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How to Graphically Interpret the Complex Roots of a Quadratic Equation

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As a secondary math teacher I have taught my students to find the roots of a quadratic equation in several ways. One of these ways is to graphically look at the quadratic and see were it crosses the *x*-axis. For example, the equation of $y = x^2 - x - 2$, as shown in Figure 1, has roots at $x = -1$ and $x = 2$. These are the two places in which the sketched graph crosses the *x*-axis.

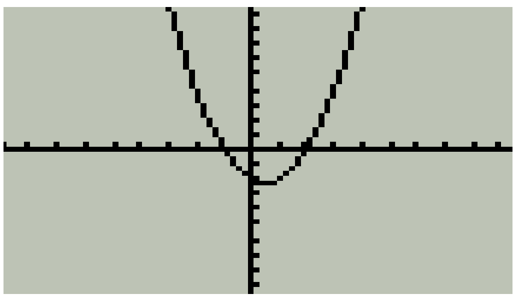

Figure 1
$y = x^2 - x - 2$

The process of uses the quadratic formula will always find the real roots of a quadratic equation. We could have also used the quadratic formula to find the roots of the this equation, $y = x^2 - x - 2$.

$$x = \frac{1 \pm \sqrt{9}}{2} = \frac{1 \pm 3}{2} = \frac{1}{2} \pm \frac{3}{2}$$

We can think of the first term (½) as a starting place for finding the two roots. Then we see that the roots are located 3/2 from the starting point in both directions.

This leads us to roots of a quadratic equation that does not cross the *x*-axis. These roots are known as complex (imaginary) roots. An example of a quadratic drawn on a coordinate plane with complex roots is shown in Figure 3. Notice that the vertex lies above the *x*-axis, and the end behavior on both sides of the graph is approaching positive infinity. The complex roots to can be found by using the quadratic formula, but it is beneficial to students to visualize a graphical connection.

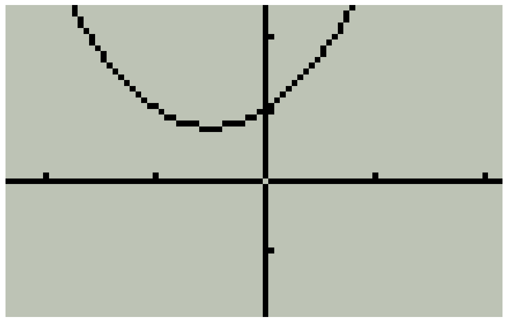

Figure 3
$y = x^2 + x + 1$

GRAPHICAL INTERPRETATION OF COMPLEX ROOTS

We know that any quadratic can be represented by $y = ax^2 + bx + c$. We also know the roots of quadratic equations can be derived from the well-known quadratic formula:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

If the roots are real we visually interpret them to cross the x-axis as shown in Figure 4a.

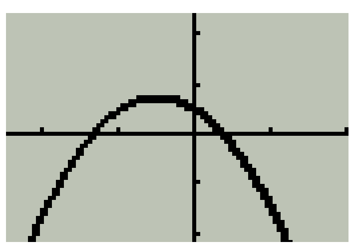

Figure 4a
$$y = -x^2 - x + \frac{1}{2}$$

But, we are interested in graphically interpreting the roots of a graph that does not cross the x-axis, as in Figure 4b.

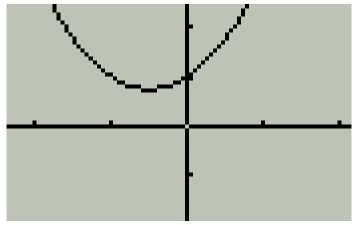

Figure 4b
$y = x^2 + x + 1$

Let's use what we graphically know about quadratics with real roots (Fig. 4a) to explain what we don't know graphically about quadratics with complex roots (Fig. 4b). Now there are infinitely many quadratics with real roots and infinitely many quadratics with complex roots. But, when comparing one to another, it would be helpful if the two quadratics were related in some way.

If $y = ax^2 + bx + c$ produces real roots (bold line in Figure 4c), a reflection of this graph upward would yield a new quadratic equation that would produce complex roots. (See Figure 4c)

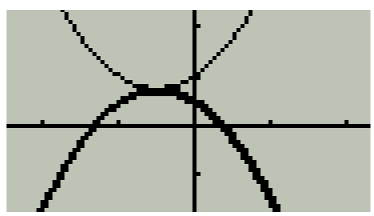

Figure 4c

To see how this is configured analytically, we will start with the general equation of a quadratic. (Remember: The quadratic that we are starting with (bold line) is known to have real roots.) First we complete the square. Then to create the flipped quadratic (which is a different equation), a negative is applied to *a*. Then I simplified the equation by multiplying.

$$y = ax^2 + bx + c$$

--------------Complete the square----------------

$$y = a(x^2 + \frac{b}{a}x + \_\_\_\_) + c$$

$$\left(\frac{b}{a} \bullet \frac{1}{2}\right)^2$$

$$y = a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) + c - \frac{b^2}{4a}$$

$$y = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

------------------Completed the square---------------

WE ARE CREATING A NEW QUADRATIC AT THIS POINT.

