University of Nebraska - Lincoln
DigitalCommons@University of Nebraska - Lincoln

Dissertations, Theses, and Student Research Papers in Mathematics
Mathematics, Department of

8-2012

An Analysis of Nonlocal Boundary Value Problems of Fractional and Integer Order

Christopher Steven Goodrich
University of Nebraska-Lincoln, s-cgoodri4@math.unl.edu

Follow this and additional works at: https://digitalcommons.unl.edu/mathstudent

Part of the Analysis Commons, and the Science and Mathematics Education Commons

Goodrich, Christopher Steven, "An Analysis of Nonlocal Boundary Value Problems of Fractional and Integer Order" (2012). *Dissertations, Theses, and Student Research Papers in Mathematics*. 35.
https://digitalcommons.unl.edu/mathstudent/35

This Article is brought to you for free and open access by the Mathematics, Department of at DigitalCommons@University of Nebraska - Lincoln. It has been accepted for inclusion in Dissertations, Theses, and Student Research Papers in Mathematics by an authorized administrator of DigitalCommons@University of Nebraska - Lincoln.

AN ANALYSIS OF NONLOCAL BOUNDARY VALUE PROBLEMS OF FRACTIONAL AND INTEGER ORDER

by

Christopher S. Goodrich

A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professors Lynn Erbe and Allan Peterson

Lincoln, Nebraska

August, 2012

AN ANALYSIS OF NONLOCAL BOUNDARY VALUE PROBLEMS OF

FRACTIONAL AND INTEGER ORDER

Christopher S. Goodrich, Ph. D.

University of Nebraska, 2012

Adviser: Lynn Erbe and Allan Peterson

In this work we provide an analysis of both fractional- and integer-order boundary value problems, certain of which contain explicit nonlocal terms. In the discrete fractional case we consider several different types of boundary value problems including the well-known right-focal problem. Attendant to our analysis of discrete fractional boundary value problems, we also provide an analysis of the continuity properties of solutions to discrete fractional initial value problems. Finally, we conclude by providing new techniques for analyzing integer-order nonlocal boundary value problems.

DEDICATION

This work is dedicated to

my late father and kindred spirit, Dr. Paul D. Goodrich, MD, a mathematician in soul and heart, if not by profession, and from whom I first glimpsed the beauty that is mathematics

and to

Maddie, for everything.

ACKNOWLEDGMENTS

Firstly, I would like to thank my advisors Drs. Lynn Erbe and Allan Peterson. One of the greatest gifts a Ph.D. advisor can give to an advisee is an interesting and tractable problem that leads to many other worthwhile results. In this sense, my advisors' decision to introduce me to the discrete fractional calculus, and their encouragement that I pursue its investigation as part of my dissertation project, was a great gift, indeed. While my mathematical interests continue to mature and evolve, their suggestion proved to be most worthwhile and has led me to a variety of interesting results, some of which are contained herein. I entered the Ph.D. program with the supposedly immutable notion that I would write the dissertation in mathematical biology, thus combining my twin interests of differential equations and biology. Thus, it is no small complement to Drs. Erbe and Peterson that through their outstanding and exciting teaching and research they stole me away from that particular area! Succinctly, I will be forever grateful for their kind advice and generous counsel.

Secondly, I would like to single out Dr. Mikil Foss for particular acknowledgement. While not my advisor proper, Mikil served as a non-reading member of my supervisory committee. However, his influence on me has gone far beyond this rather unceremonious role. Indeed, I really have come to think of Mikil as a sort-of "surrogate advisor," who has allowed my research program to develop beyond the confines of the dissertation research contained herein. He graciously allowed me to work on research with him, research that broadly described involves regularity theory for nonlinear elliptic systems of PDEs. The techniques therein are vastly different from those contained herein, and so, this represented a true commitment on Mikil's part to ensure that I could begin developing some measure of research independence. I find the techniques of regularity theory very beautiful and exciting, and I owe much of this to

Mikil.

Thirdly, I would also like to thank the department of mathematics of the University of Nebraska-Lincoln. While my choice to attend UNL was somewhat entirely dictated by family circumstance, my initial impression was that I had made a wise decision, nonetheless. Now on the precipice of completing the Ph.D. program, I am confident that indeed this was a very wise decision. The department has been very supportive of my activities as a student, and for this I will be forever grateful. In addition to Drs. Erbe, Foss, and Peterson, whom I have already acknowledged, I would like to specifically mention Drs. Mark Brittenham, Bo Deng, Allan Donsig, David Logan, David Pitts, and Mark Walker, each of whom taught me at least one course in the department and each of whom displayed outstanding and noteworthy commitment to his instructional responsibilities. In addition, I would also like to thank separately the members of my supervisory committee for graciously agreeing to serve on my committee: Drs. Kevin Cole, Allan Donsig, Lynn Erbe, Mikil Foss, Allan Peterson, and Mohammed Rammaha. I will be forever grateful to each of the aforementioned individuals for making my five years at UNL among the very happiest of my life.

Fourthly, I would like to acknowledge the important role that my family has played in my graduate school days. My mother, Barbara, my younger brothers, Nathaniel and Benjamin, and my sister-in-law Stephanie have been a source of encouragement for the duration of the Ph.D. program. It is impossible for me to express precisely the degree to which their support has buoyed me during the rigors of the program, but I believe they are well aware of my appreciation to them for said support. Furthermore, while I have not often been able to explain to them on what it is I have been working (a sad if albeit common consequence of doing abstract mathematics), I do appreciate their frequent inquiries into my mathematical activities. I hope they realize that my occasional reticence in answering had nothing to do with disinterest in their queries

but rather a sort of *de facto* dysphasia secondary to confusion at how to explain, for example, the concept of $\mathcal{C}^{0,\alpha}$ Hölder continuity or L^p theory to a non-mathematician!

Fifthly and finally, and certainly not least of all, I acknowledge the influence of my late father, Dr. Paul Goodrich. While it brings me great and palpable sadness that he did not live to see me enter the Ph.D. program let alone complete it, I would like to think that he would be proud of both my performance at UNL and my choice to pursue the doctorate in mathematics. One of the last conversations I had with him prior to his untimely death was regarding whether I should go to medical school and pursue an M.D. or go to graduate school and pursue a Ph.D. in mathematics. And so I am grateful that he was at least aware of my interest in earning a doctorate of some sort even if he did not live to see the decision made and brought to fruition. In any case, his influence on me is considerable, even though he died many years ago. I will be forever grateful for the example of intellectual curiosity, dedication, and precision, which he gave to me.

Contents

Chapter 1

Introduction

The fractional calculus has a long and storied history within the broader mathematical discipline of analysis. Indeed, research into this area was initiated in 1695 when L'Hôpital asked Leibniz about the meaning of a one-half derivative. While precise mathematical investigation of this and related concepts would not be realized for almost two centuries later, this simple question laid the initial foundation for the area. At first, it seems, the questions regarding fractional derivatives and integrals were largely academic, being as they were divorced effectively from any applicative interest. Later, however, as the fractional calculus matured, it become clear that the fractional calculus could be used effectively in a variety of modeling situations. In fact, nowadays, various tools from the fractional calculus are even used in the study of regularity of minimizers of functionals and of weak solutions to PDEs. Ostensibly, Leibniz could not possibly have envisioned the very bright and important future for the fractional calculus.

While we shall state in more detail certain of the fundamental properties of the fractional calculus in Chapter 2, let us straightaway state the definition of the fractional derivative of Riemann-Liouville type and the discrete fractional sum and dif-

ference. They are as follows.

Definition 1.1. With $\nu > 0$ and $\nu \in \mathbb{R}$, we define the ν-th Riemann-Liouville fractional derivative to be

$$D_a^\nu y(t) := \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_a^t \frac{y(s)}{(t-s)^{\nu+1-n}}\, ds,$$

where $n \in \mathbb{N}$ is the unique positive integer satisfying $n-1 \leq \nu < n$ and $t > a$.

Definition 1.2. The ν-th fractional sum of a function f, for $\nu > 0$, is $\Delta^{-\nu} f(t) = \Delta^{-\nu} f(t;a) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s)$, for $t \in \{a+\nu, a+\nu+1, \dots\} =: \mathbb{N}_{a+\nu}$. We also define the ν-th fractional difference for $\nu > 0$ by $\Delta^\nu f(t) := \Delta^N \Delta^{\nu-N} f(t)$, where $t \in \mathbb{N}_{a+\nu}$ and $N \in \mathbb{N}$ is chosen so that $0 \leq N-1 < \nu \leq N$.

We present these definitions now to highlight an interesting feature of the aformentioned operators, a feature that will, in fact, providing the unifying theme to this work. Indeed, note that each of the operators contains a *de facto* nonlocality. That is to say, the definition of the fractional derivative is not a pointwise calculation but rather involves values of y on the interval $[a,t]$. If we recall that

$$f'(t_0) := \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} \tag{1.1}$$

for a suitably differentiable function f, then we see at once that there is a considerable difference between the definitions. Whereas in (1.1) the value of $f'(t_0)$ is influenced by the behavior of f on an arbitrarily small interval about t_0 (at least heuristically), the fractional derivative is influenced by the behavior of f on the entire interval $[a,t]$. In this way, then, fractional derivatives have a sort-of memory property that is effectively ***non***local. In part, this makes the fractional derivative very useful in modeling situations wherein a memory-type effect is required. On the other hand,

this makes the mathematical theory of the fractional calculus, at times, far more complex than its integer-order counterpart.

A similar effect may be seen with the ***discrete*** fractional difference. Indeed, the integer-order fractional difference is quite straightforward, being as it is merely

$$\Delta f(t) := f(t+1) - f(t). \tag{1.2}$$

But from Definition 1.2 we see that in the fractional-order case we have a much more complicated operation. Indeed, for the sake of argument, suppose that we fix $1 < \nu < 2$. Then Definition 1.2 implies that

$$\begin{aligned}\Delta^\nu f(t) = \Delta^2\Delta^{2-\nu} f(t) &= \Delta^2\left[\frac{1}{\Gamma(\nu-2)}\sum_{s=a}^{t-\nu+2}(t-s-1)^{\underline{\nu-3}}f(s)\right]\\ &= \Delta\left[\Delta\left[\frac{1}{\Gamma(\nu-2)}\sum_{s=a}^{t-\nu+2}(t-s-1)^{\underline{\nu-3}}f(s)\right]\right]\end{aligned} \tag{1.3}$$

Once again, (1.3) is ***non***local in the sense that it involves the values of f not only at $t+1$ and t as in in (1.2), but also the values of f at $t-1$, $t-2$, $\ldots$, a. As in the continuous fractional calculus and as we shall see forthwith in Chapter 3, this makes the analysis of discrete fractional problems much more complicated than the analysis of their integer-order counterparts. This is so much so, in fact, that well developed theories in the discrete integer-order setting such as oscillation theory, have at present no known counterpart in the fractional setting due to the substantial mathematical complications encountered.

Inasmuch as the fractional calculus is concerned, there are myriad papers in the literature dealing with the existence of one or more positive solutions to fractional-order boundary values problems (FBVPs). The search for positive solutions holds a special place in the theory of BVPs due to the fact that in certain applications,

only positive (or nonnegative) solutions hold any physical meaning. As such, it is rather an industry, so to speak, in mathematics to determine conditions under which a given boundary value problem will possess at least one positive solution. As we proceed through the succeeding chapters of this work, we shall discuss the various contributions that have been made recently in this general area and just how said contributions are related to the results we present herein. Nonetheless, let us briefly mention just a few contributions so as to preliminarily contextualize our discussion thus far.

Insofar as the discrete fractional calculus is concerned, the main body of results have been presented by Atici and Eloe. In particular, in a series of papers [15, 16, 17, 18, 20] Atici and Eloe have worked out some of the basic operational properties of the discrete fractional calculus as well as applying these properties to certain boundary value problems. A paper by Atici and Şengül [19] is interesting for its development of the rudiments of the discrete fractional calculus of variations and its application to tumor growth. The treatment of discrete FBVPs has been discussed extensively by Goodrich [41, 42, 44, 45, 46, 47, 48, 51, 54, 53, 59]. Furthermore, Holm [63] has provided some additional operational properties, dual to the one's given earlier by Atici and Eloe. Other results have been given by Bastos, et al. [24, 25, 26] and Ferreira [40], and these are interesting for their attempt to generalize the fractional calculus away from the specific time scale $\mathbb{Z}$ (i.e., the difference equations case) to a completely arbitrary time scale $\mathbb{T}$; see the excellent textbook by Bohner and Peterson [29] for an introduction to the concept of the calculus on a time scale. While this endeavor is still in its infancy and while there seem to be some significant technical obstacles to bringing this endeavor to any sort of meaningful fruition, it is, nonetheless, a fascinating avenue for additional study, though one we do not address any further in this work. Conversely, insofar as the continuous fractional calculus is concerned, there

are so many works on boundary value problems that it would be impossible to cite even a modest fraction of the interesting and useful papers that have appeared recently on the subject. Indeed, the subject having been around much longer than the discrete fractional calculus, there is correspondingly a vastly greater literature available. So, we merely mention that [1, 3, 4, 13, 21, 22, 23, 27, 28, 33, 34, 38, 43, 50, 72, 74, 75, 76, 83, 93, 94, 95, 96, 97] are representative papers in this area, and, collectively, these cover the entire range of continuous fractional differential equations applications such as boundary value problems, calculus of variations, and fractional partial derivatives. In Chapters 7 and 8, which discuss results for continuous fractional boundary value problems, we shall discuss in more detail certain recent contributions to this area. We should also remark that the monographs by Oldham and Spanier [77], Podlubny [78], and Schuster [79] are excellent introductions to the theory and application of the fractional calculus; in particular, Podlubny's monograph is especially recommended.

Thus far, then, we have seen that the fractional calculus involves, at least implicitly, the notion of nonlocalities. But, in fact, the concept of nonlocalities has recently seen substantial investigation in the integer-order setting of boundary value problems. A good model problem for this strand of research is

$$\begin{aligned} -y''(t) &= f(t, y(t)),\ 0 < t < 1 \\ y(0) &= \varphi(y) \\ y(1) &= 0. \end{aligned} \tag{1.4}$$

In problem (1.4), $\varphi(y)$ is a functional, which captures the nonlocal nature of the boundary condition at $t = 0$. In particular, in most of our work (e.g., [55, 57, 58]), following the lead of Infante, Webb, Yang, and other mathematicians who have produced work on nonlocal BVPs (see, for example, [84, 85, 90]), we realize φ as a

Lebesgue-Stieltjes integral of the form

$$\varphi(y) := \int_{[0,1]} y(s)\ d\alpha(s), \tag{1.5}$$

where the measure associated to the integrator, say μ_α, may be *signed*. This leads to the interesting and nontrivial question of whether problem (1.4) may have at least one *positive* solution under such assumptions.

In fact, there has been substantial interest in such explicitly nonlocal problems lately. Once again, we shall comment more thoroughly on these contributions later in the work (cf., Chapters 9, 10, and 11), but let us just mention some of these briefly now. Principally, Infante and Webb have been instrumental in providing significant and new ideas in the study of nonlocal BVPs with linear boundary conditions – see [84, 85, 86, 87]. A paper by Graef and Webb [60] also provides some new ideas in this area. Papers by Kang and Wei [68] as well as by Yang [90, 91] provide some complementary results. More generally, nonlocal and multipoint-type BVPs have received substantial attention in the time scales setting – see, for instance, [9, 10, 11, 49, 52] and the references therein.

One can further complicate matters by supposing that φ is composed with another, possibly nonlinear function, say H. In this case, problem (1.4) becomes

$$\begin{aligned} -y''(t) &= f(t, y(t)),\ 0 < t < 1 \\ y(0) &= H(\varphi(y)) \\ y(1) &= 0. \end{aligned} \tag{1.6}$$

We then say that problem (1.6) is a BVP with nonlocal, nonlinear boundary conditions. The inclusion of the nonlinearity as well as the nonlocality further complicates the analysis of (1.6). Some recent works considering problems of this general sort

include [64, 65, 66, 67]. Of course, there are other ways in which to introduce the nonlinearity of the boundary condition. For instance, one could consider the boundary condition, say,

$$y(0) = \int_{\tau_1}^{\tau_2} F(s, y(s))\ ds, \tag{1.7}$$

for $0 \leq \tau_1 < \tau_2 \leq 1$. By then imposing certain restrictions on the integrand F, one can gain sufficient control so as to deduce the existence of at least one positive solution to the associated BVP. In any case, in this work we will only concern ourselves with the realization of the boundary condition given in (1.6).

So, in problems (1.4) and (1.6) we have ***explicit*** nonlocalities, which is in contrast to the implicit nonlocalities present in fractional derivatives and differences. Not dissimilarly, though, one is confronted with the task of modifying existing techniques in order to circumvent the difficulties caused by the presence of nonlocal terms. And, as already mentioned, if said nonlocal terms can be nonpositive, then the existence of positive solutions to the associated BVP is unclear, and this enhances the mathematical interest of the problem.

In fact, there are a great many ways to circumvent said problems. But let us focus on just one for now since it will feature prominently in Chapters 9, 10, and 11 in the sequel and since it is one of the more original ideas we present herein. In the few existing works that consider problems similar to (1.6), it seems to be a near universal assumption that H satisfy growth conditions of the sort

$$\eta_1 y \leq H(y) \leq \eta_2 y, \tag{1.8}$$

for all $y \geq 0$ and some constants $0 \leq \eta_1 < \eta_2 < +\infty$. Effectively, this means that the graph of H is bounded between the lines, say, $z = \eta_1 y$ and $z = \eta_2 y$. While not horrendously restrictive, it actually turns out that this restriction can be replaced by

a restriction that need only hold at $+\infty$. In other words, we can require of H only an asymptotic growth condition. In particular, as will be clarified later, a condition of the form

$$\lim_{y\to+\infty}|H(y)-\kappa y|=0. \tag{1.9}$$

suffices; later we will generalize this condition further, but, for the moment, this will suffice. In terms of problem (1.6), this seems to suggest something interesting. Namely, that if the boundary condition $y(0)=H(\varphi(y))$ "looks like" $y(0)=\varphi(y)$ for y very large (in a sense to made precise later), then the boundary value problem has at least one positive solution. Put differently, if problem (1.6) "looks like" problem (1.4) for y large in norm, then we can use the ideas applied to problem (1.4) to deduce the existence of at least one positive solution to problem (1.6). We discuss this idea much more extensively and thoroughly in Chapters 9, 10, and 11.

So, as can be seen from the preceding discussion the unifying theme of this work is the concept of nonlocalities and their influence in the study of boundary value problems arising in both the continuous and the discrete calculus. In the fractional problems we study herein, the influence of the nonlocalities is more subtle, affecting mostly the technical details of the proofs of our lemmas and theorems. Conversely, in the problems we study wherein the boundary conditions contain explicit nonlocal terms, the nonlocalities, unsurprisingly, have a more pervasive effect on the analysis of the problem.

Having provided now a very general outline of the ideas we consider in this work, we outline the specific plan of this work. As suggested above, the arc of our results can be summarized as follows. We first familiarize the reader with the fundamental definitions in the continuous and discrete fractional calculus. This provides a framework for the discussion that follows. Our first block of original research is then

focused on the discrete fractional calculus. Our results explicitly illustrate just how the nonlocal aspect of the fractional difference makes analyzing discrete fractional BVPs rather delicate. After presenting several results in this area, we make a subtle shift by presenting a collection of results for continuous fractional BVPs. Many of the ideas are very similar, but it is instructive to see the differences and similarities between the discrete fractional calculus and continuous fractional calculus, for, perhaps surprisingly, the discrete fractional calculus can sometimes be far more difficult to use than its continuous counterpart. Finally, having pivoted to the continuous case, we conclude this work by coming full circle and presenting some results for explicit nonlocal BVPs in the continuous case. This, then, completes the theme of nonlocality, which really is the unifying glue holding all of these seemingly disparate results together.

Chapter 2

Preliminaries

In this section we wish to collect certain results, which we shall use frequently in the sequel. In particular, we collect here the definitions of the fractional operators, which we shall use later. Moreover, we collect certain other preliminary results, such as relevant fixed point theorems, which will be of use to us in the sequel as well. The proofs of these various lemmas may be found, for instance, in certain of the recent works by Atici and Eloe [15, 16, 17, 20].

Definition 2.1. We define $t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$, for any t and ν for which the right-hand side is defined. We also appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$.

Definition 2.2. The ν-th fractional sum of a function f, for $\nu > 0$, is $\Delta^{-\nu} f(t) = \Delta^{-\nu} f(t;a) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s)$, for $t \in \{a+\nu, a+\nu+1, \dots\} =: \mathbb{N}_{a+\nu}$. We also define the ν-th fractional difference for $\nu > 0$ by $\Delta^{\nu} f(t) := \Delta^{N} \Delta^{\nu-N} f(t)$, where $t \in \mathbb{N}_{a+\nu}$ and $N \in \mathbb{N}$ is chosen so that $0 \le N-1 < \nu \le N$.

Lemma 2.3. *Let t and ν be any numbers for which $t^{\underline{\nu}}$ and $t^{\underline{\nu-1}}$ are defined. Then $\Delta t^{\underline{\nu}} = \nu t^{\underline{\nu-1}}$.*

Lemma 2.4. *Let $0 \leq N-1 < \nu \leq N$. Then $\Delta^{-\nu}\Delta^{\nu}y(t) = y(t) + C_1 t^{\underline{\nu-1}} + C_2 t^{\underline{\nu-2}} + \ldots + C_N t^{\underline{\nu-N}}$, for some $C_i \in \mathbb{R}$, with $1 \leq i \leq N$.*

Lemma 2.5. *Let f be a real-value function defined on $\mathbb{N}_a$ and let μ, $\nu > 0$. Then*

$$\Delta_{a+\mu}^{-\nu}\left[\Delta_a^{-\mu} f(t)\right] = \Delta_a^{-(\mu+\nu)} f(t) = \Delta_{a+\nu}^{-\mu}\left[\Delta_a^{-\nu} f(t)\right].$$

Lemma 2.6. *Let $0 \leq m-1 < \nu \leq m$, where m denotes a positive integer and $y(t)$ be defined on $\mathbb{N}_{\nu-m} := \{\nu-m, \nu-m+1, \ldots\}$. Then*

$$\Delta_{\nu-m}^{\nu} y(t) = \nabla_{\nu-m}^{\nu} y(t+\nu),$$

for $t \in \mathbb{Z}_{-m}$. Note that here we use the definition

$$\nabla^{-\nu} f(t) := \sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} f(s),$$

where $\nu \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$ – see [18].

Lemma 2.7. *Let $a \in \mathbb{R}$, $\mu \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, $\nu > 0$, and $(t-a)^{\underline{\mu}} : \mathbb{N}_{a+\mu} \to \mathbb{R}$. Then:*

1. $\Delta_{a+\mu}^{-\nu}(t-a)^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\nu)}(t-a)^{\underline{\mu+\nu}}$, *for $t \in \mathbb{N}_{a+\mu+\nu}$; and*
2. $\Delta_{a+\mu}^{\nu}(t-a)^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\nu)}(t-a)^{\underline{\mu-\nu}}$, *for $t \in \mathbb{N}_{a+\mu+N-\nu}$,*

where $N \in \mathbb{N}$ is the unique positive integer satisfying $N-1 < \nu \leq N$.

Remark 2.8. We remind the reader that by the notation $\Delta_{\nu-m}^{\nu} y(t)$, for instance, as occurs in the statement of Lemma 2.6 above, the subscript $\nu - m$ implies that sum defining the fractional difference (or sum) begins at $s = \nu - m$. (cf., Definition 2.2) Of course, the superscript ν implies that the order of the fractional difference is ν.

While it is crucial to keep track of domains in the discrete fractional calculus, as will be seen very shortly, if it is clear from the context, we shall omit the subscript from the fractional operator.

The next set of definitions and lemmas as well as other related results and their proofs can be found, for example, in [23, 78]. In particular, the monograph by Podlubny [78] is an excellent introduction to the theory of the *continuous* fractional calculus and certain of its attendant applications.

Definition 2.9. Let $\nu > 0$ with $\nu \in \mathbb{R}$. Suppose that $y : [a, +\infty) \to \mathbb{R}$. Then the ν-th Riemann-Liouville fractional integral is defined to be

$$D_a^{-\nu} y(t) := \frac{1}{\Gamma(\nu)} \int_a^t y(s)(t-s)^{\nu-1}\, ds,$$

whenever the right-hand side is defined. Similarly, with $\nu > 0$ and $\nu \in \mathbb{R}$, we define the ν-th Riemann-Liouville fractional derivative to be

$$D_a^{\nu} y(t) := \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_a^t \frac{y(s)}{(t-s)^{\nu+1-n}}\, ds,$$

where $n \in \mathbb{N}$ is the unique positive integer satisfying $n-1 \leq \nu < n$ and $t > a$.

Remark 2.10. In the sequel, we shall suppress the explicit dependence of D_a^{ν} on a. It will be clear from the context.

Lemma 2.11. *Let $\alpha \in \mathbb{R}$. Then $D^n D^{\alpha} y(t) = D^{n+\alpha} y(t)$, for each $n \in \mathbb{N}_0$, where $y(t)$ is assumed to be sufficiently regular so that both sides of the equality are well defined. Moreover, if $\beta \in (-\infty, 0]$ and $\gamma \in [0, +\infty)$, then $D^{\gamma} D^{\beta} y(t) = D^{\gamma+\beta} y(t)$.*

Lemma 2.12. *The general solution to $D^{\nu} y(t) = 0$, where $n-1 < \nu \leq n$ and $\nu > 0$, is the function $y(t) = c_1 t^{\nu-1} + c_2 t^{\nu-2} + \cdots + c_n t^{\nu-n}$, where $c_i \in \mathbb{R}$ for each i.*

Finally, let us also recall as a preliminary lemma Krasnosel'skiĭ's fixed point theorem – see [2]. We shall use this classical fixed point theorem frequently in the sequel.

Lemma 2.13. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{K} \subseteq \mathcal{B}$ be a cone. Assume that Ω_1 and Ω_2 are bounded open sets contained in $\mathcal{B}$ such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Assume, further, that $T : \mathcal{K} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to \mathcal{K}$ is a completely continuous operator. If either*

1. *$\|Ty\| \leq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_1$ and $\|Ty\| \geq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_2$; or*

2. *$\|Ty\| \geq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_1$ and $\|Ty\| \leq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_2$;*

then T has at least one fixed point in $\mathcal{K} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

Chapter 3

Continuity of Solutions of Discrete Fractional IVPs with Respect to Derivative Order and Initial Conditions

3.1 Introduction to the Problem

In this chapter we consider a fractional initial value problem of the sort considered in a recent paper by Atici and Eloe [17]. In that paper, the authors demonstrated that a ν-th order discrete fractional initial value problem has a unique solution, and they presented a variety of solution algorithms. However, [17] did not address a question of theoretical interest – that is, whether or not solutions to such equations satisfy a continuity condition not only with respect to initial conditions but also with respect to the order, ν, of the fractional difference. In a paper by Diethelm and Ford [34] it was shown that in the case of continuous fractional initial value problems, the

preceding two questions may be answered in the affirmative. In this paper, we argue that the same sort of continuity condition holds in the case of discrete fractional IVPs and that a number of interesting corollaries follow from this. Thus, the present work can be considered both an extension of [17] and a complement to certain of the results in [34]. Although this sort of result is not shocking, to be sure, given that a discrete analogue of the well known Gronwall inequality holds (cf., the proof in the sequel), we do believe it is nonetheless interesting since it addresses a question that cannot arise in the integer-order setting. Moreover, as will be seen in the sequel, the proof of this result, while essentially "elementary," is hardly trivial. Moreover, as we indicate in the sequel (cf., Remark 3.15), it may be interesting to attempt to refine this result in future work, and so, we believe that the result we give here may yet generate additional, interesting mathematics.

In any case, we first wish to collect two basic lemmas that will be important to us in the sequel and are rather specific in their use to this chapter. In particular, we begin with the following simple result.

Lemma 3.1. *Let $\nu \in \mathbb{R}$ and let $t,\ s \in \mathbb{R}$ such that $(t-s)^{\underline{\nu}}$ is well defined. Then $\Delta_s(t-s)^{\underline{\nu}} = -\nu(t-s-1)^{\underline{\nu-1}}$.*

Proof. Using Definition 2.1 and the fundamental properties of the gamma function, we get the following.

$$\begin{aligned}
\Delta_s(t-s)^{\underline{\nu}} &= (t-s-1)^{\underline{\nu}} - (t-s)^{\underline{\nu}} \\
&= \frac{\Gamma(t-s)}{\Gamma(t-s-\nu)} - \frac{\Gamma(t-s+1)}{\Gamma(t-s-\nu+1)} \\
&= \frac{(t-s-\nu)\Gamma(t-s) - \Gamma(t-s+1)}{\Gamma(t-s-\nu+1)} \\
&= -\nu(t-s-1)^{\underline{\nu-1}}.
\end{aligned}$$

And this completes the proof. $\square$

Secondly, we need to recall the following generalization of the Gronwall inequality to the set $\mathbb{N}_{a+\nu} := \{a+\nu, a+\nu+1, \dots\}$, where $\nu \in \mathbb{R}$. A proof of this may be found in [29], for instance.

Lemma 3.2. *Let a, $\nu \in \mathbb{R}$ be given. If y and f are functions that are defined on $\mathbb{N}_{\nu+a}$ and $\gamma > 0$ is a constant such that*

$$y(t) \leq f(t) + \gamma \sum_{\tau=\nu-1}^{t-1} y(\tau)$$

for all $t \in \mathbb{N}_{\nu+a}$, then

$$y(t) \leq f(t) + \gamma \sum_{\tau=\nu-1}^{t-1} f(\tau)(1+\gamma)^{t-\tau-1}.$$

3.2 A Continuity Result

We are now ready to prove our main theorem of this chapter. Throughout this section, we assume that $\nu \in (0, 1]$ and $f : (\mathbb{N}_{\nu-1} \cup \mathbb{N}_{\nu-\epsilon-1}) \times \mathbb{R} \to \mathbb{R}$ is given. We consider the nonlinear discrete fractional initial value problem

$$\begin{aligned} \Delta^{\nu} y(t) &= f(t+\nu-1, y(t+\nu-1)) \\ \Delta^{\nu-1} y(t)\big|_{t=0} &= y(\nu-1) = y_0, \end{aligned} \tag{3.1}$$

where $t \in \mathbb{N}_0$. Observe that

$$\Delta^{\nu-1} y(t)\big|_{t=0} = y(\nu-1) \tag{3.2}$$

holds by a completely straightforward and elementary calculation, whose proof we omit. Now, let $0 < \varepsilon_0 < \nu \leq 1$ be given. Fix an $\varepsilon > 0$ sufficiently small so that $0 < \varepsilon_0 \leq \nu - \varepsilon < \nu \leq 1$ and consider the problem

$$\begin{aligned} \Delta^{\nu-\varepsilon} z(t) &= f(t+\nu-\varepsilon-1, y(t+\nu-\varepsilon-1)) \\ \Delta^{\nu-\varepsilon-1} z(t)\big|_{t=0} &= z(\nu-\varepsilon-1) = z_0, \end{aligned} \tag{3.3}$$

where $t \in \mathbb{N}_0$. Once again, it is trivial to show that

$$\Delta^{\nu-\varepsilon-1} z(t)\big|_{t=0} = z(\nu-\varepsilon-1) \tag{3.4}$$

holds.

Note that (3.3) is the problem (3.1) perturbed both in the order of the difference (ν versus $\nu - \varepsilon$) and in the initial condition (y_0 versus z_0). Our goal is to show that under appropriate conditions on f, the solutions to the problems (3.1) and (3.3) are close in some reasonable sense as $\varepsilon \to 0^+$ and $z_0 \to y_0$. That is, problem (3.1) satisfies a continuity condition with respect to ν and y_0. To prove this result, we shall show that it is implied by Theorem 3.4 below. We now state and prove this theorem, but we require first a preliminary lemma, whose proof may be found in [17].

Lemma 3.3. *The solution to the problem (3.1) is given by*

$$y(t) = \frac{t^{\underline{\nu-1}}}{\Gamma(\nu)} a_0 + \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s+\nu-1, y(s+\nu-1)),$$

for $t \in \mathbb{N}_{\nu-1}$.

Now we state and prove Theorem 3.4.

Theorem 3.4. *Consider the discrete fractional initial value problems given by (3.1)*

and (3.3). Let $f(t,y)$, where $f:(\mathbb{N}_{\nu-1}\cup\mathbb{N}_{\nu-\varepsilon-1})\times\mathbb{R}\to\mathbb{R}$, be a function such that $f(t,y)$ satisfies a Lipschitz condition in both t and y – that is, there exists constants L, $M>0$ such that

$$\left|f\left(t_1,y_1\right)-f\left(t_2,y_2\right)\right|\leq L\left|t_1-t_2\right|+M\left|y_1-y_2\right|,$$

for all y_1, y_2, and t_1, $t_2\in\mathbb{N}_{\nu-1}\cup\mathbb{N}_{\nu-\varepsilon-1}$. Let $\xi\in\mathbb{N}_{\nu-1}$, $\xi\geq\nu$, be given. Put

$$N:=\max\left\{\max_{t\in[\nu-1,\xi]_{\mathbb{N}_{\nu-1}}}\left|\frac{t^{\underline{\nu-1}}}{\Gamma(\nu)}y_0\right|,\max_{t\in[\nu-\varepsilon-1,\xi-\varepsilon]_{\mathbb{N}_{\nu-\varepsilon-1}}}\left|\frac{(t-\varepsilon)^{\nu-\varepsilon-1}}{\Gamma(\nu-\varepsilon)}z_0\right|\right\}$$

and

$$Q_0:=\max_{(t,y)\in[\nu-1,\xi]_{\mathbb{N}_{\nu-1}}\cup[\nu-\varepsilon-1,\xi-\varepsilon]_{\mathbb{N}_{\nu-\varepsilon-1}}\times[-2N,2N]}f(t,y),$$

and assume that

$$\begin{aligned}&\max_{t\in[\nu-1,\xi]_{\mathbb{N}_{\nu-1}}}\sum_{s=0}^{t-\nu}\left|(t-s-1)^{\underline{\nu-1}}\right|,\max_{t\in[\nu-\varepsilon-1,\xi-\varepsilon]_{\mathbb{N}_{\nu-\varepsilon-1}}}\sum_{s=0}^{t-\nu+\varepsilon}\left|(t-s-1)^{\underline{\nu-\varepsilon-1}}\right|\\&\leq\frac{\Gamma(\nu-\varepsilon)}{Q_0}N.\end{aligned}$$

Then if y is a solution of (3.1) and z is a solution of (3.3), it follows that for $t\in[\nu-1,\xi]_{\mathbb{N}_{\nu-1}}$

$$|y(t)-z(t-\varepsilon)|\leq\phi(t)+\frac{MK_0}{\Gamma(\nu)}\sum_{\tau=\nu-1}^{t-1}\phi(\tau)\left(1+\frac{MK_0}{\Gamma(\nu)}\right)^{t-\tau-1},$$

where $K_0 := \max_{(t,\tau)\in[\nu-1,\xi]_{\mathbb{N}_{\nu-1}}\times[\nu-1,\xi-1]_{\mathbb{N}_{\nu-1}}} \left|(t-\tau+\nu-2)^{\underline{\nu-1}}\right|$ *and*

$$\phi(t) := \left|\frac{t^{\underline{\nu-1}}}{\Gamma(\nu)}y_0 - \frac{(t-\varepsilon)^{\underline{\nu-\varepsilon-1}}}{\Gamma(\nu-\varepsilon)}z_0\right| + \frac{1}{\Gamma(\nu)}Q_0\left|\frac{t^{\underline{\nu}}}{\nu} - \frac{(t-\varepsilon)^{\underline{\nu-\varepsilon}}}{\nu-\varepsilon}\right|$$
$$+ Q_0\left|\frac{\Gamma(\nu-\varepsilon)-\Gamma(\nu)}{\Gamma(\nu)\Gamma(\nu-\varepsilon+1)}(t-\varepsilon)^{\underline{\nu-\varepsilon}}\right| + \varepsilon\frac{L}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}.$$

Proof. By Lemma 3.1 we know that

$$y(t) = \frac{t^{\underline{\nu-1}}}{\Gamma(\nu)}y_0 + \frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}f(s+\nu-1, y(s+\nu-1)) \tag{3.5}$$

and that

$$z(t) = \frac{t^{\underline{\nu-\varepsilon-1}}}{\Gamma(\nu-\varepsilon)}z_0 + \frac{1}{\Gamma(\nu-\varepsilon)}\sum_{s=0}^{t-\nu+\varepsilon}(t-s-1)^{\underline{\nu-\varepsilon-1}}f(s+\nu-\varepsilon-1, z(s+\nu-\varepsilon-1)), \tag{3.6}$$

where we can see from (3.5) that $y(t)$ is defined on the set $\mathbb{N}_{\nu-1} := \{\nu-1, \nu, \nu+1, \dots\}$ and from (3.6) that $z(t)$ is defined on the set $\mathbb{N}_{\nu-\varepsilon-1} := \{\nu-\varepsilon-1, \nu-\varepsilon, \nu-\varepsilon+1, \dots\}$. So, at once we encounter a difficulty not encountered in the proof of the corresponding result in the continuous case – cf., [34]. Indeed, as y and z are defined on different sets, a direct comparison of the sort $|y(t)-z(t)|$ is not sensible. Therefore, we consider a shift of z, which amounts to a right shift of length ε of the graph of z; this will allow a direct comparison of the two functions.

To this end, let us put

$$\widetilde{z}(t) := z(t-\varepsilon). \tag{3.7}$$

For reference in the sequel, let us note that

$$\widetilde{z}(t) = \frac{(t-\varepsilon)^{\underline{\nu-\varepsilon-1}}}{\Gamma(\nu-\varepsilon)}z_0 + \frac{1}{\Gamma(\nu-\varepsilon)}\sum_{s=0}^{t-\nu}(t-\varepsilon-s-1)^{\underline{\nu-\varepsilon-1}}f(s+\nu-\varepsilon-1, z(s+\nu-\varepsilon-1)). \tag{3.8}$$

Note, as (3.8) demonstrates, that we leave the summand of the right-hand side of (3.8) above in terms of z, for this shall be useful in the sequel. Next, observe that (3.5) and (3.8) together imply that

$$\begin{aligned}
|y(t)-\widetilde{z}(t)| &= \Bigg|\frac{t^{\underline{\nu-1}}}{\Gamma(\nu)}y_0-\frac{(t-\varepsilon)^{\underline{\nu-\varepsilon-1}}}{\Gamma(\nu-\varepsilon)}z_0\\
&\quad+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}f(s+\nu-1,y(s+\nu-1))\\
&\quad-\frac{1}{\Gamma(\nu-\varepsilon)}\sum_{s=0}^{t-\nu}(t-\varepsilon-s-1)^{\underline{\nu-\varepsilon-1}}f(s+\nu-\varepsilon-1,z(s+\nu-\varepsilon-1))\Bigg|\\
&\le\left|\frac{t^{\underline{\nu-1}}}{\Gamma(\nu)}y_0-\frac{(t-\varepsilon)^{\underline{\nu-\varepsilon-1}}}{\Gamma(\nu-\varepsilon)}z_0\right|\\
&\quad+\Bigg|\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}f(s+\nu-1,y(s+\nu-1))\\
&\quad-\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}f(s+\nu-\varepsilon-1,z(s+\nu-\varepsilon-1))\Bigg|\\
&\quad+\Bigg|\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}f(s+\nu-\varepsilon-1,z(s+\nu-\varepsilon+1))\\
&\quad-\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\varepsilon-s-1)^{\underline{\nu-\varepsilon-1}}f(s+\nu-\varepsilon-1,z(s+\nu-\varepsilon-1))\Bigg|\\
&\quad+\Bigg|\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\varepsilon-s-1)^{\underline{\nu-\varepsilon-1}}f(s+\nu-\varepsilon-1,z(s+\nu-\varepsilon-1))\\
&\quad-\frac{1}{\Gamma(\nu-\varepsilon)}\sum_{s=0}^{t-\nu}(t-\varepsilon-s-1)^{\underline{\nu-\varepsilon-1}}f(s+\nu-\varepsilon-1,z(s+\nu-\varepsilon-1))\Bigg|.
\end{aligned}\tag{3.9}$$

We shall now analyze each of the four pairs of terms on the right-hand side of

(3.9). We consider first the term

$$\left|\frac{t^{\underline{\nu-1}}}{\Gamma(\nu)}y_0-\frac{(t-\varepsilon)^{\underline{\nu-\varepsilon-1}}}{\Gamma(\nu-\varepsilon)}z_0\right|, \tag{3.10}$$

where $t\in[\nu-1,\xi]_{\mathbb{N}_{\nu-1}}$. For the moment, we shall not rewrite (3.10) but merely notice that

$$\lim_{\varepsilon\to0^+}\left|\frac{t^{\underline{\nu-1}}}{\Gamma(\nu)}y_0-\frac{(t-\varepsilon)^{\underline{\nu-\varepsilon-1}}}{\Gamma(\nu-\varepsilon)}z_0\right|=|y_0-z_0|\frac{t^{\underline{\nu-1}}}{\Gamma(\nu)},$$

which implies that if $|y_0-z_0|<\delta$, where $\delta>0$ is fixed, then

$$\lim_{\varepsilon\to0^+}\left|\frac{t^{\underline{\nu-1}}}{\Gamma(\nu)}y_0-\frac{(t-\varepsilon)^{\underline{\nu-\varepsilon-1}}}{\Gamma(\nu-\varepsilon)}z_0\right|=|y_0-z_0|\frac{t^{\underline{\nu-1}}}{\Gamma(\nu)}<\frac{t^{\underline{\nu-1}}}{\Gamma(\nu)}\delta,$$

whence by choosing δ and ε sufficiently small, (3.10) can be made arbitrarily small for $t\in[\nu-1,\xi]_{\mathbb{N}_{\nu-1}}$.

We next focus our attention on the third term in (3.9), which is

$$\begin{aligned}&\left|\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}f(s+\nu-\varepsilon-1,z(s+\nu-\varepsilon+1))\right.\\&\left.-\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\varepsilon-s-1)^{\underline{\nu-\varepsilon-1}}f(s+\nu-\varepsilon-1,z(s+\nu-\varepsilon-1))\right|.\end{aligned} \tag{3.11}$$

Let us observe that (3.11) may be rewritten as

$$\left|\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}\left[(t-s-1)^{\underline{\nu-1}}-(t-\varepsilon-s-1)^{\underline{\nu-\varepsilon-1}}\right]f(s+\nu-\varepsilon-1,z(s+\nu-\varepsilon-1))\right|. \tag{3.12}$$

Now, let N be as given in the statement of the theorem and put

$$Q_0:=\max_{(t,y)\in[\nu-1,\xi]_{\mathbb{N}_{\nu-1}}\cup[\nu-\varepsilon-1,\xi-\varepsilon]_{\mathbb{N}_{\nu-\varepsilon-1}}\times[-2N,2N]}f(t,y). \tag{3.13}$$

Observe by the hypotheses given in the statement of the theorem that

$$|y(t)| \leq N + \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} \left|(t-s-1)^{\underline{\nu-1}}\right| |f(s+\nu-1, y(s+\nu-1))| \leq N + \frac{Q_0}{\Gamma(\nu)} \cdot \frac{\Gamma(\nu)}{Q_0} \cdot N$$

so that $|y(t)| \leq 2N$. A similar argument shows that $|z(t)| \leq 2N$, too. So, from (3.12) and (3.13), we find that

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} \left[(t-s-1)^{\underline{\nu-1}} - (t-\epsilon-s-1)^{\underline{\nu-\varepsilon-1}}\right] f(s+\nu-\varepsilon-1, z(s+\nu-\varepsilon-1))\right| \\
&\leq Q_0 \left|\frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} \left[(t-s-1)^{\underline{\nu-1}} - (t-\varepsilon-s-1)^{\underline{\nu-\varepsilon-1}}\right]\right| \\
&= \frac{1}{\Gamma(\nu)} Q_0 \left| \left[-\frac{1}{\nu}(t-s)^{\underline{\nu}}\right]_0^{t-\nu+1} + \left[\frac{1}{\nu-\varepsilon}(t-\varepsilon-s)^{\underline{\nu-\varepsilon}}\right]_0^{t-\nu+1} \right| \quad \text{(by } \textit{Lemma 3.1}\text{)} \\
&= \frac{1}{\Gamma(\nu)} Q_0 \left| \frac{t^{\underline{\nu}}}{\nu} - \frac{(t-\varepsilon)^{\underline{\nu-\varepsilon}}}{\nu-\varepsilon} \right|.
\end{aligned} \tag{3.14}$$

Let us notice, which will be important in the sequel, that

$$\lim_{\varepsilon \to 0^+} \frac{1}{\Gamma(\nu)} Q_0 \left| \frac{t^{\underline{\nu}}}{\nu} - \frac{(t-\varepsilon)^{\underline{\nu-\varepsilon}}}{\nu-\varepsilon} \right| = 0,$$

so that (3.11) tends to zero as $\varepsilon \to 0^+$.

We consider next the fourth term in (3.9), which is

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-\varepsilon-s-1)^{\underline{\nu-\varepsilon-1}} f(s+\nu-\varepsilon-1, z(s+\nu-\varepsilon-1))\right. \\
&\left. - \frac{1}{\Gamma(\nu-\varepsilon)} \sum_{s=0}^{t-\nu} (t-\varepsilon-s-1)^{\underline{\nu-\varepsilon-1}} f(s+\nu-\varepsilon-1, z(s+\nu-\varepsilon-1))\right|.
\end{aligned} \tag{3.15}$$

We wish to rewrite (3.15) in a way similar to the way in which (3.11) was rewritten

above. So, using (3.13), we have

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\varepsilon-s-1)^{\underline{\nu-\varepsilon-1}}f(s+\nu-\varepsilon-1,z(s+\nu-\varepsilon-1))\right.\\
&\left.-\frac{1}{\Gamma(\nu-\varepsilon)}\sum_{s=0}^{t-\nu}(t-\varepsilon-s-1)^{\underline{\nu-\varepsilon-1}}f(s+\nu-\varepsilon-1,z(s+\nu-\varepsilon-1))\right|\\
&\leq Q_0\left|\frac{\Gamma(\nu-\varepsilon)-\Gamma(\nu)}{\Gamma(\nu)\Gamma(\nu-\varepsilon)}\left[-\frac{1}{\nu-\varepsilon}(t-\varepsilon-s)^{\underline{\nu-\varepsilon}}\right]_0^{t-\nu+1}\right|\\
&=Q_0\left|\frac{\Gamma(\nu-\varepsilon)-\Gamma(\nu)}{\Gamma(\nu)\Gamma(\nu-\varepsilon+1)}(t-\varepsilon)^{\underline{\nu-\varepsilon}}\right|.
\end{aligned} \tag{3.16}$$

As above, if we focus on the right-hand side of (3.16), we note that

$$\lim_{\varepsilon\to 0^+} Q_0\left|\frac{\Gamma(\nu-\varepsilon)-\Gamma(\nu)}{\Gamma(\nu)\Gamma(\nu-\varepsilon+1)}(t-\varepsilon)^{\underline{\nu-\varepsilon}}\right| = 0,$$

so that (3.15) tends to 0 as $\epsilon \to 0^+$. This, too, will be important in the sequel.

Finally, let us consider the second term in (3.9),which is

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}f(s+\nu-1,y(s+\nu-1))\right.\\
&\left.-\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}f(s+\nu-\varepsilon-1,z(s+\nu-\varepsilon-1))\right|.
\end{aligned} \tag{3.17}$$

Now, using the Lipschitz condition on f, we obtain

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}f(s+\nu-1,y(s+\nu-1))\right.\\
&\left.-\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}f(s+\nu-\varepsilon-1,z(s+\nu-\varepsilon-1))\right|\\
&\leq\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}[L\varepsilon+M|y(s+\nu-1)-z(s+\nu-\varepsilon-1)|] \qquad (3.18)\\
&=\varepsilon\frac{L}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}\\
&+\frac{M}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}\left|y(s+\nu-1)-z(s+\nu-\varepsilon-1)\right|.
\end{aligned}$$

Notice that on the right-hand side of (3.18), we find that

$$\lim_{\varepsilon\to0^+}\varepsilon\frac{L}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}=0, \qquad (3.19)$$

for $t\in[\nu-1,\xi]_{\mathbb{N}_{\nu-1}}$.

We shall now summarize our results thus far. So, combining (3.10), (3.14), (3.16), and (3.18), we find that (3.9) may be rewritten as

$$\begin{aligned}
&|y(t)-z(t-\varepsilon)|\\
&=|y(t)-\widetilde{z}(t)|\\
&\leq\left|\frac{t^{\underline{\nu-1}}}{\Gamma(\nu)}y_0-\frac{(t-\varepsilon)^{\underline{\nu-\varepsilon-1}}}{\Gamma(\nu-\varepsilon)}z_0\right|+\frac{1}{\Gamma(\nu)}Q_0\left|\frac{t^{\underline{\nu}}}{\nu}-\frac{(t-\varepsilon)^{\underline{\nu-\varepsilon}}}{\nu-\varepsilon}\right|\\
&+Q_0\left|\frac{\Gamma(\nu-\varepsilon)-\Gamma(\nu)}{\Gamma(\nu)\Gamma(\nu-\varepsilon+1)}(t-\varepsilon)^{\underline{\nu-\varepsilon}}\right|+\varepsilon\frac{L}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}} \qquad (3.20)\\
&+\frac{M}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}\left|y(s+\nu-1)-z(s+\nu-\varepsilon-1)\right|.
\end{aligned}$$

Now, if we put

$$\begin{aligned}\phi(t) := &\left|\frac{t^{\underline{\nu-1}}}{\Gamma(\nu)}y_0 - \frac{(t-\varepsilon)^{\underline{\nu-\varepsilon-1}}}{\Gamma(\nu-\varepsilon)}z_0\right| + \frac{1}{\Gamma(\nu)}Q_0\left|\frac{t^{\underline{\nu}}}{\nu} - \frac{(t-\varepsilon)^{\underline{\nu-\varepsilon}}}{\nu-\varepsilon}\right| \\ &+ Q_0\left|\frac{\Gamma(\nu-\varepsilon)-\Gamma(\nu)}{\Gamma(\nu)\Gamma(\nu-\varepsilon+1)}(t-\varepsilon)^{\underline{\nu-\varepsilon}}\right| + \varepsilon\frac{L}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}},\end{aligned} \tag{3.21}$$

then we can use (3.21) together with a change of index, $\tau := s+\nu-1$, to rewrite the inequality (3.20) as

$$\begin{aligned}|y(t)-z(t-\varepsilon)| &= |y(t)-\widetilde{z}(t)| \\ &\le \phi(t) + \frac{M}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}\,|y(s+\nu-1)-z(s+\nu-\varepsilon-1)| \\ &= \phi(t) + \frac{M}{\Gamma(\nu)}\sum_{\tau=\nu-1}^{t-1}(t-\tau+\nu-2)^{\underline{\nu-1}}\,|y(\tau)-z(\tau-\varepsilon)|\,,\end{aligned} \tag{3.22}$$

for $t \in [\nu-1,\xi]_{\mathbb{N}_{\nu-1}}$.

Finally, we use the Gronwall inequality given in Lemma 3.2. In order to apply Lemma 3.2 to (3.22), let us put

$$K_0 := \max_{(t,\tau)\in[\nu-1,\xi]_{\mathbb{N}_{\nu-1}}\times[\nu-1,t-1]_{\mathbb{N}_{\nu-1}}} \left|(t-\tau+\nu-2)^{\underline{\nu-1}}\right|.$$

Thus, (3.22) becomes

$$|y(t)-z(t-\varepsilon)| \le \phi(t) + \frac{MK_0}{\Gamma(\nu)}\sum_{\tau=\nu-1}^{t-1}|y(\tau)-z(\tau-\varepsilon)|\,. \tag{3.23}$$

Finally, note that we can apply the Gronwall inequality to (3.23). Doing so, we

get that

$$|y(t) - z(t-\varepsilon)| = |y(t) - \widetilde{z}(t)| \le \phi(t) + \frac{MK_0}{\Gamma(\nu)} \sum_{\tau=\nu-1}^{t-1} \phi(\tau) \left(1 + \frac{MK_0}{\Gamma(\nu)}\right)^{t-\tau-1}, \quad (3.24)$$

which completes the proof. □

Remark 3.5. Let us make one observation regarding the statement of Theorem 3.4 and its proof. Notice that the number Q_0 is necessary if and only if $\varepsilon \neq 0$. Thus, in the case where $\varepsilon = 0$, we need not worry about the number Q_0, and, consequently, the hypotheses of Theorem 3.4 can be suitably relaxed. In the sequel, we shall not differentiate between these cases, but the reader should be aware of this difference.

Now, having proved Theorem 3.4, we deduce a number of corollaries from it.

Corollary 3.6. *Suppose that the hypotheses of Theorem 3.4 hold. Suppose, further, that* $|y_0 - z_0| := \delta$. *Then given a solution* y *of problem (3.1) and a solution* z *of problem (3.3), it follows that for each* $\eta > 0$, *we can choose* δ, $\varepsilon > 0$ *in such a way that the bound*

$$|y(t) - z(t-\varepsilon)| < \eta \quad (3.25)$$

holds for $t \in [\nu-1, \xi]_{\mathbb{N}_{\nu-1}}$, *for* $\xi > 0$ *given.*

Proof. Let us begin by noting that by writing ϕ in terms of the gamma function, we

see that

$$
\begin{aligned}
\phi(t) = & \left| \frac{\Gamma(t+1)\Gamma(\nu-\varepsilon)y_0 - \Gamma(t-\varepsilon+1)\Gamma(\nu)z_0}{\Gamma(\nu)\Gamma(t-\nu+2)\Gamma(\nu-\varepsilon)} \right| \\
& + \frac{Q_0}{\Gamma(\nu)} \left| \frac{(\nu-\varepsilon)\Gamma(t+1) - \nu\Gamma(t-\varepsilon+1)}{\nu\Gamma(t-\nu+1)} \right| \\
& + Q_0 |\Gamma(\nu-\varepsilon) - \Gamma(\nu)| \cdot \left| \frac{\Gamma(t-\varepsilon+1)}{\Gamma(\nu)\Gamma(\nu-\varepsilon+1)\Gamma(t-\nu+1)} \right| \\
& + \frac{\varepsilon L}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} \frac{\Gamma(t-s)}{\Gamma(t-s-\nu+1)}.
\end{aligned} \tag{3.26}
$$

We now argue that each term in (3.26) can be made arbitrarily small by picking δ, $\varepsilon > 0$ sufficiently small.

To this end, let $\eta_0 > 0$ be given. Notice that we can select $N_1 > 0$ such that whenever $0 < \varepsilon < N_1$, we find by the uniform continuity of $\Gamma(\cdot)$ on $[\nu - \varepsilon, +\infty)$ that

$$
|\Gamma(\nu-\varepsilon) - \Gamma(\nu)| < \frac{\eta_0}{4Q_0 \left| \frac{\Gamma(t-\varepsilon+1)}{\Gamma(\nu)\Gamma(\nu-\varepsilon+1)\Gamma(t-\nu+1)} \right| + 4} < \frac{\eta_0}{4}, \tag{3.27}
$$

for all $t \in [\nu - 1, \xi]_{\mathbb{N}_{\nu-1}}$.

Similarly, there exists a number $N_2 > 0$ such that for $0 < \varepsilon < N_2$, we find that

$$
\begin{aligned}
& \frac{Q_0}{\Gamma(\nu)} \left| \frac{(\nu-\varepsilon)\Gamma(t+1) - \nu\Gamma(t-\varepsilon+1)}{\nu\Gamma(t-\nu+1)} \right| \\
& \leq \frac{Q_0}{\Gamma(\nu)} \left[|\Gamma(t+1) - \Gamma(t+1-\varepsilon)| \cdot \frac{1}{\Gamma(t-\nu+1)} - \varepsilon \cdot \frac{\Gamma(t+1)}{\Gamma(t-\nu+1)} \right] \\
& \leq \frac{\eta_0}{8} + \frac{\eta_0}{8} \\
& = \frac{\eta_0}{4},
\end{aligned} \tag{3.28}
$$

say, where this again follows from the uniform continuity of the gamma function on the set $[\nu - \varepsilon, +\infty)$. (Note that $\nu - \varepsilon$ is, by construction, bounded away from 0.)

Moreover, for some number $N_3 > 0$, we have that whenever $0 < \varepsilon < N_3$ and

$$\delta := |y_0 - z_0| < \frac{\eta_0}{8} \cdot \max_{t \in [\nu-1,\xi]_{\mathbb{N}_{\nu-1}}} \frac{1}{\Gamma(t-\varepsilon+1)\Gamma(\nu)},$$

it follows that

$$\begin{aligned}
&\frac{1}{\Gamma(\nu)\Gamma(t-\nu+2)\Gamma(\nu-\varepsilon)} \left|\Gamma(t+1)\Gamma(\nu-\varepsilon)y_0 - \Gamma(t-\varepsilon+1)\Gamma(\nu)z_0\right| \\
&\leq |y_0| \left|\Gamma(t+1)\Gamma(\nu-\varepsilon) - \Gamma(t-\varepsilon+1)\Gamma(\nu)\right| + |y_0 - z_0| \left|\Gamma(t-\varepsilon+1)\Gamma(\nu)\right| \\
&\leq |y_0| \left[\left|\Gamma(t+1) - \Gamma(t+1-\varepsilon)\right| \cdot \left|\Gamma(\nu-\varepsilon)\right| + \left|\Gamma(t-\varepsilon+1)\right| \cdot \left|\Gamma(\nu-\varepsilon) - \Gamma(\nu)\right|\right] \\
&+ |y_0 - z_0| \left|\Gamma(t-\varepsilon+1)\Gamma(\nu)\right| \\
&\leq \frac{\eta_0}{8} + \frac{\eta_0}{8} \\
&\leq \frac{\eta_0}{4},
\end{aligned} \tag{3.29}$$

say.

Finally, it is clear that we can choose $N_4 > 0$ so that

$$\frac{\varepsilon L}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} \frac{\Gamma(t-s)}{\Gamma(t-s-\nu+1)} < \frac{\eta_0}{4} \tag{3.30}$$

whenever $0 < \varepsilon < N_4$ because both the sum in (3.30) above and the quantity $\frac{L}{\Gamma(\nu)}$ are bounded.

Now, put $N := \min\{N_1, N_2, N_3, N_4\}$. Then combining (3.27)–(3.30) implies that whenever $0 < \varepsilon < N$,

$$|\phi(t)| < \eta_0, \tag{3.31}$$

and so, for $t \in [\nu-1,\xi]_{\mathbb{N}_{\nu-1}}$, $\phi(t)$ can be made arbitrarily small.

So, now let $\eta > 0$ be given. It is clear that

$$\max_{t\in[\nu-1,\xi]_{\mathbb{N}_{\nu-1}}} \left| \frac{MK_0}{\Gamma(\nu)} \sum_{\tau=\nu-1}^{t-1} \left(1 + \frac{MK_0}{\Gamma(\nu)}\right)^{t-\tau-1} \right| \leq N_5, \tag{3.32}$$

for some number $N_5 \geq 0$. Then (3.31) and (3.32) together imply that we can choose δ and ϵ sufficiently small so that

$$\max_{t\in[\nu-1,\xi]_{\mathbb{N}_{\nu-1}}} \phi(t) < \min\left\{\frac{\eta}{2}, \frac{\eta}{2N_5+1}\right\}. \tag{3.33}$$

So, it follows, then, from (3.31)–(3.33) that for any given $\eta > 0$, we have

$$|y(t) - z(t-\varepsilon)| < \eta,$$

whenever δ and ε are chosen sufficiently small, and so, the proof is complete. $\square$

Corollary 3.7. *Suppose that the hypotheses of Theorem 3.4 hold. Suppose, further, that $|y_0 - z_0| := \delta$. Let $\xi > 0$ be given. Suppose that $\varepsilon = 0$ in (3.3). Then given a solution y of problem (3.1) and a solution z of problem (3.3), it follows that for each $\eta > 0$, we can choose $\delta > 0$ in such a way that the bound*

$$|y(t) - z(t)| < \eta \tag{3.34}$$

holds for $t \in [\nu-1,\xi]_{\mathbb{N}_{\nu-1}}$.

Proof. Note that if $\varepsilon = 0$, then we find from (3.22) that

$$\phi(t) = \frac{t^{\underline{\nu-1}}}{\Gamma(\nu)} |y_0 - z_0|.$$

Now, on the compact set $[\nu-1,\xi]_{\mathbb{N}_{\nu-1}}$, there exists a number $\xi_0 \in [\nu-1,\xi]_{\mathbb{N}_{\nu-1}}$ such

that

$$\max_{\tau\in[\nu-1,\xi-1]_{\mathbb{N}_{\nu-1}}} \phi(\tau) = \frac{\xi_0^{\underline{\nu-1}}}{\Gamma(\nu)} \left|y_0 - z_0\right|.$$

Put $K_1 := \max_{(t,\tau)\in[\nu-1,\xi]_{\mathbb{N}_{\nu-1}}\times[\nu-1,t-1]_{\mathbb{N}_{\nu-1}}} \left(1+\frac{MK_0}{\Gamma(\nu)}\right)^{t-\tau-1}$. Then we find that

$$\begin{aligned} |y(t)-z(t)| &\leq \frac{\xi_0^{\underline{\nu-1}}}{\Gamma(\nu)}\delta + \frac{MK_0}{\Gamma(\nu)} \sum_{\tau=\nu-1}^{t-1} \frac{\xi_0^{\underline{\nu-1}}}{\Gamma(\nu)} \left(1+\frac{MK_0}{\Gamma(\nu)}\right)^{t-\tau-1} \delta \\ &\leq \delta\left[\frac{\xi_0^{\underline{\nu-1}}}{\Gamma(\nu)} + \frac{MK_0K_1\xi_0^{\underline{\nu-1}}(\xi-\nu+1)}{(\Gamma(\nu))^2}\right]. \end{aligned} \tag{3.35}$$

So, pick

$$0 < \delta < \frac{\eta}{\frac{\xi_0^{\underline{\nu-1}}}{\Gamma(\nu)} + \frac{MK_0K_1\xi_0^{\underline{\nu-1}}(\xi-\nu+1)}{(\Gamma(\nu))^2}}. \tag{3.36}$$

Then (3.35) and (3.36) together imply that

$$|y(t)-z(t)| \leq \delta\left(\frac{\xi_0^{\underline{\nu-1}}}{\Gamma(\nu)} + \frac{MK_0K_1\xi_0^{\underline{\nu-1}}(\xi-\nu+1)}{(\Gamma(\nu))^2}\right) < \eta, \tag{3.37}$$

and the proof is complete. □

Corollary 3.8. *Suppose that the hypotheses of Theorem 3.4 hold. Furthermore, let $y(t)$ be a solution of (3.1) and $z(t)$ a solution of (3.3). Then in case $\nu = 1$, we get that*

$$|y(t)-z(t-\varepsilon)| \leq \phi(t) + MK_0 \sum_{\tau=0}^{t-1} \phi(\tau)\left(1+MK_0\right)^{t-\tau-1},$$

where

$$\begin{aligned} \phi(t) :=& \left|y_0 - \frac{(t-\varepsilon)^{\underline{-\varepsilon}}}{\Gamma(1-\varepsilon)} z_0\right| \\ &+ Q_0\left[\left|t - \frac{(t-\varepsilon)^{\underline{1-\varepsilon}}}{1-\varepsilon}\right| + \left|\frac{(\Gamma(1-\varepsilon)-1)(t-\varepsilon)^{\underline{-\varepsilon}}}{\Gamma(2-\varepsilon)}\right|\right] + t\varepsilon L. \end{aligned} \tag{3.38}$$

Proof. Immediate from (3.24). □

Corollary 3.9. *Suppose that the hypotheses of Theorem 3.4 hold and that $\nu = 1$. Suppose, further, that $|y_0 - z_0| := \delta$. Suppose that $\varepsilon = 0$ in (3.3). Then given a solution y of problem (3.1) and a solution z of problem (3.3), it follows that for each $\eta > 0$, we can choose $\delta > 0$ in such a way that the bound*

$$|y(t) - z(t)| < \eta \tag{3.39}$$

holds for $t \in [\nu - 1, \xi]_{\mathbb{N}_{\nu-1}}$ for $\xi > 0$ given.

