the complex representation of the stress and displacement components using the complex potentials. The stresses and displacements for each homogenous strip under in-plane deformations may be expressed in terms of two Muskhelishvili's complex potentials $\phi_j(z)$ and $\psi_j(z)$, $j = 1, 2$, of the variable $z = x + iy$, [14]. This can be further simplified by introducing a pair of commonly used potentials, $\Phi(z)$ and $\Omega(z)$, defined as

$$\Phi(z) = \phi'(z), \quad \Omega(z) = [z\phi'(z) + \psi(z)]' . \tag{7}$$

Therefore, the stress and displacement components can be derived from the following relations:

$$\begin{aligned} &\sigma_{xx} + \sigma_{yy} = 2[\Phi(z) + \overline{\Phi(z)}], \\ &\sigma_{yy} + i\sigma_{xy} = \overline{\Phi(z)} + \Omega(z) + (\bar{z} - z)\Phi'(z), \\ &-2i\mu\frac{\partial}{\partial x}(u_y + iu_x) = \kappa\overline{\Phi(z)} - \Omega(z) - (\bar{z} - z)\Phi'(z) . \end{aligned} \tag{8}$$

Using the same notations as [15], let the potentials for the two strips as shown in Figure 1a be written as

$$\begin{aligned} \Phi(z) &= \begin{cases} \Phi^a(z), & z \text{ in strip 1}, \\ \Phi^b(z), & z \text{ in strip 2}, \end{cases} \\ \Omega(z) &= \begin{cases} \Omega^a(z), & z \text{ in strip 1}, \\ \Omega^b(z), & z \text{ in strip 2}, \end{cases} \end{aligned} \tag{9}$$

where the superscripts "a" and "b" indicate the potentials for the strip above and the strip below, respectively. The stress continuity of $\sigma_{yy} + i\sigma_{xy}$ across the interface requires

$$\begin{aligned} &\overline{\Phi^a(z)} = \Omega^b(z), \qquad z \text{ in strip 2}, \\ &\overline{\Phi^b(z)} = \Omega^a(z), \qquad z \text{ in strip 1}. \end{aligned} \tag{10}$$

Substitution of (10) into (8) yields the derivative of displacement jumps across the interface, or the components of the Burgers vector

$$-2\mathrm{i}\frac{\partial}{\partial x}(\delta_y+\mathrm{i}\delta_x)=\frac{c_1+c_2}{2}[(1-\beta)\Omega^b(x)-(1+\beta)\Omega^a(x)]\ . \tag{11}$$

Furthermore, with the displacement continuity across the bonded portions of the interface, we define a function $g(z)$, which is analytic in the whole bonded strips, except on the crack line, such that

$$\begin{aligned}&\Omega^a(z)=(1-\beta)g(z),\quad z \text{ in strip } 1,\\&\Omega^b(z)=(1+\beta)g(z),\quad z \text{ in strip } 2,\end{aligned} \tag{12}$$

By inserting (12) into (11), the Burgers vector can be expressed in terms of $g(z)$as

$$-2\mathrm{i}\frac{\partial}{\partial x}(\delta_y+\mathrm{i}\delta_x)=\frac{c_1+c_2}{2}(1-\beta^2)[g^-(x)-g^+(x)]\ , \tag{13}$$

and the traction on the interface is given by

$$\sigma_{yy}+\mathrm{i}\sigma_{xy}=(1+\beta)g^-(x)+(1-\beta)g^+(x)\ . \tag{14}$$

The prescribed traction on the crack surfaces leads to the following Riemann–Hilbert problem:

$$(1+\beta)g^-(x)+(1-\beta)g^+(x)=\sigma_{yy}+\mathrm{i}\sigma_{xy}\ .\quad (\text{on crack line } \Gamma) \tag{15}$$

Now we will try to solve this Riemann–Hilbert problem in the strip domain with a cut along the negative x-axis. Consider the conformal mapping

$$\varsigma=\mathrm{e}^{\frac{\pi}{H}z}-1\ , \tag{16}$$

which maps the cracked bimaterial strip as shown in Figure 1a onto two bonded half-planes ($\zeta=\xi+\mathrm{i}\eta$) with a semi-infinite cut along the negative ξ-axis as shown in Figure 1b. Substituting (16) into (15) yields a new Riemann–Hilbert problem in two bonded half-planes with a semi-infinite cut along the negative ξ-axis as

$$(1+\beta)G^-(\xi)+(1-\beta)G^+(\xi)=\sigma_{\eta\eta}+\mathrm{i}\sigma_{\xi\eta}\,,\qquad(-\infty<\xi<0)\ . \tag{17}$$

