Summer 8-2012

Commutative Rings Graded by Abelian Groups

Brian P. Johnson
University of Nebraska-Lincoln, s-bjohns67@math.unl.edu

COMMUTATIVE RINGS GRADED BY ABELIAN GROUPS

by

Brian P. Johnson

A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professor Thomas Marley

Lincoln, Nebraska

August, 2012

COMMUTATIVE RINGS GRADED BY ABELIAN GROUPS

Brian P. Johnson, Ph.D.

University of Nebraska, 2012

Adviser: Thomas Marley

Rings graded by $\mathbb{Z}$ and $\mathbb{Z}^d$ play a central role in algebraic geometry and commutative algebra, and the purpose of this thesis is to consider rings graded by any abelian group. A commutative ring is graded by an abelian group if the ring has a direct sum decomposition by additive subgroups of the ring indexed over the group, with the additional condition that multiplication in the ring is compatible with the group operation. In this thesis, we develop a theory of graded rings by defining analogues of familiar properties—such as chain conditions, dimension, and Cohen-Macaulayness. We then study the preservation of these properties when passing to gradings induced by quotients of the grading group.

ACKNOWLEDGMENTS

I have many people to thank as I reflect on the last six years at UNL, and I would like to begin with my advisor, Tom Marley. His guidance and advice have been invaluable, and as I begin my own career, his example as a faculty member, advisor, and friend is one I will strive to emulate. I also have to thank the other members of my supervisory committee: Roger Wiegand, Mark Walker, Allan Peterson, and John Hibbing. I feel I owe Dr. Peterson a special debt of gratitude, as he was my first contact with UNL in the summer of 2005, and the REU he was advising was the reason I applied to UNL.

To my wife, Katie, I also owe many thanks. Her support and commiseration have kept me sane, and I am quite certain that without her practically serving as my personal secretary while searching for jobs, I would be joining the ranks of the unemployed rather than the faculty at Florida Gulf Coast University in the fall. But just as importantly, had Katie not decided to pursue her Ph.D. at UNL as well, I would not have spent the last six years at the only place I can possibly imagine having been so happy to do so.

Finally, I would like to thank the entire Math Department at UNL. The graduate students, faculty, and staff have made Lincoln a place I will be sad to leave and excited to visit.

GRANT INFORMATION

This work was supported in part by United States Department of Education GAANN grant #P200A060126.

Contents

Chapter 1

Introduction

In the traditional definition, a ring R is *graded* if it is a direct sum $\bigoplus_{i\in\mathbb{Z}} R_i$ of additive subgroups R_i of R, with the additional restriction that $R_iR_j \subseteq R_{i+j}$. Intuitively, the motivation comes from polynomial rings, with the subgroups R_i consisting of the homogeneous polynomials of degree i. The second condition is then simply that a degree i monomial times a degree j monomial has degree $i+j$. More recently (and even as far back as [8]), it has been common to allow what are called *multi-graded* rings, or rings that are a direct sum indexed over $\mathbb{Z}^d$. There is an obvious extension of the definition to G-graded rings—rings graded by any abelian group G—given in Chapter 2. There has been substantial work done on (not necessarily commutative) rings graded by arbitrary groups (cf. [16] or [4] for rings graded by monoids), but we restrict our focus to classical objects of study in commutative algebra.

If H is a subgroup of G, then we can induce a natural grading of R by the quotient G/H. This is also described in Chapter 2. One major theme of this thesis is to define properties of the ring strictly in terms of graded objects, and ask the following questions: When we define graded properties of the ring when graded by G, do the same properties hold when considered as a ring graded by the quotient group G/H? Under what assumptions do the properties descend (respectively, ascend) from the G-grading to the G/H-grading (respectively, to the

G-grading from the G/H-grading)? A special case of this is when $H = G$. In this case the descent of a given G-graded property to the G/H-grading is the same as descending to the ring R considered as a non-graded ring, as in [7] and [18].

In Chapter 2 we develop the basic notation and language of G-graded rings. In particular, we define G-graded analogues of many familiar properties: modules, prime ideals, localization, Nakayama's Lemma, and chain conditions, to name a few. We use a preceding "G-" to denote that a given property hold only for G-graded objects, as in [12]. While there are results in this thesis that also appear in [12], the approaches are different, and we are concerned mainly with the passage of properties between the G- and G/H-gradings rather than between just the G-grading and the trivial grading. The following theorem is indicative of the type of results we are seeking:

Theorem 2.41. *Suppose R is a G-graded ring. If H is a finitely generated subgroup of G, the following are equivalent:*

1. *R is G-Noetherian.*

2. *R is G/H-Noetherian.*

In Chapter 3 we develop a theory of primary decomposition for G-graded R-modules. It can be shown ([6, Exercise 3.5]) that in a very general setting primary decompositions and associated primes of graded modules have expected properties: all the associated primes are graded primes, and the primary modules appearing in the primary decomposition are also graded modules. For our purposes, however, this setting is not general enough, so we avoid making hypotheses on the group G and opt instead to develop the theory in the graded category. There are similarities and differences between the graded and non-graded cases. One similarity is:

Corollary 3.12. *Suppose R is a G-Noetherian ring, M is a finitely generated G-graded R-module, and $P \in \operatorname{Spec}^G(R)$. Then $P \in \operatorname{Ass}^G(M)$ if and only if $P = \operatorname{Ann}(f)$ for some homogeneous element $f \in M$. Also, the union of the associated G-primes of R is contained in the collection of zerodivisors of R, but this containment is not, in general, reversible.*

However, this corollary highlights a difference as well: the union of the associated primes of R is, in general, strictly contained in the collection of zerodivisors of R, a notable difference from the non-graded case.

Chapter 4 is devoted to dimension theory, including integrality and results on (graded) heights of graded prime ideals. Of note is an analogue of Krull's height theorem:

Corollary 4.19. *If R is G-Noetherian, and $I := (a_1, \ldots, a_n)$ is an ideal generated by G-homogeneous elements a_i, then $\operatorname{ht}^G(P) \leq n$ for any minimal G-prime P of I.*

Of the results on heights of primes, the most useful is a generalization of a result of Matijevic-Roberts [14] and Uliczka [18]:

Theorem 4.22. *Let R be a G-graded ring and H a finitely generated torsion-free subgroup of G. If $P \in \operatorname{Spec}^{G/H}(R)$, and we set $P^* := P^{*G}$, then*

$$\operatorname{ht}^{G/H}(P) = \operatorname{ht}^{G/H}(P^*) + \operatorname{ht}^{G/H}(P/P^*).$$

This and many of the other results in Chapter 4 are put to use in Chapter 5, which deals primarily with a graded version of the Cohen-Macaulay property. By defining grade in terms of Čech cohomology, we give a definition of grade and Cohen-Macaulayness for any (commutative) G-Noetherian ring. The main theorem of the chapter is:

Theorem 5.9. *Let R be a G-Noetherian graded ring, and suppose H is a finitely generated torsion-free subgroup of G. The following are equivalent:*

1. R is G-Cohen-Macaulay.

2. R is G/H-Cohen-Macaulay.

This generalizes results of [7], [14], [18].

Chapter 2

Notation and basic terminology

In this chapter we set up the basic language of commutative rings graded by an abelian group. We also give definitions for some graded analogues of common objects and properties seen in the study of commutative rings. Throughout the chapter—and the rest of this work—all rings will be commutative with identity, and G will always be an abelian group.

2.1 G-graded rings and modules

A *G-graded ring* R is a ring R with a family of subgroups $\{R_g \mid g \in G\}$ of R such that

$$R = \bigoplus_{g \in G} R_g,$$

as abelian groups, and $R_g R_h \subseteq R_{g+h}$ for all $g, h \in G$. When there is no danger of confusion, and G is understood, we may refer to the graded ring R.

Similarly, if R is a G-graded ring, a *G-graded R-module* M is an R-module M together

with a family of subgroups $\{M_g \mid g \in G\}$ of M such that

$$M = \bigoplus_{g \in G} M_g,$$

as abelian groups, and $R_g M_h \subseteq M_{g+h}$ for all $g, h \in G$. A *G-homogeneous ideal* of R is a G-graded submodule of R.

By definition, if $m \in M$ then there exist unique elements $m_g \in M_g$ for each $g \in G$, all but finitely many of which are zero, such that

$$m = \sum_{g \in G} m_g.$$

The element m_g is called the gth *homogeneous component* of m. If $m = m_g \neq 0$ for some $g \in G$ then m is called *G-homogeneous of degree g*, or just homogeneous if G is understood, and we denote the degree of m by $\deg(m)$. As before, when there is no danger of confusion, we may refer to the graded R-module M, or perhaps the graded module M.

Definition 2.1. Let R be a G-graded ring and M a graded R-module. Let $\sigma(M)$ denote the subgroup of G generated by $\{g \in G \mid M_g \neq 0\}$. If $\sigma(M) = G$ we say M is *properly* graded by G. If $\sigma(M) = 0$, we say that M is *trivially graded* or that M is *concentrated in degree 0.*

Remark 2.2. Let R be a G-graded ring and M a graded R-module. Then

1. The identity of R is homogeneous of degree 0.

2. R_0 is a commutative ring with identity.

3. For all $g \in G$, M_g is an R_0-module.

For a graded module M over the G-graded ring R, a *G-homogeneous* (or just homogeneous if G is understood) R-submodule of M is an R-submodule N of M which has a generating set

consisting of G-homogeneous elements. The next proposition characterizes such submodules in a useful way, and we omit the proof.

Proposition 2.3. *Let R be a G-graded ring and M a graded R-module. Let N be an R-submodule of M. The following are equivalent:*

1. *N is homogeneous.*

2. *For every $n \in N$, all the homogeneous components of n are in N.*

We also define graded subrings, and the following remark characterizes them in the same way as the previous proposition. A *G-graded subring* of the graded ring R is a subring S of R such that

$$S = \bigoplus_{g \in G} S_g$$

where $S_g = S \cap R_g$ for all $g \in G$.

Remark 2.4. Let R be a G-graded ring and S a subring of R. The following are equivalent:

1. S is a graded subring of R.

2. For every $s \in S$, all the homogeneous components of s are in S.

3. S is generated as an algebra over S_0 by homogeneous elements.

The next two definitions describe a fundamental operation given a G-graded ring and a subgroup H of G. In Chapters 4 and 5 we will focus a great deal on the relationship between R as a G-graded ring and R as a G/H-graded ring.

Definition 2.5. Let R be a G-graded ring and H a subgroup of G.

1. Then

$$R_H := \bigoplus_{h \in H} R_h$$

is an H-graded subring of R. One can also view R_H as a G-graded ring, and we will specify when it is considered as such. If M is a graded R-module and $g \in G$ then

$$M_{g+H} := \bigoplus_{h \in H} M_{g+h}$$

is a G-graded R_H-submodule of M (considering R_H as a G-graded ring). In particular, M_H is a graded R_H-module.

2. The family of subgroups $\{R_x \mid x \in G/H\}$ (as defined above) determines a G/H-grading on the ring R. Of course, R as a G/G-graded (equivalently, 0-graded) ring, is just R endowed with the trivial grading (where all elements have degree 0). Similarly, for a G-graded R-module M,

$$M = \bigoplus_{x \in G/H} M_x$$

defines a G/H-grading on M, and under this grading M is a G/H-graded R-module.

Definition 2.6. A homomorphism $\phi : R \to S$ of G-graded rings is called *homogeneous* if $\phi(R_g) \subseteq S_g$ for all $g \in G$. In this case, we say that S is a graded R-algebra. Note that $\phi(R)$ is a graded subring of S. We say that S is a finitely generated graded R-algebra if there exists a finite set T of homogeneous elements of S such that $S = \phi(R)[T]$. (Equivalently, S is finitely generated as an algebra over $\phi(R)$.) Similarly, a homomorphism $f : M \to N$ of graded R-modules is *homogeneous of degree h* if $f(M_g) \subseteq N_{g+h}$ for all $g \in G$.

Remark 2.7. If $f : M \to N$ is a homogeneous homomorphism of graded R-modules, then $\ker(f)$ and $\operatorname{im}(f)$ are homogeneous submodules of M and N, respectively.

Lemma 2.8. *Let R be a G-graded ring, M a G-graded R-module, H a subgroup of G, and $g \in G$. Consider R_H as a G-graded ring, and let A be a G-homogeneous R_H-submodule of M_{g+H}. If RA is the R-submodule of M generated by A, then $RA \cap M_{g+H} = A$.*

Proof. Clearly, $RA \cap M_{g+H}$ is a G-graded R_H-submodule of M_{g+H} and $A \subseteq RA \cap M_{g+H}$. Let $m \in RA \cap M_{g+H}$ be G-homogeneous of degree $g + h$ for some $h \in H$. Then there exist G-homogenous elements $a_1, \ldots, a_n$ of A such that $m = \sum r_i a_i$ for some $r_i \in R$. Say $\deg(a_i) = g + h_i$ for $h_i \in H$, $1 \leq i \leq n$. Since $\deg(m) = g + h$, we can assume each r_i is homogeneous of degree $h - h_i$. Hence, $r_i \in R_H$ for all i and so $m \in R_H A = A$. $\square$

2.2 Basic graded properties

In this section we begin defining graded analogues of common objects and properties seen in the study of commutative rings. Often we will require that a condition only holds for the homogeneous elements of a ring or module, or we will only consider properties that hold for some collection of graded objects.

As noted before, in Chapters 4 and 5 we will be studying the relationship between R as a G-graded ring and R as a G/H-graded ring. A common hypothesis we will make on the subgroup H of G is that H is torsion-free, and the following discussion is quite useful, as Proposition 2.12 is fundamental to much of Chapter 4.

Definition 2.9. A *totally ordered abelian group* is an abelian group G equipped with a total order $\leq$ with the property that whenever $a \leq b$ then $a + g \leq b + g$ for all $a, b, g \in G$.

The familiar groups $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ under addition with the usual total order are all examples of totally ordered abelian groups. If $\{G_i\}_{i \in I}$ is a collection of totally ordered abelian groups with respective orders $\leq_i$ and I is a totally ordered index set, one defines the lexicographic order on the direct sum $\bigoplus_i G_i$ by

$$(a_i) <_{\text{lex}} (b_i) \iff a_j <_j b_j \qquad \text{where } j \text{ is the smallest index } i \text{ in } I \text{ such that } a_i \neq b_i.$$

Note that such an index j exists as a_i and b_i are zero for all but finitely many i. In this way,

$\mathbb{Z}^d$ and $\bigoplus_{i=1}^{\infty}\mathbb{Z}$ are totally ordered abelian groups with the lexicographic sums of the usual order on $\mathbb{Z}$.

It is easily seen that any totally ordered abelian group must be torsion-free. The converse is also true: Any torsion-free group G can be endowed with a total order making G into a totally ordered abelian group [13]. However, for our purposes, we need only observe that this is true for finitely generated torsion-free abelian groups, a fact which is clear since any such group is isomorphic to $\mathbb{Z}^d$ for some d.

In the next definition (and many following), we incorporate the grading group as part of the definition. This allows the specificity we require when we consider multiple gradings on the same ring simultaneously (as in the following examples and Proposition 2.12).

Definition 2.10. A G-graded ring R is called a *G-field* (respectively, *G-domain*) if every nonzero G-homogeneous element of R is a unit (respectively, a non-zerodivisor). The ring R is called *G-reduced* if it has no nonzero G-homogeneous nilpotent elements. In case the ring R is a 0-field (or a 0-domain, or 0-reduced), we will simply say R is a field (a domain, reduced).

Example 2.11. The first two of the following examples show that a G-field need not be a field (or a domain, or even reduced) in the usual sense. The third shows that we can grade a field nontrivially so that it is also a G-field.

1. Let $R = k[x_1, x_1^{-1} \dots, x_d, x_d^{-1}]$ where k is a field and $x_1, \dots, x_d$ are indeterminates over k. Endow R with a $\mathbb{Z}^d$-grading by setting $\deg(x_i) = e_i \in \mathbb{Z}^d$, where e_i is zero in every component except the ith component and is 1 in the ith component. Note that under this grading, the only nonzero homogeneous elements are k-multiples of monomials in $x_1, \dots, x_d$, which are all units. Hence, R is a $\mathbb{Z}^d$-field, but R is clearly not a field, as $1 + x_1$ is not a unit, for example. However, R is both a G-domain and a domain.

2. Let $R = k[x]/(x^n - 1)$ where k is a field and x is an indeterminate over k and $n > 1$. Endow R with a $\mathbb{Z}/n\mathbb{Z}$-grading by setting $\deg(x) = \overline{1} \in \mathbb{Z}/n\mathbb{Z}$. (It is clear that $x^n - 1$ is homogeneous under this grading, so the grading passes to the quotient ring R.) Every nonzero homogenous element of R is a unit, so R is a $\mathbb{Z}/n\mathbb{Z}$-field. However, R is not a domain, as $x^n - 1$ is reducible. In fact, if n is prime and $\mathrm{char}(k) = n$, then R is not reduced, since in this case $x - 1$ is nilpotent.

3. Let $k = \mathbb{F}_p(t)$, where $\mathbb{F}_p$ is the field with p elements and t an indeterminate over $\mathbb{F}_p$. Let x be an indeterminate over k and consider $R = k[x]/(x^p - t)$. Note that as $x^p - t$ is irreducible over k, R is a field. Give R a $\mathbb{Z}/p\mathbb{Z}$-grading by setting $\deg(x) = \overline{1}$. Then R is both a field and a $\mathbb{Z}/p\mathbb{Z}$-field which is not concentrated in degree $\overline{0}$.

The previous examples show that a G-field R need not have this property descend to the ring R as a G/H graded ring, even when H is a torsion-free subgroup, as in part (1). On the other hand, when R is G-reduced or a G-domain, the next proposition prohibits such counterexamples:

Proposition 2.12. *Let R be a G-graded ring and H a torsion-free subgroup of G. Then*

1. *R is a G-domain if and only if R is a G/H-domain.*

2. *R is G-reduced if and only if R is G/H-reduced.*

Proof. We prove part (1); part (2) is proved similarly. Note that the "if" is clear, as any G-homogeneous element is also G/H-homogeneous. For the other implication, suppose $a, b \in R$ are G/H-homogeneous, and suppose $ab = 0$, but $b \neq 0$. Write $a = a_{g+h_1} + \cdots + a_{g+h_\ell}$ and $b = b_{g'+h'_1} + \cdots + b_{g'+h'_k}$, where $a_{g+h_i} \in R_{g+h_i}$, $b_{g'+h'_j} \in R_{g'+h'_j}$, and $b_{g'+h'_j} \neq 0$ for all j. Note that $F := \langle \{h_i\}, \{h'_j\} \rangle$ is a finitely generated, torsion-free subgroup of H. Therefore there exists a total order on F such that F is a totally ordered abelian group. Without loss

of generality, we can assume $h_i < h_\ell$ for all $i < \ell$ and $h'_j < h'_k$ for all $j < k$. In particular

$$h_i + h'_j = h_\ell + h'_k \text{ if and only if } i = \ell, j = k.$$

Considering $ab = \sum_{i,j}(a_{g+h_i})(b_{g'+h'_j})$, note that $g + h_i + g' + h'_j = g + h_\ell + g' + h'_k$ if and only if $h_i + h'_j = h_\ell + h'_k$, which is if and only if $i = \ell$ and $j = k$. Since $ab = 0$, this means $a_{g+h_\ell}b_{g'+h'_k} = 0$. As R is a G-domain, we must have $a_{g+h_\ell} = 0$. By induction, $a = 0$ and R is also a G/H-domain. □

While the property of being a G-field does not necessarily descend to the G/H-grading for a torsion-free subgroup H of G, given information about the structure of the group G, we can still say something about the structure of the ring in certain cases.