Proof. From (3.37), pick $0 < \delta < \frac{\eta}{1+MK_0K_1\xi}$. Then Corollary 3.7 implies the conclusion. $\square$

We conclude by first giving an example of Corollary 3.7 and then making some remarks about Theorem 3.4 and its corollaries.

Example 3.10. Suppose that we put $\varepsilon := 0$, $\nu := \frac{9}{10}$, $\eta := 2$, and $\xi := \frac{99}{10}$. Let us also suppose that $f(t, y) := t + y$. Thus, we wish to apply the result of Corollary 3.7 to the pair of FBVPs

$$\begin{aligned} \Delta^{\frac{9}{10}} y(t) &= \left(t - \frac{1}{10}\right) + y\left(t - \frac{1}{10}\right) \\ \Delta^{-\frac{1}{10}} y(t)\Big|_{t=0} &= y\left(-\frac{1}{10}\right) = y_0 \end{aligned} \tag{3.40}$$

and

$$\begin{aligned} \Delta^{\frac{9}{10}} z(t) &= \left(t - \frac{1}{10}\right) + y\left(t - \frac{1}{10}\right) \\ \Delta^{-\frac{1}{10}} z(t)\Big|_{t=0} &= z\left(-\frac{1}{10}\right) = z_0, \end{aligned} \tag{3.41}$$

and then determine how large $\delta := |y_0 - z_0|$ may be chosen so that

$$|y(t) - z(t)| < 2 = \eta \tag{3.42}$$

for all $t \in \left[-\frac{1}{10}, \frac{99}{10}\right]_{\mathbb{N}_{\nu-1}}$, where y and z are the solutions to problems (3.40) and (3.41), respectively.

To this end, we can deduce the following quantities.

$$\begin{aligned} M &:= 1 \\ \xi_0 &:= -\frac{1}{10} \\ K_0 &:= \max_{(t,\tau)\in\left[-\frac{1}{10},\frac{99}{10}\right]_{\mathbb{N}_{\nu-1}}\times\left[-\frac{1}{10},t-1\right]_{\mathbb{N}_{\nu-1}}} \left|\left(t-\tau-\frac{11}{10}\right)^{\underline{-\frac{1}{10}}}\right| \approx 1.07 \\ K_1 &:= \max_{(t,\tau)\in\left[-\frac{1}{10},\frac{99}{10}\right]_{\mathbb{N}_{\nu-1}}\times\left[-\frac{1}{10},t-1\right]_{\mathbb{N}_{\nu-1}}} \left(1+\frac{MK_0}{\Gamma(\nu)}\right)^{t-\tau-1} \approx 512 \end{aligned} \tag{3.43}$$

Thus, using estimate (3.36) together with the values given by (3.43), we find that we should take

$$\delta < \frac{\eta}{\frac{\left(-\frac{1}{10}\right)^{\underline{\frac{9}{10}-1}}}{\Gamma\left(\frac{9}{10}\right)} + \frac{1.07\cdot 512\cdot\left(-\frac{1}{10}\right)^{\underline{\frac{9}{10}-1}}\cdot\left(\frac{99}{10}-\frac{9}{10}+1\right)}{\left(\Gamma\left(\frac{9}{10}\right)\right)^2}} \approx \frac{\eta}{5121}, \tag{3.44}$$

whence by putting $\eta = 2$ into (3.44), we find that

$$\delta < 0.000391. \tag{3.45}$$

Consequently, (3.45) implies that if we wish the solutions y and z to remain within $\eta = 2$ units of each other on the interval $\left[-\frac{1}{10}, \frac{99}{10}\right]_{\mathbb{N}_{\nu-1}}$, then the initial conditions y_0 and z_0 must be within no more than approximately 0.000391 units. Clearly, if we

either shorten the interval $[\nu-1,\xi]_{\mathbb{N}_{\nu-1}}$ or relax the closeness, η, that y and z must remain to each other, then the maximum value of δ will increase.

Remark 3.11. Note that Corollary 3.6 implies that given solutions to (3.1) and (3.3), the solutions remain close (in the sense of Theorem 3.2) provided that (i) the initial conditions are sufficiently close and (ii) the orders of the differences are sufficiently close. So, this is a statement regarding continuity of solutions to two different IVPs wherein both the order of difference and the initial conditions are not (necessarily) equal.

Remark 3.12. Note that Corollary 3.7 implies that given solutions to (3.1) and (3.3) with $\varepsilon = 0$, the solutions remain close (in the sense of Theorem 3.2). So, this is a statement regarding continuity of solutions to two different IVPs wherein the order of difference is equal but the initial conditions are not (necessarily) equal.

Remark 3.13. Note that Corollary 3.8 implies that for $0 < \nu < 1$ a ν-th order initial value problem may be approximated by a first-order initial value problem (and vice versa) provided that ν is sufficiently close to (and less than) unity and that ξ is kept sufficiently close to and greater than $-\varepsilon$.

Remark 3.14. Note that Corollary 3.9 confirms the classical result – namely, that solutions to a first-order discrete initial value problem are continuous with respect to initial conditions.

Remark 3.15. In comparing our results to those that can be found in the paper by Diethelm and Ford [34], we find that our results are somewhat weaker. For example, we make some restrictions on the growth of $f(t, y)$ that Diethelm and Ford do not make. Part of the difference is that the discrete fractional difference shifts domains, and this causes some complications, as pointed out in the proof of Theorem 3.4. Moreover, our proof strategy is rather different than the one employed in [34]. It may

be possible to provide a proof more analogous to that provided in [34], and this might represent an interesting program for a future work.

As a means of concluding this chapter, we note that just as Diethelm and Ford remark in [34] and just as we mentioned at the beginning of this chapter, we point out that in this paper we have addressed a question that cannot arise in the classical theory of difference equations. Indeed, in the latter theory, we put $\nu = 1$, and so, there is no concern as to the continuity of solutions with respect to the order of the difference operator. Thus, the question that has been addressed in this chapter is one unique to the fractional difference calculus, and this makes the fractional difference equation more interesting in this respect than the integer-order counterpart. As we continue throughout this work, we will continue to see certain of these interesting differences arise in the problems we study.

Chapter 4

Sequential Properties of the Discrete Fractional Difference Operator

In the previous chapter, we considered a particular continuity property of the fractional difference operator with respect to an initial value problem. Essentially, this is an operational property of the fractional difference, and in the present chapter we consider another consequence of the operational properties of the fractional difference. Indeed, we now consider a discrete fractional boundary value problem (FBVP), for $t \in [2-\mu_1-\mu_2-\mu_3, b+2-\mu_1-\mu_2-\mu_3]_{\mathbb{N}_{2-\mu_1-\mu_2-\mu_3}}$, of the form

$$-\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}y(t) = f(t+\mu_1+\mu_2+\mu_3-1, y(t+\mu_1+\mu_2+\mu_3-1)), \tag{4.1}$$

subject to the conjugate boundary conditions

$$y(0) = 0 = y(b+2), \tag{4.2}$$

where $f : [1, b+1]_{\mathbb{N}_0} \times \mathbb{R} \to [0, +\infty)$ is a continuous function, $b \in \mathbb{N}$, and μ_1, μ_2, $\mu_3 \in (0,1)$ satisfy both

$$1 < \mu_2 + \mu_3 < 2 \tag{4.3}$$

and

$$1 < \mu_1 + \mu_2 + \mu_3 < 2. \tag{4.4}$$

The purpose of this chapter is to compare and contrast problem (4.1)–(4.2) with the non-sequential conjugate problem studied recently by Atici and Eloe [20] and to highlight the complications that arise in the sequential setting, particularly in the context of proving that (4.1)–(4.2) admits at least one positive solution. We point out that Wei, et al. [88] have addressed some of these issues in the continuous fractional setting. Indeed, because of the sequence of differences in (4.1) and the composition rules for fractional differences, it turns out that problem (4.1) is different than the simpler problem $-\Delta^{\nu} y(t) = -f(t+\nu-1, y(t+\nu-1))$, where $\nu \in (1,2]$. We shall expand on these differences. Moreover, our analysis will also yield complementary results for the delta-nabla problem

$$-\Delta^{\mu_1}\Delta^{\mu_2}\nabla^{\mu_3} y\,(t+\mu_3) = f(t+\mu_1+\mu_2+\mu_3-1, y(t+\mu_1+\mu_2+\mu_3-1)), \tag{4.5}$$

subject to (4.2), which has not yet been studied. In particular, our analysis will provide the following insights.

1. We clarify the structure of sequential fractional difference equations. Due to the lack of commutativity of the fractional difference, this represents an interesting complication that does not arise in the integer-order setting.

2. In problem (4.5), we necessarily have a composition of two fractional differences, which gives rise to a sequential problem. Consequently, while we believe (4.1)–

(4.2) to be mathematically interesting for its own sake, it is the case that one reason, among several, to be interested in problems such as (4.1)–(4.2) is due to the fact that fractional delta-nabla problems such as (4.5) are necessarily of this sort of sequential type. Now, it is the case that delta-nabla problems are not of great interest on just the time scale $\mathbb{Z}$. However, as clarified below, there are now numerous attempts to extend the discrete fractional calculus to other time scales, and so, interest in fractional delta-nabla problems may increase in these other settings.

3. We provide some connections with the recent work [88] in the discrete setting.

Since, as mentioned earlier, we shall also obtain results for problems involving the discrete fractional nabla operator, we remark that in the integer-order literature, the delta-nabla boundary value problem has received considerable attention in recent years. For example, Anderson [5] considered the problem $u^{\Delta\nabla}(t) + f(t, u(t)) = 0$, $u(0) = 0$, $\alpha u(\eta) = u(T)$, on a time scale $\mathbb{T}$. In case one puts $\mathbb{T} = \mathbb{Z}$, then one obtains an integer-order delta-nabla difference equation. Kaufmann and Raffoul [69] considered a closely related problem. Similarly, Cheung, et al. [30] considered a delta-nabla difference equation of the form $\nabla\Delta u(k) + f(k, u(k)) = 0$ together with a couple of a different specific nonlocal conditions. For some other works on delta-nabla boundary value problems on various time scales, see [6, 7, 8, 12, 14, 31, 35, 61, 83] and the references therein.

Before proceeding with our program, an operational property that we require in order to complete our program in the sequel is the following. This result, Theorem 4.1, was recently established by Holm [63] following the program in the continuous fractional calculus outlined by Podlubny [78]. In particular, one might wonder why we have chosen the domains in problem (4.1)–(4.2) as we have. Indeed, the choice of

the domain seems at odds with the choice in other recent works on discrete boundary value problems of fractional order – cf., [20]. The statement of Theorem 4.1 shall make clear why we have made this seemingly peculiar choice. As a careful examination of the proofs in [63] reveal, really all of this is a consequence of the peculiar domain requirements of the power rule in Lemma 2.7 above.

Theorem 4.1. *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given and suppose that ν, $\mu > 0$ with $N-1 < \nu \leq N$ and $M-1 < \mu \leq M$. Then for $t \in \mathbb{N}_{a+M-\mu+N-\nu}$*

$$\Delta^{\nu}_{a+M-\mu}\Delta^{\mu}_{a}f(t) = \Delta^{\nu+\mu}_{a}f(t) - \begin{cases} \sum_{j=0}^{M-1} \frac{\Delta^{j-M+\mu}f(a+M-\mu)}{\Gamma(-\nu-M+j+1)}(t-a-M+\mu)^{\underline{-\nu-M+j}}, & \nu \in (N-1,N) \\ 0, & \nu = N \end{cases}. \tag{4.6}$$

In [63], Holm did not address the meaning of the term $\Delta^{j-M+\mu}f(a+M-\mu)$ appearing in (4.6) above. In fact, in the context of our boundary value problem, this term has a special relevance, which is very easy to prove. We do so below.

Proposition 4.2. *Let $y : \mathbb{N}_0 \to \mathbb{R}$ with $\mu \in (0,1]$. Then we find that*

$$\Delta^{\mu-1}y(1-\mu) = y(0). \tag{4.7}$$

Proof. To see that this is true, observe that $\mu - 1 \leq 0$ since $\mu \in (0,1]$. By definition,

then, it follows that

$$\begin{aligned}\Delta^{\mu-1}y(1-\mu) &= \left[\frac{1}{\Gamma(1-\mu)}\sum_{s=0}^{t+\mu-1}(t-s-1)^{\underline{-\mu}}y(s)\right]_{t=1-\mu} \\ &= \frac{1}{\Gamma(1-\mu)}\sum_{s=0}^{0}(-\mu-s)^{\underline{-\mu}}y(s) \\ &= \frac{1}{\Gamma(1-\mu)}\cdot\Gamma(1-\mu)y(0) \\ &= y(0),\end{aligned} \tag{4.8}$$

as claimed. □

With these results in hand, we are now ready to analyze problem (4.1)–(4.2). We do so in the next section.

4.1 Analysis of Problem (4.1)–(4.2)

4.1.1 Green's Function Analysis

We now provide an analysis of problem (4.1)–(4.2). We begin by repeatedly using Theorem 4.1 to derive a representation of a solution to (4.1)–(4.2) as the fixed point of an appropriate operator. In the sequel, the Banach space $\mathcal{B}$ is the set of (continuous) real-valued maps from $[0, b+2]_{\mathbb{N}_0}$ when equipped with the usual maximum norm, $\|\cdot\|$, which, incidentally, is equivalent to the Banach space $\mathbb{R}^{b+3}$ equipped with the same norm. Moreover, henceforth we also put

$$\widetilde{\mu} := \mu_1 + \mu_2 + \mu_3, \tag{4.9}$$

for notational convenience. Recall, moreover, that $\mu_1+\mu_2 \in (1,2)$ and that $\widetilde{\mu} \in (1,2)$; these facts will be important in the sequel. Finally, we give the following notation, which will also be useful in the sequel.

$$\begin{aligned}T_1 := \Big\{&(t,s) \in [0,b+2]_{\mathbb{N}_0} \times [2-\widetilde{\mu}, b+2-\widetilde{\mu}]_{\mathbb{N}_{2-\widetilde{\mu}}} \ : \\ &0 \le s < t-\widetilde{\mu}+1 \le b+2\Big\}\end{aligned}$$

$$\begin{aligned}T_2 := \Big\{&(t,s) \in [0,b+2]_{\mathbb{N}_0} \times [2-\widetilde{\mu}, b+2-\widetilde{\mu}]_{\mathbb{N}_{2-\widetilde{\mu}}} \ : \\ &0 \le t-\widetilde{\mu}+1 \le s \le b+2\Big\}\end{aligned}$$

Theorem 4.3. *Let the operator $T \ : \ \mathcal{B} \to \mathcal{B}$ be defined by*

$$(Ty)(t) := \alpha(t)y(1) + \sum_{s=-\widetilde{\mu}+2}^{b+2-\widetilde{\mu}} G(t,s) f\left(s+\widetilde{\mu}-1, y\left(s+\widetilde{\mu}-1\right)\right), \tag{4.10}$$

where $\alpha \ : \ [0,b+2]_{\mathbb{N}_0} \to \mathbb{R}$ is defined by

$$\alpha(t) := \frac{(t-2+\mu_2+\mu_3)^{\underline{\mu_2+\mu_3-1}}}{\Gamma\left(\mu_2+\mu_3\right)} - \frac{(b+\mu_2+\mu_3)^{\underline{\mu_2+\mu_3-1}}}{(b+\widetilde{\mu})^{\underline{\widetilde{\mu}-1}}\Gamma\left(\mu_2+\mu_3\right)}(t+\widetilde{\mu}-2)^{\underline{\widetilde{\mu}-1}} \tag{4.11}$$

and $G \ : \ [0,b+2]_{\mathbb{N}_0} \times [-\widetilde{\mu}+2, -\widetilde{\mu}+b+2]_{\mathbb{N}_{2-\widetilde{\mu}}} \to \mathbb{R}$ is the Green's function for the non-sequential conjugate problem given by

$$G(t,s) := \begin{cases} \frac{(t+\widetilde{\mu}-2)^{\underline{\widetilde{\mu}-1}}(b+1-s)^{\underline{\widetilde{\mu}-1}}}{(b+\widetilde{\mu})^{\underline{\widetilde{\mu}-1}}} - (t-s-1)^{\underline{\widetilde{\mu}-1}}, & (t,s) \in T_1 \\ \frac{(t+\widetilde{\mu}-2)^{\underline{\widetilde{\mu}-1}}(b+1-s)^{\underline{\widetilde{\mu}-1}}}{(b+\widetilde{\mu})^{\underline{\widetilde{\mu}-1}}}, & (t,s) \in T_2 \end{cases}. \tag{4.12}$$

Then whenever $y \in \mathcal{B}$ is a fixed point of T, it follows that y is a solution of problem (4.1)–(4.2).

Proof. To prove this claim, we shall apply repeatedly Theorem 4.1. To this end, recall

both that $\mu_3 \in (0,1)$ and that $\mu_2+\mu_3 \in (1,2)$. Therefore, it follows from Lemma 2.7 and Theorem 4.1 that

$$\begin{aligned}
\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}y(t) &= \Delta^{\mu_1}\left[\Delta^{\mu_2+\mu_3}y(t) - \frac{y(0)}{\Gamma(-\mu_2)}(t-1+\mu_3)^{\underline{-\mu_2-1}}\right] \\
&= \Delta^{\mu_1}\left[\Delta^{\mu_2+\mu_3}y(t)\right] - \frac{y(0)}{\Gamma(-\mu_2)}\Delta^{\mu_1}\left[(t-1+\mu_3)^{\underline{-\mu_2-1}}\right] \\
&= \Delta^{\widetilde{\mu}}y(t) - \frac{y(0)}{\Gamma(-\mu_2)}\cdot\frac{\Gamma(-\mu_2)}{\Gamma(-\mu_2-\mu_1)}(t-1+\mu_3)^{\underline{-\mu_2-\mu_1-1}}] \\
&\quad - \sum_{j=0}^{1}\left[\frac{\Delta^{j-2+\mu_2+\mu_3}y(2-\mu_2-\mu_3)}{\Gamma(-\mu_1-2+j+1)}(t-2+\mu_2+\mu_3)^{\underline{-\mu_1-2+j}}\right] \qquad (4.13)\\
&= \Delta^{\widetilde{\mu}}y(t) - \frac{\Delta^{\mu_2+\mu_3-2}y(2-\mu_2-\mu_3)}{\Gamma(-\mu_1-1)}(t-2+\mu_2+\mu_3)^{\underline{-\mu_1-2}} \\
&\quad - \frac{\Delta^{\mu_2+\mu_3-1}y(2-\mu_2-\mu_3)}{\Gamma(-\mu_1)}(t-2+\mu_2+\mu_3)^{\underline{-\mu_1-1}} \\
&\quad - \frac{y(0)}{\Gamma(-\mu_2-\mu_1)}(t-1+\mu_3)^{\underline{-\mu_2-\mu_1-1}}.
\end{aligned}$$

Now, the same argument as in Proposition 4.2 shows that

$$\Delta^{\mu_2+\mu_3-2}y(2-\mu_2-\mu_3) = y(0). \qquad (4.14)$$

On the other hand, note that (cf., Definition 2.2)

$$\begin{aligned}
\Delta^{\mu_2+\mu_3-1}y(t) &= \Delta\Delta^{\mu_2+\mu_3-2}y(t) \\
&= \Delta_t\left[\frac{1}{\Gamma(2-\mu_2-\mu_3)}\sum_{s=0}^{t-2+\mu_2+\mu_3}(t-s-1)^{\underline{1-\mu_2-\mu_3}}y(s)\right] \\
&= \frac{1}{\Gamma(2-\mu_2-\mu_3)}\sum_{s=0}^{t-1+\mu_2+\mu_3}(t-s)^{\underline{1-\mu_2-\mu_3}}y(s) \qquad (4.15)\\
&\quad - \frac{1}{\Gamma(2-\mu_2-\mu_3)}\sum_{s=0}^{t-2+\mu_2+\mu_3}(t-s-1)^{\underline{1-\mu_2-\mu_3}}y(s).
\end{aligned}$$

So, from (4.15), it is clear that

$$\begin{aligned}
&\Delta^{\mu_2+\mu_3-1} y\left(2-\mu_2-\mu_3\right) \\
&= \frac{1}{\Gamma\left(2-\mu_2-\mu_3\right)} \sum_{s=0}^{1}\left(2-\mu_2-\mu_3-s\right)^{\underline{1-\mu_2-\mu_3}} y(s) \\
&- \frac{1}{\Gamma\left(2-\mu_2-\mu_3\right)} \sum_{s=0}^{0}\left(1-\mu_2-\mu_3-s\right)^{\underline{1-\mu_2-\mu_3}} y(s) \\
&= \frac{1}{\Gamma\left(2-\mu_2-\mu_3\right)} y(0)\left[\left(2-\mu_2-\mu_3\right)^{\underline{1-\mu_2-\mu_3}}-\left(1-\mu_2-\mu_3\right)^{\underline{1-\mu_2-\mu_3}}\right] \\
&+ \frac{1}{\Gamma\left(2-\mu_2-\mu_3\right)}\left(1-\mu_2-\mu_3\right)^{\underline{1-\mu_2-\mu_3}} y(1).
\end{aligned} \tag{4.16}$$

Putting (4.14) and (4.16) into (4.13), we find that

$$\begin{aligned}
\Delta^{\mu_1} \Delta^{\mu_2} \Delta^{\mu_3} y(t) = \Delta^{\widetilde{\mu}} y(t) &- \frac{\left[y(1)+\left(1-\mu_2-\mu_3\right) y(0)\right]}{\Gamma\left(-\mu_1\right)}\left(t-2+\mu_2+\mu_3\right)^{\underline{-\mu_1-1}} \\
&- \frac{y(0)}{\Gamma\left(-\mu_1-1\right)}\left(t-2+\mu_2+\mu_3\right)^{\underline{-\mu_1-2}} \\
&- \frac{y(0)}{\Gamma\left(-\mu_2-\mu_1\right)}\left(t-1+\mu_3\right)^{\underline{-\mu_2-\mu_1-1}},
\end{aligned} \tag{4.17}$$

where have made some routine simplifications. Now, since $y(0)=0$ by boundary condition (4.2), we find that (4.17) reduces to

$$\Delta^{\mu_1} \Delta^{\mu_2} \Delta^{\mu_3} y(t)=\Delta^{\widetilde{\mu}} y(t)-\frac{\left(t-2+\mu_2+\mu_3\right)^{\underline{-\mu_1-1}}}{\Gamma\left(-\mu_1\right)} y(1). \tag{4.18}$$

Formally inverting the problem (4.1), we find by way of (4.18) above (together

with Lemma 2.4) that

$$y(t) = -\Delta^{-\widetilde{\mu}}\left[-\frac{(t-2+\mu_2+\mu_3)^{\underline{-\mu_1-1}}}{\Gamma(-\mu_1)}y(1)\right] - \Delta^{-\widetilde{\mu}}f\left(t+\widetilde{\mu}-1, y\left(t+\widetilde{\mu}-1\right)\right)$$
$$+ c_1\left(t+\widetilde{\mu}-2\right)^{\underline{\widetilde{\mu}-1}} + c_2\left(t+\widetilde{\mu}-2\right)^{\underline{\widetilde{\mu}-2}}. \tag{4.19}$$

Before continuing further, we wish to give a careful explanation for the basis vectors $(t+\widetilde{\mu}-2)^{\underline{\widetilde{\mu}-1}}$ and $(t+\widetilde{\mu}-2)^{\underline{\widetilde{\mu}-2}}$ appearing in (4.19). (The reader should also consult Holm [63] for a detailed discussion of this point.) The reason for this choice is related to the peculiar statement of the power rule given in Lemma 2.7. Indeed, observe that if $y(t)$ is given as in (4.19) above, then in order for y to be a solution to (4.1), it must be the case that $-\Delta^{\widetilde{\mu}}y(t) = f(t+\widetilde{\mu}-1, y(t+\widetilde{\mu}-1))$, for $t \in \mathbb{N}_{2-\widetilde{\mu}}$. But, in particular, this means that both

$$\Delta^{\widetilde{\mu}}\left[(t+\widetilde{\mu}-2)^{\underline{\widetilde{\mu}-1}}\right] = 0 \tag{4.20}$$

and

$$\Delta^{\widetilde{\mu}}\left[(t+\widetilde{\mu}-2)^{\underline{\widetilde{\mu}-2}}\right] = 0 \tag{4.21}$$

must hold for each admissible t. Whether (4.20)–(4.21) hold depends upon the applicability of the power rule, namely Lemma 2.7, in this situation. Assuming that the power rule may be applied, it is straightforward to check that each of (4.20) and (4.21) does indeed hold.

So, let us explicitly check that the power rule can indeed be applied in this setting. Let us first consider (4.21). In this case, a routine calculation shows that the power rule can be applied whenever $t \in \mathbb{N}_{2-\widetilde{\mu}}$, which, of course, it is by assumption. So, (4.21) is valid. On the other hand, a similar calculation shows that (4.20) holds

provided that $t \in \mathbb{N}_{3-\widetilde{\mu}}$. This would seem to be a problem since we really desire (4.20) to hold at $t = 2 - \widetilde{\mu}$, too. However, notice that

$$\left[(t-2+\widetilde{\mu})^{\underline{\widetilde{\mu}-1}}\right]_{t=0} = 0, \tag{4.22}$$

as is easily checked, and that

$$\begin{aligned}
&\Delta^{\widetilde{\mu}}\left[(t+\widetilde{\mu}-2)^{\underline{\widetilde{\mu}-1}}\right] \\
&= \Delta^2\left[\frac{1}{\Gamma(2-\widetilde{\mu})}\sum_{s=0}^{t+\widetilde{\mu}-2}(t-s-1)^{\underline{1-\widetilde{\mu}}}(s+\widetilde{\mu}-2)^{\underline{\widetilde{\mu}-1}}\right] \\
&= \Delta^2\left[\frac{1}{\Gamma(2-\widetilde{\mu})}\sum_{s=1}^{t+\widetilde{\mu}-2}(t-s-1)^{\underline{1-\widetilde{\mu}}}(s+\widetilde{\mu}-2)^{\underline{\widetilde{\mu}-1}}\right]
\end{aligned} \tag{4.23}$$

where to get the final equality we have used (4.22) above. So, from (4.23) we conclude that (4.20) need only hold for $t \in \mathbb{N}_{3-\widetilde{\mu}}$ because when $t = 2-\widetilde{\mu}$, (4.20) holds vacuously. In summary, both (4.20) and (4.21) hold for all $t \in \mathbb{N}_{2-\widetilde{\mu}}$, as desired.

Now, continuing from (4.19), it is clear that the boundary condition $y(0) = 0$ implies that $c_2 = 0$. On the other hand, the boundary condition $y(b+2) = 0$, implies that

$$\begin{aligned}
0 &= c_1\left(b+\widetilde{\mu}\right)^{\underline{\widetilde{\mu}-1}} + \frac{y(1)}{\Gamma\left(\mu_2+\mu_3\right)}\left(b+\mu_2+\mu_3\right)^{\underline{\mu_2+\mu_3-1}} \\
&\quad - \frac{1}{\Gamma\left(\widetilde{\mu}\right)}\sum_{s=-\widetilde{\mu}+2}^{b+2-\widetilde{\mu}}(b+1-s)^{\underline{\widetilde{\mu}-1}} f\left(s+\widetilde{\mu}-1, y\left(s+\widetilde{\mu}-1\right)\right)
\end{aligned} \tag{4.24}$$

From (4.24), we deduce that

$$\begin{aligned} c_1 = &-\frac{(b+\mu_2+\mu_3)^{\underline{\mu_2+\mu_3-1}}}{(b+\widetilde{\mu})^{\underline{\widetilde{\mu}-1}}\Gamma(\mu_2+\mu_3)}y(1) \\ &+\frac{1}{\Gamma(\widetilde{\mu})}\sum_{s=-\widetilde{\mu}+2}^{b+2-\widetilde{\mu}}\frac{(b+1-s)^{\underline{\widetilde{\mu}-1}}}{(b+\widetilde{\mu})^{\underline{\widetilde{\mu}-1}}}f\left(s+\widetilde{\mu}-1, y\left(s+\widetilde{\mu}-1\right)\right). \end{aligned} \tag{4.25}$$

Finally, putting the obtained values of c_1 and c_2 back into (4.19), we find that

$$y(t) = \alpha(t)y(1) + \sum_{s=-\widetilde{\mu}+2}^{b+2-\widetilde{\mu}} G(t,s)f\left(s+\widetilde{\mu}-1, y\left(s+\widetilde{\mu}-1\right)\right), \tag{4.26}$$

where α is as defined in (4.11) above and $G(t,s)$ is as defined in (4.12) above. Now, if $(Ty)(t)$ is defined by the right-hand side of (4.26), then it is clear that T satisfies both the difference equation (4.1) and the boundary conditions (4.2). Therefore, the desired claim holds, and this completes the proof. $\square$

Remark 4.4. Observe that if $f(t,y)$ has the special form $f(t)$, that is, we consider the linear problem, then from (4.26), it is easy to show that the solution to the boundary value problem is

$$y(t) = \left[\frac{1}{1-\alpha(1)}\sum_{s=-\widetilde{\mu}+2}^{b+2-\widetilde{\mu}} G(1,s)f\left(s+\widetilde{\mu}-1\right)\right]\alpha(t) + \sum_{s=-\widetilde{\mu}+2}^{b+2-\widetilde{\mu}} G(t,s)f\left(s+\widetilde{\mu}-1\right).$$

On the other hand, if $f(t,y)$ has the form $a(t)y$, which also gives rise to a linear problem, then the analysis is rather much more delicate. In fact, there does not appear to be any results in the discrete fractional calculus literature on such linear problems. We leave this consideration, however, to future work.

We next state an easy proposition regarding the Green's function, $G(t,s)$, appearing in the operator T.

Proposition 4.5. *The Green's function $G(t,s)$ given in Theorem 4.3 satisfies:*

1. $G(t,s) \geq 0$ *for each* $(t,s) \in [0,b+2]_{\mathbb{N}_0} \times [2-\widetilde{\mu}, b+2-\widetilde{\mu}]_{\mathbb{N}_{2-\widetilde{\mu}}}$*;*

2. $\max_{t\in[0,b+2]_{\mathbb{N}_0}} G(t,s) = G(s+\widetilde{\mu}-1,s)$ *for each* $s \in [2-\widetilde{\mu}, b+2-\widetilde{\mu}]_{\mathbb{N}_{2-\widetilde{\mu}}}$*; and*

3. *there exists a number* $\gamma \in (0,1)$ *such that*

$$\min_{\left[\frac{b}{4},\frac{3b}{4}\right]_{\mathbb{N}_0}} G(t,s) \geq \gamma \max_{t\in[0,b+2]_{\mathbb{N}_0}} G(t,s) = \gamma G(s+\widetilde{\mu}-1,s),$$

for $s \in [2-\widetilde{\mu}, b+2-\widetilde{\mu}]_{\mathbb{N}_{2-\widetilde{\mu}}}$.

Proof. A straightforward modification of the proof of [20, Theorem 3.2], keeping track of the different domains, yields this result. For example, in case $(t,s) \in T_2$, it is obvious that $\Delta_t G(t,s) \geq 0$. On the other hand, in case $(t,s) \in T_1$, we find that

$$\Delta_t G(t,s) = \frac{(\widetilde{\mu}-1)(t+\widetilde{\mu}-2)^{\underline{\widetilde{\mu}-1}}(b+1-s)^{\underline{\widetilde{\mu}-1}}}{(b+\widetilde{\mu})^{\underline{\widetilde{\mu}-1}}} - (\widetilde{\mu}-1)(t-s-1)^{\underline{\widetilde{\mu}-2}}. \tag{4.27}$$

From (4.27) it is clear that $\Delta_t G(t,s) \leq 0$ if and only if

$$\frac{(t+\widetilde{\mu}-2)^{\underline{\widetilde{\mu}-2}}(b+1-s)^{\underline{\widetilde{\mu}-1}}}{(t-s-1)^{\underline{\widetilde{\mu}-2}}(b+\widetilde{\mu})^{\underline{\widetilde{\mu}-1}}} \leq 1 \tag{4.28}$$

holds. But that (4.28) holds is a consequence of the fact that $t^{\underline{\beta}}$ is increasing when $\beta \in [0,1)$ and decreasing when $\beta \in (-1,0]$, which may be easily verified by using the definition of $t^{\underline{\beta}}$. As a consequence of the fact that $G(t,s)$ is decreasing in t for all $(t,s) \in T_1$ and increasing in t for all $(t,s) \in T_2$, conclusion (2) holds. Moreover, conclusion (1) holds by combining (2) with the fact that $G(0,s) = G(b+2,s) = 0$, for each admissible s. Finally to prove (3), we can give an argument exactly similar to that of Atici and Eloe in [20]. Therefore, we omit this part of the proof. □

We next require a preliminary lemma regarding the behavior of α appearing in (4.11) above.

Lemma 4.6. *Let α be defined as in (4.11). Then $\alpha(0) = \alpha(b+2) = 0$. Moreover, $\|\alpha\| \in (0, 1)$.*

Proof. That $\alpha(0) = \alpha(b+2) = 0$ is obvious. On the other hand, to show that $0 < \|\alpha\| < 1$, we argue as follows.

We first show that $\alpha(t) > 0$, for all $t \in [1, b+1]_{\mathbb{N}}$. Let us first note that

$$\begin{aligned} \alpha(t) &= \frac{(t-2+\mu_2+\mu_3)^{\underline{\mu_2+\mu_3-1}}}{\Gamma(\mu_2+\mu_3)} - \frac{(b+\mu_2+\mu_3)^{\underline{\mu_2+\mu_3-1}}}{(b+\widetilde{\mu})^{\underline{\widetilde{\mu}-1}}\Gamma(\mu_2+\mu_3)}(t+\widetilde{\mu}-2)^{\underline{\widetilde{\mu}-1}} \\ &= \frac{\Gamma(t+\mu_2+\mu_3-1)}{\Gamma(t)\Gamma(\mu_2+\mu_3)} - \frac{\Gamma(b+\mu_2+\mu_3+1)\Gamma(t+\widetilde{\mu}-1)}{\Gamma(b+\widetilde{\mu}+1)\Gamma(\mu_2+\mu_3)\Gamma(t)} \\ &= \frac{\Gamma(t+\mu_2+\mu_3-1)\Gamma(b+\widetilde{\mu}+1) - \Gamma(t+\widetilde{\mu}-1)\Gamma(b+\mu_2+\mu_3+1)}{\Gamma(t)\Gamma(\mu_2+\mu_3)\Gamma(b+\widetilde{\mu}+1)}. \end{aligned} \tag{4.29}$$

Therefore, $\alpha(t) > 0$, for $t \in [1, b+1]_{\mathbb{N}}$, if and only if

$$\Gamma(t+\mu_2+\mu_3-1)\Gamma(b+\widetilde{\mu}+1) > \Gamma(t+\widetilde{\mu}-1)\Gamma(b+\mu_2+\mu_3+1) \tag{4.30}$$

holds for all such t. Now, (4.30) is equivalent to

$$\frac{\Gamma(t+\mu_2+\mu_3-1)\Gamma(b+\widetilde{\mu}+1)}{\Gamma(t+\widetilde{\mu}-1)\Gamma(b+\mu_2+\mu_3+1)} > 1,$$

but since

$$\frac{\Gamma(t+\mu_2+\mu_3-1)\Gamma(b+\widetilde{\mu}+1)}{\Gamma(t+\widetilde{\mu}-1)\Gamma(b+\mu_2+\mu_3+1)} = \frac{(b+\widetilde{\mu})\cdots(t+\widetilde{\mu}-1)}{(b+\mu_2+\mu_3)\cdots(t+\mu_2+\mu_3-1)} \tag{4.31}$$

and the right-hand side of (4.31) is clearly greater than unity, it follows that (4.30) holds, and so, we conclude from (4.29)–(4.31) that $\alpha(t) > 0$, for $t \in [1, b+1]_{\mathbb{N}}$, as

claimed.

On the other hand, to argue that $\alpha(t) < 1$, for $t \in [0, b+2]_{\mathbb{N}_0}$, we begin by recasting $\alpha(t)$ in an alternative form. In particular, define $\mu_0 \in (1,2)$ by

$$\mu_0 := \mu_2 + \mu_3. \tag{4.32}$$

Then it follows that

$$\widetilde{\mu} = \mu_0 + \mu_1. \tag{4.33}$$

Therefore, upon putting (4.32) and (4.33) into the definition of α given in (4.11), we find that

$$\alpha(t) = \frac{(t-2+\mu_0)^{\underline{\mu_0 - 1}}}{\Gamma(\mu_0)} - \frac{(b+\mu_0)^{\underline{\mu_0 - 1}}(t+\mu_0+\mu_1-2)^{\underline{\mu_0+\mu_1-1}}}{(b+\mu_0+\mu_1)^{\underline{\mu_0+\mu_1-1}}\Gamma(\mu_0)}. \tag{4.34}$$

Now, consider the quotient

$$\frac{(t+\mu_0+\mu_1-2)^{\underline{\mu_0+\mu_1-1}}}{(b+\mu_0+\mu_1)^{\underline{\mu_0+\mu_1-1}}} \tag{4.35}$$

appearing in (4.34) above. Since

$$\frac{(t+\mu_0+\mu_1-2)^{\underline{\mu_0+\mu_1-1}}}{(b+\mu_0+\mu_1)^{\underline{\mu_0+\mu_1-1}}} = \frac{(b+1)\cdots(t+1)(t)}{(b+\mu_0+\mu_1)\cdots(t+\mu_0+\mu_1)(t+\mu_0+\mu_1-1)}, \tag{4.36}$$

it is clear from (4.36) that for each fixed but arbitrary b, t, and μ_0, (4.35) decreases

as μ_1 increases. Consequently, for fixed but arbitrary b, t, and μ_0 we conclude that

$$
\begin{aligned}
\alpha(t) &< \frac{(t-2+\mu_0)^{\underline{\mu_0-1}}}{\Gamma(\mu_0)} - \left[\frac{(b+\mu_0)^{\underline{\mu_0-1}}(t+\mu_0+\mu_1-2)^{\underline{\mu_0+\mu_1-1}}}{(b+\mu_0+\mu_1)^{\underline{\mu_0+\mu_1-1}}\Gamma(\mu_0)}\right]_{\mu_1=1} \\
&= \frac{(t-2+\mu_0)^{\underline{\mu_0-1}}}{\Gamma(\mu_0)} - \frac{(b+\mu_0)^{\underline{\mu_0-1}}(t+\mu_0-1)^{\underline{\mu_0}}}{(b+\mu_0+1)^{\underline{\mu_0}}\Gamma(\mu_0)} \\
&= \frac{(t-2+\mu_0)^{\underline{\mu_0-1}}}{\Gamma(\mu_0)} - \frac{\Gamma(b+\mu_0+1)\Gamma(t+\mu_0)\Gamma(b+2)}{\Gamma(b+2)\Gamma(t)\Gamma(\mu_0)\Gamma(b+\mu_0+2)} \\
&= \frac{(t-2+\mu_0)^{\underline{\mu_0-1}}}{\Gamma(\mu_0)} - \frac{\Gamma(t+\mu_0)}{(b+\mu_0+1)\Gamma(t)\Gamma(\mu_0)}.
\end{aligned}
\tag{4.37}
$$

From (4.37), we see that $\alpha(t) < 1$ if and only if

$$
\frac{\Gamma(t+\mu_0-1)}{\Gamma(\mu_0)\Gamma(t)} - \frac{\Gamma(t+\mu_0)}{(b+\mu_0+1)\Gamma(t)\Gamma(\mu_0)} \leq 1 \tag{4.38}
$$

holds, which is equivalent to

$$
\frac{(b+\mu_0+1)\Gamma(t+\mu_0-1)\Gamma(t)\Gamma(\mu_0)}{\Gamma(\mu_0)\Gamma(t)\left[(b+\mu_0+1)\Gamma(\mu_0)\Gamma(t)+\Gamma(t+\mu_0)\right]} \leq 1. \tag{4.39}
$$

Observe that inequality (4.39) is equivalent to

$$
\frac{(b+\mu_0+1)\Gamma(t+\mu_0-1)}{(b+\mu_0+1)\Gamma(\mu_0)\Gamma(t)+\Gamma(t+\mu_0)} \leq 1. \tag{4.40}
$$

We claim that (4.40) holds for each admissible triple $(b, t, \mu_0) \in \mathbb{N} \times [1, b+1]_{\mathbb{N}_0} \times (1, 2)$.

Indeed, rewriting the left-hand side of inequality (4.40) yields

$$
\begin{aligned}
&\frac{(b+\mu_0+1)\Gamma(t+\mu_0-1)}{(b+\mu_0+1)\Gamma(\mu_0)\Gamma(t)+\Gamma(t+\mu_0)} \\
&= \frac{\Gamma(t+\mu_0-1)}{\Gamma(\mu_0)\Gamma(t)+\frac{\Gamma(t+\mu_0)}{b+\mu_0+1}} \\
&= \frac{1}{\frac{\Gamma(\mu_0)\Gamma(t)}{\Gamma(t+\mu_0-1)}+\frac{t+\mu_0-1}{b+\mu_0+1}},
\end{aligned}
\tag{4.41}
$$

so that inequality (4.40) is equivalent to

$$\frac{\Gamma(\mu_0)\,\Gamma(t)}{\Gamma(t+\mu_0-1)}+\frac{t+\mu_0-1}{b+\mu_0+1}\geq 1. \tag{4.42}$$

Now, each of the addends on the left-hand side of (4.42) is nonnegative. Moreover, we note that

$$\frac{\Gamma(\mu_0)\,\Gamma(t)}{\Gamma(t+\mu_0-1)}\geq 1, \tag{4.43}$$

for each admissible t and μ_0 since $t > t+\mu_0-1$. (Note that if $\mu_0 = 1$ we get equality in (4.43).) But then (4.43) implies (4.42), which in turn implies that (4.38) holds.

In summary, for each admissible triple (b, t, μ_0), we find that $\alpha(t) < 1$. In fact, based on the discussion regarding μ_1 given in (4.35)–(4.36), we have actually shown that, for fixed but arbitrary b, t, and μ_0,

$$\sup_{\mu_1\in(0,1)} \alpha(t; b, \mu_0) < 1. \tag{4.44}$$

Hence, (4.44) implies that $\alpha(t) < 1$, for each fixed but arbitrary tuple (b, t, μ_0, μ_1), it holds that $(b, t\mu_0, \mu_1) \in \mathbb{N}\times[1, b+2]_{\mathbb{N}}\times(1,2)\times(0,1)$. As we earlier showed that $\alpha(t) > 0$ whenever $t \neq 0$, $b+2$, we conclude that

$$\|\alpha\| < 1, \tag{4.45}$$

as desired. And this completes the proof. □

Remark 4.7. We emphasize that Theorem 4.3 shows that problem (4.1)–(4.2) is not the same as the conjugate problem studied in [20]. In fact, there is a *de facto* nonlocal nature to problem (4.1)–(4.2) as evidenced by the explicit appearance of $y(1)$ in the operator T.

4.1.2 An Existence Result

As an application of the preceding analysis, we now provide a typical existence theorem for problem (4.1)–(4.2). While the basic argument we employ in the sequel is by now well known, it is not entirely standard given the appearance of $y(1)$ in the operator T.

To this end, let us next provide some standard assumptions on the nonlinearity. In particular, for the sake of simplicity, in the sequel we assume that $f(t,y) := a(t)g(y)$, where we assume that a is continuous and not zero identically on $[0,b+2]_{\mathbb{N}_0}$. We also assume (H1) and (H2) below. While standard assumptions, we indicate in the sequel (cf., Remark 4.10) some potential for less standard generalizations.

H1: We find that $\lim_{y\to 0^+} \frac{g(y)}{y} = 0$.

H2: We find that $\lim_{y\to\infty} \frac{g(y)}{y} = +\infty$.

We shall also need to define a suitable cone in which to look for fixed points of T. In particular, we consider the cone $\mathcal{K} \subseteq \mathcal{B}$, defined by

$$\mathcal{K} := \left\{ y \in \mathcal{B} \ : \ y \geq 0, \ \min_{t\in\left[\frac{b}{4},\frac{3b}{4}\right]_{\mathbb{N}}} y(t) \geq \gamma^* \|y\| \right\}, \tag{4.46}$$

where $\gamma^* \in (0,1)$ is the constant $\gamma^* := \min\left\{ \frac{\min_{t\in\left[\frac{b}{4},\frac{3b}{4}\right]_{\mathbb{N}}} \alpha(t)}{\|\alpha\|}, \gamma \right\}$. Note that, therefore, the constant γ^* is ***not*** same as the constant γ appearing in part 3 of Proposition 4.5. However, it does satisfy $0 < \gamma^* < 1$, as will be demonstrated in the proof of Lemma 4.8 below. We first show that the cone $\mathcal{K}$ is invariant under the operator T. We then argue that conditions (H1)–(H2) imply, as is well known in the integer-order case (e.g., [39]), that problem (4.1)–(4.2) has at least one positive solution.

Lemma 4.8. *Let T be the operator defined in (4.10) and $\mathcal{K}$ the cone defined in (4.46). Then $T\ :\ \mathcal{K} \to \mathcal{K}$.*

Proof. It is obvious that given $y \in \mathcal{K}$, then $(Ty)(t) \geq 0$, for each admissible t. On the other hand, we observe that

$$\begin{aligned}
&\min_{t\in\left[\frac{b}{4},\frac{3b}{4}\right]_{\mathbb{N}}} (Ty)(t) \\
&\geq \gamma_0 y(1)\|\alpha\| + \gamma \sum_{s=-\widetilde{\mu}+2}^{b+2-\widetilde{\mu}} G\left(s+\widetilde{\mu}-1, s\right) f\left(s+\widetilde{\mu}-1, y\left(s+\widetilde{\mu}-1\right)\right) \\
&\geq \gamma^* \left[y(1)\|\alpha\| + \sum_{s=-\widetilde{\mu}+2}^{b+2-\widetilde{\mu}} G\left(s+\widetilde{\mu}-1, s\right) f\left(s+\widetilde{\mu}-1, y\left(s+\widetilde{\mu}-1\right)\right)\right] \\
&\geq \gamma^* \|Ty\|,
\end{aligned} \tag{4.47}$$

where the γ appearing in (4.47) is the same γ as in part 3 of Proposition 3.3, and, in addition, γ_0 is the number satisfying

$$\gamma_0 := \frac{\min_{t\in\left[\frac{b}{4},\frac{3b}{4}\right]_{\mathbb{N}}} \alpha(t)}{\|\alpha\|}, \tag{4.48}$$

with $\gamma_0 \in (0,1)$, evidently. Recall, then, that we define γ^* by

$$\gamma^* := \min\left\{\gamma_0, \gamma\right\}, \tag{4.49}$$

where γ^* obviously satisfies $0 < \gamma^* < 1$. Thus, whenever $y \in \mathcal{K}$, it follows that $Ty \in \mathcal{K}$, and so, the desired claim follows. $\square$

Theorem 4.9. *Assume that $f(t,y) := a(t)g(y)$ satisfies conditions (H1)–(H2). Then problem (4.1)–(4.2) has at least one positive solution.*

Proof. First of all, note that T is trivially completely continuous in this setting.

Second of all, recall from Lemma 4.6 that $\alpha(t) < 1$, for all $t \in [0, b+2]_{\mathbb{N}_0}$. Therefore, we may select $\epsilon > 0$ so that $\alpha(t) < \epsilon < 1$ holds for all admissible t. Given this ϵ, we may, by way of condition (H1), select $\eta_1 > 0$ sufficiently small so that both

$$g(y) \leq \eta_1 y \tag{4.50}$$

and

$$\eta_1 \sum_{s=-\widetilde{\mu}+2}^{b+2-\widetilde{\mu}} G\left(s+\widetilde{\mu}-1, s\right) a(s) \leq 1-\epsilon \tag{4.51}$$

hold for all $0 < y < r_1$, where $r_1 := r_1\left(\eta_1\right)$. Next put

$$\Omega_1 := \left\{y \in \mathcal{B} \ : \ \|y\| < r_1\right\}. \tag{4.52}$$

Then for $y \in \partial\Omega_1 \cap \mathcal{K}$ we find, upon combining (4.50)–(4.51), that

$$\begin{aligned} \|Ty\| &\leq y(1) \max_{t\in[0,b+2]_{\mathbb{N}_0}} \alpha(t) + \max_{t\in[0,b+2]_{\mathbb{N}_0}} \sum_{s=-\widetilde{\mu}+2}^{b+2-\widetilde{\mu}} G(t,s)a(s)g\left(y\left(s+\widetilde{\mu}-1\right)\right) \\ &< \epsilon y(1) + \sum_{s=-\widetilde{\mu}+2}^{b+2-\widetilde{\mu}} G\left(s+\widetilde{\mu}-1, s\right) a(s)\eta_1 y(s) \\ &\leq \epsilon\|y\| + \|y\| \cdot \eta_1 \sum_{s=-\widetilde{\mu}+2}^{b+2-\widetilde{\mu}} G\left(s+\widetilde{\mu}-1, s\right) a(s) \\ &\leq \|y\|, \end{aligned} \tag{4.53}$$

whence (4.53) implies that T is a cone contraction on $\partial\Omega_1 \cap \mathcal{K}$.

On the other hand, from condition (H2) we may select a number $\eta_2 > 0$ such that both

$$\eta_2 \sum_{s=-\widetilde{\mu}+2}^{b+2-\widetilde{\mu}} \gamma^* G\left(s+\widetilde{\mu}-1, s\right) a(s) > 1 \tag{4.54}$$

and

$$g(y) > \eta_2 y \tag{4.55}$$

hold whenever $y > r_2 > 0$, for some sufficiently large number $r_2 := r_2(\eta_2)$. Now, put

$$r_2^* := \left\{ 2r_1, \frac{r_2}{\gamma^*} \right\}. \tag{4.56}$$

Define

$$\Omega_2 := \{ y \in \mathcal{B} \ : \ \|y\| < r_2^* \}. \tag{4.57}$$

Recall that for $y \in \mathcal{K}$, we must have $y(1) \geq 0$, and that from Lemma 3.4 we know also that $\alpha(t) \geq 0$, for all $t \in [0, b+2]_{\mathbb{N}_0}$. Then it is now standard to show that

$$\|Ty\| \geq \|y\|, \tag{4.58}$$

whenever $y \in \partial\Omega_2 \cap \mathcal{K}$, so that T is a cone expansion on $\partial\Omega_2 \cap \mathcal{K}$.

In summary, we may invoke Lemma 2.13 to deduce the existence of a function $y_0 \in \mathcal{K} \cap \left(\overline{\Omega_2} \setminus \Omega_1\right)$ such that $Ty_0 = y_0$, where y_0 is a positive solution to problem (4.1)–(4.2). And this completes the proof. □

Remark 4.10. It seems quite possible to deduce a set of corresponding existence results for problem (4.1) augmented with various sorts of nonlocal boundary conditions replacing (4.2), as has been extensively investigated in the integer-order setting and, more recently, by the present author in the discrete fractional setting – see [45, 47]. We leave this investigation for future work.

4.1.3 Result for the Corresponding Delta-Nabla Problem

We conclude this section with a corollary. In particular, with the preceding analysis in hand, it is easy to study sequential problems involving one or more nabla fractional differences. In particular, using Lemma 2.6, we get the following corollary. Nearly all of the proof of Corollary 4.11 follows immediately from Theorem 4.9. While there is a modest calculation to verify the interchange of the delta and nabla differences, since this essentially follows more or less directly from Lemma 2.6, we do not present the proof of the following corollary.

Corollary 4.11. *Consider the following sequential FBVP*

$$-\Delta^{\mu_1}\Delta^{\mu_2}\nabla^{\mu_3}y\left(t+\mu_3\right)=f\left(t+\widetilde{\mu}-1,y\left(t+\widetilde{\mu}-1\right)\right) \tag{4.59}$$

subject to

$$y(0)=0=y(b+2). \tag{4.60}$$

Then supposing that $f(t,y):=a(t)g(y)$ with $g(y)$ satisfying conditions (H1)–(H2), it follows that problem (4.59)–(4.60) has at least one positive solution.

Naturally, it is possible to write down all manner of permutations of (4.59) and thus all manner of existence results. But we omit the details here.

Remark 4.12. As mentioned in Section 4.1, on the time scale $\mathbb{Z}$ there is much less reason to study delta-nabla problems than on a more general time scale. However, since the fractional calculus is beginning to progress to arbitrary time scales, a result such as Corollary 4.11 seems relevant.

Remark 4.13. As mentioned in Remark 4.10, one could write down a result dual to Corollary 4.11 in the case where boundary condition (4.2) is replaced with some sort

of nonlocal condition, but, once again, we leave this to future investigations.

4.2 Extensions

We now briefly comment on some possible extensions of the results given previously. These extensions allow us to give existence theorems analogous to Theorem 4.9 for all manner of discrete fractional sequential BVPs. In particular, let us consider the following sequential fractional difference

$$\Delta^{\mu_n} \cdots \Delta^{\mu_1} y(t). \tag{4.61}$$

where $\mu_j \in (0,1)$ for each $j = 1, \ldots, n$, under a couple of different additional assumptions on the μ_j's. For notational simplicity in the sequel, we define

$$\widetilde{\mu}_j^{+} := \sum_{k=1}^{j} \mu_k \tag{4.62}$$

and

$$\widetilde{\mu}_j^{-} := \sum_{k=n-j}^{n-1} \mu_k. \tag{4.63}$$

As before, we continue to use the symbol $\widetilde{\mu}$ to denote the sum $\sum_{j=1}^{n} \mu_j$.

Proposition 4.14. *Assume that* $0 < \sum_{j=1}^{n-1} \mu_j < 1$ *and* $1 < \sum_{j=1}^{n} \mu_j < 2$. *Then it follows that*

$$\begin{aligned}
&\Delta^{\mu_n} \cdots \Delta^{\mu_1} y(t) \\
&= \Delta^{\widetilde{\mu}_n^{+}} y(t) - \left[\frac{\left(t-1+\widetilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-1}}}{\Gamma\left(-\mu_n\right)} - \sum_{j=1}^{n-2} \frac{\left(t-1+\widetilde{\mu}_j^{+}\right)^{\underline{-\widetilde{\mu}_{n-j+1}^{-}-\mu_n-1}}}{\Gamma\left(-\widetilde{\mu}_{n-j+1}^{-}-\mu_n\right)} \right] y(0).
\end{aligned}$$

Proof. We again repeatedly appeal to Lemma 2.7 and Theorem 4.1. In particular, we

note first that

$$\begin{aligned}
&\Delta^{\mu_n}\cdots\Delta^{\mu_3}\left[\Delta^{\mu_2}\Delta^{\mu_1}y(t)\right]\\
&=\Delta^{\mu_n}\cdots\Delta^{\mu_3}\left[\Delta^{\widetilde{\mu}_2^+}y(t)-\frac{\Delta^{\mu_1-1}y\left(1-\mu_1\right)}{\Gamma\left(-\mu_2\right)}\left(t-1+\mu_1\right)^{\underline{-\mu_2-1}}\right]\\
&=\Delta^{\mu_n}\cdots\Delta^{\mu_4}\left[\Delta^{\widetilde{\mu}_3^+}y(t)-\frac{\Delta^{\mu_1+\mu_2-1}y\left(1-\mu_1-\mu_2\right)}{\Gamma\left(-\mu_3\right)}\left(t-1+\mu_1+\mu_2\right)^{\underline{-\mu_3-1}}\right.\\
&\left.-\frac{\Delta^{\mu_1-1}y\left(1-\mu_1\right)}{\Gamma\left(-\mu_2-\mu_3\right)}\left(t-1+\mu_1\right)^{\underline{-\mu_2-\mu_3-1}}\right].
\end{aligned}\tag{4.64}$$

Now, repeating this process yields by means of a straightforward induction argument

$$\Delta^{\mu_{n-1}}\cdots\Delta^{\mu_1}y(t)=\Delta^{\widetilde{\mu}_{n-1}^+}y(t)-\sum_{j=1}^{n-2}\left[\frac{\Delta^{\widetilde{\mu}_j^+-1}y\left(1-\widetilde{\mu}_j^+\right)}{\Gamma\left(-\widetilde{\mu}_{n-j-1}^-\right)}\left(t-1+\widetilde{\mu}_j^+\right)^{\underline{-\widetilde{\mu}_{n-j-1}^--1}}\right].\tag{4.65}$$

Consequently, it follows that

$$\begin{aligned}
&\Delta^{\mu_n}\cdots\Delta^{\mu_1}y(t)\\
&=\Delta^{\mu_n}\left\{\Delta^{\widetilde{\mu}_{n-1}^+}y(t)-\sum_{j=1}^{n-2}\left[\frac{\Delta^{\widetilde{\mu}_j^+-1}y\left(1-\widetilde{\mu}_j^+\right)}{\Gamma\left(-\widetilde{\mu}_{n-j-1}^-\right)}\left(t-1+\widetilde{\mu}_j^+\right)^{\underline{-\widetilde{\mu}_{n-j-1}^--1}}\right]\right\}\\
&=\Delta^{\widetilde{\mu}_n^+}y(t)-\frac{\Delta^{-1+\widetilde{\mu}_{n-1}^+}y\left(1-\widetilde{\mu}_{n-1}^+\right)}{\Gamma\left(-\mu_n\right)}\left(t-1+\widetilde{\mu}_{n-1}^+\right)^{\underline{-\mu_n-1}}\\
&+\sum_{j=1}^{n-2}\left[\frac{\Delta^{\widetilde{\mu}_j^+-1}y\left(1-\widetilde{\mu}_j^+\right)}{\Gamma\left(-\widetilde{\mu}_{n-j-1}^-\right)}\cdot\frac{\Gamma\left(-\widetilde{\mu}_{n-j-1}^-\right)}{\Gamma\left(-\widetilde{\mu}_{n-j-1}^--\mu_n\right)}\left(t-1+\widetilde{\mu}_j^+\right)^{\underline{-\widetilde{\mu}_{n-j-1}^--\mu_n-1}}\right]\\
&=\Delta^{\widetilde{\mu}_n^+}y(t)-\left[\frac{\left(t-1+\widetilde{\mu}_{n-1}^+\right)^{\underline{-\mu_n-1}}}{\Gamma\left(-\mu_n\right)}-\sum_{j=1}^{n-2}\frac{\left(t-1+\widetilde{\mu}_j^+\right)^{\underline{-\widetilde{\mu}_{n-j+1}^--\mu_n-1}}}{\Gamma\left(-\widetilde{\mu}_{n-j+1}^--\mu_n\right)}\right]y(0),
\end{aligned}\tag{4.66}$$

as claimed, which completes the proof. □

Our next proposition provides for a more direct generalization of problem (4.1) considered earlier.

Proposition 4.15. *Suppose that* $0<\sum_{j=1}^{n-2}\mu_j<1$, $1<\sum_{j=1}^{n-1}\mu_j<2$, *and* $1<$

$\sum_{j=1}^{n} \mu_j < 2$. *Then we find that*

$$\begin{aligned}
\Delta^{\mu_n} \cdots \Delta^{\mu_1} y(t) &= \Delta^{\widetilde{\mu}} y(t) - \frac{\left(t-2+\widetilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-1}}}{\Gamma\left(-\mu_n\right)} y(1) \\
&\quad - \sum_{j=1}^{n-2} \left[\frac{1}{\Gamma\left(-\widetilde{\mu}_{n-j-1}^{-} - \mu_n\right)} \left(t-1+\widetilde{\mu}_j^{+}\right)^{\underline{-\widetilde{\mu}_{n-j-1}^{-}-\mu_n-1}} \right] y(0) \\
&\quad - \left[\frac{\left(t-2+\widetilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-1}}}{\Gamma\left(-\mu_n\right)} \left(1-\widetilde{\mu}_{n-1}^{+}\right) - \frac{\left(t-2+\widetilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-2}}}{\Gamma\left(-\mu_n-1\right)} \right] y(0).
\end{aligned} \tag{4.67}$$

Proof. In this setting, observe that (4.65) still holds. Therefore, we need only make some minor modifications to the proof of Proposition 4.14. In particular, we find that

$$\begin{aligned}
&\Delta^{\mu_n} \cdots \Delta^{\mu_1} y(t) \\
&= \Delta^{\mu_n} \left\{ \Delta^{\widetilde{\mu}_{n-1}^{+}} y(t) - \sum_{j=1}^{n-2} \left[\frac{\Delta^{\widetilde{\mu}_j^{+}-1} y\left(1-\widetilde{\mu}_j^{+}\right)}{\Gamma\left(-\widetilde{\mu}_{n-j-1}^{-}\right)} \left(t-1+\widetilde{\mu}_j^{+}\right)^{\underline{-\widetilde{\mu}_{n-j-1}^{-}-1}} \right] \right\} \\
&= \Delta^{\mu_n} \Delta^{\widetilde{\mu}_{n-1}^{+}} y(t) - \sum_{j=1}^{n-2} \left[\frac{\Delta^{\widetilde{\mu}_j^{+}-1} y\left(1-\widetilde{\mu}_j^{+}\right)}{\Gamma\left(-\widetilde{\mu}_{n-j-1}^{-}\right)} \Delta^{\mu_n} \left[\left(t-1+\widetilde{\mu}_j^{+}\right)^{\underline{-\widetilde{\mu}_{n-j-1}^{-}-1}} \right] \right] \\
&= \Delta^{\widetilde{\mu}} y(t) - \sum_{k=0}^{1} \frac{\Delta^{j-2+\widetilde{\mu}_{n-1}^{+}} y\left(2-\widetilde{\mu}_{n-1}^{+}\right)}{\Gamma\left(-\mu_n-1+j\right)} \left(t-2+\widetilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-2+j}} \\
&\quad - \sum_{j=1}^{n-2} \left[\frac{\Delta^{\widetilde{\mu}_j^{+}-1} y\left(1-\widetilde{\mu}_j^{+}\right)}{\Gamma\left(-\widetilde{\mu}_{n-j-1}^{-}\right)} \cdot \frac{\Gamma\left(-\widetilde{\mu}_{n-j-1}^{-}\right)}{\Gamma\left(-\widetilde{\mu}_{n-j-1}^{-}-\mu_n\right)} \left(t-1+\widetilde{\mu}_j^{+}\right)^{\underline{-\widetilde{\mu}_{n-j-1}^{-}-\mu_n-1}} \right].
\end{aligned} \tag{4.68}$$

Now notice that

$$\frac{\Delta^{-2+\widetilde{\mu}_{n-1}^{+}} y\left(2-\widetilde{\mu}_{n-1}^{+}\right)}{\Gamma\left(-\mu_n-1\right)} \left(t-2+\widetilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-2}} = \frac{\left(t-2+\widetilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-2}}}{\Gamma\left(-\mu_n-1\right)} y(0) \tag{4.69}$$

and that

$$\begin{aligned}
&\frac{\Delta^{-1+\widetilde{\mu}_{n-1}^{+}}y\left(2-\widetilde{\mu}_{n-1}^{+}\right)}{\Gamma\left(-\mu_n\right)}\left(t-2+\widetilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-1}}\\
&=\frac{\left(t-2+\widetilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-1}}}{\Gamma\left(-\mu_n\right)}\left[\left(1-\widetilde{\mu}_{n-1}^{+}\right)y(0)+y(1)\right].
\end{aligned} \tag{4.70}$$

Therefore, it follows that

$$\begin{aligned}
\Delta^{\mu_n}\cdots\Delta^{\mu_1}y(t)&=\Delta^{\widetilde{\mu}}y(t)-\frac{\left(t-2+\widetilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-1}}}{\Gamma\left(-\mu_n\right)}y(1)\\
&\quad-\sum_{j=1}^{n-2}\left[\frac{1}{\Gamma\left(-\widetilde{\mu}_{n-j-1}^{-}-\mu_n\right)}\left(t-1+\widetilde{\mu}_j^{+}\right)^{\underline{-\widetilde{\mu}_{n-j-1}^{-}-\mu_n-1}}\right]y(0)\\
&\quad-\left[\frac{\left(t-2+\widetilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-1}}}{\Gamma\left(-\mu_n\right)}\left(1-\widetilde{\mu}_{n-1}^{+}\right)-\frac{\left(t-2+\widetilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-2}}}{\Gamma\left(-\mu_n-1\right)}\right]y(0),
\end{aligned} \tag{4.71}$$

as claimed. And this completes the proof. □

Propositions 4.14 and 4.15 again show that the sequential problems are (potentially) different than the non-sequential problems and identify, in particular, the differences. Furthermore, with Propositions 4.14 and 4.15 in hand, we can write down a multitude of results regarding the existence of positive solutions to discrete sequential fractional BVPs. But, in particular, we would need to show that the various coefficient functions of $y(0)$ and $y(1)$ appearing in Propositions 4.14 and 4.15 satisfy inequalities similar to those of Lemma 3.3. We leave this task to future work.