With the method given by Mushhelishvili, [14], we may obtain the homogenous solution of (17) in the ζ-plane as

$$\chi(\varsigma)=\varsigma^{-1/2+\mathrm{i}\varepsilon}\ . \tag{18}$$

Hence, in the (ξ, η) coordinate system, the general solution of (17) is expressed as

$$G(\varsigma)=\frac{1}{1-\beta}\frac{\chi(\varsigma)}{2\pi\mathrm{i}}\int_{\Gamma'}\frac{\sigma_{\eta\eta}(\xi)+\mathrm{i}\sigma_{\xi\eta}(\xi)}{\chi^+(\xi)(\xi-\varsigma)}\mathrm{d}\xi+\chi(\varsigma)P(\varsigma)\ , \tag{19}$$

where the integral should be taken over the union of the mapped crack and the lower and upper boundaries, and $P(\varsigma)$ is a polynomial with the form of

$$P(\varsigma)=C_0+C_1\varsigma \tag{20}$$

in which C_0 and C_1 are two unknowns to be determined.

Furthermore, noting that the mapping function (16) maps $x=\pm\infty$ onto $\xi=+\infty$ and -1, respectively, and stresses are bounded at $x=\pm\infty$, we can determine the unknowns C_0 and C_1 as

$$C_0 = -\frac{1}{1-\beta}\frac{1}{2\pi\mathrm{i}}\int\limits_{\Gamma'}\frac{\sigma_{\eta\eta}(\xi)+\mathrm{i}\sigma_{\xi\eta}(\xi)}{\chi^+(\xi)(\xi+1)}\mathrm{d}\xi, \quad C_1 = 0 \ . \tag{21}$$

Therefore, substitution of (16), (18), and (21) into (19) yields the solution of (15) in the strip domain as

$$g(z) = \frac{1}{1-\beta}\frac{\left(\mathrm{e}^{\frac{\pi}{H}z}-1\right)^{-\frac{1}{2}+\mathrm{i}\varepsilon}\mathrm{e}^{\frac{\pi}{H}z}}{2H\mathrm{i}}\int\limits_{-\infty}^{0}\frac{\sigma_{yy}(x)+\mathrm{i}\sigma_{xy}(x)}{\left(\mathrm{e}^{\frac{\pi}{H}x}-1\right)^{-\frac{1}{2}+\mathrm{i}\varepsilon}\left(\mathrm{e}^{\frac{\pi}{H}x}-\mathrm{e}^{\frac{\pi}{H}z}\right)}\mathrm{d}x \ . \tag{22}$$

Consequently, considering

$g^+(x) = g^-(x) = g(x)$

on the bonded portions of the interface and the self-equilibrated traction

$\mathbf{T}(x) = p(x) + \mathrm{i}q(x) = -(\sigma_{yy} + \mathrm{i}\sigma_{xy})$

acting at the crack surfaces, we get the SIF as

$$\mathbf{K} = \sqrt{2\pi}\lim_{x\to 0} 2x^{1/2-\mathrm{i}\varepsilon}(l_0)^{\mathrm{i}\varepsilon}g(x) = \sqrt{\frac{2}{H}}\cosh(\pi\varepsilon)\left(\frac{H}{\pi l_0}\right)^{-\mathrm{i}\varepsilon}\int\limits_{-\infty}^{0}\frac{\mathbf{T}(x)}{\left(1-\mathrm{e}^{\frac{\pi}{H}x}\right)^{\frac{1}{2}+\mathrm{i}\varepsilon}}\mathrm{d}x \ . \tag{23}$$

The corresponding ERR can be obtained by inserting (23) into (6).

3 Examples

Now let us consider two special loading cases on the crack surfaces: (1) a pair of self-equilibrated forces; (2) self-equilibrated uniform forces.