Proposition 2.13. *Let R be a G-graded ring. Suppose $G = \sigma(R) \cong A \oplus \mathbb{Z}^\ell$, where $\ell \geq 1$. If R is a G-field, then as G/A-graded rings,*

$$R \cong R_A[t_1, t_1^{-1}, \ldots, t_\ell, t_\ell^{-1}],$$

where the t_i are indeterminates over R_A.

Proof. Let $\{e_1, \ldots, e_\ell\}$ be a basis for $\mathbb{Z}^\ell$. Choose nonzero elements $x_i \in R_{e_i}$ (possible since $G = \sigma(R)$). We first need to show that the x_i are algebraically independent over R_A. Give $R_A[t_1, \ldots, t_\ell]$ a $\mathbb{Z}^\ell$-grading (a G/A-grading) by setting $\deg(t_i) = e_i$. Consider the map $\phi : R_A[t_1, \ldots, t_\ell] \to R$ which sends R_A to R_A and maps $t_i \mapsto x_i$. This is certainly homogeneous (under the $\mathbb{Z}^\ell$-grading), so $\ker(\phi)$ is $\mathbb{Z}^\ell$-homogeneous. Suppose cm is a homogeneous element in $\ker(\phi)$ with $c \in R_A$ and $m = t_1^{\alpha_1} \ldots t_\ell^{\alpha_\ell}$. Then $\phi(cm) = cx_1^{\alpha_1} \ldots x_\ell^{\alpha_\ell} = 0$, but, since R is a G-field, we see this implies that $c = 0$. Therefore $\ker(\phi) = 0$. This gives a G/A-homogeneous injection of $R_A[t_1, \ldots, t_\ell]$ into R where the t_i map to units, and this induces a map on the localization at $\{(t_1 \cdots t_\ell)^n \mid n \geq 0\}$. It is easy to see that the induced map is also an injection.

Finally, it only remains to be shown that $R_A[t_1, t_1^{-1}, \ldots, t_\ell, t_\ell^{-1}]$ maps onto R via a G/A-homogeneous map. Let $u \in R_{\overline{g}}$ for some $\overline{g} \in G/A$. We can write $\overline{g} = n_1 e_1 + \cdots + n_\ell e_\ell$ for $n_i \in \mathbb{Z}$. Let $m := x_1^{n_1} \ldots x_\ell^{n_\ell} \in R_{\overline{g}}$. Since m is a unit, we have $m^{-1}u \in R_{\overline{0}} = R_A$. Thus $u = cm$ for some $c \in R_A$. □

Corollary 2.14. *Let R be a G-field properly graded by a finitely generated torsion-free abelian group G. Then* $\dim(R) = \operatorname{rank}(G)$. *In particular, R is a field if and only if $G = 0$. (Here,* $\dim(R)$ *denotes the usual Krull dimension of the ring.)*

Proof. If $G \neq 0$ is finitely generated and torsion-free, then $G \cong \mathbb{Z}^\ell$ for some $\ell \geq 1$. By the previous proposition, $R \cong R_0[t_1, t_1^{-1}, \ldots, t_\ell, t_\ell^{-1}]$, and $\dim(R) = \ell = \operatorname{rank}(G)$. In particular, since R is a G-field, R_0 is a field, and because R is properly graded, $R = R_0$ if and only if $G = 0$. In this latter case, we also have $\dim(R) = 0 = \operatorname{rank}(G)$. □

Corollary 2.15. *Let R be a G-field, and suppose H is a subgroup of G. If H is cyclic, then every G/H-homogeneous ideal of R is principal.*

Proof. Notice that R_H is an H-field, so that R_H is isomorphic (as H-graded rings) to one of $k[t]/(t^n - 1)$ or $k[t, t^{-1}]$, for k some field, depending on whether $H \cong \mathbb{Z}/n\mathbb{Z}$ or $H \cong \mathbb{Z}$, respectively. Now suppose I is a G/H-homogeneous ideal of R. One easily checks that $I = (I \cap R_H)R$: If $a \in I$ is G/H homogeneous, write $a = a_1 + \cdots + a_k$, where for each i, $a_i \in R_{g+h_i}$ for some $h_i \in H$ and a fixed $g \in G$. Then a_1 is G-homogeneous, and therefore invertible, so $a_1^{-1}a \in I \cap R_H$. This shows $I \subseteq (I \cap R_H)R$, and the other containment is clear. Since $I \cap R_H$ is principal, I must be principal as well. □

We define the graded analogue of a free module in a natural way: a G-graded R-module M over a graded ring R is called *G-free* if it has a basis consisting of homogeneous elements. Here, a basis is a generating set which is R-linearly independent. In the non-graded case, any module over a field is free, and the analogous result is true in the G-graded setting as well:

Proposition 2.16. *Let R be a G-field and M a graded R-module. Then M is G-free. Moreover, any set of homogeneous elements which are R-linearly independent is contained in a basis consisting of homogeneous elements, and any set of homogeneous elements which generates M contains a basis.*

Proof. Let T be a set of homogeneous elements of M which are R-linearly independent. By Zorn's lemma, T is contained in a maximal set S of R-linearly independent homogeneous elements in M. We claim that S generates M. Let x be an arbitrary homogeneous element of M. It suffices to show x is in the submodule generated by S. If $x \in S$ we are done, so assume $x \notin S$. Then $S \cup \{x\}$ is R-linearly dependent. Thus, there is some nontrivial relation $r_1 s_1 + \cdots + r_n s_n + rx = 0$ where $s_i \in S$ and $r_i, r \in R$ (not all zero). Furthermore, as x and the s_i are homogeneous for all i we may assume r and the r_i are homogeneous as well. As S is linearly independent, we must have $r \neq 0$. Since R is a G-field, this means that r is a unit, which implies x is in the submodule generated by S.

The second statement is proved similarly. □

Definition 2.17. Let R be a G-graded ring and M a finitely generated graded R-module. We let $\mu_R(M)$ and $\mu_R^G(M)$, respectively, denote the least number of elements and the least number of G-homogeneous elements, respectively, needed to generate M.

Clearly, $\mu_R(M) \leq \mu_R^G(M)$ for every finitely generated graded R-module M (in general, $\mu_R^{G/H}(M) \leq \mu_R^G(M)$ for any subgroup H of G). The following is an example where $\mu_R(M) < \mu_R^G(M)$:

Example 2.18. Let $R = \mathbb{Z}[x]$ with the standard $\mathbb{Z}$-grading and consider the ideal of R defined by $I = (6, 3x, 2x^2) = (6, 3x + 2x^2)$. One easily verifies that $\mu_R(I) = 2$ and $\mu_R^{\mathbb{Z}}(I) = 3$.

A natural question to ask is if there are nontrivial cases when we have equality between $\mu_R(-)$ and $\mu_R^G(-)$. Proposition 2.16 gives a corollary describing one such case:

Corollary 2.19. *Let R be a G-field and M a finitely generated graded R-module. Then*

$$\mu_R(M) = \mu_R^G(M) = \operatorname{rank}_R^G(M).$$

Proof. As M is G-free of rank n, then M is G/H-free of rank n for every subgroup H of G. □

2.3 Prime Ideals, Localization, and Nakayama's Lemma

In this section the main topics we address are graded versions of prime ideals, graded localization, and a graded version of Nakayama's Lemma. We also define graded versions of maximal and radical ideals. The only difference in localization in a graded ring is that we restrict to multiplicatively closed sets of homogeneous elements.

Since we already have the notions of G-fields, G-domains, and G-reduced rings, we define types of homogeneous ideals in terms of these. Let R be a G-graded ring. A homogeneous ideal I is called *G-maximal* (resp., *G-prime*, *G-radical*) if R/I is a G-field (resp., a G-domain, G-reduced). We let $\operatorname{Spec}^G(R)$ denote the set of G-prime ideals of R, and we let $\operatorname{maxSpec}^G(R)$ denote the set of all G-maximal ideals. It is a routine exercise to show that a homogeneous ideal I is G-maximal if and only if there are no proper homogeneous ideals strictly containing I.

Remark 2.20. By Proposition 2.12, if R is G-graded where G is torsion-free, then any G-prime ideal of R is prime, and any G-radical ideal of R is radical. However, as seen by Proposition 2.13, a G-maximal ideal of R need not be maximal.

Remark 2.21. If R is a G-graded ring then any proper G-homogeneous ideal is contained in a G-maximal ideal of R (by Zorn's Lemma).

The next proposition is straightforward, and we omit the easy proof.

Proposition 2.22. *Let R be a G-graded ring and M, N graded R-modules. Then*

$$(N :_R M) := \{x \in R \mid xM \subseteq N\}$$

is a G-homogeneous ideal of R. In particular, $\mathrm{Ann}_R(M) := \{x \in R \mid xM = 0\} = (0 :_R M)$ *is a G-homogeneous ideal.*

Let R be a G-graded ring and H a subgroup of G. Suppose M is a G-graded R-module, and N is an R-submodule of M which is G/H-homogeneous. We let N^{*G} denote the R-submodule of M generated by all the G-homogeneous elements contained in N. Clearly, N is homogeneous if and only if $N = N^{*G}$. (Here H is somewhat irrelevant, but we state it this way to conform to the themes of the thesis.)

Remark 2.23. Let R be a G-graded ring, H a subgroup of G, and I a G/H-homogeneous ideal of R. Let $I^* = I^{*G}$.

1. If I is G/H-prime, then I^* is G-prime.

2. If I is G/H-radical, then I^* is G-radical.

Example 2.24. Let $p > 0$ be prime and $V = \mathbb{Z}_{(p)}$. Let $R = V[x]$ be a polynomial ring in one variable over V. Let R be graded by $\mathbb{Z}$ by setting $\deg(x) = 1$. It is easily seen that $I = (px - 1)R$ is a maximal ideal of R and $I^{*\mathbb{Z}} = 0$, which is not $\mathbb{Z}$-maximal.

Certainly, I^{*G} is the largest G-homogeneous ideal contained in a given G/H-homogeneous ideal I, and Proposition 2.26, used in Chapters 4 and 5, shows that we can find such an ideal "step-by-step" if we wish. First, we prove a group-theoretic lemma.

Lemma 2.25. *Let G be an abelian group. Suppose A, B are subgroups of G and $g, g' \in G$. Then $(g+A) \cap (g'+B) \subseteq g'' + (A \cap B)$ for some $g'' \in G$. In particular, if R is a G-graded ring, $x \in R$ is $G/(A \cap B)$-homogeneous if and only if x is G/A- and G/B-homogeneous.*

Proof. Suppose $(g+A) \cap (g'+B) \neq \emptyset$. Let $A' = \{a \in A \mid g + a \in g' + B\}$, and choose $a_0 \in A'$. Then $g + a_0 = g' + b_{a_0}$ for some $b_{a_0} \in B$. Similarly, for any $a \in A'$ there exists $b_a \in B$ such that $g + a = g' + b_a$. Thus $a - a_0 = b_a - b_{a_0} \in A \cap B$ for all $a \in A'$.

Let $g'' = g + a_0$. Then for all $a \in A'$,

$$\begin{aligned} g + a &= g + a_0 + (a - a_0) \\ &= g'' + (a - a_0), \end{aligned}$$

so that $g + a \in g'' + (A \cap B)$. Hence, $(g+A) \cap (g'+B) \subseteq g'' + (A \cap B)$. $\square$

Proposition 2.26. *Suppose R is G-graded. If A and B are subgroups of G, and if I is a $G/(A+B)$-homogeneous ideal, then*

$$(I^{*G/A})^{*G/B} = I^{*G/(A \cap B)}.$$

Proof. A generating set for $I^{*G/A}$ is

$$\left\{x \in I \;\middle|\; x \text{ is } G/A\text{-homogeneous}\right\},$$

and a generating set for $(I^{*G/A})^{*G/B}$ is

$$\left\{x \in I^{*G/A} \;\middle|\; x \text{ is } G/B\text{-homogeneous}\right\}.$$

Since it is clear that $(I^{*G/A})^{*G/B} \supseteq I^{*G/(A \cap B)}$, it will suffice to show (by the above generating

set) that any G/B-homogeneous element of $I^{*G/A}$ is in the ideal generated by the $G/(A \cap B)$-homogeneous elements of I.

If x is a G/B-homogeneous element of $I^{*G/A}$, we can write

$$x = x_1 + \cdots + x_n,$$

where $x_i \in (I^{*G/A})_{g_i+A} \setminus \{0\}$ and $g_i + A \neq g_j + A$ for $i \neq j$. On the other hand, since x is G/B-homogeneous, we can write

$$x = y_1 + \cdots + y_m,$$

where $y_i \in R_{g+b_i} \setminus \{0\}$ for some $g \in G$ and $b_i \neq b_j \in B$ for $i \neq j$. Reordering, if necessary, we can assume

$$\begin{aligned} x_1 &= y_1 + \cdots + y_{k_1} \\ x_2 &= y_{k_1+1} + \cdots + y_{k_2} \\ &\vdots \\ x_n &= y_{k_{n-1}+1} + \cdots + y_m, \end{aligned}$$

where the x_i are in I and both G/A- and G/B-homogeneous, and so by Lemma 2.25, $G/(A \cap B)$-homogeneous. Then $x \in I^{*G/(A\cap B)}$ (as x is a sum of $G/(A \cap B)$-homogeneous elements), and we have equality. $\square$

Let R be a G-graded ring and S a multiplicatively closed set of G-homogeneous elements not containing 0. Then the localization R_S is a G-graded ring where for $g \in G$

$$(R_S)_g := \left\{ \frac{r}{s} \;\middle|\; r \in R_h, s \in S \cap R_{h-g}, h \in G \right\}.$$

Similarly, if M is a graded R-module then M_S is a graded R_S-module. If P is a G-prime ideal of R we let (P) denote the multiplicatively closed set consisting of all G-homogeneous elements of R not in P.

Remark 2.27. Suppose M is a G-graded module over the G-graded ring R. Because the annihilator of M is G-homogeneous, one can easily show that for any G-prime ideal P of R, $M_{(P)} \neq 0$ if and only if $P \supseteq \operatorname{Ann}_R(M)$. Moreover, suppose $H \leq G$ so that M is both G- and G/H-graded. In Chapter 5 we will make use of the following fact: Define T to be the set of G/H-homogeneous elements of R not in P (in particular, $T \supseteq (P)$). One can show that $M_{(P)} = 0$ if and only if $M_T = 0$. The crux of the proof is that $\operatorname{Ann}_R(M)$ is homogeneous with respect to the finest grading M possesses (in this case, the G-grading).

Theorem 2.28. *Suppose R is G-graded and S is a multiplicatively closed set of G-homogeneous elements. The map $R \to R_S$ induces a one-to-one inclusion preserving correspondence between G-prime ideals of R whose intersection with S is empty and G-prime ideals of R_S.*

Proof. The standard proof works with minor modifications. □

We have already noted that every G-graded ring (with identity) has at least one G-maximal ideal. As in the non-graded case, a G-graded ring R is called *G-local* if it has a unique G-maximal ideal. One sees that if P is a G-prime ideal, then $R_{(P)}$ is a G-local ring with G-maximal ideal $PR_{(P)}$.

Definition 2.29. Let R be a G-graded ring. The *Jacobson G-radical* is defined to be the intersection of all G-maximal ideals of R.

Proposition 2.30 (Graded Nakayama's Lemma)**.** *Let R be a G-graded ring, M a finitely generated graded R-module and J the Jacobson G-radical of R. If $JM = M$ then $M = 0$.*

Proof. We show that if M is generated by $n > 0$ homogeneous elements, then in fact M is generated by $n-1$ homogeneous elements. Let $x_1, \ldots, x_n$ be a homogeneous generating set

for M. Then $x_n = r_1x_1 + \cdots + r_nx_n$ for some $r_1, \ldots, r_n \in J$. As each x_i is homogeneous, we may assume each r_i is homogeneous as well. In particular (so that the degrees match up), we may assume $r_n \in R_0$. Then $(1 - r_n)x_n \in Rx_1 + \cdots + Rx_{n-1}$. Now, as $r_n \in J$ and $1 - r_n$ is homogeneous, we see that $1 - r_n$ is a unit (else the homogeneous ideal $R(1 - r_n)$ is contained in some maximal ideal). Thus, $M = Rx_1 + \cdots + Rx_{n-1}$. □

Corollary 2.31. *Let R be a G-graded ring, M a finitely generated graded R-module and J the Jacobson G-radical of R. Let $x_1, \ldots, x_n$ be homogeneous elements of M. Then $\{x_1, \ldots, x_n\}$ generate M if and only if $\{x_1 + JM, \ldots, x_n + JM\}$ generate M/JM. In particular, $\mu_R^G(M) = \mu_{R/J}^G(M/JM)$.*

Proof. Suppose $\{x_1 + JM, \ldots, x_n + JM\}$ generate M/JM and let N be the (G-graded) submodule of M generated by $x_1, \ldots, x_n$. Then $N + JM = M$, which implies $M/N = J(M/N)$. By Nakayama's Lemma, $M/N = 0$. □

Corollary 2.32. *Let R be a graded G-local ring with G-maximal ideal m. Let M be a finitely generated graded R-module. Then*

$$\mu_R(M) = \mu_R^G(M) = \operatorname{rank}_{R/m}(M/mM).$$

Proof. The second equality follows from the previous corollary and Corollary 2.19. If $\mu_R(M) < \operatorname{rank}_{R/m}(M/mM)$ then $\mu_{R/m}(M/mM) < \operatorname{rank}_{R/m}(M/mM)$, contradicting Corollary 2.19. □

2.4 Chain Conditions

The ascending and descending chain conditions on ideals or submodules are also natural properties to study in a graded sense, and we define them in the obvious way.

Let R be a G-graded ring and M a graded R-module. We say that M is *G-Noetherian* (respectively, *G-Artinian*) if M satisfies the ascending (respectively, descending) chain condition on G-homogeneous submodules. The ring R is a G-Noetherian (respectively, G-Artinian) ring if it is G-Noetherian (respectively, G-Artinian) as a graded R-module. The next proposition gives useful characterizations, and the proof only requires minor modifications to the one in the non-graded case.