Remark 4.16. It is also possible to study the problem in which, say, the right-hand side of (4.1) is replaced with

$$f(t+\mu_1+\mu_2+\mu_3-1,y(t+\mu_1+\mu_2+\mu_3-1),y(1)), \tag{4.72}$$

since our analysis shows that problem (4.1)–(4.2) fits into this somewhat more general framework. Nonetheless, we feel the results of this section are still relevant, particularly in the case where $y(0) \neq 0$ as, say, would occur in the setting of a nonlocal boundary condition.

Chapter 5

Analysis of a Right-Focal Discrete Fractional BVP

We have previously considered certain of the operational properties of the fractional difference and certain of the implications of these properties. We now wish to consider these implications with rather increased specificity. To this end, In this chapter we consider existence results for a certain two-point boundary value problem of right-focal type for a fractional difference equation. A recent paper by Atici and Eloe [20] produced a well-posed fractional boundary value problem (FBVP) of the type we consider here. However, their paper considered only the case of Dirichlet or conjugate-type boundary conditions. Given the interest in right-focal BVPs in the classical literature, the present chapter can be considered an important extension of and parallel to [20].

In particular, we will be interested in the nonlinear finite discrete FBVP given by

$$\begin{cases} -\Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)) \\ y(\nu-2) = 0 = \Delta y(\nu+b) \end{cases}, \tag{5.1}$$

where $t \in [0, b+1]_{\mathbb{N}_0}$, $\nu \in (1, 2]$, $f : [\nu - 1, \nu + b]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \to \mathbb{R}$, and $b \in \mathbb{N}_0$. Thus, we shall offer results that complement and extend the exposition given in [20]. As means toward accomplishing this task, we first deduce the existence of a unique solution to the FBVP

$$\begin{cases} -\Delta^{\nu} y(t) = h(t + \nu - 1) \\ y(\nu - 2) = 0 = \Delta y(\nu + b) \end{cases}, \tag{5.2}$$

where $\nu \in (1, 2]$, $t \in [0, b+1]_{\mathbb{N}_0}$, and $h : [\nu - 1, \nu + b]_{\mathbb{N}_{\nu-1}} \to \mathbb{R}$, by means of an appropriate Green's function. We undertake this task next.

5.1 Derivation of the Green's Function

In order to help us analyze the nonlinear problem (5.1), we now wish to derive a Green's function for (5.2). Of particular note, we shall observe at the end of this section that in case $\nu = 2$, the Green's function we obtain in Theorem 5.1 below matches the Green's function obtained in the case when $\nu = 2$. Before stating this useful theorem, let us introduce the following notation, which will be important in the sequel.

$$T_1 := \left\{ (t, s) \in [\nu - 1, \nu + b + 1]_{\mathbb{N}_{\nu-1}} \times [0, b+1]_{\mathbb{N}_0} \ : \ 0 \le s < t - \nu + 1 \le b + 2 \right\}$$

$$T_2 := \left\{ (t, s) \in [\nu - 1, \nu + b + 1]_{\mathbb{N}_{\nu-1}} \times [0, b+1]_{\mathbb{N}_0} \ : \ 0 \le t - \nu + 1 \le s \le b + 2 \right\}$$

Theorem 5.1. *The unique solution of the FBVP (5.2) is given by*

$$y(t) := \sum_{s=0}^{b+1} G(t, s) h(s + \nu - 1),$$

where $G(t,s)$ is the Green's function for the problem

$$-\Delta^{\nu}y(t)=0,\ y(\nu-2)=0=\Delta y(\nu+b), \tag{*}$$

where $1<\nu\leq 2$, which is given by

$$G(t,s):=\frac{1}{\Gamma(\nu)}\begin{cases}\frac{\Gamma(b+3)t^{\underline{\nu-1}}}{\Gamma(\nu+b+1)}(\nu+b-s-1)^{\underline{\nu-2}}-(t-s-1)^{\underline{\nu-1}}, & (t,s)\in T_1\\ \frac{\Gamma(b+3)t^{\underline{\nu-1}}}{\Gamma(\nu+b+1)}(\nu+b-s-1)^{\underline{\nu-2}}, & (t,s)\in T_2\end{cases}.$$

Proof. Observe that by inverting the fractional difference operator coupled with an application of Lemma 2.4, we find that a general solution of the fractional difference equation in (5.2) is

$$y(t)=-\Delta^{-\nu}h(t+\nu-1)+C_1t^{\underline{\nu-1}}+C_2t^{\underline{\nu-2}},$$

whence we get that

$$y(t)=-\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}h(s+\nu-1)+C_1t^{\underline{\nu-1}}+C_2t^{\underline{\nu-2}}.$$

We now would like to determine the values of C_1 and C_2 so that the boundary conditions in (*) hold. To this end, applying the boundary condition $y(\nu-2)=0$, we find that

$$0=-\Delta^{-\nu}h(t+\nu-1)\Big|_{t=\nu-2}+C_1(\nu-2)^{\underline{\nu-1}}+C_2(\nu-2)^{\underline{\nu-2}}. \tag{5.3}$$

It is obvious that $(\nu-2)^{\underline{\nu-1}}=0$. Similarly, it evidently holds that $(\nu-2)^{\underline{\nu-2}}=\Gamma(\nu-1)$.

Finally,

$$-\Delta^{-\nu}h(t+\nu-1)\big|_{t=\nu-2}=-\frac{1}{\Gamma(\nu)}\sum_{s=0}^{-2}(t-s-1)^{\underline{\nu-1}}h(s+\nu-1)=0$$

by the standard convention on sums. So, in summary, we find that (5.3) implies that $C_2=0$. Similarly, we can apply the right boundary condition – namely, $\Delta y(\nu+b)=0$. Doing so, we find that

$$0=\Delta y(\nu+b)=\left\{\Delta\left[-\Delta^{-\nu}h(t+\nu-1)\right]\right\}_{t=\nu+b}+\Delta\left[C_1t^{\underline{\nu-1}}\right]_{t=\nu+b}. \tag{5.4}$$

Note that

$$\Delta\left[t^{\underline{\nu-1}}\right]_{t=\nu+b}=(\nu-1)\cdot\frac{\Gamma(\nu+b+1)}{\Gamma(b+3)} \tag{5.5}$$

and, since by definition $\Delta\Delta^{-\nu}=\Delta^{1-\nu}$, that

$$\begin{aligned}\left\{\Delta\left[\Delta^{-\nu}h(t+\nu-1)\right]\right\}_{t=\nu+b}&=\left[\Delta^{1-\nu}h(t+\nu-1)\right]_{t=\nu+b}\\&=\left[\frac{1}{\Gamma(\nu-1)}\sum_{s=0}^{t-\nu+1}(t-s-1)^{\underline{\nu-2}}h(s+\nu-1)\right]_{t=\nu+b}\\&=\frac{1}{\Gamma(\nu-1)}\sum_{s=0}^{b+1}(\nu+b-s-1)^{\underline{\nu-2}}h(s+\nu-1).\end{aligned} \tag{5.6}$$

Putting the preceding equalities together, it is a simple matter to show that

$$C_1=\frac{\left[\Delta\Delta^{-\nu}h(t+\nu-1)\right]_{t=\nu+b}}{\frac{(\nu-1)\Gamma(\nu+b+1)}{\Gamma(b+3)}}=\frac{\Gamma(b+3)}{\Gamma(\nu)\Gamma(\nu+b+1)}\sum_{s=0}^{b+1}(\nu+b-s-1)^{\underline{\nu-2}}h(s-\nu+1). \tag{5.7}$$

But with (5.7) in hand, we can determine $y(t)$ exactly. In particular, we find that

$$\begin{aligned}
y(t) &= -\Delta^{-\nu}h(t+\nu-1)+C_1t^{\underline{\nu-1}}\\
&= -\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}h(s+\nu-1)\\
&+\frac{\Gamma(b+3)t^{\underline{\nu-1}}}{\Gamma(\nu)\Gamma(\nu+b+1)}\sum_{s=0}^{b+1}(\nu+b-s-1)^{\underline{\nu-2}}h(s-\nu+1)\\
&= \frac{1}{\Gamma(\nu)}\left\{\sum_{s=0}^{t-\nu}\left[\frac{\Gamma(b+3)t^{\underline{\nu-1}}}{\Gamma(\nu+b+1)}(\nu+b-s-1)^{\underline{\nu-2}}-(t-s-1)^{\underline{\nu-1}}\right]h(s+\nu+1)\right\}\\
&+\frac{1}{\Gamma(\nu)}\sum_{s=t-\nu+1}^{b+1}\frac{\Gamma(b+3)t^{\underline{\nu-1}}}{\Gamma(\nu+b+1)}(\nu+b-s-1)^{\underline{\nu-2}}h(s-\nu+1),
\end{aligned}$$

from which it is immediately clear that we may write

$$y(t)=\sum_{s=0}^{b+1}G(t,s)h(s+\nu-1),$$

where

$$G(t,s):=\frac{1}{\Gamma(\nu)}\begin{cases}\frac{\Gamma(b+3)t^{\underline{\nu-1}}}{\Gamma(\nu+b+1)}(\nu+b-s-1)^{\underline{\nu-2}}-(t-s-1)^{\underline{\nu-1}}, & (t,s)\in T_1\\ \frac{\Gamma(b+3)t^{\underline{\nu-1}}}{\Gamma(\nu+b+1)}(\nu+b-s-1)^{\underline{\nu-2}}, & (t,s)\in T_2\end{cases},$$

is the Green's function for (*). And this completes the proof. □

Remark 5.2. Observe that $G(\nu-2,s)=0$, for each $s\in[0,b+1]_{\mathbb{N}_0}$.

Remark 5.3. Let us note for the reader that in case we put $\nu=2$ in Theorem 5.1, it follows that we get the "usual" Green's function, just as we might hope would

happen. Indeed, in case $\nu = 2$ we find that (in case $a = 0$)

$$G(t,s) = \begin{cases} s+1, & 0 \le s < t-1 \le b+2 \\ t, & 0 \le t-1 \le s \le b+2 \end{cases},$$

with $G(t,s)$ defined on $[1, b+3]_{\mathbb{N}_0} \times [0, b+1]_{\mathbb{N}_0}$, which accords with the usual results.

Remark 5.4. As is implied by the definition of the sets T_1 and T_2 as well the form of $G(t,s)$ as given in Theorem 5.1, we have that the Green's function, $G(t,s)$, is defined on the set $[\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}} \times [0, b+1]_{\mathbb{N}_0}$. Incidentally, it is easy to show that $G(t, b+2) = 0$, for each admissible t. So, G could be extended to $[\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}} \times [0, b+2]_{\mathbb{N}_0}$ without difficulty, but we do not require this in the sequel.

5.2 Properties of the Green's Function

In this section of the chapter, we wish to prove that our Green's function $G(t,s)$ satisfies, with appropriate and simple modifications, the usual classical properties. Certain of these properties will be crucial when we prove our existence theorems in the final section of this paper. We begin by stating a lemma; its proof may be found in [20].

Lemma 5.5. *Let ν be any positive real number and let a and b be two real numbers satisfying $\nu < a \le b$. Then the following hold.*

(i.) $\frac{1}{x^{\underline{\nu}}}$ *is a decreasing function for $x \in (\nu, +\infty)_{\mathbb{N}}$.*

(ii.) $\frac{(a-x)^{\underline{\nu}}}{(b-x)^{\underline{\nu}}}$ *is a decreasing function for $x \in [0, a-\nu)_{\mathbb{N}_0}$.*

We now state and prove the first of a trio of propositions regarding $G(t,s)$.

Proposition 5.6. *The function $G(t,s)$ defined in Theorem 5.1 satisfies $G(t,s) \geq 0$ for all $t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}$ and $s \in [0, b+1]_{\mathbb{N}_0}$, where $[\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}} := \{\nu-1, \nu, \ldots, \nu+b+1\}$.*

Proof. To prove this proposition, we shall show directly that $G(t,s) > 0$ for each $(t,s) \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}} \times [0, b+1]_{\mathbb{N}_0}$. For simplicity, we shall look at $\Gamma(\nu)G(t,s)$, for $\Gamma(\nu) > 0$ so that if $\Gamma(\nu)G(t,s) > 0$, then at once it follows that $G(t,s) > 0$, too.

First notice that for $(t,s) \in T_2$, we have that

$$\begin{aligned}\Gamma(\nu)G(t,s) &= \frac{\Gamma(b+3)t^{\underline{\nu-1}}}{\Gamma(\nu+b+1)}(\nu+b-s-1)^{\underline{\nu-2}} \\ &= \frac{\Gamma(b+3)\Gamma(t+1)\Gamma(\nu+b-s)}{\Gamma(\nu+b+1)\Gamma(t-\nu+2)\Gamma(b-s+2)} \\ &> 0,\end{aligned}$$

clearly.

On the other hand, for $(t,s) \in T_1$, we find that

$$\begin{aligned}\Gamma(\nu)G(t,s) &= \frac{\Gamma(b+3)t^{\underline{\nu-1}}(\nu+b-s-1)^{\underline{\nu-2}}}{\Gamma(\nu+b+1)} - (t-s-1)^{\underline{\nu-1}} \\ &= \frac{\Gamma(b+3)\Gamma(t+1)\Gamma(\nu+b-s)}{\Gamma(\nu+b+1)\Gamma(t+2-\nu)\Gamma(b-s+2)} - \frac{\Gamma(t-s)}{\Gamma(t-s-\nu+1)}.\end{aligned}$$

We claim that

$$\frac{\Gamma(b+3)\Gamma(t+1)\Gamma(\nu+b-s)}{\Gamma(\nu+b+1)\Gamma(t+2-\nu)\Gamma(b-s+2)} - \frac{\Gamma(t-s)}{\Gamma(t-s-\nu+1)} > 0.$$

To see that this is true, note that it suffices to show that

$$\frac{\Gamma(b+3)\Gamma(t+1)\Gamma(\nu+b-s)\Gamma(t-s-\nu+1)}{\Gamma(\nu+b+1)\Gamma(t+2-\nu)\Gamma(b-s+2)\Gamma(t-s)} > 1$$

whenever $(t,s) \in T_1$. To prove this latter claim, we shall show that for each admissible s and t, we have both that

$$\frac{\Gamma(b+3)\Gamma(\nu+b-s)}{\Gamma(\nu+b+1)\Gamma(b-s+2)} \geq 1 \tag{5.8}$$

and that

$$\frac{\Gamma(t+1)\Gamma(t-s-\nu+1)}{\Gamma(t+2-\nu)\Gamma(t-s)} > 1, \tag{5.9}$$

from which the desired claim will follow at once, clearly.

To see that (5.8) holds, let s_0 be an arbitrary but fixed element of $[0, b+1]_{\mathbb{N}_0}$. Then we find that

$$\begin{aligned} \frac{\Gamma(b+3)\Gamma(\nu+b-s)}{\Gamma(\nu+b+1)\Gamma(b-s+2)} &= \frac{\Gamma(b+3)\Gamma\left(\nu+b-s_0\right)}{\Gamma(\nu+b+1)\Gamma\left(b-s_0+2\right)} \\ &= \frac{(b+2)!\Gamma\left(\nu+b-s_0\right)}{\Gamma(\nu+b+1)\left(b-s_0+1\right)!} \\ &= \frac{(b+2)(b+1)\cdots\left(b-s_0+2\right)}{(b+\nu)(b+\nu-1)\cdots\left(b+\nu-s_0\right)}. \end{aligned} \tag{5.10}$$

But notice that $\frac{b+2}{b+\nu} \geq 1$, $\frac{b+1}{b+\nu-1} \geq 1$, $\ldots$, $\frac{b-s_0+2}{b+\nu-s_0} \geq 1$ in expression (5.10) above, with equality occurring if and only if $\nu = 2$. Thus, we conclude that

$$\frac{\Gamma(b+3)\Gamma(\nu+b-s)}{\Gamma(\nu+b+1)\Gamma(b-s+2)} \geq 1,$$

which establishes (5.8).

On the other hand, to see that (5.9) holds, let s_0, once again, be arbitrary but fixed such that $s_0 \in [0, b+1]_{\mathbb{N}_0}$. Then we have that for t to be admissible, $t = s_0 + k + \nu$,

for some $0 \leq k \leq b - s_0 + 1$ with $k \in \mathbb{N}_0$. But then it follows that

$$\begin{aligned}
&\frac{\Gamma(t+1)\Gamma(t-s-\nu+1)}{\Gamma(t+2-\nu)\Gamma(t-s)} \\
&= \frac{\Gamma\left(s_0+k+\nu+1\right)}{\Gamma\left(s_0+k+2\right)} \cdot \frac{\Gamma(k+1)}{\Gamma(k+\nu)} \\
&= \frac{\left(\nu+s_0+k\right)\left(\nu+s_0+k-1\right)\cdots(\nu+k)\,\Gamma(k+\nu)}{\left(s_0+k+1\right)!} \cdot \frac{k!}{\Gamma(k+\nu)} \\
&= \frac{\left(\nu+s_0+k\right)\cdots(\nu+k)\cdot k!}{\left(s_0+k+1\right)!} \\
&= \frac{\left(s_0+k+\nu\right)\left(s_0+k-1+\nu\right)\cdots(k+\nu)}{\left(s_0+k+1\right)\left(s_0+k\right)\cdots(k+1)}.
\end{aligned} \tag{5.11}$$

Notice, however, that each of the numerator and denominator in (5.11) has excatly (s_0+1)-terms. Moreover, if we consider the terms in paris, as in $\frac{s_0+k+\nu}{s_0+k+1}$, $\frac{s_0+k-1+\nu}{s_0+k}$, $\ldots$, $\frac{k+\nu}{k+1}$, then we notice that each pair is greater than unity. Indeed, as $1 < \nu \leq 2$, it follows at once, for example, that $\frac{s_0+k+\nu}{s_0+k+1} > 1$. As this argument may be applied to each of the (s_0+1)-terms in (5.11), it follows that

$$\frac{\Gamma(t+1)\Gamma(t-s-\nu+1)}{\Gamma(t+2-\nu)\Gamma(t-s)} > 1,$$

which establishes (5.9).

Finally, combining (5.8) and (5.9), we see at once that

$$\frac{\Gamma(b+3)\Gamma(t+1)\Gamma(\nu+b-s)\Gamma(t-s-\nu+1)}{\Gamma(\nu+b+1)\Gamma(t+2-\nu)\Gamma(b-s+2)\Gamma(t-s)} > 1,$$

whenever $(t,s) \in T_1$, whence $G(t,s) \geq 0$ whenever $(t,s) \in T_1$. Together with the first part of the proof, we find that $G(t,s) \geq 0$ for all $t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}$ and $s \in [0, b+1]_{\mathbb{N}_0}$, as claimed. $\square$

Before proving Proposition 5.8 below, we need an easy but important preliminary

lemma. We remark that this lemma will get used several times in later chapters – cf., Chapter 6 especially.

Lemma 5.7. *Fix $k \in \mathbb{N}$ and let $\{m_j\}_{j=1}^{k}$, $\{n_j\}_{j=1}^{k} \subseteq (0,+\infty)$ such that*

$$\max_{1 \le j \le k} m_j \le \min_{1 \le j \le k} n_j$$

and that for at least one j_0, $1 \le j_0 \le k$, we have that $m_{j_0} < n_{j_0}$. Then for fixed $\alpha_0 \in (0,1)$, it follows that

$$\left(\frac{n_1}{n_1+\alpha_0} \cdot \dots \cdot \frac{n_k}{n_k+\alpha_0}\right)\left(\frac{m_1+\alpha_0}{m_1} \cdot \dots \cdot \frac{m_k+\alpha_0}{m_k}\right) > 1.$$

Proof. Fix an index j_0, where j_0 is one of the indices, of which there exists at least one, for which $n_{j_0} > m_{j_0}$. Notice that as $n_{j_0} > m_{j_0}$ and $\alpha_0 > 0$, it follows that $n_{j_0}\alpha_0 > m_{j_0}\alpha_0$, whence $m_{j_0}n_{j_0} + n_{j_0}\alpha_0 > m_{j_0}n_{j_0} + m_{j_0}\alpha_0$, so that

$$\frac{m_{j_0}+\alpha_0}{m_{j_0}} > \frac{n_{j_0}+\alpha_0}{n_{j_0}},$$

whence

$$\frac{n_{j_0}}{n_{j_0}+\alpha_0} \cdot \frac{m_{j_0}+\alpha_0}{m_{j_0}} > 1.$$

But now the claim follows at once by repeating the above steps for each of the remaining $j_0 - 1$ terms and observing that the product of j terms, each of which is at least unity and at least one of which exceeds unity, is greater than unity. $\square$

Proposition 5.8. *For $G(t,s)$ defined in Theorem 5.1, it follows that*

$$\max_{t \in [\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}} G(t,s) = G(s+\nu-1,s),$$

whenever $s \in [0, b+1]_{\mathbb{N}_0}$.

Proof. Before beginning this proof, let us make one preliminary observation. Indeed, note that $[\Delta_t G(t,s)]_{t=\nu+b} = G(\nu+b+1,s) - G(\nu+b,s) = 0$, for each admissible s, which is easy to verify by direct computation. Of course, this must be true by virtue of the fact that G must satisfy the right-hand boundary condition in each of FBVPs (5.1) and (5.2). Practically, this means that

$$\max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}} G(t,s) = \max_{t\in[\nu-1,\nu+b]_{\mathbb{N}_{\nu-1}}} G(t,s),$$

for each admissible s. Consequently, this means that in the sequel, we can effectively ignore what happens at $t = \nu+b+1$ on account of the above noted relationship, and we do just that.

Now, let us consider the difference $\Gamma(\nu)\Delta_t G(t,s)$ for $(t,s) \in T_1$. In this case, we find that

$$\begin{aligned}
&\Gamma(\nu)\Delta_t G(t,s)\\
&= \Delta_t\left[\frac{\Gamma(b+3)t^{\underline{\nu-1}}}{\Gamma(\nu+b+1)}(\nu+b-s-1)^{\underline{\nu-2}} - (t-s-1)^{\underline{\nu-1}}\right]\\
&= \frac{\Gamma(b+3)(\nu-1)t^{\underline{\nu-2}}}{\Gamma(\nu+b+1)}(\nu+b-s-1)^{\underline{\nu-2}} - (\nu-1)(t-s-1)^{\underline{\nu-2}}\\
&= \frac{\Gamma(b+3)(\nu-1)t^{\underline{\nu-2}}(\nu+b-s-1)^{\underline{\nu-2}} - (\nu-1)\Gamma(\nu+b+1)(t-s-1)^{\underline{\nu-2}}}{\Gamma(\nu+b+1)}\\
&= \frac{\nu-1}{\Gamma(\nu+b+1)}\left[\Gamma(b+3)t^{\underline{\nu-2}}(\nu+b-s-1)^{\underline{\nu-2}} - \Gamma(\nu+b+1)(t-s-1)^{\underline{\nu-2}}\right].
\end{aligned}$$

Note that it is clear from the above expression that in case $\nu = 2$, we find that $\Delta_t G(t,s) = 0$, as expected. Consequently, let us assume in the sequel that $1 < \nu < 2$. Observe that $\frac{\nu-1}{\Gamma(\nu+b+1)} > 0$, clearly. So, it follows that $\Gamma(\nu)\Delta_t G(t,s) < 0$ (and thus

that $\Delta_t G(t,s)<0$, seeing as $\Gamma(\nu)>0$) provided that

$$\Gamma(b+3)t^{\underline{\nu-2}}(\nu+b-s-1)^{\underline{\nu-2}}<\Gamma(\nu+b+1)(t-s-1)^{\underline{\nu-2}}$$

and this is true if and only if

$$\frac{\Gamma(\nu+b+1)\Gamma(t-s)\Gamma(b-s+2)}{\Gamma(b+3)\Gamma(\nu+b-s)\Gamma(t-s-\nu+2)t^{\underline{\nu-2}}}>1. \tag{5.12}$$

To show that (5.12) holds, let us, as in the proof of Proposition 5.6, suppose that s_0 is a fixed but arbitrary element of $[0,b+1]_{\mathbb{N}_0}$. Then it follows, as before, that $t=s_0+k+\nu$, where $k\in\mathbb{N}$ such that $0\leq k\leq b-s_0$. But we then find that

$$\begin{aligned}
&\frac{\Gamma(\nu+b+1)\Gamma(t-s)\Gamma(b-s+2)}{\Gamma(b+3)\Gamma(\nu+b-s)\Gamma(t-s-\nu+2)t^{\underline{\nu-2}}}\\
&=\frac{\Gamma(\nu+b+1)\Gamma\left(s_0+k+\nu-s_0\right)\Gamma\left(b-s_0+2\right)\Gamma\left(s_0+k+\nu-\nu+3\right)}{\Gamma(b+3)\Gamma\left(\nu+b-s_0\right)\Gamma\left(s_0+k+\nu-s_0-\nu+2\right)\Gamma\left(s_0+k+\nu+1\right)}\\
&=\frac{\Gamma(\nu+b+1)\Gamma(k+\nu)\Gamma\left(b-s_0+2\right)\Gamma\left(s_0+k+3\right)}{\Gamma(b+3)\Gamma\left(\nu+b-s_0\right)\Gamma(k+2)\Gamma\left(s_0+k+\nu+1\right)}\\
&=\frac{\Gamma(\nu+b+1)\Gamma(k+\nu)\left(b-s_0+1\right)!\left(s_0+k+2\right)!}{(b+2)!\Gamma\left(\nu+b-s_0\right)(k+1)!\Gamma\left(s_0+k+\nu+1\right)}\\
&=\frac{\left[(\nu+b)(\nu+b-1)\cdots\left(\nu+b-s_0\right)\right]\left(b-s_0+1\right)!\left(s_0+k+2\right)!}{(b+2)!(k+1)!\left[\left(s_0+k+\nu\right)\left(s_0+k+\nu-1\right)\cdots(k+\nu)\right]}\\
&=\frac{(\nu+b)(\nu+b-1)\cdots\left(\nu+b-s_0\right)}{(b+2)(b+1)\cdots\left(b-s_0+2\right)}\cdot\frac{\left(s_0+k+2\right)\left(s_0+k+1\right)\cdots(k+2)}{\left(s_0+k+\nu\right)\left(s_0+k+\nu-1\right)\cdots(k+\nu)}.
\end{aligned} \tag{5.13}$$

Observe that each of the numerators and denominators of each of the two fractions in (5.13) has exactly s_0+1 factors. Moreover, observe that in the case of the first fraction, we can consider this fraction as the product of s_0+1 factors as in $\frac{\nu+b}{b+2}\cdot\frac{\nu+b-1}{b+1}\cdot\ldots\cdot\frac{\nu+b-s_0}{b-s_0+2}$. Now, put $\alpha_0:=2-\nu$ and note that $\alpha_0\in(0,1)$. Also put

$n_j := \nu + b + (1-j)$ for $1 \le j \le s_0 + 1$. Then we find that

$$\frac{(\nu+b)(\nu+b-1)\cdots(\nu+b-s_0)}{(b+2)(b+1)\cdots(b-s_0+2)} = \prod_{j=1}^{s_0+1} \frac{n_j}{n_j+\alpha_0},$$

where the finite sequence $\{n_j\}_{j=1}^{s_0+1} \subseteq (0,\infty)$ and the number α_0 satisfy the hypotheses of Lemma 5.7. In a completely similar way, if we put $m_j := k + \nu + (j-1)$, then we find that

$$\frac{(s_0+k+2)\,(s_0+k+1)\cdots(k+2)}{(s_0+k+\nu)\,(s_0+k+\nu-1)\cdots(k+\nu)} = \prod_{j=1}^{s_0+1} \frac{m_j+\alpha_0}{m_j},$$

which again is of the form in Lemma 5.7, for $\{m_j\}_{j=1}^{s_0+1} \subseteq (0,\infty)$. Consequently, with m_j, n_j, and α_0 defined as above, we note that

$$\begin{aligned}
&\frac{\Gamma(\nu+b+1)\Gamma(t-s)\Gamma(b-s+2)}{\Gamma(b+3)\Gamma(\nu+b-s)\Gamma(t-s-\nu+2)t^{\underline{\nu-2}}} \\
&= \frac{(\nu+b)(\nu+b-1)\cdots(\nu+b-s_0)}{(b+2)(b+1)\cdots(b-s_0+2)} \cdot \frac{(s_0+k+2)\,(s_0+k+1)\cdots(k+2)}{(s_0+k+\nu)\,(s_0+k+\nu-1)\cdots(k+\nu)} \\
&= \left(\prod_{j=1}^{s_0+1} \frac{n_j}{n_j+\alpha_0}\right)\left(\prod_{j=1}^{s_0+1} \frac{m_j+\alpha_0}{m_j}\right).
\end{aligned} \tag{5.14}$$

Now, in order to apply Lemma 5.7 to (5.14) above, we must consider three cases. First, it is possible, depending upon the choice of s_0, k, and b, that there are no repeated factors between the two products in (5.14). In this case, we see that $\max_j m_j < \min_j n_j$, and so, by the argument in the preceding paragraph, we may immediately apply Lemma 5.7 to deduce the bound given in (5.12).

Secondly, it is possible that some factors are repeated between the two products in (5.14). In particular, there may be p such repeated factors, with $1 \le p \le s_0$, in each of the numerators and denominators of each of the products in (5.14) that cancel. This cancellation will leave $s_0 + 1 - p$ factors – in particular, in this case it is easy to

show that

$$\begin{aligned}&\frac{\Gamma(\nu+b+1)\Gamma(t-s)\Gamma(b-s+2)}{\Gamma(b+3)\Gamma(\nu+b-s)\Gamma(t-s-\nu+2)t^{\underline{\nu-2}}}\\&=\left(\prod_{j=1}^{s_0+1}\frac{n_j}{n_j+\alpha_0}\right)\left(\prod_{j=1}^{s_0+1}\frac{m_j+\alpha_0}{m_j}\right)\\&=\left(\prod_{j=1}^{s_0+1-p}\frac{n_j}{n_j+\alpha_0}\right)\left(\prod_{j=1}^{s_0+1-p}\frac{m_j+\alpha_0}{m_j}\right).\end{aligned} \tag{5.15}$$

But then Lemma 5.7 may be applied to (5.15) above to yield the bound in (5.12) in this case, too.

Finally, if $k = b - s_0$, then it equally easy to show that product (5.13) is exactly unity – that is,

$$\frac{\Gamma(\nu+b+1)\Gamma(t-s)\Gamma(b-s+2)}{\Gamma(b+3)\Gamma(\nu+b-s)\Gamma(t-s-\nu+2)t^{\underline{\nu-2}}}=\left(\prod_{j=1}^{s_0+1}\frac{n_j}{n_j+\alpha_0}\right)\left(\prod_{j=1}^{s_0+1}\frac{m_j+\alpha_0}{m_j}\right)=1.$$

However, this corresponds to the case $\Delta_t\left[G\left(t,s_0\right)\right]_{t=\nu+b}$, and we observed at the beginning of this proof that $\Delta_t\left[G\left(t,s_0\right)\right]_{t=\nu+b}=0$, as it must from the boundary conditions.

So, in summary, in each of the three cases we can safely apply Lemma 5.7 to (5.13) to get that

$$\frac{\Gamma(\nu+b+1)\Gamma(t-s)\Gamma(b-s+2)}{\Gamma(b+3)\Gamma(\nu+b-s)\Gamma(t-s-\nu+2)t^{\underline{\nu-2}}}>1,$$

so that (5.12) holds. By the earlier observation, then, it follows that $\Delta_tG(t,s)<0$ whenever $0\le s<t-\nu+1\le b+1$, as desired.

We next argue that $\Delta_tG(t,s)>0$ for $0\le t-\nu+1\le s\le b+1$. To see that this

is true, we simply notice that for $0 \leq t - \nu + 1 \leq s \leq b + 1$,

$$\begin{aligned}
\Delta_t G(t,s) &= \Delta_t \left[\frac{\Gamma(b+3) t^{\underline{\nu-1}}}{\Gamma(\nu+b+1)} (\nu+b-s-1)^{\underline{\nu-2}} \right] \\
&= \frac{\Gamma(b+3)(\nu-1) t^{\underline{\nu-2}}}{\Gamma(\nu+b+1)} (\nu+b-s-1)^{\underline{\nu-2}} \\
&= \frac{\Gamma(b+3)(\nu-1)\Gamma(t+1)\Gamma(\nu+b-s)}{\Gamma(\nu+b+1)\Gamma(t-\nu+3)\Gamma(b-s+2)}.
\end{aligned} \tag{5.16}$$

Now, observe that each factor in (5.16) is strictly positive. Therefore, we conclude that $\Delta_t G(t,s) > 0$ in case $0 \leq t - \nu + 1 \leq s \leq b + 1$, whence $G(t,s)$ is increasing on that interval, too.

In summary, then, we have that $G(t,s)$ is increasing for $t - \nu + 1 \leq s \leq b + 1$ and decreasing for $0 \leq s < t - \nu + 1$. And from this we may conclude that

$$\max_{t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}} G(t,s) = G(s + \nu - 1, s),$$

whenever $s \in [0, b+1]_{\mathbb{N}_0}$, as desired. $\square$

Remark 5.9. Interestingly, we notice that in case $\nu \in (1,2)$, Proposition 5.8 demonstrates that $G(t,s)$ is not constant for $t > s + \nu - 1$. This contrasts with the classical case, $\nu = 2$, in which the Green's function attains its maximum at $t = s$ and then is constant for $t > s$. Furthermore, as $\nu \to 2$ from the left, our Green's function does tend to the known Green's function in case $\nu = 2$.

Before proving our final proposition, let us introduce the constants γ_1 and γ_2, which will be important not only in the following proposition but also in the final section of this paper.

$$\gamma_1 := \frac{\left(\frac{b+\nu}{4}\right)^{\underline{\nu-1}}}{(b+\nu)^{\underline{\nu-1}}}$$

$$\gamma_2 := \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}} \left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{b+1}{\Gamma(b+3)} \cdot \frac{\left(\frac{3(b+\nu)}{4} - 1\right)^{\underline{\nu-1}} \Gamma(\nu+b+1)}{(\nu+b-1)^{\underline{\nu-1}}} \right]$$

Proposition 5.10. *Assume that* $\left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right] \cap \mathbb{N}_{\nu-1} \neq \varnothing$. *For* $G(t,s)$ *defined in Theorem 5.1, it follows that there exists a number* $\gamma \in (0,1)$, *where*

$$\gamma := \min\left\{\gamma_1, \gamma_2\right\},$$

with γ_1 *and* γ_2 *as above, such that*

$$\min_{t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]} G(t,s) \geq \gamma \cdot \max_{t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}} G(t,s) = \gamma G(s+\nu-1, s),$$

for $s \in [0, b+1]_{\mathbb{N}_0}$.

Proof. Let us begin by noting that

$$\frac{G(t,s)}{G(s+\nu-1,s)} = \begin{cases} \frac{t^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}} - \frac{(t-s-1)^{\underline{\nu-1}}\Gamma(\nu+b+1)}{\Gamma(b+3)(s+\nu-1)^{\underline{\nu-1}}(\nu+b-s-1)^{\underline{\nu-2}}}, & (t,s) \in T_1 \\ \frac{t^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}}, & (t,s) \in T_2 \end{cases},$$

which is obtained by direct calculation. Now, for $s \geq t-\nu+1$ and $\frac{b+\nu}{4} \leq t \leq \frac{3(b+\nu)}{4}$, we have that

$$\frac{G(t,s)}{G(s+\nu-1,s)} = \frac{t^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}} \geq \frac{\left(\frac{b+\nu}{4}\right)^{\underline{\nu-1}}}{((b+1)+\nu-1)^{\underline{\nu-1}}} = \frac{\left(\frac{b+\nu}{4}\right)^{\underline{\nu-1}}}{(b+\nu)^{\underline{\nu-1}}}, \tag{5.17}$$

because $t^{\underline{\alpha}}$ is increasing in t for $\alpha \in (0,1)$.

On the other hand, the proof of Proposition 5.8 shows that $G(t,s)$ is decreasing in case $s < t-\nu+1$. Consequently, for $s < t-\nu+1$ and $t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]$ it follows

that

$$\begin{aligned}
&\min_{t\in\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]} \frac{G(t,s)}{G(s+\nu-1,s)}\\
&=\left[\frac{t^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}}-\frac{(t-s-1)^{\underline{\nu-1}}\Gamma(\nu+b+1)}{\Gamma(b+3)(s+\nu-1)^{\underline{\nu-1}}(\nu+b-s-1)^{\underline{\nu-2}}}\right]_{t=\frac{3(b+\nu)}{4}}\\
&=\frac{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}}-\frac{\left(\frac{3(b+\nu)}{4}-s-1\right)^{\underline{\nu-1}}\Gamma(\nu+b+1)}{\Gamma(b+3)(s+\nu-1)^{\underline{\nu-1}}(\nu+b-s-1)^{\underline{\nu-2}}}.
\end{aligned}$$

Now, put

$$\alpha(s):=\frac{1}{(s+\nu-1)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}-\frac{\left(\frac{3(b+\nu)}{4}-s-1\right)^{\underline{\nu-1}}\Gamma(\nu+b+1)}{\Gamma(b+3)(\nu+b-s-1)^{\underline{\nu-2}}}\right].$$

Notice that

$$(\nu+b-s-1)^{\underline{\nu-2}}=\frac{(\nu+b-s-1)^{\underline{\nu-1}}}{b-s+1},$$

which is a simple consequence of Definition 2.2. Furthermore, observe that by Lemma 5.5, part (ii) we find that

$$\frac{\left(\frac{3(b+\nu)}{4}-s-1\right)^{\underline{\nu-1}}}{(\nu+b-s-1)^{\underline{\nu-1}}}$$

is decreasing for $0\leq s\leq\frac{3(b+\nu)}{4}-\nu+1$. Consequently, these two observations together

with an application of Lemma 5.5, part (i) imply that

$$\begin{aligned}
&\alpha(s)\\
&=\frac{1}{(s+\nu-1)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}-\frac{\left(\frac{3(b+\nu)}{4}-s-1\right)^{\underline{\nu-1}}\Gamma(\nu+b+1)}{\Gamma(b+3)(\nu+b-s-1)^{\underline{\nu-2}}}\right]\\
&=\frac{1}{(s+\nu-1)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}-\frac{\left(\frac{3(b+\nu)}{4}-s-1\right)^{\underline{\nu-1}}\Gamma(\nu+b+1)}{\frac{\Gamma(b+3)}{b-s+1}\cdot(\nu+b-s-1)^{\underline{\nu-1}}}\right]\\
&\geq\frac{1}{(s+\nu-1)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}-\frac{b+1}{\Gamma(b+3)}\cdot\frac{\left(\frac{3(b+\nu)}{4}-1\right)^{\underline{\nu-1}}\Gamma(\nu+b+1)}{(\nu+b-1)^{\underline{\nu-1}}}\right]\\
&\geq\frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}-\frac{b+1}{\Gamma(b+3)}\cdot\frac{\left(\frac{3(b+\nu)}{4}-1\right)^{\underline{\nu-1}}\Gamma(\nu+b+1)}{(\nu+b-1)^{\underline{\nu-1}}}\right],
\end{aligned}$$

where to get the first inequality we set $s=0$ in the expression in the square brackets. As a result of this analysis, we conclude that in case $s<t-\nu+1$ and $t\in\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]$,

$$\begin{aligned}
&\frac{G(t,s)}{G(s+\nu-1,s)}\\
&\geq\frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}-\frac{b+1}{\Gamma(b+3)}\cdot\frac{\left(\frac{3(b+\nu)}{4}-1\right)^{\underline{\nu-1}}\Gamma(\nu+b+1)}{(\nu+b-1)^{\underline{\nu-1}}}\right].
\end{aligned}\tag{5.18}$$

Finally, then, upon combining (5.17) and (5.18), we deduce that

$$\min_{\frac{b+\nu}{4}\leq t\leq\frac{3(b+\nu)}{4}}G(t,s)\geq\gamma\max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}}G(t,s)=\gamma G(s+\nu-1,s),$$

where we put

$$\gamma:=\min\left\{\gamma_1,\gamma_2\right\},$$

which completes the proof. $\square$

Remark 5.11. Note that it is the case that $0<\gamma<1$ in Proposition 5.10. Indeed, it is clear that $0<\frac{\left(\frac{b+\nu}{4}\right)^{\underline{\nu-1}}}{(b+\nu)^{\underline{\nu-1}}}<1$. On the other hand, to see that

$$0<\frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}-\frac{b+1}{\Gamma(b+3)}\cdot\frac{\left(\frac{3(b+\nu)}{4}-1\right)^{\underline{\nu-1}}\Gamma(\nu+b+1)}{(\nu+b-1)^{\underline{\nu-1}}}\right]<1,$$

we may observe that

$$\begin{aligned}0&<\frac{b+1}{\Gamma(b+3)}\cdot\frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}\cdot\frac{\Gamma\left(\frac{3(b+\nu)}{4}\right)\Gamma(\nu+b+1)\Gamma(b+1)}{\Gamma\left(\frac{3(b+\nu)}{4}-\nu+1\right)\Gamma(\nu+b)}\\&=\frac{1}{b+2}\cdot\frac{\Gamma\left(\frac{3(b+\nu)}{4}\right)\Gamma(\nu+b+1)\Gamma\left(\frac{3(b+\nu)}{4}-\nu+2\right)}{\Gamma\left(\frac{3(b+\nu)}{4}+1\right)\Gamma(\nu+b)\Gamma\left(\frac{3(b+\nu)}{4}-\nu+1\right)}\\&=\frac{(b+\nu)\left(\frac{3(b+\nu)}{4}-\nu+1\right)}{(b+2)\left(\frac{3(b+\nu)}{4}\right)}\\&<1,\end{aligned}$$

which suffices to prove the claim.

Remark 5.12. In case we put $\nu=2$ in Proposition 5.10, we find by direct calculation that $\gamma:=\min\left\{\frac{1}{4},\frac{4}{3b+6}\right\}$.

Remark 5.13. It should be noted that while the right-focal problem is simpler than the Dirichlet problem in the case when $\nu=2$, it is more difficult in the fractional case (i.e., in case $1<\nu<2$) as a comparison with the above proofs to the corresponding proofs in [20] shows.

5.3 Existence and Uniqueness Theorems

In this final section of the chapter, we wish to deduce certain representative existence and uniqueness theorems. So, we now consider the nonlinear equation (5.1). We notice that y solves (5.1) if and only if y is a fixed point of the operator

$$Ty := \sum_{s=0}^{b+1} G(t,s) f(s+\nu-1, y(s+\nu-1)),$$

where G is the Green's function derived in this paper and $T : \mathcal{B} \to \mathcal{B}$, where $\mathcal{B}$ is the Banach space $\mathcal{B} := \{y : [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}} \to \mathbb{R} : y(\nu-2) = \Delta y(\nu+b) = 0\}$ equipped with the usual supremum norm, $\|\cdot\|$.

Let us also make the following declarations, which will be used in the sequel.

$$\eta := \frac{1}{\sum_{s=1}^{b+1} G(s+\nu-1, s)}$$

$$\lambda := \frac{1}{\sum_{s=\lceil \frac{\nu+b}{4} - \nu + 1 \rceil}^{\lfloor \frac{3(\nu+b)}{4} - \nu + 1 \rfloor} G\left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s\right)}$$

Let us also introduce two conditions on the behavior of f that will be useful in the sequel.

(C1) There exists a number $r > 0$ such that $f(t,y) \leq \eta r$ whenever $0 \leq y \leq r$.

(C2) There exists a number $r > 0$ such that $f(t,y) \geq \lambda r$ whenever $\gamma r \leq y \leq r$.

Remark 5.14. The technique that we use to deduce the existence of at least one positive solution is very similar to the techniques found in the classical literature on differential equations – see, for example, [39].

We now can prove the following existence result.

Theorem 5.15. *Suppose that there are distinct r_1, $r_2 > 0$ such that condition (C1)*

holds at $r = r_1$ and condition (C2) holds at $r = r_2$. Suppose also that $f(t, y) \geq 0$ and continuous. Then the FBVP (5.1) has at least one positive solution, say y_0, such that $\|y_0\|$ lies between r_1 and r_2.

Proof. We shall assume without loss of generality that $0 < r_1 < r_2$. Consider the set $\mathcal{K} := \left\{ y \in \mathcal{B} \ : \ y(t) \geq 0 \text{ and } \min_{t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]} y(t) \geq \gamma \|y\| \right\}$, which is a cone with $\mathcal{K} \subseteq \mathcal{B}$. Observe that $T \ : \ \mathcal{K} \to \mathcal{K}$, for we observe both that

$$\begin{aligned} \min_{t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]} (Ty)(t) &= \min_{t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]} \sum_{s=0}^{b+1} G(t, s) f(s + \nu - 1, y(s + \nu - 1)) \\ &\geq \gamma \sum_{s=0}^{b+1} G(s + \nu - 1, s) f(s + \nu - 1, y(s + \nu - 1)) \\ &= \gamma \max_{t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}} \sum_{s=0}^{b+1} G(t, s) f(s + \nu - 1, y(s + \nu - 1)) \\ &= \gamma \|Ty\|, \end{aligned}$$

and that $(Ty)(t) \geq 0$ whenever $y \in \mathcal{K}$, whence $Ty \in \mathcal{K}$, as claimed. Also, it is easy to see that T is a completely continuous operator.

Now, put $\Omega_1 := \{y \in \mathcal{K} \ : \ \|y\| < r_1\}$. Note that for $y \in \partial\Omega_1$, we have that $\|y\| = r_1$ so that condition (C1) holds for all $y \in \partial\Omega_1$. So, for $y \in \mathcal{K} \cap \partial\Omega_1$, we find that

$$\begin{aligned} \|Ty\| &= \max_{t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}} \sum_{s=0}^{b+1} G(t, s) f(s + \nu - 1, y(s + \nu - 1)) \\ &\leq \sum_{s=0}^{b+1} G(s + \nu - 1, s) f(s + \nu - 1, y(s + \nu - 1)) \\ &\leq \eta r_1 \sum_{s=0}^{b+1} G(s + \nu - 1, s) \\ &= r_1 = \|y\|, \end{aligned}$$

whence we find that $\|Ty\| \leq \|y\|$ whenever $y \in \mathcal{K} \cap \partial\Omega_1$. Thus we get that the operator T is a cone compression on $\mathcal{K} \cap \partial\Omega_1$.

On the other hand, put $\Omega_2 := \{y \in \mathcal{K} : \|y\| < r_2\}$. Note that for $y \in \partial\Omega_2$, we have that $\|y\| = r_2$ so that condition (C2) holds for all $y \in \partial\Omega_2$. Also note that $\left\{\left\lfloor \frac{b+1}{2} \right\rfloor + \nu\right\} \subset \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]$. So, for $y \in \mathcal{K} \cap \partial\Omega_2$, we find that

$$\begin{aligned}
Ty\left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu\right) &= \sum_{s=0}^{b+1} G\left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s\right) f(s+\nu-1, y(s+\nu-1)) \\
&\geq \sum_{s=\lceil \frac{\nu+b}{4} - \nu + 1 \rceil}^{\lfloor \frac{3(\nu+b)}{4} - \nu + 1 \rfloor} G\left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s\right) f(s+\nu-1, y(s+\nu-1)) \\
&\geq \lambda r_2 \sum_{s=\lceil \frac{\nu+b}{4} - \nu + 1 \rceil}^{\lfloor \frac{3(\nu+b)}{4} - \nu + 1 \rfloor} G\left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s\right) \\
&= r_2,
\end{aligned}$$

whence $\|Ty\| \geq \|y\|$, whenever $y \in \mathcal{K} \cap \partial\Omega_2$. Thus we get that the operator T is a cone expansion on $\mathcal{K} \cap \partial\Omega_2$. So, it follows by Lemma 2.13 that the operator T has a fixed point. But this means that (5.1) has a positive solution, say y_0, with $r_1 \leq \|y_0\| \leq r_2$, as claimed. □

Remark 5.16. Of course, it is possible to extend Theorem 5.15. In particular, one can provide conditions under which multiple positive solutions will exist. But as such extensions are standard, we omit them here.

If we assume that f satisfies a Lipschitz condition, then we can get uniqueness in addition to existence. This is the content of Theorem 5.18 below. We require first a preliminary lemma.

Lemma 5.17. *For $G(t,s)$ as defined in Theorem 5.1, we find that*

$$\max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}} \sum_{s=0}^{b+1} G(t,s) \leq \frac{(b+2)\Gamma(b+\nu+2)}{\Gamma(\nu+1)\Gamma(b+2)}.$$

Proof. By invoking Theorem 5.1 together with Proposition 5.8 we find that

$$\begin{aligned} G(s+\nu-1,s) &= \frac{\Gamma(b+3)(s+\nu-1)^{\underline{\nu-1}}\Gamma(\nu+b-s)}{\Gamma(\nu)\Gamma(\nu+b+1)\Gamma(b-s+2)} \\ &\leq \frac{(b+2)!\Gamma(b+2-s)(s+\nu-1)^{\underline{\nu-1}}}{\Gamma(\nu)(b+1)!\Gamma(b-s+2)} \\ &= \frac{(b+2)}{\Gamma(\nu)}(s+\nu-1)^{\underline{\nu-1}}, \end{aligned}$$

from which it follows that

$$\begin{aligned} \max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}} \sum_{s=0}^{b+1} G(t,s) &\leq \sum_{s=0}^{b+1} \frac{b+2}{\Gamma(\nu)}(s+\nu-1)^{\underline{\nu-1}} \\ &= \frac{b+2}{\Gamma(\nu)}\left[\frac{1}{\nu}(s+\nu-1)^{\underline{\nu}}\right]_{s=0}^{b+2} \\ &= \frac{b+2}{\Gamma(\nu)}\cdot\frac{1}{\nu}(b+\nu+1)^{\underline{\nu}} \\ &= \frac{(b+2)\Gamma(b+\nu+2)}{\Gamma(\nu+1)\Gamma(b+2)}, \end{aligned}$$

as claimed. □

Now we prove a uniqueness theorem by using the Banach contraction theorem, whose statement can be found, for example, in [92].

Theorem 5.18. *Suppose that $f(t,y)$ satisfies a Lipschitz condition in y with Lipschitz constant α – that is, $|f(t,y_2)-f(t,y_1)| \leq \alpha|y_2-y_1|$ for all (t,y_1), (t,y_2). Then it follows that if*

$$\frac{(b+2)\Gamma(b+\nu+2)}{\Gamma(\nu+1)\Gamma(b+2)} < \frac{1}{\alpha},$$

then (5.1) has a unique solution.

Proof. Let y_1, $y_2 \in \mathcal{B}$, where $\mathcal{B}$ is the Banach space described earlier. Then we find that

$$\begin{aligned}
&\|Ty_2 - Ty_1\| \\
&\leq \max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}} \sum_{s=0}^{b+1} \Big[|G(t,s)| \\
&\cdot \left| f\left(s+\nu-1, y_2(s+\nu-1)\right) - f\left(s+\nu-1, y_1(s+\nu-1)\right)\right| \Big] \\
&\leq \alpha \sum_{s=0}^{b+1} G(s+\nu-1,s)\left|y_2(s+\nu-1) - y_1(s+\nu-1)\right| \\
&\leq \alpha\|y_2 - y_1\| \sum_{s=0}^{b+1} G(s+\nu-1,s) \\
&\leq \alpha \frac{(b+2)\Gamma(b+\nu+2)}{\Gamma(\nu+1)\Gamma(b+2)}\|y_2 - y_1\|.
\end{aligned}$$

So, as $\alpha\frac{(b+2)\Gamma(b+\nu+2)}{\Gamma(\nu+1)\Gamma(b+2)} < 1$ by assumption, it follows by the Banach contraction theorem that (5.1) has a unique solution, as claimed. □

Example 5.19. Suppose that $\nu := \frac{11}{10}$ and $\alpha := \frac{1}{75}$. If $f(t,y)$ in problem (5.1) is Lipschitz with Lipschitz constant α, then Theorem 5.18 implies that (5.1) will have a unique solution provided that

$$\frac{(b+2)\Gamma\left(b+\frac{31}{10}\right)}{\Gamma\left(\frac{21}{10}\right)\Gamma(b+2)} < 75, \tag{5.19}$$

and (5.19) can be solved numerically to get that $b_{\max} \approx 5.960$, where $b_{\max}$ is the largest value of b such that the hypotheses of Theorem 5.18 is satisfied.

Remark 5.20. The bound in Theorem 5.18 can be improved if we use a more complicated bound in Lemma 5.17, which may be easily facilitated by the use of a computer.

The bound provided by Lemma 5.17 was chosen for computational simplicity.

Remark 5.21. Using the bound given by Theorem 5.18 in case $\nu = 2$, yields a unique solution provided that

$$\frac{(b+3)(b+2)^2}{2} < \frac{1}{\alpha},$$

which is not as good as the integer-order bound (cf., [70]). Once again, however, this bound can be improved by using a more complicated estimate than was used in Lemma 5.17.

Chapter 6

Analysis of a Three-Point Discrete Fractional BVP

As mentioned in Chapter 1, multipoint or, more generally, nonlocal boundary value problems have generated considerable interest in the past 20 to 30 years among mathematicians interested in ordinary differential equations and their discrete analogues. The simplest possible incarnation of this type of problem is, of course, the so-called *three-point boundary value problem.* In this Chapter, we wish to illustrate the analysis of such a three-point boundary value problem in the discrete fractional setting. As will become apparent upon reading this chapter, this provides, in some way, a generalization of the results of Chapter 5, though the results here do *not* subsume those of the last Chapter – cf., Definition 6.8 and the sequel.

Similar to the preceding chapter, we shall derive first the Green's function for the

three-point nonlinear discrete fractional boundary value problem (FBVP)

$$\begin{cases} -\Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)) \\ y(\nu-2) = 0 \\ \alpha y(\nu+K) = y(\nu+b) \end{cases}, \tag{6.1}$$

where $t \in [0,b]_{\mathbb{N}_0}$, $\nu \in (1,2]$, $\alpha \in [0,1]$, $K \in [-1,b-1]_{\mathbb{Z}}$, and $f : [\nu-1,\nu+b-1]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \to \mathbb{R}$ is continuous. We then prove that this Green's function satisfies certain properties. Verifying that the Green's function satisfies certain desirable properties is one of our primary goals in this chapter, and this verification tends to be more delicate and complicated than for continuous FBVPs.

6.1 Derivation of the Green's Function

In this section, we deduce the Green's function for the operator $-\Delta^{\nu}$ together with the boundary operators $y(\nu-2)=0$ and $\alpha y(\nu+K) = y(\nu+b)$, where $0 \le \alpha \le 1$ and $K \in [-1,b-1]_{\mathbb{Z}}$. For reference in the sequel, let us make the following declarations.

$$\begin{aligned} g_1(t,s) &:= \frac{1}{\Gamma(\nu)} \\ &\quad \times \left[-(t-s-1)^{\underline{\nu-1}} + \frac{t^{\underline{\nu-1}}}{\Omega_0}\left[(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}\right]\right] \\ g_2(t,s) &:= \frac{1}{\Gamma(\nu)}\left[\frac{t^{\underline{\nu-1}}}{\Omega_0}\left[(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}\right]\right] \\ g_3(t,s) &:= \frac{1}{\Gamma(\nu)}\left[-(t-s-1)^{\underline{\nu-1}} + \frac{t^{\underline{\nu-1}}}{\Omega_0}(b+\nu-s-1)^{\underline{\nu-1}}\right] \\ g_4(t,s) &:= \frac{1}{\Gamma(\nu)}\left[\frac{t^{\underline{\nu-1}}}{\Omega_0}(b+\nu-s-1)^{\underline{\nu-1}}\right] \\ \Omega_0 &:= (b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}} \end{aligned} \tag{6.2}$$

Theorem 6.1. *Let $h : [\nu-1, \nu+b-1]_{\mathbb{N}_{\nu-1}} \to \mathbb{R}$ be given. The unique solution of the problem*

$$\begin{cases} -\Delta^{\nu} y(t) = h(t+\nu-1) \\ y(\nu-2) = 0 \\ \alpha y(\nu+K) = y(\nu+b) \end{cases} \tag{6.3}$$

is the function

$$y(t) = \sum_{s=0}^{b} G(t,s) h(s+\nu-1),$$

where $G(t,s)$ is the Green's function for the operator $-\Delta^{\nu}$ together with the boundary conditions in (6.3), and where

$$G(t,s) := \begin{cases} g_1(t,s), & 0 \le s \le \min\{t-\nu, K\} \\ g_2(t,s), & 0 \le t-\nu < s \le K \le b \\ g_3(t,s), & 0 < K < s \le t-\nu \le b \\ g_4(t,s), & \max\{t-\nu, K\} < s \le b \end{cases},$$

with $g_i(t,s)$, $1 \le i \le 4$, are as defined in (6.2) above.

Proof. We know from previous results in this work (cf., Chapter 5) that the general solution to the equation $-\Delta^{\nu} y(t) = h(t+\nu-1)$ is the function

$$y(t) = -\Delta^{-\nu} h(t+\nu-1) + C_1 t^{\underline{\nu-1}} + C_2 t^{\underline{\nu-2}},$$

where C_1 and C_2 are constants to be determined. Now, applying the boundary condition $y(\nu-2) = 0$ implies at once that $C_2 = 0$. On the other hand, applying the

boundary condition $\alpha y(\nu+K)=y(\nu+b)$ implies that

$$\begin{aligned}0&=\alpha y(\nu+K)-y(\nu+b)\\&=\alpha\left\{-\Delta^{-\nu}h(t)\big|_{t=\nu+K}+C_1(\nu+K)^{\underline{\nu-1}}\right\}+\left\{\Delta^{-\nu}h(t)\big|_{t=\nu+b}-C_1(\nu+b)^{\underline{\nu-1}}\right\},\end{aligned}\tag{6.4}$$

and (6.3) implies that

$$C_1\left[-\alpha(\nu+K)^{\underline{\nu-1}}+(\nu+b)^{\underline{\nu-1}}\right]=C_1\Omega_0=-\alpha\Delta^{-\nu}h(t)\big|_{t=\nu+K}+\Delta^{-\nu}h(t)\big|_{t=\nu+b},\tag{6.5}$$

where Ω_0 was defined in (6.2) above. So, from (6.5) we get that

$$\begin{aligned}C_1&=\frac{1}{\Gamma(\nu)\Omega_0}\sum_{s=0}^{b}(b+\nu-s-1)^{\underline{\nu-1}}h(s+\nu-1)\\&\quad-\frac{1}{\Gamma(\nu)\Omega_0}\sum_{s=0}^{K}\alpha(K+\nu-s-1)^{\underline{\nu-1}}h(s+\nu-1).\end{aligned}\tag{6.6}$$

So, putting (6.6) into the equation for y, we find that

$$\begin{aligned}y(t)&=-\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}h(s+\nu-1)\\&\quad+\frac{t^{\underline{\nu-1}}}{\Gamma(\nu)\Omega_0}\sum_{s=0}^{b}(b+\nu-s-1)^{\underline{\nu-1}}h(s+\nu-1)\\&\quad-\frac{t^{\underline{\nu-1}}}{\Gamma(\nu)\Omega_0}\sum_{s=0}^{K}\alpha(K+\nu-s-1)^{\underline{\nu-1}}h(s+\nu-1),\end{aligned}\tag{6.7}$$

and this is the unique solution to problem (6.3).

Finally, define $G(t,s)$ as in the statement of this theorem. Then it is clear from the form of $y(t)$ given in (6.7) as well as the definition of the g_i's in (6.2) that we can write $y(t)=\sum_{s=0}^{b}G(t,s)h(s+\nu-1)$, as claimed. □

Remark 6.2. It is easy to observe that in case $\alpha = 0$, not only does problem (6.1) reduce to the usual conjugate FBVP that was considered in [20], but, moreover, the Green's function given by Theorem 6.1 reduces to the Green's function derived in [20]. Thus, our results here are, in part, a generalization of the results of [20].

6.2 Properties of the Green's Function

We now wish to prove that the Green's function $G(t,s)$ in Theorem 6.1 satisfies a variety of properties. Certain of these properties will be important in Section 6.3 when we consider existence of a solution to problem (1.1). We first prove an easy but necessary preliminary lemma.

