1975

Partial Derivatives

PARTIAL DERIVATIVES

Can you evaluate partial derivatives of functions of more than one variable? Try the following self-check test. If you get all the answers correct you should be able to handle the material in this course involving partial derivatives. If not, read the material that follows the test.

SELF-CHECK TEST

1. Suppose y is a function of the independent variables x and t:

$$y(x, t) = x^2 + t + axt^2,$$

where a is a constant. Determine the expression for

(a) $\partial y/\partial x$;

(b) $\partial y/\partial t$;

(c) $\partial^2 y/\partial t^2$.

(d) If a = 3, evaluate y at (x, t) = (2, 5).

(e) If a = 3, evaluate $\partial y/\partial t$ at (x, t) = (4, 2).

2. Try another function y of the independent variables x and t:

$$y = A \sin(\omega t - kx).$$

If the constants are $A = 3$, $\omega = 2$, $k = \pi$, evaluate:

(a) y at (x, t) = (3/2, 3π/2);

(b) $\partial y/\partial t$ at (x, t) = (-5/2, 3π/4);

(c) $\partial^2 y/\partial t^2$ at (x, t) = (1/2, -π/8).

Answers are at the bottom of this page. If you did not get them right, continue reading in this review module.

At the end of this review we shall provide a more rigorous definition of a partial derivative, but go ahead and read straight through to get a feeling for what you need to do from an operational standpoint.

Self-Check Test Answers

1. (a) $2x + at^2$; (b) $1 + 2axt$; (c) $2ax$. (d) 159; (e) 49.

2. (a) -3; (b) 6; (c) -62.

If we have a function of more than one independent variable, than we can define a partial derivative with respect to one of the variables, which is simply the derivative of the function with all the other variables fixed. The notation using ∂, which we will use below, tells you it is a partial derivative. For example, suppose we have the function y that depends on the independent variables x and t:

$$y = A \sin(kx - \omega t),$$

where A, k, and ω are constants. Then, "the partial derivative of y with respect to x" is denoted by $\partial y/\partial x$ and is found by setting t constant and differentiating with respect to x:

$$\partial y/\partial x = kA \cos(kx - \omega t).$$

Similarly, "the partial derivative of y with respect to t" is denoted by $\partial y/\partial t$ and is found by setting x constant and differentiating with respect to t:

$$\partial y/\partial t = -\omega A \cos(kx - \omega t).$$

Note that the value of either of the partial derivatives depends on both independent variables x and t as well as the constants A, k, and ω.

Here are some Exercises to practice on (answers at bottom of page):

Exercises

1. If $y(x, t) = x^2 + 4x^3t + 5t^4$, determine the expression for the following partial derivatives and their values at (x, t) = (4, 3):

   (a) $\partial y/\partial x$;

   (b) $\partial y/\partial t$;

   (c) $\partial^2 y/\partial t^2 \equiv (\partial/\partial t)(\partial y/\partial t)$ (i.e., perform the partial derivative with respect to t two times in succession).

2. If $y(x, t) = A \cos[k(x - ct) - \phi]$, determine the expression for the following partial derivatives.

   (a) $\partial y/\partial x$;

   (b) $\partial y/\partial t$;

   (c) $\partial^2 y/\partial x\, \partial t \equiv (\partial/\partial x)(\partial y/\partial t)$ (i.e., perform the partial derivative with respect to t followed by the partial derivative with respect to x).

Exercise Answers

1. (a) $2x + 12x^2t$. This is the result of setting t constant and performing the derivation with x as the variable. We evaluate this expression at (x, t) = (4, 3) to get $(2 \times 4) + (12 \times 16 \times 3) = 584$.

   (b) $4x^3 + 20t^3$; 976. (c) $60t^2$; 540.

2. (a) $-kA \sin[k(x - ct) - \phi]$. (b) $kcA \sin[k(x - ct) - \phi]$.

   (c) $k^2cA \cos[k(x - ct) - \phi]$.

As promised, here is a more formal definition of a partial derivative of a function of more than one independent variable:

Suppose that we have a function y that depends on the independent variables x_1, x_2, x_3, etc. We can write it as $y(x_1, x_2, x_3, \ldots)$. The "partial derivative of y with respect to x_1" is then denoted by $\partial y/\partial x_1$ and is defined by the expression

$$\frac{\partial y}{\partial x_1} \equiv \lim_{\Delta x \to 0} \frac{y(x_1 + \Delta x, x_2, x_3, \ldots) - y(x_1, x_2, x_3, \ldots)}{\Delta x}.$$

Similarly,

$$\frac{\partial y}{\partial x_2} \equiv \lim_{\Delta x \to 0} \frac{y(x_1, x_2 + \Delta x, x_3, \ldots) - y(x_1, x_2, x_3, \ldots)}{\Delta x}.$$

As you can see, this is the same sort of limit used to define the derivative of a function of only one variable, the difference being that the function and its partial derivatives are functions of more than one variable. You should consult a calculus textbook for more details.