Proposition 2.33. *Let R be a G-graded ring and M a graded R-module. The following conditions are equivalent:*

1. *M is G-Noetherian (respectively, G-Artinian).*
2. *Every nonempty set of G-homogeneous submodules of M has a maximal (respectively, minimal) element.*

For G-Noetherian only, the previous conditions are also equivalent to:

3. *Every G-homogeneous R-submodule of M is finitely generated.*

The next proposition and corollary are analogous to results in the non-graded case, and their proofs also require only minor modifications to work in the graded case.

Proposition 2.34. *Let R be a G-graded ring and $0 \to L \to M \to N \to 0$ a short exact sequence of graded R-modules and homogeneous maps. Then M is G-Noetherian (respectively, G-Artinian) if and only if L and N are G-Noetherian (respectively, G-Artinian).*

Corollary 2.35. *Let R be a G-Noetherian (respectively, G-Artinian) graded ring. Then every finitely generated graded R-module is G-Noetherian (respectively, G-Artinian).*

Proposition 2.36. *Let R be a G-graded ring, H a subgroup of G, and M a G-graded R-module. If M is G-Noetherian (respectively, G-Artinian), then M_{g+H} is G-Noetherian*

(respectively, G-Artinian) as an R_H-module for all $g \in G$. In particular, if R is G-Noetherian, then R_H is G-Noetherian.

Proof. We prove the assertion in the case M is G-Noetherian. The Artinian case is proved similarly. Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ be an ascending chain of G-graded R_H-submodules of M_{g+H}. Then $RA_1 \subseteq RA_2 \subseteq RA_3 \subseteq \cdots$ is an ascending chain of G-graded R-submodules of M. As M is G-Noetherian, there exists an integer n such that $RA_k = RA_{k+1}$ for all $k \geq n$. By Lemma 2.8, $A_k = RA_k \cap M_{g+H} = RA_{k+1} \cap M_{g+H} = A_{k+1}$ for all $k \geq n$. Hence, M_{g+H} is G-Noetherian. $\square$

As a consequence of the previous proposition, we also get that if M is G-Noetherian, then M_{g+H} is H-Noetherian as well. In particular, if R is G-Noetherian, then R_H is both G- and H-Noetherian.

Theorem 2.37 (Hilbert Basis Theorem)**.** *Let S be a G-graded ring and R a graded subring of S. Suppose R is G-Noetherian and S is a finitely generated R-algebra. Then S is G-Noetherian.*

Proof. It is enough to prove that if $S = R[x]$ is G-Noetherian, where x is a homogeneous element of S, then S is G-Noetherian. Let I be a homogeneous ideal of S. For each nonnegative integer n let

$$I_n := \{r_n \in R \mid r_n x^n + r_{n-1}x^{n-1} + \cdots + r_1 x + r_0 \in I \text{ for some } r_{n-1}, \ldots, r_0 \in R\}.$$

Since I and x are homogeneous, it is easily seen that I_n is a homogeneous ideal of R for all n. Furthermore, $I_n \subseteq I_{n+1}$ for all n. As R is G-Noetherian, there exists k such that $I_k = I_{k+i}$ for all $i \geq 0$. For $0 \leq i \leq k$ let $\{r_{ij} \mid j = 1, \ldots, \ell_i\}$ be a homogeneous generating set for I_i. For each such r_{ij} let f_{ij} be an element of the form $r_{ij}x^i + r_{i-1}x^{i-1} + \cdots + r_1 x + r_0 \in I$, where $r_i \in R$ for all i. Since r_{ij} is homogeneous, we can assume each f_{ij} is homogeneous as

well. Let J be the ideal of S generated by the set $\{f_{ij} \mid 1 \leq i \leq k, 1 \leq j \leq \ell_i\}$. We claim that $J = I$. Clearly, J is homogeneous and $J \subseteq I$. Let $s \in I$ be homogeneous. As $s \in R[x]$, $s = r_n x^n + r_{n-1}x^{n-1} + \cdots + r_1 x + r_0$ for some $r_0, \ldots, r_n \in R$. As s and x are homogeneous, we can assume each r_i is homogeneous as well. We induct on n to show that $s \in J$. If $n = 0$ then $s = r_0 \in I_0 = (\{f_{0j} \mid j = 1, \ldots, \ell_0\}) \subseteq J$. Suppose $0 < n \leq k$. Then $r_n \in I_n$, so there exist homogeneous $c_1, \ldots, c_{\ell_n} \in R$ such that $r_n = \sum_j c_j r_{nj}$. Let $g = \sum_j c_j f_{nj} \in J$. Then $s - g$ is homogeneous, is in I, and can be expressed as a polynomial in x of degree less than n. By the induction hypothesis, $s - g \in J$, and this implies $s \in J$. Finally, suppose $n > k$. Then $r_n \in I_n = I_k$, so there exists homogeneous $c_i \in R$ such that $r_n = \sum_j c_j r_{kj}$. Let $g = \sum_j c_j x^{n-k} f_{kj}$. Then as before, $s - g \in I$ is homogeneous and can be expressed as a polynomial in x of degree less than n. Hence, $s - g \in J$ and so $s \in J$. Thus, $I = J$ and I is finitely generated. $\square$

Theorem 2.38. *Let R be a G-graded ring and H a subgroup of G.*

1. *If R is G-Noetherian, then R_H is H-Noetherian.*
2. *If R is G-Noetherian and G/H is finitely generated, then R is a finitely generated R_H-algebra.*

Proof. Part (1) follows from Proposition 2.36. Indeed, since $(R_H)_g = 0$ for $g \in G \setminus H$, R_H is G-Noetherian if and only if R_H is H-Noetherian.

For part (2), we induct on n, the number of generators of G/H. We first consider the case when $n = 1$. Suppose $G = \langle H, g \rangle$. Again by Proposition 2.36, R_{x+H} is a finitely generated R_H-module for all $x \in G$. Hence, if $[G : H] < \infty$ then R is a finitely generated R_H-module. Thus, we can assume the image of g in G/H has infinite order. Consider the homogeneous ideal J of R generated by all R_{mg+H} where $m > 0$. Let $a_1, \ldots, a_\ell$ be a homogeneous generating set for J, where $\deg(a_i) = m_i g + h_i$ where $m_i > 0$ and $h_i \in H$. Let $N := \max\{m_i \mid i = 1, \ldots, \ell\}$.

Let $b_1, \ldots, b_k$ be homogeneous generators for the R_H-module $\bigoplus_{i=1}^{N} R_{ig+H}$. Let T denote the graded subring $R_H[a_1, \ldots, a_\ell, b_1, \ldots, b_k]$ of R. We use induction on m to show $R_{mg+H} \subseteq T$ for all $m \geq 0$. Clearly, this holds for all $0 \leq m \leq N$. Suppose $m > N$ and $x \in R_{mg+h}$ for some $h \in H$. Then $x \in J$, so there exist homogeneous elements $r_i \in R$, $1 \leq i \leq \ell$, such that $x = \sum_i r_i a_i$. Then $\deg(r_i) = (mg + h) - (m_i g + h_i) = (m - m_i)g + (h - h_i)$. As $0 < m - m_i < m$ we have by the inductive hypothesis that $r_i \in R_{(m-m_i)g+H} \subseteq T$ for all i. Thus, $x \in T$. One can make a similar argument using the ideal of R generated by all R_{mg+H} for $m < 0$ to obtain R as a finitely generated T-algebra.

Assume the theorem is true whenever G/H is generated by at most n elements and suppose G/H is generated by $n+1$ elements. Then there exists a subgroup K of G containing H such that G/K is cyclic and K/H is n-generated. By part (1), R_K is K-Noetherian. By the inductive hypothesis, R_K is a finitely generated R_H-algebra. Finally, the cyclic case above shows that R is a finitely generated R_K-algebra. The result now follows. $\square$

As a corollary, we get the following well-known result from [9, 1.1]. Note that the corollary is false if G is not finitely generated. For example, suppose $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}$, k is a field, and $\{x_i\}_{i=1}^{\infty}$ is a sequence of indeterminates. The ring $R := k[\{x_i, x_i^{-1}\}_{i=1}^{\infty}]$ has a natural G-grading by setting $\deg(x_i) = e_i$, where e_i is the vector with a 1 in the ith position and 0 elsewhere. Then R is a G-field, which is G-Noetherian, but clearly R is not Noetherian.

Corollary 2.39. *Let R be a G-graded ring where G is finitely generated. Then R is G-Noetherian if and only if R is Noetherian.*

Proof. By the theorem applied to $H = \{0\}$, R_0 is Noetherian and R is a finitely generated R_0-algebra. $\square$

Proposition 2.40. *A graded ring R is G-Noetherian if and only if every G-prime ideal is finitely generated.*

Proof. Using Zorn's Lemma, one can show that if there exists a non-finitely-generated G-homogeneous ideal, then there exists one maximal with respect to this property (via inclusion). We claim that such an ideal P is G-prime. Suppose $a, b \in R$ are homogeneous elements such that $ab \in P$, yet $a, b \notin P$. Since P is maximal with respect to being non-finitely-generated, the homogeneous ideal (P, a) must be finitely generated, say $(P, a) = (p_1 + x_1 a, \ldots, p_n + x_n a)$. Set $K = (P :_R a)$. Then K is homogeneous, and

$$K \supseteq (P, b) \supsetneq P,$$

so K must also be finitely generated, say $K = (k_1, \ldots, k_m)$. Finally, we claim

$$P = (p_1, \ldots, p_n, k_1 a, \ldots, k_m a).$$

Certainly, $(p_1, \ldots, p_n, k_1 a, \ldots, k_m a) \subseteq P$, so suppose $y \in P$. Since $P \subseteq (P, a)$, we may write

$$\begin{aligned} y &= \sum_{i=1}^{n} r_i (p_i + x_i a) \\ &= \sum_{i=1}^{n} r_i p_i + a \sum_{i=1}^{n} r_i x_i. \end{aligned}$$

That is, we can write $y = p + ak$, where $p \in P$ and $k = \sum r_i x_i$. But $ak = y - p \in P$, so $k \in K$, and we can write

$$k = \sum_{j=1}^{m} t_j k_j,$$

which gives

$$\begin{aligned} y &= \sum_{i=1}^{n} r_i p_i + a \sum_{j=1}^{m} t_j k_j \\ &= \sum_{i=1}^{n} r_i p_i + \sum_{j=1}^{m} (k_j a) t_j. \end{aligned}$$

Hence, $P = (p_1, \ldots, p_n, k_1 a, \ldots, k_m a)$, but this contradicts the assumption that P is non-finitely generated. Therefore, either $a \in P$ or $b \in P$, and P is G-prime.

It is now clear that if there exists a non-finitely-generated G-homogeneous ideal, then we can find a G-prime that is non-finitely-generated. Thus R is G-Noetherian if and only if all G-prime ideals are finitely generated. $\square$

In Chapters 4 and 5, while showing the descent of properties from the G-grading on a ring to the G/H-grading, we make several arguments via induction on the (torsion-free) rank of a subgroup H of G. It is then crucial that we have the next theorem, which is a generalization of Corollary 2.39.

Theorem 2.41. *Suppose R is a G-graded ring. If H is a finitely generated subgroup of G, the following are equivalent:*

1. *R is G-Noetherian.*

2. *R is G/H-Noetherian.*

Proof. Certainly, we only need to show (1) $\Longrightarrow$ (2). By induction, it suffices to prove the case $H = \langle a \rangle$ is cyclic. Let I be a G/H-homogeneous ideal. If $x \in I$ is G/H-homogeneous, then $x \in R_{g+H}$ for some $g \in G$. Write

$$x = x_{k_1} + \cdots + x_{k_m},$$

where $x_{k_i} \neq 0$ for each i, and $x_{k_i} \in R_{g+k_i a}$. If $|a| = \infty$ and we assume $g + k_i a \neq g + k_j a$ for $i \neq j$, this uniquely determines the k_i. If $|a| = n < \infty$ we can get uniqueness by assuming $g + k_i a \neq g + k_j a$ for $i \neq j$ and that $0 \leq k_i < n$. Let t be an indeterminate over R and consider $R[t, t^{-1}]$ as a G-graded ring where $\deg(t) = a$. By Theorem 2.37, $R[t, t^{-1}]$ is also G-Noetherian. Let

$$\widehat{x}(t) := x_{k_1} t^{k_m - k_1} + x_{k_2} t^{k_m - k_2} + \cdots + x_{k_m}.$$

Note that $\widehat{x}(t)$ is G-homogeneous of degree $g + k_m a$ and that $\widehat{x}(1) = x$. Since the ideal

$$(\{\widehat{x}(t) \mid x \text{ is } G/H\text{-homogeneous in } I\})$$

is G-homogeneous, it is finitely generated, say by $\{\widehat{s_1}(t), \ldots, \widehat{s_\ell}(t)\}$, where $s_i \in I$ is G/H-homogeneous. Then for any G/H-homogeneous $x \in I$, we can write

$$\widehat{x} = \sum_{i=1}^{\ell} r_i \widehat{s_i}(t),$$

where $r_i \in R[t]$. Evaluating this expression at $t = 1$, we get an equation in I, showing that any homogeneous element of I, and hence, any element of I can be written in terms of the s_i. Thus I is finitely generated. □

We present one final result pertaining to G-Artinian rings.

Proposition 2.42. *Suppose R is a G-Artinian ring and $J := J(R)$ is the Jacobson G-radical of R. Then $J^k = 0$ for some $k \geq 1$.*

Proof. Because R is G-Artinian, the descending chain $J \supseteq J^2 \supseteq J^3 \supseteq \cdots$ stabilizes, so we have $J^k = J^{k+1}$ for some k. Set $I := J^k$. To see that $I = 0$, we suppose $I \neq 0$ and derive a

contradiction. Set

$$\Lambda = \{K \mid K \text{ a } G\text{-homogeneous ideal of } R \text{ such that } IK \neq 0\}.$$

Because $I \neq 0$, $R \in \Lambda$, and so the descending chain condition implies there is a minimal element $K \in \Lambda$. Choose a homogeneous $y \in K$ such that $yI \neq 0$. Note that $(yI)I = yI^2 = yI \neq 0$, so $yI \in \Lambda$. The minimality of K then implies that $yI = K$, so there exists a homogeneous element $i \in I$ such that $yi = y$ (note that $i \in I_0$). Rearranging gives $(1 - i)y = 0$, and one sees that $1 - i$ is a unit because $i \in J$. This implies $y = 0$, a contradiction, and it must be the case that $I = J^k = 0$. $\square$

Chapter 3

Primary Decomposition

In Chapter 5 we will need a theory of graded associated primes. It can be shown ([6, Exercise 3.5]) that in a very general setting the usual primary decompositions and associated primes of graded modules are exactly what we get if we were to define a graded version. That is, all the associated primes are graded primes, and the primary modules appearing in the intersection are also graded modules. For our purposes, however, it is not general enough, and to avoid making hypotheses on the group G, we opt instead to develop the theory in the graded category.

Definition 3.1. Let R be a G-graded ring and M a G-graded R-module. Assume $N \subseteq M$ is a G-graded submodule. We say N is *G-irreducible* if whenever $N = N_1 \cap N_2$ with N_1, N_2 G-graded submodules of M, then we have $N_1 = N$ or $N_2 = N$.

Proposition 3.2. *Suppose $N \subseteq M$ are graded modules over the G-graded ring R, with M G-Noetherian. Then there exist G-irreducible modules $N_1, \ldots, N_\ell$ such that $N = N_1 \cap \cdots \cap N_\ell$.*

Proof. Let Λ be the collection of all graded submodules of M *not* having such a decomposition. Suppose $\Lambda \neq 0$. By the G-Noetherian property, this collection must have a maximal element. Let N be such a maximal element. Then N is not G-irreducible, so there exist G-graded

modules N_1, N_2 such that $N = N_1 \cap N_2$ with $N_i \neq N$, $i = 1, 2$. By the maximality of N (and the fact that $N_i \supsetneq N$), each N_i must have a decomposition into G-irreducible modules. But this implies $N = N_1 \cap N_2$ has a decomposition as well, a contradiction. Thus $\Lambda = \emptyset$. $\square$

Definition 3.3. Let R be a G-graded ring and $N \subseteq M$ graded R-modules. We say N is *G-primary* if for all G-homogeneous $r \in R$ the map $\mu_r : M/N \xrightarrow{\cdot r} M/N$ induced by multiplication by r is either injective or nilpotent.

Proposition 3.4. *Suppose R is a G-graded ring. If M is a G-Noetherian R-module, and $N \subseteq M$ is a G-irreducible submodule, then N is G-primary.*

Proof. Suppose, by way of contradiction, that N is not G-primary. Then there exists a G-homogeneous element $r \in R$ such that the induced map $\mu_r : M/N \to M/N$ is neither injective nor nilpotent. Consider the increasing chain of (homogeneous, as r is homogeneous) submodules of M

$$\ker(r) \subseteq \ker(r^2) \subseteq \cdots .$$

Since M is G-Noetherian, there exists n such that $\ker(r^n) = \ker(r^n) = \cdots \subseteq M/N$. Suppose $\ker(r^n) = N_1/N$ for some G-graded submodule $N_1 \subseteq M$. Note that $N \subsetneq N_1 \subsetneq M$. Indeed, the containments must be strict, since r is neither nilpotent nor injective. Now let $N_2 = r^n M + N$. We claim that $N_1 \cap N_2 = N$. First note that $r^n M + N \supsetneq N$ as $\ker(r^n) \neq M/N$. Let $x \in N_1 \cap N_2$, where $x = r^n m + d$ with $m \in M$ and $d \in N$. Also, since $x \in N_1$, we have $r^n x \in N$. But then $r^n x = r^{2n} m + r^n d$, and we see that $r^{2n} m \in N$. Since $\ker(r^{2n}) = \ker(r^n)$, $m \in N_1$, and so $r^n m \in N$ implies $x \in N$. But N was assumed to be G-irreducible, a contradiction. Thus N must be G-primary. $\square$

For a G-graded ring R and a homogeneous ideal $I \subseteq R$, we define the *G-radical* of I, denoted $\sqrt[G]{I}$, to be the (G-homogeneous) ideal generated by the homogeneous elements $r \in R$ such that $r^n \in I$ for some $n \in \mathbb{N}$. One can show that when G is torsion-free, $\sqrt[G]{I} = \sqrt{I}$.

In fact, it is true that $\sqrt[G]{I} = \sqrt[G/H]{I}$ whenever H is torsion-free. However, the ring from Example 2.11 (2) gives an example where $\sqrt[G]{0} \subsetneq \sqrt{0}$ when G has torsion. Before proceeding to developing the notion of primary decompositions in the graded case, we record some elementary results concerning G-radicals of homogeneous ideals.

Remark 3.5. Suppose R is a G-graded ring and I, J are homogeneous ideals of R.

1. $\sqrt[G]{I \cap J} = \sqrt[G]{I} \cap \sqrt[G]{J}$.