Lemma 6.3. *Let Ω_0 be as defined in (6.2). Then for each $K \in [-1, b-1]_{\mathbb{Z}}$, $\nu \in (1,2]$, and $b \in \mathbb{N}$, we find that $\Omega_0 > 0$.*

Proof. Recall from (6.2) that $\Omega_0 = (b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}}$. Clearly, this function is decreasing in α for each fixed K, ν, and b. Consequently, it suffices to show that $\Omega_0 > 0$ when $\alpha = 1$. To see that this is indeed true, note that $t^{\underline{\mu}}$ is increasing in t, whenever $0 < \mu < 1$. So, as $b + \nu > K + \nu$, it follows at once that

$$\Omega_0\Big|_{\alpha=1} = (b+\nu)^{\underline{\nu-1}} - (K+\nu)^{\underline{\nu-1}} > 0,$$

which proves the claim. (Note this holds even in case $\nu = 2$.) □

Theorem 6.4. *Let $G(t,s)$ be the Green's function given in the statement of Theorem 6.1. Then for each $(t,s) \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}} \times [0,b]_{\mathbb{N}_0}$, we find that $G(t,s) \geq 0$.*

Proof. We shall show that for each i, $1 \leq i \leq 4$, $g_i(t,s) > 0$ for each admissible pair (t,s). Let us begin by showing both that $g_2(t,s) > 0$ and that $g_4(t,s) > 0$, as these

are the easier cases. In the case of $g_2(t,s)$, observe that by the form of g_2, it suffices to show that

$$(b+\nu-s-1)^{\underline{\nu-1}}-\alpha(K+\nu-s-1)^{\underline{\nu-1}}>0. \tag{6.8}$$

Showing that (6.8) is true is equivalent to showing that

$$\frac{(b+\nu-s-1)^{\underline{\nu-1}}}{\alpha(K+\nu-s-1)^{\underline{\nu-1}}}>1. \tag{6.9}$$

But to see that (6.9) is true for each admissible pair (t,s) and each $\alpha\in(0,1]$, observe that $t^{\underline{\mu}}$ is increasing in t if $\mu\in(0,1)$. It follows that

$$\frac{(b+\nu-s-1)^{\underline{\nu-1}}}{\alpha(K+\nu-s-1)^{\underline{\nu-1}}}\geq\frac{(b+\nu-s-1)^{\underline{\nu-1}}}{(K+\nu-s-1)^{\underline{\nu-1}}}>1, \tag{6.10}$$

which proves (6.9) and hence (6.8). (Note that although $\alpha\neq 0$ in (6.9), if $\alpha=0$, then (6.8) is trivially true.) On the other hand, we note that by the form of g_4 given in (6.2), it is immediate that $g_4(t,s)>0$ since $\Omega_0>0$ by Lemma 6.3 and $b+\nu-s-1>0$ in this case. Thus, we conclude that both g_2 and g_4 are positive on their respective domains.

We next consider the function $g_3(t,s)$. Recall from (6.2) that

$$g_3(t,s)=\frac{1}{\Gamma(\nu)}\left[-(t-s-1)^{\underline{\nu-1}}+\frac{t^{\underline{\nu-1}}}{\Omega_0}(b+\nu-s-1)^{\underline{\nu-1}}\right].$$

Evidently, to prove that $g_3(t,s)>0$, we may instead just prove that $\Gamma(\nu)g_3(t,s)>0$. Now, it is clear that g_3 is increasing in α, for as α increases, Ω_0 clearly decreases. In particular, then, we deduce that

$$\Gamma(\nu)g_3(t,s)\geq-(t-s-1)^{\underline{\nu-1}}+\frac{t^{\underline{\nu-1}}(b+\nu-s-1)^{\underline{\nu-1}}}{(b+\nu)^{\underline{\nu-1}}}. \tag{6.11}$$

Note that (6.11) implies that $g_3(t,s) > 0$ if and only if

$$\frac{t^{\underline{\nu-1}}(b+\nu-s-1)^{\underline{\nu-1}}}{(t-s-1)^{\underline{\nu-1}}(b+\nu)^{\underline{\nu-1}}} > 1. \tag{6.12}$$

To prove that (6.12) holds, recall that on the domain of g_3 we have that $t \geq s+\nu > K+\nu$. So, it follows that given a fixed $s_0 > K$, we have that $t = s_0+\nu+j$, for some $0 \leq j \leq b-s_0$ with $j \in \mathbb{N}_0$. But then for this fixed but arbitrary s_0, we may rewrite the left-hand side of (6.12) as

$$\begin{aligned}
&\frac{t^{\underline{\nu-1}}(b+\nu-s-1)^{\underline{\nu-1}}}{(t-s-1)^{\underline{\nu-1}}(b+\nu)^{\underline{\nu-1}}}\\
&=\frac{\Gamma(t+1)\Gamma\left(b+\nu-s_0\right)\Gamma\left(t-s_0-\nu+1\right)\Gamma(b+2)}{\Gamma(t-\nu+2)\Gamma\left(b-s_0+1\right)\Gamma\left(t-s_0\right)\Gamma(b+\nu+1)}\\
&=\frac{\Gamma\left(s_0+\nu+j+1\right)\Gamma\left(b+\nu-s_0\right)\Gamma(j+1)\Gamma(b+2)}{\Gamma\left(s_0+j+2\right)\Gamma\left(b-s_0+1\right)\Gamma(\nu+j)\Gamma(b+\nu+1)}\\
&=\frac{j!(b+1)!\left[(\nu+j+s_0)\cdots(\nu+j)\right]}{(s_0+j+1)!\,(b-s_0)!\left[(b+\nu)\cdots(b+\nu-s_0)\right]}\\
&=\frac{(b+1)\cdots(b-s_0+1)}{(b+\nu)\cdots(b+\nu-s_0)}\cdot\frac{(\nu+j+s_0)\cdots(\nu+j)}{(s_0+j+1)\cdots(j+1)}.
\end{aligned} \tag{6.13}$$

Now, observe that each of the fractions on the right-hand side of (6.13) has exactly s_0+1 factors in each of its numerator and denominator. Moreover, by putting $\alpha_0 := \nu-1 > 0$, it is easy to see that this expression satisfies the hypotheses of Lemma 5.7. (Note that some repetition of factors may occur between the two fractions on the right-hand side of (6.13), but these may always be canceled to obtain the form required by Lemma 5.7.) Consequently, we deduce from this lemma that

$$\frac{t^{\underline{\nu-1}}(b+\nu-s-1)^{\underline{\nu-1}}}{(t-s-1)^{\underline{\nu-1}}(b+\nu)^{\underline{\nu-1}}} = \frac{(b+1)\cdots(b-s_0+1)}{(b+\nu)\cdots(b+\nu-s)}\cdot\frac{(\nu+j+s_0)\cdots(\nu+j)}{(s_0+j+1)\cdots(j+1)} > 1, \tag{6.14}$$

whence (6.12) holds. But as (6.12) holds for each admissible pair (t,s), it follows at

once that (6.11) holds, too, so that $g_3(t,s) > 0$, as claimed.

Finally, we wish to show that $g_1(t,s) > 0$ on its domain, which we recall is $0 \le s \le \min\{t-\nu, K\}$. Recall from (6.2) that

$$\Gamma(\nu)g_1(t,s) = -(t-s-1)^{\underline{\nu-1}} + \frac{t^{\underline{\nu-1}}}{\Omega_0}\left[(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}\right], \tag{6.15}$$

where we shall again use the fact that g_1 is positive if and only if $\Gamma(\nu)g_1$ is positive. Let us observe at this juncture that

$$(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}} > 0, \tag{6.16}$$

which is an important condition. Observe that (6.16) just follows from (6.8) above.

Now, observe that $g_1 > 0$ only if

$$\frac{t^{\underline{\nu-1}}}{\Omega_0}\left[(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}\right] > (t-s-1)^{\underline{\nu-1}}. \tag{6.17}$$

We shall begin by demonstrating that

$$F(\alpha) := \frac{(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}}{(b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}}} \tag{6.18}$$

is increasing in α for $0 \le \alpha \le 1$. Note that an easy calculation demonstrates that $F(\alpha)$ is increasing in α if and only if

$$\frac{(b+\nu-s-1)^{\underline{\nu-1}}(K+\nu)^{\underline{\nu-1}}}{(K+\nu-s-1)^{\underline{\nu-1}}(b+\nu)^{\underline{\nu-1}}} > 1. \tag{6.19}$$

To see that (6.19) holds, let s_0 be fixed but arbitrary such that $s_0 \in [0,b]_{\mathbb{N}_0}$ and

$0 \leq s_0 \leq \min\{t-\nu, K\}$. So, it follows that the left-hand side of (6.19) above satisfies

$$\frac{(b+\nu-s_0-1)^{\underline{\nu-1}}(K+\nu)^{\underline{\nu-1}}}{(K+\nu-s_0-1)^{\underline{\nu-1}}(b+\nu)^{\underline{\nu-1}}} = \frac{(b+1)\cdots(b-s_0+1)}{(b+\nu)\cdots(b+\nu-s_0)} \cdot \frac{(K+\nu)\cdots(K+\nu-s_0)}{(K+1)\cdots(K-s_0+1)}. \tag{6.20}$$

But it is easy to check that by putting $\alpha_0 := \nu - 1 > 0$, we may apply Lemma 5.7 to the right-hand side of (6.20) to conclude that (6.19) holds. Thus, $F(\alpha)$ is increasing in α. In particular, this implies that to prove that (6.17) is true, it suffices to check its truth in case $\alpha = 0$. In this case, we find that proving (6.17) reduces to proving that

$$\frac{t^{\underline{\nu-1}}(b+\nu-s-1)^{\underline{\nu-1}}}{(b+\nu)^{\underline{\nu-1}}(t-s-1)^{\underline{\nu-1}}} > 1 \tag{6.21}$$

holds. Observe that the same proof that was used to show that (6.12) held can be used to show that (6.21) holds, too. Thus, as (6.17) holds in case $\alpha = 0$, the result of (6.18)–(6.21) implies that (6.17) holds for each admissible α. Consequently, we conclude that $g_1(t,s) > 0$, from which it follows that $g_i(t,s) > 0$ for each i, $1 \leq i \leq 4$. Hence, $G(t,s) > 0$, which concludes the proof. □

Theorem 6.5. *Let $G(t,s)$ be the Green's function given in the statement of Theorem 6.1. In addition, suppose that for given $K \in [-1, b-1]_{\mathbb{Z}}$ and $1 < \nu \leq 2$, we have that α satisfies the inequality*

$$0 \leq \alpha \leq \min_{(t,s)\in[s+\nu,\nu+b]_{\mathbb{N}_{\nu-1}}\times[0,b]_{\mathbb{N}_0}} \left\{ \frac{(b+\nu)^{\underline{\nu-1}}}{(K+\nu)^{\underline{\nu-1}}} - \frac{t^{\underline{\nu-2}}(b+\nu-s-1)^{\underline{\nu-1}}}{(K+\nu)^{\underline{\nu-1}}(t-s-1)^{\underline{\nu-2}}} \right\} \tag{6.22}$$

Then it follows that

$$\max_{(t,s)\in[\nu-1,\nu+b]_{\mathbb{N}_{\nu-1}}\times[0,b]_{\mathbb{N}_0}} G(t,s) = G(s+\nu-1, s). \tag{6.23}$$

Proof. Our strategy is to show that $\Delta_t g_i(t,s) > 0$ for each $i = 2, 4$, and that

$\Delta_t g_i(t,s) < 0$ for $i = 1,\ 3$. From this the claim will follow, evidently. To this end, we first show that the former claim holds, as this is the easier of the two cases. For instance, when $i = 2$, we find by direct computation that

$$\Gamma(\nu)\Delta_t g_2(t,s) = \frac{(\nu-1)t^{\underline{\nu-2}}}{\Omega_0}\left[(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}\right]. \quad (6.24)$$

So, it is clear from (6.23) that $\Delta_t g_2(t,s) > 0$ if and only if

$$(b+\nu-s-1)^{\underline{\nu-1}} > \alpha(K+\nu-s-1)^{\underline{\nu-1}}. \quad (6.25)$$

But as this follows at once from (6.8)–(6.9) above, we have that $\Delta_t g_2(t,s) > 0$, as desired. On the other hand, that $\Delta_t g_4(t,s) > 0$ is immediate considering that $\Delta_t g_4(t,s) = \frac{(\nu-1)t^{\underline{\nu-2}}}{\Omega_0}(b+\nu-s-1)^{\underline{\nu-1}}$. So, this concludes the analysis of $\Delta_t g_i(t,s)$ in case i is even.

We next attend to $g_3(t,s)$, and we claim that $\Delta_t g_3(t,s) < 0$, for each admissible pair (t,s). To see that this is true, note first that

$$\Gamma(\nu)\Delta_t g_3(t,s) = -(\nu-1)(t-s-1)^{\underline{\nu-2}} + \frac{(\nu-1)t^{\underline{\nu-2}}}{\Omega_0}(b+\nu-s-1)^{\underline{\nu-1}}, \quad (6.26)$$

where we have used the fact that $\Delta_t(t-s-1)^{\underline{\nu-1}} = (\nu-1)(t-s-1)^{\underline{\nu-2}}$, which may be easily verified by definition. So, if $\Delta_t g_3$ is to be nonpositive, then it must be the case that

$$\frac{t^{\underline{\nu-2}}(b+\nu-s-1)^{\underline{\nu-1}}}{\Omega_0} < (t-s-1)^{\underline{\nu-2}} \quad (6.27)$$

holds. Note that (6.27) holds if and only if $(b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}} > \frac{t^{\underline{\nu-2}}(b+\nu-s-1)^{\underline{\nu-1}}}{(t-s-1)^{\underline{\nu-2}}}$

is true. But this latter inequality is true only if

$$-\alpha > \frac{t^{\underline{\nu-2}}(b+\nu-s-1)^{\underline{\nu-1}}}{(t-s-1)^{\underline{\nu-2}}(K+\nu)^{\underline{\nu-1}}} - \frac{(b+\nu)^{\underline{\nu-1}}}{(K+\nu)^{\underline{\nu-1}}} \tag{6.28}$$

is true. And from this, we see that by requiring α to satisfy, for each admissible K and ν, the inequality

$$0 \leq \alpha \leq \min_{(t,s)\in[s+\nu,\nu+b]_{\mathbb{N}_{\nu-1}}\times[0,b]_{\mathbb{N}_0}} \left\{ \frac{(b+\nu)^{\underline{\nu-1}}}{(K+\nu)^{\underline{\nu-1}}} - \frac{t^{\underline{\nu-2}}(b+\nu-s-1)^{\underline{\nu-1}}}{(K+\nu)^{\underline{\nu-1}}(t-s-1)^{\underline{\nu-2}}} \right\}, \tag{6.29}$$

it follows that (6.27) holds – that is, that $g_3(t,s) > 0$ for each admissible pair (t,s). Note that restriction (6.29) above is precisely restriction (6.22), which was given in the statement of this theorem. Thus, with restriction (6.22) in place, we conclude that $g_3(t,s)$ will be nonpositive on its domain, as desired.

Finally, we claim that $\Delta_t g_1(t,s) < 0$ on its domain. Observe that by the definition of g_1 given in (6.2), it follows that we must show that

$$-(\nu-1)(t-s-1)^{\underline{\nu-2}} + \frac{(\nu-1)t^{\underline{\nu-2}}}{\Omega_0}\left[(b+s-\nu-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}\right] < 0. \tag{6.30}$$

But note that

$$\begin{aligned} &-(\nu-1)(t-s-1)^{\underline{\nu-2}} + \frac{(\nu-1)t^{\underline{\nu-2}}}{\Omega_0}\left[(b+s-\nu-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}\right] \\ &\leq -(\nu-1)(t-s-1)^{\underline{\nu-2}} + \frac{(\nu-1)t^{\underline{\nu-2}}(b+s-\nu-1)^{\underline{\nu-1}}}{\Omega_0}. \end{aligned} \tag{6.31}$$

So, it follows that if

$$-(\nu-1)(t-s-1)^{\underline{\nu-2}} + \frac{(\nu-1)t^{\underline{\nu-2}}(b+s-\nu-1)^{\underline{\nu-1}}}{\Omega_0} < 0, \tag{6.32}$$

then inequality (6.30) holds. But note that we can solve for α in (6.32) to get an upper bound on α. As this calculation is exactly the same as the one given earlier in the argument, we do not repeat it here. Instead we point out that the restriction (6.32) implies that

$$0 \leq \alpha \leq \frac{(b+\nu)^{\underline{\nu-1}}}{(K+\nu)^{\underline{\nu-1}}} - \frac{t^{\underline{\nu-2}}(b+\nu-s-1)^{\underline{\nu-1}}}{(K+\nu)^{\underline{\nu-1}}(t-s-1)^{\underline{\nu-2}}}. \tag{6.33}$$

Note that the right-hand side of (6.33) is precisely restriction (6.22). Thus, by assuming (6.22) we also get that (6.30) holds. Consequently, the preceding analysis shows that (6.30) holds, whence $g_1(t,s) > 0$ on its domain. Thus, by the discussion in the first paragraph of this proof, we deduce that (6.23) holds. And this completes the proof. □

Before presenting our final theorem in this section regarding $G(t,s)$, we need to define the following constants for convenience. We shall use this in the sequel.

$$\begin{aligned}
\gamma_1 &:= \frac{\left(\frac{b+\nu}{4}\right)^{\underline{\nu-1}}}{(b+\nu)^{\underline{\nu-1}}}\\
\gamma_2 &:= \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{\left(\frac{3(b+\nu)}{4}-1\right)^{\underline{\nu-1}}\left[(b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}}\right]}{(b+\nu-1)^{\underline{\nu-1}}}\right]\\
\gamma_3 &:= \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{\left(\frac{3(b+\nu)}{4}-1\right)^{\underline{\nu-1}}(b+\nu)^{\underline{\nu-1}}}{(b+\nu-1)^{\underline{\nu-1}}}\right]
\end{aligned} \tag{6.34}$$

Theorem 6.6. *Let $G(t,s)$ be the Green's function given in the statement of Theorem*

6.1. Let γ_i, $1 \leq i \leq 3$, be defined as in (6.34) above. Then it follows that

$$\min_{t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]} G(t,s) \geq \gamma \max_{(t,s) \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}} \times [0,b]_{\mathbb{N}_0}} G(t,s) = \gamma G(s+\nu-1, s), \tag{6.35}$$

where

$$\gamma := \min\{\gamma_1, \gamma_3\}, \tag{6.36}$$

and γ satisfies the inequality $0 < \gamma < 1$.

Proof. To facilitate notation in this proof, let us put, for each $1 \leq i \leq 4$,

$$\widetilde{g}_i(t,s) := \frac{g_i(t,s)}{g_k(s+\nu-1,s)},$$

where $k = 2$ if $i = 1, 2$, and $k = 4$ if $i = 3, 4$. Observe that for $s \geq t - \nu + 1$ and $\frac{b+\nu}{4} \leq t \leq \frac{3(b+\nu)}{4}$, we find that

$$\widetilde{g}_2(t,s) = \widetilde{g}_4(t,s) = \frac{t^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}} \geq \frac{\left(\frac{b+\nu}{4}\right)^{\underline{\nu-1}}}{(b+\nu)^{\underline{\nu-1}}}, \tag{6.37}$$

whence from (6.37) it is clear that in case $s \geq t - \nu + 1$ and $t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]$, we have that $\min_{t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]} G(t,s) \geq \gamma_1 G(s+\nu-1, s)$.

On the other hand, suppose that $s < t - \nu + 1$ and $t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]$. Then we consider two cases depending upon whether or not the pair (t,s) lies in the domain

of $\widetilde{g}_1(t,s)$ or $\widetilde{g}_3(t,s)$. In the latter case, we note that by definition

$$\begin{aligned}\widetilde{g}_3(t,s) &= \frac{-(t-s-1)^{\underline{\nu-1}}\Omega_0}{(s+\nu-1)^{\underline{\nu-1}}(b+\nu-s-1)^{\underline{\nu-1}}} + \frac{t^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}}\\ &= \frac{1}{(s+\nu-1)^{\underline{\nu-1}}}\left[t^{\underline{\nu-1}} - \frac{(t-s-1)^{\underline{\nu-1}}\left[(b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}}\right]}{(b+\nu-s-1)^{\underline{\nu-1}}}\right]\\ &\geq \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}\\ &\quad\times\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{\left(\frac{3(b+\nu)}{4}-1\right)^{\underline{\nu-1}}\left[(b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}}\right]}{(b+\nu-1)^{\underline{\nu-1}}}\right].\end{aligned} \tag{6.38}$$

So, it is clear from (6.38) that in case $s < t-\nu+1$ and $t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]$, we get that $\min_{t\in\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]} G(t,s) \geq \gamma_2 G(s+\nu-1,s)$.

Finally, suppose that $s < t-\nu+1$, $t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]$, and that the pair (t,s) lies in the domain of $\widetilde{g}_1(t,s)$. By using a similar calculation as in (6.38) together with the definition of $\widetilde{g}_1(t,s)$, we find that

$$\begin{aligned}\widetilde{g}_1(t,s) &= \frac{-(t-s-1)^{\underline{\nu-1}}\Omega_0}{(s+\nu-1)^{\underline{\nu-1}}\left[(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}\right]}\\ &\quad + \frac{t^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}}\\ &\geq \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}\\ &\quad\times\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{\left(\frac{3(b+\nu)}{4}-s-1\right)^{\underline{\nu-1}}\left[(b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}}\right]}{(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}}\right].\end{aligned} \tag{6.39}$$

We now need to focus on the quotient $\frac{(b+\nu)^{\underline{\nu-1}}-\alpha(K+\nu)^{\underline{\nu-1}}}{(b+\nu-s-1)^{\underline{\nu-1}}-\alpha(K+\nu-s-1)^{\underline{\nu-1}}}$, which appears on the right-hand side of (6.39). We claim that this is a decreasing function of α.

To prove this claim, let us put

$$g(\alpha) := \frac{(b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}}}{(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}}, \tag{6.40}$$

where for each fixed but arbitrary b, s, ν, and K, we have that $g : [0,1] \to [0,+\infty)$. Now, let $F(\alpha)$ be defined as in (6.18) above. Note from (6.40) that $g(\alpha) = [F(\alpha)]^{-1}$. Recall that we already proved that $F(\alpha)$ is increasing in α, for $0 \le \alpha \le 1$. So, a routine computation demonstrates that $g(\alpha) = [F(\alpha)]^{-1}$ is thus decreasing in α, for $0 \le \alpha \le 1$, as desired.

Since g is decreasing in α, we conclude that

$$\begin{aligned}
\widetilde{g}_1(t,s) &\ge \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}} \\
&\quad \times \left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{\left(\frac{3(b+\nu)}{4}-s-1\right)^{\underline{\nu-1}}\left[(b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}}\right]}{(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}}\right] \\
&\ge \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{\left(\frac{3(b+\nu)}{4}-s-1\right)^{\underline{\nu-1}}(b+\nu)^{\underline{\nu-1}}}{(b+\nu-s-1)^{\underline{\nu-1}}}\right] \\
&\ge \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{\left(\frac{3(b+\nu)}{4}-1\right)^{\underline{\nu-1}}(b+\nu)^{\underline{\nu-1}}}{(b+\nu-1)^{\underline{\nu-1}}}\right].
\end{aligned} \tag{6.41}$$

Thus, we see that in case, $\min_{t\in\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]} G(t,s) \ge \gamma_3 G(s+\nu-1,s)$.

Finally, note that since $\gamma_2 \ge \gamma_3$, it follows that $\min\{\gamma_1,\gamma_2,\gamma_3\} = \min\{\gamma_1,\gamma_3\}$. Thus, we can put $\gamma := \min\{\gamma_1,\gamma_3\}$ as in (6.36), and the preceding part of the proof

then shows that

$$\min_{t\in\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]} G(t,s) \geq \gamma \max_{(t,s)\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}\times[0,b]_{\mathbb{N}_0}} G(t,s) = \gamma G(s+\nu-1,s), \tag{6.42}$$

and as (6.42) is (6.35), the first part of the proof is complete.

To complete the proof, it remains to show that γ, as defined in (6.36), satisfies $0<\gamma<1$. Let us first observe that $\gamma_1<1$. This follows from the fact that $t^{\underline{\nu-1}}$ is an increasing function in t whenever $\nu\in(1,2]$. Indeed, observe that

$$\Delta\left[t^{\underline{\nu-1}}\right] = (\nu-1)\cdot\frac{\Gamma(t+1)}{\Gamma(t-\nu+3)} > 0, \tag{6.43}$$

clearly. So, as $\frac{b+\nu}{4} > b+\nu$ and $\left(\frac{b+\nu}{4}\right)^{\underline{\nu-1}}$, $(b+\nu)^{\underline{\nu-1}} \neq 0$, the claim follows. In particular, this demonstrates that

$$\gamma = \min\left\{\gamma_1,\gamma_3\right\} \leq \gamma_1 < 1. \tag{6.44}$$

On the other hand, since it is clear that $\gamma_1>0$, it remains to show that $\gamma_3>0$. Observe that γ_3 is strictly positive if and only if

$$\frac{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}(b+\nu-1)^{\underline{\nu-1}}}{\left(\frac{3(b+\nu)}{4}-1\right)^{\underline{\nu-1}}(b+\nu)^{\underline{\nu-1}}} > 1. \tag{6.45}$$

But (6.45) is true if and only if

$$\frac{(b+1)\left(\frac{3(b+\nu)}{4}\right)}{(b+\nu)\left(\frac{3(b+\nu)}{4}-\nu+1\right)} > 1 \tag{6.46}$$

holds for each admissible b and ν – that is, each $b\in[2,+\infty)_{\mathbb{N}}$ and $\nu\in(1,2]$.

We claim that (6.46) does hold for each admissible b and ν. To see this, for each fixed and admissible b, put

$$H_b(\nu) := \frac{(b+1)\left(\frac{3(b+\nu)}{4}\right)}{(b+\nu)\left(\frac{3(b+\nu)}{4} - \nu + 1\right)}, \tag{6.47}$$

which is the left-hand side of inequality (6.46), and note both that

$$H_b(1) = 1 \tag{6.48}$$

and that

$$H_b(2) = \frac{(b+1)\left(\frac{3}{4}b + \frac{3}{2}\right)}{(b+2)\left(\frac{3}{4}b + \frac{1}{2}\right)} = \frac{3b+3}{3b+2}. \tag{6.49}$$

Clearly, $H_b(2) > 1$, for each admissible b. Moreover, a routine calculation shows that

$$H_b'(\nu) = \frac{3(b+1)}{(3b-\nu+4)^2}. \tag{6.50}$$

But (6.50) demonstrates that for each admissible b, we have that $H_b(\nu)$ is strictly increasing in ν. Therefore, as $H_b(1) = 1$ and $H_b(2) > 1$, we get at once that

$$H_b(\nu) > 1 \tag{6.51}$$

for each $\nu \in (1, 2]$ and $b \in [2, +\infty)_{\mathbb{N}}$. But from (6.51) we deduce that (6.46) holds for each admissible b and ν.

In summary, (6.45)–(6.51) demonstrate that $\gamma_3 > 0$. But we then find that

$$\gamma = \min\{\gamma_1, \gamma_3\} > 0. \tag{6.52}$$

Putting (6.44) and (6.52) together implies that $\gamma \in (0,1)$, as claimed. And this completes the proof. □

Remark 6.7. Note that in case $\alpha = 0$, the result of Theorem 6.6 reduces to the results obtained in [20].

We wish to conclude this section by investigating certain of the properties of the set of admissible values of α generated by condition (6.22) in Theorem 6.5. Interestingly, depending upon the magnitude of α, it may be the case that some $\alpha \in [0,1]$ are not admissible. The results in the sequel provide some analysis of this problem.

Definition 6.8. Given $b \in \mathbb{N}$, $\nu \in (1,2]$, and $K \in [-1, b-1]_{\mathbb{Z}}$, let $\Lambda_{(\nu,K,b)}$ be the set of $\alpha \in [0,1]$ such that condition (4.15) holds – that is

$$\Lambda_{(\nu,K,b)} := \left\{\alpha \in [0,1] \ : \ 0 \leq \alpha \leq \min\left\{\frac{(b+\nu)^{\underline{\nu-1}}}{(K+\nu)^{\underline{\nu-1}}} - \frac{t^{\underline{\nu-2}}(b+\nu-s-1)^{\underline{\nu-1}}}{(K+\nu)^{\underline{\nu-1}}(t-s-1)^{\underline{\nu-2}}}\right\}\right\},$$

where the min is taken over all pairs (t,s) satisfying $(t,s) \in [s+\nu, \nu+b]_{\mathbb{N}_{\nu-1}} \times [0,b]_{\mathbb{N}_0}$. Then we shall call $\Lambda_{(\nu,K,b)} \subseteq [0,1]$ the $\boldsymbol{\alpha}$**-admissible set** for problem (6.1).

Before proving a couple of results regarding the Lebesgue measure of $\Lambda_{(\nu,K,b)}$, we need to state and prove a preliminary lemma.

Lemma 6.9. *The function* $f(t,s) := \frac{t^{\underline{\nu-2}}}{(t-s-1)^{\underline{\nu-2}}}$ *is increasing in* t.

Proof. Observe that

$$\begin{aligned}\Delta_t f(t,s) &= \frac{(t+1)^{\underline{\nu-2}}}{(t-s)^{\underline{\nu-2}}} - \frac{t^{\underline{\nu-2}}}{(t-s-1)^{\underline{\nu-2}}} \\ &= \frac{\Gamma(t+2)\Gamma(t-s-\nu+3) - (t-\nu+3)(t-s)\Gamma(t+1)\Gamma(t-s-\nu+2)}{\Gamma(t-\nu+4)\Gamma(t-s+1)}.\end{aligned} \tag{6.53}$$

Now, note that the numerator of $\Delta_t f(t,s)$ on the right-hand side of (6.53) above may be rewritten as $[(t+1)(t-s-\nu+2)-(t-\nu+3)(t-s)]\Gamma(t+1)\Gamma(t-s-\nu+2)$. But the coefficient of $\Gamma(t+1)\Gamma(t-s-\nu+2)$ in this expression may be written as

$$(t+1)(t-s-\nu+2)-(t-\nu+3)(t-s)=s(2-\nu)+(2-\nu)\geq 0, \tag{6.54}$$

clearly. But from (6.53)–(6.54) it follows at once that $\Delta_t f(t,s)\geq 0$. Thus, $f(t,s)$ is increasing in t, as claimed. □

With this preliminary lemma in hand, we now prove three results regarding the measure of the set $\Lambda_{(\nu,K,b)}$. Henceforth, we consider the measurable space $\left([0,1],\mathcal{B}_{[0,1]}\right)$ equipped with the usual Lebesgue measure, denoted in the sequel by m.

Theorem 6.10. *Given $b\in\mathbb{N}$, $\nu\in(1,2]$, and $K\in[-1,b-1]_{\mathbb{Z}}$, we find that*

$$\lim_{\nu\to 1^+} m\left(\Lambda_{(\nu,K,b)}\right)=\frac{2}{b+2} \tag{6.55}$$

and that

$$\lim_{\nu\to 2^-} m\left(\Lambda_{(\nu,K,b)}\right)=\frac{1}{K+2}, \tag{6.56}$$

for each fixed K and b.

Proof. To prove (6.55), notice that when $\nu=1$, we get that

$$\begin{aligned}\Lambda_{(1,K,b)} &= \min_{[s+1,b+1]_{\mathbb{N}_0}\times[0,b]_{\mathbb{N}_0}}\left[1-\frac{t^{\underline{\nu-2}}}{(t-s-1)^{\underline{\nu-2}}}\right]_{\nu=1}\\ &=\left[1-\frac{(\nu+b)^{\underline{\nu-2}}}{(\nu+b-1)^{\underline{\nu-2}}}\right]_{\nu=1}\\ &=\left[1-\frac{b+\nu}{b+2}\right]_{\nu=1}\\ &=\frac{2}{b+2},\end{aligned}$$

where we have used the fact that Λ is continuous in ν, the result of Lemma 6.9, and the fact that $t^{\underline{\nu-2}}$ is decreasing in t. Thus, (6.55) is proved.

On the other hand, to show that (6.56) is true, note that in case $\nu = 2$, we get that

$$0 \leq \alpha \leq \min\left\{\frac{b+2}{K+2} - \frac{b+2-s-1}{K+2}\right\} = \min\left\{\frac{s+1}{K+2}\right\}$$

which proves the claim, for $s \in [0, b]_{\mathbb{N}_0}$ so that $\Lambda_{(2,K,b)} = \left[0, \frac{1}{K+2}\right]$, for each admissible triple $(2, K, b)$. (Note that we have again used the continuity of $\Lambda_{(\nu,K,b)}$ with respect to ν.) $\square$

Corollary 6.11. *For each $\varepsilon > 0$ and $K \in [-1, b-1]_{\mathbb{Z}}$ given, there is $\nu > 1$ sufficiently close to 1 and $b > 0$ sufficiently large such that*

$$m\left(\Lambda_{(\nu,K,b)}\right) < \varepsilon. \tag{6.57}$$

Proof. By using (6.55) from Theorem 6.10, we write

$$\lim_{b\to+\infty} \lim_{\nu\to 1^+} m\left(\Lambda_{(\nu,K,b)}\right) = 0. \tag{6.58}$$

Then (6.58) proves the claim. $\square$

In a similar way, we get the following corollary.

Corollary 6.12. *If $K = -1$ and $\nu = 2$, then $\Lambda_{(2,-1,b)}$ has full measure for each admissible b – that is*

$$m\left(\Lambda_{(2,-1,b)}\right) = 1. \tag{6.59}$$

Remark 6.13. Observe that (6.57) in Corollary 6.11 implies that for b sufficiently large, we can make the set of admissible α have arbitrarily small measure. In particular, then, for ν sufficiently close to and greater than unity, we find that α may be

significantly restricted, especially if b is very large. On the other hand, Corollary 6.12 implies that for ν sufficiently close to 2 and K small, α may not be very restricted; in particular, in case $\nu = 2$ and $K = -1$, $\Lambda_{(2,-1,b)} = [0,1]$ for each b, and so, α is not restricted at all.

6.3 Existence Theorems

In this section, we wish to conclude by presenting two representative existence theorems. One way to deduce existence results for the general problem (6.1) is to appeal to cone theoretic techniques. Although in light of the discussion at the end of Chapter 5 this is now rather standard, we include the relevant result for completeness. In any case, we now consider the nonlinear equation (6.1). We notice that y solves (6.1) if and only if y is a fixed point of the operator

$$(Ty)(t) := \sum_{s=0}^{b} G(t,s) f(s+\nu-1, y(s+\nu-1)),$$

where G is the Green's function derived in this paper and $T : \mathcal{B} \to \mathcal{B}$, where $\mathcal{B}$ is the Banach space consisting of all (continuous) maps from $[\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$ into $\mathbb{R}$ when equipped with the usual supremum norm, $\|\cdot\|$.

Let us also make the following declarations, which will be used in the sequel.

$$\eta := \frac{1}{\sum_{s=0}^{b} G(s+\nu-1, s)}$$

$$\lambda := \frac{1}{\sum_{s=\lceil \frac{\nu+b}{4} - \nu + 1 \rceil}^{\lfloor \frac{3(\nu+b)}{4} - \nu + 1 \rfloor} \gamma G\left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s\right)}$$

Let us also introduce two conditions on the behavior of f that will be useful in

the sequel. These are standard assumptions on the growth of the nonlinearity $f(t, y)$.

C1: There exists a number $r > 0$ such that $f(t, y) \leq \eta r$ whenever $0 \leq y \leq r$.

C2: There exists a number $r > 0$ such that $f(t, y) \geq r\lambda$ whenever $\gamma r \leq y \leq r$, where γ is the constant deduced in Theorem 6.6.

We now can prove the following existence result.

Theorem 6.14. *Suppose that there are distinct r_1, $r_2 > 0$ such that condition (C1) holds at $r = r_1$ and condition (C2) holds at $r = r_2$. Suppose also that $f(t, y) \geq 0$ and continuous. Then the FBVP (6.1) has at least one positive solution, say y_0, such that $\|y_0\|$ lies between r_1 and r_2.*

Proof. We shall assume without loss of generality that $0 < r_1 < r_2$. Consider the set

$$\mathcal{K} := \left\{ y \in \mathcal{B} \ : \ y(t) \geq 0 \text{ and } \min_{t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]} y(t) \geq \gamma \|y(t)\| \right\},$$

which is a cone with $\mathcal{K} \subseteq \mathcal{B}$. Observe that $T \ : \ \mathcal{K} \to \mathcal{K}$, for we observe that

$$\begin{aligned} \min_{t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]} (Ty)(t) &= \min_{t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]} \sum_{s=0}^{b} G(t, s) f(s + \nu - 1, y(s + \nu - 1)) \\ &\geq \gamma \sum_{s=0}^{b} G(s + \nu - 1, s) f(s + \nu - 1, y(s + \nu - 1)) \\ &= \gamma \max_{t \in [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}}} \sum_{s=0}^{b} G(t, s) f(s + \nu - 1, y(s + \nu - 1)) \\ &= \gamma \|Ty\|, \end{aligned}$$

whence $Ty \in \mathcal{K}$, as claimed. Also, it is easy to see that T is a completely continuous operator, for $T \ : \ \mathbb{R}^{b+3} \to \mathbb{R}^{b+3}$.

Now, put $\Omega_1 := \{y \in \mathcal{K} : \|y\| < r_1\}$. Note that for $y \in \partial\Omega_1$, we have that $\|y\| = r_1$ so that condition (C1) holds for all $y \in \partial\Omega_1$. So, for $y \in \mathcal{K} \cap \partial\Omega_1$, we find that

$$\begin{aligned}
\|Ty\| &= \max_{t\in[\nu-1,\nu+b]_{\mathbb{N}_{\nu-1}}} \sum_{s=0}^{b} G(t,s)f(s+\nu-1, y(s+\nu-1)) \\
&\leq \sum_{s=0}^{b} G(s+\nu-1,s)f(s+\nu-1, y(s+\nu-1)) \\
&\leq \eta r_1 \sum_{s=0}^{b} G(s+\nu-1,s) \\
&= r_1 \\
&= \|y\|,
\end{aligned}$$

whence we find that $\|Ty\| \leq \|y\|$ whenever $y \in \mathcal{K} \cap \partial\Omega_1$. Thus we get that the operator T is a cone compression on $\mathcal{K} \cap \Omega_1$.

On the other hand, put $\Omega_2 := \{y \in \mathcal{K} : \|y\| < r_2\}$. Note that for $y \in \partial\Omega_2$, we have that $\|y\| = r_2$ so that condition (C2) holds for all $y \in \partial\Omega_2$. Also note that $\left\{\lfloor \frac{b+1}{2} \rfloor + \nu\right\} \subset \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]$. So, for $y \in \mathcal{K} \cap \partial\Omega_2$, we find that

$$\begin{aligned}
Ty\left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu\right) &= \sum_{s=0}^{b} G\left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s\right) f(s+\nu-1, y(s+\nu-1)) \\
&\geq \sum_{s=\lceil \frac{\nu+b}{4} - \nu + 1 \rceil}^{\lfloor \frac{3(\nu+b)}{4} - \nu + 1 \rfloor} \gamma G\left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s\right) f(s+\nu-1, y(s+\nu-1)) \\
&\geq \lambda r_2 \sum_{s=\lceil \frac{\nu+b}{4} - \nu + 1 \rceil}^{\lfloor \frac{3(\nu+b)}{4} - \nu + 1 \rfloor} \gamma G\left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s\right) \\
&= r_2,
\end{aligned}$$

whence $\|Ty\| \geq \|y\|$, whenever $y \in \mathcal{K} \cap \partial\Omega_2$. Thus we get that the operator T is

a cone expansion on $\mathcal{K} \cap \partial\Omega_2$. So, it follows by Lemma 2.13 that the operator T has a fixed point. But this means that (1.1) has a positive solution, say y_0, with $r_1 \leq \|y_0\| \leq r_2$, as claimed. $\square$

We now provide a second existence theorem for problem (6.1).

Theorem 6.15. *Assume that there exists a constant $M > 0$ such that*

$$\max_{(t,y)\in[\nu-1,\nu+b-1]_{\mathbb{N}_{\nu-1}}\times[-M,M]} |f(t,y)| \leq \frac{M}{\sum_{s=0}^{b} G(s+\nu-1,s)}, \tag{6.60}$$

where $G(t,s)$ is the Green's function for problem (1.1). Then (1.1) has a solution, say $y_0(t)$, such that $|y_0(t)| \leq M$, for each $t \in [\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}$.

Proof. Let T be the operator defined at the beginning of this section. Moreover, let $\mathcal{B}_M$ the Banach space defined by $\mathcal{B}_M := \left\{y \in \mathbb{R}^{b+3} \ : \ \|y\| \leq M\right\}$, where M is the constant given in the statement of this theorem. Observe that

$$\begin{aligned}
\|Ty\| &\leq \max_{t\in[\nu-1,\nu+b]_{\mathbb{N}_{\nu-1}}} \sum_{s=0}^{b} |G(t,s)||f(s+\nu-1,y(s+\nu-1))| \\
&\leq \frac{M}{\sum_{s=0}^{b} G(s+\nu-1,s)} \sum_{s=0}^{b} \max_{t\in[\nu-1,\nu+b]_{\mathbb{N}_{\nu-1}}} |G(t,s)| \\
&= \frac{M}{\sum_{s=0}^{b} G(s+\nu-1,s)} \sum_{s=0}^{b} G(s+\nu-1,s) \\
&= M,
\end{aligned} \tag{6.61}$$

whence from (6.61) we conclude that $T : \mathcal{B}_M \to \mathcal{B}_M$. Consequently, we conclude by the Brouwer theorem that T has a fixed point y_0, with $y_0 \in \mathcal{B}_M$. In particular, y_0 is thus a solution of (6.1) satisfying $|y_0(t)| \leq M$, for each $t \in [\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}$. Consequently, the theorem is proved. $\square$

Chapter 7

A Note on the Green's Function for a Certain Continuous Fractional BVP

The preceding chapters have dealt exclusively with *discrete* fractional boundary value problems as well as certain operational properties of the discrete fractional difference. But, in fact, as elucidated in Chapter 1, it is the *continuous* fractional calculus which, at least at present, has inspired a much greater body of literature, owing both to its inception far before the discrete fractional calculus and to its realization as a powerful tool in applied mathematics. Consequently, in this short chapter we wish to give a simple but interesting result, which, in fact, provides a pleasant connection to the preceding results in this work and, moreover, to the results yet to come.

In particular, we are concerned with a partial extension of a problem considered in a very recent paper by Zhang [93]. Zhang considered the problem

$$D_{0^+}^{\alpha}u(t) + q(t)f\left(u, u', \ldots, u^{(n-2)}\right) = 0,\ 0 < t < 1,\ n-1 < \alpha \leq n, \tag{7.1}$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0, \tag{7.2}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order α, q may be singular at $t = 0$, and f may be singular at $u = 0$, $u' = 0$, ..., $u^{(n-2)} = 0$. As a consequence of the viewpoint assumed by Zhang, it is never addressed whether or not the Green's function associated to (7.1)–(7.2) satisfies a Harnack-like inequality. As is well-known from the existing literature, this is a crucial property when seeking the existence of positive solutions by means of cone theory. One may consult from among a great many papers the classic paper of Erbe and Wang [39] to see explicitly this fact. On the other hand and perhaps surprisingly, it was first shown by Bai and Lü [23] that the fractional analogue of the two-point conjugate BVP does *not* satisfy this property.

Here we consider, for $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ continuous, a class of (continuous) fractional boundary value problems (FBVPs) of the form

$$-D_{0+}^{\nu} y(t) = f(t, y(t)),\ 0 < t < 1,\ n - 1 < \nu \leq n, \tag{7.3}$$

$$y^{(i)}(0) = 0,\ 0 \leq i \leq n - 2, \tag{7.4}$$

$$\left[D_{0+}^{\alpha} y(t)\right]_{t=1} = 0,\ 1 \leq \alpha \leq n - 2, \tag{7.5}$$

where $y^{(i)}$ in boundary condition (7.4) represents the i^{th} (ordinary) derivative of y. Clearly, (7.4)–(7.5) generalize the boundary conditions considered in [93]. We shall assume throughout that $n \in \mathbb{N}$ is given subject to the restriction $n > 3$. Note that this problem is not unrelated to the so-called $(k, n - k)$ conjugate BVPs, which have received much attention in recent years – see, for example, the paper by Davis and Henderson [32] and the references therein. Moreover, in the special case when $\nu = 4$, problem (7.3) has been studied with a variety of boundary conditions and

nonlinearities – see, for example, [60] and the references therein.

Our primary contribution in this brief chapter is that we improve certain of Zhang's results by showing that the Green's function associated to (7.3)–(7.5) satisfies, among other properties, a Harnack-like inequality. Since by putting $\alpha = n-2$ in (7.5) above we get the boundary conditions given by (7.2), our results affirm that the Green's function associated to (7.1)–(7.2) does satisfy a Harnack-like inequality. Since our purpose here is merely to illustrate this affirmation and comment on its significance, we dispense with showing any existence-type results, which, in any case, would be standard once the requisite Green's function properties have been established.

7.1 Green's Function Properties

We begin by deriving the Green's function for the operator $-D^{\nu}$ together with the boundary conditions given in (7.4)–(7.5).

Theorem 7.1. *Let $g \in \mathcal{C}([0,1])$ be given. Then the unique solution to problem $-D^{\nu}y(t) = g(t)$ together with the boundary conditions (7.4)–(7.5) is*

$$y(t) = \int_0^1 G(t,s)g(s)\ ds, \tag{7.6}$$

where

$$G(t,s) = \begin{cases} \frac{t^{\nu-1}(1-s)^{\nu-\alpha-1}-(t-s)^{\nu-1}}{\Gamma(\nu)}, & 0 \leq s \leq t \leq 1 \\ \frac{t^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\nu)}, & 0 \leq t \leq s \leq 1 \end{cases} \tag{7.7}$$

is the Green's function for this problem.

Proof. We know from Lemma 2.12 that the general solution to our problem is

$$y(t) = c_1t^{\nu-1} + c_2t^{\nu-2} + \cdots + c_nt^{\nu-n} - D^{-\nu}g(t), \tag{7.8}$$

where we note that $-\nu < 0$. We immediately observe that boundary condition (7.4) implies that $c_2 = \cdots = c_n = 0$. On the other hand, recalling (see [78]) that $D^{\alpha}\left[t^{\nu-1}\right] = \frac{\Gamma(\nu)}{\Gamma(\nu-\alpha)}t^{\nu-\alpha-1}$, we find that boundary condition (7.5) implies that

$$0 = c_1 \cdot \frac{\Gamma(\nu)}{\Gamma(\nu-\alpha)}(1)^{\nu-\alpha-1} - \frac{1}{\Gamma(\nu-\alpha)}\int_0^1 (1-s)^{\nu-\alpha-1}g(s)\ ds, \tag{7.9}$$

where we have used Lemma 2.11. But (7.9) may be simplified to get that

$$c_1 = \frac{1}{\Gamma(\nu)}\int_0^1 (1-s)^{\nu-\alpha-1}g(s)\ ds. \tag{7.10}$$

Finally putting (7.10) into (7.8) and using the fact that $c_i = 0$ for each $i \geq 2$, we find that the general solution to the problem is

$$y(t) = \frac{t^{\nu-1}}{\Gamma(\nu)}\int_0^1 (1-s)^{\nu-\alpha-1}g(s)\ ds - \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}g(s)\ ds, \tag{7.11}$$

from which it is easy to see that (7.6) holds with $G(t,s)$ defined as in (7.2). $\square$

We now state and prove several properties of the Green's function derived in Theorem 7.1. For convenience in the sequel, let us put

$$G_1(t,s) := \frac{t^{\nu-1}(1-s)^{\nu-\alpha-1} - (t-s)^{\nu-1}}{\Gamma(\nu)},\ 0 \leq s \leq t \leq 1 \tag{7.12}$$

and

$$G_2(t,s) := \frac{t^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\nu)},\ 0 \leq t \leq s \leq 1. \tag{7.13}$$

Theorem 7.2. *Let $G(t,s)$ be as given in the statement of Theorem 7.1. Then we find that:*

1. *$G(t,s)$ is a continuous function on the unit square $[0,1]\times[0,1]$;*

2. $G(t,s) \geq 0$ *for each* $(t,s) \in [0,1] \times [0,1]$*; and*

3. $\max_{t\in[0,1]} G(t,s) = G(1,s)$*, for each* $s \in [0,1]$*.*

Proof. That property (1) holds is trivial. Indeed, it is clear that each of G_1 and G_2 are continuous on their domains and that $G_1(s,s) = G_2(s,s)$, whence (1) follows.

To prove that (3) is true, we begin by noting that for each fixed admissible s, we have $\frac{\partial G_2}{\partial t} > 0$, clearly. So, in particular, G_2 is increasing with respect to t. On the other hand, note that $\frac{\partial G_1}{\partial t} = \frac{(\nu-1)t^{\nu-2}(1-s)^{\nu-\alpha-1}-(t-s)^{\nu-2}(\nu-1)}{\Gamma(\nu)}$. Put $G^*(t,s) := \frac{\partial G_1}{\partial t}$ for each admissible (t,s). Evidently, $G^*(t,s) > 0$ on its domain if and only if $t^{\nu-2}(1-s)^{\nu-\alpha-1} - (t-s)^{\nu-2} \geq 0$. But that this latter inequality holds follows from the observation that

$$t^{\nu-2}(1-s)^{\nu-\alpha-1} - t^{\nu-2}\left(1-\frac{s}{t}\right)^{\nu-2} \geq t^{\nu-2}\left[(1-s)^{\nu-\alpha-1} - (1-s)^{\nu-2}\right] \geq 0, \quad (7.14)$$

where to get the first inequality we use the fact that $0 \leq t \leq 1$, whereas to get the final inequality we use the fact that $\nu - \alpha - 1 \leq \nu - 2$, for each admissible α. Thus, as (7.14) holds, we deduce that $G^*(t,s) \geq 0$ on its domain. In particular, then, G_1 is increasing on its domain, too. Consequently, (3) holds.

Finally, to prove that (2) holds, let us note that for each fixed and admissible s, we have that $G(0,s) = 0$. So, as (3) implies that $G(t,s)$ is increasing in t for each s, we find at once that $G(t,s) \geq 0$ at each admissible pair (t,s). Thus, (2) holds, and the proof is complete. □

Theorem 7.3. *Let* $G(t,s)$ *be as given in the statement of Theorem 7.1. Then there exists a constant* $\gamma \in (0,1)$ *such that*

$$\min_{t\in\left[\frac{1}{2},1\right]} G(t,s) \geq \gamma \max_{t\in[0,1]} G(t,s) = \gamma G(1,s). \quad (7.15)$$

Proof. Notice that Theorem 7.2 implies that

$$\begin{aligned}\min_{t\in\left[\frac{1}{2},1\right]} G(t,s) &= \begin{cases} G_1\left(\frac{1}{2},s\right), & s\in\left(0,\frac{1}{2}\right] \\ G_2\left(\frac{1}{2},s\right), & s\in\left[\frac{1}{2},1\right) \end{cases} \\ &= \begin{cases} \frac{\left(\frac{1}{2}\right)^{\nu-1}(1-s)^{\nu-\alpha-1}-\left(\frac{1}{2}-s\right)^{\nu-1}}{\Gamma(\nu)}, & s\in\left(0,\frac{1}{2}\right] \\ \frac{\left(\frac{1}{2}\right)^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\nu)}, & s\in\left[\frac{1}{2},1\right) \end{cases}.\end{aligned} \tag{7.16}$$

Moreover, observe that

$$\begin{aligned}&\lim_{s\to 0^+}\frac{\left(\frac{1}{2}\right)^{\nu-1}(1-s)^{\nu-\alpha-1}-\left(\frac{1}{2}-s\right)^{\nu-1}}{(1-s)^{\nu-\alpha-1}\left[1-(1-s)^{\alpha}\right]} \\ &\overset{\text{L'H}}{=}\lim_{s\to 0^+}\frac{-(\nu-\alpha-1)\left(\frac{1}{2}\right)^{\nu-1}(1-s)^{\nu-\alpha-2}+\left(\frac{1}{2}-s\right)^{\nu-2}(\nu-1)}{-(1-s)^{\nu-\alpha-2}(\nu-\alpha-1)+(1-s)^{\nu-2}(\nu-1)},\end{aligned} \tag{7.17}$$

whence $\lim_{s\to 0^+}\frac{\left(\frac{1}{2}\right)^{\nu-1}(1-s)^{\nu-\alpha-1}-\left(\frac{1}{2}-s\right)^{\nu-1}}{(1-s)^{\nu-\alpha-1}\left[1-(1-s)^{\alpha}\right]}=\frac{1}{\alpha}\left(\frac{1}{2}\right)^{\nu-1}(\nu+\alpha-1)>0$. It is also the case that for $0<s\leq\frac{1}{2}$

$$\begin{aligned}\frac{\left(\frac{1}{2}\right)^{\nu-1}(1-s)^{\nu-\alpha-1}-\left(\frac{1}{2}-s\right)^{\nu-1}}{(1-s)^{\nu-\alpha-1}\left[1-(1-s)^{\alpha}\right]} &\geq \frac{\left(\frac{1}{2}\right)^{\nu-1}\left(1-\frac{1}{2}\right)^{\nu-\alpha-1}-\left(\frac{1}{2}-\frac{1}{2}\right)^{\nu-1}}{\left(1-\frac{1}{2}\right)^{\nu-\alpha-1}\left[1-\left(1-\frac{1}{2}\right)^{\alpha}\right]} \\ &= \frac{\left(\frac{1}{2}\right)^{\nu-\alpha-1}}{2^{\alpha}-1}.\end{aligned} \tag{7.18}$$

Finally, observe that for $\frac{1}{2}\leq s\leq 1$, we find that

$$\frac{\left(\frac{1}{2}\right)^{\nu-1}}{1-(1-s)^{\alpha}}\geq\left(\frac{1}{2}\right)^{\nu-1}. \tag{7.19}$$

Now, define $\widetilde{\gamma}(s) \ : \ [0,1] \to (0,1)$ by

$$\widetilde{\gamma}(s) := \begin{cases} \frac{\left(\frac{1}{2}\right)^{\nu-1}(1-s)^{\nu-\alpha-1}-\left(\frac{1}{2}-s\right)^{\nu-1}}{(1-s)^{\nu-\alpha-1}[1-(1-s)^{\alpha}]}, & 0 < s \leq \frac{1}{2} \\ \frac{\left(\frac{1}{2}\right)^{\nu-1}}{1-(1-s)^{\alpha}}, & \frac{1}{2} \leq s \leq 1 \end{cases}, \tag{7.20}$$

where $\widetilde{\gamma}(0) := \lim_{s\to 0^+} \widetilde{\gamma}(s)$; note that $\widetilde{\gamma}(0) > 0$ by (7.17). Put

$$\gamma := \min \left\{ \frac{\left(\frac{1}{2}\right)^{\nu-\alpha-1}}{2^{\alpha}-1}, \left(\frac{1}{2}\right)^{\nu-1} \right\}, \tag{7.21}$$

where, evidently, $0 < \gamma < 1$. Then from (7.16)–(7.21), we find that

$$\min_{t\in\left[\frac{1}{2},1\right]} G(t,s) = \widetilde{\gamma}(s) \max_{t\in[0,1]} G(t,s) \geq \gamma \max_{t\in[0,1]} G(t,s) = \gamma G(1,s),$$

as claimed. □

Remark 7.4. Note that in great contrast to, say [23], where their γ is a function of s and satisfies $\lim_{s\to 0^+} \gamma(s) = 0$, in our Theorem 7.3 above we are able to take our γ to be a *strictly positive constant*. We believe this to be a very important difference between our results and other work (e.g., [23] and [89]) on (continuous) FBVPs. Moreover, as pointed out in the introduction to this chapter, this improves and builds on certain of the results given in [93]. Indeed, we have shown here that for the particular problem considered here, the associated Green's function *does* satisfy a typical Harnack-like inequality, which, as noted earlier, is, in fact, somewhat unusual.

Remark 7.5. Interestingly, it can be shown that for $0 \leq \alpha < 1$ in (7.5), we find that γ can no longer be taken as a constant and that, moreover, $\lim_{s\to 0^+} \gamma(s) = 0$ in this case. We omit the details, however.

Chapter 8

Systems of Nonlocal BVPs for a Continuous Fractional Operator

We continue in this chapter our transition from ***discrete fractional local problems*** to ***continuous integer-order nonlocal problems***. Consequently, we consider here a system of nonlinear differential equations of fractional order having the form

$$\begin{cases} -D_{0^+}^{\nu_1} y_1(t) = \lambda_1 a_1(t) f\left(y_1(t), y_2(t)\right), \\ -D_{0^+}^{\nu_2} y_2(t) = \lambda_2 a_2(t) g\left(y_1(t), y_2(t)\right), \end{cases} \tag{8.1}$$

where $t \in (0,1)$, ν_1, $\nu_2 \in (n-1,n]$, and λ_1, $\lambda_2 > 0$, subject to a couple of different sets of boundary conditions. In particular, we first consider problem (8.1) subject to

$$y_1^{(i)}(0) = 0 = y_2^{(i)}(0),\ 0 \leq i \leq n-2, \tag{8.2}$$

$$\left[D_{0^+}^{\alpha} y_1(t)\right]_{t=1} = 0 = \left[D_{0^+}^{\alpha} y_2(t)\right]_{t=1},\ 1 \leq \alpha \leq n-2, \tag{8.3}$$

where $y^{(i)}$ in boundary condition (8.2) represents the i-th (ordinary) derivative of y. We then consider the case in which the boundary conditions are changed to

$$y_1^{(i)}(0) = 0 = y_2^{(i)}(0),\ 0 \leq i \leq n-2, \tag{8.4}$$

$$\left[D_{0^+}^{\alpha} y_1(t)\right]_{t=1} = \phi_1(y),\ 1 \leq \alpha \leq n-2, \tag{8.5}$$

$$\left[D_{0^+}^{\alpha} y_2(t)\right]_{t=1} = \phi_2(y),\ 1 \leq \alpha \leq n-2, \tag{8.6}$$

where $\phi_1,\ \phi_2\ :\ \mathcal{C}([0,1]) \to \mathbb{R}$ are continuous functionals, where the notation $\mathcal{C}([0,1])$ means the set of continuous, real-valued functions on the unit interval $[0,1]$; even though (8.3) and (8.5)–(8.6) lose physical meaning when $\alpha \notin \mathbb{N}$, they are still mathematically meaningful. We also consider these boundary conditions in the special case in which $\lambda_1 = \lambda_2 = 1$. Note that in (8.1), (8.3), (8.5), (8.6), and, in fact throughout this chapter, $D_{0^+}^{\nu} y(t)$ represents the Riemann-Liouville fractional derivative of order ν of the function $y(t)$. We also assume throughout that $n \in \mathbb{N}$ subject to the restriction that $n > 3$. The main contribution of this chapter is to determine conditions under which either problem (8.1)–(8.3) or (8.1), (8.4)–(8.6) will exhibit at least one positive solution. In particular, we shall state conditions on λ_1, λ_2, which are eigenvalues, for which problem (8.1)–(8.3) has at least one positive solution; it ought to be noted that ***unlike in the integer-order case, the range of admissible eigenvalues depends on the choices of*** ν_1***,*** ν_2***, and*** α. In addition, we state conditions on ϕ_1, ϕ_2 such that problem (8.1), (8.4)–(8.6) has at least one positive solution.

8.1 Existence of a Positive Solution: Case - I

In this section we wish to present a general condition under which problem (8.1)–(8.3) will exhibit at least one positive solution. We first need to fix our framework for analyzing problem (8.1)–(8.3).

First of all, let $\mathcal{B}$ represent the Banach space of $\mathcal{C}([0,1])$ when equipped with the usual supremum norm, $\|\cdot\|$. Then put

$$\mathcal{X} := \mathcal{B} \times \mathcal{B}, \tag{8.7}$$

where $\mathcal{X}$ is equipped with the norm

$$\|\,(y_1, y_2)\,\| := \|y_1\| + \|y_2\|, \tag{8.8}$$

for $(y_1, y_2) \in \mathcal{X}$. Observe that $\mathcal{X}$ is also a Banach space – see Dunninger and Wang [36]. In addition, define the operators T_1, $T_2\ :\ \mathcal{X} \to \mathcal{B}$ by

$$\left(T_1\left(y_1, y_2\right)\right)(t) := \lambda_1 \int_0^1 G_1(t,s)a_1(s)f\left(y_1(s), y_2(s)\right)\ ds \tag{8.9}$$

and

$$\left(T_2\left(y_1, y_2\right)\right)(t) := \lambda_2 \int_0^1 G_2(t,s)a_2(s)g\left(y_1(s), y_2(s)\right)\ ds, \tag{8.10}$$

where $G_1(t,s)$ is the Green's function of Theorem 7.1 with ν replaced by ν_1 and, likewise, $G_2(t,s)$ is the Green's function of Theorem 7.1 with ν replaced by ν_2. Using

(8.9)–(8.10), define an operator $S : \mathcal{X} \to \mathcal{X}$ by

$$\begin{aligned}&(S(y_1, y_2))(t) := ((T_1(y_1, y_2))(t), (T_2(y_1, y_2))(t))\\&= \left(\lambda_1 \int_0^1 G_1(t,s) a_1(s) f(y_1(s), y_2(s))\ ds, \lambda_2 \int_0^1 G_2(t,s) a_2(s) g(y_1(s), y_2(s))\ ds\right).\end{aligned} \tag{8.11}$$

We claim that whenever $(y_1, y_2) \in \mathcal{X}$ is a fixed point of the operator defined in (8.11), it follows that $y_1(t)$ and $y_2(t)$ solve problem (8.1)–(8.3). This is the content of Lemma 8.1, whose proof we provide next.