3.1 Single forces on the surfaces of a semi-infinite crack

Assume the crack is opened by a complex traction $\mathbf{T} = P + \mathrm{i}Q$ at $x = -a$ on each side of the crack. Its SIF can be obtained by substituting $\mathbf{T}(x) = (P + \mathrm{i}Q)\delta(x + a)$ into (23), in which $\delta(x)$ is the Dirac's delta-function, so

$$\mathbf{K} = \sqrt{\frac{2}{H}}\cosh(\pi\varepsilon)\left(\frac{H}{\pi l_0}\right)^{-\mathrm{i}\varepsilon}\frac{P+\mathrm{i}Q}{\left(1-\mathrm{e}^{-\frac{\pi a}{H}}\right)^{\frac{1}{2}+\mathrm{i}\varepsilon}} \ . \tag{24}$$

As a check, by letting $H \to \infty$ and using the L'Hospital's rule, result (24) may cover the SIF solution for a semi-infinite crack between two bonded dissimilar half-planes

$$\mathbf{K} = \sqrt{\frac{2}{\pi a}}\cosh(\pi\varepsilon)\left(\frac{a}{l_0}\right)^{-\mathrm{i}\varepsilon}(P+\mathrm{i}Q) \ , \tag{25}$$

in accordance with the solution given in [20].

On the other hand, if letting $\varepsilon \to 0$, result (24) returns to the SIF solution for a semi-infinite crack embedded in the mid-plane of a homogenous strip as

$$\mathbf{K} = \sqrt{\frac{2}{H}}\frac{P+\mathrm{i}Q}{\left(1-\mathrm{e}^{-\frac{\pi a}{H}}\right)^{\frac{1}{2}}} \ . \tag{26}$$

3.2 Uniformly distributed forces on the surfaces of a semi-infinite crack

Now suppose the crack is opened by a complex uniformly distributed traction $\mathbf{t} = p + \mathrm{i}q$ in the interval $x \in$ [–a, 0] on each side of the crack. The SIF can be obtained by substituting $\mathbf{T}(x) = p + \mathrm{i}q$ into (23) and integrating with respect to the variable x in the interval $[-a, 0]$

$$\mathbf{K} = \sqrt{\frac{2}{H}}\cosh(\pi\varepsilon)\left(\frac{H}{\pi l_0}\right)^{-\mathrm{i}\varepsilon}\int_0^a \frac{p+\mathrm{i}q}{(1-\mathrm{e}^{-\frac{\pi}{H}x})^{\frac{1}{2}+\mathrm{i}\varepsilon}}\mathrm{d}x$$

$$= \frac{\sqrt{2H}}{\pi}\cosh(\pi\varepsilon)\left(\frac{H}{\pi l_0}\right)^{-\mathrm{i}\varepsilon}\sum_{n=0}^{\infty}\frac{\left(1-\mathrm{e}^{-\frac{\pi a}{H}}\right)^{\left(n+\frac{1}{2}\right)-\mathrm{i}\varepsilon}}{\left(n+\frac{1}{2}\right)-\mathrm{i}\varepsilon}(p+\mathrm{i}q) \ . \tag{27}$$

Again, by letting $\varepsilon \to 0$, result (27) covers the solution for a semi-infinite crack in the mid-plane of a homogenous strip

$$\mathbf{K} = \sqrt{\frac{2}{H}}\int_0^a \frac{p+\mathrm{i}q}{(1-\mathrm{e}^{-\frac{\pi}{H}x})^{\frac{1}{2}}}\mathrm{d}x = \frac{\sqrt{2H}}{\pi}\ln\left[\frac{1+\sqrt{1-\mathrm{e}^{-\frac{\pi a}{H}}}}{1-\sqrt{1-\mathrm{e}^{-\frac{\pi a}{H}}}}\right](p+\mathrm{i}q) \ , \tag{28}$$

in agreement with the solution given in [21, 22] using the Muskhelishvili's potentials and the conformal mapping technique.

The ERR of each aforementioned case may be obtained by inserting (24)–(28) into (6).

4 Conclusion

Explicit solutions for a semi-infinite interfacial crack between two bonded dissimilar elastic strips depicted in Figure 1 have been obtained in this paper. The method based on the conformal mapping technique for the Riemann-Hilbert problems has been shown to be a powerful tool in solving some collinear interfacial cracks in strips. The closed-form solutions (24) and (27) can be employed as a useful theoretical base for the assessment of numerical analysis, especially for estimating the effect of the ratio a/H on the SIF and ERR in bonded bimaterial structures. The current procedure provides an example for further investigators to find explicit solutions for interfacial cracks or interfacial edge/wedge cracks in bonded dissimilar materials with finite dimensions.

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