2. If R is G-Noetherian, I contains a power of $\sqrt[G]{I}$.

Proof. (1) is straightforward, and we prove (2): Since R is G-Noetherian, we can write $\sqrt[G]{I} = (x_1, \ldots, x_n)$, where x_i is G-homogeneous for each i. Because $x_i \in \sqrt[G]{I}$, there exists n_i such that $x_i^{n_i} \in I$. Because $(\sqrt[G]{I})^N$ is generated by elements of the form $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$ such that $\sum m_i = N$, we can choose N large enough to guarantee every generator of $(\sqrt[G]{I})^N$ will be in I. $\square$

Proposition 3.6. *Suppose R is a G-graded ring, and let I be a homogeneous ideal. Then*

$$\sqrt[G]{I} = \bigcap_{P \supseteq I} P,$$

where the intersection runs over G-prime ideals of R.

Proof. Since both ideals are G-homogeneous, it suffices to consider only homogeneous elements. Let $x \in \sqrt[G]{I}$ be homogeneous. Then $x^N \in I$, and so $x^N \in P$ for each G-prime $P \supseteq I$. Since P is G-prime, $x \in I$.

Because we are only concerned with G-prime ideals containing I, we may pass to the ring R/I and suppose that $I = 0$. Suppose $x \notin \sqrt[G]{0}$ is homogeneous. It suffices to find a G-prime ideal P such that $x \notin P$. Set $S = \{x^n \mid n \geq 0\}$. Since $x^n \neq 0$ for all $n \geq 0$, we know $R_S \neq 0$.

Therefore there exists a G-maximal ideal of R_S by Remark 2.21. This is a G-prime ideal of R_S containing 0, and its preimage P in R satisfies $P \cap S = \emptyset$. In particular, $x \notin P$. $\square$

Proposition 3.7. *Let $N \subseteq M$ be G-primary. Then $\sqrt[G]{\operatorname{Ann}_R(M/N)}$ is G-prime.*

Proof. Suppose $ab \in \sqrt[G]{\operatorname{Ann}_R(M/N)}$, where a, b are homogeneous and $b \notin \sqrt[G]{\operatorname{Ann}_R(M/N)}$. Then the map $\mu_{ab} : M/N \to M/N$ induced by multiplication by ab is nilpotent. On the other hand, N is G-primary, so every such map is either nilpotent or injective, and as $b \notin \sqrt[G]{\operatorname{Ann}_R(M/N)}$, the map $\mu_b : M/N \to M/N$ must be injective. This implies $\mu_a : M/N \to M/N$ is nilpotent, or else $\mu_{ab} : M/N \to M/N$ would be injective. Thus $a \in \sqrt[G]{\operatorname{Ann}_R(M/N)}$. $\square$

In the special case where $M = R$, we see that the G-radical of any G-primary ideal I is G-prime. In this case, if $\sqrt[G]{I} = P$, we say I is P-G-primary.

Definition 3.8. We call a decomposition $N = N_1 \cap \cdots \cap N_\ell$, where each N_i is G-primary, a *G-primary decomposition* for the module N. Given such a decomposition, if

1. the G-prime ideals $\sqrt[G]{\operatorname{Ann}_R(M/N_i)}$ are distinct for each i, and

2. for each $i = 1, \ldots, \ell$,

$$N_i \not\supseteq \bigcap_{j \neq i} N_j,$$

such a decomposition is said to be *irredundant*.

Proposition 3.9. *If a G-primary decomposition exists for $N \subseteq M$ (for example, if M is G-Noetherian), then an irredundant G-primary decomposition exists.*

Proof. First we show that if $N_1, N_2 \subseteq M$ are P-G-primary, then $N_1 \cap N_2$ is P-G-primary. To do this, first observe that

$$\begin{aligned} P &= \sqrt[G]{\operatorname{Ann}(M/N_1)} \cap \sqrt[G]{\operatorname{Ann}(M/N_2)} \\ &= \sqrt[G]{\operatorname{Ann}(M/N_1) \cap \operatorname{Ann}(M/N_2)} \\ &= \sqrt[G]{\operatorname{Ann}(M/(N_1 \cap N_2))}. \end{aligned}$$

Now, suppose that $\mu_r : M/(N_1 \cap N_2) \to M/(N_1 \cap N_2)$ is not injective. Then there exists $x \in M$ such that $rx \in N_1 \cap N_2$ and $x \notin N_1 \cap N_2$ (so that $\overline{x} \neq 0 \in M/(N_1 \cap N_2)$). This implies that for some $i = 1, 2$ we have $rx \in N_i$, yet $x \notin N_i$. Then the map $\mu_r : M/N_i \to M/N_i$ is not injective, and hence nilpotent. Therefore, $r^n M \subseteq N_i$, which implies $r^n \in \operatorname{Ann}(M/N_i)$, or that $r \in \sqrt[G]{\operatorname{Ann}(M/N_i)} = P = \sqrt[G]{\operatorname{Ann}(M/(N_1 \cap N_2))}$. In particular, $r^m M \subseteq N_1 \cap N_2$ for some m, so that $\mu_r : M/(N_1 \cap N_2) \to M/(N_1 \cap N_2)$ is nilpotent.

With this fact, given a G-primary decomposition as in Definition 3.8, we can collect the P-G-primary components together and replace them by their intersection to satisfy (1). To satisfy (2), discard any unnecessary G-primary components (which does not change the intersection). $\square$

Lemma 3.10. *Suppose R is a G-graded ring, P is a G-prime ideal, and $I_1, \dots, I_n$ are homogeneous ideals of R. If $P = \bigcap_{j=1}^n I_j$, then $P = I_j$ for some j.*

Proof. Clearly $P \subseteq I_j$ for all j, so it suffices to show that $I_j \subseteq P$ for some j. By induction, we only need to show the case $n = 2$. Suppose, by way of contradiction, that $I_1, I_2 \nsubseteq P$. Then for $j = 1, 2$, there exists a G-homogeneous element $a_j \in I_j \smallsetminus P$. On the other hand,

$$a_1 a_2 \in I_1 I_2 \subseteq I_1 \cap I_2 = P,$$

and because P is G-prime, we must have $a_1 \in P$ or $a_2 \in P$, a contradiction. Therefore

$I_j \subseteq P$ for $j = 1$ or $j = 2$. $\square$

Proposition 3.11. *Suppose R is a G-Noetherian ring and $N \subseteq M$ are graded R-modules. If $N = \bigcap N_i$ is an irredundant G-primary decomposition, then the G-prime ideals P_i that occur as G-radicals of the* $\operatorname{Ann}(M/N_i)$ *depend only on M and N.*

Proof. Without loss of generality, we may assume $N = 0$. That is, the G-primes that appear as G-radicals of the $\operatorname{Ann}(M/N_i)$ for a G-primary decomposition of $N \subseteq M$ are the same as the G-primes that appear as G-radicals of the corresponding annihilators in a G-primary decomposition of $0 \subseteq M/N$.

Assume $N = 0$ and $\bigcap_i N_i = 0$ is an irredundant G-primary decomposition. We will show that $P = \sqrt[G]{\operatorname{Ann}(M/N_i)}$ for some i if and only if $P = \operatorname{Ann}(f)$ for some G-homogeneous element $f \in M$. Suppose first that $P = \operatorname{Ann}(f) = (0 :_R f)$ is a G-prime ideal. Then

$$(0 :_R f) = (\bigcap_i N_i :_R f) = \bigcap_i (N_i :_R f).$$

By Lemma 3.10, $(0 :_R f) = (N_i :_R f)$ for some i. Let $P_i = \sqrt[G]{\operatorname{Ann}(M/N_i)}$. Because

$$(0 :_R f) = \sqrt[G]{(0 :_R f)} = \sqrt[G]{(N_i :_R f)},$$

it will suffice to show that $\sqrt[G]{(N_i :_R f)} = \sqrt[G]{\operatorname{Ann}(M/N_i)}$; i.e., this will show $\operatorname{Ann}(f) = P_i$. To see this, suppose $x \in \sqrt[G]{\operatorname{Ann}(M/N_i)}$ is homogeneous. Then $x^n \in \operatorname{Ann}(M/N_i)$ for some n, so that $x^n M \subseteq N_i$. This implies $x^n f \in N_i$, or that $x \in \sqrt[G]{(N_i :_R f)}$.

Conversely, suppose $x \in \sqrt[G]{(N_i :_R f)}$, so that $x^n \in (N_i :_R f)$ for some n. Because N_i is G-primary, x^n defines a map $\mu_{x^n} : M/N_i \to M/N_i$ that is either injective or nilpotent. Then the fact that $f \notin N_i$ (because $(N_i :_R f) = P \subsetneq R$) implies that μ_{x^n} is not injective. Thus $\mu_{x^N} = 0$ for some N, so that $x \in \sqrt[G]{\operatorname{Ann}(M/N_i)}$.

Now, suppose $P_j = \sqrt[G]{\operatorname{Ann}(M/N_j)}$. Because R is G-Noetherian, Remark 3.5 gives

$$P_j^n \subseteq \operatorname{Ann}(M/N_j)$$

for some n. Let t be the smallest (positive) integer such that $\left(\bigcap_{i\neq j} N_i\right) P_j^t \subseteq N_j$ (note that $t \geq 1$ as $\bigcap_{i\neq j} N_i \nsubseteq N_j$). This implies that

$$\left(\bigcap_{i\neq j} N_i\right) P_j^t \subseteq \bigcap_i N_i = 0,$$

yet

$$\left(\bigcap_{i\neq j} N_i\right) P_j^{t-1} \nsubseteq N_j.$$

Choose a homogeneous $y \in \left(\bigcap_{i\neq j} N_i\right) P_j^{t-1}$ such that $y \notin N_j$. Then

$$yP_j \subseteq \left(\bigcap_{i\neq j} N_i\right) P_j^t = 0,$$

so that $P_j \subseteq \operatorname{Ann}(y) = (0 :_R y)$. This gives $P_j \subseteq \sqrt[G]{(0 :_R y)}$, and we also have

$$\sqrt[G]{(0 :_R y)} = \sqrt[G]{(\bigcap_i N_i :_R y)} = \bigcap_i \sqrt[G]{(N_i :_R y)} = \sqrt[G]{(N_j :_R y)} \cap \left(\bigcap_{i\neq j} \sqrt[G]{(N_i :_R y)}\right).$$

Because $y \in \bigcap_{i\neq j} N_j$, we have $(N_i :_R y) = R$ for each $i \neq j$, and so $\bigcap_{i\neq j} \sqrt[G]{(N_i :_R y)} = R$. By an argument we have already used in this proof, we have that $\sqrt[G]{(N_j :_R y)} = P_j$. Putting all of this together, we have that

$$P_j \subseteq \operatorname{Ann}(y) \subseteq P_j,$$

which gives the desired equality. $\square$

Given G-graded R-modules $N \subseteq M$ and an irredundant G-primary decomposition of $N = \bigcap_i N_i$, the G-prime ideals that appear as G-radicals of the $\operatorname{Ann}(M/N_i)$ are called the *associated G-primes of N in M*, or the *associated G-primes of M/N*. We denote this set by $\operatorname{Ass}_R^G(M/N)$ or simply $\operatorname{Ass}^G(M/N)$ if R is understood. In the case $N = 0$, we will write $\operatorname{Ass}^G(M)$ and simply call these the *associated G-primes of M*.

Corollary 3.12. *For a G-Noetherian ring R, a finitely generated G-graded R-module M, and $P \in \operatorname{Spec}^G(R)$, we have $P \in \operatorname{Ass}^G(M)$ if and only if $P = \operatorname{Ann}(f)$ for some homogeneous element $f \in M$. In particular, $P \in \operatorname{Ass}^G(R)$ if and only if $P = \operatorname{Ann}(f)$ for some homogeneous $f \in R$. Also, the union of the associated G-primes of R is contained in the collection of zerodivisors of R, but this containment is not, in general, reversible.*

As an example, one can consider the ring R from Example 2.11 (2). As noted in the example, R is a $\mathbb{Z}/n\mathbb{Z}$-field, so the only associated G-prime ideal is 0. On the other hand, R need not even be reduced in general under the trivial grading.

Chapter 4

Dimension, Integrality, and Height

One of the main goals of this chapter will be to define the G-graded dimension of a ring R and relate it to other invariants of R or G (or both).

4.1 Dimension

Definition 4.1. Let R be a G-graded ring. We define

$$\dim^G(R) := \sup\{n \mid P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n \text{ is a chain of } G\text{-prime ideals of } R\},$$

and for any G-homogeneous ideal $I \subseteq R$

$$\mathrm{ht}^G(I) := \inf\{\dim^G(R_{(P)}) \mid P \supseteq I \text{ and } P \text{ is } G\text{-prime}\}.$$

We say that a G-prime of R is *minimal over* I if $\mathrm{ht}^G(P/I) = 0$.

Remark 4.2. Suppose R is G-graded and H is a torsion-free subgroup of G.

1. From Proposition 2.12 we have $\mathrm{Spec}^G(R) \subseteq \mathrm{Spec}^{G/H}(R)$.

2. If P is a minimal G/H-prime of R (i.e., minimal over 0), then $P^{*G} = P$.

As a preliminary example of computing dimension, consider the following. Suppose R is a G-graded ring, t is an indeterminate over R, and set $S = R[t, t^{-1}]$. Then S is a $\widetilde{G} := G \oplus \mathbb{Z}$-graded ring in the obvious way. That is, $S_{(g,n)} = R_g t^n$. Observe that if $H = G \oplus \{0\}$, then $S_H \cong R$ (as G-graded rings). We use this idea in the proof of the next proposition.

Proposition 4.3. *Let R be a G-graded ring, suppose $t_1, \ldots, t_\ell$ are indeterminates, and set $S = R[t_1, t_1^{-1}, \ldots, t_\ell, t_\ell^{-1}]$. If $\widetilde{G} = G \oplus \mathbb{Z}^\ell$, and we endow S with the obvious $\widetilde{G}$-grading, then $\dim^{\widetilde{G}}(S) = \dim^G(R)$.*

Proof. By induction, we need only show the case $\ell = 1$, so suppose t is an indeterminate and $\widetilde{G} = G \oplus \mathbb{Z}$. We want to show there is an inclusion preserving, one-to-one correspondence between the $\widetilde{G}$-prime ideals of S and the G-prime ideals of R. If P is a $\widetilde{G}$-prime of S, then certainly $P_H := P \cap R$ is a G-prime ideal of $R = S_H$. We also claim that if $Q \in \operatorname{Spec}^G(R)$, then $QS \in \operatorname{Spec}^{\widetilde{G}}(S)$. Indeed, $(QS)_{(g,n)} = Q_g t^n$, and so S/QS is a $\widetilde{G}$-domain. Finally, we only need to verify that $P_H S = P$ and $(QS)_H = Q$, but this is clear. $\square$

One consequence of the previous proposition is the following. Let $R = k[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}]$ be $\mathbb{Z}^d$-graded in the obvious way, and suppose H is the subgroup of $G := \mathbb{Z}^d$ generated by $(1, 0, \ldots, 0)$. As a G/H-graded ring, we have $R = (k[t_1, t_1^{-1}])[t_2, t_2^{-1}, \ldots, t_d, t_d^{-1}]$, where all of $k[t_1, t_1^{-1}]$ is in degree $\overline{0}$. Induction and the previous proposition then imply $\dim^{G/H}(R) = \dim(k[t_1, t_1^{-1}]) = 1$.

4.2 Integrality

Suppose $R \subseteq S$ is an extension of G-graded rings. We say that a homogenous element $x \in S$ is *integral* over R if there exists an equation of the form

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

for some n, where $a_i \in R$ for $i = 0, \dots, n-1$. Note that we may take the a_i to be homogeneous. If every homogeneous element of S is integral over R, we say that S is *integral* over R, or that $R \subseteq S$ is an *integral extension*. Because the sum and product of integral elements are also integral, this is just the usual definition of integrality for an extension of rings. However, we state it in terms of homogeneous elements to be consistent with the themes in this thesis. Much of the following is developed as in [2].

Given a homogeneous homomorphism of G-graded rings $\phi : R \to S$ and a G-homogeneous ideal $J \subseteq S$, we define the *contraction* of J, denoted $J \cap R$, to be $\phi^{-1}(J)$. With this notation, it is easy to show:

- The contraction $J \cap R$ is a homogeneous ideal of R.
- If J is G-prime, then $J \cap R$ is also G-prime.

Proposition 4.4. *Integrality is preserved by G-homogeneous quotients and localization. More precisely, suppose $R \subseteq S$ is an integral extension of G-graded rings.*

1. *If J is a G-homogeneous ideal of S, and $I = J \cap R$, then S/J is integral over R/I.*
2. *If W is a multiplicatively closed subset of G-homogeneous elements of R, then S_W is integral over R_W.*

Proof. Note that homogeneous ideals are, in particular, ideals, and homogeneous localization is a special case of the usual localization. The fact that homogeneous integrality is no

different from ordinary integrality gives us these results from the corresponding results in the non-graded case. □

Proposition 4.5. *Let $R \subseteq S$ be G-domains, and suppose S is integral over R. Then S is a G-field if and only if R is a G-field.*

Proof. Suppose R is a G-field, and let $x \in S \smallsetminus \{0\}$ be homogeneous and integral over R. Then there exists an equation of the form

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0,$$

with minimal n and the $a_i \in R$ homogeneous. Since S is a G-domain, we must have $a_0 \neq 0$. Rearranging the equation, we have

$$x(x^{n-1}a_{n-1} + \cdots + a_1) = a_0,$$

and dividing by a_0 shows that $x^{-1} = a_0^{-1}(x^{n-1}a_{n-1} + \cdots + a_1) \in S$. Thus S is a G-field.

Suppose S is a G-field. If $x \in R \smallsetminus \{0\}$ is homogeneous, then $x^{-1} \in S$, and is hence integral over R. So given an equation of integrality for x^{-1}:

$$x^{-n} + a_{n-1}x^{-n+1} + \cdots + a_0 = 0,$$

we simply multiply through by x^{n-1}, and rearrange to get

$$x^{-1} = -(a_{n-1} + a_{n-2}x + \cdots + a_0x^{n-1}),$$

so that $x^{-1} \in R$. □

Corollary 4.6. *Suppose $R \subseteq S$ is an integral extension of G-graded rings. Let Q be a*

G-prime ideal of S and $P = Q \cap R$ its contraction. Then Q is G-maximal if and only if P is G-maximal.