Lemma 8.1. *A pair of functions $(y_1, y_2) \in \mathcal{X}$ is a solution of problem (8.1)–(8.3) if and only if (y_1, y_2) is a fixed point of the operator S defined in (8.11).*

Proof. The forward implication is immediate, owing to the result given in Theorem 7.1. Conversely, suppose that $(y_1, y_2) \in \mathcal{X}$ is a fixed point of the operator S. Then, in particular, we find that

$$y_1(t) = \lambda_1 \int_0^1 G_1(t,s) a_1(s) f(y_1(s), y_2(s))\ ds. \tag{8.12}$$

Observe that the right-hand side of (8.12) may be recast as

$$\lambda_1 t^{\nu_1 - 1} \cdot \frac{\Gamma(\nu_1 - \alpha)}{\Gamma(\nu_1)} \left[D^{\alpha - \nu_1} a_1(t) f(y_1(t), y_2(t))\right]_{t=1} - \lambda_1 D^{-\nu_1}\left[a_1(t) f(y_1(t), y_2(t))\right] \tag{8.13}$$

so that, in fact,

$$\begin{aligned}y_1(t) = \lambda_1 t^{\nu_1 - 1} \cdot \frac{\Gamma(\nu_1 - \alpha)}{\Gamma(\nu_1)} &\left[D^{\alpha - \nu_1} a_1(t) f(y_1(t), y_2(t))\right]_{t=1}\\&- \lambda_1 D^{-\nu_1}\left[a_1(t) f(y_1(t), y_2(t))\right].\end{aligned} \tag{8.14}$$

We claim that $y_1(t)$ satisfies the differential equation (8.1) and the boundary conditions (8.2)–(8.3). To see that the former holds, apply the differential operator D^{ν_1} to both sides of (8.14) and recall (cf., [78]) that $D^{\nu_1}\left[t^{\nu_1-j}\right]=0$, for $1\leq j\leq n$, and that $D^{\nu_1}D^{-\nu_1}=D^0$. Then we find that

$$\begin{aligned} D^{\nu_1}y_1(t) &= \lambda_1 D^{\nu_1}\left[t^{\nu_1-1}\right]\cdot\frac{\Gamma\left(\nu_1-\alpha\right)}{\Gamma\left(\nu_1\right)}\left[D^{\alpha-\nu_1}a_1(t)f\left(y_1(t),y_2(t)\right)\right]_{t=1} \\ &\quad-\lambda_1 D^{\nu_1}\left[D^{-\nu_1}\left[a_1(t)f\left(y_1(t),y_2(t)\right)\right]\right] \\ &= -\lambda_1 a_1(t)f\left(y_1(t),y_2(t)\right), \end{aligned} \tag{8.15}$$

from which we see that $y_1(t)$ satisfies the differential equation in (8.1). On the other hand, to see that $y_1(t)$ satisfies the boundary conditions in (8.2)–(8.3), fix an i satisfying $0\leq i\leq n-2$ and note that

$$\begin{aligned} y_1^{(i)}(t) &= \lambda_1\left(\nu_1-1\right)\cdots\left(\nu_1-i\right)t^{\nu_1-1-i}\cdot\frac{\Gamma\left(\nu_1-\alpha\right)}{\Gamma\left(\nu_1\right)}\left[D^{\alpha-\nu_1}a_1(t)f\left(y_1(t),y_2(t)\right)\right]_{t=1} \\ &\quad-\lambda_1 D^iD^{-\nu_1}\left[a_1(t)f\left(y_1(t),y_2(t)\right)\right]. \end{aligned} \tag{8.16}$$

Recalling that $D^iD^{-\nu_1}=D^{i-\nu_1}$ (cf., Lemma 2.11) and that $\nu_1-1-i>0$, we find that

$$y_1^{(i)}(0)=\lambda_1\left\{0-D^{i-\nu_1}\left[a_1(t)f\left(y_1(t),y_2(t)\right)\right]\right\}_{t=0}=0, \tag{8.17}$$

so that y_1 satisfies boundary condition (8.2). (Note that we have used the continuity of a_1 and f here so that $\{D^{i-\nu_1}\left[a_1(t)f\left(y_1(t),y_2(t)\right)\right]\}_{t=0}$ has value 0.) Finally, recall that (see [78])

$$D^{\alpha}t^{\nu_1-1}=\frac{\Gamma\left(\nu_1\right)}{\Gamma\left(\nu_1-\alpha\right)}t^{\nu_1-\alpha-1}. \tag{8.18}$$

Then (8.18) together with an application of Lemma 2.11 implies that

$$\begin{aligned} D^{\alpha}y_1(t) &= \lambda_1 t^{\nu_1-\alpha-1}\left[D^{\alpha-\nu_1}a_1(t)f\left(y_1(t),y_2(t)\right)\right]_{t=1} \\ &\quad - \lambda_1 D^{\alpha}D^{-\nu_1}\left[a_1(t)f\left(y_1(t),y_2(t)\right)\right] \\ &= \lambda_1 t^{\nu_1-\alpha-1}\left[D^{\alpha-\nu_1}a_1(t)f\left(y_1(t),y_2(t)\right)\right]_{t=1} - \lambda_1 D^{\alpha-\nu_1}\left[a_1(t)f\left(y_1(t),y_2(t)\right)\right] \end{aligned} \tag{8.19}$$

so that, since $\nu_1-\alpha-1>0$,

$$\begin{aligned} \left[D^{\alpha}y_1(t)\right]_{t=1} &= \lambda_1\left[D^{\alpha-\nu_1}a_1(t)f\left(y_1(t),y_2(t)\right)\right]_{t=1} - \lambda_1 D^{\alpha-\nu_1}\left[a_1(t)f\left(y_1(t),y_2(t)\right)\right]_{t=1} \\ &= 0, \end{aligned} \tag{8.20}$$

whence y_1 satisfies the boundary condition (8.3).

Now, a completely dual calculation reveals that y_2 also satisfies boundary conditions (8.2)–(8.3) and the differential equation $-D^{\nu_2}y_2(t)=\lambda_2 a_2(t)g\left(y_1(t),y_2(t)\right)$. Therefore, we conclude that if $(y_1,y_2)\in\mathcal{X}$ is a fixed point of the operator S, then (y_1,y_2) solves the problem (8.1)–(8.3). And this completes the proof. $\square$

As a consequence of Lemma 8.1, we shall look for fixed points of the operator S, seeing as these fixed points coincide with solutions of problem (8.1)–(8.3). For use in the sequel, let γ_1 and γ_2 the constants given by Theorem 7.3 associated, respectively, to the Green's functions G_1 and G_2, and define $\widetilde{\gamma}$ by

$$\widetilde{\gamma}:=\min\left\{\gamma_1,\gamma_2\right\}, \tag{8.21}$$

and notice that $\widetilde{\gamma}\in(0,1)$. Let us next introduce some conditions on the nonlinearities as well as the eigenvalues. These are very similar to those presented by Henderson,

et al. [62].

F1: There exist numbers f^* and g^*, with f^*, $g^* \in (0,+\infty)$, such that

$$\lim_{y_1+y_2\to 0^+} \frac{f\left(y_1,y_2\right)}{y_1+y_2} = f^* \text{ and } \lim_{y_1+y_2\to 0^+} \frac{g\left(y_1,y_2\right)}{y_1+y_2} = g^*.$$

F2: There exist numbers f^{**} and g^{**}, with f^{**}, $g^{**} \in (0,+\infty)$, such that

$$\lim_{y_1+y_2\to +\infty} \frac{f\left(y_1,y_2\right)}{y_1+y_2} = f^{**} \text{ and } \lim_{y_1+y_2\to +\infty} \frac{g\left(y_1,y_2\right)}{y_1+y_2} = g^{**}.$$

L1: There are numbers Λ_1 and Λ_2, where

$$\begin{aligned}\Lambda_1 := \max\Bigg\{ & \frac{1}{2}\left[\int_{\frac{1}{2}}^{1} \widetilde{\gamma} G_1\left(1,s\right) a_1(s) f^{**}\ ds\right]^{-1}, \\ & \frac{1}{2}\left[\int_{\frac{1}{2}}^{1} \widetilde{\gamma} G_2\left(1,s\right) a_2(s) g^{**}\ ds\right]^{-1}\Bigg\}\end{aligned} \tag{8.22}$$

and

$$\Lambda_2 := \min\left\{\frac{1}{2}\left[\int_0^1 G_1(1,s)a_1(s)f^*\ ds\right]^{-1}, \frac{1}{2}\left[\int_0^1 G_2(1,s)a_2(s)g^*\ ds\right]^{-1}\right\}, \tag{8.23}$$

such that $\Lambda_1 < \lambda_1, \lambda_2 < \Lambda_2$.

Next define the cone $\mathcal{K}$ by

$$\mathcal{K} := \left\{ (y_1,y_2) \in \mathfrak{X}\ :\ y_1,\ y_2 \geq 0,\ \min_{t\in\left[\frac{1}{2},1\right]} \left[y_1(t)+y_2(t)\right] \geq \widetilde{\gamma} \|\left(y_1,y_2\right)\| \right\}. \tag{8.24}$$

We then deduce the following lemma.

Lemma 8.2. *Let S be the operator defined by (8.11). Then $S\ :\ \mathcal{K} \to \mathcal{K}$.*

Proof. Let $(y_1, y_2) \in \mathcal{X}$ be given. It is clear from the definition of S together with the fact that a_1, a_2, f, and g are nonnegative that $T_1(y_1, y_2)(t) \geq 0$ and $T_2(y_1, y_2)(t) \geq 0$, for each $t \in [0, 1]$. On the other hand, we observe that

$$\begin{aligned}
\min_{t\in\left[\frac{1}{2},1\right]} \left[T_1(y_1, y_2)(t) + T_2(y_1, y_2)(t)\right] &\geq \min_{t\in\left[\frac{1}{2},1\right]} T_1(y_1, y_2)(t) + \min_{t\in\left[\frac{1}{2},1\right]} T_2(y_1, y_2)(t) \\
&\geq \gamma_1 \|T_1(y_1, y_2)\| + \gamma_2 \|T_2(y_1, y_2)\| \\
&\geq \widetilde{\gamma}\left[\|T_1(y_1, y_2)\| + \|T_2(y_1, y_2)\|\right] \\
&= \widetilde{\gamma}\|\left(T_1(y_1, y_2), T_2(y_1, y_2)\right)\| \\
&= \widetilde{\gamma}\|S(y_1, y_2)\|.
\end{aligned} \tag{8.25}$$

So, we conclude that $S : \mathcal{K} \to \mathcal{K}$, as desired. And this completes the proof. □

We now state our existence theorem. While this theorem is similar to the existence theorem provided in [62], it is completely new in the fractional-order case. Moreover, later in this chapter we shall give results that more substantially generalize even the integer-order results presented in [62].

Theorem 8.3. *Suppose that conditions (F1)–(F2) and (L1) are satisfied. Then problem (8.1)–(8.3) has at least one positive solution.*

Proof. We have already shown in Lemma 8.2 that $S : \mathcal{K} \to \mathcal{K}$. Furthermore, a relatively straightforward application of the Arzela-Ascoil theorem, which we omit, reveals that S is a completely continuous operator.

Now, observe that by condition (L1) that there is $\epsilon > 0$ sufficiently small such

that

$$\max\left\{\frac{1}{2}\left[\int_{\frac{1}{2}}^{1}\widetilde{\gamma}G_1\left(1,s\right)a_1(s)\left(f^{**}-\epsilon\right)\;ds\right]^{-1},\right.$$
$$\left.\frac{1}{2}\left[\int_{\frac{1}{2}}^{1}\widetilde{\gamma}G_2\left(1,s\right)a_2(s)\left(g^{**}-\epsilon\right)\;ds\right]^{-1}\right\}\leq\lambda_1,\,\lambda_2 \tag{8.26}$$

and

$$\lambda_1,\,\lambda_2\leq\min\left\{\frac{1}{2}\left[\int_{0}^{1}G_1(1,s)a_1(s)\left(f^{*}+\epsilon\right)\;ds\right]^{-1},\right.$$
$$\left.\frac{1}{2}\left[\int_{0}^{1}G_2(1,s)a_2(s)\left(g^{*}+\epsilon\right)\;ds\right]^{-1}\right\}. \tag{8.27}$$

Now, given this ε, by condition (F1) it follows that there exists some number $r_1^*>0$ such that

$$f\left(y_1,y_2\right)\leq\left(f^*+\varepsilon\right)\left(y_1+y_2\right), \tag{8.28}$$

whenever $\|\left(y_1,y_2\right)\|<r_1^*$. Similarly, by condition (F1), for the same ε, there exists a number $r_1^{**}>0$ such that

$$g\left(y_1,y_2\right)\leq\left(g^*+\varepsilon\right)\left(y_1+y_2\right), \tag{8.29}$$

whenever $\|\left(y_1,y_2\right)\|<r_1^{**}$. In particular, then, by putting $r_1:=\min\left\{r_1^*,r_1^{**}\right\}$, we find that both (8.28) and (8.29) hold whenever $\|\left(y_1,y_2\right)\|<r_1$. So, define Ω_1 by

$$\Omega_1:=\left\{\left(y_1,y_2\right)\in\mathcal{X}\;:\;\|\left(y_1,y_2\right)\|<r_1\right\}. \tag{8.30}$$

Then for $(y_1, y_2) \in \mathcal{K} \cap \partial\Omega_1$ we find that

$$\begin{aligned}
\|T_1(y_1, y_2)\| &\leq \lambda_1 \int_0^1 G_1(1,s)a_1(s)f(y_1(s), y_2(s))\ ds \\
&\leq \lambda_1 \int_0^1 G_1(1,s)a_1(s)(f^* + \varepsilon)(y_1(s) + y_2(s))\ ds \\
&\leq \lambda_1 \int_0^1 G_1(1,s)a_1(s)(f^* + \varepsilon)(\|y_1\| + \|y_2\|)\ ds \\
&= \|(y_1, y_2)\| \cdot \lambda_1 \int_0^1 G_1(1,s)a_1(s)(f^* + \varepsilon)\ ds \\
&\leq \frac{1}{2}\|(y_1, y_2)\|.
\end{aligned} \tag{8.31}$$

We may deduce by an entirely dual argument that

$$\|T_2(y_1, y_2)\| \leq \frac{1}{2}\|(y_1, y_2)\|. \tag{8.32}$$

Thus, by putting (8.28)–(8.32) together we find that for $(y_1, y_2) \in \mathcal{K} \cap \partial\Omega_1$ we have

$$\begin{aligned}
\|S(y_1, y_2)\| = \|(T_1(y_1, y_2), T_2(y_1, y_2))\| &= \|T_1(y_1, y_2)\| + \|T_2(y_1, y_2)\| \\
\leq \frac{1}{2}\|(y_1, y_2)\| + \frac{1}{2}\|(y_1, y_2)\| &= \|(y_1, y_2)\|,
\end{aligned} \tag{8.33}$$

so that S is a cone compression on $\mathcal{K} \cap \partial\Omega_1$.

On the other hand, letting ϵ be the same positive number selected at the beginning of this proof, note that by virtue of condition (F2) we can find a number $\widetilde{r}_2 > 0$ such that

$$f(y_1, y_2) \geq (f^{**} - \varepsilon)(y_1 + y_2) \tag{8.34}$$

and

$$g(y_1, y_2) \geq (g^{**} - \varepsilon)(y_1 + y_2), \tag{8.35}$$

whenever $y_1 + y_2 \geq \widetilde{r}_2$. Put

$$r_2 := \max\left\{2r_1, \frac{\widetilde{r}_2}{\widetilde{\gamma}}\right\}. \tag{8.36}$$

Moreover, put

$$\Omega_2 := \left\{(y_1, y_2) \in \mathcal{X} \ : \ \|\,(y_1, y_2)\,\| < r_2\right\}. \tag{8.37}$$

Note that if $(y_1, y_2) \in \mathcal{K} \cap \partial\Omega_2$, then it follows that, for any $t \in \left[\frac{1}{2}, 1\right]$,

$$y_1(t) + y_2(t) \geq \min_{t\in\left[\frac{1}{2},1\right]} \left[y_1(t) + y_2(t)\right] \geq \widetilde{\gamma}\|\,(y_1, y_2)\,\| \geq \widetilde{r}_2. \tag{8.38}$$

In particular, (8.38) shows that whenever $(y_1, y_2) \in \mathcal{K}\cap\partial\Omega_2$, it holds that $\|\,(y_1, y_2)\,\| \geq \widetilde{r}_2$ so that (8.34)–(8.35) hold. Then for each $(y_1, y_2) \in \mathcal{K} \cap \partial\Omega_2$ we find that

$$\begin{aligned} T_1\left(y_1, y_2\right)(1) &= \lambda_1 \int_0^1 G_1\left(1, s\right) a_1(s) f\left(y_1(s), y_2(s)\right)\, ds \\ &\geq \lambda_1 \int_{\frac{1}{2}}^1 G_1\left(1, s\right) a_1(s) f\left(y_1(s), y_2(s)\right)\, ds \\ &\geq \lambda_1 \int_{\frac{1}{2}}^1 G_1\left(1, s\right) a_1(s) \left(f^{**} - \epsilon\right)\left(y_1(s) + y_2(s)\right)\, ds \\ &\geq \lambda_1 \int_{\frac{1}{2}}^1 \widetilde{\gamma} G_1\left(1, s\right) a_1(s) \left(f^{**} - \epsilon\right) \|\,(y_1, y_2)\,\|\, ds. \end{aligned} \tag{8.39}$$

Thus, we conclude from (8.39) that

$$\|T_1\left(y_1, y_2\right)\| \geq \frac{1}{2}\|\,(y_1, y_2)\,\|, \tag{8.40}$$

whenever $(y_1, y_2) \in \mathcal{K} \cap \partial\Omega_2$. Similarly, we find that

$$\|T_2\left(y_1, y_2\right)\| \geq \frac{1}{2}\|\,(y_1, y_2)\,\|. \tag{8.41}$$

Consequently, (8.34)–(8.41) imply that

$$\begin{aligned}\|S(y_1,y_2)\| &= \|(T_1(y_1,y_2),T_2(y_1,y_2))\| = \|T_1(y_1,y_2)\| + \|T_2(y_1,y_2)\| \\ &\geq \frac{1}{2}\|(y_1,y_2)\| + \frac{1}{2}\|(y_1,y_2)\| = \|(y_1,y_2)\|,\end{aligned} \tag{8.42}$$

whenever $(y_1, y_2) \in \mathcal{K} \cap \partial\Omega_2$. Thus, S is a cone expansion on $\mathcal{K} \cap \partial\Omega_2$.

In summary, each of the hypothesis of Lemma 2.13 is satisfied. Consequently, we conclude that S has a fixed point, say $(y_1^0, y_2^0) \in \mathcal{K}$. As the pair of functions $y_1^0(t)$, $y_2^0(t)$ is a positive solution of (1.1)–(1.3), the theorem is proved. □

We conclude this section with a remark about Theorem 8.3.

Remark 8.4. Evidently, by choosing t differently in (8.39), we would obtain a slightly different form for Λ_1. However, the form given in (8.22), which is the one induced by the choice of $t = 1$ in (8.39), is the optimal choice as it minimizes the value of Λ_1.

8.2 Existence of a Positive Solution: Case - II

We now wish to provide a set of conditions under which problem (8.1), (8.4)–(8.6) will have at least one positive solution. In particular, we shall consider two such cases. As remarked in Section 1, we note that although the boundary conditions in (8.5) and (8.6) do not necessarily possess any physical meaning when $\alpha \notin \mathbb{N}$, they are mathematically meaningful. Moreover, in case α is an integer, then these boundary conditions become physically meaningful.

8.2.1 Problem (8.1), (8.4)–(8.6) in the General Case

In this subsection, we consider the general problem (8.1), (8.4)–(8.6) in the sense that λ_1, λ_2 can range over a continuum of values, which we shall specify presently. We

shall still need conditions (F1)–(F2) in this setting. However, because the boundary conditions are now given by (8.4)–(8.6), we shall introduce a new condition, labeled (G1) in the sequel. Condition (G1) provides some control over the nonlocal boundary terms, ϕ_1 and ϕ_2. We state this condition now and then give a remark explicating the form and nature of these nonlocal functionals.

G1: The functionals $\phi_1(y_1)$ and $\phi_2(y_2)$ are continuous in y_1 and y_2, nonnegative for $y_1,\, y_2 \geq 0$, and satisfy

$$\lim_{\|y\|\to 0^+} \frac{\phi_1(y_1)}{\|y_1\|} = 0 \tag{8.43}$$

and

$$\lim_{\|y\|\to 0^+} \frac{\phi_2(y_2)}{\|y_2\|} = 0, \tag{8.44}$$

respectively.

L2: There are numbers Λ_3 and Λ_4, where

$$\begin{aligned}\Lambda_3 := \max\Bigg\{ &\frac{1}{2}\left[\int_{\frac{1}{2}}^{1} \gamma_0 G_1(1,s)\, a_1(s) f^{**}\, ds\right]^{-1}, \\ &\frac{1}{2}\left[\int_{\frac{1}{2}}^{1} \gamma_0 G_2(1,s)\, a_2(s) g^{**}\, ds\right]^{-1}\Bigg\},\end{aligned} \tag{8.45}$$

$$\Lambda_4 := \min\left\{ p\left[\int_0^1 G_1(1,s) a_1(s) f^{*}\, ds\right]^{-1}, p\left[\int_0^1 G_2(1,s) a_2(s) g^{*}\, ds\right]^{-1}\right\}, \tag{8.46}$$

and $p \in \left(0, \frac{1}{2}\right)$ is given, such that $\Lambda_3 < \lambda_1, \lambda_2 < \Lambda_4$ and where γ_0 is the constant defined in (8.58) in the sequel.

Remark 8.5. Let us make some additional comments regarding condition (G1) above.

First of all, we interpret these limits in the sense that (8.43) is true only if for each $\eta > 0$ there is $r > 0$ such that whenever $0 < \|y_1\| \le r$, it follows that $0 \le \frac{\phi_1(y_1)}{\|y_1\|} < \eta$. The same may be said of condition (8.4) involving ϕ_2.

Second of all, let us explicitly point out that this condition is indeed satisfied by nontrivial functionals $\phi\ :\ \mathcal{C}([0,1]) \to \mathbb{R}$. For instance, consider the functional

$$\phi_1(y) := [y(t_0)]^{\gamma}, \tag{8.47}$$

where $\phi_1\ :\ \mathcal{C}([0,1]) \to \mathbb{R}$ and $t_0 \in (0,1)$, $\gamma > 1$ are given. Let $\eta > 0$ be given. Then for nonnegative y, we find that whenever $0 < \|y\| \le \eta^{\frac{1}{\gamma-1}}$, it follows that

$$0 \le \frac{\phi_1(y)}{\|y\|} \le \frac{\|y\|^{\gamma}}{\|y\|} = \|y\|^{\gamma-1} \le \left(\eta^{\frac{1}{\gamma-1}}\right)^{\gamma-1} = \eta,$$

so that the condition described in the preceding paragraph is satisfied – note that we chose $r := \eta^{\frac{1}{\gamma-1}} > 0$ here.

We present now a trio of preliminary lemmas. These shall also be of use in Subsection 8.2.2 in the sequel.

Lemma 8.6. *A pair of functions $(y_1, y_2) \in \mathcal{X} \times \mathcal{X}$ is a solution of (8.1), (8.4)–(8.6) if and only if (y_1, y_2) is a fixed point of the operator $U\ :\ \mathcal{X} \to \mathcal{X}$ defined by*

$$\begin{aligned}(U(y_1, y_2))(t) := \Bigg(&\beta_1(t)\phi_1(y_1) + \lambda_1 \int_0^1 G_1(t,s)a_1(s)f(y_1(s), y_2(s))\ ds,\\ &\beta_2(t)\phi_2(y_2) + \lambda_2 \int_0^1 G_2(t,s)a_2(s)g(y_1(s), y_2(s))\ ds\Bigg),\end{aligned} \tag{8.48}$$

where $\beta_1,\ \beta_2\ :\ [0,1] \to [0,1]$ are defined by

$$\beta_1(t) := \frac{\Gamma(\nu_1 - \alpha)}{\Gamma(\nu_1)} t^{\nu_1 - 1} \tag{8.49}$$

and

$$\beta_2(t) := \frac{\Gamma\left(\nu_2 - \alpha\right)}{\Gamma\left(\nu_2\right)} t^{\nu_2 - 1}. \tag{8.50}$$

Proof. To prove this lemma, we can essentially repeat the proof of Lemma 8.1 given earlier together with a minor modification of the proof of Theorem 7.1 presented earlier. Indeed, define U_1, $U_2 \ : \ \mathcal{X} \to \mathcal{B}$ by, say,

$$U_1\left(y_1, y_2\right)(t) := \beta_1(t)\phi_1\left(y_1\right) + \lambda_1 \int_0^1 G_1(t,s)a_1(s)f\left(y_1(s), y_2(s)\right)\ ds \tag{8.51}$$

and

$$U_2\left(y_1, y_2\right)(t) := \beta_2(t)\phi_2\left(y_2\right) + \lambda_2 \int_0^1 G_2(t,s)a_2(s)g\left(y_1(s), y_2(s)\right)\ ds. \tag{8.52}$$

A verification very similar to the proof of Lemma 8.1 reveals that

$$\left(U_j^{(i)}\left(y_1, y_2\right)\right)(0) = 0, \tag{8.53}$$

for each $0 \leq i \leq n-2$ and each $j = 1,\ 2$, and that

$$D_{0^+}^{\alpha}\left[U_j\left(y_1, y_2\right)\right]_{t=1} = \phi_j\left(y_j\right), \tag{8.54}$$

for each $j = 1,\ 2$. Moreover, we find that, for each $j = 1,\ 2$, the operator $U_j\left(y_1, y_2\right)(t)$ satisfies the j-th equation in (8.1). Therefore, it follows that if $\left(y_1, y_2\right) \in \mathcal{X}$ is a fixed point of the operator U defined in (8.48), then the pair of functions $y_1(t)$, $y_2(t)$ is a solution of the boundary value problem (8.1), (8.4)–(8.6). And this completes the proof. $\square$

Lemma 8.7. *Let $\beta_1(t)$ and $\beta_2(t)$ be defined as in (8.49) and (8.50) above. Then*

each of $\beta_1(t)$ and $\beta_2(t)$ is strictly increasing in t and satisfy $\beta_1(0) = \beta_2(0) = 0$ and $\beta_1(1)$, $\beta_2(1) \in (0, 1)$. Moreover, there exist constants M_{β_1} and M_{β_2} satisfying M_{β_1}, $M_{\beta_2} \in (0, 1)$ such that $\min_{t\in\left[\frac{1}{2},1\right]} \beta_1(t) \geq M_{\beta_1}\|\beta_1\|$ and $\min_{t\in\left[\frac{1}{2},1\right]} \beta_2(t) \geq M_{\beta_2}\|\beta_2\|$.

Proof. It is obvious that $\beta_1(0) = \beta_2(0) = 0$. Moreover, since ν_1, $\nu_2 > 1$, it is also obvious that both β_1 and β_2 are strictly increasing for $t \in [0, 1]$. Moreover, recall that ν_1, $\nu_2 > 3$. Then as both $\nu_1 - \alpha \geq 1$ and $\nu_2 - \alpha \geq 1$, it follows $0 < \frac{\Gamma(\nu_i-\alpha)}{\Gamma(\nu_i)} < 1$, for each $i = 1$, 2. Finally, from the preceding properties, the final statement in the lemma is obviously true. And this completes the proof. □

Remark 8.8. Let us note at this juncture that

$$M_{\beta_1} := M_{\beta_1}(\nu_1) = \left(\frac{1}{2}\right)^{\nu_1-1} \tag{8.55}$$

and that

$$M_{\beta_2} := M_{\beta_2}(\nu_2) = \left(\frac{1}{2}\right)^{\nu_2-1}, \tag{8.56}$$

which may be easily verified by simply observing, for instance, that $M_{\beta_1} = \frac{\beta_1\left(\frac{1}{2}\right)}{\beta_1(1)}$.

In light of Lemma 8.7, let us define a new cone $\mathcal{K}_1$ by

$$\mathcal{K}_1 := \left\{(y_1, y_2) \in \mathfrak{X} \ : \ y_1,\ y_2 \geq 0,\ \min_{t\in\left[\frac{1}{2},1\right]} [y_1(t) + y_2(t)] \geq \gamma_0\|(y_1, y_2)\|\right\}, \tag{8.57}$$

where

$$\gamma_0 := \min\{\widetilde{\gamma}, M_{\beta_1}, M_{\beta_2}\}. \tag{8.58}$$

It is obvious that $\gamma_0 \in (0, 1)$.

Lemma 8.9. *Let U be the operator defined in (8.48). Then $U : \mathcal{K}_1 \to \mathcal{K}_1$.*

Proof. Let U_1 and U_2 be defined as in (8.51) and (8.52), respectively, above. Then whenever $(y_1, y_2) \in \mathcal{K}_1$, it is clear that $U_1 (y_1, y_2) (t)$, $U_2 (y_1, y_2) (t) \geq 0$, for each $t \in [0, 1]$. On the other hand, in light of Lemma 8.7 and the definition of γ_0 provided in (8.58), we find that

$$\begin{aligned}
&\min_{t\in\left[\frac{1}{2},1\right]} \left[U_1 (y_1, y_2) (t) + U_2 (y_1, y_2) (t)\right] \\
&\geq \min_{t\in\left[\frac{1}{2},1\right]} \beta_1(t)\phi_1 (y_1) + \min_{t\in\left[\frac{1}{2},1\right]} \lambda_1 \int_0^1 G_1(t, s)a_1(s)f (y_1(s), y_2(s)) \ ds \\
&+ \min_{t\in\left[\frac{1}{2},1\right]} \beta_2(t)\phi_2 (y_2) + \min_{t\in\left[\frac{1}{2},1\right]} \lambda_2 \int_0^1 G_2(t, s)a_2(s)g (y_1(s), y_2(s)) \ ds \\
&\geq M_{\beta_1} \max_{t\in[0,1]} \beta_1(t)\phi_1 (y_1) + \gamma_1 \max_{t\in[0,1]} \lambda_1 \int_0^1 G_1(t, s)a_1(s)f (y_1(s), y_2(s)) \ ds \\
&+ M_{\beta_2} \max_{t\in[0,1]} \beta_2(t)\phi_2 (y_2) + \gamma_2 \max_{t\in[0,1]} \lambda_2 \int_0^1 G_2(t, s)a_2(s)g (y_1(s), y_2(s)) \ ds \qquad (8.59)\\
&\geq \gamma_0 \max_{t\in[0,1]} \left[\beta_1(t)\phi_1 (y_1) + \lambda_1 \int_0^1 G_1(t, s)a_1(s)f (y_1(s), y_2(s)) \ ds\right] \\
&+ \gamma_0 \max_{t\in[0,1]} \left[\beta_2(t)\phi_2 (y_2) + \lambda_2 \int_0^1 G_2(t, s)a_2(s)g (y_1(s), y_2(s)) \ ds\right] \\
&= \gamma_0\|U_1 (y_1, y_2) \| + \gamma_0\|U_2 (y_1, y_2) \| \\
&= \gamma_0\| (U_1 (y_1, y_2) , U_2 (y_1, y_2)) \| \\
&= \gamma_0\|U (y_1, y_2) \|,
\end{aligned}$$

whence

$$\min_{t\in\left[\frac{1}{2},1\right]} \left[U_1 (y_1, y_2) (t) + U_2 (y_1, y_2) (t)\right] \geq \gamma_0\| (U_1 (y_1, y_2) , U_2 (y_1, y_2)) \|, \qquad (8.60)$$

as desired. Thus, we conclude that $U : \mathcal{K}_1 \to \mathcal{K}_1$, as claimed. And this completes the proof. □

We are now ready to state and prove our first existence theorem for problem (8.1), (8.4)–(8.6).

Theorem 8.10. *Suppose that conditions (F1)–(F2), (G1), and (L2) hold. Then problem (8.1), (8.4)–(8.6) has at least one positive solution.*

Proof. Lemma 8.9 shows that $U : \mathcal{K}_1 \to \mathcal{K}_1$. Moreover, due to the continuity of β_1, β_2, ϕ_1, and ϕ_2, it is clear that both U_1 and U_2 are completely continuous operators by a standard application of the Arzela-Ascoli theorem, which we omit.

Let p be the given number satisfying $0 < p < \frac{1}{2}$, as in the statement of condition (L2) above. Now, just as in the proof of Theorem 8.3, there is by condition (L2) a number $\varepsilon > 0$ such that

$$\begin{aligned}\Lambda_3 := \max\Bigg\{&\frac{1}{2}\left[\int_{\frac{1}{2}}^{1} \gamma_0 G_1\left(1,s\right)a_1(s)\left(f^{**}-\varepsilon\right)\ ds\right]^{-1}, \\ &\frac{1}{2}\left[\int_{\frac{1}{2}}^{1} \gamma_0 G_2\left(1,s\right)a_2(s)\left(g^{**}-\varepsilon\right)\ ds\right]^{-1}\Bigg\} \leq \lambda_1, \lambda_2\end{aligned} \tag{8.61}$$

and

$$\begin{aligned}\lambda_1, \lambda_2 \leq \min\Bigg\{&p\left[\int_{0}^{1} G_1(1,s)a_1(s)\left(f^{*}+\varepsilon\right)\ ds\right]^{-1}, \\ &p\left[\int_{0}^{1} G_2(1,s)a_2(s)\left(g^{*}+\varepsilon\right)\ ds\right]^{-1}\Bigg\}.\end{aligned} \tag{8.62}$$

Given this ε, just as before, conditions (8.28) and (8.29) remain true whenever $\|\left(y_1,y_2\right)\| < r_1$, exactly as in the proof of Theorem 8.1. In this case, however, we need to use condition (G1) as well to further refine the choice of r_1. In particular, by condition (G1) it follows that there is a number, say, $r_1^{***} > 0$ such that $\phi\left(y_1\right) \leq \eta\|y_1\|$ whenever $0 < \|y_1\| \leq r_1^{***}$. In particular and without loss of gener-

ality, let us suppose that $0 < \eta_1 < \frac{1}{2} - p$. (Note that by the choice of p, we clearly have that $\frac{1}{2} - p > 0$.) Now, put $\widetilde{r}_1 := \min\left\{r_1, r_1^{***}\right\}$. Then we find for all $(y_1, y_2) \in \mathcal{X}$ satisfying $0 < \|(y_1, y_2)\| < \widetilde{r}_1$ both that

$$\begin{cases} f(y_1, y_2) \leq (f^* + \varepsilon)(y_1 + y_2) \\ g(y_1, y_2) \leq (g^* + \varepsilon)(y_1 + y_2) \end{cases} \tag{8.63}$$

and that

$$\phi(y_1) \leq \left(\frac{1}{2} - p\right) \|y_1\|. \tag{8.64}$$

So, define Ω_1 by $\Omega_1 := \left\{(y_1, y_2) \in \mathcal{X} : \|(y_1, y_2)\| < \widetilde{r}_1\right\}$. Observe that for any $(y_1, y_2) \in \mathcal{K}$ we have that $\|y_1\|$, $\|y_2\| \leq \|(y_1, y_2)\|$. We then find for $(y_1, y_2) \in \mathcal{K}_1 \cap \Omega_1$ that

$$\begin{aligned}
&\|U_1(y_1, y_2)\| \\
&\leq \phi_1(y_1) + \lambda_1 \int_0^1 G_1(1, s) a_1(s) f(y_1(s), y_2(s))\ ds \\
&\leq \left(\frac{1}{2} - p\right) \|y_1\| + \lambda_1 \int_0^1 G_1(1, s) a_1(s) f(y_1(s), y_2(s))\ ds \\
&\leq \left(\frac{1}{2} - p\right) \|(y_1, y_2)\| + \lambda_1 \int_0^1 G_1(1, s) a_1(s)(f^* + \epsilon)(y_1(s) + y_2(s))\ ds \\
&\leq \|(y_1, y_2)\| \left[\left(\frac{1}{2} - p\right) + \lambda_1 \int_0^1 G_1(1, s) a_1(s)(f^* - \epsilon)\ ds\right] \\
&\leq \|(y_1, y_2)\| \left[\left(\frac{1}{2} - p\right) + p\right] \\
&\leq \frac{1}{2} \|(y_1, y_2)\|,
\end{aligned} \tag{8.65}$$

whence $\|U_1(y_1, y_2)\| \leq \frac{1}{2}\|(y_1, y_2)\|$. A similar analysis shows that $\|U_2(y_1, y_2)\| \leq \frac{1}{2}\|(y_1, y_2)\|$. Consequently, we conclude that whenever $(y_1, y_2) \in \mathcal{K}_1 \cap \partial\Omega_1$, it follows

that

$$\|U\left(y_1,y_2\right)\| \leq \|\left(y_1,y_2\right)\| \tag{8.66}$$

so that U is a cone compression on $\mathcal{K}_1 \cap \partial\Omega_1$.

Conversely, let ε be the same number selected at the beginning of this proof. As before, condition (F2) implies the existence of a number r_2^* such that

$$f\left(y_1,y_2\right) \geq \left(f^{**}-\epsilon\right)\left(y_1+y_2\right) \tag{8.67}$$

and

$$g\left(y_1,y_2\right) \geq \left(g^{**}-\epsilon\right)\left(y_1+y_2\right) \tag{8.68}$$

whenever $y_1+y_2 \geq r_2^*$. In addition, recall that by condition (G1) it follows that ϕ_1 and ϕ_2 are assumed to be nonnegative for $\left(y_1,y_2\right) \in \mathcal{K}_1$. Finally, if we put

$$r_2 := \max\left\{2r_1, \frac{r_2^*}{\gamma_0}\right\}, \tag{8.69}$$

similar to (8.36) earlier, then it follows that a condition like (8.38) holds whenever $\left(y_1,y_2\right) \in \mathcal{K}_1 \cap \partial\Omega_2$, where we put

$$\Omega_2 := \left\{\left(y_1,y_2\right) \in \mathcal{X} \ : \ \|\left(y_1,y_2\right)\| < r_2\right\}. \tag{8.70}$$

Thus, for each $\left(y_1,y_2\right) \in \mathcal{K}_1 \cap \partial\Omega_2$, it follows that

$$\begin{aligned} U_1\left(y_1,y_2\right)(1) &\geq \lambda_1 \int_0^1 G_1(1,s)a_1(s)f\left(y_1(s),y_2(s)\right)\ ds \\ &\geq \lambda_1 \int_{\frac{1}{2}}^1 \gamma_0 G_1(1,s)a_1(s)\left(f^{**}-\epsilon\right)\|\left(y_1,y_2\right)\|\ ds \\ &\geq \frac{1}{2}\|\left(y_1,y_2\right)\|, \end{aligned} \tag{8.71}$$

where we have used the nonnegativity of ϕ_1 to get the first inequality in (8.71). Consequently, (8.71) implies that $\|U_1(y_1, y_2)\| \geq \frac{1}{2}\|(y_1, y_2)\|$ whenever $(y_1, y_2) \in \mathcal{K}_1 \cap \partial\Omega_2$. A similar calculation reveals that $\|U_2(y_1, y_2)\| \geq \frac{1}{2}\|(y_1, y_2)\|$ whenever $(y_1, y_2) \in \mathcal{K}_1 \cap \partial\Omega_2$. Thus, we conclude that

$$\|U(y_1, y_2)\| \geq \|(y_1, y_2)\|, \tag{8.72}$$

whenever $(y_1, y_2) \in \mathcal{K}_1 \cap \partial\Omega_2$.

Finally, combining (8.66) and (8.72) and applying Lemma 2.13, we find that there exists a fixed point $(y_1^0, y_2^0) \in \mathfrak{X}$ of the operator U. As the pair of functions $y_1^0(t)$, $y_2^0(t)$ is a solution of problem (8.1), (8.4)–(8.6), the proof is complete. $\square$

Remark 8.11. Observe that the eigenvalue problem considered by Theorem 8.10 could not be handled (even in the ***integer-order case***) by the results of Henderson, et al. [62]. Thus, Theorem 8.10 is an essential generalization of problem (8.1) not only in the fractional-order case but also in the integer-order case.

8.2.2 Problem (8.1), (8.4)–(8.6) in Case $\lambda_1 = \lambda_2 = 1$

In contrast to the previous subsection, we now specialize to the case in which $\lambda_1 = \lambda_2 = 1$. In this case, problem (8.1), (8.4)–(8.6) is no longer an eigenvalue problem. Consequently, we shall no longer have any need to invoke condition (L2). Furthermore, we shall alter conditions (F1)–(F2) since their imposition was a consequence of condition (L2). In particular, then, we begin by introducing the following new conditions. Note that we retain condition (G1) as before, and so, we shall not list it separately below.

F3: We find that

$$\lim_{y_1+y_2\to 0^+} \frac{f\left(y_1, y_2\right)}{y_1+y_2} = 0 \text{ and } \lim_{y_1+y_2\to 0^+} \frac{g\left(y_1, y_2\right)}{y_1+y_2} = 0.$$

F4: We find that

$$\lim_{y_1+y_2\to +\infty} \frac{f\left(y_1, y_2\right)}{y_1+y_2} = +\infty \text{ and } \lim_{y_1+y_2\to +\infty} \frac{g\left(y_1, y_2\right)}{y_1+y_2} = +\infty.$$

We present now two preliminary lemmas. First, let us make a remark.

Remark 8.12. In the sequel, we shall represent by U^1 the operator U with $\lambda_1 = \lambda_2 = 1$. In addition, we shall represent by U_1^1 and U_2^1 the operators U_1 and U_2, respectively, with $\lambda_1 = \lambda_2 = 1$.

Lemma 8.13. *A pair of functions $(y_1, y_2) \in \mathfrak{X} \times \mathfrak{X}$ is a solution of (8.1), (8.4)–(8.6), in case $\lambda_1 = \lambda_2 = 1$, if and only if (y_1, y_2) is a fixed point of the operator U^1 defined by*

$$\left(U^1\left(y_1, y_2\right)\right)(t) := \Bigg(\beta_1(t)\phi_1\left(y_1\right) + \int_0^1 G_1(t,s)a_1(s)f\left(y_1(s), y_2(s)\right)\, ds, \quad \beta_2(t)\phi_2\left(y_2\right) + \int_0^1 G_2(t,s)a_2(s)g\left(y_1(s), y_2(s)\right)\, ds\Bigg), \tag{8.73}$$

where $\beta_1,\ \beta_2\ :\ [0,1] \to [0,1]$ are defined by (8.49) and (8.50), respectively.

Proof. The proof of this lemma is essentially the same as the proof of Lemma 8.6. Consequently, we omit it. □

Lemma 8.14. *Let U^1 be the operator defined in (8.73). Then $U^1\ :\ \mathcal{K}_1 \to \mathcal{K}_1$.*

Proof. Let U_1^1 and U_2^2 be defined as in Remark 8.12 above. Then whenever $(y_1, y_2) \in \mathcal{K}_1$, it is clear that $U_1^1(y_1, y_2)(t)$, $U_2^1(y_1, y_2)(t) \geq 0$, for each $t \in [0, 1]$.

On the other hand, in light of Lemma 8.7 and the definition of γ_0 provided in (8.48), we find that

$$\begin{aligned}
&\min_{t\in\left[\frac{1}{2},1\right]} \left[U_1^1(y_1, y_2)(t) + U_2^1(y_1, y_2)(t)\right] \\
&\geq M_{\beta_1} \max_{t\in[0,1]} \beta_1(t)\phi_1(y_1) + \gamma \max_{t\in[0,1]} \int_0^1 G_1(t, s)a_1(s)f(y_1(s), y_2(s))\ ds \\
&+ M_{\beta_2} \max_{t\in[0,1]} \beta_2(t)\phi_2(y_2) + \gamma \max_{t\in[0,1]} \int_0^1 G_2(t, s)a_2(s)g(y_1(s), y_2(s))\ ds \\
&\geq \gamma_0 \max_{t\in[0,1]} \left[\beta_1(t)\phi_1(y_1) + \int_0^1 G_1(t, s)a_1(s)f(y_1(s), y_2(s))\ ds\right] \\
&+ \gamma_0 \max_{t\in[0,1]} \left[\beta_2(t)\phi_2(y_2) + \int_0^1 G_2(t, s)a_2(s)g(y_1(s), y_2(s))\ ds\right] \\
&= \gamma_0\|U^1(y_1, y_2)\|,
\end{aligned} \tag{8.74}$$

whence

$$\min_{t\in\left[\frac{1}{2},1\right]} \left[U_1^1(y_1, y_2)(t) + U_2^1(y_1, y_2)(t)\right] \geq \gamma_0\|\left(U_1^1(y_1, y_2), U_2^1(y_1, y_2)\right)\|, \tag{8.75}$$

as desired. Thus, we conclude that $U^1 : \mathcal{K}_1 \to \mathcal{K}_1$, as claimed. And this completes the proof. □

We now present another existence theorem for problem (8.1), (8.4)–(8.6), this one in the special case when $\lambda_1 = \lambda_2 = 1$.

Theorem 8.15. *Suppose that conditions (F3)–(F4) and (G1) hold. Then problem (8.1), (8.4)–(8.6), in the case where* $\lambda_1 = \lambda_2 = 1$, *has at least one positive solution.*

Proof. Lemma 8.14 shows that $U^1 : \mathcal{K}_1 \to \mathcal{K}_1$. Moreover, due to the continuity of β_1, β_2, ϕ_1, and ϕ_2, it is clear that both U_1^1 and U_2^1 are completely continuous operators

by a standard application of the Arzela-Ascoli theorem, which we again omit.

On the other hand, choose a number $\eta_1 > 0$ such that

$$0 < \eta_1 \left[1 + \int_0^1 G_1(1,s)a_1(s)\ ds\right] < \frac{1}{2}. \tag{8.76}$$

Due condition (F3), note that there is a number $r_1^* > 0$ such that $f(y_1, y_2) \leq \eta_1 [y_1 + y_2]$ whenever $0 < \| (y_1, y_2) \| \leq r_1^*$. In addition, letting η_1 be the same number, by condition (G1) it follows that there is a number $r_1^{**} > 0$ such that $\phi_1(y_1) \leq \eta_1 \|y_1\|$ whenever $0 < \|y_1\| \leq r_1^{**}$. Now, take $r_1 := \min\{r_1^*, r_1^{**}\}$. Observe that whenever $0 < \| (y_1, y_2) \| < r_1$, it follows that $\|y_1\| < r_1 \leq r_1^{**}$. In particular, then, for all $(y_1, y_2) \in \mathfrak{X}$ satisfying $0 < \| (y_1, y_2) \| < r_1$, we find both that

$$f(y_1, y_2) \leq \eta_1 [y_1 + y_2] \tag{8.77}$$

and that

$$\phi_1(y_1) \leq \eta_1 \|y_1\| \leq \eta_1 \| (y_1, y_2) \|. \tag{8.78}$$

So, put $\Omega_1 := \{(y_1, y_2) \in \mathfrak{X} \ : \ \| (y_1, y_2) \| < r_1\}$. Then from (8.76)–(8.78), we find whenever $(y_1, y_2) \in \mathcal{K}_1 \cap \partial\Omega_1$ that

$$\begin{aligned}
\|U_1^1(y_1, y_2)\| &\leq \|\beta_1\| \phi_1(y_1) + \max_{t\in[0,1]} \int_0^1 G_1(t,s)a_1(s) f(y_1(s), y_2(s))\ ds \\
&\leq \eta_1 \|y_1\| + \int_0^1 G_1(1,s)a_1(s)\eta_1 [y_1(s) + y_2(s)]\ ds \\
&\leq \eta_1 \| (y_1, y_2) \| + \| (y_1, y_2) \| \int_0^1 \eta_1 G_1(1,s)a_1(s)\ ds \\
&\leq \| (y_1, y_2) \| \cdot \eta_1 \left[1 + \int_0^1 G_1(1,s)a_1(s)\ ds\right] \\
&\leq \frac{1}{2} \| (y_1, y_2) \|,
\end{aligned} \tag{8.79}$$

whence $\|U_1^1(y_1,y_2)\| \leq \frac{1}{2}\|(y_1,y_2)\|$. Similarly, it can be shown that

$$\|U_2^1(y_1,y_2)\| \leq \frac{1}{2}\|(y_1,y_2)\|, \tag{8.80}$$

whenever $(y_1,y_2) \in \mathcal{K}_1 \cap \partial\Omega_1$. Therefore, from (8.79)–(8.80) we conclude that whenever $(y_1,y_2) \in \mathcal{K}_1 \cap \partial\Omega_1$, it follows that

$$\|U^1(y_1,y_2)\| \leq \|(y_1,y_2)\|. \tag{8.81}$$

Conversely, recall that by assumption (G1), we have that $\phi_1(y_1)$, $\phi_2(y_2) \geq 0$, for each $(y_1,y_2) \in \mathcal{K}_1$ (because y_1, $y_2 \geq 0$ whenever $(y_1,y_2) \in \mathcal{K}_1$). In addition, choose a number $\eta_2 > 0$ such that

$$\eta_2 \int_{\frac{1}{2}}^{1} \gamma_0 G_1\left(\frac{3}{4}, s\right) a_1(s)\ ds \geq \frac{1}{2}. \tag{8.82}$$

Then condition (F4) implies the existence of a number $r_2^* > 0$ such that whenever $\|(y_1,y_2)\| \geq r_2^*$, we find that

$$f(y_1,y_2) \geq \eta_2\left[y_1+y_2\right]. \tag{8.83}$$

Now, put

$$r_2 := \max\left\{2r_1, \frac{r_2^*}{\gamma_0}\right\}, \tag{8.84}$$

and define the set Ω_2 by $\Omega_2 := \{(y_1,y_2) \in \mathcal{X} \ :\ \|(y_1,y_2)\| < r_2\}$. Then from (8.82)–

(8.84), it follows that

$$
\begin{aligned}
U_1^1(y_1, y_2)\left(\frac{3}{4}\right) &\geq \int_0^1 G_1\left(\frac{3}{4}, s\right) a_1(s) f\left(y_1(s), y_2(s)\right)\ ds \\
&\geq \int_{\frac{1}{2}}^1 G_1\left(\frac{3}{4}, s\right) a_1(s) f\left(y_1(s), y_2(s)\right)\ ds \\
&\geq \int_{\frac{1}{2}}^1 G_1\left(\frac{3}{4}, s\right) a_1(s) \eta_2\left[y_1(s) + y_2(s)\right]\ ds \qquad (8.85) \\
&\geq \|\left(y_1, y_2\right)\| \cdot \eta_2 \int_{\frac{1}{2}}^1 \gamma_0 G_1\left(\frac{3}{4}, s\right) a_1(s)\ ds \\
&\geq \frac{1}{2}\|\left(y_1, y_2\right)\|,
\end{aligned}
$$

whenever $(y_1, y_2) \in \mathcal{K}_1 \cap \partial\Omega_2$. Thus, we conclude that for any $(y_1, y_2) \in \mathcal{K}_1 \cap \partial\Omega_1$

$$\|U_1^2(y_1, y_2)\| \geq \frac{1}{2}\|(y_1, y_2)\|. \qquad (8.86)$$

A completely similar calculation shows that

$$\|U_2^1(y_1, y_2)\| \geq \frac{1}{2}\|(y_1, y_2)\|, \qquad (8.87)$$

whenever $(y_1, y_2) \in \mathcal{K}_1 \cap \partial\Omega_2$. Thus, combining (8.86)–(8.87) implies that

$$\|U^1(y_1, y_2)\| \geq \|(y_1, y_2)\|, \qquad (8.88)$$

whenever $(y_1, y_2) \in \mathcal{K}_1 \cap \partial\Omega_2$.

Finally, combining (8.81) and (8.88) and applying Lemma 2.13, we find that there exists a fixed point $(y_1^0, y_2^0) \in \mathcal{X}$ of the operator U^1. As the pair of functions $y_1^0(t)$, $y_2^0(t)$ is a solution of problem (8.1), (8.4)–(8.6), the proof is complete. □

Let us conclude this section with a final remark.

Remark 8.16. To the best of the author's knowledge, Theorems 8.10 and 8.15 provides new results not only for the fractional-order problem (8.1), (8.4)–(8.6), but also for the corresponding ***integer-order*** problem – i.e., in case $\nu_1 = \nu_2$ with $\nu_1,\ \nu_2 \in \mathbb{N}$.

8.3 Numerical Examples

We now present two numerical examples illustrating, respectively, Theorem 8.3 and Theorem 8.15.

Example 8.17. Consider the problem, for $t \in (0,1)$,

$$\begin{cases} -D_{0^+}^{5.2} y_1(t) = 12.5e^{-2t}\left(y_1(t)+y_2(t)\right)\left(20000 - \frac{19990}{(y_1(t))^2+(y_2(t))^2+1}\right) \\ -D_{0^+}^{5.95} y_2(t) = 5.75e^{-3t}\left(y_1(t)+y_2(t)\right)\left(30000 - \frac{29995}{(y_1(t))^2+(y_2(t))^2+1}\right) \end{cases}, \tag{8.89}$$

subject to the boundary conditions

$$y_1^{(i)}(0) = 0 = y_2^{(i)}(0),\ 0 \le i \le 4 \tag{8.90}$$

and

$$D_{0^+}^{1.5}\left[y_1(t)\right]_{t=1} = 0 = D_{0^+}^{1.5}\left[y_2(t)\right]_{t=1}. \tag{8.91}$$

Obviously, problem (8.89)–(8.91) fits the framework of problem (8.1)–(8.3) with $\nu_1 := 5.2$, $\nu_2 := 5.95$, $\alpha = 1.5$, $\lambda_1 = 12.5$, and $\lambda_2 = 5.75$. (Note that $n = 6$, therefore, in this case.) In addition, we have set

$$f\left(y_1,y_2\right) := \left(y_1+y_2\right)\left(20000 - \frac{19990}{y_1^2+y_2^2+1}\right), \tag{8.92}$$

$$g\left(y_1,y_2\right) := \left(y_1+y_2\right)\left(30000 - \frac{29995}{y_1^2+y_2^2+1}\right), \tag{8.93}$$

$$a_1(t) := e^{-2t} \tag{8.94}$$

and

$$a_2(t) := e^{-3t}. \tag{8.95}$$

Note that $f,\, g \,:\, [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ and are continuous. The functions $a_1(t)$ and $a_2(t)$ are obviously nonnegative.

We now check that each of the conditions of Theorem 8.3 holds. In particular, observe that

$$\lim_{y_1+y_2 \to 0^+} \frac{f\,(y_1, y_2)}{y_1 + y_2} = \lim_{y_1+y_2 \to 0^+} \left(20000 - \frac{19990}{(y_1(t))^2 + (y_2(t))^2 + 1}\right) = 10 \tag{8.96}$$

and that

$$\lim_{y_1+y_2 \to 0^+} \frac{g\,(y_1, y_2)}{y_1 + y_2} = \lim_{y_1+y_2 \to 0^+} \left(30000 - \frac{29995}{(y_1(t))^2 + (y_2(t))^2 + 1}\right) = 5. \tag{8.97}$$

Thus, put

$$f^* := 10 \tag{8.98}$$

and

$$g^* := 5. \tag{8.99}$$

On the other hand, observe that

$$\lim_{y_1+y_2 \to +\infty} \frac{f\,(y_1, y_2)}{y_1 + y_2} = \lim_{y_1+y_2 \to +\infty} \left(20000 - \frac{19990}{(y_1(t))^2 + (y_2(t))^2 + 1}\right) = 20000 \tag{8.100}$$

and that

$$\lim_{y_1+y_2 \to +\infty} \frac{g\,(y_1, y_2)}{y_1 + y_2} = \lim_{y_1+y_2 \to +\infty} \left(30000 - \frac{29995}{(y_1(t))^2 + (y_2(t))^2 + 1}\right) = 30000. \tag{8.101}$$

Thus, put

$$f^{**} := 20000 \tag{8.102}$$

and

$$g^{**} := 30000. \tag{8.103}$$

In summary, (8.96)–(8.103) show that conditions (F1) and (F2) hold, as desired.

On the other hand, to calculate the admissible range of the eigenvalues λ_1, λ_2, as given by condition (L1), observe by numerical approximation we find that

$$\Lambda_1 \approx 5.451 \tag{8.104}$$

and that

$$\Lambda_2 \approx 38.717. \tag{8.105}$$

Thus, for any λ_1, λ_2 satisfying

$$5.451 < \lambda_1, \lambda_2 < 38.717 \tag{8.106}$$

condition (L1) will be satisfied. Since it is clear from (8.89) that

$$\lambda_1, \lambda_2 \in [5.451, 38.717], \tag{8.107}$$

we find that condition (L1) is satisfied.

Thus, we see that each of conditions (F1)–(F2) and (L1) is satisfied. Consequently, (8.92)–(8.107) imply by Theorem 8.3 that problem (8.89)–(8.91) has at least one positive solution.

Example 8.18. Consider the problem, for $t \in (0,1)$,

$$\begin{cases} -D_{0^+}^{7.52} y_1(t) = e^{-2t}\left[y_1^2 + y_2^2\right] \\ -D_{0^+}^{7.31} y_2(t) = e^{-3t}\left[y_1^3 + y_2^2\right] \end{cases}, \tag{8.108}$$

subject to the boundary conditions

$$y_1^{(i)}(0) = 0 = y_2^{(i)}(0),\ 0 \le i \le 6 \tag{8.109}$$

and

$$\begin{cases} D_{0^+}^{2.25}\left[y_1(t)\right]_{t=1} = \left[y_1\left(\tfrac{1}{2}\right)\right]^6 \\ D_{0^+}^{2.25}\left[y_2(t)\right]_{t=1} = \left[y_2\left(\tfrac{3}{4}\right)\right]^{\frac{3}{2}} \end{cases}. \tag{8.110}$$

Obviously, problem (8.108)–(8.110) fits the framework of boundary value problem (8.1), (8.4)–(8.6). In particular, boundary condition (5.22) represents a nonlocal condition. Note that in this case we have selected $\nu_1 := 7.52$, $\nu_2 := 7.31$, and $\alpha = 2.25$; it is also the case that $n = 8$ here. Furthermore, we have that

$$f\left(y_1, y_2\right) := y_1^2 + y_2^2, \tag{8.111}$$

$$g\left(y_1, y_2\right) := y_1^3 + y_2^2, \tag{8.112}$$

$$a_1(t) := e^{-2t}, \tag{8.113}$$

$$a_2(t) := e^{-3t}, \tag{8.114}$$

$$\phi_1\left(y_1\right) := \left[y_1\left(\frac{1}{2}\right)\right]^6 \tag{8.115}$$

and

$$\phi_2(y_2) := \left[y_2\left(\frac{3}{4}\right)\right]^{\frac{3}{2}}. \tag{8.116}$$

We check that conditions (F3)–(F4) and (G1) hold. In particular, observe that

$$\lim_{y_1+y_2\to 0^+} \frac{y_1^2+y_2^2}{y_1+y_2} = 0, \tag{8.117}$$

$$\lim_{y_1+y_2\to 0^+} \frac{y_1^3+y_2^2}{y_1+y_2} = 0, \tag{8.118}$$

$$\lim_{y_1+y_2\to +\infty} \frac{y_1^2+y_2^2}{y_1+y_2} = +\infty, \tag{8.119}$$

$$\lim_{y_1+y_2\to +\infty} \frac{y_1^3+y_2^2}{y_1+y_2} = +\infty, \tag{8.120}$$

so that conditions (F3)–(F4) are seen to hold. On the other hand, note that

$$0 \leq \lim_{\|y_1\|\to 0^+} \frac{\phi_1(y_1)}{\|y_1\|} = \lim_{\|y_1\|\to 0^+} \frac{\left[y_1\left(\frac{1}{2}\right)\right]^6}{\|y_1\|} \leq \lim_{\|y_1\|\to 0^+} \frac{\|y_1\|^6}{\|y_1\|} = \lim_{\|y_1\|\to 0^+} \|y_1\|^5 = 0 \tag{8.121}$$

and, similarly, that

$$0 \leq \lim_{\|y_2\|\to 0^+} \frac{\phi_2(y_2)}{\|y_2\|} = \lim_{\|y_2\|\to 0^+} \frac{\left[y_2\left(\frac{3}{4}\right)\right]^{\frac{3}{2}}}{\|y_2\|} \leq \lim_{\|y_2\|\to 0^+} \frac{\|y_2\|^{\frac{3}{2}}}{\|y_2\|} = \lim_{\|y_2\|\to 0^+} \|y_2\|^{\frac{1}{2}} = 0, \tag{8.122}$$

whence by (8.121) and (8.122), respectively, we find that condition (G1) holds, too.

Thus, conditions (F3)–(F4) and (G1) hold. Therefore, by (8.121)–(8.122) together with Theorem 8.15 we conclude that problem (8.108)–(8.110) has at least one positive solution, as desired.

Remark 8.19. As implied elsewhere, to the best of the author's knowledge, the problems in Examples 8.17 and 8.18 cannot be handled by other results presently in the literature. In particular, these examples show how our results here extend those

presented in [23, 62, 80, 93], for example.

Remark 8.20. Observe that the orders of the fractional derivatives, namely ν_1, ν_2, and α, affect the admissible range of eigenvalues both in (8.22)–(8.23) and in (8.45)–(8.46). ***Thus, in the problems considered here, we really have three extra parameters affecting the problem than in the corresponding integer-order problem.***

Remark 8.21. It should be noted that in approximating the admissible range of eigenvalues in (8.107), we used the fact, which was established in Chapter 7, that

$$\gamma := \min\left\{\frac{\left(\frac{1}{2}\right)^{\nu-\alpha-1}}{2^{\alpha}-1}, \left(\frac{1}{2}\right)^{\nu-1}\right\}.$$

We conclude with two remarks regarding classes of functions satisfying conditions (F1)–(F2).

Remark 8.22. One fairly broad class of (nontrivial) functions satisfying conditions (F1)–(F2) are given by

$$f(\mathbf{x}) := C_1 e^{-g(\mathbf{x})} \nabla \cdot \mathbf{H}(\mathbf{x}),$$

where $g : \overline{\mathbb{R}^n_+} \to [0, +\infty)$, $f : \overline{\mathbb{R}^n_+} \to [0, +\infty)$, $C_1 > 0$ is a constant, $\mathbf{H} : \overline{\mathbb{R}^n_+} \to \overline{\mathbb{R}^n_+}$ is the vector field defined by

$$\mathbf{H}(\mathbf{x}) := \sum_{i=1}^{n} \frac{1}{2} x_i^2 \mathbf{e}_i,$$

where $\mathbf{e}_i$ is the i-th standard basis vector in $\mathbb{R}^n$, and by $\overline{\mathbb{R}^n_+}$ we mean the closure of the interior of the positive cone in $\mathbb{R}^n$. Obviously the class of functions $L(y_1, y_2) = ay_1 + ay_2$ trivially satisfies (F1)–(F2), for $a > 0$.

Remark 8.23. Another class of (nontrivial) functions satisfying conditions (F1)–(F2)

is

$$f(x, y) := (x + y)\left(A + \frac{B}{x^2 + y^2 + C}\right), \tag{8.123}$$

for appropriately chosen constants A, $C \in [0, +\infty)$ and $B \in \mathbb{R}$ subject to the stipulation that $f : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is continuous. Obviously the choice of function in Example 8.17 fits the framework of (8.123).

Chapter 9

A Scalar ODE with Nonlocal and Nonlinear Boundary Conditions

In this and the concluding two chapters, we now arrive at the end of the arc described in Chapter 1. Recall that in the preceding chapters of this work, we first considered discrete fractional problems, which are *de facto* nonlocal, and then continuous fractional problems with both *de facto* and explicit nonlocalities, namely Chapters 7 and 8, respectively. We now conclude this arc by deducing some results in the case of integer-order differential equations with explicit nonlocal boundary terms.

We consider first and in this chapter the problem

$$\begin{aligned} y'' &= -\lambda f(t, y(t)),\, t \in (0,1) \\ y(0) &= H\left(\phi(y) + \varepsilon_0 y\left(\xi_0\right)\right) \\ y(1) &= 0, \end{aligned} \tag{9.1}$$

where $\xi_0 \in \mathbb{R}$ satisfies $0 < \xi_0 < 1$ and is fixed and $\varepsilon_0 > 0$ is a constant to be specified later; it is worth noting that in one of our results in the sequel, we shall allow $\varepsilon_0 = 0$.

We also assume here that $f : [0,1] \times \mathbb{R} \to [0,+\infty)$ is a continuous function. We endeavor to show that problem (9.1) has at least one positive solution under relatively mild hypotheses, which we shall introduce at the beginning of subsubsection 9.2.1.1 in the sequel. Let us point out that in the context of problem (9.1) and, in fact, throughout this work, the functional ϕ represents a *linear* functional, which has the very general form

$$\phi(y) := \int_{[0,1]} y(t)\ d\alpha(t),$$

where the integral is interpreted in the Lebesgue-Stieltjes sense, and α is a function of bounded variation on $[0,1]$. In this way, there exists a unique Borel measure, say μ_α, with the property that $\mu_\alpha((-\infty,t]) = \alpha(t)$, for each $t \in \mathbb{R}$; we assume here, without loss of generality, that $\alpha \in NBV(\mathbb{R})$. As will be explained later in this section, an important assumption in this work is that μ_α may be a *signed* measure. Consequently, $\phi(y)$ may be negative for some nonnegative $y \in \mathcal{C}([0,1])$.

The novelty of our approach here is twofold. Firstly, we introduce a perturbation term – namely $\varepsilon_0 y\left(\xi_0\right)$ – in the boundary condition at $t = 0$ appearing in (1.1) above. The effect of this perturbation is that for y satisfying the condition $\min_{t\in E} y(t) \geq \gamma_0\|y\|$, for some $E \Subset (0,1)$ and $\gamma_0 \in (0,1)$, we find that

$$\varepsilon_0 y\left(\xi_0\right) \geq \varepsilon_0\gamma_0\|y\|, \tag{9.2}$$

provided that $\xi_0 \in E$. In particular, since in the sequel we shall be able to assume that $\phi(y) \geq 0$ (cf., Section 9.2), we see that (9.2) implies that the magnitude of the argument of H can be bounded below explicitly in terms of $\|y\|$. This is an essential idea in the sequel, and one that does not seem to have ever been attempted previously for this sort of problem.