-------------------Reflect the Quadratic-------------

$$y = -a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

-------------------Reflected the Quadratic-------------

----------------------------Simplify------------------------

$$y = -a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) + c - \frac{b^2}{4a}$$

$$y = -ax^2 - \frac{ab}{a}x - \frac{ab^2}{4a^2} + c - \frac{b^2}{4a}$$

$$y = -ax^2 - bx - \frac{b^2}{4a} + c - \frac{b^2}{4a}$$

$$y = -ax^2 - bx - \frac{2b^2}{4a} + c$$

$$y = -ax^2 - bx - \frac{b^2}{2a} + c$$

---------------------------Simplified------------------------

Now to be able to compare these complex roots to the real roots that we started with, plug the coefficients of the equation above into the quadratic formula.

$$x = \frac{b}{-2a} \pm \frac{\sqrt{b^2 - 4(-a)\left(c - \frac{b^2}{2a}\right)}}{-2a}$$

-------------------------Simplify----------------------

$$x = \frac{b}{-2a} \pm \frac{\sqrt{b^2 + 4a\left(c - \frac{b^2}{2a}\right)}}{-2a}$$

$$x = \frac{b}{-2a} \pm \frac{\sqrt{b^2 + 4ac - \frac{4ab^2}{2a}}}{-2a}$$

$$x = \frac{b}{-2a} \pm \frac{\sqrt{b^2 + 4ac - 2b^2}}{-2a}$$

$$x = \frac{b}{-2a} \pm \frac{\sqrt{4ac - b^2}}{-2a}$$

Because we are adding and subtracting from $\frac{b}{-2a}$ it is unnecessary to write the negative in the second denominator.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a}$$

------------------------Simplified----------------------

Since we know that we the roots are complex, we can show that by extracting a negative out of the radical.

Complex Roots of the Flipped Quadratic

$$\boldsymbol{x} = -\frac{b}{2a} \pm \frac{i\sqrt{b^2 - 4ac}}{2a}$$

Real Roots of the Quadratic We Started With

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Notice that the real roots and the complex roots of different quadratic equations yielded very similar answers. They are actually the same except for the *i* in the complex roots.

The next step is to use figure out how to use these similarities to find the complex roots graphically. First, let's review how to graph the complex number plane. Horizontal movement on the graph denotes the real part of the complex number, while vertical movement represents the imaginary part of the complex number. (See Figure 5)

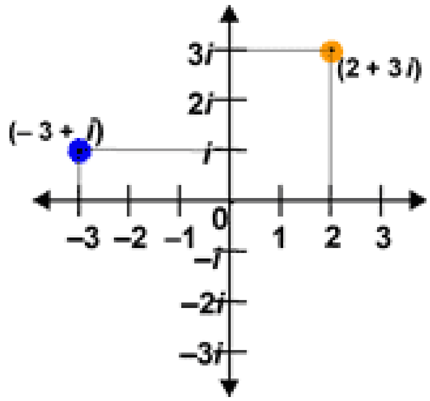

Figure 5

Now, working in three-space, imagine that these complex coordinate plane is the "floor". It is represented by the (i, x) coordinate plane in Figure 6.

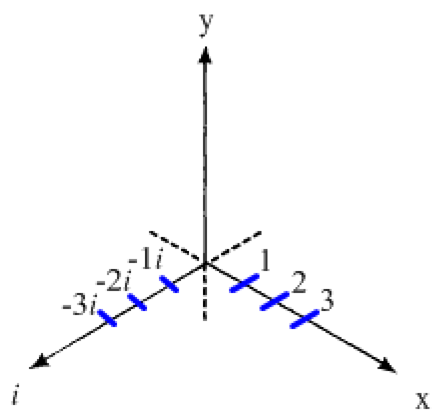

Figure 6

If we now use the (x,y) coordinate plane to draw a quadratic with complex roots we could get something that looks like Figure 7. Notice the quadratic does not cross the x-axis.

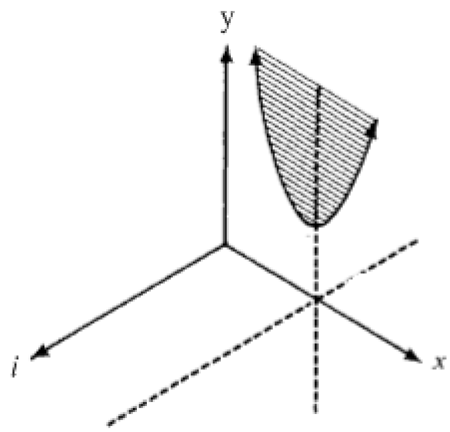

Figure 7

We currently can not see the complex roots graphically. But if we flip the quadratic horizontally over the vertex, from the proof about we should get roots that differ only by a number *i*. In order to graphically see the complex roots we need to rotate the reflected image 90 degrees to place the quadratic into the complex number plane.

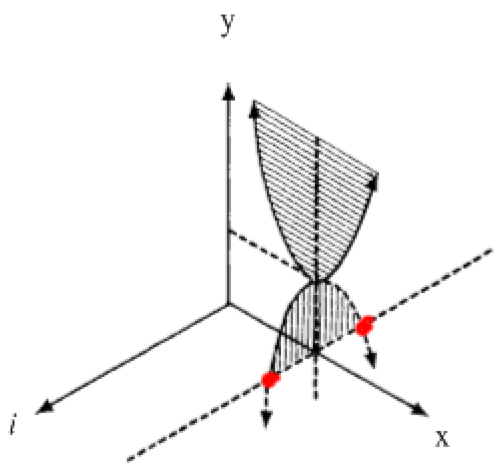

Figure 8

Notice that the two points indicated can be found by starting at x units in the real direction and *i* units in both the positive and negative direction. We are then able to graphically see the complex roots of a quadratic equation.

References

Weeks, Audrey. *Connecting Complex Roots to a Parabola's Graph.* June 20, 2007. http://my.nctm.org/eresources/view_article.asp?article_id=6168

Vest, Floyd. *The College Mathematics Journal*, Vol. 16, No. 4. (Sep., 1985), pp. 257-261.

GeoGebra demonstration created by Carmen Melliger at http://www.unl.edu/tcweb/fowler/natm2007/melliger.ggb