Proof. Apply Propositions 4.4 and 4.5. $\square$

Corollary 4.7 (Incomparability)**.** *Suppose $R \subseteq S$ is an integral extension of G-graded rings. Suppose $Q \subseteq Q'$ are G-prime ideals of S. If $Q \cap R = Q' \cap R$, then $Q = Q'$.*

Proof. This is just an application of the previous corollary. $\square$

Theorem 4.8 (Lying Over)**.** *Let $R \subseteq S$ be an integral extension of G-graded rings and P a G-prime ideal of R. There exists a G-prime ideal Q of S such that $Q \cap R = P$.*

Proof. By Proposition 4.4, localizing at (P) gives an integral extension $R_{(P)} \subseteq S_{(P)}$. Let m be a G-maximal ideal of $S_{(P)}$. By Corollary 4.6, $m \cap R_{(P)}$ is G-maximal, so we must have $m \cap R_{(P)} = PR_{(P)}$. Because m is G-prime in $S_{(P)}$, there exists $Q \in \operatorname{Spec}^G(S)$ such that $m = Q_{(P)}$, and it is an easy exercise to show that $Q \cap R = P$. $\square$

Theorem 4.9 (Going Up)**.** *Suppose $R \subseteq S$ is an integral extension of G-graded rings. If $P_1 \subseteq \cdots \subseteq P_n$ is a chain of G-prime ideals in R and $Q_1 \subseteq \cdots \subseteq Q_m$, where $m < n$, is a chain of G-prime ideals of S such that $Q_i \cap R = P_i$ for $i = 1, \ldots, m$, then there exist G-primes $Q_{m+1}, \ldots, Q_n$ of S such that $Q_1 \subseteq \cdots \subseteq Q_n$ and $Q_i \cap R = P_i$ for all i.*

Proof. By induction we need only show the case $m = 1, n = 2$. This is a straightforward application of Proposition 4.4 and Theorem 4.8. $\square$

Corollary 4.10. *Suppose $R \subseteq S$ is an extension of G-graded rings. If the extension is integral, then $\dim^G(R) = \dim^G(S)$.*

Proof. This is an easy consequence of the previous several results. $\square$

Corollary 4.11. *If R is a G-graded ring, H is a subgroup of G, and G/H is torsion, then $\dim^H(R_H) = \dim^G(R)$, and $\dim(R_H) = \dim(R)$.*

Proof. G/H torsion implies $R_H \subseteq R$ is integral, so 4.10 implies the dimensions are equal. $\square$

With this corollary we can prove a result about the preservation of the G-Artinian property when we pass to a quotient G/H and the induced grading on R. It is the analogue of Theorem 2.41, but we first show a lemma which is familiar in the case of non-graded Artinian rings.

Lemma 4.12. *Let R be a G-graded ring. Then R is G-Artinian if and only if R is G-Noetherian and* $\dim^G(R) = 0$.

Proof. First, if R is G-Noetherian and $\dim^G(R) = 0$, then all G-prime ideals of R are minimal, and so there are only finitely many G-prime ideals of R, say $\operatorname{Spec}^G(R) = \{P_1, \ldots, P_\ell\}$. By Remark 3.5,

$$\left(\prod_{i=1}^{\ell} P_i\right)^k \subseteq \left(\bigcap_{i=1}^{\ell} P_i\right)^k = \left(\sqrt[G]{0}\right)^k \subseteq 0,$$

for some k. Then we can suppose $Q_1 Q_2 \cdots Q_s = 0$ for some $Q_i \in \operatorname{Spec}^G(R)$, not necessarily distinct. Suppose $s = 1$. Then 0 is a G-maximal ideal, so R is a G-field, and clearly G-Artinian. If $s > 1$, consider the exact sequence

$$0 \to Q_1 \cdots Q_{s-1} \to R \to R/Q_1 \cdots Q_{s-1} \to 0.$$

The first module is a finitely generated graded module over R/Q_s (a G-field), so it is G-Artinian. The third is G-Artinian by induction, and so R is also G-Artinian.

Conversely, suppose R is G-Artinian. We first claim that R has only finitely many G-prime ideals, all of which are G-maximal; i.e., $\dim^G(R) = 0$. Suppose, by way of contradiction, that $m_1, m_2, \ldots$ is an infinite list of distinct G-maximal ideals. Because

$$m_1 \supseteq m_1 \cap m_2 \supseteq m_1 \cap m_2 \cap m_3 \supseteq \cdots$$

is a descending chain of G-homogeneous ideals, there exists k such that $m_1 \cap \cdots \cap m_k = m_1 \cap \cdots \cap m_k \cap m_{k+1}$. Using the proof of Lemma 3.10, one sees that $m_i \subseteq m_{k+1}$ for some i. Since both are G-maximal, we must have $m_i = m_{k+1}$, a contradiction. Therefore there are only finitely many G-maximal ideals, say $\{m_1, \ldots, m_n\}$. Now suppose $P \in \operatorname{Spec}^G(R)$ and let $J = m_1 \cap \cdots \cap m_n$ be the Jacobson G-radical of R. By Proposition 2.42, J is nilpotent, and thus contained in $\sqrt[G]{0}$, which is the intersection of all G-prime ideals of R by Proposition 3.6. In particular, $m_1 \cap \cdots \cap m_n \subseteq P$, and using the proof of Lemma 3.10 again, we see that $m_i \subseteq P$ for some i. But because m_i is G-maximal, $m_i = P$. Thus all G-prime ideals of R are G-maximal, and it now follows that $\dim^G(R) = 0$.

All that remains to be shown is that R is G-Noetherian. Using the same argument as in the beginning of the proof, we can write $Q_1 Q_2 \cdots Q_s = 0$ for some $Q_i \in \operatorname{maxSpec}^G(R)$, not necessarily distinct. We then argue by induction on s that R is G-Noetherian. If $s = 1$, then R is a G-field, which clearly satisfies the ascending chain condition. Suppose $s > 1$ and consider the exact sequence

$$0 \to Q_1 \cdots Q_{s-1} \to R \to R/Q_1 \cdots Q_{s-1} \to 0.$$

Since $R/Q_1 \cdots Q_{s-1}$ is a quotient of a G-Artinian ring, it is G-Artinian, and hence G-Noetherian by induction. The G-graded module $Q_1 \cdots Q_{s-1}$ is a graded R/Q_s-module, since Q_s annihilates it. However, since R/Q_s is a G-field, $Q_1 \cdots Q_{s-1}$ is a G-free R/Q_s-module, and therefore is G-Noetherian if and only if it is G-Artinian. By Proposition 2.34, R is G-Noetherian, and we are done. $\square$

Theorem 4.13. *Suppose R is G-graded and $H \leq G$. If H is a* finite *subgroup, the following are equivalent:*

1. *R is G-Artinian.*

2. R is G/H-Artinian.

Proof. We need only show (1) implies (2). First, suppose R is a G-field. Let

$$I_0 \supseteq I_1 \supseteq \cdots \supseteq I_k \supseteq \cdots$$

be a descending chain of G/H-homogeneous ideals of R. Suppose $|H| = n$. Then for any $k \in \mathbb{N}$, if $x \in I_k$ is G/H-homogeneous, we can write

$$x = \sum_{i=1}^{n} x_i,$$

where $x_i \in R_{g+h_i}$ for some $g \in G$ and the h_i are the distinct elements of H. If $x \neq 0$, then at least one of the x_i is nonzero (in fact, at least two are nonzero if $I_k \neq R$). Now, if $x_i \neq 0$, then x_i is invertible because R is a G-field. Therefore

$$x_i^{-1} x \in R_H. \tag{4.1}$$

Also, the chain of ideals above induces a chain

$$\widehat{I_0} \supseteq \widehat{I_1} \supseteq \cdots \supseteq \widehat{I_k} \supseteq \cdots$$

of ideals of R_H, where $\widehat{I_k} = I_k \cap R_H$. Notice the G/H-grading on R_H as a G/H-graded subring of R is the same as the 0-grading because $R_H = R_{0+H}$.

We claim R_H is Artinian. Indeed, by Corollary 4.11 with G replaced by H and H replaced by 0, we have

$$\dim(R_0) = \dim^H(R_H) \qquad \text{and} \qquad \dim(R_0) = \dim(R_H).$$

Then, since R_H is an H-field (as R is a G-field),

$$\dim(R_H) = \dim(R_0) = \dim^H(R_H) = 0.$$

Further, R_H is Noetherian (by Proposition 2.36 and Theorem 2.41), so it follows that R_H is Artinian. Therefore, there exists $N \in \mathbb{N}$ such that $\widehat{I_k} = \widehat{I_{k+1}}$ for all $k \geq N$. We claim this implies $I_k = I_{k+1}$ for all $k \geq N$.

Indeed, suppose $x \in I_k$ is G/H-homogeneous, where $k \geq N$. We'll show $x \in I_{k+1}$. If $x \neq 0$, we can write

$$x = \sum_{i=1}^{n} x_i$$

as above, and multiply by x_j^{-1} for some $x_j \neq 0$ as in (4.1). Then, since $x_j^{-1}x \in R_H$, we have

$$x_j^{-1}x \in \widehat{I_k} = \widehat{I_{k+1}} \subseteq I_{k+1}.$$

However, as I_{k+1} is an ideal $x_j(x_j^{-1}x) = x \in I_{k+1}$.

If we suppose now that (R, m) is a G-local G-Artinian ring, then for any $P \in \operatorname{Spec}^{G/H}(R)$, we have $P^{*G} = m$, so

$$\dim^{G/H}(R) = \dim^{G/H}(R/m).$$

Because R/m is a G-field, we know that R/m is G/H-Artinian, and so $\dim^{G/H}(R) = 0$. The fact that R is G/H-Noetherian comes from Theorem 2.41, as H is finitely generated and R is G-Noetherian. Thus R is G-Artinian. Finally, suppose R is any G-Artinian ring. We will show that $\operatorname{ht}^{G/H}(m) = 0$ for any $m \in \operatorname{maxSpec}^{G/H}(R)$. If $m \in \operatorname{maxSpec}^{G/H}(R)$, then $m^{*G} = P$ for some $P \in \operatorname{Spec}^G(R)$. Then the G-graded ring $R_{(P)}$ (here this is G-homogeneous localization) is G-Artinian and G-local, so $R_{(P)}$ is G/H-Artinian. Now observe that by definition, $m \smallsetminus P$ contains no G-homogeneous elements of R, so m survives in $R_{(P)}$, and we

have

$$\mathrm{ht}^{G/H}(m) = \mathrm{ht}^{G/H}(mR_{(P)}) = 0.$$

Thus, R is G/H-Noetherian and $\dim^{G/H}(R) = 0$, so R is G/H-Artinian. □

The next proposition gives another useful application of the results on integrality. In particular, it shows that for a ring graded by a finite group, the G-graded dimension and usual dimension always coincide.

Proposition 4.14. *Suppose R is a G-graded ring and H is a subgroup of G. If G/H is torsion, then $R_H \subseteq R$ is an integral extension.*

Proof. G/H being torsion is almost a nilpotence condition on G-homogeneous elements of R. Suppose $r \in R_g$ for some $g \in G$. Then there exists n such that $r^n \in R_{ng} = R_h$ for some $h \in H$. That is, r satisfies a polynomial of the form $x^n - s$, where $s \in R_H$. □

Corollary 4.15. *If R is a G-graded ring, where G is a torsion group, then* $\dim^G(R) = \dim(R_0)$.

Proof. Let $H = 0$ in Corollary 4.11. □

Theorem 4.16. *Suppose R is a properly graded G-field and H is a finitely generated subgroup of G. Then*

1. $\dim^{G/H}(R) = \dim(R_H)$.

Write $H = A \oplus B$, where A is torsion-free and B is torsion. Then

2. $\dim(R_H) = \dim(R_B) + \mathrm{rank}(H)$.

Proof. We claim that the G/H-homogeneous ideal structure of R is completely determined by—in fact, exactly the same as—that of R_H. Let $I \subseteq R$ be a G/H-homogeneous ideal, and

suppose $x \in I \setminus \{0\}$ is G/H-homogeneous. Then we can write

$$x = x_1 + \cdots + x_n \in R_{g+H}$$

for some $g \in G$, where $x_i \in R_{g+h_i} \setminus \{0\}$ for $h_i \in H$, $h_i \neq h_j$ when $i \neq j$. Since R is a G-field, each x_i is invertible, so $x_i^{-1}x \in R_H$ for each i. It's straightforward then to see that

$$I = (I \cap R_H)R.$$

Thus for G/H-homogeneous ideals $I, J \subseteq R$ we have $I \subseteq J$ if and only if $I \cap R_H \subseteq J \cap R_H$. Also, I is G/H-prime in R if and only if $I \cap R_H$ is prime in R_H. Thus $\dim^{G/H}(R) = \dim(R_H)$.

For the second claim, with $H = A \oplus B$ as above, Theorem 2.13 implies that

$$R_H \cong R_B[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}],$$

where $d = \operatorname{rank}(H)$. And now it's easy to see that

$$\dim(R_H) = \dim(R_B) + d.$$

$\square$

Corollary 4.17. *With R as in the above theorem, if H is torsion-free, we have* $\dim^{G/H}(R) = \operatorname{rank}(H)$.

Proof. It suffices to show that $\dim(R_B) = 0$. This follows from the fact that if H is torsion-free, then R_H is a properly graded H-field, and so $R_B = R_0$ is an honest-to-goodness field in the usual sense (see Corollary 2.14), so it has dimension 0. $\square$

4.3 Heights of Primes

We begin this section by proving the graded analogues of Krull's principal ideal theorem and Krull's height theorem for a G-Noetherian ring.

Theorem 4.18 (Krull's principal ideal theorem)**.** *Let R be a G-Noetherian G-graded ring and $(a) \neq R$ a homogeneous principal ideal. Then $\mathrm{ht}^G(P) \leq 1$ for any minimal G-prime P of (a), and $\mathrm{ht}^G(P) = 1$ if a is not a zerodivisor of R.*

Proof. For the first statement, consider a minimal G-prime P of (a). First note that $\mathrm{ht}^G(P) = \dim^G(R_{(P)})$ and that $PR_{(P)}$ is a minimal G-prime of $aR_{(P)}$ (in $R_{(P)}$). Thus, we can assume that R is a G-local ring whose unique G-maximal ideal m is a minimal G-prime of (a). Thus, for any $Q \in \mathrm{Spec}^G(R)$ with $Q \neq m$, we want to show $\mathrm{ht}^G(Q) = 0$. Let $Q^{(i)}$ be the inverse image of $Q^i R_{(Q)}$ in R and form the chain of homogeneous ideals

$$(a) + Q^{(1)} \supseteq (a) + Q^{(2)} \supseteq \cdots .$$

Since $\mathrm{Spec}^G(R/(a))$ has only one element, namely $m/(a)$, $R/(a)$ is G-Artinian. Hence there exists $n \in \mathbb{N}$ such that

$$(a) + Q^{(n)} = (a) + Q^{(n+1)}.$$

We write a homogeneous $s \in Q^{(n)}$ as $s = ra + s'$ with $r \in R, s' \in Q^{(n+1)}$, both homogeneous. By the definition of $Q^{(n)}$, it follows from $ra \in Q^{(n)}$, $a \notin Q$, that $r \in Q^{(n)}$. Then we get

$$Q^{(n)} = aQ^{(n)} + Q^{(n+1)},$$

and so $Q^{(n)} = Q^{(n+1)}$ by the graded Nakayama's Lemma. Then $Q^n R_{(Q)} = Q^{n+1} R_{(Q)}$ in $R_{(Q)}$ and thus $Q^n R_{(Q)} = (0)$, again by Nakayama's Lemma. Since Q is nilpotent, it follows that $\mathrm{ht}^G(Q) = \dim^G(R_{(Q)}) = 0$.

Suppose a is a non-zerodivisor. If $\mathrm{ht}^G(m) = 0$, then m is nilpotent, so a is nilpotent, a contradiction. Thus, we must have $\mathrm{ht}^G(m) > 0$. $\square$

Corollary 4.19 (Krull's height theorem)**.** *If R is G-Noetherian, and $I := (a_1, \ldots, a_n)$ is an ideal generated by G-homogeneous elements a_i, then $\mathrm{ht}^G(P) \leq n$ for any minimal G-prime P of I.*

Proof. The case $n = 1$ has already been shown, so suppose $n > 1$ and that the result holds for integers less than n. Let P be a minimal G-prime over I, and suppose, by way of contradiction, that $\mathrm{ht}^G(P) > n$. That is, there exists a chain

$$P = P_{n+1} \supsetneq P_n \supsetneq \cdots \supsetneq P_0$$

of G-primes of R. If $a_1 \in P_1$, then P is also minimal over $P_1 + (a_2, \ldots, a_n)$, and in the ring R/P_1, P/P_1 is minimal over the image of $(a_2, \ldots, a_n)$. Then the chain

$$P/P_1 = P_{n+1}/P_1 \supsetneq P_n/P_1 \supsetneq \cdots \supsetneq P_1/P_1$$

contradicts the induction hypothesis. Therefore, we may assume that $a_1 \in P_k$ and $a_1 \notin P_{k-1}$ for some $2 \leq k \leq n+1$ (certainly, $a_1 \in P_{n+1}$). It will suffice to show that there is a G-prime ideal Q such that $a_1 \in Q$ and $P_k \supsetneq Q \supsetneq P_{k-2}$, for then we can repeat the procedure until we get a chain of length $n+1$ such that $a_1 \in P_1$.

Consider the G-local G-domain $R_{(P_k)}/P_{k-2}R_{(P_k)}$. Let Q' be a minimal G-prime over the image of a_1 in this new ring. Since the image of a_1 is nonzero, $Q' \neq 0$. On the other hand, by the previous theorem, $\mathrm{ht}^G(Q') \leq 1$. Since $P_k R_{(P_k)}$ has G-height at least two, the inverse image Q of Q' in R satisfies the required properties. $\square$

The next result is a useful lemma which gives a bound on (graded) height involving the torsion-free rank of the subgroup.

Lemma 4.20. *Suppose R is a G-graded ring and H is a finitely generated torsion-free subgroup of G. Let $P \in \operatorname{Spec}^{G/H}(R)$ and set $P^* := P^{*G}$. Then $\operatorname{ht}^{G/H}(P/P^*) \leq \operatorname{rank}(H)$.*

Proof. Proposition 2.12 and Remark 2.23 imply that P^* is both G- and G/H-prime. Considering R/P^* we may assume R is both a G- and G/H-domain and that $P^* = 0$. Let W be the set of nonzero G-homogeneous elements of R. Note that $P \cap W = \emptyset$. By localizing at W, we can assume that R is a G-field, and P is G/H-prime. Applying Corollary 4.17, we have that $\dim^{G/H}(R) \leq \operatorname{rank}(H)$, which implies that $\operatorname{ht}^{G/H}(P/P^*) = \operatorname{ht}^{G/H}(P) \leq \operatorname{rank}(H)$ (the inequality rather than equality comes from the fact that we do not know if R is properly graded). □

A direct consequence of the previous lemma is the following, which is a generalization of a $\mathbb{Z}$-graded result due to Matijevic-Roberts [14]:

Proposition 4.21. *Let R be a G-graded ring and H a finitely generated torsion-free subgroup of G with $\operatorname{rank}(H) = 1$. Suppose $P \in \operatorname{Spec}^{G/H}(R)$, and set $P^* := P^{*G}$. If $P^* \neq P$, then*

$$\operatorname{ht}^{G/H}(P) = \operatorname{ht}^{G/H}(P^*) + 1.$$

Proof. Let $\operatorname{ht}(-) := \operatorname{ht}^{G/H}(-)$. Clearly, $\operatorname{ht}(P) \geq \operatorname{ht}(P^*) + 1$, so we only need to show the reverse inequality. If $\operatorname{ht}(P^*) = \infty$, there is nothing to show, so assume $n := \operatorname{ht}(P^*) < \infty$, and use induction on n. If $n = 0$, then we want to show $\operatorname{ht}(P) \leq 1$. Suppose, by way of contradiction, that $P \supsetneq Q_1 \supsetneq Q_2$ is a chain of G/H-prime ideals with Q_2 minimal. Applying $(-)^{*G}$ to the chain and using the fact that P^* is minimal, we get that $P^* = (Q_2)^{*G} = Q_2$ by (2) of Remark 4.2, as Q_2 is minimal. Then Lemma 4.20 implies that $Q_1 = P$ or $Q_1 = P^*$, a contradiction.