Secondly, the substantial upshot of this preceding observation is that inasmuch as

H is concerned, other than continuity, we need only require that

$$\lim_{z\to\infty} |H(z) - z| = 0 \tag{9.3}$$

holds, or, much more generally that

$$\lim_{z\to\infty} \frac{|H(z) - \kappa_0 z|}{|z|} = 0 \tag{9.4}$$

holds for some $\kappa_0 \geq 0$. (Note that (9.3) implies (9.4) but not conversely.) Observe that such a condition is meaningful only because (9.2) gives us explicit control over the magnitude of the argument of H in terms of $\|y\|$, which we can control by means of a cone theoretic argument.

In any case, heuristically, certain of our principal result asserts that if $H(z)$ is asymptotically linear, that is, satisfies limit condition (9.4), and certain other mild hypotheses are satisfied by f and ϕ, then problem (9.1) has at least one positive solution. As will be elucidated in the sequel, this shall provide several substantial generalizations and improvements over existing works on problems of this sort.

Indeed, these two simple yet mathematically significant modifications allow us to assume that μ_α is signed and that H need not satisfy uniform linear growth – results which do not seem to have previously appeared in the literature for both problem (9.1) as well as many broadly related problems. Moreover, as will also be clarified momentarily, this also provides a clear, straightforward, and computationally feasible connection between the linear and nonlinear boundary condition theory. As an additional and complementary result, we also show that the imposition of the dual conditions

$$\lim_{z\to 0^+} \frac{H(z)}{z} = 0 \text{ and } \lim_{z\to\infty} \frac{H(z)}{z} = +\infty \tag{9.5}$$

is also sufficient, when coupled with additional mild conditions on f and ϕ, to get the existence of at least one positive solution. This likewise provides generalizations and improvements over existing works for this type of problem. In fact, we also show that we can further weaken (9.5) to the condition

$$\limsup_{z\to 0^+}\frac{H(z)}{z}<\rho, \tag{9.6}$$

for some $\rho\in\left[0,\frac{1}{\varepsilon_1}\right)$ for some constant $\varepsilon_1\in(0,1)$ to be specified later; importantly, we show that this is sufficient even in the case where $\varepsilon_0=0$ in (9.1) – i.e., the perturbation free problem. We shall expand on these alternative conditions further in the sequel.

In [53] we have attempted to provide, in various contexts, explicit connections to the linear boundary condition theory as well as removing certain of the limiting assumptions imposed in [64, 65, 66, 67, 68, 90, 91]. Here we continue this task, and we achieve the following improvements over [64, 65, 66, 67, 68, 90, 91] and others.

1. Firstly, many previous works on nonlinear boundary conditions either use upper and lower solution techniques or assume that the equivalent of our function H is monotone – see, for example, [37, 62, 73, 81]. In our work we do not make any such assumptions. Moreover, since the use of upper and lower solution techniques do not necessarily yield the existence of a positive solution, this is an added contribution of our work as well. In fact, since we also allow H to be potentially only eventually positive and μ_α to be a signed measure, we allow for considerable flexibility and generality not found in the previous works.

   Furthermore, while it should be pointed out that many other works deal with slightly different boundary conditions, we believe that our methods extend nat-

urally to these other settings. Thus, we feel it to be no great loss that we have elected to study this problem with the simpler, Dirichlet-type boundary conditions given in (9.1) above. And we believe that it should be possible in future works to extend our techniques to the other settings previously studied.

2. Secondly, we should remark on the connection between the results herein and the results that we have given recently in [53]. In fact, it must be noted that the results given in this paper neither subsume those in [53] nor vice versa. Indeed, in [53], we focused on the *multipoint* case with H only eventually positive. Due to this specialization, we were able to obtain somewhat more general results *in that particular special setting* than we do here. Indeed, due to the presence of the perturbation term, $\varepsilon_0 y\left(\xi_0\right)$, which appears in (9.1), *in certain cases* our results here are not quite as general as those presented in [53]. On the other hand, in cases where a fixed boundary condition can subsume the extra term $\varepsilon_0 y\left(\xi_0\right)$, then our results here are considerably stronger and more general than those appearing in either [53] or any other works known to the author, and this is due to the asymptotic conditions (9.3) and (9.4), which have apparently never appeared before for this type of problem. The examples at the end of this chapter will illustrate these facts.

3. Thirdly, in nearly all works on this type of problem (cf., [64, 65, 66, 67, 68]), it seems to be assumed that H satisfies a linear bound of the sort $\alpha z \leq H(z) \leq \beta z$ for $0 \leq \alpha \leq \beta$, for all $z \geq 0$. We show that this condition is essentially unnecessary. In fact, for example, our theory allows for functions of the form $H(z) = \sqrt{z}$, for $z \geq 0$, which cannot possibly be treated by the methods in, say, [64, 65, 66, 67, 68]. We consider this insight to be an important contribution of this work. Furthermore, as already pointed out, while Yang [90, 91] introduced

a similar condition, we achieve here some substantial generalizations since the techniques there fail if the Stieltjes measured is signed.

4. Fourthly, we even show that if H satisfies instead superlinear growth at both $z=0$ and $z=+\infty$, then it is unnecessary to assume any growth condition at all on $g(y)$ at $+\infty$. This, too, provides significant improvement over existing works. Moreover, evidently, such functions cannot be incorporated into the theory developed in previous papers, for such H will not satisfy the condition $H(z) \leq \beta z$ for some $\beta > 0$ and for all $z \geq 0$ (or even all $z \geq 0$ sufficiently large). Similar to the preceding point, we also consider this observation to be an interesting insight, which apparently does not appear in most of the existing and recent literature on this type of problem.

5. Fifthly, and perhaps most importantly, we allow for the Borel measure, μ_α, associated to the integrator α to be *signed*. In this way, we make an explicit, very generally applicable connection between the nonlinear boundary condition setting and the linear theory developed by Infante, Webb, and others. In fact, as alluded to earlier, one of our principal contributions is to show that if $H(z)$ is essentially asymptotically linear, then this is sufficient, when combined with our other, relatively mild hypotheses, to deduce the existence of at least one positive solution to problem (9.1). The examples, which end this chapter, will clarify this connection further, but for simple illustration, if $H(z) := z\left(1+e^{-z}\right)$, then it is trivial to argue that condition (9.3) is satisfied by this function H. Thus, if the boundary condition at $t=0$ appearing in (9.1) is

$$y(0) = \left(\phi(y) + \varepsilon_0 y\left(\xi_0\right)\right)\left(1+e^{-\phi(y)-\varepsilon_0 y(\xi_0)}\right) \tag{9.7}$$

and if f, H, and ϕ satisfy the other conditions we shall introduce, then it will follow that problem (9.1) has at least one positive solution. Moreover, if $\phi(y)$ happens to decompose to the form $\phi(y) = \psi(y) - \varepsilon_0 y\left(\xi_0\right)$, where ψ can be realized as another Lebesgue-Stieltjes integral, then (9.7) can be recast in the form

$$y(0) = \psi(y)\left(1 + e^{-\psi(y)}\right), \tag{9.8}$$

which indicates explicitly the asymptotic linearity to which we have referred.

9.1 Preliminaries

We begin this section by observing that the operator $T : \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ defined by

$$(Ty)(t) := (1-t)H\left(\phi(y) + \varepsilon_0 y\left(\xi_0\right)\right) + \lambda \int_0^1 G(t,s)f(s,y(s))\ ds \tag{9.9}$$

may be studied as a means of deducing the existence of positive solutions to (9.1). Indeed, it is the case that a fixed point of T is simultaneously a solution to (9.1). Note that the function $G : [0,1] \times [0,1] \to \mathbb{R}$ appearing in (9.9) is the Green's function associated to the two-point conjugate problem – that is,

$$G(t,s) := \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1 \\ s(1-t), & 0 \leq s \leq t \leq 1 \end{cases}, \tag{9.10}$$

as is well known – see, for example, [71]. Let $E \Subset (0,1)$ be measurable and arbitrary but fixed. Typically, E will have the form $E := [a,b]$ for $0 < a < b < 1$. In fact, we make the following remark.

Remark 9.1. In the sequel, we shall assume that E is fixed but otherwise arbitrary,

provided that it has the form $E := [a, b]$, for $0 < a < b < 1$, so that, trivially, $E \Subset (0, 1)$.

With this it is then well-known that there is a constant $\gamma := \gamma(E)$ such that

$$\min_{t \in E} G(t, s) \geq \gamma \max_{t \in [0,1]} G(t, s) = \gamma G(s, s), \tag{9.11}$$

for each $s \in [0, 1]$. Note that $\gamma \in (0, 1)$. Inequality (9.11) will be important in the sequel.

9.2 Main Results and Numerical Examples

9.2.1 Existence Theorems for $H(z)$ Nonnegative

9.2.1.1 Asymptotically Sublinear Growth

Let us begin by stating the hypotheses which we shall impose on problem (9.1).

H1: Let $H : \mathbb{R} \to \mathbb{R}$ be a real-valued, continuous function. Moreover, $H : [0, +\infty) \to [0, +\infty)$ – i.e., H is nonnegative when restricted to $[0, +\infty)$.

H2: The functional $\phi(y)$ appearing in (9.1) is linear and, in particular, has the form

$$\phi(y) := \int_{[0,1]} y(t)\ d\alpha(t), \tag{9.12}$$

where $\alpha : [0, 1] \to \mathbb{R}$ satisfies $\alpha \in BV([0, 1])$.

H3: There is a constant $\varepsilon_1 \geq 0$ such that the functional ϕ in (9.12) satisfies the inequality

$$|\phi(y)| \leq \varepsilon_1 \|y\| \tag{9.13}$$

for all $y \in \mathcal{C}([0,1])$.

H4: There is $\kappa_0 \geq 0$ such that

$$\lim_{z \to +\infty} \frac{|H(z) - \kappa_0 z|}{|z|} = 0 \tag{9.14}$$

holds.

H5: Assume that the nonlinearity $f(t,y)$ splits in the sense that $f(t,y) = a(t)g(y)$, for continuous functions $a : [0,1] \to [0,+\infty)$ and $g : \mathbb{R} \to [0,+\infty)$ such that a is not identically zero on any subinterval $[0,1]$.

H6: We find that $\lim_{y \to +\infty} g(y) = +\infty$.

H7: We find that $\lim_{y \to +\infty} \frac{g(y)}{y} = 0$.

H8: The constants ε_0, ε_1, and κ_0 satisfy

$$0 \leq \varepsilon_0 + \varepsilon_1 < 1 \text{ and } 0 \leq \kappa_0 \left(\varepsilon_0 + \varepsilon_1\right) < 1. \tag{9.15}$$

H9: Both

$$\int_{[0,1]} (1-t)\ d\alpha(t) \geq 0 \tag{9.16}$$

and

$$\int_{[0,1]} G(t,s)\ d\alpha(t) \geq 0 \tag{9.17}$$

hold, where the latter holds for each $s \in [0,1]$.

Let us make some brief remarks regarding certain of the preceding conditions.

Remark 9.2. Regarding conditions (H2)–(H3), we point out that a wide variety of functions satisfy these conditions. Indeed, consider the following collection of func-

tionals.

$$\begin{aligned} \phi_1(y) &:= \int_F y(t)\ dt \\ \phi_2(y) &:= \sum_{i=1}^{n} a_i y\left(\xi_i\right) \\ \phi_3(y) &:= \int_{[0,1]} y(t)\ d\alpha(t) \end{aligned} \tag{9.18}$$

Since each of $(9.18)_1$–$(9.18)_3$ is linear, each satisfies (H2). On the other hand, so long as $m(F) \leq \varepsilon_1$, where m is the Lebesgue measure, then $(9.18)_1$ satisfies (H3). Provided that $\sum_{i=1}^{n} |a_i| \leq \varepsilon_1$, then $(9.18)_2$ satisfies (H3). Finally, provided that $V_{[0,1]}(\alpha)$, which is the total variation of α over $[0,1]$, satisfies $V_{[0,1]}(\alpha) \leq \varepsilon_1$, then functional $(9.18)_3$ satisfies condition (H3).

Remark 9.3. Regarding condition (H4), this is the asymptotic condition, which is key to our arguments in the sequel. Note that if it holds that

$$\lim_{z \to +\infty} |H(z) - \kappa_0 z| = 0, \tag{9.19}$$

then condition (H4) holds, too. On the other hand, there are cases where (9.19) may fail but condition (H4) nonetheless holds. In any case, some examples of functions which satisfy condition (H4) include the following.

$$\begin{aligned} H(z) &:= \ln|z+1| + z \\ H(z) &:= z^q + \left(1 - e^{-z}\right) z,\ 0 \leq q < 1 \\ H(z) &:= 3\sqrt{z}\cos\left(\frac{1}{z+1}\right) \end{aligned} \tag{9.20}$$

Indeed, it is straightforward to verify that each of $(9.20)_1$–$(9.20)_2$ satisfies condition (9.14) in case $\kappa_0 = 1$. On the other hand, the function $(9.20)_3$ satisfies condition

(9.14) in case $\kappa_0 = 0$. Incidentally, note that $(9.20)_3$ fails condition (9.19) but it does satisfy condition (H4). Finally, note that, depending upon the values of ε_0, ε_1, κ_0, and κ_1, the condition that $0 \leq \kappa_0 (\varepsilon_0 + \varepsilon_1) < 1$ holds in (9.15) may be superfluous.

Remark 9.4. Observe that we do *not* require any growth conditions on H except asymptotically as given in (9.14) above. This is in contrast to, say, Infante [64] as well as Infante and Pietramala [65, 66, 67], wherein it is assumed that there are constants, say α, $\beta \in [0, +\infty)$ with $\alpha < \beta$, such that $\alpha z \leq H(z) \leq \beta z$, for all $z \geq 0$. The paper by Kang and Wei [68] also contains such a condition locally, though they compensate for this by assuming very strong and complicated growth conditions on the equivalent of the nonlinearity $f(y)$. We are able to remove this condition since we are only concerned in the behavior of H at $+\infty$. For instance, in Example 9.24 in the sequel we give an example of a function $H(z)$ that violates the preceding condition, and so, does not fit into the theory given in those papers. However, it can be handled by our results. So, one consequence of our results here is that it is unnecessary to assume that H is sublinear at $z = 0$ as others do. Rather, we only really need sublinearity at $+\infty$, and we consider this observation to be an important contribution of this work.

Remark 9.5. Regarding condition (H5), we note that we make this assumption only for simplicity in the sequel. It is clear from other works in the literature how this condition may be successfully removed.

Let $\mathcal{B}$ represent the Banach space $\mathcal{C}([0, 1])$ when equipped with the usual supremum norm, $\| \cdot \| := \| \cdot \|_\infty$. Let γ_0 be the constant defined by

$$\gamma_0 := \min \left\{ \gamma, \min_{t \in E} (1 - t) \right\}, \tag{9.21}$$

where $\gamma_0 \in (0,1)$ and γ is the constant from (9.11) above. Then the cone, $\mathcal{K}$, we shall use in the sequel is then defined by

$$\mathcal{K} := \left\{ y \in \mathcal{B} \ : \ y \geq 0, \ \min_{t \in E} y(t) \geq \gamma_0 \|y\|, \ \phi(y) \geq 0 \right\}, \tag{9.22}$$

which was first introduced by Infante and Webb [84]. Incidentally, note that $\mathcal{K}$ is not the trivial subspace of $\mathcal{B}$, for if we put $\alpha(t) := 1 - t$, then we observe that $\alpha \in \mathcal{K}$. Finally, we shall always assume in the sequel that $\xi_0 \in E$ with E fixed as in Remark 9.1. With these observations hand, we now state and prove two straightforward preliminary lemmas.

Lemma 9.6. *For each $y \in \mathcal{K}$, we find that*

$$\phi(y) + \varepsilon_0 y\left(\xi_0\right) \geq \varepsilon_0 \gamma_0 \|y\|. \tag{9.23}$$

holds.

Proof. Since $y \in \mathcal{K}$, it follows both that $\phi(y) \geq 0$ and that $y \geq 0$. Moreover, by assumption, we have that $\xi_0 \in E$, with E fixed as above. Therefore, from these facts we deduce that

$$\phi(y) + \varepsilon_0 y\left(\xi_0\right) \geq \varepsilon_0 y\left(\xi_0\right) \geq \varepsilon_0 \min_{t \in E} y(t) \geq \varepsilon_0 \gamma_0 \|y\|, \tag{9.24}$$

as claimed. □

Lemma 9.7. *Let T be the operator defined in (9.9). Then it follows that $T : \mathcal{K} \to \mathcal{K}$.*

Proof. First of all, since $H(z) \geq 0$ for all $z \geq 0$, it is obvious that for each $y \in \mathcal{K}$ we

find that $(Ty)(t) \geq 0$, for $t \in [0,1]$. In addition, for each $y \in \mathcal{K}$, we observe that

$$\begin{aligned}
\min_{t\in E} (Ty)(t) &\geq \min_{t\in E}(1-t)H(\phi(y)+\varepsilon_0 y(\xi_0)) \\
&\quad + \min_{t\in E} \lambda \int_0^1 G(t,s)a(s)g(y(s))\ ds \\
&\geq \gamma_0 H(\phi(y)+\varepsilon_0 y(\xi_0)) + \gamma_0 \max_{t\in[0,1]} \lambda \int_0^1 G(t,s)a(s)g(y(s))\ ds \\
&\geq \gamma_0 \|Ty\|,
\end{aligned} \tag{9.25}$$

where γ_0 is defined as above. Finally, observe that

$$\begin{aligned}
&\phi(Ty) \\
&= \int_{[0,1]} (1-t)H(\phi(y)+\varepsilon_0 y(\xi_0))\ d\alpha(t) \\
&+ \lambda \int_{[0,1]} \int_0^1 G(t,s)a(s)g(y(s))\ ds\ d\alpha(t) \\
&= H(\phi(y)+\varepsilon_0 y(\xi_0)) \int_{[0,1]} (1-t)\ d\alpha(t) \\
&+ \lambda \int_{[0,1]} \int_0^1 G(t,s)a(s)g(y(s))\ ds\ d\alpha(t) \\
&= H(\phi(y)+\varepsilon_0 y(\xi_0)) \int_{[0,1]} (1-t)\ d\alpha(t) \\
&+ \lambda \int_0^1 \left[\int_{[0,1]} G(t,s)\ d\alpha(t)\right] a(s)g(y(s))\ ds \geq 0,
\end{aligned} \tag{9.26}$$

where the inequality follows from assumption (H9). Consequently, from (9.26) we deduce that $\phi(Ty) \geq 0$, whenever $y \in \mathcal{K}$. Thus, $T(\mathcal{K}) \subseteq \mathcal{K}$, and the proof is complete. $\square$

Using Lemmas 9.6 and 9.7 in tandem, we get our first existence result, which shows that for all $\lambda > 0$ sufficiently large problem (1.1) has at least one positive solution subject to the previously discussed hypotheses.

Theorem 9.8. *Let conditions (H1)–(H9) hold. Then problem* (1.1) *has at least one positive solution for all* $\lambda > 0$ *sufficiently large.*

Proof. We have already argued in Lemma 9.7 that $T : \mathcal{K} \to \mathcal{K}$. Furthermore, recalling that H is continuous, it is standard to prove that T is a completely continuous operator; so, we omit the proof of this fact.

Now, letting $t_0 \in \overset{\circ}{E}$ be fixed but arbitrary, from condition (H6) we have that

$$g(y) \geq \left[\int_E G(t_0, s)\, a(s)\ ds\right]^{-1} \tag{9.27}$$

whenever $y \geq r_1$. Let $\Omega_{\frac{r_1}{\gamma_0}} \subseteq \mathcal{B}$ denote the open, bounded, convex set $\Omega_{\frac{r_1}{\gamma_0}} := \left\{y \in \mathcal{B} : \|y\| < \frac{r_1}{\gamma_0}\right\}$. Note that for $y \in \mathcal{K} \cap \partial\Omega_{\frac{r_1}{\gamma_0}}$

$$\min_{t \in E} y(t) \geq \gamma_0 \|y\| = r_1. \tag{9.28}$$

Then as H is nonnegative, we deduce for each $y \in \mathcal{K} \cap \partial\Omega_{\frac{r_1}{\gamma_0}}$ that

$$(Ty)(t_0) \geq \lambda \int_E G(t_0, s)\, a(s) g(y(s))\ ds \geq \lambda \tag{9.29}$$

so that

$$\|Ty\| \geq \lambda. \tag{9.30}$$

Hence by selecting $\lambda := \lambda(r_1, E)$ sufficiently large, we get that T is a cone expansion on $\mathcal{K} \cap \partial\Omega_{\frac{r_1}{\gamma_0}}$.

On the other hand, since $\kappa_0(\varepsilon_0 + \varepsilon_1) < 1$, we may select $\varepsilon_2 > 0$ sufficiently small so that $\kappa_0(\varepsilon_0 + \varepsilon_1) + \varepsilon_2 < 1$. Then condition (H7) implies that there is $\eta_1 > 0$ such

that $g(y) \leq \eta_1 y$, where η_1 satisfies

$$\eta_1 \int_0^1 G(s,s)a(s)\ ds \leq \frac{\varepsilon_2}{\lambda}, \tag{9.31}$$

whenever $y \geq r_2$, with $r_2 := r_2\left(\lambda, r_1, \varepsilon_2, \kappa_0, E\right)$.

Moreover, now select another number, say $\varepsilon_3 > 0$, such that $\kappa_0\left(\varepsilon_0+\varepsilon_1\right)+\varepsilon_2+\varepsilon_3 < 1$, which is possible since $\kappa_0\left(\varepsilon_0+\varepsilon_1\right)+\varepsilon_2 < 1$. Then by condition (H4) we find that there is $r_2^* := r_2^*\left(\varepsilon_3\right)$ such that

$$\left|H\left(\phi\left(y\right)+\varepsilon_0 y\left(\xi_0\right)\right)-\kappa_0\left(\phi\left(y\right)+\varepsilon_0 y\left(\xi_0\right)\right)\right| < \varepsilon_3\|y\|, \tag{9.32}$$

whenever

$$\phi\left(y\right)+\varepsilon_0 y\left(\xi_0\right) \geq r_2^*. \tag{9.33}$$

Note that to get (9.32) we have used the fact that

$$\begin{aligned} \left|\phi\left(y\right)+\varepsilon_0 y\left(\xi_0\right)\right| &= \phi\left(y\right)+\varepsilon_0 y\left(\xi_0\right) \\ &\leq \varepsilon_1\|y\|+\varepsilon_0\|y\| = \left(\varepsilon_0+\varepsilon_1\right)\|y\| \leq \|y\|. \end{aligned} \tag{9.34}$$

By Lemma 9.6 we have that

$$\phi\left(y\right)+\varepsilon_0 y\left(\xi_0\right) \geq \varepsilon_0\gamma_0\|y\|. \tag{9.35}$$

Consequently, we deduce that whenever

$$\|y\| \geq \frac{r_2^*}{\varepsilon_0\gamma_0} \tag{9.36}$$

holds, it follows that (9.32) holds.

Now, because $g(y)$ is unbounded at $+\infty$, it is straightforward to argue that there must be a number $r_2^{**} > 0$ such that $g(y) \leq g\left(r_2^{**}\right)$, for each $y \in [0, r_2^{**}]$. Moreover, we may assume without loss of generality, that r_2^{**} satisfies

$$r_2^{**} > \max\left\{\frac{2r_1}{\gamma_0}, r_2, \frac{r_2^*}{\varepsilon_0\gamma_0}\right\}, \tag{9.37}$$

where

$$r_2^{**} = r_2^{**}\left(r_1, r_2, r_2^*, \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_0, \kappa_0, \lambda, E\right). \tag{9.38}$$

Then letting $y \in \mathcal{K} \cap \partial\Omega_{r_2^{**}}$, we thus estimate

$$\begin{aligned}
\|Ty\| &\leq H\left(\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right) + \lambda\int_0^1 G(s,s)a(s)g\left(y(s)\right)\ ds\\
&\leq \left|H\left(\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right) - \kappa_0\left(\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right)\right|\\
&+ \kappa_0\left(\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right) + \lambda\int_0^1 G(s,s)a(s)g\left(y(s)\right)\ ds\\
&\leq \varepsilon_3\|y\| + \kappa_0\left[\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right] + \lambda\int_0^1 G(s,s)a(s)g\left(y(s)\right)\ ds\\
&\leq \varepsilon_3\|y\| + \kappa_0\left[\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right] + \lambda\int_0^1 G(s,s)a(s)g\left(r_2^{**}\right)\ ds\\
&\leq \varepsilon_3\|y\| + \kappa_0\left[\varepsilon_1\|y\| + \varepsilon_0\|y\|\right] + \lambda\int_0^1 G(s,s)a(s)\eta_1 r_2^{**}\ ds\\
&\leq \varepsilon_3\|y\| + \kappa_0\left(\varepsilon_0 + \varepsilon_1\right)\|y\| + \varepsilon_2\|y\|\\
&\leq \left(\kappa_0\left(\varepsilon_0 + \varepsilon_1\right) + \varepsilon_2 + \varepsilon_3\right)\|y\|,
\end{aligned} \tag{9.39}$$

Observe that since we have assumed that

$$0 \leq \kappa_0\left(\varepsilon_0 + \varepsilon_1\right) + \varepsilon_2 + \varepsilon_3 < 1, \tag{9.40}$$

it follows from (9.39) that

$$\|Ty\| \leq \|y\|, \tag{9.41}$$

for each $y \in \mathcal{K} \cap \partial\Omega_{r_2^{**}}$, whence T is a cone compression on $\mathcal{K} \cap \partial\Omega_{r_2^{**}}$.

Finally, we may invoke Lemma 2.13 to deduce the existence of a function y_0 satisfying

$$y_0 \in \mathcal{K} \cap \left(\overline{\Omega}_{r_2^{**}} \setminus \Omega_{\frac{r_1}{\gamma_0}}\right) \tag{9.42}$$

such that $Ty_0 = y_0$. In fact, it holds that

$$0 < \frac{r_1}{\gamma_0} \leq \|y_0\| \leq r_2^{**} < +\infty.$$

Consequently, the function y_0 is a positive solution of problem (9.1), and so, the proof is complete. □

Remark 9.9. Although not explicitly mentioned in the preceding proof, the range of admissible eigenvalues is, in fact, explicitly computable. Indeed, if we set

$$\lambda_0 := \frac{1}{\gamma_0} \inf\left\{x \in [0, +\infty) \ : \ g(y) \geq \left[\int_E G\left(t_0, s\right) a(s)\ ds\right]^{-1} \text{ for all } y \in [x, +\infty)\right\}, \tag{9.43}$$

then the interval $[\lambda_0, +\infty)$ is the range of admissible eigenvalues for problem (9.1). Note that (9.43) is computable since each of a, g, G, and γ_0 is known *a priori*.

Now, since we are assuming in the preceding work that $H(z) \geq 0$ for all $z \geq 0$, we can modify conditions (H6)–(H7) rather easily. Indeed, if we are willing to assume that g is sublinear at 0, then we may remove the assumption that g is unbounded at $+\infty$. In fact, in this setting we may also assume that $\lambda = 1$. In this way, then, we

show that problem (9.1) with $\lambda = 1$ and a standard assumption of the sublinearity of g at both 0 and $+\infty$ also suffices to deduce the existence of a positive solution. Since the proof of this result proceeds essentially the same as the preceding result, we only outline the differences.

Theorem 9.10. *Assume that conditions (H1)–(H5) and (H7)–(H9) hold. In addition, assume that*

$$\lim_{y\to 0^+} \frac{g(y)}{y} = +\infty \tag{9.44}$$

holds. Then in case $\lambda = 1$ problem (9.1) has at least one positive solution.

Proof. As before, we have that $T : \mathcal{K} \to \mathcal{K}$. Moreover, T is a completely continuous operator. Now, select the number $\eta_1 > 0$ such that

$$\eta_1 \int_E \gamma_0 G\left(t_0, s\right) a(s)\ ds \geq 1, \tag{9.45}$$

where t_0 is any point satisfying $t_0 \in \overset{\circ}{E}$. Then owing to the sublinearity condition given in (9.44), we find that there exists a number $r_1 > 0$ such that whenever $0 < y < r_1$, it follows that $g\left(y\right) \geq \eta_1 y$. Let $y \in \mathcal{K} \cap \partial\Omega_{r_1}$. Then we find that

$$\begin{aligned}
(Ty)\left(t_0\right) &\geq \int_0^1 G\left(t_0, s\right) a(s) g\left(y(s)\right)\ ds \\
&\geq \eta_1 \int_E G\left(t_0, s\right) a(s) \gamma_0 \|y\|\ ds \\
&\geq \|y\|,
\end{aligned} \tag{9.46}$$

whence $\|Ty\| \geq \|y\|$, for each $y \in \mathcal{K} \cap \partial\Omega_{r_1}$.

On the other hand, given condition (H7) there exist two possibilities: either g is bounded at $+\infty$ or it is not. Let us assume first that the former case holds. Then we may find a number $r_2 > 0$ sufficiently large such that $g(y) \leq r_2$, for all $y \geq 0$. In

fact, without loss of generality, we may assume that r_2 is chosen such that

$$g(y) \leq \frac{r_2}{\int_0^1 G(s,s)a(s)\,ds} \tag{9.47}$$

holds. Next, and as in the proof of Theorem 9.8, we may choose numbers ε_2, $\varepsilon_3 > 0$ such that $\kappa_0\left(\varepsilon_0 + \varepsilon_1\right) + \varepsilon_2 + \varepsilon_3 < 1$ holds. Then condition (H4) implies that there is $r_2^* > 0$ such that

$$\left|H\left(\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right) - \kappa_0\left(\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right)\right| < \varepsilon_3\|y\|, \tag{9.48}$$

holds provided that

$$\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\ dt \geq r_2^*. \tag{9.49}$$

But just as in the proof of Theorem 9.8, by selecting

$$\|y\| \geq \frac{r_2^*}{\varepsilon_0\gamma_0}, \tag{9.50}$$

we find that (9.48) holds. So, put

$$r_2^{**} := \max\left\{2r_1, \frac{r_2^*}{\varepsilon_0\gamma_0}, \frac{r_2}{\varepsilon_2}\right\}. \tag{9.51}$$

Observe that $r_2^{**} := r_2^{**}\left(r_1, r_2, r_2^*, \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_0, \kappa_0, E\right)$. Consequently, for each $y \in$

$\mathcal{K} \cap \partial\Omega_{r_2^{**}}$, it follows that

$$\begin{aligned} \|Ty\| &\leq H\left(\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right) + \int_0^1 G(s,s)a(s)g\left(y(s)\right)\ ds \\ &\leq \left|H\left(\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right) - \kappa_0\left(\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right)\right| \\ &+ \kappa_0\left(\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right) + r_2 \\ &\leq \left(\kappa_0\left(\varepsilon_0 + \varepsilon_1\right) + \varepsilon_2 + \varepsilon_3\right)\|y\|, \end{aligned} \tag{9.52}$$

where, as in the proof of Theorem 9.8, we assume, without loss of generality, that $r_2^{**} \geq 1$. We have also used the fact in (9.52) that $r_2 < \varepsilon_2 r_2^{**} = \varepsilon_2\|y\|$. Consequently, from (9.52), we deduce that $\|Ty\| \leq \|y\|$, whenever $y_j \in \mathcal{K} \cap \partial\Omega_{r_2^{**}}$. On the other hand, if g is unbounded at $+\infty$, then we may give a proof identical to that given in the proof of Theorem 9.8. So, in either case, we conclude that T is a cone compression on $\mathcal{K} \cap \partial\Omega_{r_2^{**}}$.

We now invoke Theorem 2.13 as in the previous proof. Consequently, problem (9.1) has at least one positive solution, and the proof is complete. □

Finally, we evidently get the following corollary, which is perhaps of some independent interest and is, to the best of the author's knowledge, a new result. Note that in the statement of Corollary 9.11 below, we use the standard notation

$$⨍_G f(t)\ dt := \frac{1}{m(G)} \int_G f(t)\ dt, \tag{9.53}$$

where G is measurable and m is the usual Lebesgue measure.

Corollary 9.11. *Let $F \subseteq [0,1]$ be a measurable set with F not m-null. Suppose that either conditions (H1)–(H9) hold or conditions (H1)–(H5), (H7)–(H9), and (9.44)*

hold. Then the boundary value problem

$$\begin{aligned} y'' &= -\lambda f(t, y(t)),\ t \in (0,1) \\ y(0) &= H\left(\phi(y) + \varepsilon_0 \fint_F y(s)\ ds\right) \\ y(1) &= 0 \end{aligned} \tag{9.54}$$

has at least one positive solution for all $\lambda > 0$ sufficiently large under the former set of assumptions and for $\lambda = 1$ under the latter set of assumptions.

Proof. Pick F_0 measurable such that $E \supseteq F_0$ with $F \supseteq F_0$ and F_0 not m-null. Then since, in addition, $y \in \mathcal{K}$, it holds that

$$\varepsilon_0 \fint_F y(s)\ ds \geq \varepsilon_0 \fint_{F_0} y(s)\ ds \geq \varepsilon_0 \fint_{F_0} \gamma_0 \|y\|\ ds = \varepsilon_0 \gamma_0 \|y\|. \tag{9.55}$$

But from (9.55) it is evident that we may use in an obvious way the proof techniques previously introduced. Therefore, we omit the remainder of the proof of this result. □

Incidentally, since, assuming that F_0 is not a null set,

$$\varepsilon_0 \int_{F_0} y(s)\ ds \geq m\left(F_0\right) \varepsilon_0 \gamma_0 \|y\| > 0,$$

we may replace in (9.54) the term $\varepsilon_0 \fint_F y(s)\ ds$ with the term $\varepsilon_0 \int_F y(s)\ ds$.

9.2.1.2 Asymptotically Superlinear Growth

In this subsection, we show that provided we are willing to slightly modify certain of the growth assumptions on the nonlinearity g, then, in fact, it is not even necessary

to assume that H is asymptotically sublinear at ∞. In particular, we introduce the following new conditions.

H10: It holds that

$$\lim_{z\to+\infty}\frac{H(z)}{z}=+\infty. \tag{9.56}$$

Furthermore, it holds that

$$\lim_{z\to 0^+}\frac{H(z)}{z}=0. \tag{9.57}$$

H11: We find that

$$\lim_{y\to 0^+}\frac{g(y)}{y}=0. \tag{9.58}$$

We first state two remarks. We then state and prove our next existence theorem.

Remark 9.12. Observe that condition (H10) allows for a substantial range of nonlinearities not previously allowed in any existing work on problem (9.1) or even many related problems with signed measures, at least to the best of the author's knowledge. For example, the function

$$H(z):=z^2 \tag{9.59}$$

satisfies (9.56)–(9.57). Similarly, for $q>1$, the function

$$H(z):=z^q\cos\left(\frac{1}{z+1}\right) \tag{9.60}$$

satisfies (9.56)–(9.57). Finally, and more generally, for $r>1$ and any function $\zeta:\mathbb{R}\to[0,+\infty)$ satisfying $\sup_{z\in[0,+\infty)}\zeta(z)<+\infty$, the function

$$H(z):=z^r\zeta(z) \tag{9.61}$$

satisfies (9.56)–(9.57).

Now, none of these functions can be incorporated into the results presented in subsubsection 9.2.1.1. Furthermore, none of these can be incorporated into any previous results such as [64, 65, 66, 67, 68] due to the fact that each of these functions grows superlinearly (or even superquadratically, etc.) at $+\infty$, and so, there is no $\beta > 0$ such that $H(z) \leq \beta z$, for all $z \geq 0$ or even merely for all $z \geq 0$ sufficiently large. These cannot be incorporated into even the results of [90, 91] since the results therein do not apply to problem (9.1). Thus, our results in this subsection together with those in the preceding subsection definitively show that by requiring asymptotic conditions, we discover that many previous restrictions on the nonlinear boundary terms inasmuch as their uniform linear growth and which appear in almost every recent work on nonlocal BVPs with nonlinear BCs are concerned are completely unnecessary. As previously mentioned, we consider this to be an interesting insight.

Remark 9.13. Let $P(z) : \mathbb{R} \to \mathbb{R}$ be any polynomial satisfying $\mathsf{deg}(P) \geq 2$. Assume that P has positive leading coefficient and that, in addition, if ζ_0 is a real zero of P, then $\zeta_0 \leq 0$ – i.e., all real zeros of P lie in the left half-plane. Then it follows easily that

$$\lim_{z \to +\infty} \frac{P(z)}{z} = +\infty \tag{9.62}$$

so that condition (9.56)–(9.57) is satisfied. In particular, our theory here allows for all manner of polynomials which are superlinear at $+\infty$.

Theorem 9.14. *Assume that conditions (H1)–(H3), (H5), and (H9)–(H11) hold. In addition, suppose that* $0 \leq \varepsilon_0 + \varepsilon_1 < 1$. *Then in case* $\lambda = 1$, *problem (9.1) has at least one positive solution.*

Proof. As before the operator T is completely continuous and satisfies $T(\mathcal{K}) \subseteq \mathcal{K}$. Moreover, a fixed point of T is a positive solution of problem (9.1).

Now, by condition (H11) we have that there is $r_1 > 0$ such that whenever $0 < y < r_1$ it follows that

$$g(y) \leq \eta_1 y, \tag{9.63}$$

where the number η_1 is chosen so that

$$\eta_1 \int_0^1 G(s,s)a(s)\ ds = \frac{1}{2} \tag{9.64}$$

holds. In addition, since by condition (H10) we have that $\lim_{z\to 0^+} \frac{H(z)}{z} = 0$, we estimate

$$H\left(\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right) < \frac{1}{2}\left(\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right) \tag{9.65}$$

whenever

$$\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right) < \varepsilon_2, \tag{9.66}$$

for $\varepsilon_2 > 0$ chosen sufficiently small. Since we estimate

$$\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right) \leq \varepsilon_1 \|y\| + \varepsilon_0 \|y\| = \left(\varepsilon_0 + \varepsilon_1\right)\|y\|, \tag{9.67}$$

we deduce that (9.63) and (9.65) simultaneously hold provided that we require

$$0 < \|y\| < \min\left\{\frac{\varepsilon_2}{\varepsilon_0 + \varepsilon_1}, r_1\right\}. \tag{9.68}$$

Moreover, the estimate in (9.67) reveals that, in fact,

$$H\left(\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right) < \frac{1}{2}\left(\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right) < \frac{\varepsilon_0 + \varepsilon_1}{2}\|y\| \leq \frac{1}{2}\|y\|, \tag{9.69}$$

provided that $\|y\| < \frac{\varepsilon_2}{\varepsilon_0+\varepsilon_1}$. Put $r_1^* := \frac{1}{2}\min\left\{\frac{\varepsilon_2}{\varepsilon_0+\varepsilon_1}, r_1\right\}$ and define

$$\Omega_{r_1^*} := \{y \in \mathcal{B} \ : \ \|y\| < r_1^*\}.$$

Then for $y \in \mathcal{K} \cap \partial\Omega_{r_1^*}$, we estimate

$$\begin{aligned}
\|Ty\| &\leq H\left(\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right) + \int_0^1 G(s,s)a(s)g\left(y(s)\right)\ ds \\
&\leq H\left(\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right) + \eta_1 \int_0^1 G(s,s)a(s)y(s)\ ds \\
&\leq H\left(\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right) + \frac{1}{2}\|y\| \\
&\leq \frac{1}{2}\|y\| + \frac{1}{2}\|y\| \\
&= \|y\|.
\end{aligned} \tag{9.70}$$

Consequently, (9.70) proves that the operator T is a cone compression on $\mathcal{K} \cap \partial\Omega_{r_1^*}$.

On the other hand, let $t_0 \in \overset{\circ}{E}$ be any fixed but arbitrary point. Then condition (H10) implies that

$$H\left(\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right) \geq \eta_3\left(\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right), \tag{9.71}$$

where we put

$$\eta_3 := \frac{1}{1-t_0}\left(\varepsilon_0\gamma_0\right)^{-1}. \tag{9.72}$$

provided that

$$\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right) \geq r_2^*, \tag{9.73}$$

for some number $r_2^* > 0$ sufficiently large. From previous estimates (cf., Lemma 9.6) we find that

$$\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right) \geq \varepsilon_0\gamma_0\|y\| \tag{9.74}$$

so that, much as before, inequality (9.71) holds provided that

$$\|y\| \geq \frac{r_2^*}{\varepsilon_0 \gamma_0}. \tag{9.75}$$

Now, put

$$r_2^{**} := \max\left\{2r_1^*, \frac{r_2^*}{\varepsilon_0\gamma_0}\right\}. \tag{9.76}$$

Then for $y \in \mathcal{K} \cap \partial\Omega_{r_2^{**}}$ we find that

$$\begin{aligned}(Ty)(t_0) &\geq (1-t_0)\, H\left(\phi(y) + \varepsilon_0 y(\xi_0)\right)\\ &\geq (1-t_0)\,\eta_3\left(\phi(y) + \varepsilon_0 y(\xi_0)\right)\\ &\geq (1-t_0)\,\eta_3\left(\varepsilon_0\gamma_0\|y\|\right)\\ &= \|y\|,\end{aligned} \tag{9.77}$$

Furthermore, we note that in (9.77) we have used the fact that

$$\int_0^1 G(t_0, s)\, a(s) g(y(s))\ ds \geq 0. \tag{9.78}$$

In any case, (9.77) implies that for each $y \in \mathcal{K} \cap \partial\Omega_{r_2^{**}}$

$$\|Ty\| \geq \|y\|, \tag{9.79}$$

whence T is a cone expansion on $\mathcal{K} \cap \partial\Omega_{r_2^{**}}$.

Consequently, we may now invoke Theorem 2.13 to deduce the existence of a fixed point of T, which is simultaneously a positive solution of problem (9.1). And this completes the proof of the theorem. □

Remark 9.15. Observe that in Theorem 9.14 we do *not* require any particular growth

assumption on $g(y)$ at $+\infty$. While (9.58) implies that g is superlinear at $y = 0$, it may be that g is either sublinear or superlinear at $+\infty$. This is a consequence of the superlinearity of the nonlinear boundary function, $H(z)$, at $+\infty$, and we believe this to be another interesting insight that our results here yield.

Remark 9.16. We believe it possible to prove an analogue of Theorem 9.14 in the case where we assume only condition (9.57) but impose the condition $\frac{g(y)}{y} \to +\infty$ as $y \to +\infty$ in addition to (9.58). We do not explicitly write down this result, however.

Let us conclude this subsection by showing that, in fact, the weaker condition, which we label as condition (H10a) below, can be used, provided that we also assume the superlinearity of g at $+\infty$, which we label as condition (H11a) below.

H10a: There exists a constant $\rho \geq 0$ satisfying

$$0 \leq \rho < \frac{1}{\varepsilon_1} \tag{9.80}$$

such that

$$\limsup_{z \to 0^+} \frac{H(z)}{z} \leq \rho \tag{9.81}$$

holds. Here the number ε_1 is from condition (H3).

H11a: It holds that

$$\lim_{y \to +\infty} \frac{g(y)}{y} = +\infty. \tag{9.82}$$

In fact, we shall show that if we impose condition (H10a) instead, then we may take $\varepsilon_0 = 0$; that is, no perturbation term is required in (9.1). Of course, in addition, condition (H10a) is weaker since once may consider, for instance, the continuous

function $H : [0,+\infty) \to [0,+\infty)$ defined by

$$H(z) := \begin{cases} z\cos^2\left(\frac{1}{z}\right), & z > 0 \\ 0, & z = 0 \end{cases}. \tag{9.83}$$

Indeed, it is easy to see that

$$\limsup_{z\to 0^+} \frac{H(z)}{z} = 1, \tag{9.84}$$

whereas $\lim_{z\to 0^+} \frac{H(z)}{z}$ does not exist. Moreover, with regard to condition (H10a), we do not require any additional hypotheses on $\frac{H(z)}{z}$ at $z = +\infty$. Finally, condition (H10a) makes a clearer connection with the results produced by Yang – especially, [90]. While, as previously indicated, the results in [90] regard a somewhat different problem than problem (9.1), Yang's works seem to be the only other ones which employ asymptotic conditions that are related to the ones we employ herein. Therefore, we feel it to be of interest to demonstrate as clear a connection as possible with the results presented therein.

Theorem 9.17. *Assume that (H1)–(H5), (H9), (H10a), (H11), and (H11a) hold. In addition, suppose that $\varepsilon_0 = 0$. Then in case $\lambda = 1$, problem (9.1) has at least one positive solution.*

Proof. Let $\rho < \frac{1}{\varepsilon_1}$ be given. Then we may select $k \in \mathbb{N}$ sufficiently large such that

$$0 \le \rho < \frac{2^k - 1}{2^k \varepsilon_1} < \frac{1}{\varepsilon_1} \tag{9.85}$$

holds. Select the number $\eta_1 > 0$ such that

$$\eta_1 \int_0^1 G(s,s)a(s) \le \frac{1}{2^k} \tag{9.86}$$

holds. Then condition (H11) implies the existence of a number $r_1 > 0$ such that

$$g(y) \leq \eta_1 y \tag{9.87}$$

provided that $0 \leq y < r_1$. Furthermore, due to condition (H10a), we may select $\varepsilon > 0$ such that

$$H(y) < (\rho - \varepsilon)y, \tag{9.88}$$

holds for all $y \in [0, r_1^*]$, with ε chosen in such a way that

$$0 < \rho - \varepsilon < \frac{2^k - 1}{2^k \varepsilon_1} \tag{9.89}$$

holds. Importantly, note that since, by assumption, we have that

$$\rho < \frac{2^k - 1}{2^k \varepsilon_1}, \tag{9.90}$$

it follows that any ε satisfying

$$0 < \varepsilon \leq \rho \tag{9.91}$$

is admissible; in particular, ε may be chosen to be arbitrarily close to 0. By condition (H3), it thus follows that

$$\phi(y) \leq \varepsilon_1 \|y\|. \tag{9.92}$$

In particular, then, for each $y \in \mathcal{K}$ satisfying

$$0 \leq \|y\| < \min\{r_1, r_1^*\}, \tag{9.93}$$

it follows that

$$\phi(y) \leq \varepsilon_1 \|y\| < \varepsilon_1 \min\{r_1, r_1^*\} \leq \varepsilon_1 r_1^* < r_1^*. \tag{9.94}$$

Thus, combining (9.85)–(9.94), we conclude that if we put $r_1^{**} := \frac{1}{2}\min\{r_1, r_1^*\}$ and put

$$\Omega_{r_1^{**}} := \{y \in \mathcal{K} \ : \ \|y\| < r_1^{**}\}, \tag{9.95}$$

then it follows that

$$H(\phi(y)) < (\rho - \varepsilon)\phi(y), \tag{9.96}$$

for each $y \in \mathcal{K} \cap \partial\Omega_{r_1^{**}}$. Putting each of the preceding estimates together, we deduce that

$$\begin{aligned}
\|Ty\| &\leq H(\phi(y)) + \int_0^1 G(s,s)a(s)g(y(s))\ ds \\
&\leq (\rho - \varepsilon)\phi(y) + \eta_1 \int_0^1 G(s,s)a(s)y(s)\ ds \\
&\leq (\rho - \varepsilon)\varepsilon_1\|y\| + \frac{1}{2^k}\|y\| \\
&\leq \frac{2^k - 1}{2^k\varepsilon_1} \cdot \varepsilon_1\|y\| + \frac{1}{2^k}\|y\| \\
&\leq \frac{2^k - 1}{2^k}\|y\| + \frac{1}{2^k}\|y\| \\
&= \|y\|,
\end{aligned} \tag{9.97}$$

whence $\|Ty\| \leq \|y\|$, for each $y \in \mathcal{K} \cap \partial\Omega_{r_1^{**}}$.

On the other hand, by condition (H11a) we may find a number $r_2 > 0$ such that

$$g(y) \geq \eta_2 y, \tag{9.98}$$

whenever $y \geq r_2$ and where η_2 satisfies

$$\eta_2 \int_E \gamma_0^2 G(s,s)a(s)\ ds \geq 1. \tag{9.99}$$

Clearly, since $y \in \mathcal{K}$, we find that $H(\phi(y)) \geq 0$, for each $y \in \mathcal{K}$. Therefore, upon

setting

$$r_2^* := \max\left\{2r_1^{**}, \frac{r_2}{\gamma_0}\right\} \tag{9.100}$$

and observing that for $y \in \mathcal{K} \cap \partial\Omega_{r_2^*}$ it holds that

$$\min_{t\in E} y(t) \geq \gamma_0\|y\| \geq r_2, \tag{9.101}$$

we deduce that

$$\min_{t\in E}(Ty)(t) \geq \eta_2\|y\| \int_E \gamma_0^2 G(s,s)a(s)\ ds \geq \|y\|, \tag{9.102}$$

for each $y \in \mathcal{K} \cap \partial\Omega_{r_2^*}$, whence $\|Ty\| \geq \|y\|$.

Consequently, we may now invoke Lemma 2.13 to deduce that there is $y_0 \in \mathcal{K}$ and satisfying

$$0 < r_1^{**} \leq \|y_0\| \leq r_2^* \tag{9.103}$$

such that $Ty_0 = y_0$. Thus, problem (9.1) has at least one positive solution, as claimed. $\square$

Remark 9.18. To summarize, the result of Theorem 9.17 shows that if we impose the condition

$$0 \leq \limsup_{z\to 0^+} \frac{H(z)}{z} \leq \rho, \tag{9.104}$$

for some finite constant $\rho \geq 0$ satisfying $\rho \in \left[0, \frac{1}{\varepsilon_1}\right)$ together with the superlinearity of g at 0 and at $+\infty$, then the *unperturbed* problem

$$\begin{aligned} y'' &= -f(t, y(t)),\ t \in (0,1) \\ y(0) &= H(\phi(y)) \\ y(1) &= 0 \end{aligned} \tag{9.105}$$

has at least one positive solution.

Remark 9.19. It is worth noting that this condition – namely, (H10a) – is, at least ostensibly, slightly less general than the related condition given by Yang [90]. Indeed, the analogue of certain of Yang's results in [90], it would seem at least, would be to require ρ to satisfy the inequality

$$0 \leq \rho \leq \frac{1}{\int_{[0,1]}(1-t)\ d\alpha(t)}. \tag{9.106}$$

Noting that

$$\int_{[0,1]}(1-t)\ d\alpha(t) = \phi(1-t) \leq \varepsilon_1\|1-t\| = \varepsilon_1, \tag{9.107}$$

we see that (9.106) would imply that ρ satisfy

$$\rho \in [0, \rho_0]\,, \tag{9.108}$$

with

$$\rho_0 \geq \frac{1}{\varepsilon_1}. \tag{9.109}$$

Remark 9.20. We perhaps ought to remark that we were not able to produce a result analogous to Theorem 9.17 in the case of sublinear growth at $+\infty$ – i.e., a condition such as

$$\limsup_{z\to+\infty}\frac{H(z)}{z} = 0. \tag{9.110}$$

Indeed, it seems that imposing (9.110) and then attempting an argument analogous to those given in Yang [90, 91] runs into considerable trouble due to the fact that μ_α may be a *signed* measure in our setting. In particular, the trouble thus encountered seems to be a consequence of the fact that if μ_α is a signed measure, then the order relationship $0 \leq f \leq g$ does *not necessarily* imply the order relationship

$\int_{[0,1]} f \; d\alpha(t) \leq \int_{[0,1]} g \; d\alpha(t)$ even if it is known *a priori* that each of these integrals is nonnegative. This is why we used condition (H4) in the preceding subsection rather than a condition such as (9.110) even though this necessitates the inclusion of the perturbation term $\varepsilon_0 y\left(\xi_0\right)$ in the argument of H in problem (9.1). In any case, this would be an interesting avenue for future investigation.

9.2.2 Existence Theorem for $H(z)$ Eventually Positive

In the previous subsection we assumed that the function $H(z)$, which captures the nonlinearity of the boundary condition at $t = 0$, is nonnegative for all $z \geq 0$. It is possible to relax this condition somewhat if we utilize a slightly different approach in our proofs. In this subsection we indicate briefly in what way this may be accomplished. In particular, we shall no longer assume that H is nonnegative for all $z \geq 0$ but rather that H is only eventually positive in the following sense, which we label as condition (H12).

H12: Let $H : \mathbb{R} \to \mathbb{R}$ be a real-valued, continuous function. Moreover, there exists a real number $\zeta_0 \geq 0$ such that $H(z) \geq 0$ for all $z > \zeta_0$.

Now, in this setting we will again require the parameter λ, which is not necessarily unity. Indeed, the essential idea in the following, which, incidentally, we also utilized in [53], is to choose a number $\rho > 0$ in such a way that the set $\mathcal{K} \setminus \Omega_\rho$ contains y with the property that $\min_{t\in E} y(t)$ is sufficiently large so as to guarantee that H is nonnegative on $\mathcal{K} \setminus \Omega_\rho$. In fact, we can only achieve this because the result of Lemma 9.6 gives control of the argument of H in terms of $\|y\|$. We begin with the following result.

Lemma 9.21. *Assume that conditions (H1)–(H5) and (H12) hold. Then it follows*

that $T : \mathcal{K} \setminus \Omega_{\frac{\zeta_0}{\epsilon_0\gamma_0}} \to \mathcal{K}$.

Proof. Let $y \in \mathcal{K} \setminus \Omega_{\frac{\zeta_0}{\varepsilon_0\gamma_0}}$. Since $y \in \mathcal{K}$, we estimate

$$\min_{t\in E} y(t) \geq \gamma_0\|y\| \geq \gamma_0 \cdot \frac{\zeta_0}{\varepsilon_0\gamma_0} = \frac{\zeta_0}{\varepsilon_0}. \tag{9.111}$$

Recall that $\xi_0 \in E$. Consequently, we estimate

$$\phi(y) + \varepsilon_0 y\left(\xi_0\right) \geq \varepsilon_0 y\left(\xi_0\right) \geq \varepsilon_0 \cdot \frac{\zeta_0}{\varepsilon_0} = \zeta_0. \tag{9.112}$$

Thus, we conclude that for $y \in \mathcal{K} \setminus \Omega_{\frac{\zeta_0}{\varepsilon_0\gamma_0}}$, we have

$$H\left(\phi(y) + \varepsilon_0 y\left(\xi_0\right)\right) \geq 0. \tag{9.113}$$

From (9.113), therefore, it follows that $(Ty)(t) \geq 0$ whenever $y \in \mathcal{K} \cap \Omega_{\frac{\zeta_0}{\varepsilon_0\gamma_0}}$ and $t \in [0,1]$. Moreover, observe that

$$\begin{aligned} \min_{t\in E}(Ty)(t) &\geq \min_{t\in E}(1-t)H\left(\phi(y) + \varepsilon_0 y\left(\xi_0\right)\right) + \gamma_0 \int_0^1 G(s,s)a(s)g(y(s))\ ds \\ &\geq \gamma_0\|Ty\|, \end{aligned} \tag{9.114}$$

just as in the proof of Lemma 9.7. Finally, we note that

$$\begin{aligned} \phi\left(Ty\right) &= H\left(\phi(y) + \varepsilon_0 y\left(\xi_0\right)\right) \int_{[0,1]} (1-t)\ d\alpha(t) \\ &\quad + \int_{[0,1]} \int_0^1 G(t,s)a(s)g(y(s))\ ds\ d\alpha(t) \\ &= H\left(\phi(y) + \varepsilon_0 y\left(\xi_0\right)\right) \int_{[0,1]} (1-t)\ d\alpha(t) \\ &\quad + \int_0^1 \left[\int_{[0,1]} G(t,s)\ d\alpha(t)\right] a(s)g(y(s))\ ds \geq 0, \end{aligned} \tag{9.115}$$

holds, for each $y \in \mathcal{K} \setminus \Omega_{\frac{\zeta_0}{\varepsilon_0\gamma_0}}$. Therefore, we conclude that $T(\mathcal{K}) \subseteq \mathcal{K}$, as desired. And this completes the proof. $\square$

We argue next that problem (9.1) has at least one positive solution under these new hypotheses.

Theorem 9.22. *Assume that conditions (H1)–(H9) and (H12) hold. Then for all* $\lambda > 0$ *sufficiently large, problem (9.1) has at least one positive solution.*

Proof. We have already argued that $T : \mathcal{K} \setminus \Omega_{\frac{\zeta_0}{\varepsilon_0\gamma_0}} \to \mathcal{K}$. Moreover, as before, we have that T is a completely continuous operator.

So, we first observe that from condition (H6), there is $r_1 > 0$ and $\eta_1 > 0$ such that $g(y) \geq \eta_1$ for all $y \geq r_1$ where η_1 satisfies

$$\eta_1 \int_E G\left(t_0, s\right) a(s)\ ds = 1, \tag{9.116}$$

where t_0 is some (arbitrary but fixed) point in $\mathring{E}$. Put

$$r_1^* := \max\left\{r_1, \frac{\zeta_0}{\varepsilon_0\gamma_0}\right\}. \tag{9.117}$$

Observe that for $y \in \mathcal{K} \setminus \Omega_{r_1^*}$, it follows that

$$\min_{t\in E} y(t) \geq \gamma_0\|y\| \geq \frac{\zeta_0}{\varepsilon_0}. \tag{9.118}$$

Accordingly, for such y, we have that

$$H\left(\phi\left(y\right) + \varepsilon_0 y\left(\xi_0\right)\right) \geq 0. \tag{9.119}$$

Consequently, we estimate

$$(Ty)(t_0) \geq \lambda \int_E G(t_0, s)\, a(s) g(y(s))\, ds \geq \lambda. \tag{9.120}$$

for each $y \in \mathcal{K} \setminus \Omega_{r_1^*}$. Clearly, then, by selecting $\lambda > 0$ sufficiently large, we get that

$$\|Ty\| \geq \|y\|, \tag{9.121}$$

for each $y \in \mathcal{K} \setminus \Omega_{r_1^*}$.

On the other hand, let $\eta_2 > 0$ be a number such that

$$\eta_2 \int_0^1 G(s,s) a(s)\, ds \leq \frac{\varepsilon_2}{\lambda}, \tag{9.122}$$

where $\varepsilon_2 > 0$ is, as before, selected appropriately so that $\kappa_0(\varepsilon_0 + \varepsilon_1) + \varepsilon_2 < 1$. Then from condition (H7), we find that there is a number $r_2 > 0$ such that $g(y) \leq \eta_2 y$ whenever $y \geq r_2$. Now, as in the proof of Theorem 9.8, we may eventually select a number r_2^{**} satisfying

$$r_2^{**} > \max\left\{2r_1^*, r_2, \frac{r_2^*}{\varepsilon_0 \gamma_0}\right\}. \tag{9.123}$$

Then, once again as in the proof of Theorem 9.8, we conclude that for $y \in \mathcal{K} \cap \partial\Omega_{r_2^{**}}$, we have that

$$\|Ty\| \leq \|y\|. \tag{9.124}$$

Consequently, we may invoke Lemma 2.13 to deduce the existence of a function

$$y_0 \in \mathcal{K} \cap \left(\overline{\Omega}_{r_2^{**}} \setminus \Omega_{r_1^*}\right) \tag{9.125}$$

such that $Ty_0 = y_0$. Since this function is a positive solution of problem (9.1), the

proof is complete. $\square$

As before, we may give the following corollary, whose proof we omit. Note that, as before, it is also possible to replace $\fint_F y(s)\ ds$ with $\int_F y(s)\ ds$.

Corollary 9.23. *Let $F \subseteq [0,1]$ be measurable. Suppose that conditions (H1)–(H9) and (H12) hold. Then the boundary value problem*

$$\begin{aligned} y'' &= -\lambda f(t, y(t)),\ t \in (0,1) \\ y(0) &= H\left(\phi(y) + \fint_F y(s)\ ds\right) \\ y(1) &= 0 \end{aligned} \tag{9.126}$$

has at least one positive solution for all $\lambda > 0$ sufficiently large.

Proof. Omitted. $\square$

9.2.3 Numerical Examples

We conclude this section and this chapter by providing three numerical examples, which shall explicate our results. In particular, these results show explicitly how our results here greatly extend and generalize existing results on nonlocal boundary value problems. Moreover, we indicate in what ways all known results are not applicable to the following problems, as least to the best of the author's knowledge.

Example 9.24. Consider the boundary value problem

$$\begin{aligned} y''(t) &= -\lambda(2t+1)\ln\left(y(t)+1\right) \\ y(0) &= \left[\phi(y) + \frac{1}{15}y\left(\frac{9}{20}\right)\right]^q + \left(1 - e^{-\phi(y) - \frac{1}{15}y\left(\frac{9}{20}\right)}\right)\left[\phi(y) + \frac{1}{15}y\left(\frac{9}{20}\right)\right] \\ y(1) &= 0, \end{aligned} \tag{9.127}$$

where $0 \leq q < 1$ is fixed and $\phi(y)$ is the linear functional defined by

$$\phi(y) := \frac{1}{6}y\left(\frac{2}{5}\right) - \frac{1}{15}y\left(\frac{9}{20}\right) - \frac{1}{20}y\left(\frac{1}{2}\right) + \int_{\left[\frac{11}{20},\frac{7}{10}\right]} y(s)\ ds. \tag{9.128}$$

Observe that if we define the integrator $\alpha\ :\ \mathbb{R} \to \mathbb{R}$ by

$$\alpha := \begin{cases} 0, & t < \frac{2}{5} \\ \frac{1}{6}, & \frac{2}{5} \leq t < \frac{9}{20} \\ \frac{1}{10}, & \frac{9}{20} \leq t < \frac{1}{2} \\ \frac{1}{20}, & \frac{1}{2} \leq t < \frac{11}{20} \\ t - \frac{1}{2}, & \frac{11}{20} \leq t < \frac{7}{10} \\ \frac{1}{5}, & t \geq \frac{7}{10} \end{cases}, \tag{9.129}$$

then $\alpha \in NBV(\mathbb{R})$ and, in particular, we can write

$$\phi(y) = \int_{[0,1]} y(s)\ d\alpha(s), \tag{9.130}$$

where

$$\begin{aligned} \mu_\alpha((-\infty,t]) := \frac{1}{6}\delta_{\frac{2}{5}}((-\infty,t]) - \frac{1}{15}\delta_{\frac{9}{20}}((-\infty,t]) - \frac{1}{20}\delta_{\frac{1}{2}}((-\infty,t]) \\ + m\left((-\infty,t] \cap \left(\frac{11}{20},\frac{7}{10}\right)\right) \end{aligned} \tag{9.131}$$

is the *signed* Borel measure associated to the integrator $\alpha(t)$. Furthermore, by comparing (9.127) to (9.1), we see that here we have put

$$\xi_0 := \frac{9}{20} \tag{9.132}$$

and

$$\varepsilon_0 := \frac{1}{15}. \tag{9.133}$$

Finally, let us set $E := \left[\frac{1}{4}, \frac{3}{4}\right] \Subset (0, 1)$ here, which, it is seen, is a valid choice in this setting.