Suppose $n > 0$. Let Q be any G/H-prime ideal properly contained in P. It suffices to show that $\operatorname{ht}(Q) \leq n$. Certainly, $Q^{*G} \subseteq P^*$. If $Q^{*G} = P^*$, then by Lemma 4.20, $Q = P^*$. If

$Q^{*G} \subsetneq P^*$, then $\mathrm{ht}(Q^{*G}) \leq n-1$, and by induction $\mathrm{ht}(Q) \leq n$. □

Much along the lines of a recent result by Uliczka [18], we can generalize this as follows:

Theorem 4.22. *Let R be a G-graded ring and H a finitely generated torsion-free subgroup of G. If $P \in \mathrm{Spec}^{G/H}(R)$, and we set $P^* := P^{*G}$, then*

$$\mathrm{ht}^{G/H}(P) = \mathrm{ht}^{G/H}(P^*) + \mathrm{ht}^{G/H}(P/P^*).$$

Proof. Throughout this proof, set $\mathrm{ht}(-) := \mathrm{ht}^{G/H}(-)$. Use induction on $n := \mathrm{rank}(H)$. The case $n = 1$ is the previous proposition. If $n > 1$, write $H = A \oplus B$, where $\mathrm{rank}(A) = n-1$ and $\mathrm{rank}(B) = 1$. Define $G' = G/B$ and $H' = H/B$. Since $G'/H' \cong G/H$, we have $P \in \mathrm{Spec}^{G/H}(R) = \mathrm{Spec}^{G'/H'}(R)$ and $\mathrm{ht}(-) = \mathrm{ht}^{G'/H'}(-)$. Set $Q = P^{*G'}$. Now, H' is a torsion-free subgroup of G' with rank $n-1$, so by induction

$$\mathrm{ht}(P) = \mathrm{ht}(Q) + \mathrm{ht}(P/Q).$$

If $P^* = Q$, there is nothing more to show, so suppose $P^* \subsetneq Q$. Using induction again, now applied to $H'' := H/A$ as a subgroup of $G'' := G/A$, and noting that $\mathrm{ht}^{G''/H''}(-) = \mathrm{ht}(-)$, we have

$$\mathrm{ht}(Q) = \mathrm{ht}(Q^{*G''}) + \mathrm{ht}(Q/Q^{*G''}).$$

Using Proposition 2.26 and the fact that $A \cap B = \{0\}$, we have $Q^{*G''} = P^*$. Since $P^* \subsetneq Q$, $\mathrm{ht}(Q/P^*) = 1$ by Proposition 4.21. Therefore $\mathrm{ht}(Q) = \mathrm{ht}(P^*) + 1$, and this gives

$$\mathrm{ht}(P) = \mathrm{ht}(P^*) + 1 + \mathrm{ht}(P/Q).$$

Also, since $P^* \subsetneq Q$, we have $1 + \mathrm{ht}(P/Q) \leq \mathrm{ht}(P/P^*)$. From this it follows that

$$\mathrm{ht}(P) \leq \mathrm{ht}(P^*) + \mathrm{ht}(P/P^*),$$

and the reverse inequality is trivial. □

Up until this point, none of the results we have established on height compare what happens when we consider the same ring with different gradings. However, if we add the hypothesis that R is G-Noetherian, we can apply Theorem 4.18 and get the following proposition.

Proposition 4.23. *Suppose R is G-graded and G-Noetherian, H is a finitely generated torsion-free subgroup of G, and $P \in \mathrm{Spec}^G(R)$. Then*

$$\mathrm{ht}^{G/H}(P) = \mathrm{ht}^G(P).$$

Proof. Notice that since $\mathrm{Spec}^G(R) \subseteq \mathrm{Spec}^{G/H}(R)$, we always have $\mathrm{ht}^G(P) \leq \mathrm{ht}^{G/H}(P)$. If $\mathrm{rank}(H) = 1$, use induction on $n := \mathrm{ht}^{G/H}(P)$. Suppose $n = 0$. Then $\mathrm{ht}^G(P) \leq \mathrm{ht}^{G/H}(P)$ implies $\mathrm{ht}^G(P) = 0$. Suppose $n > 0$. Let $Q \in \mathrm{Spec}^{G/H}(R)$ be contained in P with $\mathrm{ht}^{G/H}(Q) = n-1$. If $Q \in \mathrm{Spec}^G(R)$, we're done by induction. If not, then $\mathrm{ht}^{G/H}(Q^{*G}) = n-2$ by Proposition 4.21. Now, by passing to R/Q^{*G} we can assume R is a G-domain and $P \in \mathrm{Spec}^G(R)$ has $\mathrm{ht}^{G/H}(P) = 2$. Then it will be enough to show that $\mathrm{ht}^G(P) \geq 2$. Let $f \in P$ be a nonzero G-homogeneous element of P. Then P is not (G/H-)minimal over (f) by Theorem 4.18. Since $\mathrm{ht}^{G/H}(P) = 2$, there exists $Q' \in \mathrm{Spec}^{G/H}(R)$ such that $P \supsetneq Q' \supseteq (f) \supsetneq (0)$. Then $(Q')^{*G} \in \mathrm{Spec}^G(R)$ and $P \supsetneq (Q')^{*G} \supsetneq (0)$, so that $\mathrm{ht}^G(P) \geq 2$. This establishes the result in the case $\mathrm{rank}(H) = 1$.

Suppose $\mathrm{rank}(H) > 1$. Write $H = A \oplus B$, where $\mathrm{rank}(A) = 1$. From the previous case,

we know

$$\mathrm{ht}^G(P) = \mathrm{ht}^{G/A}(P).$$

By Theorem 2.41, R is G/A-Noetherian. Notice the subgroup H/A of G/A is torsion-free and has rank one less than that of H. By induction on the rank of H (and the fact that $(G/A)/(H/A) \cong G/H$),

$$\mathrm{ht}^{G/H}(P) = \mathrm{ht}^{G/A}(P),$$

and it now follows that $\mathrm{ht}^G(P) = \mathrm{ht}^{G/H}(P)$. $\square$

With this proposition and the previous theorem, we immediately get the following:

Corollary 4.24. *Let R be a G-Noetherian graded ring and H a finitely generated torsion-free subgroup of G. If $P \in \mathrm{Spec}^{G/H}(R)$, then*

$$\mathrm{ht}^{G/H}(P) = \mathrm{ht}^G(P^{*G}) + \mathrm{ht}^{G/H}(P/P^{*G}).$$

There is another natural corollary which relates the G/H-graded dimension and the G-graded dimension of a G-Noetherian ring which follows from the previous results:

Corollary 4.25. *If R is a G-graded G-Noetherian ring, and H is a finitely generated torsion-free subgroup of G, then*

$$\dim^G(R) \leq \dim^{G/H}(R) \leq \dim^G(R) + \mathrm{rank}(H).$$

Proof. The first inequality simply follows from the fact that H being torsion-free implies $\mathrm{Spec}^G(R) \subseteq \mathrm{Spec}^{G/H}(R)$.

For the second inequality, if the rank of H is infinite, there is nothing to show. Therfore, we may assume $\mathrm{rank}(H) < \infty$. Let m be any G/H-maximal ideal of R, and set $m^* := m^{*G}$.

Applying (in order) Theorem 4.22, Lemma 4.20, and Theorem 4.23, we get

$$\begin{aligned}\mathrm{ht}^{G/H}(m) &= \mathrm{ht}^{G/H}(m^*) + \mathrm{ht}^{G/H}(m/m^*) \\ &\leq \mathrm{ht}^{G/H}(m^*) + \mathrm{rank}(H) \\ &= \mathrm{ht}^{G}(m^*) + \mathrm{rank}(H).\end{aligned}$$

Because this holds for every G/H-maximal ideal and $\mathrm{ht}^G(m^*) \leq \dim^G(R)$, we have

$$\dim^{G/H}(R) \leq \dim^G(R) + \mathrm{rank}(H).$$

□

Example 4.26. To see that the inequality is, in some sense, the best possible, let k be a field and x an indeterminate and consider $R := k[x]$ and $S := k[x, x^{-1}]$. Set $G = H = \mathbb{Z}$ and give both rings a G-grading by setting $\deg(x) = 1$. Note that

$$\dim^{G/H}(R) = 1 = \dim^G(R).$$

On the other hand, because S is a G-field,

$$\dim^{G/H}(S) = 1 = \dim^G(S) + 1.$$

In particular, both the upper and lower bounds can be attained.

The final two results of this section describe a situation in which $\dim^{G/H}(R) = \dim^G(R)$. Corollary 4.28 is similar to the fact that in a non-negatively $\mathbb{Z}$-graded ring R, the dimension of R can be found by examining only homogeneous prime ideals. In our situation, we have to generalize the condition that R is non-negatively graded, however.

Proposition 4.27. *Suppose R is a G-field and H is a torsion-free subgroup of G. Then R is a G/H-field if and only if $R_H = R_0$.*

Proof. Suppose first that $R_H = R_0$, and let f be any G/H-homogeneous element of R. Since $f \in R_{g+H}$ for some $g \in G$, we can write

$$f = \sum_{i=0}^{k} f_i,$$

where the f_i are in distinct homogeneous components R_{g+h_i}. Since R is a G-field, we can multiply f by f_0^{-1} and we see that $f_0^{-1} f \in R_H$. Since $R_H = R_0$, $f_0^{-1} f = f_0^{-1} f_i$ for some $1 \leq i \leq k$. But this implies that $f = 0$ or that f is G-homogeneous. Thus every nonzero G/H-homogeneous element of R is invertible.

Conversely, suppose R is a G/H-field such that $R_H \neq R_0$. Then there exists $h \in H \setminus \{0\}$ such that $R_h \neq 0$. Choose $f \in R_h \setminus \{0\}$. Since f is a unit and H is torsion-free, it is easily seen that $1 + f$ cannot be a unit, a contradiction (note that since $1 + f$ is G/H-homogeneous, if $(1 + f)^{-1}$ exists, it must be in R_H).

$\square$

Corollary 4.28. *Let R be a G-graded ring and H a finitely generated torsion-free subgroup of G. Let $K = \{h \in H \mid R_h \neq 0\}$. If there exists a total order on H such that K is a well-ordered subset, then*

1. $\mathrm{maxSpec}^G(R) \subseteq \mathrm{maxSpec}^{G/H}(R)$.

If, in addition, R is G-Noetherian, then

2. $\dim^{G/H}(R) = \dim^G(R)$

Proof. First assume $m \in \mathrm{maxSpec}^G(R)$. Then $m \in \mathrm{maxSpec}^{G/H}(R)$ if and only if R/m is a G/H-field. But the previous proposition states this is if and only if $(R/m)_H = (R/m)_0$. So to

prove (1), it suffices to show that our hypothesis on K implies that for each $m \in \operatorname{maxSpec}^G(R)$, we have $(R/m)_h = 0$ for all $h \in H \smallsetminus \{0\}$. Suppose then that $(R/m)_h \neq 0$ for some $h \neq 0$, say $f \in R_h \smallsetminus \{0\}$. Since R/m is a G-field, we know f^{-1} exists and is nonzero. Then, using either powers of f or f^{-1}, we can find nonzero elements of $(R/m)_H$ of arbitrarily high or low degree (with respect to the total order given on H), which is a contradiction. This proves (1).

Suppose that R is also G-Noetherian. We use induction on $\operatorname{rank}(H) = n$. Suppose $n = 1$ and $m \in \operatorname{maxSpec}^{G/H}(R) \smallsetminus \operatorname{maxSpec}^G(R)$. By Proposition 4.21,

$$\operatorname{ht}^{G/H}(m) = \operatorname{ht}^{G/H}(m^{*G}) + 1.$$

But by Remark 2.21, $m^{*G} \subseteq N$ for some $N \in \operatorname{maxSpec}^G(R)$. Note that in fact $m^{*G} \subsetneq N$. Indeed, if $m^{*G} = N$, then $m^{*G} \in \operatorname{maxSpec}^{G/H}(R)$ by part (1), and so $m = m^{*G}$, a contradiction. Therefore, by Propositions 4.21 and 4.23,

$$\operatorname{ht}^G(N) = \operatorname{ht}^{G/H}(N) \geq \operatorname{ht}^{G/H}(m^{*G}) + 1 = \operatorname{ht}^{G/H}(m).$$

This shows $\dim^{G/H}(R) \leq \dim^G(R)$, and the reverse inequality is obvious. Suppose now the result holds for all such subgroups of rank less than n. Write $H = A \oplus B$, where $\operatorname{rank}(A) = 1$ and $\operatorname{rank}(B) = n - 1$.

Before using induction, note the hypothesis that there exists a total order on H such that K is a well-ordered subset passes to any subgroup $H' \leq H$ and the subset $H' \cap K$, since a subset of a well-ordered set is well-ordered. Thus, we can apply induction to H/A as a subgroup of rank $n - 1$ of G/A (because $H/A \cong B \leq H$) and to A as a subgroup of G of rank 1 to get:

$$\dim^{G/H}(R) = \dim^{G/A}(R) = \dim^G(R).$$

□

4.4 Primes Extended to Polynomial Rings

It is well-known that a prime ideal $p \subseteq R$ extends to a prime ideal $p[t] := pR[t]$ in a polynomial ring over R. Our situation will be the following: R is a G-graded ring, and we introduce a natural $\widetilde{G} := G \oplus \mathbb{Z}$-grading on $R[t]$ by setting $\deg(t) = (0, 1)$; i.e., $(R[t])_{(g,n)} = R_g t^n$. As noted in the discussion preceding Proposition 4.3, it is clear that if R is a G-domain, then $R[t]$ is a $\widetilde{G}$-domain. Thus, if $P \in \operatorname{Spec}^G(R)$, then $PR[t] \in \operatorname{Spec}^{\widetilde{G}}(R[t])$.

Our goal now is to prove a generalization of the fact that in a Noetherian ring the heights of these extended primes remain the same.

Remark 4.29. For Proposition 4.31 (and in Lemma 5.6 as well), common proofs often rely on the familiar prime avoidance lemma. This fails—in one sense—in a very particular way in the graded setting:

1. Let $R = \mathbb{Z}[t]$, where t is an indeterminate, and give R a $\mathbb{Z}$-grading by setting $\deg(t) = 1$. Set $I = (2, t)$, $P_1 = (2)$, and $P_2 = (t)$. Then P_1 and P_2 are $\mathbb{Z}$-prime ideals, and $I \nsubseteq P_1 \cup P_2$, but there is no $\mathbb{Z}$-*homogeneous* element $x \in I$ such that $x \notin P_1 \cup P_2$.

On the other hand, the following statement still holds in general in the graded setting:

2. Suppose $P_1, \ldots, P_n$ are G-homogeneous ideals of R, at most two of which are not G-prime. If I is a G-homogenous ideal of R such that $I \subseteq \bigcup P_i$, then $I \subseteq P_i$ for some i.

Proof of (2). We use induction on n, the number of ideals. For $n = 1$, it is trivially true, and for $n = 2$, it is a matter of group theory. Suppose $n > 2$. Then (reordering if necessary) we may assume P_1 is G-prime. If $I \subseteq \bigcup_{j \neq i} P_j$ for some i, we are done by induction. Therefore, for each $i = 1, \ldots, n$, we may assume $I \nsubseteq \bigcup_{j \neq i} P_j$; that is, there exists $a_i \in P_i$ such that $a_i \notin \bigcup_{j \neq i} P_j$.

Now, each a_i is a sum of G-homogeneous components, say $a_i = \sum_k b_{i,k}$. Because $a_i \notin \bigcup_{j \neq i} P_j$, we know that for each $i = 2, \ldots, n$, at least one of the homogeneous components

is not in P_1. Without loss of generality, we can assume (reordering if necessary) $b_{i,1} \notin P_1$, for $i = 2, \ldots, n$. Because each P_i is homogeneous, $b_{i,1} \in P_i$ for each $i = 1, \ldots, n$. Consider the element

$$b = b_{1,1} + b_{2,1} b_{3,1} \cdots b_{n,1}.$$

Certainly $b \in I$. If $b \in P_1$, then $b_{2,1} \cdots b_{n,1} \in P_1$, and because P_1 is G-prime, we get that $b_{i,1} \in P_1$ for some i, a contradiction. If $b \in P_i$ for $i = 2, \ldots, n$, then $b_{1,1} \in P_i$, also a contradiction. Thus, we have constructed an element $b \in I$, yet $b \notin \bigcup P_i$, a contradiction. $\square$

We first isolate in a lemma an argument that allows us to avoid using prime avoidance.

Lemma 4.30. *Suppose R is a G-Noetherian graded ring and $P \in \operatorname{Spec}^G(R)$. Let $f \in P$ be homogeneous. Then $\operatorname{ht}^G_{R/(f)}(P/(f)) \geq \operatorname{ht}^G(P) - 1$.*

Proof. Using G-homogeneous localization at P, we can assume (R, m) is G-Noetherian and G-local, with $f \in m$. If $\dim^G(R)$ is 0 or 1, there is nothing to show, so assume $\dim^G(R) = n \geq 2$. Suppose, by way of contradiction, that there exists a homogeneous $f \in R$ such that $\dim^G(R/(f)) \leq n - 2$. Let

$$m \supsetneq Q_{n-1} \supsetneq \cdots \supsetneq Q_1 \supsetneq Q_0$$

be a saturated chain of G-prime ideals. Note that $f \notin Q_1$. Indeed, if $f \in Q_1$, then $\dim^G(R/(f)) \geq n - 1$. Therefore, modulo Q_1, f is nonzero so $\dim^G(R/(Q_1, f)) \leq n - 2$. If $\dim^G(R/(Q_1, f)) < n - 2$, induction implies $\dim^G(R/Q_1) < n - 1$, a contradiction. So assume $\dim^G(R/(Q_1, f)) = n - 2$ (this last condition is forced if $n = 2$). Thus, we have

$$n - 2 = \dim^G(R/(Q_1, f)) \leq \dim^G(R/(f)) \leq n - 2,$$

which gives $\dim^G(R/(f)) = \dim^G(R/(Q_1, f))$. This implies (Q_1, f) is contained in some

minimal G-prime of (f), but the minimal G-primes of (f), by Theorem 4.18, must have G-height less than or equal to 1. Therefore, Q_1 is minimal, a contradiction. $\square$

Proposition 4.31. *Let R be a G-Noetherian graded ring, and suppose $\widetilde{G}$ is as in the discussion at the beginning of this section. For any $P \in \operatorname{Spec}^G(R)$, $\operatorname{ht}^G(P) = \operatorname{ht}^{\widetilde{G}}(P[t])$.*

Proof. Since $\operatorname{ht}^G(P) \leq \operatorname{ht}^{\widetilde{G}}(P[t])$ is clear, we only need to show the reverse inequality. Since G-primes of R extend to $\widetilde{G}$-primes of $\widetilde{R}$, one can easily show that the minimal $\widetilde{G}$-primes of $R[t]$ are precisely those extended from minimal G-primes of R, so if $\operatorname{ht}^G(P) = 0$, we are done. If $n := \operatorname{ht}^G(P) > 0$, it then suffices to show that $\operatorname{ht}^G(P/Q) \geq \operatorname{ht}^{\widetilde{G}}((P/Q)[t])$ for all minimal $Q \in \operatorname{Spec}^G(R)$. Thus, we can assume R is a G-domain and

$$P \supsetneq Q_{n-1} \supsetneq \cdots \supsetneq Q_1 \supsetneq (0)$$

is a saturated chain of G-prime ideals. Since R is a G-domain, there exists a homogeneous non-zerodivisor $f \in Q_1$. By Lemma 4.30, $\operatorname{ht}^G(P/(f)) = n-1$, and by induction, $\operatorname{ht}^{\widetilde{G}}(P[t]/(f)[t]) = n-1$. Lemma 4.30 (with $G = \widetilde{G}$) also gives $\operatorname{ht}^{\widetilde{G}}(P[t]) \leq n$ (combining Theorems 2.37 and 2.41 shows $R[t]$ is $\widetilde{G}$-Noetherian). $\square$

In Chapter 5 we will find it convenient to consider $R[t]$ graded still by G, rather than by $\widetilde{G}$. If we set $\deg(t) = g$ for some $g \in G$, the next corollary guarantees that the G-heights of the extended G-primes still remain the same.