Now, let us first observe that

$$H(z) := z^q + \left(1 - e^{-z}\right) z \tag{9.134}$$

satisfies condition (H4) with $\kappa_0 = 1$. Indeed, we estimate

$$\lim_{z\to\infty} \frac{|(z^q + (1 - e^{-z})\, z) - z|}{z} = \lim_{z\to\infty} \frac{|z^q - ze^{-z}|}{z} = \lim_{z\to\infty} \left|z^{q-1} - e^{-z}\right| = 0. \tag{9.135}$$

On the other hand, routine numerical calculations reveal both that

$$\int_{[0,1]} 1 - t \ d\alpha(t) \approx 0.0946 > 0 \tag{9.136}$$

and that

$$\int_{[0,1]} G(t, s) \ d\alpha(t) \geq 0 \tag{9.137}$$

hold, where the latter holds for each $s \in [0, 1]$, whence condition (H9) holds. Condition (H1) is obviously satisfied. Finally, condition (H3) is satisfied, for

$$\begin{aligned} &\left|\frac{1}{6}y\left(\frac{2}{5}\right) - \frac{1}{15}y\left(\frac{9}{20}\right) - \frac{1}{20}y\left(\frac{1}{2}\right) + \int_{\left[\frac{11}{20}, \frac{7}{10}\right]} y(s)\ ds\right| \\ &\leq \frac{1}{6}\|y\| + \frac{1}{15}\|y\| + \frac{1}{20}\|y\| + \frac{3}{20}\|y\| = \frac{1}{3}\|y\|. \end{aligned} \tag{9.138}$$

So, in particular, we can take

$$\varepsilon_1 := \frac{1}{3}. \tag{9.139}$$

Thus, we evidently have $\varepsilon_0 + \varepsilon_1 \in [0, 1)$ here. Since $\kappa_0 = 1$, condition (H8) holds.

Since it is easy to see that conditions (H5)–(H7) are satisfied, we conclude that each of conditions (H1)–(H9) is satisfied. Therefore, we may apply the result of Theorem 9.8 to deduce that problem (9.127) has at least one positive solution for all $\lambda > 0$ sufficiently large. In fact, by (9.43), we compute

$$\begin{aligned}\lambda_0 = \frac{1}{\frac{1}{4}} \inf \Bigg\{ x \in [0, +\infty) \ : \ \ln(y+1) \geq \left[\int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right)(2s+1)\ ds \right]^{-1}, \\ \text{for all } y \in [x, +\infty) \Bigg\} \approx 824.509\end{aligned} \tag{9.140}$$

so that, more precisely, for each λ satisfying

$$824.509 < \lambda < +\infty, \tag{9.141}$$

we find that problem (9.127) has at least one positive solution. Finally, if we put

$$\psi(y) := \frac{1}{6} y\left(\frac{2}{5}\right) - \frac{1}{20} y\left(\frac{1}{2}\right) + \int_{\left[\frac{11}{20}, \frac{7}{10}\right]} y(s)\ ds, \tag{9.142}$$

then let us point out that we can recast (9.127) in the form

$$\begin{aligned} y''(t) &= -\lambda(2t+1)\ln(y(t)+1) \\ y(0) &= [\psi(y)]^q + \left(1 - e^{-\psi(y)}\right)\psi(y) \\ y(1) &= 0, \end{aligned} \tag{9.143}$$

by using the definition of ϕ given in (9.128). Thus, problem (9.143) has at least one positive solution, too.

Example 9.25. Consider the boundary value problem

$$
\begin{aligned}
y''(t) &= -(\cos t)\sqrt[3]{y(t)} \\
y(0) &= \left(\phi(y) + \frac{1}{30}y\left(\frac{1}{3}\right)\right)\cos\left(\frac{1}{\phi(y) + \frac{1}{30}y\left(\frac{1}{3}\right) + 1}\right) \\
y(1) &= 0,
\end{aligned}
\tag{9.144}
$$

where $\phi(y)$ is the linear functional defined by

$$
\phi(y) := -\frac{1}{30}y\left(\frac{1}{3}\right) + \frac{1}{10}y\left(\frac{7}{20}\right) - \frac{1}{25}y\left(\frac{3}{5}\right) + \int_{\left[\frac{7}{10},\frac{3}{4}\right]} y(s)\ ds. \tag{9.145}
$$

Here we may define the integrator $\alpha : \mathbb{R} \to \mathbb{R}$ by

$$
\alpha(t) := \begin{cases}
0, & t < \frac{1}{3} \\
-\frac{1}{30}, & \frac{1}{3} \leq t < \frac{7}{20} \\
\frac{1}{15}, & \frac{7}{20} \leq t < \frac{3}{5} \\
\frac{2}{75}, & \frac{3}{5} \leq t < \frac{7}{10} \\
t - \frac{101}{150}, & \frac{7}{10} \leq t < \frac{3}{4} \\
\frac{23}{300}, & t \geq \frac{3}{4}
\end{cases}. \tag{9.146}
$$

Then $\alpha \in NBV(\mathbb{R})$,

$$
\phi(y) = \int_{[0,1]} y(s)\ d\alpha(s), \tag{9.147}
$$

and

$$
\begin{aligned}
\mu_\alpha((-\infty,t]) := -\frac{1}{30}\delta_{\frac{1}{3}}((-\infty,t]) + \frac{1}{10}\delta_{\frac{7}{20}}((-\infty,t]) - \frac{1}{25}\delta_{\frac{3}{5}}((-\infty,t]) \\
+ m\left((-\infty,t] \cap \left(\frac{7}{10},\frac{3}{4}\right)\right)
\end{aligned}
\tag{9.148}
$$

is the *signed* Borel measure associated to the integrator α. Finally, note that here we have selected

$$\xi_0 := \frac{1}{3} \tag{9.149}$$

and

$$\varepsilon_0 := \frac{1}{30}. \tag{9.150}$$

Finally, as in Example 9.24, it is permissible here to select $E := \left[\frac{1}{4}, \frac{3}{4}\right] \Subset (0,1)$.

Now, in this example we have

$$H(z) := z\cos\left(\frac{1}{z+1}\right). \tag{9.151}$$

The function H satisfies condition (H4) with $\kappa_0 = 1$, for we estimate

$$\lim_{z\to\infty} \frac{\left|z\cos\left(\frac{1}{z+1}\right) - z\right|}{z} = \lim_{z\to\infty}\left|\cos\left(\frac{1}{z+1}\right) - 1\right| = 0. \tag{9.152}$$

On the other hand, it is easy to argue that

$$\int_{[0,1]} 1 - t\ d\alpha(t) \approx 0.041 > 0, \tag{9.153}$$

whereas

$$\int_{[0,1]} G(t,s)\ d\alpha(t) \geq 0, \tag{9.154}$$

for each $s \in [0,1]$. Thus condition (H9) is satisfied. Furthermore, we estimate

$$\begin{aligned}&\left|-\frac{1}{30}y\left(\frac{1}{3}\right) + \frac{1}{10}y\left(\frac{7}{20}\right) - \frac{1}{25}y\left(\frac{3}{5}\right) + \int_{\left[\frac{7}{10},\frac{3}{4}\right]} y(s)\ ds\right| \\ &\leq \frac{1}{30}\|y\| + \frac{1}{10}\|y\| + \frac{1}{25}\|y\| + \frac{1}{20}\|y\| = \frac{67}{300}\|y\|,\end{aligned} \tag{9.155}$$

so that condition (H3) is satisfied with

$$\varepsilon_1 := \frac{67}{300}. \tag{9.156}$$

Therefore, combining (9.149) and (9.156), we see that $\varepsilon_0 + \varepsilon_1 \in [0, 1)$. Furthermore, since $\kappa_0 = 1$ here, it also follows that $\kappa_0 (\varepsilon_0 + \varepsilon_1) < 1$ holds, too. Finally, it is easy to show that $g(y) := \sqrt[3]{y}$ satisfies conditions (H7) and (9.44).

In summary, then, each of the hypotheses of Theorem 9.10 holds. Therefore, we conclude that problem (9.144) has at least one positive solution. Finally, as in Example 9.24, by putting

$$\psi(y) := \frac{1}{10} y \left(\frac{7}{20}\right) - \frac{1}{25} y \left(\frac{3}{5}\right) + \int_{\left[\frac{7}{10}, \frac{3}{4}\right]} y(s)\ ds, \tag{9.157}$$

we may conclude that, in fact, the problem

$$\begin{aligned} y''(t) &= -\cos t \sqrt[3]{y(t)} \\ y(0) &= \psi(y) \cos\left(\frac{1}{\psi(y)+1}\right) \\ &= \left[\frac{1}{10} y \left(\frac{7}{20}\right) - \frac{1}{25} y \left(\frac{3}{5}\right) + \int_{\left[\frac{7}{10}, \frac{3}{4}\right]} y(s)\ ds\right] \\ &\quad \times \cos\left(\frac{1}{\left[\frac{1}{10} y \left(\frac{7}{20}\right) - \frac{1}{25} y \left(\frac{3}{5}\right) + \int_{\left[\frac{7}{10}, \frac{3}{4}\right]} y(s)\ ds\right] + 1}\right) \\ y(1) &= 0 \end{aligned} \tag{9.158}$$

has at least one positive solution.

Example 9.26. Consider the boundary value problem

$$\begin{aligned}
&y''(t) = -t^2[y(t)]^3 \sin y(t)\\
&y(0) = \left[\phi(y) + \frac{1}{15}y\left(\frac{9}{20}\right)\right]^5 + \left[\phi(y) + \frac{1}{15}y\left(\frac{9}{20}\right)\right]^3 + 3\left[\phi(y) + \frac{1}{15}y\left(\frac{9}{20}\right)\right]^2\\
&y(1) = 0.
\end{aligned} \tag{9.159}$$

Here, in (9.159), the linear functional $\phi(y)$ is defined by

$$\phi(y) := \frac{1}{6}y\left(\frac{2}{5}\right) - \frac{1}{15}y\left(\frac{9}{20}\right) - \frac{1}{20}y\left(\frac{1}{2}\right) + \int_{\left[\frac{11}{20},\frac{7}{10}\right]} y(s)\ ds \tag{9.160}$$

and, evidently, we have here that

$$H(z) := z^5 + z^3 + 3z^2. \tag{9.161}$$

Note, that for simplicity, we have selected $\phi(y)$ here to be the same functional as in Example 9.24. Obviously, if we put

$$\psi(y) := \frac{1}{6}y\left(\frac{2}{5}\right) - \frac{1}{20}y\left(\frac{1}{2}\right) + \int_{\left[\frac{11}{20},\frac{7}{10}\right]} y(s)\ ds, \tag{9.162}$$

then we may recast the boundary value problem (9.159) as

$$\begin{aligned}
&y''(t) = -t^2[y(t)]^3 \sin y(t)\\
&y(0) = [\psi(y)]^5 + [\psi(y)]^3 + 3[\psi(y)]^2\\
&y(1) = 0,
\end{aligned} \tag{9.163}$$

where

$$\begin{aligned} y(0) = &\left[\frac{1}{6}y\left(\frac{2}{5}\right) - \frac{1}{20}y\left(\frac{1}{2}\right) + \int_{\left[\frac{11}{20}, \frac{7}{10}\right]} y(s)\ ds\right]^5 \\ &+ \left[\frac{1}{6}y\left(\frac{2}{5}\right) - \frac{1}{20}y\left(\frac{1}{2}\right) + \int_{\left[\frac{11}{20}, \frac{7}{10}\right]} y(s)\ ds\right]^3 \\ &+ 3\left[\frac{1}{6}y\left(\frac{2}{5}\right) - \frac{1}{20}y\left(\frac{1}{2}\right) + \int_{\left[\frac{11}{20}, \frac{7}{10}\right]} y(s)\ ds\right]^2. \end{aligned} \tag{9.164}$$

Now, since the functional ϕ here is the same as in Example 9.24, we need not recheck that conditions (H2)–(H4) and (H9) hold. Moreover, since ε_0 and ε_1 are the same here as in Example 9.24, condition (H8) obviously holds, too. So, we check instead that conditions (H10) and (H11) hold. That these do hold follows from the following simple observations.

$$\begin{aligned} \lim_{z\to 0^+} \frac{z^5 + z^3 + 3z^2}{z} &= 0 \\ \lim_{z\to +\infty} \frac{z^5 + z^3 + 3z^2}{z} &= +\infty \\ \lim_{z\to 0^+} \frac{y^3 \sin y}{y} &= 0 \end{aligned} \tag{9.165}$$

Therefore, from (9.165) we deduce that conditions (H10)–(H11) hold.

In summary, then, each of the hypotheses of Theorem 9.14 hold. Consequently, we deduce that problem (9.159) or, equivalently, problem (9.163) has at least one positive solution. And this completes the example.

Let us conclude with some remarks.

Remark 9.27. Note that, to the best of the author's knowledge, none of these examples can be treated by any of the results in the existing literature. Indeed, regarding Examples 9.24 and 9.25, none of the results in [37, 62, 64, 65, 73, 81] can be easily

modified to apply to either of these examples. Specifically, the results of [64, 65, 66, 67, 68, 90] cannot be easily modified since the measure μ_α is signed in each example; the results of [65, 66, 68] cannot be easily modified since the function H in Example 9.24 does not satisfy linear growth at $z = 0$; the results of [66, 68] cannot be easily modified since in Example 9.24 we are only assuming a growth condition on the nonlinearity $g(y)$ at $+\infty$. Finally, while Yang [90, 91] employs some asymptotic conditions, the results there do not apply to our problem, not least of which because of the fact that μ_α is signed here.

Similarly, regarding Example 9.26, none of the results in [37, 62, 64, 65, 73, 81, 90, 91] can be applied to this problem. In this case, the fact that $H(z)$ is superlinear at $+\infty$ prevents any straightforward modification of the results in [64, 65, 66, 67, 68] because there is no number $\beta > 0$ such that $H(z) \leq \beta z$, for all $z \geq 0$ or, even less strictly, eventually. Moreover, we require no growth condition on g at $+\infty$ whatsoever. This also completely eliminates the same works from straightforward modifications. Of course, that μ_α is signed removes other works from consideration, too.

Succinctly, we believe that these three examples illustrate that our techniques here recover considerably more generality and flexibility than currently exists. Furthermore, they provide useful insight into certain of the growth conditions required for existence and the superfluousness of certain of these in previous works in the literature.

Remark 9.28. Notice that for the function H given in Example 9.24 we compute

$$H'(z) = qz^{q-1} + 1 - e^{-z} + ze^{-z}. \tag{9.166}$$

Since $q-1<0$, it follows that

$$\lim_{z\to 0^+} H(z) = +\infty. \tag{9.167}$$

Consequently, this type of function cannot be incorporated into the theories discussed in the papers by Infante [64] or by Infante and Pietramala [65, 66, 67]. Indeed, their theory requires the existence of a $\beta > 0$ such that $H(z) \leq \beta z$, for all $z \geq 0$. But clearly no such β exists since (9.167) holds. Nonetheless, our results here are able to handle this case. We consider this to be an important observation regarding the generality of our results as compared to others.

Remark 9.29. As intimated in Section 1, our results here leave some room for future work and improvement due to the presence of the term $\varepsilon_0 y\left(\xi_0\right)$ in (9.1). While the preceding examples have demonstrated that, in general, this is no great loss, it would be better to recover, if possible, the case where ε_0 may be set equal to zero. This would yield a more general result for BVPs with nonlinear boundary conditions with asymptotic growth conditions. In the case of Theorems 9.17, we were able to recover this more general setting by modifying some techniques due to Yang [90, 91]. However, as mentioned earlier (cf., Remark 9.20), in the sublinear growth setting it does not seem that the techniques of [90, 91] can be easily modified. And this leaves some possibility for additional investigation.

Another possible improvement concerns the nonlinear function $H(z)$ itself. In particular, by the way we have set up problem (9.1), all of the nonlocality $\phi(y)$ must necessarily occur at each occurrence in the boundary condition at $t=0$ – that is, we have the composition $H \circ \phi$ as the boundary condition at $t=0$. It would be more general if only some of the terms in ϕ need appear in any particular part of H, as this would allow for an even more general and flexible boundary condition at $t=0$ – for

instance, if in (9.160) the four addends, which comprise $\phi(y)$, were allowed to occur in the boundary condition at $t = 0$ in (9.159) in mixed combinations rather than all together. In any case, we leave these questions for possible future work on these sorts of problems.

Chapter 10

A System of BVPs with Nonlocal, Nonlinear Boundary Conditions with Superlinear Growth

In the preceding chapter we considered some results for *scalar* valued nonlocal, nonlinear boundary value problems. Now, we wish to consider, in this the penultimate chapter as well as in the final chapter of this work, the *vectorial* setting. That is to say, we shall now focus on the setting of a *system* of ordinary differential equations together with a specified collection of nonlocal, nonlinear boundary conditions. In particular, as in the preceding chapter, this shall yield some relatively substantial generalizations over the existing literature by, also as in the preceding chapter, making some relatively simple mathematical modifications to the techniques used to deduce existence of solution.

Specifically, here we consider as our model problem the nonlinear system of bound-

ary value problems

$$\begin{aligned}
x''(t) &= -a_1(t)g_1(x(t), y(t)),\ t \in (0,1)\\
y''(t) &= -a_2(t)g_2(x(t), y(t)),\ t \in (0,1)\\
x(0) &= 0 = y(0)\\
x(1) &= H_1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right), \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^1\right)\right)\\
y(1) &= H_2\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^2\right), \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^2\right)\right)
\end{aligned} \tag{10.1}$$

where ε_0^1, $\varepsilon_0^2 > 0$ are constants, which shall be specified later, $\xi_0^1, \xi_0^2 \in (0,1)$ are fixed, $\phi_1,\ \phi_2\ :\ \mathcal{C}([0,1]) \to \mathbb{R}$ are linear functionals, which capture the nonlocal nature of the boundary conditions, and $H_1,\ H_2\ :\ \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, which capture the nonlinear nature of the boundary conditions. We also assume that the nonlinearities $g_1,\ g_2\ :\ [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are continuous functions. The nonlocal terms here are quite general since they are realized as Lebesgue-Stieltjes integrals – that is,

$$\phi_1(x) := \int_{[0,1]} x(t)\ d\alpha_1(t) \text{ and } \phi_2(y) := \int_{[0,1]} y(t)\ d\alpha_2(t), \tag{10.2}$$

with $\alpha_1,\ \alpha_2 \in BV([0,1])$. Since it may be assumed without loss that, in fact, α_1, $\alpha_2 \in NBV([0,1])$, we get that associated to each of α_1, α_2 there exists a unique Borel measure, say μ_{α_1} and μ_{α_2}, respectively. In our context, importantly, these measures may be *signed*.

Here we study the existence of at least one positive solution to problem (10.1). To accomplish this task, we use the perturbation terms in (10.1) – namely, $\varepsilon_0^1 x\left(\xi_0^1\right)$, $\varepsilon_0^2 y\left(\xi_0^1\right)$, $\varepsilon_0^1 x\left(\xi_0^2\right)$, and $\varepsilon_0^2 y\left(\xi_0^2\right)$ – as well as a new condition on the nonlinear functions H_1 and H_2. These novelties reveal, in a way that shall be delineated momentarily,

that many of the restrictions previous authors have imposed on the various terms appearing in other problems similar to (10.1) are, in fact, unnecessary in our setting. Our principal condition on these functions is to require that, for each $i = 1, 2$,

$$\lim_{z_1+z_2\to+\infty} \frac{H_i\left(z_1, z_2\right)}{z_1^{p_i^{\infty}} + z_2^{q_i^{\infty}}} = +\infty \tag{10.3}$$

holds for some p_i^{∞}, $q_i^{\infty} \in (0, 1]$ with at least one of p_i^{∞} and q_i^{∞}, for each $i = 1, 2$, able to be taken equal to unity. In particular (cf., Remark 10.2), condition (10.3) implies that each of H_1 and H_2 may enjoy asymptotically superlinear growth in at least one of the two coordinate directions (cf., Remark 10.3). We will even give an existence result associated to the somewhat more relaxed condition

$$\limsup_{z_1+z_2\to 0^+} \frac{H_i\left(z_1, z_2\right)}{z_1 + z_2} < \rho_i, \tag{10.4}$$

for each $i = 1, 2$, with ρ_i a positive constant to be selected later; importantly, the result associated to condition (10.4) will even be applicable in the unperturbed case – i.e., $\varepsilon_0^1 = \varepsilon_0^2 = 0$. It should be pointed out that, in fact, Yang [90, 91] introduced an asymptotic condition similar to (10.4), though in the context of a slightly different problem. Regardless, Yang imposes a number of other hypotheses – such as complicated conditions on the equivalent of our nonlinearities g_1 and g_2 as well as the assumption that the equivalent of μ_{α_1} and μ_{α_2} be positive – with which we completely dispense here.

To summarize, we provide here the following generalizations over preceding works.

1. We allow for each of μ_{α_1} and μ_{α_2} to be *signed* measures rather than merely positive. This is an improvement over the preceding works, as intimated above.

2. We do *not* assume a uniform linear growth condition on either H_1 or H_2. We

instead assume either the asymptotic condition given in (10.3) together with an assumption that these functions possess superlinear growth as $z_1 + z_2 \to 0$ or condition (10.4). In particular, this shows that superlinear growth at $(+\infty, +\infty)$ is allowable. More generally, one need *not* assume a uniform linear growth condition as seems to appear in nearly all works on this sorts of problems – cf., [64, 65, 66, 67] – since in our setting there may be no $\beta > 0$ such that $H_i(z_1, z_2) \leq \beta(z_1 + z_2)$, for all z_1, $z_2 \geq 0$.

3. Specifically regarding Yang's works [90, 91], we point out that our results here even provide some interesting generalizations of the methods contained therein. In particular, while the results of [90, 91] concern different problems than (10.1), those works do appear to be among the only ones to consider an asymptotic condition with respect to the nonlinear boundary functions, at least to the best of the author's knowledge. A close examination of the proofs in those works, however, reveals that they use in a very explicit way the positivity of the respective Stieltjes measures. Lacking this positivity, as we do here, we must search for alternative approaches. Consequently, we feel that our results here represent an interesting advancement over those presented in [90, 91].

4. We believe that our techniques even allow H to be only eventually positive, though we do not prove such a theorem here – see [53] for an exemplar of this extension in a context somewhat different from this one.

5. We show that the assumption of asymptotic superlinearity of the functions H_1 and H_2 allows for neither g_1 nor g_2 to have any particular type of growth (e.g., sub- or superlinearity) as $\|(x, y)\| \to +\infty$. In particular, this means that g_1 and g_2 can have completely different limiting behavior. For example, g_1 could be sublinear as $\|(x, y)\| \to +\infty$, whilst g_2 is superlinear as $\|(x, y)\| \to +\infty$. While

Yang also allowed for mixed asymptotic behavior of the nonlinearities in [90], a cursory examination of that paper indicates that a number of complicated conditions are required to deduce that result. By contrast, our conditions are quite simple and relatively easy to check computationally.

10.1 Main Result and Numerical Example

We begin by listing the various structural conditions we impose on the constituent parts of problem (10.1). These conditions are the following.

H1: For each i, let $H_i : \mathbb{R}^2 \to [0,+\infty)$ be a real-valued, continuous function. Moreover, $H_i : [0,+\infty)\times[0,+\infty) \to [0,+\infty)$ – i.e., H_i is nonnegative when restricted to $[0,+\infty)\times[0,+\infty)$.

H2: For each i, the functional $\phi_i(y)$ appearing in (10.1) is linear and, in particular, has the realization

$$\phi_i(y) := \int_{[0,1]} y(t)\ d\alpha_i(t), \tag{10.5}$$

where $\alpha_i : [0,1] \to \mathbb{R}$ satisfies $\alpha_i \in BV([0,1])$.

H3: For each i, there is a constant $\varepsilon_1^i \in \left[0, \frac{1}{2}\right)$ such that the functional ϕ_i in (10.1) satisfies the inequality

$$|\phi_i(y)| \leq \varepsilon_1^i \|y\| \tag{10.6}$$

for all $y \in \mathcal{C}([0,1])$.

H4: For each i, there are $p_i^\infty \in (0,1]$ and $q_i^\infty \in (0,1]$, where for each i at least one of p_i^∞ and q_i^∞ is equal to unity, such that

$$\lim_{z_1+z_2\to+\infty} \frac{H_i(z_1,z_2)}{z_1^{p_i^\infty} + z_2^{q_i^\infty}} = +\infty \tag{10.7}$$

holds. Furthermore, for each i it holds that

$$\lim_{z_1+z_2\to 0^+} \frac{H_i\left(z_1, z_2\right)}{z_1+z_2} = 0. \tag{10.8}$$

H5: We find that

$$\lim_{z_1+z_2\to 0^+} \frac{g_1(x,y)}{x+y} = 0 \text{ and } \lim_{z_1+z_2\to 0^+} \frac{g_2(x,y)}{x+y} = 0. \tag{10.9}$$

H6: The constants ε_0^1, ε_0^2, ε_1^1, and ε_1^2 satisfy

$$0 \leq \varepsilon_0^1 + \varepsilon_0^2 + \varepsilon_1^2 + \varepsilon_1^2 < \frac{1}{2}. \tag{10.10}$$

H7: For each i, each of

$$\int_{[0,1]} t \; d\alpha_i(t) \geq 0 \tag{10.11}$$

and

$$\int_{[0,1]} G(t,s) \; d\alpha_i(t) \geq 0 \tag{10.12}$$

holds, where the latter holds for each $s \in [0,1]$.

Let us make some brief remarks regarding certain of the preceding conditions.

Remark 10.1. As in Chapter 9, regarding conditions (H2)–(H3), we point out that a wide variety of functions satisfy these conditions, such as the following.

$$\begin{aligned} \phi_1^i(y) &:= \int_F y(t) \; dt \\ \phi_2^i(y) &:= \sum_{k=1}^{n} a_k y\left(\xi_k\right) \end{aligned} \tag{10.13}$$

Remark 10.2. Regarding condition (H4) and specifically (10.7) therein, this is the asymptotic superlinear condition which, in part, distinguishes our methods here from others. On the other hand, (10.7) appearing in condition (H4) implies that H is also superlinear as $(x, y) \to (0^+, 0^+)$. Some functions, $H : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$, satisfying condition (H4), then, are the following. (In each case, $p_i^\infty = q_i^\infty = 1$, for each i.)

$$\begin{aligned}
&H(z_1, z_2) := z_1^{r_1} + z_2^{r_2},\ r_1, r_2 > 1 \\
&H(z_1, z_2) := (z_1 + z_2)^r \cos\left(\frac{1}{z_1 + z_2 + 1}\right),\ r > 1 \\
&H(z_1, z_2) := \begin{cases} (z_1 + z_2)^2, & 0 \le z_1 + z_2 \le 1 \\ e^{z_1 + z_2 - 1}, & z_1 + z_2 > 1 \end{cases}
\end{aligned} \tag{10.14}$$

It is easy to check that each of $(10.14)_1$–$(10.14)_3$ satisfies each part of condition (H4).

Furthermore, we should mention that each of the functions above cannot be incorporated into the theory of either [65] or [68] due to the superlinear growth at $(+\infty, +\infty)$. In fact, such nonlinear boundary functions could not be incorporated into any of the results given in [64, 66, 67, 68] for that matter. So, condition (H4) allows for a vastly different variety of nonlinear boundary functions than other recent works on these sorts of problems. Moreover, as shall be explicated in the proof of Theorem 10.5, which is our first existence result, this asymptotic superlinear growth condition also allows for the mixed growth of the nonlinearities g_1 and g_2, as mentioned in earlier.

Remark 10.3. Also regarding condition (H4), we point out that this condition allows for H_i to have different types of growth in the different coordinate directions. For example, consider the continuous function $H : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ defined

by

$$H\left(z_1, z_2\right) := \begin{cases} \left(z_1+z_2\right)^2\left(z_1^2+\sqrt{z_2}\right), & 0 \leq z_1+z_2 \leq 1 \\ z_1^2+\sqrt{z_2}, & z_1+z_2 \geq 1 \end{cases}. \tag{10.15}$$

In the z_1-coordinate direction, we find that H grows superlinearly as $z_1+z_2 \to +\infty$. On the other hand, in the z_2-coordinate direction, we find that H grows sublinearly as $z_1+z_2 \to +\infty$. Finally, it holds that

$$\lim_{z_1+z_2 \to 0^+} \frac{H\left(z_1, z_2\right)}{z_1+z_2} = 0 \text{ and } \lim_{z_1+z_2 \to +\infty} \frac{H\left(z_1, z_2\right)}{z_1+z_2^{0.3}} = +\infty. \tag{10.16}$$

Remark 10.4. As remarked in earlier, we believe that the conditions imposed on H_i by condition (H4) may be changed in a manner similar to the argument presented in [53]. But we leave such investigations for future work.

Now, let γ_0 be the constant defined by

$$\gamma_0 := \min\left\{\gamma, \min_{t \in E} t\right\}, \tag{10.17}$$

where $\gamma_0 \in (0,1)$. Then the cone, $\mathcal{K}$, we shall use in the sequel is then defined by

$$\mathcal{K} := \left\{(x,y) \in \mathcal{X} \ : \ x, y \geq 0,\ \min_{t \in E}[x(t)+y(t)] \geq \gamma_0\|(x,y)\|,\ \phi_1(x), \phi_2(y) \geq 0\right\}, \tag{10.18}$$

which is a simple modification of a cone first introduced by Infante and Webb [84]. Let us point out at this juncture that $\mathcal{K}$ does not contain only the neutral element of $\mathcal{X}$. Indeed, if we put, say, $\boldsymbol{\beta}_1(t) := (t,0)$, $\boldsymbol{\beta}_2(t) := (0,t)$, and $\boldsymbol{\beta}_3(t) := \left(\boldsymbol{\beta}_1+\boldsymbol{\beta}_2\right)(t) = (t,t)$, then it is easy to see that $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$, $\boldsymbol{\beta}_3 \in \mathcal{K}$ so that $\mathcal{K}$ contains infinitely many nontrivial elements of $\mathcal{X}$.

In any case, with these preliminary observations, we now state and prove our main result. We note, however, that in the statement of this theorem we assume that $p_1^{\infty}=p_2^{\infty}=1$. In other words, it is the numbers q_1^{∞}, q_2^{∞} that can be potentially less than unity. We do this only for definiteness and ease of exposition in the sequel.

Theorem 10.5. *Assume that ξ_0^1, $\xi_0^2\in E$, where E is a fixed set satisfying $E\Subset(0,1)$ as in Section 2. Then there exists a number $\delta\in(0,1)$ such that if both q_1^{∞}, $q_2^{\infty}\in(1-\delta,1]$ and (H1)–(H7) hold, then problem (1.1) has at least one positive solution.*

Proof. To begin, we consider the operator $S\ :\ \mathcal{X}\rightarrow\mathcal{X}$ defined by

$$S\left(x,y\right)(t):=\left(T_1\left(x,y\right),T_2\left(x,y\right)\right)\tag{10.19}$$

where, for each $i=1,2$, we have that $T_i\ :\ \mathcal{X}\rightarrow\mathcal{B}$ is defined by

$$\begin{aligned}T_i\left(x,y\right):=&\ tH_i\left(\phi_1\left(x\right)+\varepsilon_0^1x\left(\xi_0^i\right),\phi_2\left(y\right)+\varepsilon_0^2y\left(\xi_0^i\right)\right)\\&+\int_0^1G(t,s)a_i(s)g_i\left(x(s),y(s)\right)\ ds.\end{aligned}\tag{10.20}$$

We shall first argue that $S\ :\ \mathcal{K}\rightarrow\mathcal{K}$. To this end, it is obvious that for $(x,y)\in\mathcal{K}$, it follows that $T_i(x,y)(t)\geq 0$, for each $t\in[0,1]$ and $i=1,2$. We also note from the definition of γ_0 in (10.17) that

$$\begin{aligned}\min_{t\in E}T_i\left(x,y\right)\geq&\ \gamma_0H_i\left(\phi_1\left(x\right)+\varepsilon_0^1x\left(\xi_0^i\right),\phi_2\left(y\right)+\varepsilon_0^2y\left(\xi_0^i\right)\right)\\&+\gamma\max_{t\in[0,1]}\int_0^1G(t,s)a_i(s)g_i\left(x(s),y(s)\right)\ ds\\\geq&\ \gamma_0\|T_i\left(x,y\right)\|.\end{aligned}\tag{10.21}$$

We conclude that

$$\min_{t\in E}\left[\left(T_1\left(x,y\right)\right)(t)+\left(T_2\left(x,y\right)\right)(t)\right]\geq\gamma_0\|S\left(x,y\right)\|. \tag{10.22}$$

Finally, we observe that

$$\begin{aligned}\phi_1\left(T_1\left(x,y\right)\right)&=H_1\left(\phi_1\left(x\right)+\varepsilon_0^1x\left(\xi_0^1\right),\phi_2\left(y\right)+\varepsilon_0^2y\left(\xi_0^1\right)\right)\int_{[0,1]}t\ d\alpha_1(t)\\&+\int_{[0,1]}\int_0^1G(t,s)a_1(s)g_1\left(x(s),y(s)\right)\ ds\ d\alpha_1(t)\\&=H_1\left(\phi_1\left(x\right)+\varepsilon_0^1x\left(\xi_0^1\right),\phi_2\left(y\right)+\varepsilon_0^2y\left(\xi_0^1\right)\right)\int_{[0,1]}t\ d\alpha_1(t)\\&+\int_0^1\left[\int_{[0,1]}G(t,s)\ d\alpha_1(t)\right]a_1(s)g_1\left(x(s),y(s)\right)\ ds\\&\geq0,\end{aligned} \tag{10.23}$$

where the final inequality follows from assumption (H7). In a similar way, it follows that $\phi_2\left(T_2(x,y)\right)\geq0$. Thus, $S\ :\ \mathcal{K}\to\mathcal{K}$, as claimed. Let us also point out at this juncture that, by a standard argument involving the Arzela-Ascoli theorem (recall here that H_i is assumed to be continuous, for each $i=1,2$), we find that the operator S is completely continuous; we omit the details of this argument, however.

Now, by condition (H5) we find that there is a number $r_1>0$ such that

$$g_1(x,y)\leq\eta_1(x+y) \tag{10.24}$$

whenever $\|(x,y)\|\leq r_1$ and where $\eta_1>0$ satisfies

$$\eta_1\max\left\{\int_0^1G(s,s)a_1(s)\ ds,\int_0^1G(s,s)a_2(s)\ ds\right\}\leq\frac{1}{4}. \tag{10.25}$$

In addition, condition (H4) – i.e., equation (10.8) – implies the existence of a number

$r_1^* > 0$ such that, for each $i = 1, 2$,

$$H_i\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^i\right), \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^i\right)\right) < \eta_2\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^i\right) + \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^i\right)\right) \tag{10.26}$$

whenever

$$\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^i\right) + \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^i\right) < r_1^*, \tag{10.27}$$

and where $\eta_2 > 0$ is defined by

$$\eta_2 := \frac{1}{8\max\left\{\varepsilon_0^1, \varepsilon_0^2, \varepsilon_1^2, \varepsilon_1^2\right\}}. \tag{10.28}$$

Notice that

$$\begin{aligned} \phi_1(x) + \varepsilon_0^1 x\left(\xi_0^i\right) + \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^i\right) &\le \varepsilon_1^1\|x\| + \varepsilon_1^2\|y\| + \varepsilon_0^1\|x\| + \varepsilon_0^2\|y\| \\ &\le \left[\max\left\{\varepsilon_1^1, \varepsilon_1^2\right\} + \max\left\{\varepsilon_0^1, \varepsilon_0^2\right\}\right]\|(x,y)\| \\ &\le 2\max\left\{\varepsilon_0^1, \varepsilon_0^2, \varepsilon_1^2, \varepsilon_1^2\right\}\|(x,y)\|. \end{aligned} \tag{10.29}$$

So, in particular, if $(x, y) \in \mathcal{K}$ satisfies

$$\|(x,y)\| < \frac{r_1^*}{2\max\left\{\varepsilon_0^1, \varepsilon_0^2, \varepsilon_1^2, \varepsilon_1^2\right\}}, \tag{10.30}$$

then it follows that (10.28) holds.

So, set

$$r_1^{**} := \min\left\{r_1, \frac{r_1^*}{2\max\left\{\varepsilon_0^1, \varepsilon_0^2, \varepsilon_1^2, \varepsilon_1^2\right\}}\right\}. \tag{10.31}$$

Put

$$\Omega_{r_1^{**}} := \left\{(x,y) \in \mathfrak{X} \;:\; \|(x,y)\| < r_1^{**}\right\}. \tag{10.32}$$

Then for each $(x, y) \in \mathcal{K} \cap \partial\Omega_{r_1^{**}}$, we have that

$$
\begin{aligned}
&\|T_1(x,y)\| \\
&\leq H_1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right), \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^1\right)\right) + \int_0^1 G(s,s)a_1(s)g_1(x(s),y(s))\ ds \\
&\leq \eta_2\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right) + \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^1\right)\right) + \eta_1 \int_0^1 G(s,s)a_1(s)(x(s)+y(s))\ ds \\
&\leq \eta_2\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right) + \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^1\right)\right) + \frac{1}{4}\|(x,y)\| \\
&\leq \frac{1}{4}\|(x,y)\| + \frac{1}{4}\|(x,y)\| \\
&= \frac{1}{2}\|(x,y)\|.
\end{aligned}
\tag{10.33}
$$

Thus, we conclude that

$$\|T_1(x,y)\| \leq \frac{1}{2}\|(x,y)\|, \tag{10.34}$$

for each $(x, y) \in \mathcal{K} \cap \partial\Omega_{r_1^{**}}$. A similar argument holds for the operator T_2. Consequently, we deduce that

$$\|S(x,y)\| \leq \|(x,y)\|, \tag{10.35}$$

for each $(x, y) \in \mathcal{K} \cap \partial\Omega_{r_1^{**}}$.

On the other hand, let us assume without loss of generality that $p_i^{\infty} = 1$ for each i so that $q_i^{\infty} \in (0, 1]$, for each i. Then condition (H4) – i.e., equation (10.7) – implies the existence of a number $r_2^* := r_2^*(\eta_3) > 0$ such that

$$
\begin{aligned}
&H_1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right), \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^1\right)\right) \\
&\geq \eta_3\left(\left[\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right)\right] + \left[\phi_2(y) + \varepsilon_0^2 y\left(\xi_0^1\right)\right]^{q_1^{\infty}}\right)
\end{aligned}
\tag{10.36}
$$

whenever

$$\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right) + \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^1\right) \geq r_2^* \tag{10.37}$$

for some number r_2^*. Note that by picking r_2^* sufficiently large, the same type of estimate likewise holds for H_2; we assume henceforth that this is so. Here, in (10.36), we choose η_3 to be the number

$$\eta_3 := \frac{1}{t_0\gamma_0 \min\left\{\varepsilon_0^1, \varepsilon_0^2, \varepsilon_0^2\right\}}, \tag{10.38}$$

where $t_0 \in \overset{\circ}{E}$ is fixed but arbitrary; since $E \Subset (0,1)$, it holds that $t_0 \neq 0$, and so, $\eta_3 > 0$. Importantly, η_3 depends *neither* on q_1^∞ *nor* on q_2^∞. Now, notice that for $(x,y) \in \mathcal{K}$ since $\phi_1(x)$, $\phi_2(y) \geq 0$ and $\xi_0^1 \in E$, we may estimate

$$\begin{aligned}
\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right) + \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^1\right) &\geq \min\left\{\varepsilon_0^1, \varepsilon_0^2\right\}\left[x\left(\xi_0^1\right) + y\left(\xi_0^1\right)\right] \\
&\geq \min\left\{\varepsilon_0^1, \varepsilon_0^2\right\}\min_{t\in E}[x(t) + y(t)] \qquad (10.39) \\
&\geq \gamma_0 \min\left\{\varepsilon_0^1, \varepsilon_0^2\right\}\|(x,y)\|.
\end{aligned}$$

Consequently, if (x,y) satisfies

$$\|(x,y)\| \geq \frac{r_2^*}{\gamma_0 \min\left\{\varepsilon_0^1, \varepsilon_0^2\right\}}, \tag{10.40}$$

then (10.36) holds.

We next interrupt to prove an easy lemma. Suppose that x, $y \geq 0$ with x, $y \leq M$ for some $M \geq 1$ and finite. Let q satisfy $0 < q \leq 1$. Choose the constant c such that

$$c := \min\left\{1, M^{q-1}\right\}; \tag{10.41}$$

note that $-1 < q - 1 \leq 0$. Obviously, $c \in (0,1]$ since $M \geq 1$ and $q - 1 \leq 0$. Then it follows that

$$x + y^q \geq c(x+y), \tag{10.42}$$

for all $(x, y) \in [0, M] \times [0, M]$. Indeed, we merely notice that, for $(x, y) \in [0, M] \times [0, M]$

$$cx \le x \tag{10.43}$$

and

$$cy \le y^q, \tag{10.44}$$

since $y \mapsto y^{q-1}$ is decreasing for $y > 0$, whereupon adding (10.43)–(10.44) we estimate

$$cx + cy \le x + y^q, \tag{10.45}$$

which evidently proves inequality (10.42).

Now continuing with the proof, let us put

$$r_2^{**} := \max\left\{1, 2r_1^{**}, \frac{r_2^*}{\gamma_0 \min\left\{\varepsilon_0^1, \varepsilon_0^2\right\}}\right\}, \tag{10.46}$$

which is *independent* of each of q_1^∞ and q_2^∞. Define $\Omega_{r_2^{**}}$ by

$$\Omega_{r_2^{**}} := \left\{(x, y) \in \mathcal{X} \ : \ \|(x, y)\| < r_2^{**}\right\}. \tag{10.47}$$

Using estimate (10.42), then, and the fact that

$$\int_0^1 G\left(t_0, s\right) a_1(s) g_1(x(s), y(s)) \ ds \ge 0, \tag{10.48}$$

we deduce that for each $(x, y) \in \mathcal{K} \cap \partial\Omega_{r_2^{**}}$

$$
\begin{aligned}
(T_1(x,y))(t_0) &= t_0 H_1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right), \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^1\right)\right) \\
&\quad + \int_0^1 G(t_0, s)\, a_1(s) g_1(x(s), y(s))\, ds \\
&\geq t_0 H_1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right), \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^1\right)\right) \\
&\geq t_0 \eta_3 \left(\left[\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right)\right] + \left[\phi_2(y) + \varepsilon_0^2 y\left(\xi_0^1\right)\right]^{q_1^\infty}\right) \\
&\geq t_0 \eta_3 \left[\varepsilon_0^1 x\left(\xi_0^1\right) + \left(\varepsilon_0^2\right)^{q_1^\infty}\left[y\left(\xi_0^1\right)\right]^{q_1^\infty}\right] \\
&\geq t_0 \eta_3 \left[\varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2\left[y\left(\xi_0^1\right)\right]^{q_1^\infty}\right] \\
&\geq t_0 \eta_3 \min\left\{\varepsilon_0^1, \varepsilon_0^2\right\}\left[\left[x\left(\xi_0^1\right)\right] + \left[y\left(\xi_0^1\right)\right]^{q_1^\infty}\right] \\
&\geq t_0 \eta_3 \min\left\{\varepsilon_0^1, \varepsilon_0^2\right\} c_1 \left[x\left(\xi_0^1\right) + y\left(\xi_0^1\right)\right] \\
&\geq t_0 \eta_3 \min\left\{\varepsilon_0^1, \varepsilon_0^2\right\} \gamma_0 c_1 \|(x,y)\| \\
&\geq c_1 \|(x,y)\|,
\end{aligned}
\tag{10.49}
$$

where we have used the lemma of the previous paragraph to get the third-to-last inequality, and so, here $c_1 := \min\left\{1, (r_2^{**})^{q_1^\infty - 1}\right\}$. We have also used both the fact that $\varepsilon_0^2 \in \left[0, \frac{1}{2}\right)$ and that $q_1^\infty \in (0, 1]$ so that $(\varepsilon_0^2)^{q_1^\infty} \geq \varepsilon_0^2$. In summary, it follows that

$$\|T_1(x,y)\| \geq c_1 \|(x,y)\|. \tag{10.50}$$

Likewise, for each $(x, y) \in \mathcal{K} \cap \partial\Omega_{r_2^{**}}$ we deduce that for $c_2 := \min\left\{1, (r_2^{**})^{q_2^\infty - 1}\right\}$

$$\|T_2(x,y)\| \geq c_2 \|(x,y)\|. \tag{10.51}$$

We now conclude the argument by considering cases. If $q_1^\infty = q_2^\infty = 1$, then from (10.41), it is obvious that $c_1 = c_2 = 1$. In this case we deduce from (10.50)–(10.51)

that

$$\|S(x,y)\| \geq 2\|(x,y)\| > \|(x,y)\|, \tag{10.52}$$

for each $(x,y) \in \mathcal{K} \cap \partial\Omega_{r_2^{**}}$. On the other hand, in case $0 < \max\{q_1^\infty, q_2^\infty\} < 1$, then

$$c_1 := (r_2^{**})^{q_1^\infty - 1} \text{ and } c_2 := (r_2^{**})^{q_2^\infty - 1}. \tag{10.53}$$

In order that $c_1 + c_2 \geq 1$ be satisfied, at a minimum we must have that

$$\min\left\{2^{\frac{1}{1-q_1^\infty}}, 2^{\frac{1}{1-q_2^\infty}}\right\} \geq r_2^{**}. \tag{10.54}$$

Evidently, since r_2^{**} is finite and $(1-q_i^\infty)^{-1} \to +\infty$ as $q_i^\infty \to 1^-$, there exists a $\delta > 0$ sufficiently small such that for each q_1^∞, $q_2^\infty \in (1-\delta, 1]$ we have that (10.54) holds. In this case, we again deduce that (10.52) holds with, say, the factor 2 replaced by 1. Importantly, we point out that r_2^{**} does *not* depend on q_i^∞ for either i. Consequently, we may, in inequality (10.54) above, freely increase q_i^∞, for each i, without changing the previously selected and fixed value of r_2^{**}.

Finally, putting the preceding paragraphs together, we make two conclusions. Firstly, if $q_1^\infty = q_2^\infty = 1$, then by Lemma 2.13 and inequality (10.52) we deduce the existence of a function $(x_0, y_0) \in \mathcal{K}$ such that $S(x_0, y_0) = (x_0, y_0)$, where $x_0(t)$, $y_0(t)$ forms a positive solution of problem (10.1). Secondly, if q_1^∞, $q_2^\infty \leq 1$, then there exists a $\delta > 0$ sufficiently small such that if q_1^∞, $q_2^\infty \in (1-\delta, 1]$, then problem (10.1) still has at least one positive solution. And as these cases are exhaustive this completes the proof. $\square$

We now prove a second result that demonstrates an alternative approach to problem (10.1). In particular, we begin by introducing the following condition.

H8: For each $i = 1, 2$, there is a constant $\rho_i > 0$ such that

$$\limsup_{(z_1,z_2)\to(0^+,0^+)} \frac{H_i(z_1,z_2)}{z_1+z_2} < \rho_i \tag{10.55}$$

holds, where $\rho_i \in \left[0, \frac{1}{2\max\left\{\varepsilon_1^1,\varepsilon_1^2\right\}}\right)$.

On the one hand, condition (H8) is certainly more general than condition (H4). For instance, the continuous function $H : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ defined by

$$H(z_1,z_2) := \begin{cases} (z_1+z_2)\cos\left(\frac{1}{z_1+z_2}\right), & z_1+z_2 \neq 0 \\ 0, & z_1 = z_2 = 0 \end{cases} \tag{10.56}$$

satisfies

$$\limsup_{(z_1,z_2)\to(0^+,0^+)} \frac{H(z_1,z_2)}{z_1+z_2} = 1 \tag{10.57}$$

but $\lim_{(z_1,z_2)\to(0^+,0^+)} \frac{H(z_1,z_2)}{z_1+z_2}$ does not exist. On the other hand, in order to prove the next result, we shall have to impose growth conditions on the nonlinearities g_1 and g_2 at infinity. Thus, we introduce condition (H9) below.

H9: We find that

$$\lim_{(x,y)\to(+\infty,+\infty)} \frac{g_1(x,y)}{x+y} = +\infty \text{ and } \lim_{(x,y)\to(+\infty,+\infty)} \frac{g_2(x,y)}{x+y} = +\infty. \tag{10.58}$$

With condition (H8) and (H9) in hand we state and prove the following theorem. We first give two preliminary remarks.

Remark 10.6. We note that condition (H8) is more closely related to certain of the conditions given by Yang [90, 91], to which was alluded earlier. In particular, however, we note that *unlike* the results Yang gives, which admittedly were for a slightly

different problem than (10.1), we do *not* require complicated conditions on the nonlinearities g_1 and g_2. Indeed, conditions (H5) and (H9) are quite straightforward and standard. Moreover, the measures here are signed. So, we consider these observations to be both interesting and noteworthy.

Remark 10.7. We also note, as will become clear in the statement and proof of Theorem 10.8 in the sequel, that with this particular assumption – namely (H8) – we may dispense with the perturbation terms appearing in (10.1). In particular and importantly, then, we may set $\varepsilon_0^1 = \varepsilon_0^2 = 0$.

Theorem 10.8. *Suppose that conditions (H1)–(H3) and (H5)–(H9) hold. In addition, suppose that* $\varepsilon_0^1 = \varepsilon_0^2 = 0$. *Then the unperturbed problem (10.1) has at least one positive solution.*

Proof. Due to the assumptions given in the statement of this theorem, it is still the case that $T : \mathcal{K} \to \mathcal{K}$ and that T is a completely continuous operator. So, we proceed directly to the cone theoretic part of the argument.

To this end, let $\rho_i < \frac{1}{2\max\left\{\varepsilon_1^1,\varepsilon_1^2\right\}}$ be given, for each $i = 1, 2$. Evidently, we may select $k \in \mathbb{N}$ sufficiently large such that

$$0 \leq \rho_i < \frac{2^k - 1}{2^{k+1}\max\left\{\varepsilon_1^1, \varepsilon_1^2\right\}} < \frac{1}{2\max\left\{\varepsilon_1^1, \varepsilon_1^2\right\}} \tag{10.59}$$

holds for each i. Moreover, for each i, select the number $\eta_i > 0$ such that

$$\eta_i \int_0^1 G(s, s)a_i(s)\ ds \leq \frac{1}{2^{k+1}} \tag{10.60}$$

holds. Condition (H5) implies the existence of a number $r_1 > 0$ such that $g_i(x, y) \leq \eta_i(x + y)$ for all $0 \leq x + y < r_1$ and for each i. On the other hand, from condition

(H8), we may select a number $0 < \varepsilon < \min\{\rho_1, \rho_2\}$ sufficient small such that

$$H_i(z_1, z_2) < (\rho_i - \varepsilon)(z_1 + z_2) \tag{10.61}$$

holds whenever $0 \leq z_1 + z_2 < r_1^*$ for some number $r_1^* > 0$, for each $i = 1, 2$. In addition, since (10.59) holds, for each i, it evidently holds that

$$0 < \rho_i - \varepsilon < \frac{2^k - 1}{2^{k+1}\max\{\varepsilon_1^1, \varepsilon_1^2\}}. \tag{10.62}$$

Now, condition (H3) implies that

$$\phi_1(x) \leq \varepsilon_1^1\|x\| \text{ and that } \phi_2(y) \leq \varepsilon_1^2\|y\|. \tag{10.63}$$

Consequently, for each $(x, y) \in \mathcal{K}$ satisfying

$$0 \leq \|(x, y)\| < \min\{r_1, r_1^*\}, \tag{10.64}$$

it follows that

$$\phi_1(x) \leq \varepsilon_1^1\|x\| \leq \varepsilon_1^1\|(x, y)\| < \frac{1}{2}r_1^* \text{ and that } \phi_2(y) \leq \varepsilon_1^2\|y\| \leq \varepsilon_1^2\|(x, y)\| < \frac{1}{2}r_1^*. \tag{10.65}$$

Now, select $r_1^{**} > 0$ such that

$$r_1^{**} < \min\{r_1, r_1^*\} \tag{10.66}$$

and put $\Omega_{r_1^{**}} := \{(x, y) \in \mathcal{K} \ : \ \|(x, y)\| < r_1^{**}\}$. Upon combining (10.63)–(10.66), we

may then estimate

$$H_i\left(\phi_1(x), \phi_2(y)\right) < \left(\rho_i - \varepsilon\right)\left(\phi_1(x) + \phi_2(y)\right), \tag{10.67}$$

for each $(x, y) \in \mathcal{K} \cap \partial\Omega_{r_1^{**}}$ and $i = 1, 2$. So, combining all of these estimates, we deduce that

$$\begin{aligned} \|T_1(x,y)\| &\leq H_1\left(\phi_1(x), \phi_2(y)\right) + \int_0^1 G(s,s)a_1(s)g_1(x(s), y(s))\ ds \\ &\leq \left(\rho_1 - \varepsilon\right)\left(\phi_1(x) + \phi_2(y)\right) + \frac{1}{2^{k+1}}\|(x,y)\| \\ &\leq \frac{2^k - 1}{2^{k+1}\max\left\{\varepsilon_1^1, \varepsilon_1^2\right\}}\left(\varepsilon_1^1\|x\| + \varepsilon_1^2\|y\|\right) + \frac{1}{2^{k+1}}\|(x,y)\| \\ &\leq \frac{2^k - 1}{2^{k+1}\max\left\{\varepsilon_1^1, \varepsilon_1^2\right\}}\max\left\{\varepsilon_1^1, \varepsilon_1^2\right\}\left(\|x\| + \|y\|\right) + \frac{1}{2^{k+1}}\|(x,y)\| \\ &= \frac{2^k - 1}{2^{k+1}\max\left\{\varepsilon_1^1, \varepsilon_1^2\right\}}\max\left\{\varepsilon_1^1, \varepsilon_1^2\right\}\|(x,y)\| + \frac{1}{2^{k+1}}\|(x,y)\| \\ &= \frac{1}{2}\|(x,y)\|. \end{aligned} \tag{10.68}$$

Similarly, we deduce that

$$\|T_2(x,y)\| \leq \frac{1}{2}\|(x,y)\| \tag{10.69}$$

whence

$$\|S(x,y)\| \leq \|(x,y)\|, \tag{10.70}$$

for each $(x, y) \in \mathcal{K} \cap \partial\Omega_{r_1^{**}}$.

On the other hand, select the number $\eta_3 > 0$ to satisfy

$$\eta_3 \max\left\{\int_E \gamma_0^2 G(s,s)a_1(s)\ ds, \int_E \gamma_0^2 G(s,s)a_2(s)\ ds\right\} \geq \frac{1}{2}. \tag{10.71}$$

Then by condition (H9), we have that

$$g_i(x, y) \geq \eta_3(x + y), \tag{10.72}$$

for all $x + y \geq r_2$ and for each $i = 1, 2$. Put

$$r_2^* := \max\left\{\frac{r_2}{\gamma_0}, 2r_1^{**}\right\}. \tag{10.73}$$

Then since $H_1(z_1, z_2) \geq 0$, for all $(z_1, z_2) \in [0, +\infty) \times [0, +\infty)$, we deduce that

$$\begin{aligned}\min_{t\in E}\left(T_1(x, y)\right)(t) &\geq \eta_3 \int_E \gamma_0 G(s, s)a_1(s)[x(s) + y(s)]\ ds \\ &\geq \|(x, y)\|\eta_3 \int_E \gamma_0^2 G(s, s)a_1(s)\ ds \\ &\geq \frac{1}{2}\|(x, y)\|,\end{aligned} \tag{10.74}$$

whence

$$\|T_1(x, y)\| \geq \frac{1}{2}\|(x, y)\|, \tag{10.75}$$

for each $(x, y) \in \mathcal{K} \cap \partial\Omega_{r_2^*}$. Similarly,

$$\|T_2(x, y)\| \geq \frac{1}{2}\|(x, y)\|, \tag{10.76}$$

so that $\|S(x, y)\| \geq \|(x, y)\|$, for $(x, y) \in \mathcal{K} \cap \partial\Omega_{r_2^*}$. Consequently, we may invoke Lemma 2.13 to deduce the existence of at least one positive solution to problem (10.1). $\square$

We conclude with an explicit numerical example together with some final remarks.

Example 10.9. Consider the boundary value problem

$$
\begin{aligned}
-x''(t) &= (2t+1)g_1(x(t), y(t)) \\
-y''(t) &= e^{-3t+1}g_2(x(t), y(t)) \\
x(0) &= H_1\left(\phi_1(x) + \frac{1}{40}x\left(\frac{1}{2}\right), \phi_2(y) + \frac{1}{300}y\left(\frac{2}{5}\right)\right) \\
y(0) &= H_2\left(\phi_1(x) + \frac{1}{40}x\left(\frac{1}{2}\right), \phi_2(y) + \frac{1}{300}y\left(\frac{2}{5}\right)\right) \\
x(1) &= 0 = y(1),
\end{aligned}
\tag{10.77}
$$

where we make the following declarations.

$$
\begin{aligned}
H_1\left(z_1, z_2\right) &:= \left(z_1 + z_2\right)^3 \\
H_2\left(z_1, z_2\right) &:= \left(z_1 + z_2\right)^2 \cos\left(\frac{1}{z_1 + z_2 + 1}\right) \\
\phi_1(x) &:= \frac{1}{8}x\left(\frac{1}{3}\right) - \frac{1}{40}x\left(\frac{1}{2}\right) - \frac{1}{12}x\left(\frac{3}{5}\right) + \frac{1}{2}\int_{\left[\frac{13}{20}, \frac{3}{4}\right]} x(s)\ ds \\
\phi_2(y) &:= -\frac{1}{300}y\left(\frac{2}{5}\right) + \frac{1}{15}y\left(\frac{9}{20}\right) - \frac{1}{100}y\left(\frac{11}{20}\right) + \frac{1}{10}\int_{\left[\frac{3}{5}, \frac{7}{10}\right]} y(s)\ ds \\
g_1(x, y) &:= \begin{cases} (x+y)^2, & x + y \le 1 \\ \sqrt{x+y}, & x + y \ge 1 \end{cases} \\
g_2(x, y) &:= (x+y)^3
\end{aligned}
\tag{10.78}
$$

Interestingly, note that g_1 is sublinear as $(x, y) \to (+\infty, +\infty)$, whereas g_2 is superlinear. Furthermore, let us observe at this juncture that on account of the definitions of ϕ_1 and ϕ_2 given in (10.78), we may recast the boundary conditions at $t = 0$ in (10.77)

in the somewhat simpler form

$$\begin{aligned} x(0) &= H_1\left(\psi_1(x), \psi_2(y)\right) = \left[\psi_1(x) + \psi_2(y)\right]^3 \\ y(0) &= H_2\left(\psi_1(x), \psi_2(y)\right) = \left[\psi_1(x) + \psi_2(y)\right]^2 \cos\left(\frac{1}{\psi_1(x) + \psi_2(y) + 1}\right), \end{aligned} \tag{10.79}$$

where we have put $\psi_1(x) := \phi_1(x) + \frac{1}{40}x\left(\frac{1}{2}\right)$ and $\psi_2(y) := \phi_2(y) + \frac{1}{300}y\left(\frac{2}{5}\right)$. Incidentally, though we do not show this explicitly, let us also remark that it is easy to show that the Stieltjes measures μ_{α_1} and μ_{α_2} are *signed* for this problem.

It is now easy to check that each of conditions (H1)–(H7) is satisfied. In particular, note that we may select $\varepsilon_1^1 := \frac{17}{60}$, $\varepsilon_1^2 := \frac{9}{100}$, $\varepsilon_0^1 := \frac{1}{40}$, and $\varepsilon_0^2 := \frac{1}{300}$. Moreover, we note that $\int_{[0,1]} t\ d\alpha_1(t) = \frac{17}{1200} \geq 0$ and that $\int_{[0,1]} t\ d\alpha_2(t) = \frac{89}{3000} \geq 0$. In any case, we conclude that we may invoke Theorem 10.5 to deduce that problem (10.77) has at least one positive solution. Likewise, problem (10.79) has at least one positive solution, too.

Remark 10.10. We note that problem (10.77) could not be addressed by any existing results. This is true for a variety of reasons, among which are the following: problem (10.77) involves a system of equations; it imposes no growth conditions on g_1 and g_2 for (x, y) large in norm; it allows for each of H_1 and H_2 to have superlinear growth as $(x, y) \to (+\infty, +\infty)$; and it allows for each of ϕ_1 and ϕ_2 to be have associated signed Borel measures. In short, we are not aware that any results in the existing literature can be applied to problem (10.77). And this is the advantage of the asymptotic conditions (H4) and (H8), which we have introduced in this work.

Remark 10.11. We have elected not to give an example of Theorem 10.8 since its application would proceed in a very similar manner to Example 10.9. Nonetheless, we emphasize that in the case of Theorem 10.8, we may take the perturbation terms in (10.1) equal to zero and, hence, in this case we are recovering solutions to the

unperturbed (i.e., $\varepsilon_0^1 = \varepsilon_0^2 = 0$) problem (10.1).

Chapter 11

A System of BVPs with Nonlocal, Nonlinear Boundary Conditions with Sublinear Growth

In this concluding chapter we wish to illustrate how techniques similar (though not identical) to the preceding chapter may still be used in the vectorial setting with an assumption of sublinear growth in the nonlinear boundary terms. As shall be seen in the sequel, the assumption of asymptotic sublinearity requires some modifications of the techniques used to deduce the existence of at least one positive solution.

In particular, in this chapter our model problem is

$$\begin{aligned}
x''(t) &= -\lambda_1 a_1(t) g_1(x(t), y(t)),\ t \in (0,1)\\
y''(t) &= -\lambda_2 a_2(t) g_2(x(t), y(t)),\ t \in (0,1)\\
x(0) &= H_1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2 y\left(\xi_0^1\right)\right)\\
y(0) &= H_2\left(\phi_2(y) + \varepsilon_0^1 x\left(\xi_0^2\right) + \varepsilon_0^2 y\left(\xi_0^2\right)\right)\\
x(1) &= 0 = y(1),
\end{aligned} \tag{11.1}$$

where $\lambda_1,\ \lambda_2 > 0$ are eigenvalues, $\varepsilon_0^1,\ \varepsilon_0^2 > 0$ are constants, which shall be specified later, $\xi_0^1, \xi_0^2 \in (0,1)$ are fixed, $\phi_1,\ \phi_2\ :\ \mathcal{C}([0,1]) \to \mathbb{R}$ are functionals, which are realizations of the nonlocal nature of the boundary conditions, $H_1,\ H_2\ :\ \mathbb{R} \to \mathbb{R}$ are continuous functions, which are realizations of the nonlinear nature of the boundary conditions, and $g_1,\ g_2\ :\ [0,+\infty)\times[0,+\infty) \to [0,+\infty)$ are continuous. The nonlocal terms here are, once again as in Chapters 9 and 10, very general, being as they are realized by Lebesgue-Stieltjes integrals – that is,

$$\phi_1(x) := \int_{[0,1]} x(t)\ d\alpha_1(t) \text{ and } \phi_2(y) := \int_{[0,1]} y(t)\ d\alpha_2(t), \tag{11.2}$$

with $\alpha_1,\ \alpha_2 \in BV([0,1])$. It may be assumed without loss that, in fact, $\alpha_1,\ \alpha_2 \in NBV([0,1])$. Consequently, we observe that to each of $\alpha_1,\ \alpha_2$, there exists a unique Borel measure, say μ_{α_1} and μ_{α_2}, respectively. In our context, these measures may be *signed*, which is one of the key contributions of this work.