Corollary 4.32. *With the setup as above, suppose $P \in \operatorname{Spec}^G(R)$, and consider $R[t]$ as a G-graded ring by setting $\deg(t) = g$ for some $g \in G$. Then $\operatorname{ht}^G(P) = \operatorname{ht}^G(P[t])$.*

Proof. Since we know that $\operatorname{ht}^G(P) = \operatorname{ht}^{\widetilde{G}}(P[t])$, we only need to use Proposition 4.23 with $G = \widetilde{G}$ and $H = \langle (g, -1) \rangle$, as $\widetilde{G}/H \cong G$ and the $\widetilde{G}/H$-grading on $R[t]$ is the same as the G-grading. $\square$

Chapter 5

Grade and Cohen-Macaulayness

For a commutative ring R, one common way of defining $\mathrm{grade}_I(R)$, the grade of an ideal $I \subseteq R$, is to let it be the maximum length of an R-regular sequence contained in I. Recall that an element $x \in R$ is regular (on R) if x is a non-zerodivisor on R and $xR \neq R$. A sequence $x_1, \ldots, x_n \in R$ is regular if x_1 is regular on R and x_i is regular on $R/(x_1, \ldots, x_{i-1})R$. Obviously, one would simply like to define grade in a G-graded ring by the following: if I is a homogeneous ideal, set $\mathrm{grade}_I(R)$ to be the maximum length of a regular sequence of homogeneous elements. However, the following example illustrates a problem with that definition:

Example 5.1. Let k be a field, and consider the $\mathbb{Z}$-graded ring $R := k[x, y]/(xy)$, where we set $\deg x = 1$ and $\deg y = 0$. Then $(x, y)R$ has positive grade under the trivial grading (i.e., in the usual sense), but every $\mathbb{Z}$-homogeneous element of $(x, y)R$ is a zerodivisor.

This is essentially the same problem one has when working with grade in non-Noetherian rings: it can happen that a (finitely generated) ideal has annihilator 0, but no single element has annihilator 0. In Example 5.1, the ideal $(x, y)R$ has annihilator 0, but every homogeneous element of $(x, y)R$ has non-zero annihilator. As we will see in Proposition 5.3, this problem

can be rectified by adjoining indeterminates.

Instead we define grade in terms of Čech cohomology, motivated in part by work on non-Noetherian rings (cf. [1], [3], [10], [11]). This approach coincides with the classical notion of grade in the case that the underlying ring is Noetherian, but we will not necessarily assume this. Later, we will assume that R is G-Noetherian, but if G is not finitely generated, there is no guarantee that R will be Noetherian with respect to the trivial grading.

The Čech complex can be defined in the following way. If $f \in R$, $C(f)$ is the cochain complex

$$0 \to R \underset{r \mapsto \frac{r}{1}}{\to} R_f \to 0.$$

For $\mathbf{f} = f_1, \ldots, f_n$, we inductively define $C(\mathbf{f}) := C(f_1, \ldots, f_{n-1}) \otimes_R C(f_n)$. The ith Čech cohomology of R with respect to $\mathbf{f}$ is $H^i_{\mathbf{f}}(R) := H^i(C(\mathbf{f}))$. Notice that if R is G-graded and $\mathbf{f}$ is a sequence of G-homogeneous elements, then the differentials in $C(\mathbf{f})$ have degree 0. Hence $H^i(C(\mathbf{f}))$ is G-graded for each i.

Suppose R is a G-graded ring, and $I = (f_1, \ldots, f_n)$ is a G-homogeneous ideal generated by the homogeneous sequence of elements $\mathbf{f} := f_1, \ldots, f_n$. Define

$$\mathrm{grade}^G_I(R) = \min\{i \mid H^i_{\mathbf{f}}(R) \neq 0\}.$$

Remark 5.2.

1. The modules $H^i_{\mathbf{f}}(R)$ are independent of the generating set $\mathbf{f}$ chosen for I.

2. If I is G-homogeneous, then $\mathrm{grade}^G_I(R) = \mathrm{grade}^{G/H}_I(R)$ for all subgroups H of G. Hence, we often drop the superscript G and simply write $\mathrm{grade}_I(R)$.

3. If $R \to S$ is a faithfully flat ring homomorphism of G-graded rings and I is G-homogeneous, then $\mathrm{grade}_I(R) = \mathrm{grade}_{IS}(S)$.

The advantage of having a homogeneous regular sequence $\mathbf{x} = x_1, \ldots, x_n$ contained in I is that by combining Propositions 2.6 and 2.7 in [10], we get

$$\mathrm{grade}_{I/(\mathbf{x})}(R/(\mathbf{x})) = \mathrm{grade}_I(R) - n.$$

Thus, we can make inductive arguments while still staying in the graded category; this is not possible if there are no homogeneous regular elements. One solution is to create homogeneous elements (in fact, this is mentioned in [14, Remark 1]). In particular:

Proposition 5.3. *If R is a G-graded ring and $I \subsetneq R$ is a finitely generated homogeneous ideal, there exist $d \geq 1$ and indeterminates $t_1, \ldots, t_d$ with $\deg t_i = g_i$ for some $g_i \in G$, $i = 1, \ldots, d$, such that $IR[t_1, \ldots, t_d]$ contains a homogeneous $R[t_1, \ldots, t_d]$-regular sequence of length* $\mathrm{grade}_I(R)$.

Proof. Notice that since the grade is determined by Čech cohomology, and the map $R \to R[t_1, \ldots, t_d]$ is faithfully flat, we have

$$\mathrm{grade}_I(R) = \mathrm{grade}_{IR[t_1,\ldots,t_d]}(R[t_1, \ldots, t_d]).$$

Therefore, if we show that by adding some (finite) number of indeterminates, we can force the existence of one homogeneous regular element, we can iterate. Suppose $\mathrm{grade}_I(R) > 0$. Consider the G-graded ring $R[x]$, where x is an indeterminate with $\deg(x) := 0$. By Chapter 5, Theorem 7 of [17], there exists a non-zerodivisor $g \in IR[x]$. Since $\mathrm{grade}_I(R) = \mathrm{grade}_{IR[x]}(R[x])$, by replacing R with $R[x]$ and I with $IR[x]$, we may assume I contains a non-zerodivisor:

$$g = f_1 + \cdots + f_n \in I,$$

where $f_i \in R_{g_i} \smallsetminus \{0\}$ for some $g_i \in G$. We assume $g_i \neq g_j$ for $i \neq j$ (and presumably that

$n \geq 2$, or else we need not do the following construction, but $n \geq 2$ is irrelevant to the proof). Consider the subgroup of G generated by the g_i, say $K = \langle g_1, \ldots, g_n \rangle$. Then K is a finitely generated abelian group, so we can write

$$K \cong \mathbb{Z}^m \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_r\mathbb{Z},$$

where $m, r \geq 0$. Consider the following ring, where the t_j are indeterminates:

$$R[\mathbf{t}] := R[t_1, \ldots, t_m, t_{m+1}, \ldots, t_{m+r}].$$

For $j = 1, \ldots, m$, we set $\deg(t_j) = e_j$ with $e_i \in K$ the vector of length $m + r$ with a $1 \in \mathbb{Z}$ in the jth position and 0 elsewhere. For $a = 1, \ldots, r$, we set $\deg(t_{m+a}) = \varepsilon_a$, with ε_a the vector of length $m + r$ with a $\overline{1} \in \mathbb{Z}/m_a\mathbb{Z}$ in the $(m + a)$th position and 0 elsewhere. Identifying $\mathbb{Z}^m \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_r\mathbb{Z}$ with $K \leq G$, this ring now has a G-grading.

Each g_i can be identified with

$$\mathbf{k}_i = \sum_{j=1}^{m} k_{i,j} e_j + \sum_{a=1}^{r} \ell_{i,a} \varepsilon_a,$$

where $k_{i,j}, \ell_{i,a} \in \mathbb{Z}$ and we can choose $0 \leq \ell_{i,a} < m_a$ (here $1 \leq j \leq m$ and $1 \leq a \leq r$, and the restriction of the $\ell_{i,a}$ gives a unique choice). Define $N := \max\{|k_{i,j}| \mid 1 \leq i \leq n, 1 \leq j \leq m\}$, and set

$$\mathbf{k} = (\underbrace{N, \ldots, N}_{m}, m_1, \ldots, m_r).$$

For arbitrary $\mathbf{h} = \sum_{j=1}^{m} h_j e_j + \sum_{a=1}^{r} h_{m+a} \varepsilon_a$, define

$$\mathbf{t}^{\mathbf{h}} = t_1^{h_1} t_2^{h_2} \cdots t_{m+r}^{h_{m+r}}.$$

One sees that for each $i = 1, \dots, n$, the term $f_i \mathbf{t}^{\mathbf{k}-\mathbf{k}_i}$ is homogeneous of degree $\mathbf{k}$. Further, by the choice of N and the $\ell_{i,a}$, $\mathbf{k} - \mathbf{k}_i$ has all positive entries so that $f_i \mathbf{t}^{\mathbf{k}-\mathbf{k}_i} \in R[\mathbf{t}]$. And as $f_i \in I$, we have $f_i \mathbf{t}^{\mathbf{k}-\mathbf{k}_i} \in IR[\mathbf{t}]$. Then

$$g(\mathbf{t}) := \sum_{i=1}^{n} f_i \mathbf{t}^{\mathbf{k}-\mathbf{k}_i}$$

is homogeneous of degree $\mathbf{k}$ in $IR[\mathbf{t}]$. All that remains to be shown is that $g(\mathbf{t})$ is $R[\mathbf{t}]$-regular. Indeed, since $IR \neq R$, we have $IR[\mathbf{t}] \neq R[\mathbf{t}]$. Further, $g(\mathbf{t})$ is a zerodivisor if and only if there exists $r \in R \smallsetminus \{0\}$ such that $rg(\mathbf{t}) = 0$, which is if and only if $rf_i = 0$ for $i = 1, \dots, n$. This implies $g = \sum f_i$ is a zerodivisor, contradicting the choice of g, so $g(\mathbf{t})$ is a G-homogeneous non-zerodivisor, as desired. $\square$

If (R, m) is a G-local G-Noetherian ring, we set

$$\operatorname{depth}^G(R) = \operatorname{grade}_m(R).$$

Then, by making minor modifications to the proof of [10, Prop. 2.4] (Proposition 5.4 below), we have that

$$\operatorname{depth}^G(R) \leq \dim^G(R),$$

and we say that R is *G-Cohen-Macaulay* if

$$\operatorname{depth}^G(R) = \dim^G(R).$$

For a graded ring that is not necessarily G-local, we say R is G-Cohen-Macaulay if $R_{(n)}$ is G-Cohen-Macaulay for each G-maximal ideal n of R.

At this point it is worth noting that a G-local ring (R, m) is G-Cohen Macaulay if and only if $H^i_{\mathbf{x}}(R) = 0$ for all $i < \operatorname{ht}^G(m)$, where $m = (\mathbf{x}) = (x_1, \dots, x_r)$. Indeed, the "only if" is

clear. The vanishing of the Čech cohomology forces $\operatorname{depth}^G(m) \geq \dim^G(R) = \operatorname{ht}^G(m)$. Thus, one only needs to see that $\operatorname{depth}^G(R) \leq \dim^G(R)$, but as mentioned above, this is just a generalization of [10, Prop. 2.4]:

Proposition 5.4 (Prop. 2.4 in [10])**.** *Let R be a G-graded ring with $\dim^G(R) = d < \infty$ and $\mathbf{x} := x_1, \ldots, x_n$ a sequence of elements from R. Then $H^i_{\mathbf{x}}(R) = 0$ for all $i > d$.*

Proof. Use induction on d, and note that we may assume (R, m) is G-local and $(\mathbf{x})R \subseteq m$. Indeed, $H^i_{\mathbf{x}}(R) = 0$ if and only if

$$H^i_{\mathbf{x}}(R)_{(m)} = H^i_{\frac{\mathbf{x}}{1}}(R_{(m)}) = 0,$$

and if $(\mathbf{x})R \nsubseteq m$, then (d) of Proposition 2.1 in [10] implies a unit annihilates $H^i_{\frac{\mathbf{x}}{1}}(R_{(m)})$. If $d = 0$, then $\sqrt[G]{I} = m$, so every element of m is nilpotent. Thus $H^i_{\mathbf{x}}(R) = 0$ for $i > 0$.

Suppose $d > 0$ and that the result holds for rings of dimension less than d. If $n \leq d$, then there is nothing to show. Suppose $n > d$ and let j be the largest integer such that $H^j_{\mathbf{x}}(R) \neq 0$; we may assume $j > d$. Set $\mathbf{x}' = x_1, \ldots, x_{n-1}$, so that by induction on n, $H^j_{\mathbf{x}'}(R) = 0$. From the exact sequence

$$\cdots \to H^{j-1}_{\mathbf{x}'}(R) \to H^{j-1}_{\mathbf{x}'}(R)_{x_n} \to H^j_{\mathbf{x}}(R) \to 0,$$

we obtain $H^{j-1}_{\mathbf{x}'}(R_{x_n}) \neq 0$. As R is G-local with $\dim^G(R) = d$, $\dim^G(R_{x_n}) \leq d - 1$, and so by induction on d, $j - 1 \leq d - 1$, a contradiction. □

Now we introduce a useful construction. Suppose (R, m) is a G-local G-Noetherian ring. By Proposition 5.3 above, we can adjoin a finite number of indeterminates $\mathbf{t} = t_1, \ldots, t_d$, so that

$$\operatorname{grade}_{mR[\mathbf{t}]}(R[\mathbf{t}]) = \operatorname{depth}^G(R),$$

and by assigning the proper degrees to the t_i (as described in the proof) we get a homogeneous

regular sequence of length $\operatorname{depth}^G(R)$ as well. But now $R[\mathbf{t}]$ is no longer local. Using Corollary 4.32, we know that $\operatorname{ht}^G(m) = \operatorname{ht}^G(mR[\mathbf{t}])$. Setting $\widetilde{R} = R[\mathbf{t}]_{(mR[\mathbf{t}])}$ and $\widetilde{m} = m\widetilde{R}$, we then have that $(\widetilde{R}, \widetilde{m})$ is a G-local ring with the same G-graded dimension as R. In fact, since $\widetilde{m} = m\widetilde{R}$ and the extension $R \to \widetilde{R}$ is faithfully flat, $\operatorname{depth}^G(R) = \operatorname{depth}^G(\widetilde{R})$, so that R is G-Cohen-Macaulay if and only if $\widetilde{R}$ is, the advantage being that $\widetilde{m}$ now contains a G-homogeneous regular sequence of length $\operatorname{depth}^G(\widetilde{R})$. This is useful in the proof of the next proposition and Proposition 5.7, which are basic results in the study of Cohen-Macaulay rings. Comparing to the proofs in [5], we see that we can sometimes follow the same basic idea if we can reduce to the case where homogeneous regular elements exist.

Proposition 5.5. *Let (R, m) be a G-local, G-Noetherian, and G-Cohen-Macaulay ring. Then*

$$\dim^G(R/P) = \dim^G(R)$$

for all $P \in \operatorname{Ass}^G(R)$.

Proof. We obviously only need to show that $\dim^G(R) \leq \dim^G(R/P)$ for all $P \in \operatorname{Ass}^G(R)$. Because R is G-Cohen-Macaulay, it suffices to show that $\operatorname{depth}^G(R) \leq \dim^G(R/P)$. If $\operatorname{depth}^G(R) = 0$, there is nothing to show, so assume $\operatorname{depth}^G(R) > 0$. We first prove the result in the case where there exists a homogeneous regular element $x \in m$. Setting $\overline{R} := R/(x)$, we have that $R/(x)$ is G-Cohen-Macaulay. By induction

$$\operatorname{depth}^G(\overline{R}) \leq \dim^G(\overline{R}/\overline{Q})$$

for all $\overline{Q} \in \operatorname{Ass}^G(\overline{R})$. It suffices to show that for each $P \in \operatorname{Ass}^G(R)$, there exists $\overline{Q} \in \operatorname{Ass}^G(\overline{R})$ such that $\dim^G(R/P) > \dim^G(\overline{R}/\overline{Q})$, because then we have

$$\dim^G(R/P) > \operatorname{depth}^G(\overline{R}) = \operatorname{depth}^G(R) - 1.$$

Given $P \in \mathrm{Ass}^G(R)$, choose $(z) \subseteq R$ that is maximal among the homogeneous principal ideals annihilated by P. Suppose $z \in (x)$. Then $z = xy$ for some $y \in R$, and $Py = 0$, because x is a non-zerodivisor. Moreover, we claim that $(z) \subsetneq (y)$. Indeed, if $y = az$ for some $a \in R$, then we can multiply by x to get $z = axz$; note here that a, x, and z are all homogeneous, so that $\deg(ax) = 0$. This gives $z(1 - ax) = 0$, but $1 - ax$ is a (homogeneous) unit, so $z = 0$ (this is a contradiction provided $P \neq 0$, and there is nothing to show if $P = 0$). Hence $(z) \subsetneq (y)$. But (z) was chosen to be maximal, and so we must have $z \notin (x)$. This implies that in the quotient ring $\overline{R}$, the ideal $(P + (x))/(x)$ consists of zerodivisors on $\overline{R}$; in fact, this ideal is contained in $(0 :_{\overline{R}} \overline{z})$. Thus $(P + (x))/(x) \subseteq \overline{Q}$ for some $\overline{Q} \in \mathrm{Ass}^G(\overline{R})$. In R, the fact that $x \notin P$ implies that $P \subsetneq Q$, so we have

$$\dim^G(R/P) > \dim^G(R/Q) = \dim^G(\overline{R}/\overline{Q}).$$

If $\mathrm{depth}^G(R) > 0$, but there does not exist a homogeneous regular element in m, then we form the ring $(\widetilde{R}, \widetilde{m})$ described above so that there is a homogeneous regular sequence in $\widetilde{m}$ of length $\mathrm{depth}^G(R)$. One first needs to show that the extension of the associated G-primes of R are precisely the associated G-primes of $\widetilde{R}$. This is straightforward using the characterization that associated G-primes of R are precisely G-prime ideals of the form $\mathrm{Ann}_R(f)$ for G-homogeneous elements $f \in R$. Using this and the fact that

$$\dim^G(R) = \dim^G(\widetilde{R}) = \dim^G(\widetilde{R}/P\widetilde{R})$$

for any $P \in \mathrm{Ass}^G(R)$ (note that the second equality follows from the previous argument, as $\mathrm{grade}_m(\widetilde{R}) > 0$, so m contains a homogeneous non-zerodivisor), we only need to show that $\dim^G(R/P) = \dim^G(\widetilde{R}/P\widetilde{R})$. This last equality follows from the construction of $\widetilde{R}$: Because $\widetilde{R}$ is constructed by adjoining variables and then localizing, we have an isomorphism

$\widetilde{R}/P\widetilde{R} \cong \widetilde{(R/P)}$, and the equality follows from the fact that the operation $\widetilde{(-)}$ preserves the dimension of the original ring. $\square$

The next lemma is a basic result, and it has a clean proof using the Čech complex.