Our novel approach to problem (11.1) is twofold. We first introduce the perturbation terms $\varepsilon_0^1 x\left(\xi_0^1\right)$, $\varepsilon_0^2 y\left(\xi_0^1\right)$, $\varepsilon_0^1 x\left(\xi_0^2\right)$, and $\varepsilon_0^2 y\left(\xi_0^2\right)$ appearing in (11.1). These perturbation terms allows us in turn to introduce a second novelty – namely, to utilize much less restrictive growth conditions on each of H_1 and H_2 appearing in (11.1). Indeed, we require that, for each $i = 1, 2$,

$$\lim_{z\to\infty} \frac{|H_i(z) - \kappa_0^i z|}{|z|} = 0, \tag{11.3}$$

for some $\kappa_0^i \in [0,+\infty)$. Note that condition (11.3) implies that H_i may grow either sub- or superlinearly at $z = 0$ – e.g., each of $H(z) = \sqrt[3]{z}$ and $H(z) = z^2$ is an admissible function for small z. These two relatively simple modifications allow for considerably weaker conditions on problem (11.1), for we may now assume that each

of the measures μ_{α_1} and μ_{α_2} is signed and that neither H_1 nor H_2 is sublinear at $z = 0$, assumptions that seem to be made in most problems related to (11.1) as we indicate in the sequel. Furthermore, it turns out that we do not even require the perturbation terms provided that we assume that each of $H_1(z)$ and $H_2(z)$ is monotone increasing for $z \geq 0$.

Closely related to these observations, we should point out at this juncture that Yang [90, 91] actually introduced asymptotic conditions in those works not entirely dissimilar to (11.3) above. In particular, in [90] a system of equations, which are very similar to (11.1), was studied. Among a variety of other conditions, Yang was able to employ an asymptotic condition of the general form

$$\limsup_{z\to\infty} \frac{H(z)}{z} < \frac{1}{\varphi}, \tag{11.4}$$

for some positive, finite constant φ. Certainly, (11.4) is more general than our condition (11.3). However, a careful examination of the proof in [90] reveals that the positivity of the measures μ_{α_1} and μ_{α_2} is essential. Consequently, it does not seem possible at present to give a simple modification of Yang's techniques in the case where the measures may be signed (i.e., our situation here).

Thus, we employ two different strategies to overcome these difficulties. Our first strategy is via condition (11.3) and the perturbation terms in (11.1), whereas our second strategy is via a monotonicity assumption on each of H_1 and H_2. In any case, we should also point out that although Yang [90] achieves a more general condition in (11.4), in [90] much more complicated structural conditions are instead assumed on the nonlinearities g_1, g_2 than we assume here, and the eigenvalue problem is not studied in [90] either.

Prior to enumerating specifically the contributions of this paper, let us briefly

review the relevant existing literature on problems similar to (11.1). Recently, Infante and Webb [84] provided an elegant theory for nonlocal BVPs in the case where the boundary conditions are *linear*; furthermore, one may consult the introduction of [84] for a thorough review of the recent literature on multipoint BVPs prior to the contribution of Infante and Webb. Related extensions may be found in recent papers by Webb [85, 86, 87] as well as by Graef and Webb [60].

On the other hand, recently there has been some attempts by Infante [64], Infante and Pietramala [65, 66, 67], Kang and Wei [68], and Yang [90, 91] to consider in fairly general contexts BVPs with nonlinear BCs. However, insofar as these papers are concerned, while they do make a connection to the linear boundary condition theory, they do so under some limiting assumptions, namely that H, which is the function capturing the nonlinearity of the BCs, is strictly positive, that the Borel measure associated to the Lebesgue-Stieltjes integral $\phi(y) = \int_E y(t)\ d\alpha(t)$ is *positive*, and, in nearly all cases ([90, 91] being partial exceptions), that H satisfies a uniform growth condition of the form $\zeta_1 z \leq H(z) \leq \zeta_2 z$, for $0 \leq \zeta_1 \leq \zeta_2 < +\infty$, *for all* $z \geq 0$.

In particular, our work here directly generalizes and improves [65, 90] since those works are very closely related to our work here. Indeed, Infante and Pietramala [65] and Yang [90] each considered a system almost identical to (11.1) but with the nonlocal condition at $t = 1$ rather than at $t = 0$, which is a trivial difference. Here we achieve in the particular case of problem (11.1) the following generalizations over various aspects of the results presented in [65, 90] and, more tangentially, in [64, 66, 67, 68, 91]. We enumerate these generalizations and improvements as follows.

1. For the first of our two existence results, we do not assume that either H_1 or H_2 is monotone, unlike some works in the literature involving nonlinear boundary conditions. Where we do assume monotonicity, this assumption, as noted above,

allows us to dispense with the perturbation terms appearing in (11.1) above.

2. We allow for each of μ_{α_1} and μ_{α_2} to be *signed* measures rather than merely positive. This is a notable generalization over preceding works on related problems – specifically, [64, 65, 66, 67, 90, 91].

3. We do *not* assume a uniform linear growth condition on either H_1 or H_2. While condition (11.3) does imply linear growth of the H_i's at $+\infty$, this is only an asymptotic condition, which is much weaker than the uniform condition proposed in other works on related problems – specifically, [64, 65, 66, 67].

4. We believe that our techniques here allow for H to be only eventually positive, though we do not prove such a theorem here – see [53] for an exemplar of this extension.

5. While we present our results in the somewhat simpler setting of Dirichlet-type boundary conditions, we believe that our techniques can be extended to include some of the other types of boundary conditions considered by other authors.

6. Finally, we exhibit an explicit and direct connection to the linear BC theory developed originally in [84]. Indeed, condition (11.3) essentially shows that if the boundary conditions merely possess asymptotically sublinear growth at $+\infty$ (i.e., are asymptotically similar to the sorts of conditions considered in [84]), then this is sufficient, together with some other relatively standard assumptions, to deduce that problem (11.1) has at least one positive solution. Heuristically, then, if $\phi(y)$ is a linear functional to which the theory of [84] applies and if $H_i(\phi(y)) \approx \phi(y)$ for $\phi(y) \gg 1$, for each i, then we recover the existence of at least one positive solution to problem (11.1). We feel that this is both a novel and interesting observation.

11.1 Main Results and Numerical Example

Before stating and proving our two main results, which are Theorem 11.6 and Theorem 11.8, we introduce some structural conditions on the various functions and functionals in (11.1). They are as follows.

H1: For each i, let $H_i : \mathbb{R} \to \mathbb{R}$ be a real-valued, continuous function. Moreover, $H_i : [0, +\infty) \to [0, +\infty)$ – i.e., H_i is nonnegative when restricted to $[0, +\infty)$.

H2: For each i, the functional $\phi_i(y)$ appearing in (11.1) is linear and, in particular, has the form

$$\phi_i(y) := \int_{[0,1]} y(t)\ d\alpha_i(t), \tag{11.5}$$

where $\alpha_i : [0, 1] \to \mathbb{R}$ satisfies $\alpha_i \in BV([0, 1])$.

H3: For each i, there is a constant ε_1^i such that the functional ϕ_i in (11.1) satisfies the inequality

$$|\phi_i(y)| \leq \varepsilon_1^i \|y\| \tag{11.6}$$

for all $y \in \mathcal{C}([0, 1])$.

H4: For each i, there is $\kappa_0^i \geq 0$ such that

$$\lim_{z \to +\infty} \frac{|H_i(z) - \kappa_0^i z|}{|z|} = 0 \tag{11.7}$$

holds.

H5: We find that

$$\lim_{x+y \to +\infty} g_1(x, y) = +\infty \text{ and } \lim_{x+y \to +\infty} g_2(x, y) = +\infty. \tag{11.8}$$

H6: We find that

$$\lim_{x+y\to+\infty}\frac{g_1(x,y)}{x+y}=0 \text{ and } \lim_{x+y\to+\infty}\frac{g_2(x,y)}{x+y}=0. \tag{11.9}$$

H7: Each of the following holds.

$$\begin{aligned}
&0\leq \varepsilon_0^1+\varepsilon_0^2+\varepsilon_1^2+\varepsilon_1^2<\frac{1}{2}\\
&0\leq \kappa_0^1\left(\varepsilon_0^1+\varepsilon_0^2+\varepsilon_1^1\right)<\frac{1}{2}\\
&0\leq \kappa_0^2\left(\varepsilon_0^1+\varepsilon_0^2+\varepsilon_1^2\right)<\frac{1}{2}
\end{aligned} \tag{11.10}$$

H8: For each i, each of

$$\int_{[0,1]}(1-t)\ d\alpha_i(t)\geq 0 \tag{11.11}$$

and

$$\int_{[0,1]}G(t,s)\ d\alpha_i(t)\geq 0 \tag{11.12}$$

holds, where the latter holds for each $s\in[0,1]$.

H9: The nonlinearities g_1 and g_2 satisfy either the relationship $g_1(x,y)\leq g_2(x,y)$ or the relationship $g_2(x,y)\leq g_1(x,y)$, for all x, $y\geq 0$.

Let us make some brief remarks regarding certain of the preceding conditions.

Remark 11.1. Regarding conditions (H2)–(H3), the same sorts of functionals that have been admissible in Chapter 9 and 10 remain admissible here.

Remark 11.2. Regarding condition (H4), this is the asymptotic condition, which is key to our arguments in the sequel. Note that if the condition

$$\lim_{z\to+\infty}|H(z)-z|=0, \tag{11.13}$$

which implies that $H(z)$ converges to z at $+\infty$, holds, then it follows that condition (H4) holds, too. It should also be noted that there are many nontrivial functions which do not satisfy condition (11.4) but do satisfy condition (H4) for some κ_0^i. For instance, consider the function $H_1 : [0, +\infty) \to [0, +\infty)$ defined by

$$H_1(z) := 2\sqrt{z}\cos\left(\frac{1}{1+z}\right) \tag{11.14}$$

Then it is clear that H_1 satisfies (11.7) in case $\kappa_0^1 = 0$ but fails to satisfy the condition (11.13).

Remark 11.3. Note that in (11.10) above, depending upon the values of the various constants, it may be that each of conditions $(11.10)_2$ and $(11.10)_3$ is superfluous.

Remark 11.4. Observe that we do *not* require any growth conditions on H_i except asymptotically as given in (11.7) above. This is in contrast to nearly all other recent papers on BVPs with nonlinear, nonlocal boundary conditions – see, for instance, [64, 65, 66, 67, 68]. Indeed, as mentioned in Section 1, it seems to be assumed frequently that the function capturing the nonlinear aspect of the boundary conditions satisfy a condition of the sort $\alpha z \leq H_i(z) \leq \beta z$, for $0 \leq \alpha \leq \beta$ and all $z \geq 0$. Here we remove such restrictions entirely. Indeed, we only really need sublinearity at $+\infty$, and we consider this observation to be an interesting contribution of this work.

Remark 11.5. Observe that no growth conditions are required of either H_1 or H_2 at 0. In particular, $H_1(z)$ could be sublinear at $z = 0$, whilst $H_2(z)$ is superlinear at $z = 0$. In particular, the nonlinearities H_1, H_2 may exhibit mixed behavior at $z = 0$. The same comment may be given for the nonlinearities g_1 and g_2.

Now, let γ_0 be the constant defined by

$$\gamma_0 := \min\left\{\gamma, \min_{t\in E}(1-t)\right\}, \tag{11.15}$$

where $\gamma_0 \in (0,1)$ and γ is the constant from Chapter 9. Then the cone, $\mathcal{K}$, we shall use in the sequel is defined by

$$\begin{aligned}\mathcal{K} := \Bigg\{(x,y) \in \boldsymbol{\mathcal{X}}\ :\ & x, y \geq 0,\ \min_{t\in E}[x(t)+y(t)] \geq \gamma_0\|(x,y)\|,\\ & \phi_1(x), \phi_2(y) \geq 0\Bigg\},\end{aligned} \tag{11.16}$$

which is a simple modification of a cone first introduced by Infante and Webb [84]. Let us point out at this juncture that $\mathcal{K}$ is not just the trivial subspace of $\mathcal{X}$. Indeed, it is easy to verify that if we put $\boldsymbol{\beta}(t) := (1-t, 1-t)$, then $\boldsymbol{\beta} \in \mathcal{K}$. In fact, it is also true, of course, that if we put $\boldsymbol{\beta}_1(t) := (1-t, 0)$ and $\boldsymbol{\beta}_2(t) := (0, 1-t)$, then $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2 \in \mathcal{K}$. With this in hand, we now state and prove our main result.

Theorem 11.6. *Let conditions (H1)–(H9) hold. Assume that ξ_0^1, $\xi_0^2 \in E$, where the set E is fixed as in Section 2. Then for all λ_1, $\lambda_2 > 0$ sufficiently large problem (11.1) has at least one positive solution.*

Proof. We consider the problem

$$\begin{aligned}x''(t) &= -\lambda_1 a_1(t) g_1\left(x(t), y(t)\right),\ t \in (0,1)\\ y''(t) &= -\lambda_2 a_2(t) g_2\left(x(t), y(t)\right),\ t \in (0,1)\\ x(0) &= H_1\left(\phi_1\left(x_j\right) + \varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2 y\left(\xi_0^1\right)\right)\\ y(0) &= H_2\left(\phi_2\left(y_j\right) + \varepsilon_0^1 x\left(\xi_0^2\right) + \varepsilon_0^2 y\left(\xi_0^2\right)\right)\\ x(1) &= 0 = y(1).\end{aligned} \tag{11.17}$$

We first show that $S(\mathcal{K}) \subseteq \mathcal{K}$. To this end, let $(x,y) \in \mathcal{K}$. Then it is obvious that $T_i\left(x,y\right)(t) \geq 0$, for each $t \in [0,1]$ and for each $i = 1,2$. On the other hand, note that

$$\begin{aligned}\min_{t\in E} T_1\left(x,y\right)(t) &\geq \gamma_0 H_1\left(\phi_1\left(x\right)+\varepsilon_0^1 x\left(\xi_0^1\right)+\varepsilon_0^2 y\left(\xi_0^1\right)\right)\\ &\quad + \lambda_1\gamma \max_{t\in[0,1]}\int_0^1 G(t,s)a_1(s)g_1\left(x(s),y(s)\right)\ ds\\ &\geq \gamma_0\|T_1\left(x,y\right)\|.\end{aligned} \tag{11.18}$$

It similarly holds that $\min_{t\in E} T_2\left(x,y\right)(t) \geq \gamma_0\|T_2\left(x,y\right)\|$. We thus conclude that

$$\min_{t\in E}\left[\left(T_1\left(x,y\right)\right)(t)+\left(T_2\left(x,y\right)\right)(t)\right] \geq \gamma_0\|S\left(x,y\right)\|. \tag{11.19}$$

Finally, note that

$$\begin{aligned}\phi_1\left(T_1\left(x,y\right)\right) &= H_1\left(\phi_1\left(x\right)+\varepsilon_0^1 x\left(\xi_0^1\right)+\varepsilon_0^2 y\left(\xi_0^1\right)\right)\int_{[0,1]}(1-t)\ d\alpha_1(t)\\ &\quad + \lambda_1\int_{[0,1]}\int_0^1 G(t,s)a_1(s)g_1\left(x(s),y(s)\right)\ ds\ d\alpha_1(t)\\ &= H_1\left(\phi_1\left(x\right)+\varepsilon_0^1 x\left(\xi_0^1\right)+\varepsilon_0^2 y\left(\xi_0^1\right)\right)\int_{[0,1]}(1-t)\ d\alpha_1(t)\\ &\quad + \lambda_1\int_0^1\left[\int_{[0,1]} G(t,s)\ d\alpha_1(t)\right]a_1(s)g_1\left(x(s),y(s)\right)\ ds\\ &\geq 0,\end{aligned} \tag{11.20}$$

where the final inequality from assumption (H8). Similarly, $\phi_2\left(T_2\left(x,y\right)\right) \geq 0$. Thus, $S : \mathcal{K} \to \mathcal{K}$, as claimed. Furthermore, since it is standard to show that S is a completely continuous operator, we omit the proof of this claim.

We next make a simple observation. For each $(x,y) \in \mathcal{K}$, we have that

$$\min_{t\in E}[x(t)+y(t)] \geq \gamma_0\|(x,y)\|, \tag{11.21}$$

and, thus, since $\phi_1(x) \geq 0$ it follows that

$$\begin{aligned}\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2 y\left(\xi_0^1\right) &\geq \varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2 y\left(\xi_0^1\right) \\ &\geq \min\left\{\varepsilon_0^1, \varepsilon_0^2\right\} \min_{t\in E}\left[x(t)+y(t)\right] \\ &\geq \min\left\{\varepsilon_0^1, \varepsilon_0^2\right\} \gamma_0 \|(x,y)\|.\end{aligned} \tag{11.22}$$

Of course, the same inequality holds if we replace $\phi_1(x)$ with $\phi_2(y)$, $x\left(\xi_0^1\right)$ with $x\left(\xi_0^2\right)$, and $y\left(\xi_0^1\right)$ with $y\left(\xi_0^2\right)$. In any case, observation (11.22) will be very important in the sequel.

Now, we note that by condition (H5), there is $r_1 > 0$ sufficiently large such that whenever $x + y \geq r_1$, we find that

$$g_1(x,y) \geq \frac{1}{\int_E G\left(t_0, s\right) a_1(s)\ ds}, \tag{11.23}$$

where $t_0 \in \mathring{E}$ is any fixed but otherwise arbitrary point; note that since $E \Subset (0,1)$, it follows that $0 < t_0 < 1$. Similarly, there is $r_1^* > 0$ such that for $x + y \geq r_1^*$, it follows that

$$g_2(x,y) \geq \frac{1}{\int_E G\left(t_0, s\right) a_2(s)\ ds}. \tag{11.24}$$

Define the number $r_1^{**} > 0$ by

$$r_1^{**} := \max\left\{\frac{r_1}{\gamma_0}, \frac{r_1^*}{\gamma_0}\right\} \tag{11.25}$$

and the set $\Omega_{r_1^{**}} \subset \mathcal{X}$ by

$$\Omega_{r_1^{**}} := \left\{(x,y) \in \mathcal{X}\ :\ \|(x,y)\| < r_1^{**}\right\}. \tag{11.26}$$

Observe that for $(x, y) \in \mathcal{K} \cap \partial\Omega_{r_1^{**}}$ it follows that

$$\min_{t \in E} [x(t) + y(t)] \geq \gamma_0 \| (x, y) \| = \gamma_0 r_1^{**} = \max \{r_1, r_1^*\}. \tag{11.27}$$

In particular, both (11.23) and (11.24) hold. Therefore, it follows that for each $(x, y) \in \mathcal{K} \cap \partial\Omega_{r_1^{**}}$ we have

$$\begin{aligned} T_1 (x, y) (t_0) &\geq \lambda_1 \int_0^1 G (t_0, s) \, a_1(s) g_1 (x(s), y(s)) \, ds \\ &\geq \lambda_1 \int_E G (t_0, s) \, a_1(s) g_1 (x(s), y(s)) \, ds \\ &\geq \lambda_1, \end{aligned} \tag{11.28}$$

where we have used the fact that $H(z) \geq 0$, for each $z \geq 0$. By now making λ_1 sufficiently large, we get

$$\|T_1 (x, y) \| \geq \frac{1}{2} \| (x, y) \|. \tag{11.29}$$

Similarly, by making λ_2 sufficiently large we deduce that

$$\|T_2 (x, y) \| \geq \frac{1}{2} \| (x, y) \|. \tag{11.30}$$

So, from (11.29)–(11.30) we conclude that

$$\|S (x, y) \| \geq \| (x, y) \|, \tag{11.31}$$

for each $(x, y) \in \mathcal{K} \cap \partial\Omega_{r_1^{**}}$.

On the other hand, select numbers ε_2^1, $\varepsilon_2^2 > 0$ sufficiently small such that each of

the following inequalities holds.

$$\begin{aligned} \kappa_0^1\left(\varepsilon_0^1+\varepsilon_0^2+\varepsilon_1^1\right)+\varepsilon_2^1 &< \frac{1}{2} \\ \kappa_0^1\left(\varepsilon_0^1+\varepsilon_0^2+\varepsilon_1^2\right)+\varepsilon_2^2 &< \frac{1}{2} \end{aligned} \tag{11.32}$$

Evidently, these inequalities may be satisfied because condition (H7) holds. Then condition (H6) implies that there is $r_2>0$ sufficiently large such that for each $i=1,2$ it holds that

$$g_i(x,y) \leq \eta_1(x+y), \tag{11.33}$$

whenever $x+y \geq r_2$, where η_1 satisfies both

$$\eta_1 \int_0^1 G(s,s)a_1(s)\ ds \leq \frac{\varepsilon_2^1}{2\lambda_1} \text{ and } \eta_1 \int_0^1 G(s,s)a_2(s)\ ds \leq \frac{\varepsilon_2^2}{2\lambda_2}. \tag{11.34}$$

Additionally, for a given number $\varepsilon_3^1>0$ condition (H4) implies the existence of a number $r_2^* := r_2^*\left(\varepsilon_3^1\right)>0$ such that

$$\begin{aligned} &\left|H_1\left(\phi_1(x)+\varepsilon_0^1 x\left(\xi_0^1\right)+\epsilon_0^2 y\left(\xi_0^1\right)\right)-\kappa_0^1\left(\phi_1(x)+\varepsilon_0^1 x\left(\xi_0^1\right)+\varepsilon_0^2 y\left(\xi_0^1\right)\right)\right| \\ &< \varepsilon_3^1 \|(x,y)\| \end{aligned} \tag{11.35}$$

whenever

$$\phi_1(x)+\varepsilon_0^1 x\left(\xi_0^1\right)+\varepsilon_0^2 y\left(\xi_0^1\right) \geq r_2^*. \tag{11.36}$$

Note that to get (11.35) we have used the fact that

$$\begin{aligned} &0 \leq \phi_1(x)+\varepsilon_0^1 x\left(\xi_0^1\right)+\varepsilon_0^2 y\left(\xi_0^1\right) \\ &\quad \leq \varepsilon_1^1\|x\|+\varepsilon_0^1\|x\|+\varepsilon_0^2\|y\| \leq \max\left\{\varepsilon_0^1,\varepsilon_0^2,\varepsilon_1^1\right\}\|(x,y)\| < \|(x,y)\|. \end{aligned} \tag{11.37}$$

Furthermore, in the same manner as in the preceding paragraph, we may select ε_3^1 in

such a way so that

$$\kappa_0^1\left(\varepsilon_0^1+\varepsilon_0^2+\varepsilon_1^1\right)+\varepsilon_2^1+\varepsilon_3^1<\frac{1}{2} \tag{11.38}$$

holds. In any case, recalling (11.22) and the fact that $\phi_1(x)\geq 0$ since $(x,y)\in\mathcal{K}$, we have that (11.35) is satisfied provided that

$$\|\,(x,y)\,\|\geq\frac{r_2^*}{\gamma_0\min\{\varepsilon_0^1,\varepsilon_0^2\}} \tag{11.39}$$

holds. A dual argument reveals that (11.35) also holds for the function H_2 whenever (11.39) holds replacing r_2^* with some (possibly larger) constant r_2^{**}, by making the obvious changes in the various subscripts appearing in (11.35)–(11.36), and changing ε_3^1 to some ε_3^2 – i.e., provided that

$$\|\,(x,y)\,\|\geq\frac{r_2^{**}}{\gamma_0\min\{\varepsilon_0^1,\varepsilon_0^2\}} \tag{11.40}$$

holds. Here, of course, analogous to (11.38) we choose ε_3^2 so that

$$\kappa_0^1\left(\varepsilon_0^1+\varepsilon_0^2+\varepsilon_1^2\right)+\varepsilon_2^2+\varepsilon_3^2<\frac{1}{2} \tag{11.41}$$

is satisfied. So, both conditions hold provided that

$$\|\,(x,y)\,\|\geq\max\left\{\frac{r_2^*}{\gamma_0\min\{\varepsilon_0^1,\varepsilon_0^2\}},\frac{r_2^{**}}{\gamma_0\min\{\varepsilon_0^1,\varepsilon_0^2\}}\right\}. \tag{11.42}$$

Now, assume by condition (H9) and without loss of generality that $g_2(x,y)\leq g_1(x,y)$, for all x, $y\geq 0$. Then because g_1 is unbounded at infinity in the sense of condition (H5), we may select a number $R_1>0$, where R_1 satisfies

$$R_1>\max\left\{2r_1^{**},r_2,\frac{r_2^*}{\gamma_0\min\{\varepsilon_0^1,\varepsilon_0^2\}},\frac{r_2^{**}}{\gamma_0\min\{\varepsilon_0^1,\varepsilon_0^2\}}\right\}=:\vartheta, \tag{11.43}$$

such that

$$g_1(x,y) \leq g_1(\rho_1, \rho_2), \tag{11.44}$$

for all $(x,y) \in [0,R_1] \times [0,R_1]$, where either $\rho_1 = R_1$ and $0 \leq \rho_2 \leq R_1$ or $0 \leq \rho_1 \leq R_1$ and $\rho_2 = R_1$. To prove this claim, pick a number $\theta^* > 0$ such that

$$\theta^* > \vartheta. \tag{11.45}$$

By the extreme value theorem, the function g_1 attains its maximum on the square $[0,\theta^*] \times [0,\theta^*]$, say

$$\max_{(x,y)\in[0,\theta^*]\times[0,\theta^*]} g_1(x,y) = g_1(x_0, y_0). \tag{11.46}$$

Now, if

$$(x_0, y_0) \in [0,\theta^*]^2 \setminus [0,\vartheta]^2 \tag{11.47}$$

holds, then we may put $R_1 := \max\{x_0, y_0\}$; for instance, if $x_0 > y_0$, then $\rho_1 = x_0 = R_1$ and $\rho_2 = y_0 \leq R_1$. On the other hand, if (11.46) is not true, then because of condition (H5), there must be a number $h > 0$ sufficiently large and a point (x_1, y_1) satisfying

$$(x_1, y_1) \in [0,\theta^* + h]^2 \setminus [0,\vartheta]^2 \tag{11.48}$$

such that $g_1(x_1, y_1) \geq g_1(x_0, y_0)$. In this case, put $R_1 := \max\{x_1, y_1\}$, with $\vartheta < R_1 \leq \theta^* + h$. We then have that, say, $\rho_1 := \max\{x_1, y_1\}$ and $\rho_2 := \min\{x_1, y_1\}$. We conclude, therefore, that we can always construct a square $[0,R_1] \times [0,R_1]$ with R_1 chosen sufficiently large such that either

$$g_1(x,y) \leq g_1(R_1, \rho_2) \tag{11.49}$$

holds for some $0 \leq \rho_2 \leq R_1$ or

$$g_1(x, y) \leq g_1(\rho_1, R_1) \tag{11.50}$$

holds for some $0 \leq \rho_1 \leq R_1$, and such that R_1 satisfies the inequality

$$R_1 > \vartheta. \tag{11.51}$$

Notice, then, for x, $y \leq R_1$, it follows that if in (11.44) we have that $\rho_1 = R_1$ and $0 \leq \rho_2 \leq R_1$, then for all $(x, y) \in [0, R_1] \times [0, R_1]$ it holds that

$$g_1(x, y) \leq g_1(\rho_1, \rho_2) = g_1(R_1, \rho_2) \leq \eta_1(R_1 + \rho_2) \leq 2\eta_1 R_1, \tag{11.52}$$

where the second-to-last inequality follows from invoking (3.30), which is valid since $R_1 > \vartheta$, whence $R_1 + \rho_2 \geq r_2$. On the other hand, if $0 \leq \rho_1 \leq R_1$ and $\rho_2 = R_1$ in (11.33), then inequality (11.52) still holds. Inequality (11.52) is the key observation, for we observe that if $\|(x, y)\| = R_1$, then

$$g_1(x(t), y(t)) \leq 2\eta_1 R_1 \tag{11.53}$$

holds for $t \in [0, 1]$. Since, by assumption, $g_2(x, y) \leq g_1(x, y)$ for each x, $y \geq 0$, it also follows that for $\|(x, y)\| = R_1$ the inequality

$$g_2(x(t), y(t)) \leq 2\eta_1 R_1 \tag{11.54}$$

holds.

So, let R_1 be the number constructed in the previous paragraph. Define the set

Ω_{R_1} by

$$\Omega_{R_1} := \{(x,y) \in \mathcal{X} \ : \ \|(x,y)\| < R_1\}. \tag{11.55}$$

Then for each $(x,y) \in \mathcal{K} \cap \partial\Omega_{R_1}$ we find that

$$\begin{aligned}
&\|T_1(x,y)\| \\
&\leq H_1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2 y\left(\xi_0^1\right)\right) + \lambda_1 \int_0^1 G(s,s)a_1(s)g_1(x(s),y(s))\ ds \\
&\leq \Big| H_1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right)\ ds + \varepsilon_0^2 y\left(\xi_0^1\right)\right) \\
&- \kappa_0^1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2 y\left(\xi_0^1\right)\right)\Big| + \kappa_0^1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2 y\left(\xi_0^1\right)\right) \\
&+ \lambda_1 \int_0^1 G(s,s)a_1(s)g_1(x(s),y(s))\ ds \\
&\leq \varepsilon_3^1\|(x,y)\| + \kappa_0^1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2 y\left(\xi_0^1\right)\right) \\
&+ \lambda_1 \int_0^1 G(s,s)a_1(s)2\eta_1 R_1\ ds \\
&\leq \varepsilon_3^1\|(x,y)\| + \kappa_0^1\left(\varepsilon_1^1\|x\| + \varepsilon_0^1\|x\| + \varepsilon_0^2\|y\|\right) \\
&+ 2\eta_1 R_1 \lambda_1 \int_0^1 G(s,s)a_1(s)\ ds \\
&\leq \varepsilon_3^1\|(x,y)\| + \kappa_0^1\left(\varepsilon_0^1 + \varepsilon_0^2 + \varepsilon_1^1\right)\|(x,y)\| + \varepsilon_2^1 R_1 \\
&= \left(\varepsilon_3^1 + \kappa_0^1\left(\varepsilon_0^1 + \varepsilon_0^2 + \varepsilon_1^1\right) + \varepsilon_2^1\right)\|(x,y)\| \\
&< \frac{1}{2}\|(x,y)\|,
\end{aligned} \tag{11.56}$$

where we have used the fact that

$$0 \leq \varepsilon_3^1 + \kappa_0^1\left(\varepsilon_0^1 + \varepsilon_0^2 + \varepsilon_1^1\right) + \varepsilon_2^1 < \frac{1}{2} \tag{11.57}$$

by construction. Similarly, we estimate

$$\|T_2(x,y)\| \leq \frac{1}{2}\|(x,y)\|, \tag{11.58}$$

for each $(x,y) \in \mathcal{K} \cap \partial\Omega_{R_1}$. Consequently, from (11.56) and (11.58) we conclude that

$$\|S\,(x,y)\,\| \leq \|\,(x,y)\,\|, \tag{11.59}$$

for each $(x,y) \in \mathcal{K} \cap \partial\Omega_{R_1}$.

Putting the preceding part of the proof together, we see that may thus invoke Lemma 2.13 to deduce the existence of a function

$$(x_0, y_0) \in \mathcal{K} \cap \left(\overline{\Omega}_{R_1} \setminus \Omega_{r_1^{**}}\right) \tag{11.60}$$

such that $S\,(x_0, y_0) = (x_0, y_0)$. The functions $x_0(t)$ and $y_0(t)$ from (11.60) represent a positive solution to problem (1.1); in fact, it satisfies the *a priori* bounds

$$0 < r_1^{**} \leq \|\,(x_0, y_0)\,\| \leq R_1 < +\infty. \tag{11.61}$$

Thus, in particular, we have shown that problem (11.1) has at least one positive solution. And this completes the proof. □

Remark 11.7. Although not explicitly stated in either the statement or the proof of Theorem 11.6, it is possible to write an explicit formula for the admissible range of the eigenvalues, λ_1 and λ_2. In particular, put

$$\begin{aligned}\alpha_1 := \frac{1}{\gamma_0} \inf \Bigg\{ & y \in [0, +\infty) \;:\; g_1\,(z_1, z_2) \geq \left[\int_E G\,(t_0, s)\,a_1(s)\;ds\right]^{-1}, \\ & \text{for all } z_1 + z_2 \in [y, +\infty) \Bigg\}\end{aligned} \tag{11.62}$$

and

$$\alpha_2 := \frac{1}{\gamma_0} \inf \Bigg\{ y \in [0, +\infty) \ : \ g_2(z_1, z_2) \geq \left[\int_E G(t_0, s)\, a_2(s)\, ds \right]^{-1}, \\ \text{for all } z_1 + z_2 \in [y, +\infty) \Bigg\}. \tag{11.63}$$

Now, define α_0 by

$$\alpha_0 := \frac{1}{2\gamma_0} \max\{\alpha_1, \alpha_2\}. \tag{11.64}$$

Then it follows that whenever

$$\lambda_1,\ \lambda_2 \in [\alpha_0, +\infty) \tag{11.65}$$

we have that the pair λ_1, λ_2 is a pair of admissible eigenvalues for problem (11.1). In particular, (11.62)–(11.63) demonstrate that the range of admissible eigenvalues for problem (11.1) is explicitly computable.

We next state our second existence theorem, which provides an alternative approach to problem (11.1). Indeed, as intimated in the introduction to this chapter, here we give up the assumption that H need not be monotone increasing. In return, however, we are able to recover an existence theorem for the *unperturbed* problem (11.1) – i.e., the case in which $\varepsilon_0^1 = \varepsilon_0^2 = 0$. Moreover, we may still retain the other upshots of Theorem 11.6 such as the fact that the measures μ_{α_1} and μ_{α_2} are possibly signed and that H need only be asymptotically sublinear.

Theorem 11.8. *Suppose that conditions (H1)–(H9) hold but with $\kappa_0^i = 0$ for each i in condition (H4). In addition, assume that each of $H_1(z)$ and $H_2(z)$ is a monotone increasing function for all $z \geq 0$. Let $\varepsilon_0^1 = \varepsilon_0^2 = 0$. Then problem (11.1) has at least one positive solution.*

Proof. As in the proof of Theorem 11.6, the operator S is completely continuous and satisfies $S(\mathcal{K}) \subseteq \mathcal{K}$. So, since these facts still hold, we need only show that S has at least one nontrivial fixed point in $\mathcal{K}$.

To this end, observe that the first part of the proof of Theorem 11.6 may be repeated *verbatim* in spite of the fact that $\varepsilon_0^1 = \varepsilon_0^2 = 0$ here. Indeed, this is because estimate (11.22) was not used in the first part of the proof of Theorem 11.6 but rather only in the second part. In any case, in the same exact way as in the proof of Theorem 11.6, we arrive at a number r_1^{**} such that inequality (11.31) holds for each $(x, y) \in \mathcal{K} \cap \partial\Omega_{r_1^{**}}$, provided that the numbers λ_1, λ_2 are chosen sufficiently large, say according to (11.65).

We next diverge somewhat with respect to the proof of Theorem 11.6. Indeed, because each of H_1 and H_2 is monotone increasing by assumption, by means of assumption (H3) we may estimate both

$$H_1\left(\phi_1(x)\right) \leq H_1\left(\varepsilon_1^1\|x\|\right) \leq H_1\left(\varepsilon_1^1\|(x, y)\|\right) \tag{11.66}$$

and

$$H_2\left(\phi_2(y)\right) \leq H_2\left(\varepsilon_1^2\|y\|\right) \leq H_2\left(\varepsilon_1^2\|(x, y)\|\right), \tag{11.67}$$

for each each $(x, y) \in \mathcal{K}$. Next, as in the proof of Theorem 11.6, we may assume that inequalities (11.33)–(11.34) hold whenever $x + y \geq r_2$. Moreover, by assumption (H4) with $\kappa_0^i = 0$, there is a number $r_2^* > 0$ sufficiently large such that, for each i,

$$H_i(z) \leq z, \tag{11.68}$$

provided that $z \geq r_2^*$. Now, define the number r_2^{**} by

$$r_2^{**} := \max\left\{2r_1^{**}, r_2, \frac{r_2^*}{\min\left\{\varepsilon_1^1, \varepsilon_1^2\right\}}\right\}. \tag{11.69}$$

Note that for $\|(x,y)\| = r_2^{**}$, by means of (11.66) and (11.68) we may thus estimate

$$H_1\left(\phi_1(x)\right) \leq H_1\left(\varepsilon_1^1\|(x,y)\|\right) = H_1\left(\varepsilon_1^1 r_2^{**}\right) \leq \varepsilon_1^1 r_2^{**} \tag{11.70}$$

since

$$\varepsilon_1^1 r_2^{**} \geq \frac{\varepsilon_1^1 r_2^*}{\min\left\{\varepsilon_1^1, \varepsilon_1^2\right\}} \geq r_2^*. \tag{11.71}$$

Reasoning similarly, we also deduce the estimate

$$H_2\left(\phi_2(y)\right) \leq \varepsilon_1^2 r_2^{**} \tag{11.72}$$

whenever $\|(x,y)\| = r_2^{**}$. Finally, we may assume that r_2^{**} is chosen sufficiently large such that inequality (11.53) and hence inequality (11.54) hold for the number r_2^{**}.

Now, define the set $\Omega_{r_2^{**}} \subseteq \mathfrak{X}$ by

$$\Omega_{r_2^{**}} := \left\{(x,y) \in \mathfrak{X} \ : \ \|(x,y)\| < r_2^{**}\right\}. \tag{11.73}$$

Then, for each $(x, y) \in \mathcal{K} \cap \partial\Omega_{r_2^{**}}$, similar to inequality (11.56) we estimate

$$\begin{aligned} \|T_1(x,y)\| &\le H_1\left(\phi_1(x)\right) + \lambda_1 \int_0^1 G(s,s)a_1(s)g_1(x(s),y(s))\ ds \\ &\le \varepsilon_1^1 r_2^{**} + \lambda_1 \int_0^1 G(s,s)a_1(s)g_1(x(s),y(s))\ ds \\ &\le \varepsilon_1^1 r_2^{**} + 2\eta_1 r_2^{**}\lambda_1 \int_0^1 G(s,s)a_1(s)\ ds \\ &\le \varepsilon_1^1 r_2^{**} + \varepsilon_2^1 r_2^{**} \\ &= \left(\varepsilon_1^1 + \varepsilon_2^1\right)\|(x,y)\| \\ &\le \frac{1}{2}\|(x,y)\|. \end{aligned} \tag{11.74}$$

In a completely similar manner, we deduce that

$$\|T_2(x,y)\| \le \frac{1}{2}\|(x,y)\|, \tag{11.75}$$

whence

$$\|S(x,y)\| \le \|(x,y)\|, \tag{11.76}$$

for each $(x, y) \in \mathcal{K} \cap \partial\Omega_{r_2^{**}}$.

Consequently, we may invoke Lemma 2.13 to deduce the existence of a fixed point $(x_0, y_0) \in \mathcal{K} \cap \left(\overline{\Omega}_{r_2^{**}} \setminus \Omega_{r_1^{**}}\right)$ of the operator S. And this completes the proof. □

Remark 11.9. We note that inequality (11.74) reveals that the slightly weaker condition

$$0 \le \max\left\{\varepsilon_1^1, \varepsilon_1^2\right\} < \frac{1}{2} \tag{11.77}$$

may replace the slightly stronger hypothesis (H7) in the statement of Theorem 11.8. In this way, it is unnecessary to assume that $\varepsilon_1^1 + \varepsilon_1^2 \in \left[0, \frac{1}{2}\right)$ since as long as inequality (11.77) holds, we may always choose $\varepsilon_2^i > 0$ sufficiently small such that $\varepsilon_1^i + \varepsilon_2^i \in \left(0, \frac{1}{2}\right]$.

However, we omit the statement of this slightly more general result.

We conclude with an explicit numerical example, which explicates the use of Theorem 11.6, together with some final remarks.

Example 11.10. Consider the boundary value problem

$$\begin{aligned}
x''(t) &= -\lambda_1 \left(e^t - 1\right) \left(\sqrt{x+y} + 2\right) \\
y''(t) &= -\lambda_2 \left(t^2 + 1\right) \sqrt{x+y} \\
x(0) &= H_1 \left(\phi_1(x) + \frac{1}{200} x\left(\frac{2}{5}\right) + \frac{1}{30} y\left(\frac{2}{5}\right)\right) \\
y(0) &= H_2 \left(\phi_2(y) + \frac{1}{200} x\left(\frac{2}{5}\right) + \frac{1}{30} y\left(\frac{2}{5}\right)\right) \\
x(1) &= 0 = y(1),
\end{aligned} \tag{11.78}$$

where we make the following declarations.

$$\begin{aligned}
\phi_1(x) &:= \frac{1}{60} x\left(\frac{1}{3}\right) - \frac{1}{200} x\left(\frac{2}{5}\right) - \frac{1}{120} x\left(\frac{3}{5}\right) + \frac{1}{20} \int_{\left[\frac{13}{20}, \frac{3}{4}\right]} x(s)\ ds \\
\phi_2(y) &:= \frac{1}{6} y\left(\frac{3}{10}\right) - \frac{1}{30} y\left(\frac{2}{5}\right) - \frac{1}{15} y\left(\frac{11}{20}\right) + \frac{2}{3} \int_{\left[\frac{3}{5}, \frac{7}{10}\right]} y(s)\ ds \\
H_1(z) &:= z \cos\left(\frac{1}{z+1}\right) \\
H_2(z) &:= z^{\frac{1}{3}} + z
\end{aligned} \tag{11.79}$$

Obviously, each of H_1 and H_2 satisfies condition (H4) with $\kappa_0^1 = \kappa_0^2 = 1$. Moreover, it is clear that each of $g_1(x, y) := \sqrt{x+y} + 2$ and $g_2(x, y) := \sqrt{x+y}$ satisfies conditions

(H5), (H6), and (H9). Incidentally, we remark that if we define $\alpha_1,\, \alpha_2 \,:\, \mathbb{R} \to \mathbb{R}$ by

$$\alpha_1(t) := \begin{cases} 0, & t < \frac{1}{3} \\ \frac{1}{60}, & \frac{1}{3} \leq t < \frac{2}{5} \\ \frac{7}{600}, & \frac{2}{5} \leq t < \frac{3}{5} \\ \frac{1}{300}, & \frac{3}{5} \leq t < \frac{13}{20} \\ t - \frac{97}{150}, & \frac{13}{20} \leq t < \frac{3}{4} \\ \frac{31}{300}, & t \geq \frac{3}{4} \end{cases} \tag{11.80}$$

and

$$\alpha_2(t) := \begin{cases} 0, & t < \frac{3}{10} \\ \frac{1}{6}, & \frac{3}{10} \leq t < \frac{2}{5} \\ \frac{2}{15}, & \frac{2}{5} \leq t < \frac{11}{20} \\ \frac{1}{15}, & \frac{11}{20} \leq t < \frac{3}{5} \\ t - \frac{8}{15}, & \frac{3}{5} \leq t < \frac{7}{10} \\ \frac{1}{6}, & t \geq \frac{7}{10} \end{cases}, \tag{11.81}$$

then we may write

$$\phi_1(x) := \int_{[0,1]} x(s)\ d\alpha_1(s) \text{ and } \phi_2(y) := \int_{[0,1]} y(s)\ d\alpha_2(s), \tag{11.82}$$

where the unique Borel measures associated to the Lebesgue-Stieltjes integrals in

(11.82) are

$$\begin{aligned}\mu_{\alpha_1}((-\infty,t]) :=&\frac{1}{60}\delta_{\frac{1}{3}}((-\infty,t])-\frac{1}{200}\delta_{\frac{2}{5}}((-\infty,t])\\&-\frac{1}{120}\delta_{\frac{3}{5}}((-\infty,t])+\frac{1}{20}m\left((-\infty,t]\cap\left(\frac{13}{20},\frac{3}{4}\right)\right)\end{aligned}\tag{11.83}$$

and

$$\begin{aligned}\mu_{\alpha_2}((-\infty,t]) :=&\frac{1}{6}\delta_{\frac{3}{10}}((-\infty,t])-\frac{1}{30}\delta_{\frac{2}{5}}((-\infty,t])\\&-\frac{1}{15}\delta_{\frac{11}{20}}((-\infty,t])+\frac{2}{3}m\left((-\infty,t]\cap\left(\frac{3}{5},\frac{7}{10}\right)\right),\end{aligned}\tag{11.84}$$

respectively. Importantly, we observe that each of the measures μ_{α_1} and μ_{α_2} is *signed*.

Now, it is easy to check numerically that condition (H8) holds. Furthermore, we may select $\varepsilon_1^1 := \frac{7}{200}$ here and $\varepsilon_1^2 := \frac{1}{3}$ here; for instance, we observe both that

$$\phi_1(x)\le\frac{1}{60}\|x\|+\frac{1}{200}\|x\|+\frac{1}{120}\|x\|+\frac{1}{20}\left[\frac{3}{4}-\frac{13}{20}\right]\|x\|\le\frac{7}{200}\|x\|\tag{11.85}$$

and that

$$\phi_2(y)\le\frac{1}{6}\|y\|+\frac{1}{30}\|y\|+\frac{1}{15}\|y\|+\frac{2}{3}\left[\frac{7}{10}-\frac{3}{5}\right]\|y\|\le\frac{1}{3}\|y\|.\tag{11.86}$$

Since, in addition, $\varepsilon_0^1 = \frac{1}{200}$ and $\varepsilon_0^2 = \frac{1}{30}$, it follows that condition (H7) holds, too. Moreover, we find that

$$\int_{[0,1]}1-t\;d\alpha_1(t)=\frac{87}{900}\text{ and }\int_{[0,1]}1-t\;d\alpha_2(t)=\frac{9}{100}.\tag{11.87}$$

Since the remaining conditions clearly hold, it follows that problem (11.78) has at least one positive solution. Finally, we remark that problem (11.78) may be recast in

the form

$$\begin{aligned}
x''(t) &= -\lambda_1\left(e^t-1\right)\left(\sqrt{x+y}+2\right)\\
y''(t) &= -\lambda_2\left(t^2+1\right)\sqrt{x+y}\\
x(0) &= \left[\psi_1(x)+\frac{1}{30}y\left(\frac{2}{5}\right)\right]\cos\left(\frac{1}{1+\psi_1(x)+\frac{1}{30}y\left(\frac{2}{5}\right)}\right)\\
y(0) &= \left[\psi_2(y)+\frac{1}{200}x\left(\frac{2}{5}\right)\right]^{\frac{1}{3}}+\left[\psi_2(y)+\frac{1}{200}x\left(\frac{2}{5}\right)\right]\\
x(1) &= 0 = y(1),
\end{aligned} \tag{11.88}$$

if we put

$$\psi_1(x) := \frac{1}{60}x\left(\frac{1}{3}\right)-\frac{1}{120}x\left(\frac{3}{5}\right)+\frac{1}{20}\int_{\left[\frac{13}{20},\frac{3}{4}\right]} x(s)\ ds \tag{11.89}$$

and

$$\psi_2(y) := \frac{1}{6}y\left(\frac{3}{10}\right)-\frac{1}{15}y\left(\frac{11}{20}\right)+\frac{2}{3}\int_{\left[\frac{3}{5},\frac{7}{10}\right]} y(s)\ ds \tag{11.90}$$

Remark 11.11. We note that problem (11.78) could not be addressed by any existing results in the literature. This is due to several reasons, among which are the following. Firstly, as (11.83)–(11.84) demonstrate, each of the measures μ_{α_1} and μ_{α_2} is signed; contenting ourselves with the papers on systems with nonlocal, nonlinear boundary conditions, this removes from straightforward modification the results of [65, 66, 90]. Secondly, since

$$H_2'(z) = \frac{1}{3}z^{-\frac{2}{3}}+1, \tag{11.91}$$

it is clear that there is no $\beta \in \mathbb{R}$ satisfying $+\infty > \beta > 0$ such that $H_2(z) \leq \beta z$, for all $z \geq 0$. Thus, in particular, the results of [65] (and related works) cannot straightforwardly modified. In summary, the fact that the measures are signed rather than positive and that H_2 does not satisfy uniform linear growth seems to remove from consideration any simple modification of the existing results in the literature.

Remark 11.12. We have elected not to give an explicit example of Theorem 11.8. However, we emphasize that this theorem recovers at least one positive solution to the *unperturbed* problem, namely

$$\begin{aligned}
x''(t) &= -\lambda_1 a_1(t) g_1(x(t), y(t)),\ t \in (0,1)\\
y''(t) &= -\lambda_2 a_2(t) g_2(x(t), y(t)),\ t \in (0,1)\\
x(0) &= H_1\left(\phi_1(x)\right)\\
y(0) &= H_2\left(\phi_2(y)\right)\\
x(1) &= 0 = y(1).
\end{aligned} \tag{11.92}$$

It ought to be noted that problem (11.92) is very nearly the problem studied by Infante and Pietramala [65] as well as Yang [90]. Consequently, we have here obtained a direct generalization and improvement of their results.

Bibliography

[1] O. P. Agrawal, *Formulation of Euler-Lagrange equations for fractional variational problems*, J. Math. Anal. Appl. **272** (2002), 368–379.

[2] R. Agarwal, M. Meehan, and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge, 2001.

[3] R. P. Agarwal, V. Lakshmikantham, and J. J. Nieto, *On the concept of solution for fractional differential equations with uncertainty*, Nonlinear Anal. **72** (2009), 2859–2862.

[4] R. Almeida and D. F. M. Torres, *Calculus of variations with fractional derivatives and fractional integrals*, Appl. Math. Lett. **22** (2009), 1816–1820.

[5] D. R. Anderson, *Solutions to second-order three-point problems on time scales*, J. Difference Equ. Appl. **8** (2002), 673–688.

[6] D. R. Anderson and J. Hoffacker, *A stacked delta-nabla self-adjoint problem of even order*, Math. Comput. Modelling **38** (2003), 481–494.

[7] D. R. Anderson and R. I. Avery, *An even-order three-point boundary value problem on time scales*, J. Math. Anal. Appl. **291** (2004), 514–525.

[8] D. R. Anderson and J. Hoffacker, *Even order self adjoint time scale problems*, Electron. J. Differential Equations (2005), no. 24, 9 pp.

[9] D. R. Anderson and C. C. Tisdell, *Third-order nonlocal problems with sign-changing nonlinearity on time scales*, Electron. J. Differential Equations (19) (2007), pp. 1–12.

[10] D. R. Anderson, *Existence of solutions for first-order multi-point problems with changing-sign nonlinearity*, J. Difference Equ. Appl. **14** (2008), 657–666.

[11] D. R. Anderson, C. Zhai, *Positive solutions to semi-positone second-order three-point problems on time scales*, Appl. Math. Comput. **215** (2010), 3713–3720.

[12] D. R. Anderson and J. Hoffacker, *Existence of solutions for a cantilever beam problem*, J. Math. Anal. Appl. **323** (2006), 958–973.

[13] A. Arara, et al., *Fractional order differential equations on an unbounded domain*, Nonlinear Anal. **72** (2010), 580–586.

[14] F. M. Atici and G. Guseinov, *On Green's functions and positive solutions for boundary value problems on time scales*, J. Comput. Appl. Math. **141** (2002), 75–99.

[15] F. M. Atici and P. W. Eloe, *Fractional q-calculus on a time scale*, J. Nonlinear Mathematical Physics, **14** (2007), 333–344.

[16] F. M. Atici and P. W. Eloe, *A transform method in discrete fractional calculus*, Int. J. of Difference Equ., **2** (2007), 165–176.

[17] F. M. Atici and P. W. Eloe, *Initial value problems in discrete fractional calculus*, Proc. Amer. Math. Soc., **137** (2009), 981–989.

[18] F. M. Atici and P. W. Eloe, *Discrete fractional calculus with the nabla operator*, Electron. J. Qual. Theory Differ. Equ. (Spec. Ed. I) **3** (2009), 1–12.

[19] F. M. Atici and S. Şengül, *Modeling with fractional difference equations*, J. Math. Anal. Appl. **369** (2010), 1–9.

[20] F. M. Atici and P. W. Eloe, *Two-point boundary value problems for finite fractional difference equations*, J. Difference Equ. Appl. **17** (2011), 445–456.

[21] A. Babakhani and V. Daftardar-Gejji, *Existence of positive solutions of nonlinear fractional differential equations*, J. Math. Anal. Appl. **278** (2003), 434–442.

[22] Z. Bai, *On positive solutions of nonlocal fractional boundary value problem*, Nonlinear Anal. **72** (2010), 916–924.

[23] Z. Bai and H. Lü, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl. **311** (2005), 495–505.

[24] N. R. O. Bastos, et al., *Fractional derivatives and integrals on time scales via the inverse generalized Laplace transform*, Int. J. Math. Comput. **11** (2011), 1–9.

[25] N. R. O. Bastos, et al., *Necessary optimality conditions for fractional difference problems of the calculus of variations*, Discrete Cont. Dyn. Syst. **29** (2011), 417–437.

[26] N. R. O. Bastos, et al., *Discrete-time fractional variational problems*, Signal Process. **91** (2011), 513–524.

[27] M. Benchohra, et al., *Existence results for fractional order functional differential equations with infinite delay*, J. Math. Anal. Appl. **338** (2008), 1340–1350.

[28] M. Benchohra, S. Hamani, and S. K. Ntouyas, *Boundary value problems for differential equations with fractional order and nonlocal conditions*, Nonlinear Anal. **71** (2009), 2391–2396.

[29] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Application*, Birkhäuser, Boston, MA, 2001.

[30] W. Cheung, J. Ren, P. J. Y. Wong, and D. Zhao, *Multiple positive solutions for discrete nonlocal boundary value problems*, J. Math. Anal. Appl. **330** (2007), 900–915.

[31] J. J. DaCunha, J. M. Davis, and P. K. Singh, *Existence results for singular three point boundary value problems on time scales*, J. Math. Anal. Appl. **295** (2004), 378–391.

[32] J. M. Davis and J. Henderson, *Triple positive solutions for $(k, n-k)$ conjugate boundary value problems*, Math. Slovaca 51 (2001), no. 3, 313–320.

[33] J. V. Devi, *Generalized monotone method for periodic boundary value problems of Caputo fractional differential equation*, Commun. Appl. Anal. **12** (2008), 399–406.

[34] K. Diethelm and N. Ford, *Analysis of fractional differential equations*, J. Math. Anal. Appl. **265**, 229–248 (2002).

[35] O. Došlý, *Reciprocity principle for even-order dynamic equations with mixed derivatives*, Dynam. Systems Appl. **16** (2007), no. 4, 697–708.

[36] D. Dunninger and H. Wang, *Existence and multiplicity of positive solutions for elliptic systems*, Nonlinear Anal. **29** (1997), 1051–1060.

[37] J. Ehme, P. Eloe, and J. Henderson, *Upper and lower solution methods for fully nonlinear boundary value problems*, J. Differential Equations **180** (2002), 51–64.

[38] S. Eidelman and A. Kochubei, *Cauchy problem for fractional diffusion equations*, J. Differential Equations **199** (2004), 211–255.

[39] L. H. Erbe and H. Wang, *On the existence of positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc. **120** (1994), 743–748.

[40] R. A. C. Ferreira, *Positive solutions for a class of boundary value problems with fractional q-differences*, Comput. Math. Appl. **61** (2011), 367–373.

[41] C. S. Goodrich, *Solutions to a discrete right-focal boundary value problem*, Int. J. Difference Equ. **5** (2010), 195–216.

[42] C. S. Goodrich, *Continuity of solutions to discrete fractional initial value problems*, Comput. Math. Appl. **59** (2010), 3489-3499.

[43] C. S. Goodrich, *Existence of a positive solution to a class of fractional differential equations*, Appl. Math. Lett. **23** (2010), 1050–1055.

[44] C. S. Goodrich, *Some new existence results for fractional difference equations*, Int. J. Dyn. Syst. Differ. Equ. **3** (2011), 145–162.

[45] C. S. Goodrich, *Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions*, Comput. Math. Appl. **61** (2011), 191–202.

[46] C. S. Goodrich, *A comparison result for the fractional difference operator*, Int. J. Difference Equ. **6** (2011), 17–37.

[47] C. S. Goodrich, *On positive solutions to nonlocal fractional and integer-order difference equations*, Appl. Anal. Discrete Math. **5** (2011), 122–132.

[48] C. S. Goodrich, *Existence of a positive solution to a system of discrete fractional boundary value problems*, Appl. Math. Comput. **217** (2011), 4740–4753.

[49] C. S. Goodrich, *Existence of a positive solution to a first-order p-Laplacian BVP on a time scales*, Nonlinear Anal. **74** (2011), 1926–1936.

[50] C. S. Goodrich, *Existence of a positive solution to systems of differential equations of fractional order*, Comput. Math. Appl. **62** (2011), 1251–1268.

[51] C. S. Goodrich, *On discrete sequential fractional boundary value problems*, J. Math. Anal. Appl. **385** (2012), 111–124.

[52] C. S. Goodrich, *The existence of a positive solution to a second-order p-Laplacian BVP on a time scale*, Appl. Math. Lett. **25** (2012), 157–162.

[53] C. S. Goodrich, *Positive solutions to boundary value problems with nonlinear boundary conditions*, Nonlinear Anal. **75** (2012), 417–432.

[54] C. S. Goodrich, *On a fractional boundary value problem with fractional boundary conditions*, Appl. Math. Lett. **25** (2012), 1101–1105.

[55] C. S. Goodrich, *Nonlocal systems of BVPs with asymptotically superlinear boundary conditions*, Comment. Math. Univ. Carolin. **53** (2012), 79–97.

[56] C. S. Goodrich, *On a discrete fractional three-point boundary value problem*, J. Difference Equ. Appl. **18** (2012), 397–415.

[57] C. S. Goodrich, *On nonlocal BVPs with boundary conditions with asymptotically sublinear or superlinear growth*, Math. Nachr., doi: 10.1002/mana.201100210

[58] C. S. Goodrich, *Nonlocal systems of BVPs with asymptotically sublinear boundary conditions*, Appl. Anal. Discrete Math., doi: 10.2298/AADM120329010G

[59] C. S. Goodrich, *On discrete fractional boundary value problems with nonlocal, nonlinear boundary conditions*, Commun. Appl. Anal., to appear

[60] J. Graef and J. R. L. Webb, *Third order boundary value problems with nonlocal boundary conditions*, Nonlinear Anal. **71** (2009), 1542–1551.

[61] G. Guseinov, *Self-adjoint boundary value problems on time scales and symmetric Green's functions*, Turkish J. Math. **29** (2005), no. 4, 365–380.

[62] J. Henderson, et al., *Positive solutions for systems of generalized three-point nonlinear boundary value problems*, Commen. Math. Univ. Carolin. **49** (2008), 79–91.

[63] M. Holm, *Sum and difference compositions and applications in discrete fractional calculus*, Cubo **13** (2011), 153–184.

[64] G. Infante, *Nonlocal boundary value problems with two nonlinear boundary conditions*, Commun. Appl. Anal. **12** (2008), 279–288.

[65] G. Infante and P. Pietramala, *Existence and multiplicity of non-negative solutions for systems of perturbed Hammerstein integral equations*, Nonlinear Anal. **71** (2009), 1301–1310.

[66] G. Infante and P. Pietramala, *Eigenvalues and non-negative solutions of a system with nonlocal BCs*, Nonlinear Stud. **16** (2009), 187–196.

[67] G. Infante and P. Pietramala, *A third order boundary value problem subject to nonlinear boundary conditions*, Math. Bohem. **135** (2010), 113–121.

[68] P. Kang and Z. Wei, *Three positive solutions of singular nonlocal boundary value problems for systems of nonlinear second-order ordinary differential equations*, Nonlinear Anal. **70** (2009), 444–451.

[69] E. R. Kaufmann and Y. N. Raffoul, *Positive solutions for a nonlinear functional dynamic equation on a time scale*, Nonlinear Anal. **62** (2005), 1267–1276.

[70] W. G. Kelley and A. C. Peterson, *Difference Equations: An Introduction with Applications, Second Edition*, Academic Press, New York, 2000.

[71] W. G. Kelley and A. C. Peterson, *The Theory of Differential Equations: Classical and Qualitative*, Prentice Hall, Upper Saddle River, 2004.

[72] M. Kirane and S. A. Malik, *The profile of blowing-up solutions to a nonlinear system of fractional differential equations*, Nonlinear Anal. **73** (2010), 3723–3736.

[73] L. Kong and Q. Kong, *Positive solutions of higher-order boundary-value problems*, Proc. Edinb. Math. Soc. **48** (2005), 445–464.

[74] V. Lakshmikantham and A. S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Anal. **69** (2008), 2677–2682.

[75] A. Malinowska and D. F. M. Torres, *Generalized natural boundary conditions for fractional variational problems in terms of the Caputo derivative*, Comput. Math. Appl. **59** (2010), 3110–3116.

[76] J. J. Nieto, *Maximum principles for fractional differential equations derived from Mittag-Leffler functions*, Appl. Math. Lett. **23** (2010), 1248–1251.

[77] K. Oldham and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Dover Publications, Inc., Mineola, New York, 2002.

[78] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[79] H. G. Schuster (ed.), *Reviews of Nonlinear Dynamics and Complexity*, Wiley-VCH (2008).

[80] X. Su, *Boundary value problems for a coupled system of nonlinear fractional differential equations*, Appl. Math. Lett. **22** (2009), 64–69.

[81] G. Wang, et al., *Fourth-order problems with fully nonlinear boundary conditions*, J. Math. Anal. Appl. **325** (2007), 130–140.

[82] J. Wang and Y. Zhou, *A class of fractional evolution equations and optimal controls*, Nonlinear Anal. Real World Appl. **12** (2011), 262–272.

[83] P. Wang and Y. Wang, *Existence of positive solutions for second-order m-point boundary value problems on time scales*, Acta Math. Appl. Sin. Engl. Ser. **22** (2006), 457–468.

[84] J. R. L. Webb and G. Infante, *Positive solutions of nonlocal boundary value problems: a unified approach*, J. Lond. Math. Soc. (2) **74** (2006), 673–693.

[85] J. R. L. Webb, *Nonlocal conjugate type boundary value problems of higher order*, Nonlinear Anal. **71** (2009), 1933–1940.

[86] J. R. L. Webb, *Solutions of nonlinear equations in cones and positive linear operators*, J. Lond. Math Soc. (2) **82** (2010), 420–436.

[87] J. R. L. Webb, *Remarks on a non-local boundary value problem*, Nonlinear Anal. **72** (2010), 1075–1077.

[88] Z. Wei, Q. Li, and J. Che, *Initial value problems for fractional differential equations involving Riemann-Liouville sequential fractional derivative*, J. Math. Anal. Appl. **367** (2010), 260–272.

[89] X. Xu, D. Jiang, C. Yuan, *Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation*, Nonlinear Anal. **71** (2009), 4676–4688.

[90] Z. Yang, *Positive solutions to a system of second-order nonlocal boundary value problems*, Nonlinear Anal. **62** (2005), 1251–1265.

[91] Z. Yang, *Positive solutions of a second-order integral boundary value problem*, J. Math. Anal. Appl. **321** (2006), 751–765.

[92] E. Zeidler (P. R. Wadsack, translator), *Nonlinear Functional Analysis and its Applications: Part 1: Fixed-Point Theorems*, Springer, New York, 1986.

[93] S. Zhang, *Positive solutions to singular boundary value problem for nonlinear fractional differential equation*, Comput. Math. Appl. **59** (2010), 1300–1309.

[94] X. Zhao and W. Ge, *Unbounded solutions for a fractional boundary value problem on the infinite interval*, Acta Appl. Math. **109** (2010), 495–505.

[95] Y. Zhou, F. Jiao, and J. Li, *Existence and uniqueness for fractional neutral differential equations with infinite delay*, Nonlinear Anal. **71** (2009), 3249–3256.

[96] Y. Zhou, F. Jiao, and J. Li, *Existence and uniqueness for p-type fractional neutral differential equations*, Nonlinear Anal. **71** (2009), 2724–2733.

[97] Y. Zhou and F. Jiao, *Nonlocal Cauchy problem for fractional evolution equations*, Nonlinear Anal. Real World Appl. **11** (2010), 4465–4475.