Lemma 5.6. *For a G-graded G-Noetherian ring R and a G-homogeneous ideal $I \subseteq R$, we have* $\operatorname{grade}_I(R) = 0$ *if and only if $I \subseteq P$ for some $P \in \operatorname{Ass}^G(R)$. In particular,* $\operatorname{depth}^G(R) = 0$ *if and only if $m \in \operatorname{Ass}^G(R)$.*

Proof. Suppose $I \subseteq P$ for some $P \in \operatorname{Ass}^G(R)$ and I is generated by the (homogeneous) sequence $\mathbf{a} = a_1, \ldots, a_k$. Since $P = (0 : r)$ for some homogeneous $r \in R$, the first map in the Čech complex

$$0 \to R \to \bigoplus_{i=1}^{k} R_{a_i} \to \cdots \to R_{a_1 \cdots a_k} \to 0$$

is not injective, as each a_i is a zerodivisor. It follows that $H^0_{\mathbf{a}}(R) \neq 0$.

To see the other implication, let $\mathbf{a}$ be as before. Then $H^0_{\mathbf{a}}(R) \neq 0$ implies that there exist a homogeneous $r \in R \setminus \{0\}$ and an ℓ such that $ra_i^\ell = 0$ for each i. This gives $(0 : (a_1^\ell, \ldots, a_n^\ell)) \neq 0$, so that $(0 : I^n) \neq 0$ for some n. Then $I^n \subseteq (0 : r)$, and $(0 : r)$ is contained in an associated G-prime. The second statement now follows easily. $\square$

The final result we establish before we undertake the real work in proving the main theorem in this chapter is the familiar localization of the Cohen-Macaulay property.

Proposition 5.7. *Suppose R is G-Noetherian. Then R is G-Cohen-Macaulay if and only if $R_{(P)}$ is G-Cohen-Macaulay for all $P \in \operatorname{Spec}^G(R)$.*

Proof. Clearly, we only need to show the "only if" direction. If R is G-Cohen-Macaulay, we may assume (R, m) is G-local: each ring $R_{(P)}$ is a further localization of $R_{(m)}$ for some G-maximal ideal m of R. We use induction on $\operatorname{depth}^G(R_{(P)})$ to show $R_{(P)}$ is G-Cohen-Macaulay. Suppose $\operatorname{depth}^G(R_{(P)}) = 0$. Then $PR_{(P)} \in \operatorname{Ass}^G(R_{(P)})$, but this is if and only if $P \in \operatorname{Ass}^G(R)$.

By Proposition 5.5,

$$\dim^G(R/P) = \dim^G(R),$$

whence $\dim^G(R_{(P)}) = 0$.

Suppose $\operatorname{depth}^G(R_{(P)}) > 0$. We want to show $\operatorname{grade}_P(R) > 0$. Suppose, by way of contradiction, that $\operatorname{grade}_P(R) = 0$. The previous lemma implies that $P \subseteq Q$ for some $Q \in \operatorname{Ass}^G(R)$. Since R is G-local and G-Cohen-Macaulay, Proposition 5.5 implies Q is a minimal prime, so that $P \in \operatorname{Ass}^G(R)$. Certainly, if $P \in \operatorname{Ass}^G(R)$ then $PR_{(P)} \in \operatorname{Ass}^G(R_{(P)})$, giving $\operatorname{depth}^G(R_{(P)}) = 0$. Therefore, we must have $\operatorname{grade}_P(R) > 0$.

Form the ring $(\widetilde{R}, \widetilde{m})$ above, so $P\widetilde{R}$ contains a regular sequence of length $\operatorname{grade}_P(R) > 0$. By construction, $\widetilde{R}$ is G-Cohen-Macaulay. We also know that

$$\dim^G(R_{(P)}) = \operatorname{ht}^G(P) = \operatorname{ht}^G(P\widetilde{R}) = \dim^G(\widetilde{R}_{(P\widetilde{R})}),$$

and to see that $\operatorname{depth}^G(R_{(P)}) = \operatorname{depth}^G(\widetilde{R}_{(P\widetilde{R})})$, we only have to note that $\widetilde{R}_{(P\widetilde{R})}$ is a faithfully flat $R_{(P)}$-algebra. Replacing $\widetilde{R}$ by R and $P\widetilde{R}$ by P, we may assume now that P contains a homogeneous regular element x.

The ring $R/(x)$ is also G-Cohen-Macaulay, and because we have $\operatorname{depth}^G(R_{(P)}/xR_{(P)}) = \operatorname{depth}^G(R_{(P)}) - 1$, induction implies

$$\operatorname{depth}^G(R_{(P)}) - 1 = \dim^G(R_{(P)}) - 1,$$

so that $R_{(P)}$ is G-Cohen-Macaulay. $\square$

The following lemma (with $G = H = \mathbb{Z}$) is well-known for $\mathbb{Z}$-graded rings, and it was shown recently for $\mathbb{Z}^d$-graded rings [18]. It is integral in the proof of the next theorem, and the proof requires substantially different tools, since the underlying ring is not necessarily Noetherian.

Lemma 5.8. *Let R be a G-Noetherian graded ring, suppose H is a finitely generated torsion-free subgroup of G, suppose $P \in \operatorname{Spec}^{G/H}(R)$, and set $P^* := P^{*G}$. Then $R_{(P)}$ is G/H-Cohen-Macaulay if and only if $R_{(P^*)}$ is G/H-Cohen-Macaulay (where the homogeneous localizations are with respect to the G/H-grading).*

Proof. For the "only if" direction, note that $R_{(P^*)}$ is just a further localization of $R_{(P)}$, so by Proposition 5.7, $R_{(P^*)}$ is G/H-Cohen-Macaulay.

For the other implication, we may assume that $P \notin \operatorname{Spec}^G(R)$, or there is nothing to show. Suppose $\mathbf{x}'$ is a G-homogeneous sequence of elements that generates P^* and $\mathbf{x}$ is the extension of $\mathbf{x}'$ to a G/H-homogeneous generating set of P. If $R_{(P^*)}$ is G/H-Cohen-Macaulay, then

$$0 = H^i_{\mathbf{x}' R_{(P^*)}}(R_{(P^*)}) \cong H^i_{\mathbf{x}'}(R)_{(P^*)}$$

for all $i < \operatorname{ht}^{G/H}(P^*) = \operatorname{ht}^G(P^*)$. We want to show that $H^i_{\mathbf{x}}(R)_{(P)} = 0$ for all $i < \operatorname{ht}^{G/H}(P)$.

Reset notation: Suppose (R, m) is a G-Noetherian G-local graded ring. Let $P \in \operatorname{Spec}^{G/H}(R)$ such that $P^* = m \subsetneq P$, and assume $R_{(m)}$ is G/H-Cohen-Macaulay. We want to show that $R_{(P)}$ is G/H-Cohen-Macaulay. Let $\mathbf{x}'$ and $\mathbf{x}$ be as above; that is, $\mathbf{x}'$ generates m and $\mathbf{x}$ is the extension to a generating set for P. We use induction on the rank of H. Suppose $\operatorname{rank}(H) = 1$ (i.e, $H \cong \mathbb{Z}$).

Define S to be the multiplicatively closed set of G/H-homogeneous elements of R not in m. By Remark 2.27 and the fact that $H^i_{\mathbf{x}'}(R)$ is G-graded over the G-local ring (R, m),

$$H^i_{\mathbf{x}'}(R)_{(m)} = 0 \text{ if and only if } H^i_{\mathbf{x}'}(R) = 0,$$

This means the condition $H^i_{\mathbf{x}'}(R)_{(m)} = 0$ for $i < \operatorname{ht}^{G/H}(m)$ is equivalent to $H^i_{\mathbf{x}'}(R) = 0$ for

$i < \mathrm{ht}^{G/H}(m) = \mathrm{ht}^G(m)$. Because H has rank 1 and $P \notin \mathrm{Spec}^G(R)$, we have

$$\mathrm{ht}^{G/H}(P) = \mathrm{ht}^{G/H}(m) + 1 = \mathrm{ht}^G(m) + 1.$$

So now we just need to show that $H^i_{\mathbf{x}'}(R) = 0$ for $i < \mathrm{ht}^G(m)$ implies that $H^i_{\mathbf{x}}(R)_{(P)} = 0$ for $i < \mathrm{ht}^G(m) + 1$. By Corollary 2.15, we know that $P = (\mathbf{x}', f)$ for some $f \in P \smallsetminus m$ that is G/H-homogeneous in R (that is, we can explicitly take $\mathbf{x} = \mathbf{x}', f$). Therefore, we have a long exact sequence in cohomology:

$$\cdots \to H^i_{\mathbf{x}}(R) \to H^i_{\mathbf{x}'}(R) \to H^i_{\mathbf{x}'}(R)_f \to H^{i+1}_{\mathbf{x}}(R) \to \cdots$$

and we know that $H^i_{\mathbf{x}'}(R) = 0 = H^i_{\mathbf{x}'}(R)_f$ for $i < h := \mathrm{ht}^G(m)$. This gives an exact sequence

$$0 \to H^h_{\mathbf{x}}(R) \to H^h_{\mathbf{x}'}(R) \xrightarrow{\phi} H^h_{\mathbf{x}'}(R)_f \to H^{h+1}_{\mathbf{x}}(R) \to 0.$$

We want to show that $H^h_{\mathbf{x}}(R) = 0$, so it suffices to show ϕ is injective. Since ϕ is (induced by) the natural localization map (up to a sign), we have $\phi(x) = 0$ if and only if $xf^n = 0$ for some $n \in \mathbb{N}$. But this implies $f^n \in \mathrm{Ann}(x) \subseteq m$ (since x is G-homogeneous), and m is G/H-prime in R, so $f \in m$, a contradiction. Therefore ϕ is injective and $H^h_{\mathbf{x}}(R) = 0$, which implies $H^i_{\mathbf{x}}(R) = 0$ for $i < h + 1 = \mathrm{ht}^{G/H}(P)$. In particular, $H^i_{\mathbf{x}}(R)_{(P)} = 0$ for $i < \mathrm{ht}^{G/H}(P)$.

If $\mathrm{rank}(H) > 1$, write $H = A \oplus B$, where A and B are also torsion-free with $\mathrm{rank}(A) = 1$ and $\mathrm{rank}(B) = \mathrm{rank}(H) - 1$. Now, H/A is a finitely generated torsion-free subgroup of G/A of rank less than $\mathrm{rank}(H)$, so induction (and the fact that $(G/A)/(H/A) \cong G/H$) implies that $R_{(P)}$ is G/H-Cohen-Macaulay if and only if $R_{(P^{*(G/A)})}$ is G/H-Cohen-Macaulay. Notice that $P^{*(G/A)} \in \mathrm{Spec}^{G/H}(R)$. Applying induction again, this time to H/B as a subgroup of G/B, we get $R_{(P^{*(G/A)})}$ is G/H-Cohen-Macaulay if and only if $R_{((P^{*(G/A)})^{*(G/B)})}$ is G/H-Cohen-Macaulay. By Proposition 2.26, $(P^{*(G/A)})^{*(G/B)} = P^{*G} = P^*$, so we are done. $\square$

We now prove the main theorem of Chapter 5. In some sense it is an analogue to a conjecture of Nagata from [15]: for an $\mathbb{N}$-graded ring, is knowing R_m is Cohen-Macaulay for all homogeneous maximal ideals sufficient to imply R is Cohen-Macaulay? This was answered affirmatively (for $\mathbb{Z}$-graded rings) in [14]. We recover this result by setting $G = H = \mathbb{Z}$ in the following theorem.

Theorem 5.9. *Let R be a G-Noetherian graded ring, and suppose H is a finitely generated torsion-free subgroup of G. The following are equivalent:*

1. *R is G-Cohen-Macaulay.*

2. *R is G/H-Cohen-Macaulay.*

Proof. (2) $\implies$ (1): Suppose R is G/H-Cohen-Macaulay. If $P \in \operatorname{Spec}^G(R)$, then because H is torsion-free, we have $P \in \operatorname{Spec}^{G/H}(R)$, and $R_{(P)}$ is G/H-Cohen-Macaulay, where the (P) denotes the set of G/H-homogeneous elements of R not in P. Let $P = (\mathbf{x})$ for some (finite) G-homogeneous generating set, and let W be the set of G-homogeneous elements of R not in P. To see that R_W is G-Cohen-Macaulay, it is enough to show that $H^i_{\mathbf{x}}(R)_W = 0$ for all $i < \operatorname{ht}^G(P) = \operatorname{ht}^{G/H}(P)$. We already know that $H^i_{\mathbf{x}}(R)_{(P)} = 0$ for all $i < \operatorname{ht}^{G/H}(P)$, and because $H^i_{\mathbf{x}}(R)$ is G-graded, Remark 2.27 gives

$$H^i_{\mathbf{x}}(R)_{(P)} = 0 \text{ if and only if } H^i_{\mathbf{x}}(R)_W = 0.$$

Therefore $R_{(P)}$ being G/H-Cohen-Macaulay implies R_W is G-Cohen-Macaulay.

(1) $\implies$ (2): Throughout, $R_{(-)}$ denotes G/H-homogeneous localization at a G/H-prime ideal. From Proposition 5.7 we know that R is G/H-Cohen-Macaulay if and only if $R_{(P)}$ is G/H-Cohen-Macaulay for all $P \in \operatorname{Spec}^{G/H}(R)$. And by the previous lemma, $R_{(P)}$ is G/H-Cohen-Macaulay if and only if $R_{(P^*)}$ is G/H-Cohen-Macaulay, where $P^* := P^{*G}$. Further, since R is G-Cohen-Macaulay, we know R_S is G-Cohen-Macaulay, where S is the set of

G-homogeneous elements of R not in P^*. Finally, because $R_{(P^*)}$ is just a further localization of R_S, we can reset notation and assume that (R, m) is G-local and G-Cohen-Macaulay, and we want to show that $R_{(m)}$ is G/H-Cohen-Macaulay.

Since R is G-Cohen-Macaulay, we know $\dim^G(R) = \operatorname{depth}^G(R)$. But because R is G-Noetherian and H is finitely generated and torsion-free,

$$\dim^G(R) = \operatorname{ht}^G(m) = \operatorname{ht}^{G/H}(m) = \dim^{G/H}(R_{(m)}).$$

Therefore, it will suffice to show that $\operatorname{depth}^G(R) = \operatorname{depth}^{G/H}(R_{(m)})$, or equivalently, that

$$\operatorname{grade}_m(R) = \operatorname{grade}_{mR_{(m)}}(R_{(m)}),$$

but this is really just another application of Remark 2.27. $\square$

Bibliography

[1] Bernard Alfonsi. Grade non-noethérien. *Comm. Algebra*, 9(8):811–840, 1981.

[2] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[3] S. Floyd Barger. A theory of grade for commutative rings. *Proc. Amer. Math. Soc.*, 36:365–368, 1972.

[4] Nicolas Bourbaki. *Algebra I. Chapters 1–3*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation [MR0979982 (90d:00002)].

[5] Winfried Bruns and H. Jürgen Herzog. *Cohen-Macaulay Rings (Cambridge Studies in Advanced Mathematics)*. Cambridge University Press, 1998.

[6] David Eisenbud. *Commutative algebra with a view toward algebraic geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[7] Shiro Goto and Keiichi Watanabe. On graded rings. I. *J. Math. Soc. Japan*, 30(2):179–213, 1978.

[8] Shiro Goto and Keiichi Watanabe. On graded rings. II. (Z^n-graded rings). *Tokyo J. Math.*, 1(2):237–261, 1978.

[9] Shiro Goto and Kikumichi Yamagishi. Finite generation of Noetherian graded rings. *Proc. Amer. Math. Soc.*, 89(1):41–44, 1983.

[10] Tracy Dawn Hamilton and Thomas Marley. Non-Noetherian Cohen-Macaulay rings. *J. Algebra*, 307(1):343–360, 2007.

[11] M. Hochster. Grade-sensitive modules and perfect modules. *Proc. London Math. Soc. (3)*, 29:55–76, 1974.

[12] Yuji Kamoi. Noetherian rings graded by an abelian group. *Tokyo J. Math.*, 18(1):31–48, 1995.

[13] F. W. Levi. Ordered groups. *Proc. Indian Acad. Sci., Sect. A.*, 16:256–263, 1942.

[14] Jacob Matijevic and Paul Roberts. A conjecture of Nagata on graded Cohen-Macaulay rings. *J. Math. Kyoto Univ.*, 14:125–128, 1974.

[15] Masayoshi Nagata. Some questions on Cohen-Macaulay rings. *J. Math. Kyoto Univ.*, 13:123–128, 1973.

[16] Constantin Năstăsescu and Freddy Van Oystaeyen. *Methods of graded rings*, volume 1836 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2004.

[17] D. G. Northcott. *Finite free resolutions*. Cambridge University Press, Cambridge, 1976. Cambridge Tracts in Mathematics, No. 71.

[18] Jan Uliczka. A note on the dimension theory of $\mathbb{Z}^n$-graded rings. *Comm. Algebra*, 37(10):3401–3409, 2009.